Chapter 6

Trapezoidal interval type-2 fuzzy soft stochastic set and its application in stochastic multi-criteria decision-making

[1]

6.1 Introduction

Usually, fuzzy set theory [183] is one of the significant mathematical approaches for dealing with uncertainty. However, in our real life, defining an object by using type-1 fuzzy set is not adequate. Sometimes, the membership value of a type-1 fuzzy set (i.e., plane fuzzy set) is itself fuzzy. Then, to handle such type of uncertainties, Zadeh [184] introduced an effective idea of type-2 fuzzy set by adding a secondary membership function in a type-1 fuzzy set. Therefore, type-2 fuzzy set can handle more difficult types of real-life problems. Though, type-2 fuzzy set can formulate more and more uncertainties, but in reality, it is very difficult to assign it in practical situation due to its computational complexity in defining a second membership function. Then to make type-2 fuzzy set more user friendly, Mendel et al. [115] introduced the notion of interval type-2 fuzzy set(IT2FS) where, every second membership value is equals to 1. In existing literature, we have seen that, interval type-2 fuzzy set has been widely used in solving many real-life related problems [116, 135].

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Furthermore, Molodtsov [118] proposed the notion of soft set by using the idea of parameterization to avoid the introduction of a membership function likewise fuzzy set. Then, researchers have developed several types of generalizations of soft set theory by considering parameters under different uncertain environments like, fuzzy, intuitionistic fuzzy, neutrosophic, etc. First, Maji et el. [104] amalgamated soft set theory with fuzzy set theory and addressed the new notion of fuzzy soft set theory where, all the parameters are in fuzzy sense. Later, Majumdar et al. [111] extended the notion of fuzzy soft set theory to the generalized fuzzy soft set theory. Then, Yang et al. [177] proposed the notion of interval-valued fuzzy soft set by harmonizing soft set and interval-valued fuzzy set. Jiang et al. [81] established interval-valued intuitionistic fuzzy soft set by combining interval-valued intuitionistic fuzzy set with soft set. Moreover, Xiao et al. [170] addressed the satisfaction of the parameters with respect to the objects in terms of trapezoidal fuzzy numbers and established the notion of trapezoidal fuzzy soft set. Further, Zhang et al. [187] proposed the notion of trapezoidal interval type-2 fuzzy soft set by taking all the considered parameters in trapezoidal interval type-2 fuzzy sense so that, the evaluation of an alternative is in terms of trapezoidal interval type-2 fuzzy number.

In a multi-criteria decision making(MCDM) problem, the ranking of the effective alternatives based on some associated parameters is the main theme which will have to be solved. Due to the variety of decision making fields, MCDM problems may be based on several environments such as, rough set theory [128], vague set theory [68], fuzzy set [164, 165, 167], intuitionistic fuzzy set [18, 151, 180, 192], interval-valued intuitionistic fuzzy set [175], etc. Besides, soft set theory has also been used to solve decision-making problems. Readers can follow the references [22, 50, 81, 92, 111, 170] to study on decision-making by using soft set theory. However the evaluations of the alternatives over some corresponding parameters may not be fixed, but may be determined randomly. Such types of MCDM problems are known as stochastic multi-criteria decision making (SMCDM) problems. Cao et al. [39] proposed an approach to solve stochastic multi-criteria decision making problems by using set pair analysis where all the data are in terms of interval-valued intuitionistic fuzzy numbers. Then, Zhou et al. [193] proposed a method to solve stochastic multi-criteria decision making problems based on grey numbers by using the idea of regret theory and TOPSIS approach. After that, Wang et al. [161] established a formulation based on interval-valued rough fuzzy number for handling stochastic multi-criteria decision making problems. But, till now, no researchers have worked on stochastic multi criteria decision making under interval type-2 fuzzy environment.

Based on the aforementioned discussion, we have seen that, there is a research gap on solving stochastic multi criteria decision making under interval type-2 fuzzy environment. Moreover, soft set theory has been used very successfully in solving decision-making problems. Then, to full fill this research gap, in this chapter, we have proposed a methodological approach to solve stochastic multi criteria decision-making based on soft set theory under interval type-2 fuzzy environment. Moreover, in many practical problems, due to the increasing complexity in our real life problems, decision maker is not always able to provide a complete information about the weights of the attributes. Few researches have been done related to the incomplete information of the weights of the parameters [172, 189] to derive exact weights of the parameters. Therefore, in our trapezoidal interval type-fuzzy soft set approach, we have added parameters weight finding procedure. Therefore, the main aims of this chapter are as follows:

- Firstly, we have introduced the notion of trapezoidal interval type-2 fuzzy soft stochastic set.
- Then, we have proposed a new methodology to solve MCDM problems under stochastic environment based on trapezoidal interval type-2 fuzzy soft stochastic set.
- Additionally, in this chapter, we have considered that, weights of the parameters are unknown.

The remaining part of this chapter has been organized as follows. Section 6.2 recalls some basic definitions and properties. In Section 6.3, we have given a conceptualized definition of trapezoidal interval type-2 fuzzy soft stochastic set with an illustrative example. Then, in Section 6.4, firstly we have defined the concept of expected trapezoidal interval type-2 fuzzy soft set and then, an algorithm has been provided to solve trapezoidal interval type-2 fuzzy soft stochastic set based MCDM problems under stochastic environment. In Section 6.5, we have given a numerical illustration to verify our proposed approach. Finally Section 6.6 presents the conclusion of this chapter.

6.2 Some basic relevant notions

(i) **Type-2 fuzzy set (T2FS) [115].**

A type-2 fuzzy set $\tilde{A}$ over X can be defined by a type-2 membership function $\mu_{\tilde{A}}$ as follows,

$$\tilde{A} = \{((x, v), \mu_{\tilde{A}}(x, v))|\ \forall x \in X, v \in J_x \subseteq [0, 1]\}.$$

Here $x \in X$ is the primary variable, J_x is the primary membership of x, $v \in J_x \subseteq [0, 1]$ is the secondary variable and $\mu_{\tilde{A}}(x, v) \in [0, 1]$ is referred to as a secondary membership degree of x. For the continuous case the type-2 fuzzy set $\tilde{A}$ can be written as,

$$\tilde{A} = \int_{x\in X} \int_{v\in J_x\subseteq[0,1]} \mu_{\tilde{A}}(x, v)/(x, v) = \int_{x\in X} \int_{v\in J_x\subseteq[0,1]} [\mu_{\tilde{A}}(x, v)/v]/x .$$

Here $\int$ indicates the union of all the possible values of x and v. Figure 6.1 provides a type-2 fuzzy set.

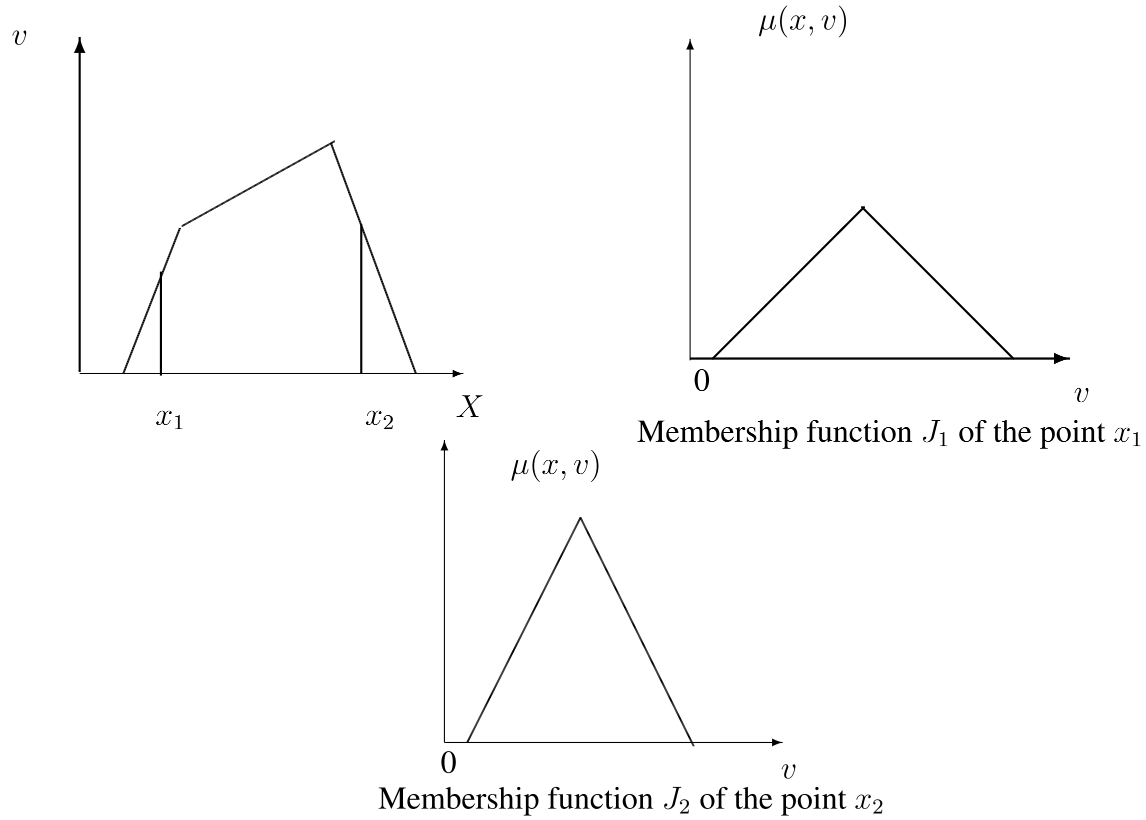


Figure 6.1: Graphical representation of a T2FS

(ii) **Interval type-2 fuzzy set (IT2FS) [42, 115].**

A type-2 fuzzy set $\tilde{A}$ is called an interval type-2 fuzzy set over if every corresponding secondary membership degree is equals to 1 i.e., $\forall x \in X$ and $v \in J_x$, $\mu_{\tilde{A}}(x, v) = 1$. Mathematically it can be written as,

$$\tilde{A} = \int_{x\in X} \int_{v\in J_x \subseteq [0,1]} 1/(x, v) = \int_{x\in X} [\int_{v\in J_x \subseteq [0,1]} 1/v]/x \quad .$$

