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New Results in Bipolar Fuzzy Graphs with an Application

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ABSTRACT

Fuzzy graph models take on the presence being ubiquitous in environmental and fabricated structures by humans, specifically the vibrant processes in physical, biological, and social systems. Owing to the unpredictable and indiscriminate data which are intrinsic in real life, problems are often ambiguous, so it is very challenging for an expert to exemplify those problems by applying a fuzzy graph. Bipolar fuzzy graphs, belonging to the fuzzy graphs family have good capabilities when facing problems that cannot be expressed by fuzzy graphs. Therefore, in this paper, we have introduced the degree and total degree of an edge in the cartesian product of two bipolar fuzzy graphs. Likewise, $\mu-$complement, self $\mu-$complement, and self weak $\mu-$complement on bipolar fuzzy graphs have been presented. Finally, an application of bipolar fuzzy digraphs in social relations has been given.

Keyword: Bipolar fuzzy graph, cartesian product, total edge degree, $\mu-$complement, self weak $\mu-$complement.

Mathematical Subject Classification (2010): 05C72

1. Introduction

The origin of graph theory started with the problem of Konigsberg bridge, in 1735. This problem leads to the concept of the Eulerian graph. Euler studied the problem of the Knigsberg bridge and constructed a structure that solves the problem called the Eulerian graph. In 1840, Mobius gave the idea of the complete graph and bipartite graph and Kuratowski proved that they are planar by means of recreational problems. At present, graph theoretical concepts are highly utilized by computer science applications. Especially in research areas of computer science including data mining, image segmentation, clustering, image capturing, and networking.

In 1965, Zadeh [24] introduced the notion of a fuzzy subset of a set. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines including medical and life sciences, management sciences, social sciences, engineering, statistic, signal processing, decision making, and automata theory. In 1994, Zhang [25] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. A bipolar fuzzy set in an extension of Zadeh's fuzzy set theory whose membership degree range is $[-1,1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0,1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1,0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. The fuzzy graph concept serves as one of the most dominant and extensively employed tools for multiple real-word problem representations, modeling, and analysis. To specify the objects and the relations between them, the graph vertices or nodes and edges or arcs are applied, respectively. Graphs have long been used to describe objects and the relationships between them. In 1975, Rosenfeld [8] discussed the concept of fuzzy graphs whose basic idea was introduced by Kauffman [4] in 1973. Bhattacharya [2] gave some remarks on fuzzy graphs. Mordeson and Peng [5] introduced some operations on fuzzy graphs. The complement of a fuzzy graph was defined by Mordeson [6]. Jun in [3] introduced intuitionistic fuzzy graphs. Gani and Radha investigated the degree of a vertex in some fuzzy graphs [7]. Akram et al. [1] introduced the concepts of bipolar fuzzy graphs. Rashmanlou and Jun defined complete interval-valued fuzzy graphs [9]. Rashmanlou et al. [10, 11,12,13,14,15] introduced several properties of bipolar fuzzy graphs, vague graphs, and interval-valued fuzzy graphs. Samanta and Pal investigated fuzzy tolerance graphs [18], bipolar fuzzy hypergraphs [19], fuzzy k-competition graphs, and p-competition fuzzy graphs [20]. Radha and Kumaravel [17] defined the degree of an edge in cartesian product and composition of two fuzzy graphs. Talebi et al. [21,22,23] investigated new results in interval-valued intuitionistic fuzzy graphs. In this paper, we defined the degree of an edge in the cartesian product of two bipolar fuzzy graphs G_1 and G_2 in some particular cases. Furthermore, $\mu-$complement, self $\mu-$complement, and self weak $\mu-$complement on bipolar fuzzy graphs have been described. Finally, an application of bipolar fuzzy digraphs in social relations has given Throughout this paper, $G_1=(A_1,B_1)$ and $G_2=(A_2,B_2)$ are two bipolar fuzzy graphs with underlying crisp graphs $G_1^*=(V_1,E_1)$ and $G_2^*=(V_2,E_2)$ with $|V_i|=P_i \quad (i=1,2)$.

2. Preliminaries

A fuzzy subset of a set V is a mapping σ from V to $[0,1]$. A fuzzy graph G is a pair of functions $G=(\sigma,\mu)$ where σ is a fuzzy subset of a non-empty set V and μ is a symmetric fuzzy relation on σ, i.e. $\mu(u,v)\le\sigma(u)\wedge\sigma(v)$, where $\sigma(u)\wedge\sigma(v)=\min\{\sigma(u),\sigma(v)\}$. The underlying crisp graph of $G=(\sigma,\mu)$ is denoted by $G^*=(V,E)$ where $E\subseteq V\times V$. The degree of a vertex u is $d_G(u)=\sum_{u\neq v}\mu(uv)$. Since $\mu(uv)>0$ for $uv\in E$, then $\mu(uv)=0$ for $uv\not\in E$. This is equivalent to $d_G(u)=\sum_{u,v\in V}\mu(uv)$. The minimum degree of G is $\delta(G)=\wedge\{d_G(v),\forall\ v\in V\}$ and the maximum degree of G is $\Delta(G)=\vee\{d_G(v),\forall v\in V\}$. The total degree of a vertex $u\in V$ is defined by $td_G(u)=\sum_{u\neq v}\mu(uv)+\sigma(u)$. Since $\mu(u,v)>0$ for $uv\in E$, then $\mu(u,v)>0$ for $uv\not\in E$. This is equivalent to $td_G(u)=d_G(u)+\sigma(u)$. The order and size of a fuzzy graph G are defined by $O(G)=\sum_{u\in V}\sigma(u)$ and $S(G)=\sum_{uv\in E}\mu(uv)$. Let $e=uv$ be an edge in G^*. Then the degree of an edge $e=uv\in E$ is defined by $d_{G^*}(uv)=d_{G^*}(u)+d_{G^*}(v)-2$.

