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A further Extension of Gamma and Beta Functions involving Generalized Mittag-Leffler Function and its Applications

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ABSTRACT

Recently, many researchers have proposed different extensions of gamma, beta, confluent hypergeometric, and Gauss hypergeometric functions. Our aim is to study another extension of gamma and beta functions related to the generalized Mittag-Leffler function. We also study some functional relations, integral representations and Mellin transforms. Some special cases of obtained results are considered and shown to further reduce to some known results.

Keywords: Gamma function, Beta function, Mittage -Leffler function, Integral representations

Mathematical Subject Classification (2010): 46F30

1. Introduction

In recent years, several extensions of many of the familiar special functions have been considered by several authors [1-7]. In 1994, Chaudhary and Zubair [1] introduced the following extension of the gamma function

$$\Gamma_\rho(\lambda_1) = \int_0^\infty t^{\lambda_1-1} exp(-t-\rho t^{-1})dt, \quad (Re(\rho) > 0, Re(\lambda_1) > 0). \tag{1.1}$$

In 1997, Chaudhary et al. [2] presented the following extension of Euler's beta function

$$B_\rho(\lambda_1,\lambda_2) = \int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1} exp\left(-\frac{\rho}{t(1-t)}\right)dt, \tag{1.2}$$
$$(\mathrm{Re}(\rho) > 0, Re(\lambda_1) > 0, Re(\lambda_2) > 0),$$

and they demonstrated that this extension has connections to the Macdonald, Error and Whittakers functions. Apparently it seems that $\Gamma_0(\lambda_1) = \Gamma(\lambda)$ and $B_0(\lambda_1,\lambda_2)$.
In 2011, Ozergin [8] (see also Ozergin et al. [9] introduced and studied a further potentially useful extension of the gamma and beta functions as follows:

$$\Gamma_\rho^{k_1,k_2}(\lambda_1) = \int_0^\infty t^{\lambda_1-1}\, {}_1F_1(k_1,k_2;-t-\rho t^{-1})\,dt, \tag{1.3}$$
$$(Re(k_1) > 0, Re(k_2) > 0, Re(\lambda_1) > 0, Re(\lambda_2) > 0 \; and \; Re(\rho) > 0),$$

and

$$B_{\rho}^{k_1,k_2}(\lambda_1,\lambda_2)=\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}\,{}_1F_1\left(k_1;k_2;-\frac{\rho}{t(1-t)}\right)dt, \tag{1.4}$$

$$(Re(k_1)>0, Re(k_2)>0, Re(\lambda_1)>0, Re(\lambda_2)>0\ and\ Re(\rho)>0),$$

correspondingly, where ${}_1F_1$represents the confluent hypergeometric function given in [10].

We clearly have

$$\Gamma_{\rho}^{k_1,k_1}(\lambda_1)=\Gamma_{\rho}(\lambda_1), \tag{1.5 a}$$

$$\Gamma_{0}^{k_1,k_1}(\lambda_1)=\Gamma(\lambda_1), \tag{1.5 b}$$

and

$$B_{\rho}^{k_1,k_1}(\lambda_1,\lambda_2)=B_{\rho}(\lambda_1,\lambda_2), \tag{1.5 c}$$

$$B_{0}^{k_1,k_2}(\lambda_1,\lambda_2)=B(\lambda_1,\lambda_2). \tag{1.5 d}$$

Another extension of the beta and gamma functions in terms of integrals with the kernel containing the Mittage-Leffler function has been introduced by Pucheta [11] in the following forms:

$$\Gamma^{k_1}(\lambda_1)=\int_0^{\infty} t^{\lambda_1-1}E_{K_1}(-t)dt\ (k_1\in\mathbb{R}^+, Re(\lambda_1)>0), \tag{1.6}$$

and

$$B_{\rho}^{k_1}(\lambda_1,\lambda_2)=\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}E_{k_1}\big(-\rho t(1-t)\big)dt, \tag{1.7}$$

$$(Re(\lambda_1)>0, Re(\lambda_2)>0, k_1\in\mathbb{R}^+, \rho\geq 0),$$

where the Mittage -Leffler function $E_{K_1}(z)$is defined by [12]

$$E_{k_1}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\,k_1+1)}, (z\in\mathbb{C}; Re(k_1)>0). \tag{1.8}$$

