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On the Exponential Diophantine Equation $23^x - 19^y = z^2$

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ABSTRACT

This article attempts to solve the exponential Diophantine equation $23^x - 19^y = z^2$ where x, y and z are non-negative integers using Catalan's conjecture, factorisation methods, modular arithmetic, and elementary mathematical concepts. This equation has exactly two solutions $(x, y, z) = (0,0,0), (1,1,2)$.

Keywords: Exponential Diophantine equation; Integer Solution; Catalan's Conjecture; Non-linear equation; modular arithmetic.

AMS Mathematics Subject Classification (2020): 11D61**,** 11D72

1. Introduction

Search for non-negative integer solutions of the Diophantine equations [4] is of greater interest to mathematicians all over the world for decades. Of them, the exponential Diophantine equations of the form $a^x + b^y = z^2$ are studied by so many mathematicians [5-12]. While solving these Diophantine equations, a well-known conjecture proposed in the year 1844 by Catalan **[2]** plays a key role. Later in the year 2004, it was proved by Mihailescu **[**3**]**. A new kind of Diophantine equation [9-12] of the form $a^x - b^y = z^2$ together with non-negative integer solutions are of special interest in this paper. Thongnak et al. [9, 11, 12] solved it for$(a, b) = (2, 3), (7,5), (15, 13)$.The non-negative integer solution sets for these equations are $\{(0,0,0), (1,0,1), (2,1,1)\}$, $\{(0,0,0)\}$ and $\{(0,0,0)\}$ respectively. Buosi et al. [10] investigated the exponential Diophantine equation $p^x - 2^y = z^2$ with $p = k^2 + 2$ a prime number for the integer solutions. In this paper, the exponential Diophantine equation $23^x - 19^y = z^2$ is investigated for the non-negative integer solutions. As there are no general methods, Catalan's lemma, factoring method, modular arithmetic, and some basic mathematical concepts [1] are used to solve it for non-negative integer solutions.

2. Primary results

Lemma 2.1. (Catalan's Conjecture [2] or Mihailescu's Theorem [3**])**
The quadruple $(a, x, b, y) = (3,2,2,3)$ is the only integer solution for the Diophantine equation $a^x - b^y = 1$, where a, x, b, y are integers with $min\{a, x, b, y\} > 1$.

Lemma 2.2. The exponential Diophantine equation $1-19^{y}=z^{2}$ has non-negative integer solution$(y,z)=(0,0)$.

Proof: let y and z be non-negative integers.

Consider the exponential Diophantine equation $1-19^{y}=z^{2}$ (1)

Case-1: $y=0$. If $y=0$ then we get $z=0$ from (1) so that $(y,z)=(0,0)$ is a solution of (1)

Case-2: $y>0$**.** Then $z^{2}=1-19^{y}$ is a negative integer for $y>0$ which is a contradiction to the fact that z^{2} is a non-negative integer. Hence the only possible non-negative integer solution is $z=0$ and $y=0$.

Lemma 2.3. The exponential Diophantine equation $23^{x}-1=z^{2}$ has only one non-negative integer solution$(x,z)=(0,0)$.

Proof: let x and z be non-negative integers.

Consider the exponential Diophantine equation $23^{x}-1=z^{2}$ (2)

Case-1: $x=0$.

If $x=0$ then we get $z=0$ from (2) so that $(x,z)=(0,0)$ is a solution of (2)

Case-2: $x>0$. If $x=1$, then $z^{2}=22$ this has no integer solution.

For $z=1$, $23^{x}=2$ this is impossible.

Thus $x>1$ and $z>1$, so that $min\{23,x,z,2\}>1$.

Then by Lemma 2.1 the exponential Diophantine equation $23^{x}-z^{2}=1$ has no solutions. Therefore there is only one non-negative integer solution$(x,z)=(0,0)$.

3. Main result

Theorem 3.1. Let x, y and z be non-negative integers.

The exponential Diophantine equation $23^{x}-19^{y}=z^{2}$ has two non-negative integer solutions $(x,y,z)=(0,0,0)and$ $(1,1,2)$.

Proof: let x,y and z be non-negative integers such that $23^{x}-19^{y}=z^{2}$. (3)

Case-1: $x=0$ in (3), we get $1-19^{y}=z^{2}$.

Then by lemma 2.1 a solution $(y,z)=(0,0)$ is obtained. Thus $(x,y,z)=(0,0,0)$ is a solution of (3).