Definition 2.1. *[5] The cartesian product of two fuzzy graphs $G_1=(\sigma_1,\mu_1)$ and $G_2=(\sigma_2,\mu_2)$ is defined as a fuzzy graph $G=G_1\times G_2=(\sigma_1\times\sigma_2,\mu_1\times\mu_2)$ on $G^*=(V,E)$ where $V=V_1\times V_2$ and*

$$E=\{(u_1,u_2)(v_1,v_2)|u_1=v_1 \ \ and \ \ u_2v_2\in E_2 \quad or \quad u_2=v_2 \ \ and \ \ u_1v_1\in E_1\}$$

with $(\sigma_1\times\sigma_2)(u_1,u_2)=\sigma_1(u_1)\wedge\sigma_2(u_2)$, for all $(u_1,u_2)\in V_1\times V_2$ and

$$(\mu_1\times\mu_2)((u_1,u_2)(v_1,v_2))=$$

$$\begin{cases}\sigma_1(u_1) \wedge \mu_2(u_2v_2) & if \quad u_1 = v_1 \quad and \;\; u_2v_2 \in E_2,\\ \sigma_2(u_2) \wedge \mu_1(u_1v_1) & if \quad u_2 = v_2 \quad and \;\; u_1v_1 \in E_1.\end{cases}$$

Definition 2.2. *[25] Let X be a non-empty set. A bipolar fuzzy set B in X is an object having the form*

$$B = \{(x, \mu_B^P(x), \mu_B^N(x)) | x \in X\}$$

where $\mu_B^P: X \to [0,1]$ and $\mu_B^N: X \to [-1,0]$. We say $A = (\mu_{A^P}, \mu_{A^N}): X \times X \to [0,1] \times [-1,0]$ is a bipolar fuzzy relation on X, where $\mu_{A^P}(x,y) \in [0,1]$ and $\mu_{A^N}(x,y) \in [-1,0]$.

Definition 2.3. *[10] Let $A = (\mu_{A^P}, \mu_{A^N})$ and $B = (\mu_{B^P}, \mu_{B^N})$ be bipolar fuzzy sets on set X. If $A = (\mu_{A^P}, \mu_{A^N})$ is a bipolar fuzzy relation on set X, then $A = (\mu_{A^P}, \mu_{A^N})$ is called a bipolar fuzzy relation on $B = (\mu_{B^P}, \mu_{B^N})$ if $\mu_{A^P}(x,y) \le min(\mu_{B^P}(x), \mu_{B^P}(y))$ and $\mu_{A^N}(x,y) \ge max(\mu_{B^N}(x), \mu_{B^N}(y))$, for all $x, y \in X$. A bipolar fuzzy relation A on X is called symmetric if $\mu_{A^P}(x,y) = \mu_{A^P}(y,x)$ and $\mu_{A^N}(x,y) = \mu_{A^N}(y,x)$, for all $x, y \in X$. By a bipolar fuzzy graph $G = (A, B)$ of a graph $G^* = (V, E)$ we mean a pair $G = (A, B)$, where $A = (\mu_{A^P}, \mu_{A^N})$ is a bipolar fuzzy set on V and $B = (\mu_{B^P}, \mu_{B^N})$ is a bipolar fuzzy relation on E such that for all $xy \in E$*

$$\mu_{B^P}(xy) \le \min(\mu_{A^P}(x), \mu_{A^P}(y)) \;\; and \;\; \mu_{B^N}(xy) \ge \max(\mu_{A^N}(x), \mu_{A^N}(y))$$

Definition 2.4. *[1] Given a bipolar fuzzy graph $G = (A, B)$ with the underlying set V. Then the order of G is defined by $O(G) = (\sum_{x \in V} \mu_{A^P}(x), \sum_{x \in V} \mu_{A^N}(x))$. Moreover, the size of a bipolar fuzzy graph G is defined by $S(G) = (\sum_{\substack{x \ne y \\ x,y \in V}} \mu_{B^P}(xy), \sum_{\substack{x \ne y \\ x,y \in V}} \mu_{B^N}(xy))$. The open degree of a vertex u is defined as $deg(u) = (d^P(u), d^N(u))$, where $d^P(u) = \sum_{\substack{u \ne v \\ u \in V}} \mu_{B^P}(uv)$ and $d^N(u) = \sum_{\substack{u \ne v \\ u \in V}} \mu_{B^N}(uv)$. If all the vertices have the same open neighborhood degree n, then G is called an n-regular bipolar fuzzy graph.*

Definition 2.5. *[17] Let $G = (\sigma, \mu)$ be a fuzzy graph on $G^* = (V, E)$. The degree of an edge uv is $d_G(uv) = d_G(u) + d_G(v) - 2\mu(uv)$. The minimum degree and maximum degree of G are $\delta_E(G) = \wedge \{d_G(uv), \forall \;\; uv \in E\}$ and $\Delta_E(G) = \vee \{d_G(uv), \forall \;\; uv \in E\}$.*

Definition 2.6. *[17] Let $G = (\sigma, \mu)$ be a fuzzy graph. The total degree of an edge $uv \in E$ is defined by $td_G(uv) = d_G(u) + d_G(v) - \mu(uv)$.*

Definition 2.7. *[1] The cartesian product $G = G_1 \times G_2$ of two bipolar fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is defined as a pair $(A_1 \times A_2, B_1 \times B_2)$ such that:*

$$(i) \begin{cases}(\mu_{A_1^P} \times \mu_{A_2^P})(u_1, u_2) = \min(\mu_{A_1^P}(u_1), \mu_{A_2^P}(u_2))\\ (\mu_{A_1^N} \times \mu_{A_2^N})(u_1, u_2) = \max(\mu_{A_1^N}(u_1), \mu_{A_2^N}(u_2))\end{cases} \quad for \;\; all \quad (u_1, u_2) \in V,$$

$$(ii)\begin{cases}(\mu_{B_1^P}\times\mu_{B_2^P})((u,u_2)(u,v_2))=\min(\mu_{A_1^P}(u),\mu_{B_2^P}(u_2v_2))\\(\mu_{B_1^N}\times\mu_{B_2^N})((u,u_2)(u,v_2))=\max(\mu_{A_1^N}(u),\mu_{B_2^N}(u_2v_2))\end{cases}\quad for\ \ all\quad u \in V_1 \quad and \quad u_2v_2\in E_2,$$

$$(iii)\begin{cases}(\mu_{B_1^P}\times\mu_{B_2^P})((u_1,z)(v_1,z))=\min(\mu_{B_1^P}(u_1v_1),\mu_{A_2^P}(z))\\(\mu_{B_1^N}\times\mu_{B_2^N})((u_1,z)(v_1,z))=\max(\mu_{B_1^N}(u_1v_1),\mu_{A_2^N}(z))\end{cases}\quad for\ \ all\quad z \in V_2 \quad and \quad u_1v_1\in E_1.$$

Theorem 2.8. *If* $G_1=(A_1,B_1)$ *and* $G_2=(A_2,B_2)$ *are two bipolar fuzzy graphs such that* $\mu_{A_1^P}\le\mu_{B_2^P}$ and $\mu_{A_1^N}\ge\mu_{B_2^N}$. Then $\mu_{A_2^P}\ge\mu_{B_1^P}$, $\mu_{A_2^N}\le\mu_{B_1^N}$ and vice versa.

3. Degree of an edge in the cartesian product

In this section, we define the total degree of a vertex, degree and total degree of an edge in bipolar fuzzy graphs and prove two basic theorems that help us for deriving the degree of an edge in the cartesian product.