Other properties and applications of this type of beta function were also discussed in [13, 14]

Abubakar et al., [15] presented the following extended forms of the extended beta function

$$\Gamma^{k_1,k_2}(\lambda_1)=\int_0^{\infty} t^{\lambda_1-1}E_{k_1,k_2}(-t)dt\ (k_1\in\mathbb{R}^+, Re(\lambda_1)>0), \tag{1.9}$$

and

$$B_{\rho}^{k_1,k_2}(\lambda_1,\lambda_2)=\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}E_{k_1}\big(-t\rho(1-t)\big)dt, \tag{1.10}$$

$$(Re(\lambda_1)>0, Re(\lambda_2)>0, k_1\in\mathbb{R}^+, \rho\geq 0),$$

where the Mittage -Leffler function $E_{k_1,k_2}(z)$ is defined by [16-18]

$$E_{k_1,k_2}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\,k_1+k_2)}, (z\in\mathbb{C}; Re(k_1)>0, Re(k_2)>0). \tag{1.11}$$

Obviously,

$$\Gamma^{k_1,1}(\lambda_1)=\Gamma^{k_1}(\lambda_1), \tag{1.12 a}$$

$$\Gamma^{1,1}(\lambda_1)=\Gamma(\lambda_1), \tag{1.12 b}$$

and

$$B_{\rho}^{k_1,1}(\lambda_1,\lambda_2)=B_{\rho}^{k_1}(\lambda_1,\lambda_2), \tag{1.12 c}$$

$$B_{0}^{1,1}(\lambda_1,\lambda_2)=B(\lambda_1,\lambda_2). \tag{1.12 d}$$

The Caputo fractional integral, Riemann-Liouville fractional derivative and integral have also been introduced in [19, 20] by looking at equations (1.9) and (1.10).

Also, Shadab et al. [21] have introduced another form of the extended beta function as follows:

$$B_{k_1}^{\rho}(\lambda_1,\lambda_2)=\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}E_{k_1}\left(-\frac{\rho}{t\,(1-t)}\right)dt, \tag{1.13}$$
$$(Re(\lambda_1)>0, Re(\lambda_2)>0, k_1\in R_0^+, Re(\rho)>0).$$

Goyal et al., [22] provided the following extension of the beta function by treating the Wiman function as the kernel:

$$B_{k_1,k_2}^{\rho}(\lambda_1,\lambda_2)=\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}E_{k_1,k_2}\left(-\frac{\rho}{t\,(1-t)}\right)dt, \tag{1.14}$$
$$\left(Re(\lambda_1)>0, Re(\lambda_2)>0, k_1,k_2\in R_0^+, Re(\rho)>0\right),$$

they also study some important properties such derivative formulas, integral representations, functional relation and $E_{k_1,k_2}(;)$ is as defined in (1.11).

Al-Gonal and Mohammed, [23] used the Prabhakar function to present the following extended gamma and beta functions:

$$\Gamma_{k_1,k_2}^{k_2,\rho}(\lambda_1)=\int_0^1 t^{\lambda_1-1}E_{k_1,k_2}^{k_3}(-t-\rho t^{-1})dt, \tag{1.15}$$
$$\left(Re(\lambda_1)>0, k_1,k_2,k_3\in R_0^+,, Re(\rho)>0\right),$$

and

$$B_{k_1,k_2}^{k_3,\rho}(\lambda_1,\lambda_2)=\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}E_{k_1,k_2}^{k_3}\left(-\frac{\rho}{t\,(1-t)}\right)dt, \tag{1.16}$$
$$\left(Re(\lambda_1)>0, Re(\lambda_2)>0, k_1,k_2,k_3\in R_0^+,, Re(\rho)>0\right)$$

where $E_{k_1,k_2}^{k_3}(;)$ is the Prabhakar function defined as [24, 25]

$$E_{k_1,k_2}^{k_3}(z)=\sum_{n=0}^{\infty}\frac{(k_3)_n}{n!\,\Gamma(k_1n+k_2)}z^n, \tag{1.17}$$
$$(k_1,k_2,k_3\in\mathbb{C}; Re(k_1)>0, Re(k_2)>0, Re(k_3)>0).$$

Other properties of the extended gamma and beta functions in equations (1.16) and (1.17) such as extended hypergeometric functions, integral transforms, multi-index beta and hypergeometric functions were discussed in Al-Gonah and Mohammed [26, 27], Abubakar and Kabara [28] and Ali et al., [29].

