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# The Subgroups for the Finite *p*-Group of the Structure $D_2^4 \times C_2^5$

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**ABSTRACT**

Any finite nilpotent group can be uniquely written as a direct product of $p$-groups In this paper, an attempt for the computation of $D_{2^4} \times C_{2^5}$ was made. This happens as the computation of the number of distinct fuzzy subgroups of Cartesian product of the dihedral group of order $2^4$ with another cyclic group having an order $2^5$

***Keywords:*** Finite $p$-groups, nilpotent group, fuzzy subgroups, dihedral group, inclusion-exclusion principle, maximal subgroups

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## 1. Introduction

Dynamism is always the order of the day, most generally in the aspect of pure Mathematics over the years. For instance, many researchers have treated cases of finite abelian groups. Since then, the study has been extended to other important classes of finite abelian and nonabelian groups, such as the dihedral, quaternion, semi dihedral, and hamiltonian groups. This work actuiscalculate the exact number of distinct fuzzy subgroups for nilpotent groups derived from p-groups. The use of the exclusion – inclusion method have been involved in a number of related research directions, such as we had in [1]. Our esteemed reader can as well consult [ 3 to 17 ] of our references.

## 2. Methodology

The method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite $p$-group $G$ is described. Suppose that $M_1, M_2, \dots, M_t$ are the maximal subgroups of $G$, and denote by $h(G)$ the number of chains of subgroups of $G$ which ends in $G$. By simply applying the technique of computing $h(G)$, using the application of the Inclusion-Exclusion Principle, we have that:

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1\le r_1<r_2\le t} h(M_{r_1}\cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right)(\#)$$

In [2], (#) was used to obtain the explicit formulas for some positive integers $n$.

**Theorem 2.1.** (*) [Marius]: The number of distinct fuzzy subgroups of a finite $p$-group of order $p^n$ which have a cyclic maximal subgroup is:

1. $h(\mathbb{Z}_{p^n}) = 2^n$
2. $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2 + (n-1)p]$.

## 3. The number of fuzzy subgroups for $\mathbb{Z}_8 \times \mathbb{Z}_8$

**Lemma 3.1.** $(\gamma)$ : Let $G$ be abelian such that $G = \mathbb{Z}_4 \times \mathbb{Z}_4$.

Then, $h(G) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 48$

**Proof:** By the use of GAP (Group Algorithms and Programming), $G$ has three maximal subgroups in which each of them is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^2}$. Hence, we have that: $\frac{1}{2}h(G) = 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = h(\mathbb{Z}_2 \times \mathbb{Z}_4)$.

And by theorem (*), $h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 24. \Rightarrow h(\mathbb{Z}_4 \times \mathbb{Z}_4) = 48$

**Corrolary 3.2.** Following the last lemma, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^5}), h(\mathbb{Z}_4 \times \mathbb{Z}_{2^6}), h(\mathbb{Z}_4 \times \mathbb{Z}_{2^7})$ and $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^8})$ = 1536, 4096, 10496 and 26112 respectively.

**Theorem 3.3.** Let $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_8$, then $h(G) = \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)$

**Proof:** The three maximal subgroups of $G$ have the following properties :

one is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}$), while two are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^n}$) .

We have : $\frac{1}{2}h(G) = 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 3h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) = 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) = h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})$

Hence, $h(G) = 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) = 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) - 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) + 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) - 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) + 16h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-4}}) = 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) + 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) + 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) - 64h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-5}}) + 32h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-5}}) + \cdots - 2^{j+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-j}}) + 2^j h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-j}})$ , for $n - j = 3$

$$= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 2^{n-3}h(\mathbb{Z}_8 \times \mathbb{Z}_{2^3}) - 2^{n-1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^3}) + \sum_{k=1}^{n-3} [2^{k+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}})$$

$= 2^{n+2}[n^2 + 5n + 3] + \sum_{k=1}^{n-3} h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}}) = 2^{n+2}((n^2 + 5n + 3) + \frac{1}{6}(n-3)(n^2 + 9n + 14)) = \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24), n > 2.$

**Theorem 3.4.** Suppose that $G = D_{2^3} \times \mathbb{C}_8$. Then, $h(G) = 5376$

**Proof:** $\frac{1}{2}h(G) = h(D_{2^3} \times Z_4) + 2h(Z_{2^3} \times Z_2 \times Z_2) - 4h(Z_{2^2} \times Z_2 \times Z_2) + h(Z_8 \times Z_4) - 6h(Z_8 \times Z_2) - 2h(Z_4 \times Z_4) + 8h(Z_4 \times Z_2) + h(Z_{2^3}) = 2688. \therefore h(G) = 2.2688 = 5376.$