Definition 3.1. *Let* $G=(A,B)$ *be a bipolar fuzzy graph. Then the total degree of a vertex* $u\in V$ *is defined by* $td_G(u)=(td_G^P(u),td_G^N(u))$ *where*

$$td_G^P(u)=\sum_{u\neq v}\mu_{B^P}(uv)+\mu_{A^P}(u)\qquad and \qquad td_G^N(u)=\sum_{u\neq v}\mu_{B^N}(uv)+\mu_{A^N}(u).$$

Definition 3.2. *Let* $G=(A,B)$ *be a bipolar fuzzy graph on* $G^*=(V,E)$. *The degree of an edge* uv *is* $d_G(uv)=(d_G^P(uv),d_G^N(uv))$ *where* $d_G^P(uv)=d_G^P(u)+d_G^P(v)-2\mu_{B^P}(uv)$ *and* $d_G^N(uv)=d_G^N(u)+d_G^N(v)-2\mu_{B^N}(uv)$. *This is equivalent to*

$$d_G^P(uv)=\sum_{\substack{uw\in E\\ w\neq v}}\mu_{B^P}(uw)+\sum_{\substack{wv\in E\\ w\neq u}}\mu_{B^P}(wv),\qquad d_G^N(uv)=\sum_{\substack{uw\in E\\ w\neq v}}\mu_{B^N}(uw)+\sum_{\substack{wv\in E\\ w\neq u}}\mu_{B^N}(wv).$$

The minimum degree and maximum degree of G are

$$\delta_E(G)=\wedge\{d_G(uv),\forall\ \ uv\in E\}\quad and \quad \Delta_E(G)=\vee\{d_G(uv),\forall\ \ uv\in E\}.$$

Definition 3.3. *Let* $G=(A,B)$ *be a bipolar fuzzy graph on* $G^*=(V,E)$. *Then the total degree of an edge* uv *is defined by* $td_G(uv)=(td_G^P(uv),td_G^N(uv))$ *where*

$$td_G^P(uv)=d_G^P(u)+d_G^P(v)-\mu_{B^P}(uv)\quad and \quad td_G^N(uv)=d_G^N(u)+d_G^N(v)-\mu_{B^N}(uv).$$

This is equivalent to

$$td_G^P(uv)=\sum_{\substack{uw\in E\\ w\neq v}}\mu_{B^P}(wu)+\sum_{\substack{wv\in E\\ w\neq u}}\mu_{B^P}(wv)+\mu_{B^P}(uv)=d_G^P(uv)+\mu_{B^P}(uv),$$

$$td_G^N(uv)=\sum_{\substack{uw\in E\\ w\neq v}}\mu_{B^N}(uw)+\sum_{\substack{wv\in E\\ w\neq u}}\mu_{B^N}(wv)+\mu_{B^N}(uv)=d_G^N(uv)+\mu_{B^N}(uv).$$

Examples 3.4. *Consider bipolar fuzzy graph* G *defined as follow.*

$d_G(u)=(0.6,-0.8)$, $td_G(u)=(0.9,-1.2)$, $\delta(G)=(0.6,-0.8)=d_G(u)$ and $\Delta(G)=$

$(1.4, -1.7) = d_G(w)$, $d_G(uv) = (0.7, -0.9)$, $td_G(uv) = (1, -1.3)$, $\delta_E(G) = (0.7, -0.9) = d_G(uv) = d_G(wz)$, $\Delta_E(G) = (1.4, -1.7) = d_G(uw)$.

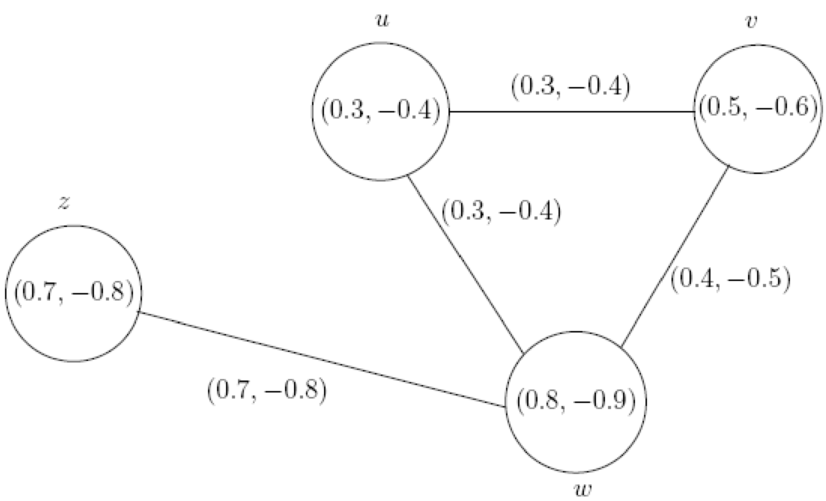

Figure 1: Bipolar fuzzy graph G

Note: By the definition of cartesian product, for any $(u_1, u_2) \in V_1 \times V_2$ and $((u_1, u_2)(v_1, v_2)) \in E$ with $u_1 = v_1,\ u_2 \neq v_2$ or $u_1 \neq v_1,\ u_2 = v_2$

$$d^P_{G_1 \times G_2}((u_1, u_2)(v_1, v_2)) = \sum_{(u_1,u_2)(z_1,z_2) \in E_{(z_1,z_2) \neq (v_1,v_2)}} (\mu_{B_1^P} \times \mu_{B_2^P})((u_1, u_2)(z_1, z_2)) + \sum_{(z_1,z_2)(v_1,v_2) \in E_{(z_1,z_2) \neq (u_1,u_2)}} (\mu_{B_1^P} \times \mu_{B_2^P})((z_1, z_2)(v_1, v_2)).$$

Also, we have

$$d^N_{G_1 \times G_2}((u_1, u_2)(v_1, v_2)) = \sum_{(u_1,u_2)(z_1,z_2) \in E_{(z_1,z_2) \neq (v_1,v_2)}} (\mu_{B_1^N} \times \mu_{B_2^N})((u_1, u_2)(z_1, z_2)) + \sum_{(z_1,z_2)(v_1,v_2) \in E_{(z_1,z_2) \neq (u_1,u_2)}} (\mu_{B_1^N} \times \mu_{B_2^N})((z_1, z_2)(v_1, v_2)).$$