Abubakar and Kabara [30] study and studied the following extended gamma and beta function by using the four-parameter Mittag-Laffler function:

$$\Gamma_{k_1,k_2,k_3}^{k_4,\rho}(\lambda_1)=\int_0^1 t^{\lambda_1-1}E_{k_1,k_2,k_3}^{k_4}(-t-\rho t^{-1})dt, \tag{1.18}$$
$$\left(Re(\lambda_1)>0, k_1,k_2,k_3,k_4\in R_0^+,, Re(\rho)>0\right),$$

and

$$B_{k_1,k_2,k_3}^{k_3,\rho}(\lambda_1,\lambda_2)=\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}E_{k_1,k_2,k_3}^{k_4}\left(-\frac{\rho}{t\,(1-t)}\right)dt, \tag{1.19}$$
$$\left(Re(\lambda_1)>0, Re(\lambda_2)>0, k_1,k_2,k_3,k_4\in R_0^+,, Re(\rho)>0\right).$$

Here $E_{k_1,k_2,k_3}^{k_4}(;)$ is the Salim function defined as [32]

$$E_{k_1,k_2,k_4}^{k_3}(z)=\sum_{n=0}^{\infty}\frac{(k_3)_n}{(k_4)_n\,\Gamma(k_1n+k_2)}z^n, \tag{1.20}$$
$$(k_1,k_2,k_3\in\mathbb{C}; Re(k_1)>0, Re(k_2)>0, Re(k_3)>0, Re(k_4)>0).$$

Mainly driven by some interesting recent extensions of gamma and beta functions. In particular, by learning from the of work of Shadab et al. [21], Goyal et al., [22], Al-Gonal and Mohammed, [23] and Abubakar and Kabara [30]. In this work, we will introduce and study a new extension of extended gamma and beta functions in terms of integral whose kernel contains the generalized Mittage-Leffler function. To do this, we recall that the generalized Mittage-Leffler function is introduced by Salim et al., [32] is as follows:

$$E_{k_1,k_2,p}^{k_3,k_4,q}(z) = \sum_{n=0}^{\infty} \frac{(k_3)_{q\,n}}{(k_4)_{pn}\,\Gamma(k_1 n+k_2)} z^n, \tag{1.21}$$

$(k_1, k_2, k_3, k_4 \in \mathbb{C}; min\{Re(k_1), Re(k_2), Re(k_3), Re(k_4)\} > 0, p, q > 0 \;\; and \; q < Re(k_1) + p),$

It is obvious that we have

$$E_{k_1,k_2,1}^{k_3,k_4,q}(z) = E_{k_1,k_2}^{k_3,k_4,q}(z), \qquad E_{k_1,k_2,1}^{k_3,k_4,1}(z) = E_{k_1,k_2}^{k_3,k_4}(z), \qquad E_{k_1,k_2,1}^{k_3,1,1}(z) = E_{k_1,k_2}^{k_3}(z),$$

$$E_{1,k_2,1}^{k_3,1,1}(z) = \frac{1}{\Gamma(k_2)}\; {}_1F_1(k_2; k_3 z), \quad E_{k_1,k_2,1}^{1,1,1}(z) = E_{k_1,k_2}(z), \quad E_{k_1,1,1}^{1,1,1}(z) = E_{k_1}(z),$$

$$E_{k_1,1,1}^{1,1,1}(z) = exp(z).$$

2. Main result

In this section, we have introduced a new further extension of the gamma and beta functions with their properties such as functional relations.

