2022

M.Sc.

4th Semester Examination

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER—MTM-401

FUNCTIONAL ANALYSIS

Full Marks : 50

Time : 2 Hours

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Answer any *four* questions : 4×2

(a) Let X and Y be normed spaces. If X is finite dimensional, then show that every linear map from X to Y is continuous.

(b) Give an example of a normal operator which is not self-adjoint.

(c) Let X be a normed space. Show that $x_n \to x$ weakly in X does not imply $x_n \to x$ in X in general.

(Turn Over)

(d) Show that every normed space can be embedded as a dense subspace of a Banach space.

(e) Let X be a normed space and $\phi(x) = \phi(y)$ for every $\phi \in X^*$. Show that $x = y$.

(f) Let $T \in BL(H)$ be self-adjoint. Show that $Ker(T) = Ker(T^*)$.

2. Answer any *four* questions : 4×4

(a) Let $X = C[0, 1]$ with the supremum norm.

Consider the sequence $x_n(t) = \frac{t^n}{n^5}$, $t \in [0, 1]$.

Check whether the series $\sum_{n=1}^{\infty} x_n$ is summable in X.

(b) Show that $\langle Ae_j, e_i \rangle = (i + j + 1)^{-1}$ for $0 \le i, j \le \infty$ defines a bounded operator on $l^2(N \cup \{0\})$ with $\|A\| \le \pi$.

(c) Let Y be a normed space and Y_0 be a dense subspace of Y. Suppose Z is a Banach space and $T \in BL(Y_0, Z)$. Prove that there exist a unique $\tilde{T} \in BL(Y, Z)$ such that $\tilde{T} | Y_0 = T$.

(d) Let $S \in BL(H)$, where H is a Hilbert space. Prove that for all $x, y \in H$,

$$\langle Sx, y\rangle = \frac{1}{4}\sum_{n=0}^{3} i^n \left\langle S\left(x + i^n y\right), \left(x + i^n y\right)\right\rangle.$$

(e) Let X and Y be normed spaces and $\psi : X \to Y$ be linear. Show that ψ is continuous if and only if for every Cauchy sequence $\{x_n\}$ in X, the sequence $\{\psi(x_n)\}$ is Cauchy in Y.

(f) Give an example to show that the completeness of the domain is an essential requirement in the Uniform Boundedness Principle.

3. Answer any *two* questions : 2×8

(a) Let X and Y be Banach spaces and $A \in BL(X, Y)$. Show that there is a constant $c > 0$ such that $\|Ax\| \geq c\|x\|$ for all $x \in X$ if and only if $\text{Ker}(A) = \{0\}$ and $\text{Ran}(A)$ is closed in X.

(b) Let $F \in BL(X, Y)$ and $Z(F) = \{x \in X : F(x) = 0\}$. Define $\tilde{F} : X/Z(F) \to Y$ by $\tilde{F}(x + Z(F)) = F(x)$, $x \in X$. Show that $\|\tilde{F}\| = \|F\|$, where X, Y are normed spaces. 3+5

4. (a) Show that a Banach space is a Hilbert space if and only if the Parallelogram law holds.

(b) Let S be the Unilateral shift operator. Show that $(S^*)^n \xrightarrow{S} 0$ but not uniformly. 5+3

5. (a) Let $P \in BL(H)$ be a non-zero projection on a Hilbert space H and $\|P\| = 1$. Then show that P is an orthogonal projection.

(b) Show that Ran(T) = Ran(T^*) if $T \in BL(H)$ is normal and H is a Hilbert space. 4+4

6. (a) Let the space $l^2(Z)$ be defined as the space of all two-sided square summable sequences and the bilateral shift is the operator W on $l^2(Z)$ defined by

$$W(\ldots, a_{-2}, a_{-1}, \hat{a}_0, a_1, a_2, \ldots) = (\ldots, a_{-3}, a_{-2}, \hat{a}_{-1}, a_0, a_2, \ldots).$$

(i) W is unitary, and

(ii) the adjoint W^* of W is given by

$$W^*(\ldots, a_{-2}, a_{-1}, \hat{a}_0, a_1, a_2, \ldots) = (\ldots, a_{-1}, a_0, \hat{a}_1, a_2, a_3, \ldots).$$

(b) Let H be a Hilbert space and $E \subset H$. Prove that $\overline{\text{span}(E)} = E^{\perp\perp}$. 5+3

[Internal Assessment – 10]

———