If $u_1 = v_1,\ u_2 \neq v_2$, then

$$d^P_{G_1 \times G_2}((u_1, u_2)(u_1, v_2)) = \sum_{(u_1,u_2)(z_1,z_2) \in E_{(z_1,z_2) \neq (u_1,v_2)}} (\mu_{B_1^P} \times \mu_{B_2^P})((u_1, u_2)(z_1, z_2)) + \sum_{(z_1,z_2)(u_1,v_2) \in E_{(z_1,z_2) \neq (u_1,u_2)}} (\mu_{B_1^P} \times \mu_{B_2^P})((z_1, z_2)(u_1, v_2)).$$

$$= \sum_{(u_1,u_2)(u_1,z_2) \in E, u_1 = z_{1_{z_2 \neq v_2}}} (\mu_{B_1^P} \times \mu_{B_2^P})((u_1, u_2)(u_1, z_2))$$

$$+ \sum_{(u_1,u_2)(z_1,u_2) \in E_{u_2 = z_2}} (\mu_{B_1^P} \times \mu_{B_2^P})((u_1, u_2)(z_1, u_2))$$

$$+\sum_{(u_1,z_2)(u_1,v_2)\in E_{z_1=u_1,z_2\neq u_2}} (\mu_{B_1^P}\times\mu_{B_2^P})((u_1,z_2)(u_1,v_2))$$
$$+\sum_{(z_1,v_2)(u_1,v_2)\in E_{z_2=v_2}} (\mu_{B_1^P}\times\mu_{B_2^P})((z_1,v_2)(u_1,v_2))$$
$$=\sum_{u_2z_2\in E_2,u_1=z_{1_{z_2\neq v_2}}} \mu_{A_1^P}(u_1)\wedge\mu_{B_2^P}(u_2z_2)+$$
$$\sum_{u_1z_1\in E,u_2=z_2} \mu_{B_1^P}(u_1z_1)\wedge\mu_{A_2^P}(u_2)$$
$$+\sum_{z_2v_2\in E_2,z_1=u_{1_{z_2\neq u_2}}} \mu_{A_1^P}(u_1)\wedge\mu_{B_2^P}(z_2v_2)+$$
$$\sum_{z_1u_1\in E_1,z_2=v_2} \mu_{B_1^P}(z_1u_1)\wedge\mu_{A_2^P}(v_2).$$

So,

$$d^P_{G_1\times G_2}((u_1,u_2)(u_1,v_2))=\sum_{z_2\in V_2,z_2\neq v_2}\mu_{A_1^P}(u_1)\wedge\mu_{B_2^P}(u_2z_2)+\sum_{z_1\in V_1}\mu_{B_1^P}(u_1z_1)\wedge\mu_{A_2^P}(u_2)$$
$$+\sum_{z_2\in V_2,z_2\neq u_2}\mu_{A_1^P}(u_1)\wedge\mu_{B_2^P}(z_2v_2)+\sum_{z_1\in V_1}\mu_{B_1^P}(z_1u_1)\wedge\mu_{A_2^P}(v_2). \tag{3.1}$$

In the same way we can show that

$$d^N_{G_1\times G_2}((u_1,u_2)(u_1,v_2))=\sum_{z_2\in V_2,z_2\neq v_2}\mu_{A_1^N}(u_1)\vee\mu_{B_2^N}(u_2z_2)+\sum_{z_1\in V_1}\mu_{B_1^N}(u_1z_1)\vee\mu_{A_2^N}(u_2)$$
$$+\sum_{z_2\in V_2,z_2\neq u_2}\mu_{A_1^N}(u_1)\vee\mu_{B_2^N}(z_2v_2)+\sum_{z_1\in V_1}\mu_{B_1^N}(z_1u_1)\vee\mu_{A_2^N}(v_2). \tag{3.2}$$

If $u_1\neq v_1,\ u_2=v_2$, by routine computations it is easy to see that

$$d^P_{G_1\times G_2}((u_1,u_2)(v_1,u_2))=\sum_{z_2\in V_2}\mu_{A_1^P}(u_1)\wedge\mu_{B_2^P}(u_2z_2)+\sum_{z_1\in V_1,z_1\neq v_1}\mu_{B_1^P}(u_1z_1)\wedge\mu_{A_2^P}(u_2)$$
$$+\sum_{z_2\in V_2}\mu_{A_1^P}(v_1)\wedge\mu_{B_2^P}(z_2u_2)+\sum_{z_1\in V_1,z_1\neq u_1}\mu_{B_1^P}(z_1v_1)\wedge\mu_{A_2^P}(u_2). \tag{3.3}$$

$$d^N_{G_1\times G_2}\big((u_1,u_2)(v_1,v_2)\big)=\sum_{z_2\in V_2}\mu_{A_1^N}(u_1)\vee\mu_{B_2^N}(u_2z_2)+\sum_{z_1\in V_1,z_1\neq v_1}\mu_{B_1^N}(u_1z_1)\vee\mu_{A_2^N}(u_2)$$
$$+\sum_{z_2\in V_2}\mu_{A_1^N}(v_1)\vee\mu_{B_2^N}(z_2v_2)+\sum_{z_1\in V_1,z_1\neq u_1}\mu_{B_1^N}(z_1v_1)\vee\mu_{A_2^N}(u_2). \tag{3.4}$$

In the following theorems, we find the degree of $((u_1,u_2)(u_1,v_2))$ and $((u_1,u_2)(v_1,u_2))$ in $G_1\times G_2$ in terms of those in G_1 and G_2 in some particular cases.

Theorem 3.5. *Let $G_1=(A_1,B_1)$ and $G_2=(A_2,B_2)$ be two bipolar fuzzy graphs. If* $\mu_{A_1^P}\geq\mu_{B_2^P},\ \mu_{A_1^N}\leq\mu_{B_2^N},\ \mu_{A_2^P}\geq\mu_{B_1^P}$ and $\mu_{A_2^N}\leq\mu_{B_1^N}$, then

- $d^P_{G_1\times G_2}((u_1,u_2)(u_1,v_2))=2d^P_{G_1}(u_1)+d^P_{G_1}(u_2v_2),$ $\quad if \quad ((u_1,u_2)(u_1,v_2))\in E,$
  $d^N_{G_1\times G_2}((u_1,u_2)(u_1,v_2))=2d^N_{G_1}(u_1)+d^N_{G_1}(u_2v_2),$ $\quad if \quad ((u_1,u_2)(u_1,v_2))\in E.$
- $d^P_{G_1\times G_2}((u_1,u_2)(v_1,u_2))=d^P_{G_1}(u_1v_1)+2d^P_{G_2}(u_2), \quad if \quad ((u_1,u_2)(v_1,u_2))\in E,$

$d^N_{G_1\times G_2}((u_1,u_2)(v_1,u_2)) = d^N_{G_1}(u_1v_1) + 2d^N_{G_2}(u_2), \qquad if \quad ((u_1,u_2)(v_1,u_2)) \in E.$