Definition 2.1. The further extended gamma function is given by:

$$\Gamma_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1) = \int_0^{\infty} t^{\lambda_1-1} E_{k_1,k_2,p}^{k_3,k_4,q}(-t-\rho t^{-1})\,dt \tag{2.1}$$

$(Re(\lambda_1) > 0, min\{Re(k_1), Re(k_2), Re(k_3), Re(k_4)\} > 0, p, q > 0 \;\; and \; q < Re(k_1) + p\,).$

Clearly, we have

$$\Gamma_{k_1,k_2,1;\rho}^{k_3,k_4,q}(\lambda_1) = \Gamma_{k_1,k_2;\rho}^{k_3,k_4,q}(\lambda_1), \tag{2.2 a}$$

$$\Gamma_{k_1,k_2,1;\rho}^{k_3,k_4,1}(\lambda_1) = \Gamma_{k_1,k_2;\rho}^{k_3,k_4}(\lambda_1), \tag{2.2 b}$$

$$\Gamma_{k_1,k_2,1;\rho}^{k_3,1,1}(\lambda_1) = \Gamma_{k_1,k_2;\rho}^{k_3}(\lambda_1), \tag{2.2 c}$$

$$\Gamma_{1,k_2,1;\rho}^{k_3,1,1}(\lambda_1) = \frac{1}{\Gamma(k_2)}\Gamma_{\rho}^{k_2,k_3}(\lambda_1), \tag{2.2 d}$$

$$\Gamma_{k_1,k_2,1;\rho}^{1,1,1}(\lambda_1) = \Gamma_{k_1,k_2}^{\rho}(\lambda_1), \tag{2.2 e}$$

$$\Gamma_{k_1,1,1;\rho}^{1,1,1}(\lambda_1) = \Gamma_{k_1,}^{\rho}(\lambda_1), \tag{2.2 f}$$

$$\Gamma_{1,1,1;\rho}^{1,1,1}(\lambda_1) = \Gamma_{\rho}(\lambda_1), \tag{2.2 g}$$

$$\Gamma_{1,1,1;0}^{1,1,1}(\lambda_1) = \Gamma(\lambda_1). \tag{2.2 h}$$

Definition 2.2. The new further extended beta function is given by:

$$B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2) = \int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1} E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{\rho}{t\,(1-t)}\right)dt \tag{2.3}$$

$(Re(\lambda_1) > 0, min\{Re(k_1), Re(k_2), Re(k_3), Re(k_4)\} > 0, p, q > 0 \;\; and \; q < Re(k_1) + p\;\;),$

which can be reduces to

$$B_{k_1,k_2,1;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2) = B_{k_1,k_2;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2), \tag{2.4 a}$$

$$\mathrm{B}_{k_1,k_2,1;\rho}^{k_3,k_4,1}(\lambda_1,\lambda_2) = \mathrm{B}_{k_1,k_2;\rho}^{k_3,k_4}(\lambda_1,\lambda_2), \tag{2.4 b}$$

$$\mathrm{B}_{k_1,k_2,1;\rho}^{k_3,1,1}(\lambda_1,\lambda_2) = \mathrm{B}_{k_1,k_2;\rho}^{k_3}(\lambda_1,\lambda_2), \tag{2.4 c}$$

$$\mathrm{B}_{1,k_2,1;\rho}^{k_3,1,1}(\lambda_1,\lambda_2) = \frac{1}{\Gamma(k_2)}\mathrm{B}_{\rho}^{k_2,k_3}(\lambda_1,\lambda_2), \tag{2.4 d}$$

$$B_{k_1,k_2,1;\rho}^{1,1,1}(\lambda_1,\lambda_2) = \mathrm{B}_{k_1,k_2}^{\rho}(\lambda_1,\lambda_2), \quad (2.4\text{ e})$$
$$\mathrm{B}_{k_1,1,1;\rho}^{1,1,1}(\lambda_1,\lambda_2) = \mathrm{B}_{k_1,}^{\rho}(\lambda_1,\lambda_2), \quad (2.4\text{ f})$$
$$B_{1,1,1;\rho}^{1,1,1}(\lambda_1,\lambda_2) = \mathrm{B}_{\rho}(\lambda_1,\lambda_2), \quad (2.4\text{ g})$$
$$\mathrm{B}_{1,1,1;0}^{1,1,1}(\lambda_1,\lambda_2) = \mathrm{B}(\lambda_1,\lambda_2). \quad (2.4\text{ h})$$

Theorem 2.2. (Functional relation) The new further extended functional relation for the beta function is given by

$$B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2+1) + B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1+1,\lambda_2) = B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2). \quad (2.5)$$

