Chapter 4

A new algorithmic approach to linguistic valued soft multi-criteria group decision-making problems using linguistic scale function

[1]

4.1 Introduction

Uncertainty or vagueness is undergoing as an emergent research area not only in mathematics but also in other fields including, social science, economics, medical science, engineering, etc. Some well known existing theories to manage different types of uncertainties are, probability theory [85], fuzzy set theory [183], theory of interval mathematics [95], etc. Besides these approaches, soft set theory is the recent developed approach proposed by Prof. D. Molodtsov [118] where, parameterization has been used as a key feature to define an object under uncertainty. Afterwards, different generalizations of soft set theory have been proposed by using various parameters under different uncertain environments such as, fuzzy soft set theory [104], intuitionistic fuzzy soft set theory [105], type-2 fuzzy soft set theory [186], etc. theory. With these inaugurations, these approaches

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have been used in several types of application fields, such as, in medical science [22], decision-making [187], etc. In the previous two chapters we have worked on fuzzy soft sets where, in Chapter 2, we have used fuzzy soft set theory in solving group decision-making while, in Chapter 3, we have developed some properties of group theory through fuzzy soft set theory.

However, due to the presence of complexity in our real-life, some times linguistic variables are more acceptable than any crisp value or fuzzy value. Naturally, linguistic variables are much more closer to our human cognitive process. In references ([55, 57, 77, 184]), some linguistic-based models have have been developed. Basically, in literature, there exist several computational techniques to handle linguistic variable as given below:

(1) The first method is the transformation of the linguistic variables into some fuzzy numbers [77, 184] (trapezoidal fuzzy number, triangular fuzzy number, type-2 fuzzy number, trapezoidal interval type-2 fuzzy number etc.) by using the extension principle. But in that case, consideration of a suitable membership function is a difficult task to a decision maker.

(2) Second approach is the symbolic based computational process which is based on the indices of the associated linguistic terms [31, 76, 119, 171]. Though, the operating system of this process is very easy to handle however, there is a possibility of data loss.

(3) Third way is the 2-tuple linguistic representation model, where a linguistic evaluation s_x is expressed as a pair (s_α, a) where s_α is a original linguistic term and a is a numerical value supports the linguistic information s_x. But this representation is applicable only on the equidistant linguistic term sets. Then based on this idea, Wang et al. [160] generalized this notion to the linguistic scale function. In this process, a linguistic variable is converted to a numerical value by using a strictly monotonically increasing and continuous function [96, 100, 160]. This model can be applicable to all types of linguistic variables.

(4) Another new approach is the granulation of linguistic terms. In this computational process, for granulating the linguistic variables, a multi-objective optimization task has been formulated [30, 32, 129].

Then, by using Molodtsov's soft set theory, some new models have been produced by mixing a variety of linguistic information with the soft set theory. Tao et al. [156] proposed the notion of 2-tuple linguistic soft set theory by combining the notion of soft set with the 2-tuple linguistic information and further applied this model in solving group decision-making problems. Then, the notion of uncertain linguistic fuzzy soft set has been introduced together with an application in decision-making. Moreover, Guan et al. [71] proposed the notion of intuitionistic fuzzy linguistic soft set by incorporating soft set and intuitionistic fuzzy linguistic set. Zhao et al. [191] established the concept of fuzzy-valued linguistic soft set where the performance of an alternative with respect a parameter is in terms of linguistic variable and also each of the linguistic variables has a fuzzy membership degree. Recently, the notion of linguistic valued soft set theory has been founded by Sun et al. [153]. They also offered an approach to get a solution from linguistic valued soft set based multi-criteria group decision-making problems through the symbolic-based computational process.

But in Sun's algorithm we have observed that, their approach is not true for all types of linguistic valued soft set based group decision-making problems. Our argument has been tested by some counter examples. Then, to overcome this limitation, in this chapter, we have provided a new model to deal with the linguistic valued soft set based multi-criteria group decision-making problems through the notion of linguistic scale function. Our approach is composed into three parts: determination of the satisfaction level of each of the decision makers, aggregation of the opinions all the decision makers and then selection of the best or optimal alternative. With this aim, a new definition of a similarity measure for linguistic valued sets has been introduced. Then, a modified version of ranking function of an alternative in a linguistic valued soft set has been proposed. Finally, a step-wise decision-making methodology has been established for handling linguistic valued soft set based group decision-making problems. Moreover, we have verified our proposed approach by comparing it with the Sun's method [153].

The outline of this chapter is as follows. Section 4.2 provides some fundamental definitions and related operational laws which have been used in our subsequent discussions. In Section 4.3, we have discussed some drawbacks of Sun's [153] approach. In Section 4.4, we have proposed a new approach for solving linguistic valued soft set based multi-criteria group decision-making problems. The optimality test and performance measure have also been included in this section. Section 4.5 provides a discussion to justify our proposed approach. In section 4.6, we have provided an application of our proposed approach. Section 4.7 gives the conclusion of this chapter.

4.2 Some basic relevant notions

In this section, we have briefly reviewed some essential ideas which have been widely used in the rest of this chapter.

(i) **Operations on linguistic variables [171].**

Let, $S = \{s_\alpha | \alpha = -\tau, .., -1, 0, 1, .., \tau\}$ be a totally ordered discrete linguistic term set and $\bar{S} = \{s_x | -\tau \leq x \leq \tau\}$ be its continuous form. If $s_x \in S$, then it is referred to as original linguistic term otherwise, it is called virtual linguistic term.
Now assume that, $s_x, s_y \in \bar{S}$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$. Then, the following properties hold truly:
(1) $s_x + s_y = s_{x+y}$; (2) $\lambda s_x = s_{\lambda x}$;

(3) $(\lambda_1 + \lambda_2)s_x = \lambda_1 s_x + \lambda_2 s_x = s_{\lambda_1 x} + s_{\lambda_2 x}$;
(4) $\lambda(s_x + s_y) = \lambda s_x + \lambda s_y = s_{\lambda x} + s_{\lambda y}$.

(ii) **Linguistic scale function [96, 160].**

$S = \{s_\alpha | \alpha = -\tau, .., -1, 0, 1, .., \tau\}$ be a discrete linguistic term set and $f : S \to \mathbb{R}^+$ be a monotonically increasing and continuous function on S such that,

$$f(s_\alpha) = \vartheta_\alpha; \alpha = -\tau, .., -1, 0, 1, .., \tau$$

where, $\vartheta_\alpha \in [0, 1]$. Then, the function f is called the linguistic scale function which obeys the following properties: (P1) $f(s_{-\tau}) = 0$, $f(s_\tau) = 1$; (P2) $f(s_x) \geq f(s_y)$ if and only if $x \geq y$.

Now, based on the reference [96], the linguistic scale function can be defined by the following three ways,
(1) **Neutral linguistic scale function.** $\tilde{f}_1(s_x) = \frac{1}{2}(\frac{x}{\tau} + 1), x \in [-\tau, \tau]$
(2) **Optimistic linguistic scale function.** $\tilde{f}_2(s_x) = (\frac{1}{2}(\frac{x}{\tau} + 1))^\tau, x \in [-\tau, \tau]$.
(3) **Pessimistic linguistic scale function.** $\tilde{f}_3(s_x) = (\frac{1}{2}(\frac{x}{\tau} + 1))^{\frac{1}{\tau}}, x \in [-\tau, \tau]$.

The neutral linguistic scale function is used when the absolute deviation between two adjacent linguistic subscripts remains unchanged, optimistic linguistic scale function is used when the absolute deviation is going to increase and the pessimistic linguistic scale function is applied when the absolute deviation is going to decrease.

The above function can be extended to the continuous linguistic term set $\bar{S}$ to keep up all the above information and to facilitate the calculation as follows:

$$f : \bar{S} \to \mathbb{R}^+ such\ that\ f(s_x) = \vartheta_x; x \in [-\tau, \tau]$$

The inverse function of f exists and is denoted by f^{-1}.
Operational laws of linguistic variables through linguistic scale function [161].
Let s_x, $s_y \in \bar{S}$ and f be a linguistic scale function. Then, based on a linguistic scale function f, the basic operational laws of the linguistic variables can be defined as follows:
(1) $s_x \oplus s_y = f^{-1}(f(s_x) + f(s_y))$;
(2) $s_x \otimes s_y = f^{-1}(f(s_x)f(s_y))$;
(3) $\lambda s_x = f^{-1}(\lambda f(s_x))$, $\lambda \geq 0$;
(4) $(s_x)^\lambda = f^{-1}((f(s_x))^\lambda)$, $\lambda \geq 0$;
(5) $neg(s_x) = f^{-1}(f(s_{-x}))$; where f^{-1} the inverse function of the function f.

Example 4.1. Let us considered a set of seven linguistic terms as,
$S = \{s_{-3}(very\ poor(VP)), s_{-2}(poor(P)), s_{-1}(slightly\ poor(SP)), s_0(fair(F)), s_1(slightly\ good(SG)), s_2(good(G)), s_3(very\ good(VG))\}$.
Now, consider a linguistic scale function over S as, $f(s_x) = \frac{1}{2}(\frac{x}{3} + 1); x \in [-3, 3]$ to

convert all the linguistic variables into a numerical values. Then, $f(s_{-3}) = 0, f(s_{-2}) = 0.17, f(s_{-1}) = 0.33, f(s_0) = 0.5, f(s_1) = 0.67, f(s_2) = 0.83, f(s_3) = 1$.
Graphical representation of rating of all the seven linguistic terms based on above defined linguistic scale function has been given in Figure 4.1.

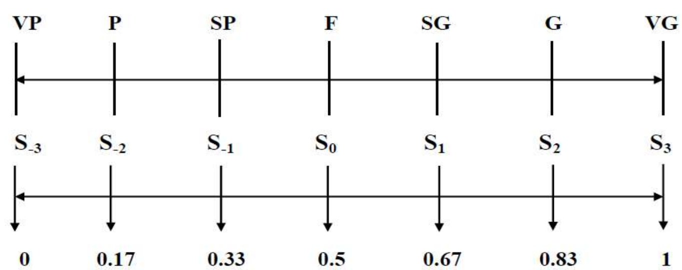

Figure 4.1: Rating of the linguistic variables by using a linguistic scale function

(iii) **Mathematical representation of a LVSS [153].**

If, $X = \{x_1, x_2, .., x_m\}$ and $E = \{e_1, e_2, .., e_n\}$ which are in linguistic sense. Then, a linguistic valued soft set (F^L, E) over X is defined as follows:

$$\begin{aligned}(F^L, E) &= \left\{\left(e_1, F^L(e_1)\right), \left(e_2, F^L(e_2)\right), .., \left(e_n, F^L(e_n)\right)\right\} \\ &= \{(e_1, (x_1, F^Le_1(x_1)), (x_2, F^Le_1(x_2)), .., (x_m, F^Le_1(x_m))), (e_2, (x_1, F^Le_2(x_1)), \\ &(x_2, F^Le_2(x_2)), ..., (x_m, F^Le_2(x_m))), .., (e_n, (x_1, F^Le_n(x_1)), \\ &(x_2, F^Le_n(x_2)), .., (x_m, F^Le_n(x_m)))\}\end{aligned}$$

Here, $F^Le_j(x_s)$ is the linguistic valued rating of an alternative x_s over a parameter e_j, $s = 1, 2, .., m$, $j = 1, 2, .., n$. Tabular form of a linguistic valued soft set (F^L, E) has been illustrated in Table 4.1.