**Theorem 3.5.** Let $G = D_{2^5} \times \mathbb{Z}_8$. Then, $h(G) = 111136$

**Proof:** $\because \frac{1}{2}h(G) = h(D_{2^5} \times Z_{2^2}) + 2h(D_{2^4} \times Z_{2^3}) - 4h(D_{2^4} \times Z_{2^2}) + h(\mathbb{Z}_{2^4} \times Z_{2^3}) - 2h(\mathbb{Z}_{2^4} \times Z_{2^2}) - 2h(\mathbb{Z}_{2^3} \times Z_{2^3}) + 8h(\mathbb{Z}_{2^3} \times Z_{2^2}) + h(\mathbb{Z}_{2^4}) - 4h(\mathbb{Z}_{2^3}) = 55568.$
$\therefore h(G) = 2 \times 55568 = 111136.$

**Theorem 3.6.** Suppose that $G = D_{2^6} \times \mathbb{Z}_8$. Then, $h(G) = 492864$

**Proof:** $\frac{1}{2}h(G) = h(D_{2^6} \times Z_4) + 2h(D_{2^5} \times Z_{2^3}) - 4h(D_{2^5} \times Z_4) + h(\mathbb{Z}_{2^5} \times Z_{2^3}) - 2h(\mathbb{Z}_{2^5} \times Z_{2^2}) - 2h(\mathbb{Z}_{2^4} \times Z_{2^3}) + 8h(\mathbb{Z}_{2^4} \times Z_{2^2}) + h(\mathbb{Z}_{2^5}) - 4h(\mathbb{Z}_{2^4}) = 246432.$
$\therefore h(G) = 2 \times 246432 = 492864.$

**Theorem 3.7**. Let $\boldsymbol{G} = \boldsymbol{D_{2^n}} \times \mathbb{C}_2$, the nilpotent group formed by the cartesian product of the dihedral group of order $\boldsymbol{2^n}$ and a cyclic group of order 2. Then, the number of distinct fuzzy subgroups of $\boldsymbol{G}$ is given by : $h(G) = 2^{2n}(2n+1) - 2^{n+1}, n > 3$

## 4. The number of fuzzy subgroups for $D_{2^n} \times \mathbb{C}_8$

**Proposition 4.1.** Suppose that $G = D_{2^n} \times \mathbb{C}_8$. Then, the number of distinct fuzzy subgroups of $G$ is given by :

$$2^{2(n-1)}(6n+113) + 2^n[13 - 6n - 2n^2 + 3\sum_{j=1}^{n-3} 2^{(j-1j)}(2n+1-2j)]$$
$$+\frac{1}{3}(2^{n+2})[(n-1)^3 + (n-2)^3 + 24n^2 - 38n - 30 + \sum_{k=1}^{n-5} 2^k[(n-2-k)^3 + 12(n-2-k)^2 + 17(n-k) - 58]]$$

**Proof:** $h(D_{2^n} \times C_8) = 2h(\mathbb{Z}_{2^{n-1}}) + 2h(D_{2^n} \times Z_4) + 2h(D_{2^{n-1}} \times C_8) + 4h(Z_{2^{n-2}} \times C_8) + 2^4h(Z_{2^{n-3}} \times C_8) + 2^6h(\mathbb{Z}_{2^{n-4}} \times C_8) - 2^8h(\mathbb{Z}_{2^{n-5}} \times \mathbb{Z}_{2^3}) - 4h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) + 2^{10}h(\mathbb{Z}_{2^{n-5}}) \times \mathbb{Z}_{2^2} - 2^9h(\mathbb{Z}_{2^{n-5}}) - 2^9h(D_{2^{n-4}} \times C_{2^2}) + 2^8h(D_{2^{n-4}} \times C_{2^3}) = 2^n + 2h(D_{2^n} \times C_4) + 2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^3}) + 2^2h(\mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_{2^3}) - 2^{2(n-3)}h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3}) + 2^{2(n-2)}h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} - 2^2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) - 2^{2n-5}h(\mathbb{Z}_{2^2}) - 2^{2n-5}h(D_{2^3} \times \mathbb{Z}_{2^2}) + 2^{2(n-3)}h(D_{2^3} \times \mathbb{Z}_{2^3}) + \quad 3\sum_{i=1}^{n-5} 2^{2ij}h(\mathbb{Z}_{2^{n-2-i}} \times \mathbb{Z}_{2^3})$
as required. ■

**Proposition 4.2.** ( see [16] ) Suppose that $\boldsymbol{G} = \boldsymbol{D_{2^n}} \times \mathbb{C}_8$. Then, the number of distinct fuzzy subgroups of $G$ is given by :

$$2^{2(n-1)}(6n+113) + 2^n[13 - 6n - 2n^2 + 3\sum_{j=1}^{n-3} 2^{(j-1j)}(2n+1-2j)]$$
$$+\frac{1}{3}(2^{n+2})[(n-1)^3 + (n-2)^3 + 24n^2 - 38n - 30 + \sum_{k=1}^{n-5} 2^k[(n-2-k)^3 + 12(n-2-k)^2 + 17(n-k) - 58]]$$