Case-2: $y=0$ in (3), we get $23^{x}-1=z^{2}$. Then by lemma 2.2 a solution $(x,z)=(0,0)$ is obtained. Thus we get a solution $(x,y,z)=(0,0,0)$.

Case-3: $x=1$ and $y=1$ in (3), it gives a solution $(x,y,z)=(1,1,2)$ of (3)

Case-4: $x>1$ and $y>1$

We have $23^{x}\equiv 3^{x}\equiv\begin{cases}1mod4\,, if\ x\ is\ even\\ 3mod\ 4,\ \ if\ x\ is\ odd\end{cases}$ and $19^{y}\equiv 3^{y}\equiv\begin{cases}1mod4\,, if\ y\ is\ even\\ 3mod\ 4,\ \ if\ y\ is\ odd\end{cases}$

Thus $z^2 \equiv 23^x - 19^y \equiv 3^x - 3^y \equiv \begin{cases} 0\ mod4\ if\ both\ x\ and\ y\ are\ even & (4) \\ 0\ mod4\ if\ both\ x\ and\ y\ are\ odd & (5) \\ -2mod4\ if\ x\ is\ even\ and\ y\ is\ odd & (6) \\ 2mod4\ if\ x\ is\ odd\ and\ y\ is\ even & (7) \end{cases}$

Neglect (6) and (7) as always $z^2 \equiv 0mod4$ or $z^2 \equiv 1mod4$.
Hence both x and y are even or both x and y are odd.

Subcase-1: suppose that both x and y are even non-negative integers.
Let $x = 2m$ and $y = 2n$. Here m, n are non-negative integers.
Then (3) becomes $23^{2m} - 19^{2n} = z^2$ (8)
$19^{2n} = 23^{2m} - z^2 = (23^m - z)(23^m + z).$ (9)
Let $(23^m - z) = 19^u$, u is a non-negative integer. (10)
From (9) and (10), we get $(23^m + z) = 19^{2n-u}$. (11)
Adding (10) and (11), we get $2(23^m) = 19^u + 19^{2n-u} = 19^u(1 + 19^{2n-2u})$.
It follows that $19^u = 1$ and $2(23^m) = (1 + 19^{2n-2u})$.
From this we must have $u = 0, m = 0, n = 0$ only. Thus again we get the zero solution.

Subcase-2: Suppose that x and y are odd non-negative integers.
Let $x = 2m + 1$ and $y = 2n + 1$, for some non-negative integers m, n.
Then from (3), $23^{2m+1} - 19^{2n+1} = z^2$. (12)
$19^{2n+1} = 23^{2m+1} - z^2 = 23^{2m}(16 + 7) - z^2$ (13)
From (13), $19^{2n+1} - 23^{2m}(7) = 23^{2m}(16) - z^2 = (23^m(4) - z)(23^m(4) + z)$ (14)
Let $(23^m(4) - z) = 19^u$, u is a non-negative integer. (15)
Then $(23^m(4) + z) = \left(19^{2n+1} - 23^{2m}(7)\right)19^{-u}$ (16)
Adding (15) and (16) we get $23^m(8) = 19^u + \left(19^{2n+1} - 23^{2m}(7)\right)19^{-u}$
$23^m(8) = 19^u\left[1 + \left(19^{2n+1} - 23^{2m}(7)\right)19^{-2u}\right]$ (17)
From this $19^u = 1$ and $1 + \left(19^{2n+1} - 23^{2m}(7)\right)19^{-2u} = 23^m(8)$
Hence $u = 0$ only, so that $1 + \left(19^{2n+1} - 23^{2m}(7)\right) = 23^m(8)$ (18)
As $\left[1 + \left(19^{2n+1} - 23^{2m}(7)\right)\right] \equiv 1\ mod4$ and $23^m(8) \equiv 0\ mod4$.
From (18) we get $1 \equiv 0mod4$ this is a contradiction.
Hence there is no solution in this case.
Therefore (3) has exactly two non-negative integer solutions $(x, y, z) = (0,0,0), (1,1,2)$ only.

4. Open problem

Search for the non-negative integer solutions of the Diophantine equations of the form $p^x - q^y = z^2$ where p and q are prime numbers with $p > q$.

5. Conclusion

In this paper it is proved that the exponential Diophantine equation $23^x - 19^y = z^2$ has exactly two non-negative integer solutions $(x, y, z) = (0,0,0), (1,1,2)$.

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