Table 4.1: Tabular form of a LVSS (F^L, E) (in general case)

	e_1	e_2	$\ldots$	e_n
x_1	$F^Le_1(x_1)$	$F^Le_2(x_1)$	$\ldots$	$F^Le_n(x_1)$
x_2	$F^Le_1(x_2)$	$F^Le_2(x_2)$	$\ldots$	$F^Le_n(x_2)$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
x_m	$F^Le_1(x_m)$	$F^Le_2(x_m)$	$\ldots$	$F^Le_1(x_m)$

(iv) **Complement of a linguistic valued soft set [153].**

Let $A \subseteq E$ and (F^L, A) be a linguistic valued soft set over X. Then the complement of (F^L, A) is denoted by $(F^L, A)^c$ and is defined as,
$(F^L, A)^c = \{(e, (F^L(e))^c)|\forall e \in A\}$,
where, $(F^L(e))^c = \{(x, neg(F^L(e)(x)))|\forall e \in A, x \in X\}$. $neg(F^L(e)(x))$ is the inverse of the linguistic evaluation $F^L(e)(x)$.

(v) **Union of two linguistic valued soft sets [153].**

Let $X = \{x_1, x_2, .., x_m\}$ be a set of m objects and $E = \{e_1, e_2, .., e_n\}$ be a set of n parameters which are in linguistic valued sense. Let, (F_1^L, E) and (F_2^L, E) be two linguistic valued soft sets over X. Then the union of (F_1^L, E) and (F_2^L, E) is denoted by, $(F_1^L, E) \cup (F_2^L, E) = (H^L, E)$ and is defined as follows:

Union: $(H^L, E) = \{(e_j, H^L(e_j))|\forall e_j \in E\} = \{(e_j, (F_1^L(e_j) \cup F_2^L(e_j)))|\forall e_j \in E\} = \{(e_j, (x_s, max(F_1^L(e_j)(x_s), F_2^L(e_j)(x_s))))|\forall e_j \in E, x_s \in X\}; j = 1, 2, .., n; s = 1, 2, .., m.$

Example 4.2. Let, S={s_{-2}=very low, s_{-1}=low, s_0=fair, s_1=high, s_2=very high} be a linguistic term set. Consider, $X = \{x_1, x_2, x_3\}$ be a set of three houses and $E = \{e_1, e_2, e_3, e_4\}$ be a set of four parameters related to the elements of X where, e_1 stands for the parameter 'price', e_2 stands for the parameter 'in green surroundings', e_3 stands for the parameter 'outlook' and e_4 stands for the parameter 'wooden quality'. Now consider two linguistic valued soft sets (F_1^L, E) and (F_2^L, E) over X as follows:
$(F_1^L, E) = \{(e_1, ((x_1, s_{-2}), (x_2, s_1), (u_3, s_0))), (e_2, ((x_1, s_{-1}), (x_2, s_2), (u_3, s_1))),$
$(e_3, ((x_1, s_{-1}), (x_2, s_0), (x_3, s_2))), (e_4, ((x_1, s_{-2}), (x_2, s_1), (x_3, s_2)))\}$,
$(F_2^L, E) = \{(e_1, ((x_1, s_{-1}), (x_2, s_2), (x_3, s_1))), (e_2, ((x_1, s_{-2}), (x_2, s_1), (x_3, s_1))),$
$(e_3, ((x_1, s_1), (x_2, s_0), (x_3, s_{-2}))), (e_4, ((x_1, s_{-2}), (x_2, s_0), (x_3, s_{-2})))\}$.

Then the union of (F_1^L, E) and (F_2^L, E) is denoted by, $(F_1^L, E) \cup (F_2^L, E) = (H^L, E)$ where, $H^L(e_1) = ((x_1, s_{-1}), (x_2, s_2), (x_3, s_1))$; $H^L(e_2) = ((x_1, s_{-1}), (x_2, s_2), (x_3, s_1))$;
$H^L(e_3) = ((x_1, s_1), (x_2, s_0), (x_3, s_2))$; $H^L(e_4) = ((x_1, s_{-2}), (x_2, s_1), (x_3, s_2))$.

(vi) **Difference of two soft sets [11].**

Let (F_1, E) and (F_2, E) be two soft sets over the universe X. Then the restricted difference of (F_1, E) and (F_2, E) is denoted by, $(F_1, E) \smile_R (F_2, E)$ and is defined as, $(F_1, E) \smile_R (F_2, E) = (H, E)$ where $\forall e \in E$, $H(e) = F_1(e) - F_2(e)$. '$-'$ indicates the difference between two e-approximate elements $F_1(e)$ and $F_2(e)$.

(vii) **Choice matrix [153].**

Consider $X = \{x_1, x_2, .., x_m\}$ and $E = \{e_1, e_2, .., e_n\}$. Let, (F^L, E) be a linguistic valued soft set over X and $A, B \subseteq E$. Then the choice value matrix $C_{(A,B)}$, given by the two different experts, is denoted by $C_{(A,B)} = (\alpha_{ij})_{n\times n}$ and is defined as follows:

$$\alpha_{ij} = \begin{cases} 1, & e_i \in A, e_j \in B; \\ \\ -1, & e_i \in A, e_j \notin B; \\ & e_i \notin A, e_j \in B; \\ & e_i \notin A, e_j \notin B. \end{cases}$$

For converting all the entries into linguistic variables, the following transformations have been used, $-1 \triangleq s_{-\tau}$ and $1 \triangleq s_{\tau}$.

$(viii)$ **Product operation on LVSSs [153]**

The product operation of $C_{(A,B)} = (\alpha_{jk})_{n\times l}$ and $(F^L, E) = F^L = (u_{sj})_{m\times n}$ is denoted by $F^L \otimes C_{(A,B)}$ and is defined as,

$$F^L \otimes C_{(A,B)} = (max_{s=1}^{m} min_{j=1}^{n}(F^L(e_j)(x_s), \alpha_{jk}))_{m\times l}$$

(ix) **Ranking function in a LVSSs [153].**

Consider $X = \{x_1, x_2, .., x_m\}$ and $E = \{e_1, e_2, .., e_n\}$. Now assume (F^L, E) be a linguistic valued soft set over X. Then the ranking function of an alternative x_s over all the parameters is derived by the following equation,

$$R_{F^L}(x_s) = \sum_{j=1}^{n} F^L(e_j)(x_s), s = 1, 2, .., m$$

where $F^L(e_j)(x_s)$ is the linguistic valued evaluation of an alternative x_s with respect to a parameter e_j.

(x) **Bounded difference of fuzzy sets [154].**

Let $A = \{(x, \mu_A(x))|x \in X\}$ and $B = \{(x, \mu_B(x))|x \in X\}$ be two fuzzy sets over X. Then the bounded difference of A and B is denoted by $A \ominus B$ and is defined as follows:

$$A \ominus B = 0 \vee (\mu_A(x) - \mu_B(x)) = max\{0, (\mu_A(x) - \mu_B(x))\}.$$

4.3 Linguistic valued soft set based multi-criteria group decision-making

4.3.1 Problem description

Let us consider a set of m alternatives, $X = \{x_1, x_2, .., x_m\}$, and a set of n corresponding parameters, $E = \{e_1, e_2, .., e_n\}$ which are in linguistic sense. Now assume a set of k decision makers, $D = \{d_1, d_2, .., d_k\}$, who has taken some parameters from the set E in favor of their preferences. The k selected parameter subsets of the set E, considered by k decision makers, are $d_1^E, d_2^E, .., d_k^E$ respectively. It is assumed that, every decision makers expresses his/her opinion in terms of linguistic information such as, 'high', 'very high', 'extremely low', etc. *Now the problem is to select the best alternative or ranking the m alternatives based on all the decision makers' opinions*.

To handle this problem, we have followed 'soft set theory' under linguistic environment through the notion of linguistic valued soft set.

The evaluations of m alternatives over n parameters provided by k decision makers have been expressed in k linguistic valued soft sets, $(F_1^L, d_1^E), (F_2^L, d_2^E), .., (F_k^L, d_k^E)$ respectively.

To solve such type of decision-making problems, Sun et al. [153] proposed a methodology through index-based operational laws as given in the following subsection.

4.3.2 Sun's Algorithm [153]

Step 1. Input k linguistic valued soft sets $(F_1^L, d_1^E), (F_2^L, d_2^E), .., (F_k^L, d_k^E)$.
Step 2. Construct the combined choice value matrix $C(d_l^E, \cap_{l'=1,l'\neq l}^{|D|} d_{l'}^E)$ between parameter subset d_l^E given by the decision maker d_l and the combination of other decision-makers in D.
Step 3. Determine the weight of a decision maker d_l by the following equation:

$$w_l = \frac{\sum_{\alpha_{tj} \in C(d_l^E, \cap_{l'=1,l'\neq l}^{|D|} d_{l'}^E), \alpha_{tj}=1} \alpha_{tj}}{\sum_{l=1}^{k} \sum_{\alpha_{tj} \in C(d_l^E, \cap_{l'=1,l'\neq l}^{|D|} d_{l'}^E), \alpha_{tj}=1} \alpha_{tj}}, 1 \leq l \leq k, 1 \leq t, j \leq n \tag{4.1}$$

Step 4. Derive the product values (given in point $(viii)$ in Section 4.2) as follows,
$(P_l^L, E) = (F_l^L, d_l^E) \otimes C(d_l^E, \cap_{l'=1,l'\neq l}^{|D|} d_{l'}^E) = (p_{sj}^l)_{m\times n}^k$.
Step 5. Evaluate the result of the sum for, $(P^L, E) = (\sum_{l=1}^{k} w_l p_{sj}^l)_{m\times n} = (p_{sj})_{m\times n}$.
Step 6. By using point (ix) of Section 4.2, derive the ranking value of an alternative x_s based on the linguistic valued soft set (P^L, E).
Step 7. The optimal solution is x_o if $R_{F^L}(x_o) = max_{s=1}^{m}\{R_{F^L}(x_s)\}$.

But some limitations have grown from this algorithm. In the next subsection, we have discussed these drawbacks by using some counter examples.

4.3.3 Limitations in Sun's approach [153]

First Limitation

In Sun's algorithm, at Step 3, the weight of a decision maker d_l has been derived. Therefore, at Step 2, they constructed a combined choice value matrix $C(d_l^E, \cap_{l'=1,\neq l}^{|D|} d_{l'}^E)$ between the parameter subset d_l^E considered by the decision maker d_l and the commonly selected parameters by other decision makers except d_l in D, i.e., $\cap_{l'=1,l'\neq l}^{|D|} d_{l'}^E$. Then, they determined the weight of a decision maker d_l by using Equation 4.1.

Now if in case, except the parameter subset d_l^E, the combination of the parameters for all other decision makers is null, i.e., $\cap_{l'=1,l'\neq l}^{|D|} d_{l'}^E = \phi$, $\forall l$, then, from the definition of the combined choice value matrix, each of the entries of the matrix $C(d_l^E, \cap_{l'=1,l'\neq l}^{|D|} d_{l'}^E)$ is equals to -1, $\forall l = 1, 2, .., k$. But, according to Sun's approach, for evaluating the weight of a decision maker, we shall take only the entry 1 as given in Equation 4.1. So, when each of the entries is equals to -1, then what will be the weight of a decision maker d_l? Now we have clarified this question through the following example.