Proof: We have $\mu_{A_1^P} \geq \mu_{B_2^P}$, $\mu_{A_1^N} \leq \mu_{B_2^N}$, $\mu_{A_2^P} \geq \mu_{B_1^P}$ and $\mu_{A_2^N} \leq \mu_{B_1^N}$. From (3.1) and (3.2), for any $(u_1,u_2)(u_1,v_2) \in E$,

$$d^P_{G_1\times G_2}((u_1,u_2)(u_1,v_2)) = \sum_{z\in V_2, z_2\neq v_2} \mu_{A_1^P}(u_1) \wedge \mu_{B_2^P}(u_2z_2) + \sum_{z_1\in V_1} \mu_{B_1^P}(u_1z_1) \wedge \mu_{A_2^P}(u_2)$$
$$+ \sum_{z_2\in V_2, z_2\neq u_2} \mu_{A_1^P}(u_1) \wedge \mu_{B_2^P}(z_2v_2) + \sum_{z_1\in V_1} \mu_{B_1^P}(z_1u_1) \wedge \mu_{A_2^P}(v_2)$$
$$= \sum_{z_2\in V_2, z_2\neq v_2} \mu_{B_2^P}(u_2z_2) + \sum_{z_1\in V_1} \mu_{B_1^P}(u_1z_1) + \sum_{z_2\in V_2, z_2\neq u_2} \mu_{B_2^P}(z_2v_2)$$
$$+ \sum_{z_1\in V_1} \mu_{B_1^P}(z_1u_1)$$
$$= 2d^P_{G_1}(u_1) + d^P_{G_2}(u_2v_2).$$

By the similar way, we have $d^N_{G_1\times G_2}((u_1,u_2)(u_1,v_2)) = 2d^N_{G_1}(u_1) + d^N_{G_2}(u_2v_2)$.

From (3.3) and (3.4), for any $(u_1,u_2)(v_1,u_2) \in E$,

$$d^P_{G_1\times G_2}((u_1,u_2)(v_1,u_2)) = \sum_{z_2\in V_2} \mu_{A_1^P}(u_1) \wedge \mu_{B_2^P}(u_2z_2) + \sum_{z_1\in V_1, z_1\neq v_1} \mu_{B_1^P}(u_1z_1) \wedge \mu_{A_2^P}(u_2)$$
$$+ \sum_{z_2\in V_2} \mu_{A_1^P}(v_1) \wedge \mu_{B_2^P}(z_2v_2) + \sum_{z_1\in V_1, z_1\neq u_1} \mu_{B_1^P}(z_1v_1) \wedge \mu_{A_2^P}(u_2)$$
$$= \sum_{z_2\in V_2} \mu_{B_2^P}(u_2z_2) + \sum_{z_1\in V_1, z_1\neq v_1} \mu_{B_1^P}(u_1z_1) + \sum_{z_2\in V_2} \mu_{B_2^P}(z_2u_2)$$
$$+ \sum_{z_1\in V_1, z_1\neq u_1} \mu_{B_1^P}(z_1v_1) = d^P_{G_1}(u_1v_1) + 2d^P_{G_2}(u_2).$$

Similarly, we can show that $d^N_{G_1\times G_2}((u_1,u_2)(v_1,u_2)) = d^N_{G_1}(u_1v_1) + 2d^N_{G_2}(u_2)$.

Examples 3.6. *Consider the bipolar fuzzy graphs G_1, G_2, $G_1 \times G_2$ shown in Figure 2.*

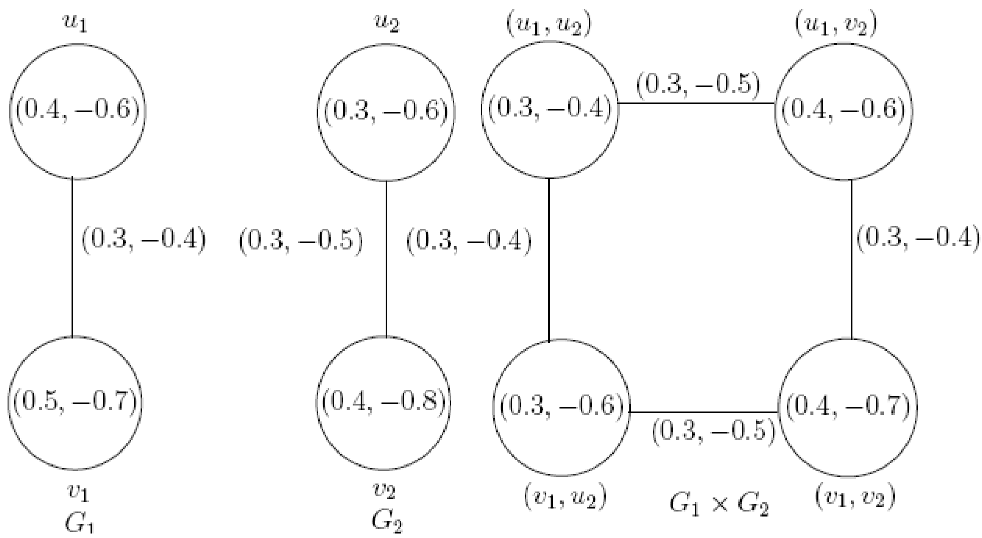

Figure 2: Cartesian product $(G_1 \times G_2)$ of G_1 and G_2

Since, $\mu_{A_1^P} \geq \mu_{B_2^P}$, $\mu_{A_1^N} \leq \mu_{B_2^N}$ and $\mu_{A_2^P} \geq \mu_{B_1^P}$, $\mu_{A_2^N} \leq \mu_{B_1^N}$. Then by Theorem 3.5 we have

$$d^P_{G_1\times G_2}((u_1,u_2)(u_1,v_2)) = 2d^P_{G_1}(u_1) + d^P_{G_2}(u_2v_2) = 2(0.3) + 0 = 0.6,$$
$$d^N_{G_1\times G_2}((u_1,u_2)(u_1,v_2)) = 2d^N_{G_1}(u_1) + d^N_{G_2}(u_2v_2) = 2(-0.4) + 0 = -0.8,$$
$$d^P_{G_1\times G_2}((v_1,u_2)(v_1,v_2)) = 2d^P_{G_1}(v_1) + d^P_{G_2}(u_2v_2) = 2(0.3) + 0 = 0.6,$$
$$d^N_{G_1\times G_2}((v_1,u_2)(v_1,v_2)) = 2d^N_{G_1}(v_1) + d^N_{G_2}(u_2v_2) = 2(-0.4) + 0 = -0.8,$$
$$d^P_{G_1\times G_2}((u_1,u_2)(v_1,u_2)) = d^P_{G_1}(u_1v_1) + 2d^P_{G_2}(u_2) = 0 + 2(0.3) = 0.6,$$
$$d^N_{G_1\times G_2}((u_1,u_2)(v_1,u_2)) = d^N_{G_1}(u_1v_1) + 2d^N_{G_2}(u_2) = 0 + 2(-0.5) = -1.$$

Similarly, we can find the degrees of all the edges in $G_1 \times G_2$. This can be verified in Figure 2.

