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M.Sc. Part-I Examination, 2013

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER – II

Full Marks : 100

Time : 4 hours

The figures in the right-hand margin indicate marks

GROUP – A

(*Algebra*)

[*Marks* : 50]

Answer **Q. No. 6** and any **three** from the rest

1. (*a*) Define Kernel of a group homomorphism. If f be a homomorphism from the group G to G', show that Kerf is a normal subgroup of G. 1 + 4

(*Turn Over*)

(*b*) Define duality of a lattice. Show that the dual of a lattice is a lattice. 1 + 4

(*c*) Define digraph, in degree and out degree of a graph. Find the adjacency matrix of a complete bipartite graph $K_{3,3}$. 2 + 3

2. (*a*) The inner automorphisms of any group *G* form a normal subgroup of the group of all automorphism of *G*. 5

(*b*) Derive the necessary condition for a simple connected planar graph. Show that simple planar graph has at least a vertex of degree 5 or less. 3 + 2

(*c*) Let $f: R \to S$ and $f': R \to S'$ be two epimorphisms of ring with the same domain *R* and kernel *H*. Show that there exists a isomorphism from the ring *S* to S'. 5

3. (*a*) Define minimal spanning tree. Write an algorithm to find a minimal spanning tree of a graph. 1 + 4

(*b*) Let *H* be a normal subgroup of a group *G* and *G*/*H* is the set of all cosets of *H* in *G*. Show that $(G/H, *)$ is a group, where $*$ defines on *G*/*H* by $(aH) * (bH) = abH$. 5

(*c*) Define and explain with example : Simple graph and multi-graph. Show that the minimum number of edges in a simple graph with *n* vertices is $\frac{n(n-1)}{2}$. $\left(1\frac{1}{2}+1\frac{1}{2}\right)+2$

4. (*a*) Let *G* be a finite group and suppose that *p* is a prime number such that $p/o(G)$. Show that there exists $x \in G$ such that $o(x) = p$. 8

(*b*) Define forest. How many edges are there in a forest with *n* vertices and *K* components. 1 + 2

(*c*) Show that a commutative ring with unity has no proper ideals if and only if it is a field. 4

5. (*a*) Show that in any graph sum of degrees of all vertices is equal to twice the number of edges. Deduce from this result that the number of odd degree vertices is always even. $\left(2\frac{1}{2}+2\frac{1}{2}\right)$

(*b*) Let D be the set of all divisers of 12. Show that D is a poset with respect to the relation $\leq$, where $a \leq b$ means a divides b. Draw Hasse diagram of $(D, \leq)$. 3 + 2

(*c*) Define bipartite graph. Show that if a bipartite graph has any circuits, they must be of all even length. 1 + 4

6. Answer any *one* question : 5 × 1

(*a*) Show that any connected graph G is an Eulerian graph if and only if all vertices of G are of even degree. 5

(*b*) Show that any group of order p^a has non-trivial center, where p is a prime number and a is an integer. 5

GROUP – B

(*Functional Analysis*)

[*Marks* : 50]

Answer **Q. No. 7** and any **three** from the rest

7. Answer any *one* : 2 × 1

(*a*) Let H be a Hilbert space and y be a fixed element of H. Define $f(x)$ as $f(x) = <x, y>$, $x \in H$. Find $\|f\|$.

(*b*) Is the set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 9\}$ compact in $\mathbb{R}^3$?

8. (*a*) Define a complete metric space. Show that $C[a, b]$ is a complete metric space w.r.t. the metric

$$d(f, g) = \max_{x \in [a, b]} |f(x) - g(x)| \text{ for } f, g, \in c[a, b].$$ 8

(*b*) Let X be an infinite set and d be the discrete metric defined on X. Show that X is closed and bounded but not compact. 6

(*c*) Define a no where dense set. Give an example. 2

9. (*a*) State and prove Banach fixed-point theorem. 7

(*b*) If $T : X \to X$ is a contraction, show that $T^n (n \in \mathbb{N})$ is a contraction, where X is a metric space. Also, if T^n is a contraction for $n > 1$, show by an example that T need not be a contraction. 5

(*c*) Consider $X = \{x \mid 1 \leq x \leq \infty\}$, taken with the usual metric of the real line, and $T : X \to X$ be defined by $T(x) = x + \frac{1}{x}$. Show that $|Tx - Ty| < |x - y|$ when $x \neq y$, but the mapping has no fixed points. 4

10. (*a*) Show that every normed space is a metric space. Give an example to show that the converse is not necessarily true. 5

(*b*) Define a bounded linear operator and norm of a bounded linear operator between two normed spaces. Also, give an example of a linear operator which is not bounded. 5

(*c*) Let $T : X \to Y$ be a linear operator where X and Y are normed spaces. Then prove that T is continuous if and only if T is bounded. 6

11. (*a*) Prove that if W is a Banach space, then so is $B(V, W)$, where V is a normed space. 6

(*b*) If $f(x) = f(y)$ for every bounded linear functional f on a normed space X, show that $x = y$. 4

(*c*) If $\{e_1, e_2, ..., e_n\}$ is a finite orthonormal set in an inner product space X and $x \in X$ then prove that

$$\sum_{i=1}^{n} |< x, e_i >|^2 \leq \|x\|^2 \text{ and } \left(x - \sum_{i=1}^{n} < x, e_i > e_i \right)$$

is orthogonal to l_j for all $j = 1, 2, 3, ..., n$. 6

12. (*a*) State and prove Riesz representation theorem. 8

(*b*) Let H be a Hilbert space. Show that $T \in B(H)$ is self-adjoint if $< Tx, x > \in \mathbb{R},\ \forall x \in H$. 3

(*c*) Let X be an inner product space ; $A, B \subset X$. Then prove the following : 5

(*i*) $A \subset B \Rightarrow B^{\perp} \subset A^{\perp}$

(*ii*) $A \subset A^{\perp\perp}$

(*iii*) $A^{\perp} = A^{\perp\perp\perp}$.