Proof:

$$B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2+1) + B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1+1,\lambda_2)$$

$$= \int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2} E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{\rho}{t\,(1-t)}\right)dt + \int_0^1 t^{\lambda_1}(1-t)^{\lambda_2-1} E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{\rho}{t\,(1-t)}\right)dt$$
$$= \int_0^1 \left[t^{\lambda_1-1}(1-t)^{\lambda_2} + t^{\lambda_1}(1-t)^{\lambda_2-1}\right] E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{\rho}{t\,(1-t)}\right) dt$$
$$= \int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1} E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{\rho}{t\,(1-t)}\right)dt$$
$$= B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2).$$

3. Integral representations

In this section, we derive integral representations for the new further extended gamma and beta functions in the form of the following theorems:

Theorem 3.1. The following integral representations are true for the new further extended gamma function

$$\Gamma_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1) = \int_0^{\frac{\pi}{2}} \frac{sin^{\lambda_1-1}\theta}{cos^{\lambda_1+1}\theta} E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{(sin^2\theta+\rho\, cos^2\theta)}{sin\theta\, cos\theta}\right)d\theta, \quad (3.1)$$
$$\Gamma_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1) = -\int_{-1}^{0} \frac{u^{\lambda_1-1}}{(u+1)^{\lambda_1+1}} E_{k_1,k_2,p}^{k_3,k_4,q}\left(\frac{-u^2-\rho(u+1)^2}{u(u+1)}\right)du, \quad (3.2)$$
$$\Gamma_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1) = -\int_0^1 \frac{u^{\lambda_1-1}}{(u-1)^{\lambda_1+1}} E_{k_1,k_2,p}^{k_3,k_4,q}\left(\frac{-u^2-\rho(u-1)^2}{u(u-1)}\right)du. \quad (3.3)$$

Proof: In definition (2.1), by setting $t = tan\theta, t = \frac{u}{u+1}$ and $t = \frac{u}{u-1}$, we obtained the results (3.1),(3.2) and (3.3) respectively.

Remark 3.1. In equation (3.3) put $u = \frac{t-\alpha}{\beta-\alpha}$, $u = \frac{1+t}{2}$ and $u = \frac{t}{\gamma}$ we get the following integral representations:

Corollary 3.1. The following integral representations are true:

$$\Gamma_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1) = (\alpha-\beta)\int_{\alpha}^{\infty} \frac{(t-\alpha)^{\lambda_1-1}}{(t-\beta)^{\lambda_1+1}} E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{(t-\alpha)^2-\rho(t-\beta)^2}{(t-\alpha)(t-\beta)}\right)dt, \quad (3.4)$$
$$\Gamma_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1) = -2\int_{-1}^{1} \frac{(t+1)^{\lambda_1-1}}{(t-1)^{\lambda_1+1}} E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{(t+1)^2-\rho(t-1)^2}{(t+1)(t-1)}\right)dt, \quad (3.5)$$

$$\Gamma_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1) = -\int_0^{\gamma} \frac{t^{\lambda_1-1}}{(t-\gamma)^{\lambda_1+1}} E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{t^2-\rho(t-\gamma)^2}{t\,(t-\gamma)}\right) dt. \tag{3.6}$$

Theorem 3.2. The following integral representations holds true for the new extended beta function:

$$B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2)=2\int_0^{\frac{\pi}{2}} cos^{2\lambda_1-1}\theta\, sin^{2\lambda_2-1}\theta\, E_{k_1,k_2,p}^{k_3,k_4,q}(-\rho sec^2\theta csc^2\theta)d\theta, \tag{3.7}$$

$$B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2)=\int_0^{\infty} \frac{u^{\lambda_1-1}}{(1+u)^{\lambda_1+\lambda_2}} E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\rho\left(2+u+\frac{1}{u}\right)\right) du, \tag{3.8}$$

$$B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2)=(\beta-\alpha)^{1-\lambda_1-\lambda_2}\int_{\alpha}^{\beta}(u-\alpha)^{\lambda_1-1}(\beta-u)^{\lambda_2-1} E_{k_1,k_2,p}^{k_3,k_4,q}\left(\frac{-\rho\,(\beta-\alpha)^2}{(u-\alpha)(\beta-u)}\right) du\,, \tag{3.9}$$

$$B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2) = \gamma^{1-\lambda_1-\lambda_2}\int_0^{\gamma} u^{\lambda_1-1}(\gamma-u)^{\beta-1} E_{k_1,k_2,p}^{k_3,k_4,q}\left(\frac{-\rho\gamma^2}{u(\gamma-u)}\right) du\,, \tag{3.10}$$

Proof: In definition (2.3), putting $t = u = \frac{u}{u+1}$ and $t = \frac{u-\alpha}{\beta-\alpha}$ and $t = \frac{u}{\gamma}$, we obtained the results (3.7),(3.8) , (3.9) and (3.10) respectively.