Example 4.3. Let us consider a set of four elements, $X = \{x_1, x_2, x_3, x_4\}$, as the universal set and a set of four elements, $E = \{e_1, e_2, e_3, e_4\}$, as the set of corresponding parameters. Now consider a set of nine linguistic terms as, $S = \{s_{-4} = extremely\ poor, s_{-3} = very\ poor, s_{-2} = poor, s_{-1} = slightly\ poor, s_0 = fair, s_1 = slightly\ good, s_2 = good, s_3 = very\ good,$ $s_4 = extremely\ good\}$, to express the evaluation of an alternative over the parameters. Assume that, four decision makers have been selected, $D = \{d_1, d_2, d_3, d_4\}$, to obtain the best alternative from the set of four alternatives. The parameter subsets considered by the four decision makers are as follows: d_1 consider the parameter subset as, $d_1^E = \{e_1\}$; d_2 consider the parameter subset as, $d_2^E = \{e_2\}$; d_3 consider the parameter subset as, $d_3^E = \{e_3\}$ and d_4 consider the parameter subset as, $d_4^E = \{e_4\}$. The opinions of the four decision makers about the alternatives over the corresponding parameters have been expressed by the four linguistic valued soft sets as, (F_1^L, d_1^E), (F_2^L, d_2^E), (F_3^L, d_3^E), (F_4^L, d_4^E), as given in Tables 4.2, 4.3, 4.4, 4.5.
Here we have seen that, $\cap_{l'=1,2,3} d_{l'}^E = \phi$, $\cap_{l'=1,3,4} d_{l'}^E = \phi$, $\cap_{l'=2,3,4} d_{l'}^E = \phi$, $\cap_{l'=1,2,4} d_{l'}^E = \phi$.
Then, based on Sun's approach, the combined choice value matrices $C(d_1^E, \cap_{l'=2,3,4} d_{l'}^E)$, $C(d_2^E, \cap_{l'=1,3,4} d_{l'}^E)$, $C(d_3^E, \cap_{l'=1,2,4} d_{l'}^E)$, $C(d_4^E, \cap_{l'=1,2,3} d_{l'}^E)$ have been given in Tables 4.6, 4.7, 4.8, 4.9.
From Tables 4.6, 4.7, 4.8, 4.9 it is observed that, all the entries of each of the matrices are

Table 4.2: LVSS (F_1^L, d_1^E) (Example 4.3)

	e_1	e_2	e_3	e_4
x_1	s_{-2}	s_0	s_0	s_0
x_2	s_4	s_0	s_0	s_0
x_3	s_0	s_0	s_0	s_0
x_4	s_2	s_0	s_0	s_0

Table 4.3: LVSS (F_2^L, d_2^E) (Example 4.3)

	e_1	e_2	e_3	e_4
x_1	s_0	s_3	s_0	s_0
x_2	s_0	s_2	s_0	s_0
x_3	s_0	s_3	s_0	s_0
x_4	s_0	s_1	s_0	s_0

Table 4.4: LVSS (F_3^L, d_3^E) (Example 4.3)

	e_1	e_2	e_3	e_4
x_1	s_0	s_0	s_{-1}	s_0
x_2	s_0	s_0	s_1	s_0
x_3	s_0	s_0	s_0	s_0
x_4	s_0	s_0	s_2	s_0

Table 4.5: LVSS (F_4^L, d_4^E) (Example 4.3)

	e_1	e_2	e_3	e_4
x_1	s_0	s_0	s_0	s_1
x_2	s_0	s_0	s_0	s_0
x_3	s_0	s_0	s_0	s_1
x_4	s_0	s_0	s_0	s_1

Table 4.6: Table of $C(d_1^E, \cap_{l'=2,3,4} d_{l'}^E)$

	e_1	e_2	e_3	e_4
e_1	-1	-1	-1	-1
e_2	-1	-1	-1	-1
e_3	-1	-1	-1	-1
e_4	-1	-1	-1	-1

Table 4.7: Table of $C(d_2^E, \cap_{l'=1,3,4} d_{l'}^E)$

	e_1	e_2	e_3	e_4
e_1	-1	-1	-1	-1
e_2	-1	-1	-1	-1
e_3	-1	-1	-1	-1
e_4	-1	-1	-1	-1

Table 4.8: Table of $C(d_3^E, \cap_{l'=1,2,4} d_{l'}^E)$

	e_1	e_2	e_3	e_4
e_1	-1	-1	-1	-1
e_2	-1	-1	-1	-1
e_3	-1	-1	-1	-1
e_4	-1	-1	-1	-1

Table 4.9: Table of $C(d_4^E, \cap_{l'=1,2,3} d_{l'}^E)$

	e_1	e_2	e_3	e_4
e_1	-1	-1	-1	-1
e_2	-1	-1	-1	-1
e_3	-1	-1	-1	-1
e_4	-1	-1	-1	-1

equals to -1. So in that case, by using Sun's approach, the weights of each of the decision makers is being indeterminate. Therefore, in this situation, we shall never get the weight of a decision maker.

Second Limitation.

In their algorithm, there exists an another limitation. Now consider an example as follows:

Example 4.4. Let us consider $X = \{x_1, x_2, x_3\}$ be a set of initial universal set with $E = \{e_1, e_2, e_3\}$ be a set of corresponding parameters and $D = \{d_1, d_2\}$ be a set of decision makers whose evaluations have given in Tables 4.10 and 4.11. Here each of the decision makers has taken all parameters of the set E. The linguistic term set, which has been used in Example 4.3 has also been used here to represent the performance of an alternative over a parameter.
Now by applying Sun's approach, weights of the decision makers are $w_1 = 0.5$, $w_2 = 0.5$. Then, the resultant linguistic valued soft set (P^L, E) has been given in Table 4.12.
From the above table it is observed that, each of the objects has the same ranking value. So, according to their algorithm, we have to select any one of them.

Now in general, in a decision-making problem, the selected alternative must have a maximum degree of similarity with the ideal solution. But in this case, by using their algorithm, we may select an alternative which has a minimum degree of similarity with the

Table 4.10: LVSS (F_1^L, E) (Example 4.4)

	e_1	e_2	e_3
x_1	s_{-1}	s_4	s_1
x_2	s_3	s_{-1}	s_2
x_3	s_1	s_2	s_1

Table 4.11: LVSS (F_2^L, E) (Example 4.4)

	e_1	e_2	e_3
x_1	s_3	s_{-2}	s_0
x_2	s_{-4}	s_2	s_3
x_3	s_2	s_{-1}	s_0

Table 4.12: Resultant linguistic valued soft set (P^L, E) (Example 4.4)

	e_1	e_2	e_3	R_{P^L} (Ranking Value)
x_1	s_1	s_1	$s_{0.5}$	$s_{2.5}$
x_2	$s_{-0.5}$	$s_{0.5}$	$s_{2.5}$	$s_{2.5}$
x_3	$s_{1.5}$	$s_{0.5}$	$s_{0.5}$	$s_{2.5}$

ideal solution. Therefore, in this situation, Sun's algorithm is not suitable to solve this problem.

Then to overcome these drawbacks, we have proposed a new approach to solve linguistic valued soft set based multi-criteria group decision-making problems.

4.4 A new approach to solve linguistic valued soft set based multi-criteria group decision-making problems

In this section, we have proposed a new methodological approach to get a solution for a linguistic valued soft set based multi-criteria group decision-making. In our problem, our aim is to detect the best alternative based on the opinion of all experts where each of them has taken several parameters according to their choices.

Sun et al. [153] used index-based operational laws to for handling linguistic information. Though this computational technique is very easy to operate, but there is a possibility of data loss. For instance, the difference between the linguistic terms 'slightly good' and 'good' may be distinct from the difference between the linguistic terms 'good' and 'very good'. Such type of effective situation is not considered in index-based operational laws, rather equal distance between the linguistic variables is considered. Therefore, to get better and more realistic result, in this chapter, we have used the notion of linguistic scale function for handling the linguistic information.

Let, $S = \{s_{-\tau}, ..s_{-1}, s_0, s_1, .., s_\tau\}$ be a linguistic term set and $f : s_\alpha \to \vartheta_\alpha$ be a linguistic scale function where $\alpha = -\tau, .., -1, 0, 1, .., \tau$ and $0 = \vartheta_{-\tau} < .. < \vartheta_\tau = 1$.

Now consider, $X = \{x_1, x_2, .., x_m\}$ be a set of m alternatives as a universal set,

$E = \{e_1, e_2, .., e_n\}$ be a set of n corresponding parameters. Let, $W = \{W_1, W_2, .., W_n\}$ be the weighted vectors of the parameters such that, $0 \leq W_j \leq 1, \forall j$ and $\sum_{j=1}^{n} W_j = 1$. Consider, k decision makers as, $D = \{d_1, d_2, .., d_k\}$ who have taken some parameters from the set E in favor of their preferences. The k selected parameter subsets of the set E, considered by k decision makers, are $d_1^E, d_2^E, .., d_k^E$ respectively. Then, the k linguistic valued soft sets $(F_1^L, d_1^E), (F_2^L, d_2^E), .., (F_k^L, d_k^E)$ given by the k decision makers are as follows:

$(F_l^L, d_l^E) = \{(e_1, F_l^L(e_1)), (e_2, F_l^L(e_2)), .., (e_n, F_l^L(e_n))\} =$
$\{(e_1, ((x_1, F_l^L(e_1)(x_1)), (x_2, F_l^L(e_1)(x_2)), .., (x_m, F_l^L(e_1)(x_m)))),$
$(e_2, ((x_1, F_l^L(e_2)(x_1)), (x_2, F_l^L(e_2)(x_2)), .., (x_m, F_l^L(e_2)(x_m)))), ..,$
$(e_n, ((x_1, F_l^L(e_n)(x_1)), (x_2, F_l^L(e_n)(x_2)), .., (x_m, F_l^L(e_n)(x_m))))\}, l = 1, 2, .., k$

where, $F_l^L(e_j)(x_s) = u_{sj}^l \in S; s = 1, 2, .., m; j = 1, 2, .., n; l = 1, 2, .., k$ is the satisfaction of an alternative x_s over a parameter e_j given by a d_l decision maker. If a parameter e_j is not selected by a decision maker d_l, then associated evaluations of each of the alternatives has been considered as s_0, i.e., if $e_j \notin d_l^E, j = 1, 2, .., n; l = 1, 2, .., k$, then $\forall, s = 1, 2, .., m, u_{sj}^l = s_0$.

The matrix form of a linguistic valued soft set $(F_l^L, d_l^E)|l = 1, 2, .., k$ is denoted by, $(F_l^L, d_l^E) = (u_{sj}^l)_{m \times n}^k$, which has been given in Table 4.13.

Table 4.13: LVSS $(F_l^L, d_l^E); l = 1, 2, .., k$

	e_1	e_2	$\dots$	e_n
x_1	u_{11}^l	u_{12}^l	$\dots$	u_{1n}^l
.	.	.	$\dots$	.
.	.	.	$\dots$	.
x_m	u_{m1}^l	u_{m2}^l	$\dots$	u_{mn}^l

4.4.1 A method for computing the satisfaction level of a decision maker

Now suppose that, a purchaser wants to buy the best house over two attributes such as outlook and wooden quality in a big city. For selecting an appropriate house, he/she (purchaser) has appointed three decision makers as d_1, d_2, and d_3. Here, his/her (purchaser) goal is to select a house with a very good outlook and a very good wooden quality, i.e., satisfaction level of a house over the attributes to be maximum. Moreover, since he/she (purchaser) employs three decision makers for taking this decision, so based on the three decision makers' opinion he/she will select the best house. Therefore, in this case, to select the best house, the satisfaction level of a house over the attributes should be as much as close to the maximum satisfaction level among all the houses over the attributes by all decision makers.