(iii) **Interval type-2 fuzzy number (IT2FN) [47, 48].**

Let $\tilde{A}$ be an interval type-2 fuzzy set over X. Now, if for some $x \in X$, $\tilde{A}(x)$ is defined in a closed and bounded interval and also if it is a convex set then, $\tilde{A}$ is called an interval type-2 fuzzy number (IT2FN).
An interval type-2 fuzzy number $\tilde{A}(x)$ can be written as, $\tilde{A}(x) = [\tilde{A}^L(x), \tilde{A}^U(x)]$ where, $\tilde{A}^L(x), \tilde{A}^U(x) \in [0, 1]$ are the lower and upper fuzzy numbers (in type-1 form) with $\tilde{A}^L(x) \leq \tilde{A}^U(x)$.

(iv) **Trapezoidal interval type-2 fuzzy number (TrIT2FN) [47].**

In a interval type-2 fuzzy number $\tilde{A}(x) = [\tilde{A}^L(x), \tilde{A}^U(x)]$, if both the lower and upper fuzzy numbers are in trapezoidal fuzzy number then, the interval type-2 fuzzy number $\tilde{A}$ is called a trapezoidal interval type-2 fuzzy number.
A trapezoidal interval type-2 fuzzy number $\tilde{A}$, as given in Figure 6.2, can be written as, $\tilde{A}(x) = [\tilde{A}^L(x), \tilde{A}^U(x)]$ where,
$\tilde{A}^L = (a_1^L, a_2^L, a_3^L, a_4^L; H_1(\tilde{A}^L), H_2(\tilde{A}^L))$; $\tilde{A}^U = (a_1^U, a_2^U, a_3^U, a_4^U; H_1(\tilde{A}^U), H_2(\tilde{A}^U))$ and $H_i(\tilde{A}^L)$ is the membership value of the element a_{i+1}^L, $1 \leq i \leq 2$ and $H_i(\tilde{A}^U)$ is the membership value of the element a_{i+1}^U, $1 \leq i \leq 2$ with $0 \leq H_i(\tilde{A}^L) \leq H_i(\tilde{A}^U) \leq 1$ and $a_1^L \leq a_2^L \leq a_3^L \leq a_4^L, a_1^U \leq a_2^U \leq a_3^U \leq a_4^U, a_1^U \leq a_1^L, a_4^L \leq a_4^U$.

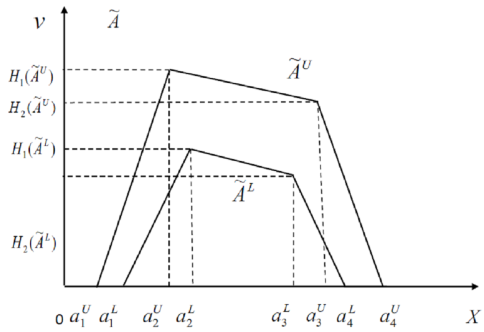


Figure 6.2: Graphical representation of a TrIT2FN

(v) **Signed distance of a TrIT2FN [47, 48].**

Sometimes, we have to compare two trapezoidal interval type-2 fuzzy numbers in many decision making problems. Signed distance is one of the useful tools that can help to make a comparison between two TrIT2FNs. It can also be referred as the oriented or directed distance of TrIT2FN from y-axis.

Let $\tilde{A} = ((a_1^L, a_2^L, a_3^L, a_4^L; H_1(\tilde{A}^L), H_2(\tilde{A}^L)), (a_1^U, a_2^U, a_3^U, a_4^U; H_1(\tilde{A}^U), H_2(\tilde{A}^U)))$ be a TrIT2FN and the level-1 fuzzy number that maps onto y-axis at $x = 0$ is denoted by $\tilde{0}_1$ and can be defined as, $\tilde{0}_1 = ((0, 0, 0, 0; 1, 1), (0, 0, 0, 0; 1, 1))$. Then the signed distance of $\tilde{A}$ from $\tilde{0}_1$ is defined as follows:

$$\begin{aligned}\tilde{d}(\tilde{A}, \tilde{0}_1) &= \frac{1}{8}(a_1^L + a_2^L + a_3^L + a_4^L + \frac{2H(\tilde{A}^U) + 3H(\tilde{A}^L)}{H(\tilde{A}^U)}(a_2^U + a_3^U) + \\ &\frac{4H(\tilde{A}^U) - 3H(\tilde{A}^L)}{H(\tilde{A}^U)}(a_1^U + a_4^U)) \qquad (6.1)\end{aligned}$$

where $H(\tilde{A}^U) = max(H_1(\tilde{A}^U), H_2(\tilde{A}^U))$ and $H(\tilde{A}^L) = max(H_1(\tilde{A}^L), H_2(\tilde{A}^L))$ such that, $0 \leq H(\tilde{A}^U) \leq 1, 0 \leq H(\tilde{A}^L) \leq 1$.
Ranking of two TrIT2FNs [47, 48].
Let $\tilde{A} = ((a_1^L, a_2^L, a_3^L, a_4^L; H_1(\tilde{A}^L), H_2(\tilde{A}^L)), (a_1^U, a_2^U, a_3^U, a_4^U; H_1(\tilde{A}^U), H_2(\tilde{A}^U)))$ and $\tilde{B} = ((b_1^L, b_2^L, b_3^L, b_4^L; H_1(\tilde{B}^L), H_2(\tilde{B}^L)), (b_1^U, b_2^U, b_3^U, b_4^U; H_1(\tilde{B}^U), H_2(\tilde{B}^U)))$ be two TrIT2FNs. Then by using the signed distance measure the ranking of $\tilde{A}$ and $\tilde{B}$ can be defined as follows,
(i) $\tilde{d}(\tilde{A}, \tilde{0}_1) \geq \tilde{d}(\tilde{B}, \tilde{0}_1)$ if and only if $\tilde{A} \tilde{\geq} \tilde{B}$,
(ii) $\tilde{d}(\tilde{A}, \tilde{0}_1) = \tilde{d}(\tilde{B}, \tilde{0}_1)$ if and only if $\tilde{A} \tilde{=} \tilde{B}$,
(iii) $\tilde{d}(\tilde{A}, \tilde{0}_1) \leq \tilde{d}(\tilde{B}, \tilde{0}_1)$ if and only if $\tilde{A} \tilde{\leq} \tilde{B}$.

(vi) **Trapezoidal interval type-2 fuzzy soft set (TrIT2FSS) [187].**

Trapezoidal interval type-2 fuzzy soft set is an extension of soft set where, satisfaction of every parameter over every object is in terms of trapezoidal interval type-2 fuzzy number.

Let $\tilde{P}_{\text{TrIT2}}(X)$ denotes the set of all trapezoidal interval type-2 fuzzy subsets of the set X. Then, a pair $(\tilde{f}_{\text{TrIT2}}, E)$ is said to be a trapezoidal interval type-2 fuzzy soft set over X, if $\tilde{f}_{\text{TrIT2}}$ is a mapping given by $\tilde{f}_{\text{TrIT2}} : E \to \tilde{P}_{\text{TrIT2}}(X)$ i.e., for each $e_j \in E$, $\tilde{f}_{\text{TrIT2}}(e_j)$ is considered as the set of e_j-approximate elements of the trapezoidal interval type-2 fuzzy soft set $(\tilde{f}_{\text{TrIT2}}, E)$ over X. Mathematically, it can be written as follows,

$$\tilde{f}_{\text{TrIT2}}(e_j) = \{(x_s, u_{sj}) | u_s \in X\},$$

where u_{sj} is the membership degree of x_s with respect to the parameter e_j in terms of trapezoidal interval type-2 fuzzy number.

Example 6.1. Now consider, $X = \{u_1, u_2, u_3\}$ be a set of initial universal set and $E = \{e_1, e_2, e_3, e_4\}$ be the set of corresponding parameters with $A = \{e_2, e_3\} \subseteq E$. Now consider $\tilde{f}_{\text{TrIT2}}$ is a mapping given by,
$\tilde{f}_{\text{TrIT2}}(e_2) = (u_1/((0.6, 0.7, 0.8, 0.85; 0.9, 0.9), (0.5, 0.65, 0.8, 1; 1, 1)),$
$u_2/((0.56, 0.59, 0.6, 0.68; 0.9, 0.9), (0.45, 0.55, 0.7, 0.95; 1, 1))$
$, u_3/((0.6, 0.65, 0.7, 0.75; 0.9, 0.9), (0.55, 0.60, 0.82, 0.95; 1, 1))),$
$\tilde{f}_{\text{TrIT2}}(e_3) = (u_1/((0.45, 0.52, 0.75, 0.78; 0.9, 0.9), (0.6, 0.65, 0.74, 0.89; 1, 1)),$
$u_2/((0.55, 0.65, 0.78, 0.85; 0.9, 0.9), (0.35, 0.46, 0.75, 0.86; 1, 1)),$
$u_3/((0.78, 0.79, 0.8, 0.82; 0.9, 0.9), (0.6, 0.75, 0.8, 0.9; 1, 1))).$
Then $(\tilde{f}_{\text{TrIT2}}, A)$ forms a trapezoidal interval type-2 fuzzy soft set over X. Its tabular representation has been given in Table 6.1.

