2022

M.Sc.

2nd Semester Examination

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER—MTM-203

Full Marks : 50

Time : 2 Hours

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

MTM-203 (UNIT-1) ABSTRACT ALGEBRA

1. Answer any *two* questions : 2×2

(a) An element x in a group G is called commutator if $x = aba^{-1}b^{-1}$ for some $a, b \in G$. By H we mean the subgroup generated by all the commutators. Show that G/H is abelian.

(Turn Over)

(b) Prove that if $|G| = 1365$, then G is not a simple group.

(c) Show that every irreducible real polynomial is either linear or

$$ax^2 + bx + c \text{ with } b^2 - 4ac < 0.$$

(d) Write down the class equation for the symmetric group S_3.

2. Answer any *two* questions : 2×4

(a) Prove that the group $SL(2, \mathbb{R})$ is a normal subgroup of the group $GL(2, \mathbb{R})$ and the quotient group $GL(2, \mathbb{R})/SL(2, \mathbb{R}) \cong \mathbb{R}^*$ where $\mathbb{R}^*$ is the multiplicative group of all non-zero real numbers.

(b) Define direct product of the groups $G_1, G_2, \ldots, G_n$. Then show that the center of a direct product is the direct product of the centers, i.e. $Z(G_1 \times G_2 \times \ldots \times G_n) = Z(G_1) \times Z(G_2) \times \ldots \times Z(G_n)$.

(c) Let $K \subseteq F$ be a field extension and $\alpha \in F$ be algebraic over K. Then show that there exists a unique monic irreducible polynomial $f(x) \in K[x]$ such that $f(\alpha) = 0$.

(d) Suppose G be a group of order p^2q where p and q are distinct primes. Then show that G is not simple.

3. Answer any *one* question : 1×8

(a) (i) State and prove the Sylow's first theorem.

(ii) If G is an abelian group having subgroups $H_1, H_2, \ldots, H_t$ such that $|H_i \cap H_j| = 1$, for all $i \neq j$, then prove that $K = H_1H_2 \ldots H_t$ is a subgroup of G of order $|H_1| \times |H_2| \times \ldots \times |H_t|$ and $K \cong H_1 \times H_2 \times \ldots \times H_t$.

(b) (i) Show that if R is a principal ideal domain, then it is a unique factorization domain. Give an example to show that the converse is not true.

(ii) If $K \subseteq F \subseteq L$ is a tower of fields then show that $[L : F]\,[F : K] = [L : K]$ where $[L : F]$ denotes the degree of L over F.

[Internal Assessment – 05]

MTM–203 (UNIT–2) LINEAR ALGEBRA

1. Answer any *two* questions : 2×2

(a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator defined by $T(x, y) = (x, 0)$ for all $(x, y) \in \mathbb{R}^2$. Find the minimal polynomial of T.

(b) Define linear functional on a vector space. Let T be a linear operator on a finite-dimensional vector space V. When you say that T is diagonalizable ?

(c) What do you mean by a normal operator ? Are all self-adjoint operators normal ? Justify.

(d) Let V be the space of all real valued continuous functions. Define T : V → V by $(Tf)(x) = \int_0^x f(t)\,dt$.

Show that T has no eigen value.

2. Answer any *two* questions : 2×4

(a) (i) Determine all possible Jordan canonical forms for a linear operator T whose characteristic polynomial is $(t - 2)^5$ and minimal polynomial is $(t - 2)^2$.

(ii) Let V(R) be a vector space of polynomials with inner product defined by $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. If $f(x) = x^2 + x - 4$, $g(x) = x - 1$, then find $\langle f, g\rangle$ and $\|g\|$. 2+2

(b) Define quotient space. Let P_n be the vector space of all polynomials of degree $\leq$ n over R. Exhibit a basis of P_4/P_2. Hence verify that

$$\dim\left(\frac{P_4}{P_2}\right) = \dim(P_4) - \dim(P_2).$$

(c) Let V and V′ be two vector spaces over the same field F and T : V → V′ be a linear mapping. Show that V/KerT ≅ ImT. Hence derive the rank nullity theorem if V is a finite dimensional vector space. 2+2

(d) Let V be a finite dimensional vector space over a field F and T : V → V be a linear operator. Show that there exists a unique monic polynomial m(x) of least degree such that m(T) = T_0.

3. Answer any *one* question : 1×8

(a) (i) Let T be a linear operator on a finite dimensional vector space V and let $c_1, c_2, \ldots, c_k$ be the distinct characteristic values of and let W_i be the null space of $(T - c_i I)$, i = 1, 2, ..., k. Then prove that the following are equivalent :

(1) T is diagonalizable.

(2) The characteristic polynomial for T is

$$f = (x - c_1)^{d_1} (x - c_2)^{d_2} \ldots, (x - c_k)^{d_k}$$

and dim W_i = d_i, i = 1, 2, ..., k.

(3) dim W_1 + dim W_2 + ... + dim W_k = dim V.

(ii) Let T be a linear operator on V. If $T^2 = 0$, what can you say about the relation of the range of T to the null space of T ? Give an example of linear operator of T on R^2 such that $T^2 = 0$, but $T \neq 0$. 5+3

(b) (i) Find the rational canonical form of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$. Is the rational canonical form of the given matrix A coincides with the Jordan canonical form of A ? Justify your answer.

(ii) Let V be a vector space over a field F and $f : V \to V$ be an idempotent linear operator. Show that $V = Kerf \oplus Imf$. (3+2)+3

[Internal Assessment – 05]