Consequently, in this subsection, we have evaluated the satisfaction level of all the decision makers in giving their opinions. In this regard, firstly we have constructed the ideal soft set (F_*^L, E) by taking the union of all the linguistic valued soft sets, $(F_1^L, d_1^E), (F_2^L, d_2^E), .., (F_k^L, d_k^E)$. The reason for taking their union operation is that, union operation has taken the maximum value among individuals as given in point (v) in Section 4.2. This ideal soft set (F_*^L, E) is called here as the linguistic soft reference decision set. Then the closeness between linguistic soft reference decision set (F_*^L, E) and each of the individual linguistic valued soft sets (F_l^L, d_l^E); $l = 1, 2, .., k$ has been derived. Finally, we have obtained the satisfaction level of each of the decision makers d_l; $l = 1, 2, .., k$. A comprehensive procedure has been provided below.

I. *Construction of the linguistic soft reference decision set* (F_*^L, E).

Since the target of this decision problem is to select an alternative with a maximum linguistic evaluation over all the parameters, therefore, to achieve this goal, at first we have defined the linguistic soft reference decision set which is denoted by, (F_*^L, E) and is defined as follows:

Definition 4.1. The linguistic soft reference decision set is defined as the union of all the linguistic valued soft sets. Mathematically, it can be written as follows:

$$(F_*^L, E) = \cup_{l=1}^{k} (F_l^L, d_l^E)$$

$$\text{i.e.,} (F_*^L, E) = \{(e_j, F_*^L(e_j)) | \forall e_j \in E\}$$
$$= \{(e_j, (F_1^L(e_j) \cup F_2^L(e_j) \cup .... \cup F_k^L(e_j))) | \forall e_j \in E\}$$

The significance of this soft set is that, this soft set is the ideal soft set for this group decision-making problem with respect to all decision makers over all the parameters.

Now according to the definition of the union of two soft sets, it is obtained that,

$$F_1^L(e_j) \cup F_2^L(e_j) \cup .... \cup F_k^L(e_j) = \{max(F_1^L(e_j)(x_s), F_2^L(e_j)(x_s), .., F_k^L(e_j)(x_s))\}$$
$$= \{max(u_{sj}^1, u_{sj}^2, .., u_{sj}^k) = u_{sj}^* | \forall e_j \in E, x_s \in X\}$$

The tabular form of the soft set (F_*^L, E) has been Table 4.14.

II. *Calculation of the relative closeness between* (F_*^L, E) *and* (F_l^L, d_l^E), $l = 1, 2, .., k$.

The linguistic soft reference decision set (F_*^L, E) is the optimistic soft set that gives the ideal evaluations of all alternatives with respect to all the parameters for all decision makers. Therefore we have evaluated the relative closeness between (F_*^L, E) and each of the individuals $(F_l^L, d_l^E); l = 1, 2, .., k$ to measure how close is each of the soft sets $(F_l^L, d_l^E); l = 1, 2, .., k$ to the ideal or best soft set (F_*^L, E).

Table 4.14: Tabular form of the linguistic soft reference decision set (F_*^L, E)

	e_1	e_2	$\ldots$	e_n
u_1	u_{11}^*	u_{12}^*	$\ldots$	u_{1n}^*
u_2	u_{21}^*	u_{22}^*	$\ldots$	u_{2n}^*
.	.	.	$\ldots$	.
.	.	.	$\ldots$	.
u_m	u_{m1}^*	u_{m2}^*	$\ldots$	u_{mn}^*

(i) Firstly, we have transformed all the linguistic variables into numerical values by using a linguistic scale function $f : s_\alpha \to \vartheta_\alpha$ in such a way that, $f(s_{-\tau}) = 0$ and $f(s_\tau) = 1$. After this transformation, the linguistic valued soft sets, (F_*^L, E) and $(F_l^L, d_l^E), l = 1, 2, .., k$ take the form of fuzzy soft sets.

(ii) Then, by inspiring the references [11, 154], we have defined the restricted bounded difference of the fuzzy-valued soft sets (F_*^L, E) and $(F_l^L, d_l^E);\ l = 1, 2, .., k$ as follows:

Definition 4.2. The restricted bounded difference of the fuzzy-valued soft sets (F_*^L, E) and (F_l^L, d_l^E) is denoted by, $(F_*^L, E) \smile_{RB} (F_l^L, d_l^E) = (H_l^L, E);\ l = 1, 2, .., k$ and is defined as,

$$(H_l^L, E) = \{(e_j, H_l^L(e_j)) | \forall e_j \in E\} = \{(e_j, (F_*^L(e_j) \ominus F_l^L(e_j))) | e_j \in E\}$$

where $\ominus$ represents the fuzzy bounded difference operation as given in point (x) in Section 4.2. Now from the definition of the bounded difference of fuzzy sets [154], it is obtained that,

$$\begin{aligned}\forall e_j \in E, F_*^L(e_j) \ominus F_l^L(e_j) &= \{max(0, (F_*^L(e_j)(x_s) - F_l^L(e_j)(x_s))) | x_s \in X\} \\ &= \{max(0, (u_{sj}^* - u_{sj}^l)) | s = 1, 2, .., m; j = 1, 2, .., n\} \\ &= \{\breve{u}_{sj}^l | s = 1, 2, .., m; j = 1, 2, .., n\}\end{aligned}$$

(iii) After that, we have calculated the grey relational degree of each of the fuzzy valued evaluations $(\breve{u}_{sj}^l)$ in the soft set (H_l^L, E), which is denoted by γ_{sj}^l and is defined as follows:

$$\gamma_{sj}^l = \frac{\min\limits_{1 \le j \le n} \min\limits_{1 \le s \le m} \breve{u}_{sj}^l + \rho \max\limits_{1 \le j \le n} \max\limits_{1 \le s \le m} \breve{u}_{sj}^l}{\breve{u}_{sj}^l + \rho \max\limits_{1 \le j \le n} \max\limits_{1 \le s \le m} \breve{u}_{sj}^l}$$

The significance of the grey relational degree is that, the large value of γ_{sj}^l implies the much closeness of the evaluation u_{sj}^l to the evaluation u_{sj}^*, $\rho \in [0, 1]$ is the distinguishing coefficient which has been used to expand or compress the range of the grey relational degree. In this chapter, we have considered the value of ρ as 0.5.

(iv) Finally, we have derived the relative closeness between the ideal soft set (F_*^L, E) and the each of the individual soft sets $(F_l^L, d_l^E) : l = 1, 2, .., k$ by the following equation.

$$\Gamma((F_*^L, E), (F_l^L, d_l^E)) = \frac{\sum_{s=1}^{m}\sum_{j=1}^{n}\gamma_{sj}^{l}}{m \times n}, l = 1, 2, .., k$$

III. *Evaluation of the satisfaction level of a decision maker* d_l.

The satisfaction level of a decision maker d_l is derived as follows:

$$\lambda_l = \frac{\Gamma((F_*^L, E), (F_l^L, d_l^E))}{\sum_{l=1}^{k}\Gamma((F_*^L, E), (F_l^L, d_l^E))}$$

The greater value of λ_l indicates that, the satisfaction level of a decision maker d_l in giving his/her opinion about the evaluations of all alternatives over all parameters is high for this decision-making problem.

4.4.2 A new version of ranking function through the linguistic scale function

Let, $X = \{x_1, x_2, .., x_m\}$ be a set of m alternatives and $E = \{e_1, e_2, .., e_n\}$ be a set of n corresponding parameters. Assume that, $W = \{W_1, W_2, .., W_n\}$ be the weights of the parameters $e_1, e_2, .., e_n$ respectively such that, $0 \leq W_j \leq 1; \forall j = 1, 2, .., n$ and $\sum_{j=1}^{n} W_j = 1$.
In reference [153], Sun et al. proposed the notion of ranking function for evaluating the best alternative in a linguistic valued soft set as given in point (ix) Section 4.2. But in this case, resultant linguistic variable may exceed the boundary of the associated linguistic term set. Therefore, to keep the resultant linguistic variable between upper bound and lower bound of the associated linguistic term set, now we have redefined the notion of ranking function of an alternative in a linguistic valued soft set through the notion of linguistic scale function as follows:

Let $S = \{s_\alpha | \alpha = -\tau, .., -1, 0, 1, .., \tau\}$ be a linguistic term set and $f : s_\alpha \to \vartheta_\alpha$ be a linguistic scale function over S where, $0 = \vartheta_{-\tau} < ... < \vartheta_0 ... < \vartheta_\tau = 1$. Let $(F^L, E) = (u_{sj})_{m \times n}$, $s = 1, 2, .., m$; $j = 1, 2, .., n$ be a linguistic valued soft set over the universe X as given in Table 4.15.
where, $u_{sj} \in S$ is the linguistic evaluation of an alternative x_s with respect to a parameter e_j.

Then the ranking function of an alternative x_s with respect to all the parameters is

Table 4.15: Tabular form of linguistic valued soft set (F^L, E)

	e_1	e_2	$\ldots$	e_n
x_1	u_{11}	u_{12}	$\ldots$	u_{1n}
x_2	u_{21}	u_{22}	$\ldots$	u_{2n}
.	.	.	$\ldots$	.
.	.	.	$\ldots$	.
x_m	u_{m1}	u_{m2}	$\ldots$	u_{mn}

denoted by $\tilde{R}_{F^L}$ and is defined by the following equation.

$$\tilde{R}_{F^L}(x_s) = f^{-1}(\sum_{j=1}^{n} W_j f(u_{sj})) \tag{4.2}$$

where, f^{-1} is the inverse of the linguistic scale function f and W_j is the weight of the parameter e_j.

- If the weights of the parameters are unknown in a decision-making problem, then we can determine the weight (W_j) of a parameter $e_j; j = 1, 2, .., n$ by computing the total deviation of the evaluation values of one alternative to other with respect to e_j. Because, over a parameter e_j, if the total difference of satisfaction of one alternative to other alternative is small, then it means that, such a parameter has a little importance during decision-making. On the other hand, if the total difference of satisfaction of one alternative to other alternative is high, then such a parameter has a strong importance during decision-making. Now, based on this idea, we have determined the weight W_j of a parameter e_j as follows:

- Firstly, derive the total deviation of satisfaction of one alternative to other with respect to the parameter e_j which is denoted by $W_j^{'}$ and is defined as follows:

$$W_j^{'} = \frac{\sum_{s'=1}^{m} \sum_{s=1, s \neq s'}^{m} d(u_{sj}, u_{s'j})}{m(m-1)} \tag{4.3}$$

  where $d(u_{sj}, u_{s'j}) = |f(u_{sj}) - f(u_{s'j})|$ is the deviation measure of between two linguistic variables u_{sj} and $u_{s'j}$ based on the linguistic scale function f.

- Then the weight of a parameter e_j, W_j, is evaluated by the following equation,

$$W_j = \frac{W_j^{'}}{\sum_{j=1}^{n} W_j^{'}} \tag{4.4}$$

4.4.3 A new definition of similarity measure for linguistic valued sets

In this subsection, we have proposed a new definition of similarity measure for linguistic valued sets through the notion of linguistic scale function.

Let us consider a linguistic term set as, $S = \{s_\alpha | \alpha = -\tau, .., -1, 0, 1, .., \tau\}$ and a linguistic scale function as, $f : s_\alpha \to \vartheta_\alpha$, where, $0 = \vartheta_{-\tau} < ... < \vartheta_0 ... < \vartheta_\tau = 1$.

Now consider two linguistic valued sets L_1 and L_2 having n number of linguistic variables as follows,
$L_1 = \{s_1^1, s_2^1, .., s_n^1 | s_j^1 \in S, j = 1, 2, .., n\}$,
$L_2 = \{s_1^2, s_2^2, .., s_n^2 | s_j^2 \in S, j = 1, 2, .., n\}$.