(vii) **Choice value and choice parameters [22].**

According to the decision making problem, the parameters of a decision maker's choice or requirement which forms a subset of the whole parameter set of the problem, are known as choice parameters.

Choice value of an object is the sum of the membership values of that object corresponding to their respective choice parameters.

Table 6.1: TrIT2FSS $(\tilde{f}_{\text{TrIT2}}, A)$ (Example 6.1)

	e_2
u_1	((0.6,0.7,0.8,0.85;0.9,0.9),(0.5,0.65,0.8,1;1,1))
u_2	((0.56,0.59,0.6,0.68;0.9,0.9),(0.45,0.55,0.7,0.95;1,1))
u_3	((0.6,0.65,0.7,0.75;0.9,0.9),(0.55,0.60,0.82,0.95;1,1))

	e_3
u_1	((0.45,0.52,0.75,0.78;0.9,0.9),(0.6,0.65,0.74,0.89;1,1))
u_2	((0.55,0.65,0.78,0.85;0.9,0.9),(0.35,0.46,0.75,0.86;1,1))
u_3	((0.78,0.79,0.8,0.82;0.9,0.9),(0.6,0.75,0.8,0.9;1,1))

6.3 Trapezoidal interval type-2 fuzzy soft stochastic set

The notion of the trapezoidal interval type-2 fuzzy soft set has a broad utility in decision making problems. However, due to the increasing uncertainty in decision-making problems, some times the performances of an alternative over a parameter may be changed randomly with respect to some possible states. In that case, the satisfaction of an alternative over a parameter is deflected to stochastic. By using this idea, in this section we have introduced the notion of trapezoidal interval type-2 fuzzy soft stochastic set(TrIT2FSSS).

Definition 6.1. Trapezoidal interval type-2 fuzzy soft stochastic set (TrIT2FSSS).

Let $X = \{u_1, u_2, .., u_m\}$ be a finite initial universal set and $E = \{e_1, e_2, .., e_n\}$ be a set of parameters. Then, a pair $(\tilde{\tilde{f}}_{\text{TrIT2}}, E)$ is said to be a trapezoidal interval type-2 fuzzy soft stochastic set over X if and only if, there exist finite number of natural states $(S_t,\ t = 1, 2, .., l,\ l$ denotes the number of states) and for each different existing state, there exist a trapezoidal interval type-2 fuzzy soft set $(\tilde{\tilde{f}}_{\text{TrIT2}}, E)_t,\ t = \{1, 2, .., l\}$ over X.
In this model, we can not predict which state will occur but, we can decide the probability of occurrence of each of the states S_t, associated with the trapezoidal interval type-2 fuzzy soft set$(\tilde{\tilde{f}}_{\text{TrIT2}}, E)_t,\ t = \{1, 2, .., l\}$.

Here the random variable S, defined over the universal set X, describes ‘different states’ in which the evaluations (i.e., the satisfaction of the parameters associated with the objects) have been taken. Now let p_t be the probability of a possible state S_t for getting the t^{th} trapezoidal interval type-2 fuzzy soft set where, $\forall t,\ 0 \leq p_t \leq 1$ and $\sum_{t=1}^{l} p_t = 1$,

$t = 1, 2, .., l$. Here the random variable S takes up a discrete set of values $S_1, S_2, .., S_l$ with probability $P(S = S_t) = p_t, t = 1, 2, .., l$. The distribution function of the random variable S is as follows:

$$F(S_t) = (\tilde{\tilde{f}}_{\text{TrIT2}}, E)_t, t = 1, 2, .., l \tag{6.2}$$

Thus, a trapezoidal interval type-2 fuzzy soft stochastic set (TrIT2FSSS) is a collection of trapezoidal interval type-2 fuzzy soft sets $((\tilde{\tilde{f}}_{\text{TrIT2}}, E)_t, t = \{1, 2, .., l\})$ corresponding to some natural states where, the satisfactory level of a parameter over an object is a trapezoidal interval type-2 fuzzy number.

Mathematically, the trapezoidal interval type-2 fuzzy soft stochastic set can be defined as follows:

$$(\tilde{\tilde{f}}_{\text{TrIT2}}, E) = \{(S_t, (\tilde{\tilde{f}}_{\text{TrIT2}}, E)_t)\} = \{(S_t, ((e_j, \tilde{\tilde{f}}_{\text{TrIT2}}(e_j)), \forall e_j \in E))\}$$
$$=\{(S_t, ((e_j, (u_s, u^t_{sj})), \ u_s \in X, \ e_j \in A))\}, s = 1, 2, .., m, \ j = 1, 2, .., n, \ t = 1, 2, .., l.$$

u^t_{sj} is a trapezoidal interval type-2 fuzzy number which represents the evaluation of the alternative u_s with respect to the parameter e_j at the state S_t. In Table 6.2, its tabular form has been given.

Table 6.2: Tabular representation of a TrIT2FSSS $(\tilde{\tilde{f}}_{\text{TrIT2}}, E)$ (in general case)

	e_1	e_2	$\ldots$	e_n
	$(\tilde{\tilde{f}}_{\text{TrIT2}}, E)_1,$	$P(S_1) = p_1$		
u_1	u^1_{11}	u^1_{12}	$\ldots$	u^1_{1n}
u_2	u^1_{21}	u^1_{22}	$\ldots$	u^1_{2n}
.	.	.	$\ldots$	.
.	.	.	$\ldots$	.
u_m	u^1_{m1}	u^1_{m2}	$\ldots$	u^1_{mn}
	$(\tilde{\tilde{f}}_{\text{TrIT2}}, E)_2,$	$P(S_2) = p_2$		
u_1	u^2_{11}	u^2_{12}	$\ldots$	u^2_{1n}
u_2	u^2_{21}	u^2_{22}	$\ldots$	u^2_{2n}
.	.	.	$\ldots$	.
.	.	.	$\ldots$	.
u_m	u^2_{m1}	u^2_{m2}	$\ldots$	u^2_{mn}
.	.	.	$\ldots$	.
.	.	.	$\ldots$	.
	$(\tilde{\tilde{f}}_{\text{TrIT2}}, E)_l,$	$P(S_l) = p_l$		
u_1	u^l_{11}	u^l_{12}	$\ldots$	u^l_{1n}
u_2	u^l_{21}	u^l_{22}	$\ldots$	u^l_{2n}
.	.	.	$\ldots$	.
.	.	.	$\ldots$	.
u_m	u^l_{m1}	u^l_{m2}	$\ldots$	u^l_{mn}

To clear out the concept of trapezoidal interval type-2 fuzzy soft stochastic set, an example has been provided as given below.

Example 6.2. Let us consider three biscuit factories $X = \{f_1, f_2, f_3\}$ as the universal set and $E = \{e_1 = tasty, \ e_2 = low\ price, \ e_3 = healthy, e_4 = good\ packing\}$ be the

corresponding four criteria of the objects of X.Consider, $A = \{e_1,\ e_2,\ e_3\} \subseteq E$.
Now assume that, each of these three factories has possible three states such as, $S = \{S_1$ =fast-developing, S_2=medium-developing, S_3=low-developing$\}$ and the probability of occurrence of these three states are $0.3, 0.4, 0.3$ respectively.

Now consider that, at the state S_1 with probability 0.3, the satisfactory levels of the parameters over the alternatives are as follows:
$\tilde{\tilde{f}}_{TrIT2}(e_1) = \{f_1/((0.3, 0.4, 0.5, 0.6; 0.9, 0.9), (0.1, 0.2, 0.3, 0.7; 1, 1)),$
$f_2/((0.4, 0.45, 0.56, 0.6; 0.8, 0.9), (0, 0.47, 0.6, 0.78; 1, 1)),$
$f_3/((0.56, 0.58, 0.61, 0.66; 0.6, 0.7), (0.16, 0.24, 0.76, 0.87; 0.9, 1))\},$
$\tilde{\tilde{f}}_{TrIT2}(e_2) = \{f_1/((0.31, 0.42, 0.56, 0.58; 0.9, 0.9), (0.25, 0.45, 0.52, 0.7; 1, 1)),$
$f_2/((0.29, 0.43, 0.49, 0.54; 0.8, 0.9), (0.15, 0.46, 0.71, 0.80; 1, 1)),$
$f_3/((0.56, 0.57, 0.58, 0.59; 0.9, 0.9), (0.45, 0.46, 0.56, 0.76; 1, 1))\},$
$\tilde{\tilde{f}}_{TrIT2}(e_3) = \{f_1/((0.22, 0.25, 0.26, 0.34; 0.6, 0.8), (0.13, 0.5, 0.6, 0.74; 1, 1)),$
$f_2/((0.42, 0.43, 0.53, 0.62; 0.8, 0.8), (0.3, 0.7, 0.7, 0.8; 1, 1)),$
$f_3/((0.4, 0.45, 0.56, 0.58; 0.5, 0.6), (0.3, 0.6, 0.6, 0.7; 1, 1))\}.$