**Proof:** $h(D_{2^n} \times C_8) = 2h(\mathbb{Z}_{2^{n-1}}) + 2h(D_{2^n} \times Z_4) + 2h(D_{2^{n-1}} \times C_8) + 4h(Z_{2^{n-2}} \times C_8) + 2^4h(Z_{2^{n-3}} \times C_8) + 2^6h(\mathbb{Z}_{2^{n-4}} \times C_8) - 2^8h(\mathbb{Z}_{2^{n-5}} \times \mathbb{Z}_{2^3}) - 4h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) +$

$$2^{10}h(\mathbb{Z}_{2^{n-5}}) \times \mathbb{Z}_{2^2} - 2^9 h(\mathbb{Z}_{2^{n-5}}) - 2^9 h(D_{2^{n-4}} \times C_{2^2}) + 2^8 h(D_{2^{n-4}} \times C_{2^3})$$

$$= 2^n + 2h(D_{2^n} \times C_4) + 2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^3}) + 2^2 h(\mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_{2^3}) - 2^{2(n-3)} h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3}) + 2^{2(n-2)} h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} - 2^2 h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) - 2^{2n-5} h(\mathbb{Z}_{2^2}) - 2^{2n-5} h(D_{2^3} \times \mathbb{Z}_{2^2}) + 2^{2(n-3)} h(D_{2^3} \times \mathbb{Z}_{2^3}) + 3\sum_{i=1}^{n-5} 2^{2i} h(\mathbb{Z}_{2^{n-2-i}} \times \mathbb{Z}_{2^3})$$

as required.

**Theorem 4.3.** (see [3] ) Let $G = D_{2^4} \times \mathbb{C}_{2^4}$. Then , $h(G) = 61384$.

**Proof:** There exist seven maximal subgroups . Two isomorphic to $\boldsymbol{D_{2^4}} \times \mathbb{C}_{\boldsymbol{2^3}}$. two isomorphic to $D_{2^3} \times \mathbb{C}_{2^4}$. two isomorphic to $\boldsymbol{D_{2^4}} \times \mathbb{C}_{\boldsymbol{2^2}}$, while the seventh is isomorphic to $\mathbb{Z}_{\boldsymbol{2^4}}$.

Hence , we have that : $\frac{1}{2}h(G) = 2h(D_{2^4} \times Z_{2^2}) + 2h(D_{2^4} \times Z_{2^3}) + 2h(D_{2^3} \times Z_{2^4}) - 6h(D_{2^3} \times \mathbb{Z}_{2^3}) - 6h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3}) - 6h(\mathbb{Z}_{2^4}) + 2h(D_{2^3} \times \mathbb{Z}_{2^3}) + 28h(\mathbb{Z}_{2^3} \times Z_{2^2}) + 2h(Z_{2^4} \times Z_{2^2}) + 2h(\mathbb{Z}_{2^4}) + h(Z_{2^3} \times \mathbb{Z}_{2^3}) - 35h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) + 21h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) - 7h(\mathbb{Z}_{2^3}\mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2})$

$= 2[h(D_{2^4} \times Z_{2^2}) + h(D_{2^4} \times Z_{2^3}) + h(D_{2^3} \times Z_{2^4}) - 2h(D_{2^3} \times \mathbb{Z}_{2^3}) - 2h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) - h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3}) + 4h(D_{2^3} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^4}) + \frac{1}{2}h(Z_{2^4})]$

$\therefore h(G) = 4[700 + 8416 + 10744 - 10752 – 1088 + 162 + 704 – 40] = 4[15346]$

$= 61384$ ■

## 5. The number of fuzzy subgroups for $D_{2^5} \times \mathbb{C}_{2^4}$

**Proposition 5.1.** Let $G = (D_{2^4} \times C_{2^5})$ . Then , we are going to have that

$$\frac{1}{2}h(G) = h(D_{2^4} \times C_{2^4}) + 3h(C_{2^3} \times C_{2^5}) + 2h(C_{2^5} \times C_{2^3}) - 4h(D_{2^4} \times C_{2^3}) - 6h(C_{2^4} \times C_{2^3}) + 8h(C_{2^3} \times C_{2^3}) - 3h(C_{2^4})$$

$= 3h(D_{2^5} \times C_{2^3}) + 3h(D_{2^4} \times C_{2^4}) - 4h(D_{2^4} \times C_{2^3}) - 6h(C_{2^3} \times C_{2^4}) + 8h(C_{2^3} \times C_{2^3}) - 3h(C_{2^4})$

Now, $h(G) = 6h(D_{2^5} \times C_{2^3}) + 6h(D_{2^4} \times C_2^4) - 8h(D_{2^4} \times C_{2^3}) - 12h(C_{2^3} \times C_{2^4}) + 16h(C_{2^3}) \times C_{2^3}) - 6h(C_{2^4}.)$

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