Then, the degree of similarity of the set L_1 with the set L_2 is defined by the following equation,

$$\hat{S}(L_1, L_2) = \frac{\sum_{j=1}^{m} min(f(s_j^1), f(s_j^2))}{\sum_{j=1}^{m} max(f(s_j^1), f(s_j^2))}$$

If $w = \{w_1, w_2, .., w_n\}$ be the associated weighted vectors of the elements of L_1 and L_2 in such a way that $0 \leq w_j \leq 1; \forall j = 1, 2, .., n$ and $\sum_{j=1}^{n} w_j = 1$, then the weighted similarity measure between L_1 and L_2 is defined as,

$$\hat{S}_w(L_1, L_2) = \frac{\sum_{j=1}^{n} w_j min(f(s_j^1), f(s_j^2))}{\sum_{j=1}^{n} w_j max(f(s_j^1), f(s_j^2))} \tag{4.5}$$

This similarity measure $\hat{S}_w$ satisfies the following properties.

Property 1. $0 \leq \hat{S}_w(L_1, L_2) \leq 1$.

Proof.
$0 \leq min(f(s_j^1), f(s_j^2)) \leq max(f(s_j^1), f(s_j^2))$
$\Rightarrow 0 \leq w_j min(f(s_j^1), f(s_j^2)) \leq w_j max(f(s_j^1), f(s_j^2))$
$\Rightarrow 0 \leq \sum_{j=1}^{n} w_j min(f(s_j^1), f(s_j^2)) \leq \sum_{j=1}^{n} w_j max(f(s_j^1), f(s_j^2))$
$\Rightarrow 0 \leq \frac{\sum_{j=1}^{n} w_j min(f(s_j^1), f(s_j^2))}{\sum_{j=1}^{n} w_j max(f(s_j^1), f(s_j^2))} \leq 1$
$\Rightarrow 0 \leq \hat{S}_w(L_1, L_2) \leq 1$

Property 2. $\hat{S}_w(L_1, L_2) = 1$ *if and only if for all* $j = 1, 2, .., n$, $s_j^1 = s_j^2$ *where* $s_j^1 \in L_1$ *and* $s_j^2 \in L_2$.

Proof. $\hat{S}_w(L_1, L_2) = 1 \Leftrightarrow \frac{\sum_{j=1}^{n} w_j min(f(s_j^1), f(s_j^2))}{\sum_{j=1}^{n} w_j max(f(s_j^1), f(s_j^2))} = 1$
$\Leftrightarrow min(f(s_j^1), f(s_j^2)) = max(f(s_j^1), f(s_j^2)) \Leftrightarrow f(s_j^1) = f(s_j^2) \Leftrightarrow s_j^1 = s_j^2, j = 1, 2, .., n.$

Property 3. $\hat{S}_w(L_1, L_2) = \hat{S}_w(L_2, L_1)$.
Proof. It is straightforward.

Property 4. Let $L_3 = \{s_1^3, s_2^3, .., s_n^3 | s_j^3 \in S, j = 1, 2, .., n\}$ be an another linguistic valued set over U. Now if, $L_1 \subseteq L_2 \subseteq L_3$, then $\hat{S}_w(L_1, L_2) \supseteq \hat{S}_w(L_1, L_3)$ and $\hat{S}_w(L_2, L_3) \supseteq \hat{S}_w(L_1, L_3)$.

Proof. If $L_1 \subseteq L_2 \subseteq L_3$ then $s_j^1 \subseteq s_j^2 \subseteq s_j^3, \forall j = 1, 2, .., n$, which implies that, $\hat{S}_w(L_1, L_2) = \frac{\sum_{j=1}^{n} w_j min(f(s_j^1), f(s_j^2))}{\sum_{j=1}^{n} w_j max(f(s_j^1), f(s_j^2))} = \frac{\sum_{j=1}^{n} w_j f(s_j^1)}{\sum_{j=1}^{n} w_j f(s_j^2)} \supseteq \frac{\sum_{j=1}^{n} w_j f(s_j^1)}{\sum_{j=1}^{n} w_j f(s_j^3)} = \hat{S}_w(L_1, L_3)$.

In the similar way, the second inequality can also be proved.

Example 4.5. Let, $S = \{s_{-3}, s_{-2}, s_{-1}, s_0, s_1, s_2, s_3\}$ be a linguistic term set and $L_1 = \{s_{-3}, s_0, s_{-1}, s_0\}$, $L_2 = \{s_1, s_{-2}, s_1, s_2\}$ be two linguistic valued sets with $w = \{0.25, 0.25, 0.25, 0.25\}$ be the associated weighted vectors of the elements of L_1 and L_2.
Now consider a linguistic scale function as, $f(s_x) = \frac{1}{2}(\frac{x}{3} + 1); x \in [-3, 3]$. Then, $f(s_{-3}) = 0$, $f(s_{-2}) = 0.17$, $f(s_{-1}) = 0.33$, $f(s_0) = 0.5$, $f(s_1) = 0.67$, $f(s_2) = 0.83$, $f(s_3) = 1$.
Now by using Equation 4.5, the degree of similarity between L_1 and L_2 is,

$$\hat{S}_w(L_1, L_2) = \frac{0.25min(f(s_{-3}),f(s_1))+0.25min(f(s_0),f(s_{-2}))+0.25min(f(s_-),f(s_1))+0.25min(f(s_0),f(s_2))}{0.25max(f(s_{-3}),f(s_1))+0.25max(f(s_0),f(s_{-2}))+0.25max(f(s_{-1}),f(s_1))+0.25max(f(s_0),f(s_2))}$$
$$= \frac{0.25min(0,0.67)+0.25min(0.5,0.17)+0.25min(0.33,0.67)+0.25min(0.5,0.83)}{0.25max(0,0.67)+0.25max(0.5,0.17)+0.25max(0.33,0.67)+0.25max(0.5,0.83)}$$
$$= \frac{0.25\times0+0.25\times0.17+0.25\times0.33+0.25\times0.5}{0.25\times0.67+0.25\times0.5+0.25\times0.67+0.25\times0.83} = 0.37$$

4.4.4 Linguistic valued soft set based approach to multi-criteria group decision-making problems

Now, we have established a step-wise algorithm for linguistic valued soft set based multi-criteria group decision-making problems by utilizing our above proposed ideas. In Figure 4.2, flow chart of our proposed algorithm has been given. Corresponding steps of our methodology have been discussed in the following.

Step 1. Input, the universal set as, $X = \{x_1, x_2, .., x_m\}$ and the associated parameter set as, $E = \{e_1, e_2, .., e_n\}$. Input the weighted vectors of the parameters as, $W = \{W_1, W_2, .., W_n\}$ (if given). Input k decision makers as, $D = \{d_1, d_2, .., d_k\}$ and their corresponding choice parameter subsets of the set E as, $d_1^E, d_2^E, .., d_k^E$. Construct k linguistic valued soft sets as, $\{(F_l^L, d_l^E) | l = 1, 2, .., k\} = (u_{sj}^l)_{m\times n}^k$ over X. The tabular form of the l^{th} linguistic valued soft set has been given in Table 4.13. Consider a suitable linguistic scale function

$f : s_\alpha \rightarrow \vartheta_\alpha$, where $\alpha = -\tau, .., -1, 0, 1, .., \tau$ and $\vartheta_\alpha \in [0, 1]$.

Step 2. Categorize the parameter set E into two subsets A and B in such a way that, $A \cup B = E$ and $A \cap B = \phi$ where A is a set of benefited attributes and B is a set of cost attributes. Now to equate the sense of all the parameters, we have to take the complement of all the evaluations with respect to each of the cost parameters as follows.

$$(F_l^L(e_j))^c = \{(x_s, (u_{sj}^l)^c) \mid e_j \in B \ and \ x_s \in X\};\ l = 1, 2, .., k$$

where c denotes the complement or negation of a linguistic evaluation u_{sj}^l.

Step 3. Determine the satisfaction level (λ_l) of a decision maker d_l, $l = 1, 1, .., k$, by using Subsection 4.4.1.

Step 4. Aggregate k linguistic valued soft sets, $\{(F_l^L, d_l^E) = (\tilde{u}_{sj}^l)_{m\times n}^k \mid l = 1, 2, .., k\}$ into a resultant linguistic valued soft set $(\tilde{F}^L, E) = \{(u_{sj})_{m\times n} \mid s = 1, 2, .., m; j = 1, 2, .., n\}$ by the following equation:

$$u_{sj} = f^{-1}(\sum_{l=1}^{k} \lambda_l(f(\tilde{u}_{sj}^l))) \tag{4.6}$$

i.e., Aggregation of $((F_1^L, d_1^E), (F_2^L, d_2^E), .., (F_k^L, d_k^E)) = (\tilde{F}^L, E)$

Step 5. By using Subsection 4.4.2, derive the ranking function $\tilde{R}_{\tilde{F}^L}$ of each of the alternatives x_s, $s = 1, 2, .., m$ based on the resultant linguistic valued soft set $(\tilde{F}^L, E)$.

If, the weights of the parameters are not provided in the associated decision-making problem, then we have to determine the weight (W_j) of a parameter e_j by using Equations 4.3 and 4.4.

It is noted that, all the resultant ranking values are in terms of linguistic variables.

Step 6. The alternative with maximum ranking value is the best or optimal alternative (x_O) for this decision-making problem i.e.,

$$Optimal\ Solution\ (x_O) = max_{s=1}^{m} \tilde{R}_{\tilde{F}^L}(x_s) \tag{4.7}$$

If x_O is not unique, then we will go to the next step.

Step 7. Determine the ideal alternative or ideal solution x_* from the ideal soft set (F_*^L, E) (given in Table 4.14) by taking the satisfaction of each of the parameters as the maximum satisfaction over all alternatives. Mathematically it can be defined as follows:

$$\begin{aligned} x_* &= \{max(u_{11}^*, u_{21}^*, .., u_{m1}^*), max(u_{12}^*, u_{22}^*, .., u_{m2}^*), .., max(u_{1n}^*, u_{2n}^*, .., u_{mn}^*)\} \\ &= \{u_{*1}, u_{*2}, .., u_{*n}\} \end{aligned}$$

where u^*_{sj} is the evaluation of an alternative x_s with respect to a parameter e_j corresponding to the ideal soft set (F^L_*, E).

Step 8. Evaluate the degree of similarity between x_* and each of the alternatives which has the equal ranking values associated with the resultant linguistic valued soft set $(\tilde{F}^L, E)$ as follows,

$$\hat{S}_w(x_*, x_{s'}) = \frac{\sum_{j=1}^{n} W_j min(f(u_{s'j})f(u_{*j}))}{\sum_{j=1}^{n} W_j max(f(u_{s'j}), f(u_{*j}))} \tag{4.8}$$

where $s^{'} = 1, 2, .., m^{'}$ is the number of alternatives having equal ranking values getting from *Step 6* and W_j is the weight of the parameter e_j.

Step 9. The alternative having maximum degree of similarity with the ideal solution x_* should be selected as an optimal solution (which may or may not be unique). Optimal Solution (x_O)=$max^m_{s'=1}\hat{S}_w(u_*, u_{s'})$.

Example 4.6. Now, we have solved an example by using our proposed approach. Let, $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be the set of alternatives and $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the set of associated parameters. A set of four experts as, $D = \{d_1, d_2, .., d_k\}$, have been assigned to select the best alternative. The selected parameters of these experts are, $d^E_1\{e_1, e_2, e_3, e_4\}$, $d^E_2 = \{e_1, e_2, e_3, e_5\}$, $d^E_3 = \{e_1, e_2, e_4, e_5\}$, $d^E_4 = \{e_1, e_2, e_3, e_4, e_5\}$. Now, the corresponding linguistic valued soft sets (F^L_1, d^E_1), (F^L_2, d^E_2), (F^L_3, d^E_3) and (F^L_4, d^E_4) provided by the four experts have been given in Tables 4.16, 4.17, 4.18 and 4.19 respectively.