Similarly for the state S_2 with the probability 0.4, the satisfactory levels of the parameters over the alternatives are as follows:
$\tilde{\tilde{f}}_{TrIT2}(e_1) = \{f_1/((0.32, 0.46, 0.51, 0.65; 0.91, 0.92), (0.1, 0.25, 0.3, 0.75; 1, 1)),$
$f_2/((0.24, 0.35, 0.46, 0.56; 0.8, 0.9), (0.12, 0.37, 0.46, 0.58; 1, 1)),$
$f_3/((0.46, 0.48, 0.51, 0.56; 0.6, 0.9), (0.16, 0.34, 0.56, 0.67; 0.9, 1))\},$
$\tilde{\tilde{f}}_{TrIT2}(e_2) = \{f_1/((0.21, 0.32, 0.36, 0.38; 0.7, 0.9), (0.15, 0.35, 0.42, 0.57; 1, 1)),$
$f_2/((0.29, 0.43, 0.49, 0.54; 0.9, 0.9), (0.15, 0.46, 0.71, 0.78; 1, 1)),$
$f_3/((0.46, 0.47, 0.58, 0.59; 0.9, 0.9), (0.45, 0.46, 0.56, 0.76; 1, 1))\},$
$\tilde{\tilde{f}}_{TrIT2}(e_3) = \{f_1/((0.32, 0.25, 0.26, 0.44; 0.7, 0.8), (0.13, 0.5, 0.6, 0.74; 1, 1)),$
$f_2/((0.32, 0.43, 0.53, 0.62; 0.8, 0.8), (0.3, 0.7, 0.7, 0.78; 1, 1)),$
$f_3/((0.4, 0.45, 0.58, 0.61; 0.5, 0.6), (0.3, 0.63, 0.65, 0.67; 1, 1))\}.$
For the state S_3 with probability 0.3, the satisfactory levels of the parameters over the alternatives are as follows:
$\tilde{\tilde{f}}_{TrIT2}(e_1) = \{f_1/((0.51, 0.54, 0.55, 0.56; 0.1, 0.2), (0.45, 0.55, 0.58, 0.75; 0.5, 0.6)),$
$f_2/((0.4, 0.45, 0.56, 0.6; 0.8, 0.9), (0, 0.47, 0.6, 0.78; 1, 1)),$
$f_3/((0.56, 0.58, 0.61, 0.66; 0.6, 0.7), (0.16, 0.24, 0.76, 0.87; 0.9, 1))\},$
$\tilde{\tilde{f}}_{TrIT2}(e_2) = \{f_1/((0.33, 0.46, 0.49, 0.58; 0.9, 0.9), (0.25, 0.45, 0.5, 0.7; 1, 1)),$
$f_2/((0.29, 0.43, 0.46, 0.54; 0.8, 0.9), (0.1, 0.46, 0.71, 0.80; 1, 1)),$
$f_3/((0.56, 0.57, 0.58, 0.59; 0.9, 0.9), (0.45, 0.46, 0.6, 0.76; 1, 1))\},$
$\tilde{\tilde{f}}_{TrIT2}(e_3) = \{f_1/((0.22, 0.25, 0.26, 0.34; 0.6, 0.8), (0.13, 0.5, 0.6, 0.73; 1, 1)),$
$f_2/((0.4, 0.43, 0.5, 0.62; 0.7, 0.8), (0.3, 0.7, 0.7, 0.8; 1, 1)),$
$f_3/((0.4, 0.45, 0.46, 0.5; 0.5, 0.6), (0.3, 0.6, 0.6, 0.7; 1, 1))\}.$
So, for each of the different states, we get different trapezoidal interval type-2 fuzzy soft sets. For the first state S_1, we get $(\tilde{\tilde{f}}_{\text{TrIT2}}, A)_1$, for the second state S_2 we get $(\tilde{\tilde{f}}_{\text{TrIT2}}, A)_2$ and

for the third state S_3 we get $(\tilde{\tilde{f}}_{\text{TrIT2}}, A)_3$. Now, the tabular representation of the trapezoidal interval type-2 fuzzy soft stochastic set $(\tilde{\tilde{f}}_{\text{TrIT2S}}, A)$ has been given in Table 6.3.

Table 6.3: TrIT2FSSS $(\tilde{\tilde{f}}_{\text{TrIT2}}, E)$ (Example 6.2)

	e_1	$e_2,$	e_3
		$(\tilde{\tilde{f}}_{\text{TrIT2}}, A)_1,$	$P(S_1) = 0.3$
f_1	((0.3,0.4,0.5,0.6;0.9,0.9),(0.1,0.2,0.3,0.7;1,1))	((0.31,0.42,0.56,0.58;0.9,0.9),(0.25,0.45,0.52,0.7;1,1))	((0.22,0.25,0.26,0.34;0.6,0.8),(0.13,0.5,0.6,0.74;1,1))
f_2	((0.4,0.45,0.56,0.6;0.8,0.9),(0,0.47,0.6,0.78;1,1))	((0.29,0.43,0.49,0.54;0.8,0.9),(0.15,0.46,0.71,0.80;1,1))	((0.42,0.43,0.53,0.62;0.8,0.8),(0.3,0.7,0.7,0.8;1,1))
f_3	((0.56,0.58,0.61,0.66;0.6,0.7),(0.16,0.24,0.76,0.87;0.9,1))	((0.56,0.58,0.61,0.66;0.6,0.7),(0.16,0.24,0.76,0.87;0.9,1))	((0.4,0.45,0.56,0.58;0.5,0.6),(0.3,0.6,0.6,0.7;1,1))
		$(\tilde{\tilde{f}}_{\text{TrIT2}}, A)_2,$	$P(S_2) = 0.4$
f_1	((0.32,0.46,0.51,0.65;0.91,0.92),(0.1,0.25,0.3,0.75;1,1))	((0.21,0.32,0.36,0.38;0.7,0.9),(0.15,0.35,0.42,0.57;1,1))	((0.32,0.25,0.26,0.44;0.7,0.8),(0.13,0.5,0.6,0.74;1,1))
f_2	((0.24,0.35,0.46,0.56;0.8,0.9),(0.12,0.37,0.46,0.58;1,1))	((0.29,0.43,0.49,0.54;0.9,0.9),(0.15,0.46,0.71,0.78;1,1))	((0.32,0.43,0.53,0.62;0.8,0.8),(0.3,0.7,0.7,0.78;1,1))
f_3	((0.46,0.48,0.51,0.56;0.6,0.9),(0.16,0.34,0.56,0.67;0.9,1))	((0.46,0.47,0.58,0.59;0.9,0.9),(0.45,0.46,0.56,0.76;1,1))	((0.4,0.45,0.58,0.61;0.5,0.6),(0.3,0.63,0.65,0.67;1,1))
		$(\tilde{\tilde{f}}_{\text{TrIT2}}, A)_3,$	$P(S_3) = 0.3$
f_1	((0.51,0.54,0.55,0.56;0.1,0.2),(0.45,0.55,0.58,0.75;0.5,0.6))	((0.33,0.46,0.49,0.58;0.9,0.9),(0.25,0.45,0.5,0.7;1,1))	((0.22,0.25,0.26,0.34;0.6,0.8),(0.13,0.5,0.6,0.73;1,1))
f_2	((0.4,0.45,0.56,0.6;0.8,0.9),(0,0.47,0.6,0.78;1,1))	((0.29,0.43,0.46,0.54;0.8,0.9),(0.1,0.46,0.71,0.80;1,1))	((0.4,0.43,0.5,0.62;0.7,0.8),(0.3,0.7,0.7,0.8;1,1))
f_3	((0.56,0.58,0.61,0.66;0.6,0.7),(0.16,0.24,0.76,0.87;0.9,1))	((0.56,0.57,0.58,0.59;0.9,0.9),(0.45,0.46,0.6,0.76;1,1))	((0.4,0.45,0.46,0.5;0.5,0.6),(0.3,0.6,0.6,0.7;1,1))

From this example we have seen that. the evaluations of the alternatives with respect to the parameters are not fixed. They changes randomly over the states S_1, S_2, S_3 and the probability of occurrence of these states are, 0.3, 0.4, 0.3 respectively.

6.4 A novel approach to solve SMCDM problems based on trapezoidal interval type-2 fuzzy soft stochastic set

Molodtsov's soft set theory is one of modern concepts by which MCDM problems can be dealt by using different parameterizations. Up-to-date there exist many algorithms based on different types of soft sets (such as, fuzzy soft set, interval-valued fuzzy soft set, intuitionistic fuzzy soft set, interval-valued intuitionistic fuzzy soft set, trapezoidal interval type-2 fuzzy soft set, etc.) for solving MCDM problems. However in reality, in a MCDM problem, all the data may not be always fixed, they may be changed randomly. Such type of problems is known as stochastic multi criteria decision-making (SMCDM) problems. There exist several atricles on SMCDM under different fields such as, interval-valued fuzzy soft set [130], interval-valued intuitionistic fuzzy set [39], grey numbers [193], etc. Now, in this section, we have handled trapezoidal interval type-2 fuzzy soft set based multi criteria decision-making problems under stochastic situation.

