M.Sc. Part-I Examination, 2013

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER – I

Full Marks : 100

Time : 4 hours

The figures in the right hand margin indicate marks

GROUP – A

(*Real Analysis*)

[*Marks* : 40]

Answer **Q.No. 1** and any **three** from **Q.No. 2** to **Q.No. 6**

1. Answer any *one* of the following : 1 × 1

(*a*) Define outer measure for any bounded subset of $[a, b]$.

(*b*) Define a function of bounded variation defined on a closed interval.

2. (*a*) (*i*) Show that the set of all functions of bounded variation on $[a, b]$ form a real vector space. 3

(*ii*) Give an example of a function which is not continuous on $[0, 3]$ but is a function of bounded variation in $[0, 3]$. 4

(*b*) Let $f(x) = x^2 - 2x + 2, x \in [0, 2]$. Show that f is a function of bounded variation on $[0, 2]$. Find the variation function V on $[0, 2]$. Also, express f as the difference of two monotone functions on $[0, 2]$. 6

3. (*a*) (*i*) Show that the function

$$f(x)=\begin{cases} x\sin\dfrac{3\pi}{4x} & , \quad x\neq 0 \\ 0 & , \quad x\neq 0 \end{cases}$$

is continuous but not of bounded variation over the interval $[0, 1]$. 4

(*ii*) If f is continuous and α is of bounded variation on (a, b), then show that $\int_a^b f dx$ exists. 3

(*b*) Let $f(x) = e^x + 3x + 2,\ x \in [-4, 5]$ and

$$\alpha(x)=\begin{cases} 0, & -2\leq x<0 \\ \dfrac{1}{2}, & x=0 \\ 1 & 0<x\leq 5 \end{cases}$$

Show that f is R-S integrable w.r.t. α over $[-4, 5]$. Find $\int_{-4}^{5} f d\alpha$. 6

4. (*a*) (*i*) If $A_n; n = 1, 2, 3, ...$ are measurable subsets of $[a, b]$ and if $A_n \subset A_{n+1}$ for all $n \in N$ then $\bigcup_{n=1}^{\infty} A_n$ is measurable and

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n).$$ 4

(*ii*) Prove that any countable set of $\mathbb{R}$ is measurable and its measure is 0. 4

(*b*) How to construct Cantor set ? Also, find its measure. 5

5. (*a*) If $f(x)$ is monotonic increasing and $g(x)$ is continuous on $[a, b]$ then prove that there exists a number $\xi \in [a, b]$ such that

$$\int_a^b f(x)dg(x) = f(a)\int_a^{\xi} dg(x) + f(b)\int_{\xi}^b dg(x).$$ 7

(*b*) Evaluate the *RS*-integral

$$\int_0^4 (3x^2 + 6x - 4)d[3x + 2[x] - 4].$$ 6

6. (*a*) (*i*) State the following theorem : Lebesgue Dominated Convergence Theorem, Monotone Convergence Theorem. 4

(*ii*) Let f be a real measurable function on $[a, b]$. Then show that the set $\{x : f(x) \geq \alpha\}$ is a measurable set for every real α. 4

(*b*) Let $f(x)$ be defined on $[0, 1]$ as

$$f(x) = \frac{2}{3x^{3/4}}, 0 < x \leq 1$$
$$= 0 \quad , x = 0$$

Show that $f(x)$ is Lebesgue integrable on $[0, 1]$. 5

GROUP – B

(*Complex Analysis*)

[*Marks* : 30]

Answer any two questions

7. (*a*) If

$$f(z) = \frac{x^3 y^5 (x + iy)}{x^4 + y^{10}}, \quad z \neq 0$$
$$= 0 \quad , \quad z = 0$$

verify where or not :

(*i*) Cauchy-Riemann relations are satisfied at the origin.