Table 4.16: LVSS (F^L_1, d^E_1) (Example 4.6)

	e_1	e_2	e_3	e_4	e_5
x_1	s_{-2}	s_{-1}	s_{-2}	s_{-2}	s_0
x_2	s_{-1}	s_{-4}	s_1	s_{-1}	s_0
x_3	s_1	s_{-1}	s_0	s_1	s_0
x_4	s_{-1}	s_0	s_0	s_2	s_0
x_5	s_1	s_{-3}	s_0	s_3	s_0
x_6	s_2	s_1	s_2	s_3	s_0

Table 4.17: LVSS (F^L_2, d^E_2) (Example 4.6)

	e_1	e_2	e_3	e_4	e_5
x_1	s_{-1}	s_1	s_{-2}	s_0	s_3
x_2	s_{-2}	s_{-2}	s_{-2}	s_0	s_{-1}
x_3	s_0	s_0	s_{-3}	s_0	s_2
x_4	s_1	s_{-1}	s_1	s_0	s_2
x_5	s_{-1}	s_{-3}	s_{-1}	s_0	s_{-1}
x_6	s_1	s_2	s_1	s_0	s_3

Solution.

- ***By our proposed approach.***

  Consider a linguistic scale function as, $f : s_\alpha \rightarrow \vartheta_\alpha$ as, $f(s_\alpha) = \frac{1}{2}(\frac{\alpha}{4} + 1)$; $\alpha \in [-4, 4]$.

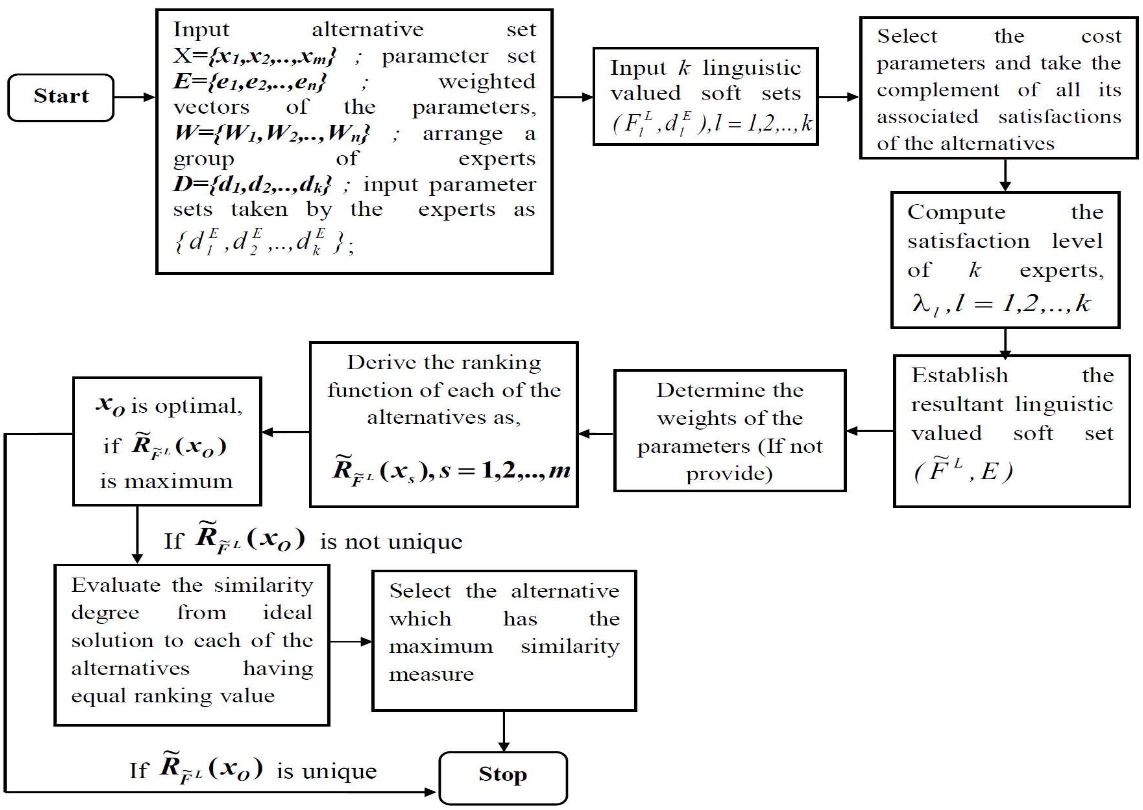

Figure 4.2: Flow chart of our proposed algorithm

Table 4.18: LVSS (F_3^L, d_3^E) (Example 4.6)

	e_1	e_2	e_3	e_4	e_5
x_1	s_0	s_{-3}	s_0	s_{-2}	s_{-3}
x_2	s_{-4}	s_4	s_0	s_{-1}	s_3
x_3	s_0	s_1	s_0	s_2	s_4
x_4	s_1	s_{-1}	s_0	s_1	s_{-2}
x_5	s_2	s_{-2}	s_0	s_0	s_0
x_6	s_3	s_1	s_0	s_1	s_1

Table 4.19: LVSS (F_4^L, d_4^E) (Example 4.6)

	e_1	e_2	e_3	e_4	e_5
x_1	s_0	s_{-3}	s_{-1}	s_4	s_{-3}
x_2	s_{-2}	s_0	s_{-2}	s_{-1}	s_3
x_3	s_1	s_0	s_{-3}	s_2	s_4
x_4	s_1	s_{-1}	s_{-1}	s_1	s_{-2}
x_5	s_{-1}	s_{-2}	s_3	s_0	s_0
x_6	s_2	s_1	s_2	s_2	s_1

Step 1. $f(s_{-4}) = 0$, $f(s_{-3}) = \frac{1}{8} = 0.125$, $f(s_{-2}) = \frac{2}{8} = 0.25$, $f(s_{-1}) = \frac{3}{8} = 0.375$, $f(s_0) = \frac{4}{8} = 0.5$, $f(s_1) = \frac{5}{8} = 0.625$, $f(s_2) = \frac{6}{8} = 0.75$, $f(s_3) = \frac{7}{8} = 0.875$, $f(s_4) = 1$.
Since, all the parameters are in same sense, therefore, Step 2 has been omitted here.
Step 2. *Determine the satisfaction level of each of the decision makers.*
The linguistic soft reference decision set (F_*^L, E) has been given in Table 4.20.

Table 4.20: Tabular form of (F_*^L, E) (Example 4.6)

	e_1	e_2	e_3	e_4	e_5
x_1	s_0	s_1	s_0	s_4	s_3
x_2	s_{-1}	s_4	s_1	s_0	s_3
x_3	s_1	s_1	s_0	s_2	s_4
x_4	s_1	s_0	s_1	s_2	s_2
x_5	s_2	s_{-2}	s_3	s_3	s_0
x_6	s_3	s_2	s_2	s_3	s_3

The relative closeness between (F_*^L, E) and each of the individuals (F_1^L, d_1^E), (F_2^L, d_2^E), (F_3^L, d_3^E) and (F_4^L, d_4^E) is as follows:
$\Gamma(F_1^L, F_*^L) = 0.771$, $\Gamma(F_2^L, F_*^L) = 0.704$, $\Gamma(F_3^L, F_*^L) = 0.769$, $\Gamma(F_4^L, F_*^L) = 0.75$.
Then, the satisfaction level of each of the decision makers is, $\lambda_1 = 0.26$, $\lambda_2 = 0.23$, $\lambda_3 = 0.26$, $\lambda_4 = 0.25$ respectively.
Step 3. *Aggregate four linguistic valued soft sets (F_1^L, d_1^E), (F_2^L, d_2^E), (F_3^L, d_3^E) and (F_4^L, d_4^E) by using Equation 4.6.*
Now we have,

$$\lambda_1 \begin{pmatrix} f(s_{-2}) & f(s_{-1}) & f(s_{-2}) & f(s_{-2}) & f(s_0) \\ f(s_{-1}) & f(s_{-4}) & f(s_1) & f(s_{-1}) & f(s_0) \\ f(s_1) & f(s_{-1}) & f(s_0) & f(s_1) & f(s_0) \\ f(s_{-1}) & f(s_0) & f(s_0) & f(s_2) & f(s_0) \\ f(s_1) & f(s_{-3}) & f(s_0) & f(s_3) & f(s_0) \\ f(s_2) & f(s_1) & f(s_2) & f(s_3) & f(s_0) \end{pmatrix} + \lambda_2 \begin{pmatrix} f(s_{-1}) & f(s_1) & f(s_{-2}) & f(s_0) & f(s_3) \\ f(s_{-2}) & f(s_{-2}) & f(s_{-2}) & f(s_0) & f(s_{-1}) \\ f(s_0) & f(s_0) & f(s_{-3}) & f(s_0) & f(s_2) \\ f(s_1) & f(s_{-1}) & f(s_1) & f(s_0) & f(s_2) \\ f(s_{-1}) & f(s_{-3}) & f(s_{-1}) & f(s_0) & f(s_{-1}) \\ f(s_1) & f(s_2) & f(s_1) & f(s_0) & f(s_3) \end{pmatrix} +$$

$$\lambda_3 \begin{pmatrix} f(s_0) & f(s_{-3}) & f(s_0) & f(s_{-2}) & f(s_{-3}) \\ f(s_{-4}) & f(s_4) & f(s_0) & f(s_{-1}) & f(s_3) \\ f(s_0) & f(s_1) & f(s_0) & f(s_2) & f(s_4) \\ f(s_1) & f(s_{-1}) & f(s_0) & f(s_1) & f(s_{-2}) \\ f(s_2) & f(s_{-2}) & f(s_0) & f(s_0) & f(s_0) \\ f(s_3) & f(s_1) & f(s_0) & f(s_1) & f(s_1) \end{pmatrix} + \lambda_4 \begin{pmatrix} f(s_0) & f(s_{-3}) & f(s_{-1}) & f(s_4) & f(s_{-3}) \\ f(s_{-2}) & f(s_0) & f(s_{-2}) & f(s_{-1}) & f(s_3) \\ f(s_1) & f(s_0) & f(s_{-3}) & f(s_2) & f(s_4) \\ f(s_1) & f(s_{-1}) & f(s_{-1}) & f(s_1) & f(s_{-2}) \\ f(s_{-1}) & f(s_{-2}) & f(s_3) & f(s_0) & f(s_0) \\ f(s_2) & f(s_1) & f(s_2) & f(s_2) & f(s_1) \end{pmatrix}$$

$$= \begin{pmatrix} 0.4062 & 0.305 & 0.3462 & 0.495 & 0.395 \\ 0.2175 & 0.4425 & 0.4125 & 0.4037 & 0.6625 \\ 0.5637 & 0.5 & 0.32 & 0.66 & 0.8125 \\ 0.5600 & 0.4075 & 0.4975 & 0.6287 & 0.43 \\ 0.5375 & 0.1887 & 0.565 & 0.5975 & 0.4712 \\ 0.7537 & 0.6537 & 0.6562 & 0.6925 & 0.65 \end{pmatrix}$$

Then, the resultant linguistic valued soft set $(\tilde{F}^L, E)$ is given in Table 4.21.