6.4.1 SMCDM based on TrIT2FSSS

Let us consider that, $X = \{u_1, u_2, .., u_m\}$ be the finite set of alternatives and $E = \{e_1, e_2, .., e_n\}$ be the set of corresponding parameters with $W = \{w_1, w_2, ., w_n\}$ be the weighted vector of the parameters such that, $\forall i = 1, 2, .., n, w_j \geq 0$ and $\sum_{j=1}^{n} w_i = 1$. Here the set E is classified by two distinct subsets A and B of E in the sense of positive parameter set and negative parameter set where, $A \cup B = E$ and $A \cap B = \phi$.
Now consider, l possible states $S = \{S_1, S_2, .., S_l\}$ and p_t be the corresponding probability of the state S_t such that $0 \leq p_t \leq 1$ and $\sum_{t=1}^{l} p_t = 1$. Now consider $(\tilde{\tilde{f}}_{\text{TrIT2}}, E)$ be a trapezoidal interval type-2 fuzzy soft stochastic set over X where for each of the states (S_t), we get a trapezoidal interval type-2 fuzzy soft set $(\tilde{\tilde{f}}_{\text{TrIT2}}, E)_t$, $t = 1, 2, .., l$. Therefore, we get total l trapezoidal interval type-2 fuzzy soft decision matrix as, $M = [u^t_{sj}]_{m\times n}$, $s = 1, 2, .., m, j = 1, 2, .., n, t = 1, 2, .., l$ as given in Table 6.2 where, u^t_{ij} represents the performance of an alternative $u_s \in X$ with respect to a parameter e_j at the state S_t in terms of trapezoidal interval type-2 fuzzy number. Now the problem is to select the best/optimal alternative from a set of m alternatives.

Firstly, we have introduced the necessary definition of expected trapezoidal interval type-2 fuzzy soft set (ETrIT2FSS) as follows:

Definition 6.2. Expected trapezoidal interval type-2 fuzzy soft set (ETrIT2FSS).

Let $(\tilde{\tilde{f}}_{\text{TrIT2}}, E)$ be a trapezoidal interval type-2 fuzzy soft stochastic set over the universal set X where, E is the set of parameters. Let us assume that, $S = \{S_1, S_2, .., S_l\}$ be the set of possible states and $(\tilde{\tilde{f}}_{\text{TrIT2}}, A)_t$ be the corresponding trapezoidal interval type-2 fuzzy soft set for the state S_t over X.
Then, the expected value or mean value of the distribution of S is denoted as follows:

$$E\{F(S)\} = \sum_{t=1}^{l} p_t \times F(S_t) = \sum_{t=1}^{l} p_t \times [u^t_{sj}]_{m\times n} = [\breve{u}_{sj}]_{m\times n} \tag{6.3}$$

Here, the matrix $[u^t_{sj}]_{m\times n}$ represents the trapezoidal interval type-2 fuzzy soft set $(\tilde{\tilde{f}}_{\text{TrIT2}}, E)_t$, $t = 1, 2, .., l$ and $[\breve{u}_{sj}]_{m\times n}$ represents the expected trapezoidal interval type-2 fuzzy soft set over the universal set X which can be denoted as $(\tilde{\tilde{f}}_{\text{ETrIT2}}, E)$.

Expected trapezoidal interval type-2 fuzzy soft set (ETrIT2FSS) $(\tilde{\tilde{f}}_{\text{ETrIT2}}, E)$ is also a trapezoidal interval type-2 fuzzy soft set which can be defined as follows,

$$(\tilde{\tilde{f}}_{\text{ETrIT2}}, E) = (\{e_j, (u_s, \tilde{\tilde{f}}_{\text{ETrIT2}}(e_j)(u_s))\} = \{e_j, (u_s, \breve{u}_{sj})\}\} .$$

Mathematical form of $[\breve{u}_{sj}]_{m\times n}$ is as follows:

$$[\breve{u}_{sj}]_{m\times n} = \begin{bmatrix} p_1u_{11}^1 + p_2u_{11}^2 + \cdots + p_lu_{11}^l & p_1u_{12}^1 + p_2u_{12}^2 + \cdots + p_lu_{12}^l & \dots & p_1u_{1n}^1 + p_2u_{1n}^2 + \cdots + p_lu_{1n}^l \\ p_1u_{21}^1 + p_2u_{21}^2 + \cdots + p_lu_{21}^l & p_1u_{22}^1 + p_2u_{22}^2 + \cdots + p_lu_{22}^l & \dots & p_1u_{2n}^1 + p_2u_{2n}^2 + \cdots + p_lu_{2n}^l \\ \vdots & \vdots & \dots & \vdots \\ p_1u_{m1}^1 + p_2u_{m1}^2 + \cdots + p_lu_{m1}^l & p_1u_{m2}^1 + p_2u_{m2}^2 + \cdots + p_lu_{m2}^l & \dots & p_1u_{mn}^1 + p_2u_{mn}^2 + \cdots + p_lu_{mn}^l \end{bmatrix}$$

Algorithm.

Step 1. Input a trapezoidal interval type-2 fuzzy soft stochastic set.

Input a trapezoidal interval type-2 fuzzy soft stochastic set $(\tilde{\tilde{f}}_{\text{TrIT2}}, E) = [u_{sj}^t]_{m\times n}^l$ over the universal set X as given in Table 6.2 where, the entry u_{sj}^t represents the evaluation of an alternative $u_s; s = 1, 2, .., m$ corresponding to the parameter $e_j; j = 1, 2, .., n$ at the state $S_t; t = 1, 2, .., l$ in terms of trapezoidal interval type-2 fuzzy number as,
$u_{sj}^t = ((a_{1sj}^L, a_{2sj}^L, a_{3sj}^L, a_{4sj}^L; H_1(\tilde{A}_{sj}^L), H_2(\tilde{A}_{sj}^L)), (a_{1sj}^U, a_{2sj}^U, a_{3sj}^U, a_{4sj}^U; H_1(\tilde{A}_{sj}^U), H_2(\tilde{A}_{sj}^U)))$.

Step 2. Obtain the ETrIT2FSS.

By using the Equation 6.3, construct the expected trapezoidal interval type-2 fuzzy soft set $(\tilde{\tilde{f}}_{\text{ETrIT2}}, E) = (\{e_j, (u_s, \tilde{\tilde{f}}_{\text{ETrIT2}}(e_j)(u_s))\} = \{e_j, (u_s, \breve{u}_{sj}))\}$ over X.

Step 3. Normalization of the data.

Due to the different physical situations, all the parameters do not have same sense. Some of them may have positive concept and some of them may have negative concept. Therefore, we have characterized the parameter set E into two subsets A and B such that, $A \cup B = E$ and $A \cap B = \phi$.
Then, to equalize all the evaluations, we have normalized all the data. After normalization, the evaluation $(\breve{u}_{sj})$ of an alternative u_s over the parameter e_j corresponding to the expected trapezoidal interval type-2 fuzzy soft set is denoted by $\breve{u}_{sj}^{'}$ where, for $e_j \in A$,

$$\begin{aligned} \breve{u}_{sj}^{'} = \quad & ((\frac{a_{1sj}^L}{\tilde{a}_j^+}, \frac{a_{2sj}^L}{\tilde{a}_j^+}, \frac{a_{3sj}^L}{\tilde{a}_j^+}, \frac{a_{4sj}^L}{\tilde{a}_j^+}; H_1(\tilde{A_{sj}}^L), H_2(\tilde{A_{sj}}^L)), \\ & (\frac{a_{1sj}^U}{\tilde{a}_j^+}, \frac{a_{2sj}^U}{\tilde{a}_j^+}, \frac{a_{3sj}^U}{\tilde{a}_j^+}, \frac{a_{4sj}^U}{\tilde{a}_j^+}; H_1(\tilde{A_{sj}}^U), H_2(\tilde{A_{sj}}^U))) \end{aligned} \tag{6.4}$$

where $\tilde{a}_j^+ = max_s\{a_{4sj}^U : s = 1, 2, .., m\}, j = 1, 2, .., n$.

For $e_j \in B$,

$$\begin{aligned} \breve{u}'_{sj} &= ((\frac{\tilde{a}_j^-}{a_{4sj}^L}, \frac{\tilde{a}_j^-}{a_{3sj}^L}, \frac{\tilde{a}_j^-}{a_{2sj}^L}, \frac{\tilde{a}_j^-}{a_{1sj}^L}; H_1(\tilde{A_{sj}}^L), H_2(\tilde{A_{sj}}^L)), \\ &(\frac{\tilde{a}_j^-}{a_{4sj}^U}, \frac{\tilde{a}_j^-}{a_{3sj}^U}, \frac{\tilde{a}_j^-}{a_{2sj}^U}, \frac{\tilde{a}_j^-}{a_{1sj}^U}; H_1(\tilde{A_{sj}}^U), H_2(\tilde{A_{sj}}^U))) \end{aligned} \tag{6.5}$$

where $\tilde{a}_j^- = min_s\{a_{1sj}^L : s = 1, 2, .., m\}, j = 1, 2, .., n.$
Normalized expected trapezoidal interval type-2 fuzzy soft set is denoted by, $(\tilde{\tilde{f}}'_{\text{ETrIT2}}, E)$.

Step 4. Evaluate the optimal weights of the criteria.

Due to the increasing complexity in decision-making problems, a complete information about criterion weights may not be always possible to provide. Based on the reference [172], the information about the weights of the parameters may be incomplete or inconsistent provided by the decision maker. Based on this reference, let us suppose that, $H = K_1 \cup K_2 \cup K_3 \cup K_4 \cup K_5$ be the set of all information about the criterion weights where,
For $i \neq j \neq k,\ where\ i, j, k \in N(the\ set\ of\ natural\ numbers)$,
(i) $K_1 = \{w_i \geq w_j | i, j \in N\}$ (***weak order***).
(ii) $K_2 = \{w_i - w_j \geq \alpha, \alpha \geq 0\}$ (***strict order***).
(iii) $K_3 = \{w_i - w_j \geq w_k - w_l\}$, for $i \neq j \neq k \neq l$ (***difference order***).
(iv) $K_4 = \{\alpha_i \leq w_i \leq \alpha_i + \varepsilon_i, 0 \leq \alpha_i < \alpha_i + \varepsilon_i \leq 1\}$ (***interval bound***).
(v) $K_5 = \{w_i \leq \beta w_j, 0 \leq \beta \leq 1\}$ (***ratio bound***).