(*ii*) The function is differentiable at the origin. 5

(*b*) If $z = re^{i\theta}$ and $f(z) = u(r, \theta) + iv(r, \theta)$ obtain the Cauchy-Riemann relation in terms of r and θ. 5

(*c*) Show that $W = \dfrac{5-4z}{4z-2}$ transforms. $|z| = 1$ into a circle in the W-plane, find the centre and the radius of the circle. 5

8. (*a*) Show that, under suitable conditions, to be stated,

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2}.$$

where C is a closed contour surrounding the point $z = a$. 5

(*b*) When is a function $f(z)$ said to have a pole of order m at z_0? If a function $f(z)$ has a pole of order m at z_0, prove that $\dfrac{1}{f(z)}$ has a zero of order m at z_0. 5

(*c*) Prove that all roots of the equation

$$z^5 - 12^2 + 14 = 0$$

lie between the circles $|z| = 1$ and $|z| = \dfrac{5}{2}$. How many lie inside $|z| = 2$? 5

9. (*a*) State and prove Cauchy's residue theorem. 5

(*b*) Evaluate any *two* of the following by the method of contour integration. 5 + 5

(*i*) $\displaystyle\int_0^{2\pi} \frac{dx}{3+\cos x}$

(*ii*) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$

(*iii*) $\displaystyle\int_0^{\infty} \frac{\sin \pi x}{x(1-x^2)}$

(*iv*) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$

GROUP – C

(*Ordinary Differential Equations*)

[*Marks* : 30]

Answer any **two** questions

10. (*a*) Show that

$$J_0^2(z) + 2\sum_{n=0}^{\infty} J_n^2(z) = 1$$

and prove that for real z. $|J_0(z)| \le 1$ and $|J_n(z)| \le \frac{1}{\sqrt{2}}$ for all $n \ge 1$. 5

(*b*) The points $Z = \pm 1$ are singular points of the ODE

$$(1 - z^2)w''(z) + (1 + z)w'(z) - w = 0.$$

Show that one of them is a regular singular point and the other is an irregular singular point. 3

(*c*) Let $P_n(z)$ denotes the Legendre polynomial of degree n. Show that

$$\int_0^{\pi} P_m(\cos\theta)\, P_n(\cos\theta)\, \sin\theta\, d\theta = \begin{cases} 0 , & \text{if } m \ne n \\ \dfrac{2}{2n+1}, & \text{if } m = n. \end{cases}$$ 5

(*d*) Prove that

$$\tan^{-1} z = z\, F\left(\frac{1}{2}, 1, \frac{3}{2}, -z^2\right).$$ 2

11. (*a*) If $f(z)$ is continuous and has continuous derivatives in $[-1, 1]$ then prove that $f(z)$ has unique Lagrange's series expansion given by 8

$$f(z) = \sum_{n=0}^{\infty} C_n P_n(z)$$

where P_n's are Legendre's polynomials and

$$C_n = \frac{2n+1}{2}\int_{-1}^{1} f(z) P_n(z)\, dz, \quad n = 1, 2, 3, \ldots$$

(*b*) Find the general solution of the ODE

$$2zw''(z) + (1 + z)w'(z) - kw = 0$$

(where k is a real constant) in power series form. For which values of k is there a polynomial solution ? 7

12. (*a*) If

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+n}}{2^{2m+n} m!(m+n)!},$$

show that

$$e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{m=-\infty}^{\infty} J_n(z)t^n.$$ 5

(*b*) If $\lambda_1, \lambda_2, \ldots$, are the positive zeros of the Bessel's function $J_n(z)$, then 7

$$\int_0^1 z\, J_n(\lambda_m z)\, J_n(\lambda_p z)\, dz = \begin{cases} 0, & \text{if } m \neq p \\ \frac{1}{2} J_{n+1}^2(\lambda p), & \text{if } m = p. \end{cases}$$

(*c*) Deduce the integral representation of confluent Hypergeometric function $F(\alpha, \gamma; z)$. 3