Theorem 3.7. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two bipolar fuzzy graphs. If $\mu_{A_1^P} \leq \mu_{B_2^P}$, $\mu_{A_1^N} \geq \mu_{B_2^N}$ and $\mu_{A_1^P}$, $\mu_{A_1^N}$ be constant function with $\mu_{A_1^P}(u) = c_1$, $\mu_{A_1^N}(u) = c_1$, for all $u \in V_1$, then*

(i) For any $(u_1,u_2)(u_1,v_2) \in E$, $d^P_{G_1\times G_2}((u_1,u_2),(u_1,v_2)) = 2d^P_{G_1}(u_1) + c_1(d^P_{G_2^*}(u_2) + d^P_{G_2^*}(v_2) - 2)$ and

$$d^N_{G_1\times G_2}((u_1,u_2),(u_1,v_2)) = 2d^N_{G_1}(u_1) + c_1(d^N_{G_2^*}(u_2) + d^N_{G_2^*}(v_2) - 2).$$

(ii) For any $(u_1,u_2)(v_1,u_2) \in E$, $d^P_{G_1\times G_2}((u_1,u_2),(v_1,u_2)) = d^P_{G_1}(u_1v_1) + 2c_1d^P_{G_2^*}(u_2)$ and

$$d^N_{G_1\times G_2}((u_1,u_2),(v_1,u_2)) = d^N_{G_1}(u_1v_1) + 2c_1d^N_{G_2^*}(u_2).$$

Proof: Since $\mu_{A_1^P} \leq \mu_{B_2^P}$ and $\mu_{A_1^N} \geq \mu_{B_2^N}$, then Theorem 2.8 shows, $\mu_{A_2^P} \geq \mu_{B_1^P}$ and $\mu_{A_2^N} \leq \mu_{B_1^N}$. From (3.1) and (3.2), for any $(u_1,u_2),(u_1,v_2) \in E$,

$$d^P_{G_1\times G_2}((u_1,u_2),(u_1,v_2)) = \sum_{z_2\in V_2, z_2\neq v_2} \mu_{A_1^P}(u_1) \wedge \mu_{B_2^P}(u_2z_2) + \sum_{z_1\in V_1} \mu_{B_1^P}(u_1z_1) \wedge \mu_{A_2^P}(u_2)$$
$$+ \sum_{z_2\in V_2, z_2\neq u_2} \mu_{A_1^P}(u_1) \wedge \mu_{B_2^P}(z_2v_2) + \sum_{z_1\in V_1} \mu_{B_1^P}(z_1u_1) \wedge \mu_{A_2^P}(v_2).$$

Since $\mu_{A_1^P}(u) = c_1$ and $\mu_{A_1^N}(u) = c_1$, for all $u \in V_1$.

$$d^P_{G_1\times G_2}((u_1,u_2),(u_1,v_2))$$
$$= \sum_{z_2\in V_2, u_2z_2\in E_{2_{z_2\neq v_2}}} c_1 + \sum_{z_1\in V_1} \mu_{B_1^P}(u_1z_1)$$
$$+ \sum_{z_2\in V_2, z_2v_2\in E_{2_{z_2\neq u_2}}} c_1 + \sum_{z_1\in V_1} \mu_{B_1^P}(z_1u_1)$$
$$= \sum_{u_2z_2\in E_{2,_{z_2\neq v_2}}} c_1 + \sum_{z_1\in V_1} \mu_{B_1^P}(u_1z_1) + \sum_{z_2v_2\in E_{2_{z_2\neq u_2}}} c_1 + \sum_{z_1\in V_1} \mu_{B_1^P}(z_1u_1)$$
$$= c_1(d^P_{G_2^*}(u_2) - 1) + d^P_{G_1}(u_1) + c_1(d^P_{G_2^*}(v_2) - 1)$$

$+d^P_{G_1}(u_1) = 2d^P_{G_1}(u_1) + c_1(d^P_{G_2^*}(u_2) + d^P_{G_2^*}(v_2) - 2)$. By the similar way we have $d^N_{G_1\times G_2}((u_1,u_2),(u_1,v_2)) = 2d^N_{G_1}(u_1) + c_1(d^N_{G_2^*}(u_2) + d^N_{G_2^*}(v_2) - 2)$. The prove of (ii) is similar to above argument.

4. μ −complement and self μ −complement bipolar fuzzy graphs

In this section we defined $G^{\mu}:(A,B^{\mu})$, μ −complement of a bipolar fuzzy graph G.

Definition 4.1. *Let $G=(A,B)$ be a bipolar fuzzy graph. The μ −complement of G is defined as $G^{\mu}:(A,B^{\mu})$, where $B^{\mu}=(\mu^{\mu}_{B^P},\mu^{\mu}_{B^N})$ and we have:*

$$\mu^{\mu}_{B^P}(xy)=\begin{cases}\mu^P_A(x)\wedge\mu^P_A(y)-\mu^P_B(xy) & if \quad \mu^P_B(xy)>0\\ 0 & if \quad \mu^P_B(xy)=0\end{cases}$$

$$\mu^{\mu}_{B^N}(xy)=\begin{cases}\mu^N_A(x)\vee\mu^N_A(y)-\mu^N_B(xy) & if \quad \mu^N_B(xy)<0\\ 0 & if \quad \mu^N_B(xy)=0\end{cases}$$

Definition 4.2. *A bipolar fuzzy graph G is said to be a self μ −complement bipolar fuzzy graph if $G\cong G^{\mu}$.*

Theorem 4.3. *Let $G=(A,B)$ be a self μ −complement bipolar fuzzy graph of a graph $G^*=(V,E)$. Then,*

$$\sum_{\substack{x\neq y\\ \mu^P_B(xy)>0}}\mu^P_B(xy)=\frac{1}{2}\sum_{\substack{x\neq y\\ \mu^P_B(xy)>0}}\mu^P_A(x)\wedge\mu^P_A(y),$$

$$\sum_{x\neq y}\mu^N_B(xy)=\frac{1}{2}\sum_{x\neq y}\mu^N_A(x)\vee\mu^N_A(y).$$

Proof: Let $G=(A,B)$ be a self μ −complement bipolar fuzzy graph of a graph $G^*=(V,E)$. Then, there exists an isomorphism $g:V\to V$ such that $\mu^P_A(x)=\mu^{\mu}_{A^P}(g(x))$, $\mu^N_A(x)=\mu^{\mu}_{A^N}(g(x))$, for all $x\in V$ and $\mu^P_B(xy)=\mu^{\mu}_{B^P}(g(x)g(y))$, $\mu^N_B(xy)=\mu^{\mu}_{B^N}(g(x)g(y))$, for all $x,y\in V$.