Now by using signed distance measure we have determined the optimal weights of each of the parameters as follows.

Step 4.1 Determine the relative closeness of the alternative u_s to $\tilde{0}_1$ with respect to the parameter e_j as following,

$$\tilde{D}_{sj} = \tilde{d}(\breve{u}_{sj}, \tilde{0}_1) \tag{6.6}$$

where $\tilde{d}$ is the signed distance measure of the trapezoidal interval type-2 fuzzy number $\breve{u}_{sj}$ and $\tilde{0}_1$ is the level-1 fuzzy number that maps onto y-axis at $x = 0$.

Step 4.2 Calculate the total deviation degree of all the alternatives to $\tilde{0}_1$ with respect to the parameter $e_j \in E$ by using the following formula,

$$\tilde{D}_j = \sum_{s=1}^{m} \tilde{d}(\breve{u}_{sj}, \tilde{0}_1) \tag{6.7}$$

Step 4.3 Now we have Constructed a linear programming model to obtain the attribute weights which maximizes the total deviation degree as follows.

$$Max\ D(w) = \sum_{j=1}^{n} (\frac{\tilde{D}_j}{\sum_{j=1}^{n} \tilde{D}_j}) w_j$$

$$s.t,\ w \in H, \sum_{j=1}^{n} w_j = 1, w_j \geq 0, j = 1, 2, .., n. \tag{6.8}$$

By solving this linear programming model, we will get the optimal value of the each of the $w_1, w_2, .., w_n$.

Step 5. Construct the weighted expected trapezoidal interval type-2 fuzzy soft set (WETrIT2FSS).

Derive the weighted expected trapezoidal interval type-2 fuzzy soft set (WEIrIT2FSS) $(\tilde{\tilde{f}}'_{\text{WETrIT2}}, E)$ from the normalized expected trapezoidal interval type-2 fuzzy soft set $(\tilde{\tilde{f}}'_{\text{ETrIT2}}, E)$ by the following formula where, for any $e_j \in E$,

$$\tilde{\tilde{f}}'_{\text{WETrIT2}}(e_j) = \{(u_s, \breve{u}'_{sj}) | u_s \in X\} = \{(u_s, w_j \times \breve{u}_{sj}) | u_s \in X, s = 1, 2, .., m\} \tag{6.9}$$

Step 6. Select the optimal alternative of the WETrIT2FSS.

Now, to select the best alternative based on the weighted expected trapezoidal interval type-2 fuzzy soft set $(\tilde{\tilde{f}}'_{\text{WETrIT2}}, E)$, we have followed the following steps.

Step 6.1 Determine the expected mean potentiality (EMP) for the WETrIT2FSS $(\tilde{\tilde{f}}'_{\text{WETrIT2}}$ as follows:
Here,
$\breve{u}'_{sj} = ((a^L_{1sj}, a^L_{2sj}, a^L_{3sj}, a^L_{4sj}; H_1(A^L_{sj}), H_1(A^L_{sj})), (a^U_{1sj}, a^U_{2sj}, a^U_{3sj}, a^U_{4sj}; H_1(A^U_{sj}), H_1(A^U_{sj}))),$
where $s = 1, 2, .., m\ and\ j = 1, 2, .., n.$
Then, the expected mean potentiality $\tilde{m}_p$ can be derived as follows:

$$\tilde{m}_p = ((\frac{\sum_{s=1}^{m}\sum_{j=1}^{n} a^L_{1sj}}{m \times n}, \frac{\sum_{s=1}^{m}\sum_{j=1}^{n} a^L_{2sj}}{m \times n}, \frac{\sum_{s=1}^{m}\sum_{j=1}^{n} a^L_{3sj}}{m \times n}, \frac{\sum_{s=1}^{m}\sum_{j=1}^{n} a^L_{4sj}}{m \times n}; min(H_1(A^L_{sj})), min(H_2(A^L_{sj})))$$
$$, (\frac{\sum_{s=1}^{m}\sum_{j=1}^{n} a^U_{1sj}}{m \times n}, \frac{\sum_{s=1}^{m}\sum_{j=1}^{n} a^U_{2sj}}{m \times n}, \frac{\sum_{s=1}^{m}\sum_{j=1}^{n} a^U_{3sj}}{m \times n}, \frac{\sum_{s=1}^{m}\sum_{j=1}^{n} a^U_{4sj}}{m \times n}; min(H_1(A^U_{sj})), min(H_1(A^L_{sj})))) \tag{6.10}$$

Step 6.2 Construct the tabular representation of $\tilde{m}_p$-level soft set of WETrIT2FSS $(\tilde{\tilde{f}}'_{\text{WETrIT2}}, A)$. The $\tilde{m}_p$-level soft set is a crisp soft set which can be denoted by, (f_{m_p}, E) and is defined as follows:

$$f_{m_p}(e_j) = \{x \in X : \tilde{\tilde{f}}'_{\text{WETrIT2}}(e_j)(x) \tilde{\geq} \tilde{m}_p, \forall e_j \in E\}.$$

Note: We have used the signed distance for comparing the trapezoidal interval type-2 fuzzy numbers.

Step 6.3 Compute the choice value C_s of each of an object u_s and then evaluate the maximum choice value(C_k) among $C_1, C_2, .., C_m$. If the maximum choice value C_k exists uniquely, then the corresponding object u_k will be the optimal solution. If not then go to the next step.

Step 6.4 Determine the largest and the smallest membership value in each row and column which are in terms of TrIT2FNs and then evaluate the difference between them. The resultant in each row and column are denoted by β_s, α_j respectively, where $s = 1, 2, .., m$ and $j = 1, 2, .., n$.
Note: Here the signed distance has been used for comparing the trapezoidal interval type-2 fuzzy numbers.

Step 6.5 Calculate the average value of α_j's, $j = 1, 2, .., n$ which is denoted by α. Then construct the α-level soft set of the WETrIT2FSS $(\tilde{\tilde{f}}'_{\text{WETrIT2}}, E)$.

Step 6.6 From the α-level soft set, determine the choice value C'_s of an alternative u_s. If the maximum choice value C'_k exists uniquely among C'_1,C'_2,..,C'_m, then the associated alternative u_k will be optimal solution. If C'_k is not unique then go to the next step.

If C'_k is not unique then, select the object consisting of the minimum value of β_s among those alternatives which have the maximum equal choice value at the α-level soft set. If, the optimal solution is not unique then, we will select any one of them.

6.5 Case study

In this section, we have illustrated a numerical example by using reference [39, 193] to verify the feasibility and effectiveness of our proposed approach.
Investment bank is a financial intermediary that performs several assignments such as, (i) raising capital and security underwriting (ii) Mergers and Acquisitions (iii) Sales and Trading and (iv) Retail and Commercial Banking. Investment bank earns profit by charging fees and commissions for providing these services and other kinds of financial and business advice. So invest bank has a significant role in improving of an industry/company.

Now let us assume that an investment bank corporation wants to invest four companies such as $\{u_1, u_2, u_3, u_4\}$. The bank authority invites an expert to select the best company for investment based on the three criteria such as, annual product income(e_1), social benefit(e_2) and environmental pollution degree(e_3). Here the criteria e_1 and e_2 have positive sense and the criterion (e_3) has negative sense. i.e., the big evaluation of a company with respect to the both e_1 and e_2 indicates the goodness of the company whereas, big evaluation of a company

Table 6.4: Linguistic evaluation (based on our case study)

	e_1, w_1	e_2, w_2	e_3, w_3
			$P(S_1) = 0.6$
u_1	VL	L	VH
u_2	L	VH	L
u_3	MH	M	H
u_4	H	H	MH
			$P(S_2) = 0.3$
u_1	L	M	H
u_2	VL	VH	H
u_3	VH	H	VL
u_4	L	MH	VH
			$P(S_3) = 0.1$
u_1	H	VL	VH
u_2	M	VH	L
u_3	VH	M	H
u_4	VH	VH	MH

Table 6.5: Seven linguistic terms and their corresponding TrIT2FNs [42]

Linguistic terms(the attribute values)	Trapezoidal interval type-2 fuzzy numbers
very low(VL)	((0,0,0,0.05;0.9,0.9),(0,0,0,0.1;1,1))
Low(L)	((0.05,0.1,0.1,0.2;0.9,0.9),(0,0.1,0.1,0.3;1,1))
Medium low(ML)	((0.2,0.3,0.3,0.4;0.9,0.9),(0.1,0.3,0.3,0.5;1,1))
Medium(M)	((0.4,0.5,0.5,0.6;0.9,0.9),(0.3,0.5,0.5,0.7;1,1))
Medium high(MH)	((0.6,0.7,0.7,0.8;0.9,0.9),(0.5,0.7,0.7,0.9;1,1))
High(H)	((0.8,0.9,0.9,0.95;0.9,0.9),(0.7,0.9,0.9,1;1,1))
Very High(VH)	((0.95,1,1,1;0.9,0.9),(0.9,1,1,1;1,1))