Table 4.21: Tabular form of $(\tilde{F}^L, E)$ (Example 4.6)

	e_1	e_2	e_3	e_4	e_5
x_1	$s_{-0.75}$	$s_{-0.56}$	$s_{-1.23}$	$s_{-0.04}$	$s_{-0.84}$
x_2	$s_{-2.26}$	$s_{-0.46}$	$s_{-0.7}$	$s_{-0.77}$	$s_{1.3}$
x_3	$s_{0.5637}$	s_0	$s_{-1.44}$	$s_{1.28}$	$s_{2.5}$
x_4	$s_{0.48}$	$s_{-0.74}$	$s_{-0.02}$	$s_{1.03}$	$s_{-0.56}$
x_5	$s_{0.3}$	$s_{-2.49}$	$s_{0.52}$	$s_{0.78}$	$s_{-0.23}$
x_6	$s_{2.03}$	$s_{1.23}$	$s_{1.25}$	$s_{1.54}$	$s_{1.2}$

Step 4. *Derive the ranking function of each of the alternatives from the resultant linguistic valued soft set $(\tilde{F}^L, E)$.*

Since, weights of the parameters are unknown in this problem, so based on Equations 4.3 and 4.4, the weights of the parameters can be evaluated as follows:

$W_1 = 0.24, W_2 = 0.18, W_3 = 0.16, W_4 = 0.17, W_5 = 0.25.$

So, by using Equation 4.2, the ranking values of each of the alternatives is as following:

$\tilde{R}_{\tilde{F}^L}(x_1) = f^{-1}(0.24f(s_{-0.75}) + 0.18f(s_{-0.56}) + 0.16f(s_{-1.23}) + 0.17f(s_{-0.04}) + 0.25f(s_{-0.84})) = f^{-1}(0.3907) = s_{-0.87}$;

$\tilde{R}_{\tilde{F}^L}(x_2) = f^{-1}(0.24f(s_{-2.26}) + 0.18f(s_{-0.46}) + 0.16f(s_{-0.7}) + 0.17f(s_{-0.77}) + 0.25f(s_{1.3})) = f^{-1}(0.4321) = s_{-0.54}$;

$\tilde{R}_{\tilde{F}^L}(x_3) = f^{-1}(0.24f(s_{0.5637}) + 0.18f(s_0) + 0.16f(s_{-1.44}) + 0.17f(s_{1.28}) + 0.25f(s_{2.5})) = f^{-1}(0.5918) = s_{0.73}$;

$\tilde{R}_{\tilde{F}^L}(x_4) = f^{-1}(0.24f(s_{0.48}) + 0.18f(s_{-0.74}) + 0.16f(s_{-0.02}) + 0.17f(s_{1.03}) + 0.25f(s_{-0.56})) = f^{-1}(0.5017) = s_{0.01}$;

$\tilde{R}_{\tilde{F}^L}(x_5) = f^{-1}(0.24f(s_{0.3}) + 0.18f(s_{-2.49}) + 0.16f(s_{0.52}) + 0.17f(s_{0.78}) + 0.25f(s_{-0.23})) = f^{-1}(0.4727) = s_{-0.22}$;

$\tilde{R}_{\tilde{F}^L}(x_6) = f^{-1}(0.24f(s_{2.03} + 0.18f(s_{1.23}) + 0.16f(s_{1.25}) + 0.17f(s_{1.54}) +$

$0.25f(s_{1.2})) = f^{-1}(0.6838) = s_{1.47}$.
Step 5. From the above ranking values, the order of the alternatives is,
$x_6 > x_3 > x_4 > x_5 > x_2 > x_1$.
Hence, x_6 is the optimal solution.

- ***By Sun's approach.*** By using Sun's approach, ranking values of the alternatives are, $\tilde{R}_{F^L}(x_1) = s_{-5.566}, \tilde{R}_{F^L}(x_2) = s_{-4.434}, \tilde{R}_{F^L}(x_3) = s_{-1.651}, \tilde{R}_{F^L}(x_4) = s_{-4.519}$, $\tilde{R}_{F^L}(x_5) = s_{-4.434}, \tilde{R}_{F^L}(x_6) = s_{0.953}$.
So, the order of the alternatives is, $x_6 > x_3 > x_2 = x_5 > x_4 > x_1$.
Hence, based on Sun's approach, the optimal alternative is x_6.

Example 4.7. Now, we have solved Example 4.4.
Solution.

- ***By our proposed approach.***
Step 1. Consider a linguistic scale function $f : s_\alpha \rightarrow \vartheta_\alpha$ as $f(s_\alpha) = \frac{1}{2}(\frac{\alpha}{4} + 1)$; $\alpha \in [-4, 4]$. Then, $f(s_{-4}) = 0$, $f(s_{-3}) = \frac{1}{8} = 0.125$, $f(s_{-2}) = \frac{2}{8} = 0.25$, $f(s_{-1}) = \frac{3}{8} = 0.375$, $f(s_0) = \frac{4}{8} = 0.5$, $f(s_1) = \frac{5}{8} = 0.625$, $f(s_2) = \frac{6}{8} = 0.75$, $f(s_3) = \frac{7}{8} = 0.875$, $f(s_4) = 1$.
Since, all the parameters are in same sense, therefore, we have omitted Step 2.
Step 2. Now, linguistic soft reference decision set (F_*^L, E) for this group decision-making has been given in Table 4.22.

Table 4.22: Tabular form of (F_*^L, E) (Example 4.7)

	e_1	e_2	e_3
x_1	s_3	s_4	s_1
x_2	s_3	s_2	s_3
x_3	s_2	s_2	s_1

The satisfaction level of the decision makers are, $\lambda_1 = 0.51, \lambda_2 = 0.49$.
Step 3. Then, the resultant linguistic valued soft set $(\tilde{F}^L, E)$ has been given in Table 4.23.

Table 4.23: Tabular form of $(\tilde{F}^L, E)$ (Example 4.7)

	e_1	e_2	e_3
x_1	$s_{0.96}$	$s_{1.04}$	$s_{0.48}$
x_2	$s_{-0.4}$	$s_{0.48}$	$s_{2.48}$
x_3	$s_{1.52}$	$s_{0.56}$	$s_{0.48}$

Step 4. The weights of the parameters are, $W_1 = 0.34, W_2 = 0.3, W_3 = 0.36$.
The ranking function of the alternatives are, $\tilde{R}_{\tilde{F}^L}(x_1) = s_{0.80}$; $\tilde{R}_{\tilde{F}^L}(x_2) = s_{0.88}$;

$\tilde{R}_{\tilde{F}^L}(x_3) = s_{0.88}$.
Step 5. Here we have seen that, alternatives x_2 and x_3 have the same ranking value. Therefore, to obtain the best alternative, we have to determine the degree of similarity of each of the alternatives x_2 and x_3 to the ideal solution.
Step 6. From the ideal soft set (F^L_*, E) as given in Table 4.22, the ideal solution is, $x_* = \{s_3, s_4, s_3\}$.
Step 7. Now, by using Equation 4.8, the degrees of similarity of x_2 and x_3 to the ideal solution x_* are as follows:
$\hat{S}_w(x_*, x_2) = 0.671$ $\hat{S}_w(x_*, x_3) = 0.665$.
Step 8. Hence, alternative x_2 is the optimal solution.

- ***By Sun's approach.***
  All the alternatives have the same ranking value as given in Example 4.4. So, any one of them will be selected as an optimal alternative.

4.4.5 Optimality test for linguistic valued soft set based group decision-making problems

In this subsection, we have derived the optimality criteria for linguistic valued soft set based multi-criteria group decision-making problems.

(i) The optimal object (x_O) will satisfy all the parameters with a maximum level of linguistic evaluation, i.e., in this decision problem, at least one of the alternative x_O must have a maximum ranking value $\tilde{R}_{\tilde{F}^L}(x_O)$, $O \in \{1, 2, .., m\}$.

(ii) The optimal alternative must have a the maximum degree of similarity with the ideal solution or ideal object.

4.4.6 Measure of performance

The performance measurement of a method is a process by which the efficiency and effectiveness of a method can be discussed. Now, the measure of performance of a method for solving linguistic valued soft set based decision making problem, that satisfies the above optimality criteria, is defined as follows:

$$\tilde{\Upsilon}_M = \hat{S}_w(x_*, x_O) \times f(\tilde{R}_{\tilde{F}^L}(x_O)) \quad (4.9)$$

where x_* is the ideal solution, x_O is the optimal object corresponding to the resultant linguistic valued soft set $(\tilde{F}^L, E)$, $\tilde{R}_{\tilde{F}^L}(x_O)$ is the ranking value of the optimal alternative x_O, f is a linguistic scale function and $\hat{S}_w(x_*, x_O)$ measures the similarity degree between x_* and x_O.

4.5 A comparative discussion between Sun's approach and our proposed approach

In order to illustrate the effectiveness of our proposed approach, a comparative analysis has been provided in this section.
Comparison based on the experimental analysis.

- In Example 4.3, we have seen that, by using Sun's approach the optimal solution can not be determined. But by applying our proposed approach, we get the object x_2 as an optimal solution as given in Table 4.24. From this table, it is also observed that, the performance measure $\tilde{\Upsilon}_M$ of our proposed approach is 0.38 whereas for Sun's approach, this value is 0. Therefore, it is concluded that, our approach is more suitable than Sun's approach.

Table 4.24: Performance analysis of two methods based on Example 4.3

	Rank of the alternatives	Value of the performance measure
Sun's Approach	not found	0
Our Approach	$x_2 > x_4 > x_3 > x_1$	0.38

- In Example 4.6, both the methods give the same optimal solution, x_6, as given in Table 4.25. Consequently, the measure of performance of our proposed approach and Sun's approach is same. Here, the value of the measure of performance is 0.56.

Table 4.25: Performance analysis of two methods based on Example 4.6

	Rank of the alternatives	Value of the performance measure
Sun's Approach	$x_6 > x_3 > x_2 = x_5 > x_4 > x_1$	0.56
Our Approach	$x_6 > x_3 > x_4 > x_5 > x_2 > x_1$	0.56

- Furthermore, from Examples 4.4 (Example 4.7) we have seen that, based on Sun's approach, all objects have same ranking value. Therefore, according to their algorithm, any one of them has to be selected as an optimal solution, whereas, by applying our proposed approach, we have obtained the object x_2 as an optimal solution as given in Table 4.26.
  Now, in general, in a decision-making problem, the alternative which has the maximum degree of similarity with the ideal solution should be best alternative for the decision problem. Now we have derived the degree of similarity of each of the objects to the ideal solution based on Table 4.12 as follows:
  $\hat{S}(x_1, x_*) = 0.661$, $\hat{S}(x_2, x_*) = 0.664$, $\hat{S}(x_3, x_*) = 0.664$.

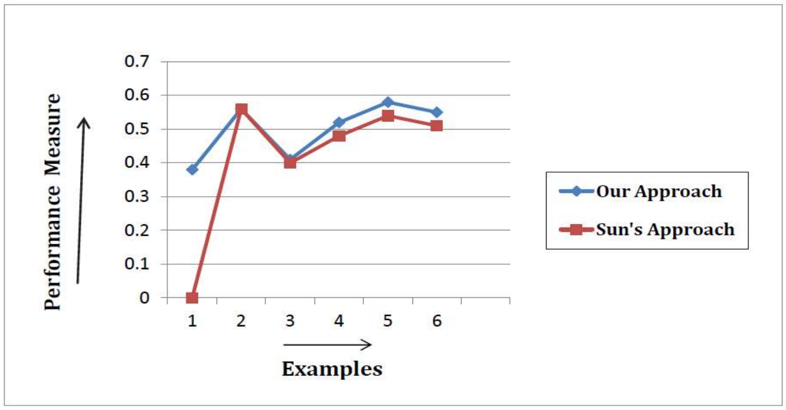

Figure 4.3: The performance measure of these two approaches

From these values it is observed that, the objects x_2 and x_3 have the same as well as maximum degrees of similarity with the ideal solution. Therefore, the optimal solution will be any one of them among x_2 and x_3. But, by using Sun's approach, we may choose an alternative (x_1) whose similarity value with the ideal solution is low. Therefore, in this situation, our approach gives more precise and best result than Sun's approach. Thus, we can conclude that, our approach is more effective than Sun's approach.