Now by definition of G^{μ}, for all $x,y\in V$ which $\mu^P_B(xy)>0$ we have

$$\mu^{\mu}_{B^P}(g(x)g(y))=\mu^{\mu}_{A^P}(g(x))\wedge\mu^{\mu}_{A^N}(g(y))-\mu^{\mu}_{B^P}(g(x)g(y))$$

$$i.e., \mu^P_B(xy)=\mu^{\mu}_{A^P}(g(x))\wedge\mu^P_A(g(y))-\mu^{\mu}_{B^P}(g(x)g(y)).$$

Hence,

$$\sum_{\substack{x\neq y\\ \mu^P_B(xy)>0}}\mu^P_B(xy)+\sum_{\substack{x\neq y\\ \mu^P_B(xy)>0}}\mu^P_B(g(x)g(y))=\sum_{\substack{x\neq y\\ \mu^P_B(xy)>0}}\mu^P_A(x)\wedge\mu^P_A(y).$$

It follows that

$$2\sum_{\mu^P_B(xy)>0}\mu^P_B(xy)-\sum_{\mu^P_B(xy)=0}\mu^P_B(g(x)g(y))=\sum_{\mu^P_B(xy)>0}\mu^P_A(x)\wedge\mu^P_A(y).$$

In other hand, if $\mu^P_B(xy)=0$ and $\mu^P_B(g(x)g(y))\neq 0$ for some $x,y\in V$, then $\mu^P_B(g(x)g(y))=\mu_{A^P}(g(x))\wedge\mu^P_A(g(y))=\mu_{A^P}(x)\wedge\mu^P_A(y)$. Thus

$$2\sum_{x\neq y}\mu^P_B(xy)=\sum_{x\neq y}\mu^P_A(x)\wedge\mu^P_A(y)$$

$$i.e., \sum_{x\neq y}\mu^P_B(xy)=\frac{1}{2}\sum_{x\neq y}\mu^P_A(x)\wedge\mu^P_A(y).$$

Similarly we can show that

$$\sum_{x\neq y}\mu^N_B(xy)=\frac{1}{2}\sum_{x\neq y}\mu^N_A(x)\vee\mu^N_A(y).$$

Definition 4.4. *A bipolar fuzzy graph $G=(A,B)$ is said to be a self weak μ −complement bipolar fuzzy graph if G is weak isomorphic with G^{μ}.*

Theorem 4.5. *Let $G=(A,B)$ be a self weak μ −complement bipolar fuzzy graph of a graph $G^*=(V,E)$. Then,*

$$\sum_{x\neq y} \mu_B^P(xy) \leq \frac{1}{2}\sum_{x\neq y} \mu_A^P(x) \wedge \mu_A^P(y),$$
$$\sum_{x\neq y} \mu_B^N(xy) \geq \frac{1}{2}\sum_{x\neq y} \mu_A^N(x) \vee \mu_A^N(y).$$

Proof: Let $G=(A,B)$ be a self weak μ −complement bipolar fuzzy graph of a graph $G^*=(V,E)$. Then there exists a weak isomorphism $h: G \to G^{\mu}$ such that for all $x,y \in V$ we have

$\mu_A^P(x)=\mu_{A^P}^{\mu}(h(x))=\mu_A^P(h(x))$, $\mu_A^N(x)=\mu_{A^N}^{\mu}(h(x))=\mu_A^N(h(x))$, for all $x \in V$.

$\mu_B^P(xy) \leq \mu_{B^P}^{\mu}(h(x)h(y))$, $\mu_B^N(xy) \geq \mu_{B^N}^{\mu}(h(x)h(y))$, for all $x,y \in V$.

If $\mu_B^P(x)>0$, then $\mu_{B^P}^{\mu}(x)(h(x)h(y))>0$ and using the definition of complement in the above inequality, we have

$$\mu_B^P(xy) \leq \mu_{B^P}^{\mu}(h(x)h(y)) = \mu_A^P(h(x)) \wedge \mu_A^P(h(y)) - \mu_B^P(h(x)h(y))$$
$$\mu_B^P(xy) + \mu_B^P(h(x)h(y)) \leq \mu_A^P(h(x)) \wedge \mu_A^P(h(y))$$

$$\sum_{x\neq y_{\mu_B^P(xy)\neq 0}} \mu_B^P(xy) + \sum_{x\neq y_{\mu_B^P(xy)>0}} \mu_B^P(h(x)h(y)) \leq \sum_{x\neq y_{\mu_B^P(xy)>0}} \mu_A^P(h(x)) \wedge \mu_{A^P}(h(y)).$$

So,

$$\sum_{x\neq y} \mu_B^P(xy) + \sum_{x\neq y} \mu_B^P(h(x)h(y)) - \sum_{x\neq y_{\mu_B^P(xy)=0}} \mu_B^P(h(x)h(y)) \leq \sum_{\mu_B^P(xy)>0} \mu_A^P(h(x)) \wedge \mu_A^P(h(y)).$$

Now $\mu_B^P(h(x)h(y)) \leq \mu_A^P(h(x)) \wedge \mu_A^P(h(y))$, implies that

$$2\sum_{x\neq y} \mu_B^P(xy) \leq \sum_{x\neq y} \mu_A^P(h(x)) \wedge \mu_A^P(h(y))$$
$$= \sum_{x\neq y} \mu_A^P(x) \wedge \mu_A^P(y).$$

Hence,

$$\sum_{x\neq y} \mu_B^P(xy) \leq \frac{1}{2}\sum_{x\neq y} \mu_A^P(x) \wedge \mu_A^P(y).$$

Similarly, we can show that

$$\sum_{x\neq y} \mu_B^N(xy) \geq \frac{1}{2}\sum_{x\neq y} \mu_A^N(x) \vee \mu_A^N(y).$$

Theorem 4.6. *Let $G=(A,B)$ be a bipolar fuzzy graph of a graph $G^*=(V,E)$, if*

$\mu_B^P(xy) \leq \frac{1}{2}(\mu_A^P(x) \wedge \mu_A^P(y))$ and $\mu_B^N(xy) \geq \frac{1}{2}(\mu_A^N(x) \vee \mu_A^N(y))$, then $G=(A,B)$ is a self weak μ −complement bipolar fuzzy graph.

Proof: Consider the identity map $I: V \to V$, $\mu_A^P(x)=\mu_A^P(I(x))$, $\mu_A^N(x)=\mu_A^N(I(x))$, for all $x \in V$.