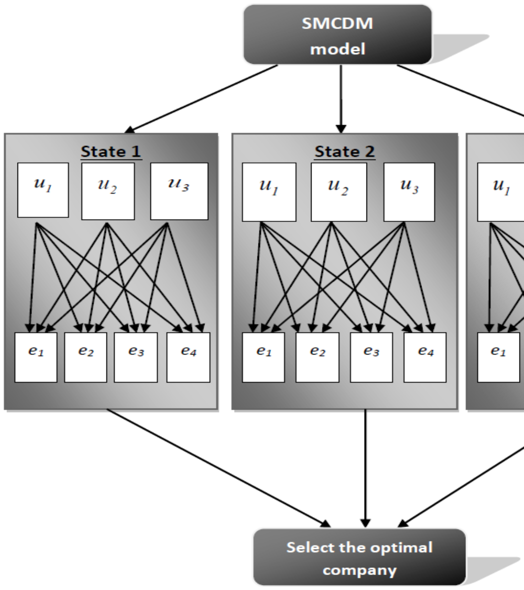
with respect to the criterion e_3 indicates the wickedness of the company. Now consider that, all the four companies have three possible states named as, fast-developing (S_1), medium developing (S_2) and low developing (S_3). The probability of each of the states is, $P(S_1) = 0.6$, $P(S_2) = 0.3$, $P(S_3) = 0.1$. Let w_1, w_2, w_3 be the weighted vectors of the parameters e_1, e_2, e_3 respectively and the information about them is given as below:

$$H = \{0.11 \leq w_1 \leq 0.25, 0.45 \leq w_2 \leq 0.47, 0.15 \leq w_3 \leq 0.35, \sum_{i=1}^{n} w_i = 1\}.$$

Now with respect to each of the states, we get different evaluations of an alternative corresponding to a parameter. Here, all the evaluations are in terms of linguistic information as given in Table 6.4. Nor, by transforming these linguistic values into trapezoidal interval type-2 fuzzy numbers with the help of Table 6.5, the corresponding trapezoidal interval type-2 fuzzy soft stochastic set has been given in Table 6.6.

Table 6.6: TrIT2FSSS $(\tilde{\tilde{f}}_{\text{TrIT2}}, E)$ (based on our case study)

	e_1, w_1	e_2,w_2	e_3,w_3
		$(\tilde{\tilde{f}}_{\text{TrIT2}}, E)_1$,	$P(S_1) = 0.6$
u_1	((0,0,0,0.05;0.9,0.9),(0,0,0,0.1;1,1))	((0.05,0.1,0.15,0.2;0.9,0.9),(0,0.1,0.15,0.3;1,1))	((0.95,1,1,1;0.9,0.9),(0.9,1,1,1;1,1))
u_2	((0.05,0.1,0.15,0.2;0.9,0.9),(0,0.1,0.15,0.3;1,1))	((0.95,1,1,1;0.9,0.9),(0.9,1,1,1;1,1))	((0.05,0.1,0.15,0.2;0.9,0.9),(0,0.1,0.15,0.3;1,1))
u_3	((0.2,0.3,0.35,0.4;0.9,0.9),(0.1,0.3,0.35,0.5;1,1))	((0.4,0.5,0.55,0.6;0.9,0.9),(0.3,0.5,0.55,0.7;1,1))	((0.8,0.85,0.9,0.95;0.9,0.9),(0.7,0.85,0.9,1;1,1))
u_4	((0.8,0.85,0.9,0.95;0.9,0.9),(0.7,0.85,0.9,1;1,1))	((0.8,0.85,0.9,0.95;0.9,0.9),(0.7,0.85,0.9,1;1,1))	((0.2,0.3,0.35,0.4;0.9,0.9),(0.1,0.3,0.35,0.5;1,1))
		$(\tilde{\tilde{f}}_{\text{TrIT2}}, E)_2$,	$P(S_2) = 0.3$
u_1	((0.05,0.1,0.15,0.2;0.9,0.9),(0,0.1,0.15,0.3;1,1))	((0.4,0.5,0.55,0.6;0.9,0.9),(0.3,0.5,0.55,0.7;1,1))	((0.8,0.85,0.9,0.95;0.9,0.9),(0.7,0.85,0.9,1;1,1))
u_2	((0,0,0,0.05;0.9,0.9),(0,0,0,0.1;1,1))	((0.95,1,1,1;0.9,0.9),(0.9,1,1,1;1,1))	((0.8,0.85,0.9,0.95;0.9,0.9),(0.7,0.85,0.9,1;1,1))
u_3	((0.95,1,1,1;0.9,0.9),(0.9,1,1,1;1,1))	((0.8,0.85,0.9,0.95;0.9,0.9),(0.7,0.85,0.9,1;1,1))	((0,0,0,0.05;0.9,0.9),(0,0,0,0.1;1,1))
u_4	((0.05,0.1,0.15,0.2;0.9,0.9),(0,0.1,0.15,0.3;1,1))	((0.2,0.3,0.35,0.4;0.9,0.9),(0.1,0.3,0.35,0.5;1,1))	((0.95,1,1,1;0.9,0.9),(0.9,1,1,1;1,1))
		$(\tilde{\tilde{f}}_{\text{TrIT2}}, E)_3$,	$P(S_3) = 0.1$
u_1	((0.8,0.85,0.9,0.95;0.9,0.9),(0.7,0.85,0.9,1;1,1))	((0,0,0,0.05;0.9,0.9),(0,0,0,0.1;1,1))	((0.95,1,1,1;0.9,0.9),(0.9,1,1,1;1,1))
u_2	((0.4,0.5,0.55,0.6;0.9,0.9),(0.3,0.5,0.55,0.7;1,1))	((0.95,1,1,1;0.9,0.9),(0.9,1,1,1;1,1))	((0.05,0.1,0.15,0.2;0.9,0.9),(0,0.1,0.15,0.3;1,1))
u_3	((0.95,1,1,1;0.9,0.9),(0.9,1,1,1;1,1))	((0.4,0.5,0.55,0.6;0.9,0.9),(0.3,0.5,0.55,0.7;1,1))	((0.8,0.85,0.9,0.95;0.9,0.9),(0.7,0.85,0.9,1;1,1))
u_4	((0.95,1,1,1;0.9,0.9),(0.9,1,1,1;1,1))	((0.95,1,1,1;0.9,0.9),(0.9,1,1,1;1,1))	((0.2,0.3,0.35,0.4;0.9,0.9),(0.1,0.3,0.35,0.5;1,1))



The hierarchical structure of our illustrated case study

Now our goal is to detect the best company among u_1, u_2, u_3, u_4 over the respective parameters based on the TrIT2FSSS $(\tilde{\tilde{f}}_{TrIT2}, E)$ as given in Table 6.6. To solve this problem, we have used our proposed trapezoidal interval type-2 fuzzy soft stochastic set based decision making methodology. The hierarchical structure of our illustrative example has been given in Figure 6.5.

Solution:

Step 1. The corresponding trapezoidal interval type-2 fuzzy soft stochastic set has been given in Table 6.6.

Step 2. Then, by using Equation 6.3, the expected trapezoidal interval type-2 fuzzy soft set $(\tilde{\tilde{f}}_{\text{ETrIT2}}, E)$ has been given in Table 6.7.

Step 3. Since, the parameters e_1 and e_2 have the positive sense and e_3 has negative sense, then, by using Equations 6.4 and 6.5 we have normalized all data. The normalized expected trapezoidal interval type-2 fuzzy soft set $(\tilde{\tilde{f}}'_{\text{ETrIT2}}, E)$ which are given in Table 6.8.

Table 6.7: ETrIT2FSS $(\tilde{\tilde{f}}_{\text{ETrIT2}}, E)$ (based on our case study)

	e_1, w_1	e_2, w_2
u_1	((0.095,0.115,0.135,0.185;0.9,0.9),(0.07,0.115,0.135,0.25;1,1))	((0.15,0.21,0.255,0.305;0.9,0.9),(0.09,0.21,0.255,0.4;1,1))
u_2	((0.07,0.11,0.145,0.195;0.9,0.9),(0.03,0.11,0.145,0.28;1,1))	((0.95,1,1,1;0.9,0.9),(0.9,1,1,1;1,1))
u_3	((0.5,0.58,0.61,0.64;0.9,0.9),(0.42,0.58,0.61,0.7;1,1))	((0.52,0.605,0.655,0.70;0.9,0.9),(0.42,0.605,0.655,0.79;1,1))
u_4	((0.59,0.64,0.68,0.73;0.9,0.9),(0.51,0.64,0.68,0.79;1,1))	((0.635,0.7,0.745,0.79;0.9,0.9),(0.54,0.7,0.745,0.85;1,1))

	e_3, w_3
u_1	((0.905,0.955,0.97,1;0.9,0.9),(0.84,0.955,0.97,1;1,1))
u_2	((0.275,0.325,0.375,0.425;0.9,0.9),(0.21,0.325,0.375,0.51;1,1),)
u_3	((0.56,0.595,0.63,0.68;0.9,0.9),(0.49,0.595,0.63,0.78;1,1))
u_4	((0.425,0.51,0.545,0.58;0.9,0.9),(0.34,0.51,0.545,0.65;1,1))

Step 4. According to our illustrative example, pieces of information about the weights of the parameters are $0.11 \leq w_1 \leq 0.25$, $0.45 \leq w_2 \leq 0.47$ and $0.15 \leq w_3 \leq 0.35$ such that $\sum_{i=1}^{3} w_i = 1$. Now we have determined the optimal values of w_1, w_2 and w_3 as follows:

Table 6.8: Normalized ETrIT2FSS $(\tilde{\tilde{f}}'_{\text{ETrIT2}}, E)$

	e_1, w_1	e_2, w_2
u_1	((0.12,0.14,0.17,0.23;0.9,0.9),(0.09,0.14,0.17,0.32;1,1))	((0.15,0.21,0.25,0.35;0.9,0.9),(0.09,0.21,0.25,0.4;1,1))
u_2	((0.09,0.14,0.18,0.25;0.9,0.9),(0.04,0.14,0.18,0.35;1,1))	((0.95,1,1,1;0.9,0.9),(0.9,1,1,1;1,1))
u_3	((0.63,0.73,0.77,0.81;0.9,0.9),(0.53,0.73,0.77,0.89;1,1))	((0.52,0.60,0.65,0.70;0.9,0.9),(0.42,0.60,0.65,0.79;1,1))
u_4	((0.,75,0.81,0.86,0.92;0.9,0.9)(0.64,0.81,0.86,1;1,1))	((0.63,0.7,0.74,0.79;0.9,0.9),(0.54,0.7,0.74,0.85;1,1))

	e_3, w_3
u_1	((0.21,0.22,0.22,0.23;0.9,0.9),(0.21,0.22,0.22,0.25;1,1))
u_2	((0.49,0.56,0.65,0.76;0.9,0.9),(0.41,0.56,0.65,1;1,1))
u_3	((0.31,0.33,0.35,0.37;0.9,0.9),(0.27,0.33,0.35,0.43;1,1))
u_4	((0.36,0.38,0.41,0.49;0.9,0.9),(0.32,0.38,0.41,0.62;1,1))

Step 4.1 Now, the signed distance of each of the entries in the normalized expected trapezoidal interval type-2 fuzzy soft set $(\tilde{\tilde{f}}'_{\text{ETrIT2}}, E)$ (given in Table 6.8 has been given in Table 6.9.

Step 4.2 By using Equation 6.7, the total deviation degree of all the alternatives to with respect to e_1, e_2 and e_3 are $\tilde{D}_1 = 3.77$, $\tilde{D}_2 = 5.12$, $\tilde{D}_3 = 3.33$ respectively.

Step 4.3 Then, to derive the weights of the parameters, we have solved the following mathematical formulation.

$$Max\ D(w) = 0.31w_1 + 0.42w_2 + 0.27w_3$$

Table 6.9: Signed distance values (based on Table 6.8)

	e_1, w_1	e_2, w_2	e_3, w_3
u_1	0.33	0.47	0.44
u_2	0.46	1.98	1.36
u_3	1.32	1.24	0.68
u_4	1.66	1.43	0.85

$$\text{s.t } w_1 + w_2 + w_3 = 1$$

with the conditions ,

$$\begin{aligned} 0.11 \leq w_1 \leq 0.25 \\ 0.45 \leq w_2 \leq 0.47, \\ 0.15 \leq w_3 \leq 0.35. \end{aligned} ,$$

We have solved this linear programming by using **Maple-18 software**. The optimal solution of w_1, w_2 *and* w_3 are, $w_1 = 0.25$, $w_2 = 0.47$, $w_3 = 0.28$.

Step 5. Construct the weighted expected trapezoidal interval type-2 fuzzy soft set $(\tilde{\tilde{f}}'_{\text{WETrIT2}}, E)$ from the Normalized ETrIT2FSS $(\tilde{\tilde{f}}'_{\text{ETrIT2}}, E)$ as given in Table 6.10.

Table 6.10: WETrIT2FSS $(\tilde{\tilde{f}}'_{\text{WETrIT2}}, E)$

	e_1	e_2
u_1	((0.03,0.03,0.04,0.06;0.9,0.9),(0.02,0.03,0.04,0.08;1,1))	((0.07,0.10,0.12,0.16;0.9,0.9),(0.04,0.1,0.12,0.19;1,1))
u_2	((0.02,0.03,0.04,0.06;0.9,0.9),(0.01,0.03,0.04,0.09;1,1))	((0.45,0.47,0.47,0.47;0.9,0.9),(0.42,0.47,0.47,0.47;1,1))
u_3	((0.16,0.18,0.19,0.20;0.9,0.9),(0.13,0.18,0.19,0.22;1,1))	((0.24,0.28,0.30,0.33;0.9,0.9),(0.20,0.28,0.30,0.37;1,1))
u_4	((0.19,0.20,0.21,0.23;0.9,0.9),(0.16,0.20,0.21,0.25;1,1))	((0.30,0.33,0.35,0.37;0.9,0.9),(0.25,0.33,0.35,0.4;1,1))

	e_3
u_1	((0.06,0.06,0.06,0.06;0.9,0.9),(0.06,0.06,0.06,0.07;1,1))
u_2	((0.14,0.16,0.18,0.21;0.9,0.9),(0.11,0.16,0.18,0.28;1,1))
u_3	((0.09,0.09,0.1,0.1;0.9,0.9),(0.07,0.09,0.1,0.12;1,1))
u_4	((0.1,0.11,0.11,0.14;0.9,0.9),(0.09,0.11,0.11,0.17;1,1))

Step 6. The expected mean potentiality($\tilde{m}_p$) of the WETrIT2FSS $(\tilde{\tilde{f}}'_{\text{WETrIT2}}, E)$ is $\tilde{m}_p = ((0.15, 0.17, 0.18, 0.2; 0.9, 0.9), (0.13, 0.17, 0.18, 0.22; 1, 1))$.

Step 6.1 With the help of the signed distance measure, the tabular representation of the $\tilde{m}_p$-level soft set has been given in Table 6.11.

Table 6.11: $\tilde{m}_p$-level soft set of $(\tilde{\tilde{f}}'_{\text{WETrIT2}}, E)$

	e_1	e_2	e_3	Choice value
u_1	0	1	0	1
u_2	0	1	1	2
u_3	1	1	0	2
u_4	1	1	0	2

Table 6.12: α-level soft set of $(\tilde{\tilde{f}}'_{\text{WETrIT2}}, E)$

	e_1	e_2	e_3	Choice value
u_1	0	0	0	0
u_2	0	1	1	2
u_3	0	1	0	1
u_4	1	1	0	2

Step 6.2 Here the objects u_2, u_3 and u_4 have the same choice value 2. So, we have tried to get an unique optimal alternative as follows:

Step 6.3 Obtain the values of $\beta_s, s = 1, 2, 3, 4$ and $\alpha_j, j = 1, 2, 3$ as follows:
$\beta_1 = ((0.01, 0.06, 0.09, 0.13; 0.9, 0.9), (-0.04, 0.06, 0.09, 0.17; 1, 1))$,
$\beta_2 = ((0.39, 0.43, 0.44, 0.45; 0.9, 0.9), (0.33, 0.43, 0.44, 0.46; 1, 1))$,
$\beta_3 = ((0.14, 0.18, 0.21, 0.24; 0.9, 0.9), (0.08, 0.18, 0.21, 0.3; 1, 1))$,
$\beta_4 = ((0.16, 0.22, 0.24, 0.27; 0.9, 0.9), (0.08, 0.22, 0.24, 0.31; 1, 1))$,
$\alpha_1 = ((0.12, 0.16, 0.18, 0.21; 0.9, 0.9), (0.06, 0.16, 0.18, 0.24; 1, 1))$,
$\alpha_2 = ((0.29, 0.35, 0.37, 0.4; 0.9, 0.9), (0.23, 0.35, 0.37, 0.43; 1, 1))$,
$\alpha_3 = ((0.08, 0.1, 0.12, 0.15; 0.9, 0.9), (0.04, 0.1, 0.12, 0.22; 1, 1))$.

Step 6.4 The average value of α_1, α_2 and α_3, which is denoted by α, is as follows:
$\alpha = ((0.16, 0.20, 0.22, 0.25; 0.9, 0.9), (0.11, 0.20, 0.22, 0.30; 1, 1))$. Then, the α-level soft set has been given in Table 6.12.

Step 6.5 Here the alternatives u_2 and u_4 have the same choice 2. Moreover, from Step 6.3, among β_2 and β_4, β_4 is minimum and unique. So, by our proposed approach, the corresponding object u_4 is the optimal alternative.

Hence, it can be concluded that, u_4 is best company for investment.

6.6 Conclusion

In this chapter, we have offered a new uncertain controlling tool to deal with the stochastic multi-criteria decision making problems. In this regard, we have proposed the notion of trapezoidal interval type-2 fuzzy soft stochastic set and then of expected trapezoidal interval type-2 fuzzy soft stochastic set (ETrIT2FSSS). Further, a new methodology has been provided to solve stochastic multi-criteria decision making(SMCDM) problems based on our proposed trapezoidal interval type-2 fuzzy soft stochastic set. In addition, we have proposed a methodological approach to derive the weights of the parameters when, weights of the parameters in a decision-making in partially unknown. Now, the main advantages of our proposed approach can be pointed out as follows:

(i) It is a parameterized family of subsets of the universal set. It can be molded according to our human perception.

(ii) This notion can solve the multi-criteria decision making problems under stochastic environment.

Therefore we can conclude that, our approach is more closer to the human perception process and has a wide application in decision making fields.

In further research, one can develop the other types of decision making approaches such as, TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution) method, TODIM method, VIKOR method, etc. to solve trapezoidal interval type-2 fuzzy soft stochastic set based SMCDM problems. Moreover, complex fuzzy soft set theory is a recent developed generalization of soft set theory which can handle the problems having an additional information with respect to every parameter. So, as a further research, one can extend this decision-making approach to complex fuzzy soft set theory.