Remark 3.2. Substituting $\alpha = -1$ and $\beta = 1$ in equation (3.9), we get the following integral representation:

Corollary 3.2. The following integral representations holds true:

$$B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2)=2^{1-\lambda_1-\lambda_2}\int_{-1}^{1}(u+1)^{\lambda_1-1}(1-u)^{\lambda_2-1} E_{k_1,k_2,p}^{k_3,k_4,q}\left(\frac{-4\rho}{1-u^2}\right) du. \tag{3.11}$$

Theorem 3.4. The following integral formula for the new extended gamma functions:

$$\Gamma_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1)\Gamma_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_2) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} s^{2(\lambda_1+\lambda_2)-1}cos^{2\lambda_1-1}\theta sin^{2\lambda_2-1}\theta$$
$$\times E_{k_1,k_2,p}^{k_3,k_4,q}\left(-s^2cos^2\theta - \frac{-\rho}{s^2cos^2\theta}\right) E_{k_1,k_2,p}^{k_3,k_4,q}\left(-s^2sin^2\theta - \frac{\rho}{s^2sin^2\theta}\right) dsd\theta. \tag{3.12}$$

Proof: Putting $t = \eta^2$ in definition (2.1), we get

$$\Gamma_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1) = 2\int_0^{\infty}\eta^{2\lambda_1-1}E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\eta^2-\frac{\rho}{\eta^2}\right)d\eta. \tag{3.13}$$

Such that

$$\Gamma_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1)\Gamma_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_2)$$
$$= 4\int_0^{\infty}\int_0^{\infty}\eta^{2\lambda_1-1}\xi^{2\lambda_2-1}E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\eta^2-\frac{\rho}{\eta^2}\right)E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\xi^2-\frac{\rho}{\xi^2}\right)d\eta d\xi. \tag{3.14}$$

Putting $\eta = s\cos\theta$ and $\xi = s\, sin\theta$ in (3.14), we get the result in (3.12).

4. Connection with the other special functions

Theorem 4.1.

$$B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2) = \frac{\Gamma(k_4)}{\Gamma(k_3)}\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}\ {}_2\psi_2\left[\begin{matrix}(k_3,q), & (1,1)\\ (k_4,q), & (k_2,k_1)\end{matrix}; -\frac{\rho}{t\,(1-t)}\right]dt. \tag{4.1}$$

Proof: Applying the result from Salim et al., [32]

$$E_{k_1,k_2,p}^{k_3,k_4,q}(z) = \frac{\Gamma(k_4)}{\Gamma(k_3)}\ {}_2\psi_2\left[\begin{matrix}(k_3,q), & (1,1)\\ (k_4,q), & (k_2,k_1)\end{matrix}; z\right],$$

On setting $z \to -\frac{\rho}{t(1-t)}$, multiplying both sides by $t^{\lambda_1-1}(1-t)^{\lambda_2-1}$ and integrating with respect to t limit from 0 to 1, gives the desired result in equation (4.1).

Theorem 4.2.

$$B_{n,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2)$$
$$= \frac{1}{\Gamma(k_2)}\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}\,{}_{q+1}F_{p+n}\left[\begin{matrix}1, & \Delta(q,k_3) \\ \Delta(n,k_2), & (p,k_4)\end{matrix}; -\frac{\rho}{t\,(1-t)}\frac{q^q}{p^p n^n}\right]dt, \quad (4.2)$$

where $k_1 = n \in \mathbb{N}$.

