PG/IIIS/MATH/301/23
(New & Old)

M.Sc. 3rd Semester Examination, 2023

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

(*Partial Differential Equations and Generalized Functions*)

PAPER – MTM-301(New & Old)

Full Marks : 50

Time : 2 hours

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

1. Answer any *four* questions from the following : 2×4

(*a*) What are the main differences between an ODE and PDE ?

(*Turn Over*)

(*b*) Show by using weak maximum and weak minimum principle that the Dirichlet problem for the Poisson's equation has unique solution.

(*c*) Define characteristic curve and base characteristics of a first order quasi linear PDE.

(*d*) Check the validity of the maximum principle for the harmonic function

$$u(x,y)=\frac{1-x^2-y^2}{1-2x+x^2+y^2}$$

in the disk $\bar{D}=\{(x,y):x^2+y^2\leq 1\}$. Explain.

(*e*) Define Dirac delta function.

(*f*) Find the solution of

$$p^2y(1+x^2)=qx^2.$$

2. Answer any *four* questions from the following : 4×4

(*a*) Solve the PDE $u_x + 3u_y = u$ subject to Cauchy condition $u = \cos x$ on the line $y = \alpha x$. Find the value of α for which the method fails.

(*b*) Let u be a harmonic function on the whole plane such that $u = 3\sin 2\theta + 1$ on the circle $x^2 + y^2 = 2$. Without finding the concrete form of the solution, find the value of u at the origin.

(*c*) Let $u \in C^2(D)$ be harmonic in D. Show that u satisfies the mean value property in D.

(*d*) Solve the problem

$$u_{tt} - u_{xx} = xt, -\infty < x < \infty, t > 0$$
$$u(x,0) = 0, -\infty < x < \infty$$
$$u_t(x,0) = e^x, -\infty < x < \infty.$$

(*e*) Using the method of separation of variables find a formal solution of the problem

$$u_{tt} = u_{xx}, 0 < x < \pi, t > 0,$$
$$u(0, t) = u(\pi, t) = 0, t \geq 0,$$
$$u(x, 0) = \sin^3 x, 0 \leq x \leq \pi,$$
$$u_t(x, 0) = \sin 2x, 0 \leq x \leq \pi.$$

(*f*) Show that the Green function for the Laplace equation is symmetric.

3. Answer any *two* questions from the following : 8 × 2

(*a*) (*i*) Solve the following :

$$(D^2 + 5DD' + D'^2)z = 0$$

where $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$. 4

(*ii*) Let $\Omega \subseteq \mathbb{R}^2$ be given by

$$\Omega = \{(x, y) \in \mathbb{R}^2 : y > 0\}.$$

Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a harmonic function in Ω. Further assume than u is bounded above in $\bar{\Omega}$. Prove that

$$\operatorname*{Sup}_{\bar{\Omega}} u = \operatorname*{Sup}_{\partial\Omega} u.$$ 4

(*b*) (*i*) State and prove the strong maximum principle. 5

(*ii*) Establish the Laplace equation in polar coordinates. 3

(*c*) (*i*) Prove that

$$\delta(at) = \frac{1}{a}\delta(t), a > 0.$$

Symbols have their usual meanings. 3

(*ii*) Solve the following problem :

$$u_t = 12u_{xx}, 0 < x < \pi, t > 0$$
$$u_x(0,t) = u_x(\pi,t) = 0, t \geq 0$$
$$u(x,0) = 1 + \sin^3 x, 0 \leq x \leq \pi.$$
5

(*d*) Consider the equation

$$u_{xx} + 4u_{xy} + u_x = 0.$$

(*i*) Bring the equation to a canonical form. 4

(*ii*) Find the general solution $u(x, y)$ of the equation. 2

(*iii*) Find the solution $u(x, y)$ which satisfies $u(x, 8x) = 0$ and $u_x(x, 8x) = 4\,e^{-2x}$ for all $x \in \mathbb{R}$. 2

[Internal Assessment — 10 Marks]