By definition of μ_B^P, for all $x,y \in V$ such that $\mu_B^P(xy)>0$, we have

$\mu^{\mu}_{B^P}(xy) = \mu^P_A(x) \wedge \mu^P_A(y) - \mu^P_B(xy)$. Hence,

$$\mu^{\mu}_{B^P}(xy) \geq \mu^P_A(x) \wedge \mu^P_A(y) - \frac{1}{2}(\mu^P_A(x) \wedge \mu^P_A(y)) = \frac{1}{2}(\mu^P_A(x) \wedge \mu^P_A(y)) \geq \mu^P_B(xy).$$

Also, if $\mu^P_B(xy) = 0$, it is clear that $\mu^{\mu}_{B^P}(xy) \geq \mu^P_B(xy)$. Hence, $\mu^{\mu}_{B^P}(xy) \geq \mu^P_B(xy)$, for all $x, y \in V$.

Similarly, we can prove that $\mu^{\mu}_{B^N}(xy) \geq \mu^N_B(xy)$, for all $x, y \in V$.

Notation 4.7. *We denote by* $Aut(G)$, *the automorphism group of a bipolar fuzzy graph* G.

Theorem 4.8. *Let* $G = (A, B)$ *be a bipolar fuzzy graph. Then, the automorphism group of* G *and* G^{μ} *are identical.*

Proof: We show that for any injective map $h: V \to V$, $h \in Aut(G)$ if and only if $h \in Aut(G^{\mu})$. We have

$$\mu^{\mu}_{A^P}(h(x)) = \mu_{A^P}(h(x)) = \mu_{A^P}(x) = \mu^{\mu}_{A^P}(x), for \quad all \quad x \in V,$$

$$\mu^{\mu}_{A^N}(h(x)) = \mu^N_A(h(x)) = \mu^N_A(x) = \mu^{\mu}_{A^N}(x), for \quad all \quad x \in V.$$

Also, for all $x, y \in V$, $\mu^{\mu}_{B^P}(h(x)h(y)) = \mu^{\mu}_{B^P}(xy)$

$\Leftrightarrow \mu^P_A(h(x)) \wedge \mu^P_A(h(y)) - \mu_{B^P}(h(x)h(y)) = \mu^P_A(x) \wedge \mu^P_A(y) - \mu^P_B(xy)$

$\Leftrightarrow \mu^P_B(h(x)h(y)) = \mu^P_B(xy), \qquad and \qquad \mu^{\mu}_{B^N}(h(x)h(y)) = \mu^{\mu}_{B^N}(xy)$

$\Leftrightarrow \mu^N_A(h(x)) \wedge \mu^N_A(h(y)) - \mu_{B^N}(h(x)h(y)) = \mu^N_A(x) \wedge \mu^N_A(y) - \mu^N_B(xy)$

$\Leftrightarrow \mu^N_B(h(x)h(y)) = \mu^N_B(xy)$.

This completes the proof.

5. Application of bipolar fuzzy digraphs in social relations

Today, social relations in associations and service centers are very important issues that can play a significant role in the development of that community. As we know, if the employees of a service center have a higher level of communication and eloquence, then they can be more accountable and solve people's problems more quickly. These days, we see that graph model have many applications in different sciences such as computer science, topology, operations research, biological and social sciences. If we consider group behavior, it is observed that in a social group some people can influence the thinking of others. Now with help of a directed graph which is an influence graph, we can use to model this behavior. We consider each person of a group as a vertex.

Now let us consider a fuzzy influence graph of a social group. In Figure 3, the nodes are depicting the degree of power of a person who belongs to a set of a social groups. The degree of power of a person is defined in terms of membership. The degree of membership can be interpreted as how much power a person possesses, e.g., Mehdi has 50% power within the social group. The edges of a graph show the influence of one person on another person. The degree of membership of edges can be interpreted as the percentage of positive, e.g., Hassan follows 30% of Mehdi's suggestions.

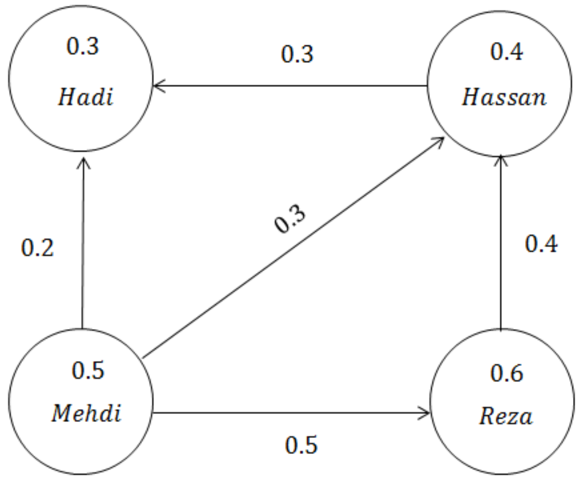

Figure 3: Fuzzy influence graph

It is clear that the person's speech in a group may not be impressive. For example, may be two persons in a group have some conflicts. Hence, we present a bipolar fuzzy influence graph for such case. In Figure 4, the nodes are depicting the degree of power of a person belongs to a set of social group. The degree of power of a person is defined in terms of its positive membership and negative membership. Degree of positive membership can be interpreted as how much power a person possess and negative membership can be interpreted as how much power a person losses, Mehdi has 50% power within the social group but he losses 30% power in the same group. The edges of a graph show the influence of one person onto another person. The degree of positive membership and negative membership of edges can be interpreted as the percentage of positive and negative influence, e.g., Hassan follows 30% Mehdi's suggestions but he does not follows 20% his suggestions.

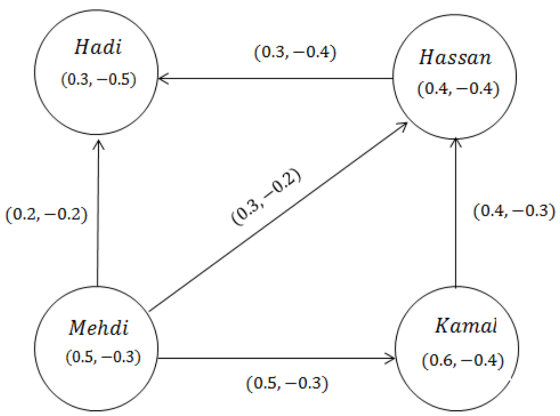

Figure 4: Bipolar fuzzy influence graph

6. Conclusion

The fuzzy graph has various uses in modern science and technology, especially in the fields of neural networks, computer science, operation research, and decision making. Bipolar fuzzy graphs have more precision, flexibility and compatibility, as compared to fuzzy graphs. Today, bipolar fuzzy graphs play an important role in social networks and allow users to find the most effective person in a group or organization. So, in this paper, we have given the degree and total degree of an edge in the cartesian product of two bipolar fuzzy graphs. Also, $\mu-$complement, self $\mu-$complement, and self weak $\mu-$complement on bipolar fuzzy graphs have been presented. Finally, an application of bipolar fuzzy digraphs in social relations has been introduced. In our future work, we will define different kinds of Energy in bipolar fuzzy graph structure with some examples.

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