Proof: Using the formula from Salim et al., [32]

$$E_{n,k_2,p}^{k_3,k_4,q}(z) = \frac{1}{\Gamma(k_2)}\,{}_{q+1}F_{p+n}\left[\begin{matrix}1, & \Delta(q,k_3) \\ \Delta(n,k_2), & (p,k_4)\end{matrix}; \frac{zq^q}{p^p n^n}\right]dt,$$

On setting $z \to -\frac{\rho}{t(1-t)}$, multiplying both sides by $t^{\lambda_1-1}(1-t)^{\lambda_2-1}$ and integrating with respect to t limit from 0 to 1, gives the desired result in equation (4.2).

Theorem 4.3.

$$B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2)$$
$$= \frac{\Gamma(k_4)}{\Gamma(k_3)}\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1} H_{2,3}^{1,2}\left[\frac{\rho}{t\,(1-t)}\middle|\begin{matrix} & (0,1) & (1-k_3,q) \\ (0,1), & (1-k_2,k_1), & (1-k_4,p)\end{matrix}\right]dt. \quad (4.3)$$

Proof: Applying the result from Salim et al., [32]

$$E_{k_1,k_2,p}^{k_3,k_4,q}(z) = \frac{\Gamma(k_4)}{\Gamma(k_3)} H_{2,3}^{1,2}\left[-z\middle|\begin{matrix} & (0,1) & (1-k_3,q) \\ (0,1), & (1-k_2,k_1), & (1-k_4,p)\end{matrix}\right],$$

On setting $z \to -\frac{\rho}{t(1-t)}$, multiplying both sides by $t^{\lambda_1-1}(1-t)^{\lambda_2-1}$ and integrating with respect to t limit from 0 to 1, gives the desired result in equation (4.3).

5. Mellin transform

Mellin transform representation of the new extended beta function in the form of following theorem:

Theorem 5.1. Mellin transform representation of the new extended beta function is given by

$$\int_0^\infty \rho^{l-1} B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2)d\rho = B(\lambda_1+l,\lambda_2+l)\Gamma_{k_1,k_2,p;0}^{k_3,k_4,q}(l), \quad (5.1)$$

$(Re(l) > 0, Re(\lambda_1+l) > 0, Re(\lambda_2+l) > 0, Re(\rho) > 0, min\{Re(k_1), Re(k_2), Re(k_3), Re(k_4)\} > 0, p, q > 0 \ \ and\ q < Re(k_1)+p).$

Proof: Multiplying equation (2.3) by ρ^{l-1} and integrating the equation with respect to ρ limit from $\rho = 0$ to $\rho = \infty$, we get

$$\int_0^\infty \rho^{l-1} B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2)d\rho = \int_0^\infty \rho^{l-1}\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1} E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{\rho}{t\,(1-t)}\right)dt d\rho. \quad (5.2)$$

Interchanging the order of integration in the right-hand side of equation (5.2), we have

$$\int_0^\infty \rho^{l-1} B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2)d\rho = \int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}\left\{\int_0^\infty \rho^{l-1} E_{k_1,k_2,p}^{k_3,k_4,q}\left(-\frac{\rho}{t\,(1-t)}\right)d\rho\right\}dt. \quad (5.3)$$

Now using the one-to-one transformation (except possibly at the boundaries and maps the region onto itself) $\rho = \nu t(1-t)$ in equation(5.3), we get

$$\int_0^\infty \rho^{l-1} B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2)d\rho = \int_0^1 t^{(\lambda_1+l)-1}(1-t)^{(\lambda_2+l)-1}dt\left\{\int_0^\infty \nu^{l-1} E_{k_1,k_2,p}^{k_3,k_4,q}(-\nu)d\nu\right\}, \tag{5.4}$$

which on using definition (2.1) (for $\rho = 0$)in the right-hand side yields the desired result in (5.1).

Theorem 5.2. The inverse Mellin transform is given as

$$B_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} B(\lambda_1+l,\lambda_2+l)\Gamma_{k_1,k_2,p;0}^{k_3,k_4,q}(l)\,\rho^{-l}dl. \tag{5.5}$$

Proof. Equation (5.5) follows as a consequence of (5.1).

6. Special cases

In this section, we derive some results for some forms of the new further extended gamma and beta functions in as special cases of the results derived in the previous sections.

(I) Setting $p = q = 1$ in equations (2.1) and (2.3) reduces extended gamma and beta functions in (2.2 b) and (2.4 b) presented by Abubakar and Kabara [23], if $p = q = 1$ and $k_4 = 1$, equations (2.2 c) and (2.4 c) can be obtained, see Al-Gonah and Mohammed [23], letting $p = q = 1$ and $k_1 = k_4 = 1$ Ozarslan and Orzergin [8] extended gamma and beta functions in equations (2.2 d) and (2.4 d) can be established putting $p = q = 1$ and $k_3 = k_4 = 1$ one can obtained equation (2.4 e) introduced by Goyal et al., [22], substituting $p = q = 1$ and $k_2 = k_3 = k_4 = 1$, the new introduced by Shadab et al., [21], Chaudhry and Zubair [1] and Chaudhry et al., [2] in equations (2.2 g) and (2.4 g), moreover, classical gamma and beta functions in equations (2.2 h) and (2.4 h) can be obtained by setting $p = q = 1$, $k_1 = k_2 = k_3 = k_4 = 1$ and $\rho = 0$, see [34].

(II) Setting $p = 1$ in equations (2.1) and (2.3) and using (2.2 b) and (2.4 d) reduced to the following (presumably) new extended gamma and beta functions

$$\Gamma_{k_1,k_2;\rho}^{k_3,k_4,q}(\lambda_1) = \int_0^\infty t^{\lambda_1-1} E_{k_1,k_2}^{k_3,k_4,q}(-t-\rho t^{-1})\,dt,$$

$(Re(\lambda_1) > 0, min\{Re(k_1), Re(k_2), Re(k_3), Re(k_4)\} > 0\ and\ q \in (0,1) \cup \mathbb{N})$, and

$$B_{k_1,k_2;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2) = \int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1} E_{k_1,k_2}^{k_3,k_4,q}\left(-\frac{\rho}{t\,(1-t)}\right)dt,$$

$(Re(\lambda_1) > 0, min\{Re(k_1), Re(k_2), Re(k_3), Re(k_4)\} > 0,\ q \in (0,1) \cup \mathbb{N})$,

here $E_{k_1,k_2}^{k_3,k_4,q}(z)$ is the generalized Mittag-Leffler function introduced by Khan and Ahmad [35, 36]

$$E_{k_1,k_2}^{k_3,k_4,q}(z) = \sum_{n=0}^\infty \frac{(k_3)_{q\,n}}{(k_4)_n \Gamma(k_1 n + k_2)} z^n,$$

$(k_1, k_2, k_3, \in \mathbb{C}; min\{Re(k_1), Re(k_2), Re(k_3),\} > 0,\ and\ q \in (0,1) \cup \mathbb{N})$.

(III) Setting $p = 1$ and $k_4 = 1$ in equations (2.1) and (2.3) and using (2.2 b) and (2.4 d) reduced to the following (presumably) new extended gamma and beta functions

$$\Gamma_{k_1,k_2,p;\rho}^{k_3,k_4,q}(\lambda_1) = \int_0^\infty t^{\lambda_1-1} E_{k_1,k_2,p}^{k_3,k_4,q}(-t-\rho t^{-1})\, dt,$$

$$(Re(\lambda_1) > 0, min\{Re(k_1), Re(k_2), Re(k_3), Re(k_4)\} > 0 \; and \; q \in (0,1) \cup \mathbb{N}),$$

and

$$B_{k_1,k_2;\rho}^{k_3,k_4,q}(\lambda_1,\lambda_2) = \int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1} E_{k_1,k_2}^{k_3,k_4,q}\left(-\frac{\rho}{t\,(1-t)}\right) dt,$$

$$(Re(\lambda_1) > 0, min\{Re(k_1), Re(k_2), Re(k_3), Re(k_4)\} > 0, and \; q \in (0,1) \cup \mathbb{N}),$$

here $E_{k_1,k_2}^{k_3,k_4,q}(z)$ is the generalized Mittag-Leffler function introduced by Shukla and Prajapati [33]

$$E_{k_1,k_2}^{k_3,k_4,q}(z) = \sum_{n=0}^{\infty} \frac{(k_3)_{q\,n}}{n!\,\Gamma(k_1 n + k_2)} z^n,$$

$$(k_1, k_2, k_3, \in \mathbb{C}; min\{Re(k_1), Re(k_2), Re(k_3),\} > 0, \; and \; q \in (0,1) \cup \mathbb{N}).$$

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