Table 4.26: Performance analysis of two methods based on Example 4.4 (Example 4.7)

	Rank of the alternatives	Value of the performance measure
Sun's Approach	$x_1 = x_2 = x_3$	0.40
Our Approach	$x_2 > x_3 > x_1$	0.41

In Figure 4.3, the measures of performance of our proposed approach and Sun's approach have been given. Three extra examples have been used here for drawing this graph which are not cited in this chapter. From this figure it is seen that, the performance of our proposed approach is high than Sun's approach.

Comparison based on the ranking function.

- Sun et al. [153] proposed the notion of ranking function to determine the best or optimal alternative of a linguistic valued soft set. But in this case, resultant may not lie between the upper bound and the lower bound of the associated linguistic term set S. For instance, in Example 4.6, we have seen that, by applying Sun's method, the resultant values of the alternatives x_1, x_2, x_4 and x_5 are $s_{-5.56}, s_{-4.434}, s_{-4.519}$ and $s_{-4.434}$ respectively, which are lying out side of the associated linguistic term set

$[s_{-4}, s_4]$.
Then, to overcome this difficulty, we have redefined the ranking function in Subsection 4.4.2 through the notion of linguistic scale function. By using our proposed ranking function, all resultant linguistic values will lie in the corresponding linguistic term set $\bar{S}$. For instance, in the Examples 4.6 and 4.7 it is observed that, by utilizing our proposed ranking function, the resultant values do not exceed the boundary of the linguistic term set $[s_{-4}, s_4]$.

Therefore, from the above illustration it is concluded that, our proposed is more suitable to solve such type of decision-making problems.

4.6 A plant location selection problem for implementation of our proposed methodology

In this section, we have illustrated a decision-making problem regarding the plant location selection of a manufacturing company to demonstrate the implementation of our proposed approach.

Example 4.8. Suitable location selection plays a very significant role to improve the economic development and to enhance the stability of existence of a company. In several types of plants, such as fuel plant, industrial power plant etc., it is very difficult to reverse the location after taking a decision about the location of a plant construction. Therefore, for a company, it is very necessary to select a suitable location for constructing a manufacturing plant where the workers would produce the good. In this regard, based on four parameters such as investment cost, availability of skilled workers, availability of acquirement material, expansion possibility etc., a group of three experts have been appointed to choose the best location among three locations selected by the company.

Now assume that, X={u_1, u_2, u_3} be a set of three places and E={investment cost (e_1), availability of skilled workers (e_2), availability of acquirement material (e_3), expansion possibility (e_4)} be a set of four associated parameters with respect to the elements of X. Among these four parameters, e_1 has the cost character and other three have the profit character. So, the parameter e_1 will be minimized and other three will be maximized. Now, $D = \{d_1, d_2, d_3\}$ be the three decision makers which have been employed to complete this selection process. The hierarchical structure of this plant location selection problem has been given in Figure 4.4.

A set of nine linguistic variables as, $S = \{s_{-4} = extremely\ poor, s_{-3} = very\ poor, s_{-2} = poor, s_{-1} = slightly\ poor, s_0 = fair, s_1 = slightly\ good, s_2 = good, s_3 = very\ good, s_4 = extremely\ good\}$, have been considered to express the satisfaction level of a location $(u_i, i = 1, 2, 3)$ over a parameter (e_j).

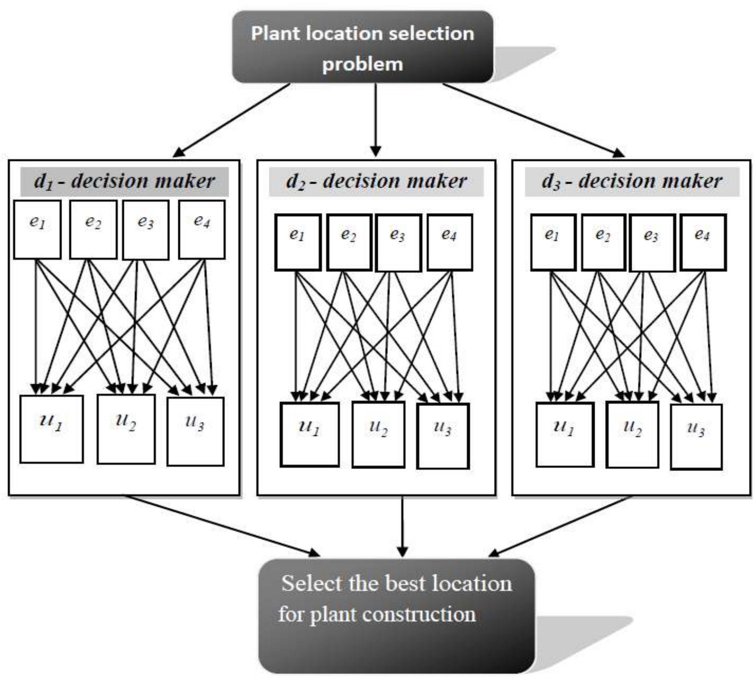

Figure 4.4: Hierarchical structure of this plant location selection problem

The opinions of the three decision makers have been given in three linguistic valued soft sets as, (F_1^L, E),(F_2^L, E), (F_3^L, E) as given in Tables 4.27, 4.28, 4.29 respectively. Here, it is considered that, each of the decision makers has taken the same parameter set as E.

Table 4.27: LVSS (F_1^L, E) (Example 4.8)

	e_1	e_2	e_3	e_4
u_1	s_0	s_{-3}	s_{-1}	s_4
u_2	s_{-1}	s_0	s_{-2}	s_2
u_3	s_2	s_0	s_{-1}	s_3

Table 4.28: LVSS (F_2^L, E) (Example 4.8)

	e_1	e_2	e_3	e_4
u_1	s_1	s_{-2}	s_{-3}	s_2
u_2	s_2	s_0	s_{-1}	s_{-3}
u_3	s_3	s_2	s_{-2}	s_2

Table 4.29: LVSS (F_3^L, E) (Example 4.8)

	e_1	e_2	e_3	e_4
u_1	s_0	s_{-3}	s_{-1}	s_2
u_2	s_{-2}	s_0	s_{-3}	s_{-1}
u_3	s_2	s_0	s_{-2}	s_2

Solution.
Step 1. Consider a linguistic scale function as, $f(s_\alpha) = \frac{1}{2}(\frac{\alpha}{4} + 1)$, $\alpha \in [-4, 4]$.
Then, $f(s_{-4}) = 0$, $f(s_{-3}) = 0.125$, $f(s_{-2}) = 0.25$, $f(s_{-1}) = 0.375$, $f(s_0) = 0.5$, $f(s_1) = 0.625$, $f(s_2) = 0.75$, $f(s_3) = 0.875$, $f(s_4) = 1$.
Step 2. Since e_1 is the cost parameter, so for equating the sense of all parameters, firstly we have to take the complement of the evaluations of all alternatives for all the decision makers with respect to e_1. In Tables 4.30, 4.31 and 4.32, the complement values of all the alternatives for each of the three decision makers with resect to e_1 have been given.

Table 4.30: LVSS (F_1^L, E) after equalization

	e_1	e_2	e_3	e_4
u_1	s_0	s_{-3}	s_{-1}	s_4
u_2	s_1	s_0	s_{-2}	s_2
u_3	s_{-2}	s_0	s_{-1}	s_3

Table 4.31: LVSS (F_2^L, E) after equalization

	e_1	e_2	e_3	e_4
u_1	s_{-1}	s_{-2}	s_{-3}	s_2
u_2	s_{-2}	s_0	s_{-1}	s_{-3}
u_3	s_{-3}	s_2	s_{-2}	s_2

Table 4.32: LVSS (F_3^L, E) after equalization

	e_1	e_2	e_3	e_4
u_1	s_0	s_{-3}	s_{-1}	s_2
u_2	s_2	s_0	s_{-3}	s_{-1}
u_3	s_{-2}	s_0	s_{-2}	s_2

Step 3. *Determine the satisfaction level of each of the decision makers.*
The linguistic soft reference decision set (F_*^L, E) has been given in Table 4.33.

Table 4.33: Tabular form of (F_*^L, E) (Example 4.8)

	e_1	e_2	e_3	e_4
u_1	s_0	s_{-2}	s_{-1}	s_4
u_2	s_2	s_0	s_{-1}	s_2
u_3	s_{-2}	s_2	s_{-1}	s_3

The relative closeness between (F_*^L, E) and each of the individuals

$(F_1^L, E), (F_2^L, E), (F_3^L, E)$ is as follows:
$\Gamma(F_1^L, F_*^L) = 0.819$, $\Gamma(F_2^L, F_*^L) = 0.721$, $\Gamma(F_3^L, F_*^L) = 0.702$.
Then, the satisfaction level of the three decision makers d_1, d_2, and d_3 are, $\lambda_1 = 0.37$, $\lambda_2 = 0.32$ and $\lambda_3 = 0.31$ respectively.
Step 4. Aggregate three linguistic valued soft sets $(F_1^L, E), (F_2^L, E)$ and (F_3^L, E) (which are given in Tables 4.30, 4.31 and 4.32) as follows:

$$\lambda_1 \begin{pmatrix} f(s_0) & f(s_{-3}) & f(s_{-1}) & f(s_4) \\ f(s_1) & f(s_0) & f(s_{-2}) & f(s_2) \\ f(s_{-2}) & f(s_0) & f(s_{-1}) & f(s_3) \end{pmatrix} + \lambda_2 \begin{pmatrix} f(s_{-1}) & f(s_{-2}) & f(s_{-3}) & f(s_2) \\ f(s_{-2}) & f(s_0) & f(s_{-1}) & f(s_{-3}) \\ f(s_{-3}) & f(s_2) & f(s_{-2}) & f(s_2) \end{pmatrix} +$$
$$\lambda_3 \begin{pmatrix} f(s_0) & f(s_{-3}) & f(s_{-1}) & f(s_2) \\ f(s_2) & f(s_0) & f(s_{-3}) & f(s_{-1}) \\ f(s_{-2}) & f(s_0) & f(s_{-2}) & f(s_2) \end{pmatrix} = \begin{pmatrix} 0.46 & 0.17 & 0.30 & 0.84 \\ 0.54 & 0.5 & 0.25 & 0.43 \\ 0.21 & 0.58 & 0.3 & 0.8 \end{pmatrix}$$

Then, the resultant linguistic valued soft set is $(\tilde{F}^L, E) = \begin{pmatrix} s_{-0.32} & s_{-2.64} & s_{-1.6} & s_{2.72} \\ s_{0.32} & s_0 & s_{-2} & s_{-0.56} \\ s_{-2.32} & s_{0.64} & s_{-1.6} & s_{2.4} \end{pmatrix}$

Step 5. *Derive the ranking function of each of the alternatives.*
Since, weights of the parameters are not given here, so firstly, we have to determine the weights of each of the parameters by using Equations 4.3 and 4.4 as follows,
$W_1 = 0.28, W_2 = 0.34, W_3 = 0.04, W_4 = 0.34$.
Then, the ranking values of all the alternatives are,
$\tilde{R}_{\tilde{F}^L}(u_1) = s_{-0.13}; \tilde{R}_{\tilde{F}^L}(u_2) = s_{-0.18}; \tilde{R}_{\tilde{F}^L}(u_3) = s_{0.32}$.
Step 6. The order of the alternatives is, $u_3 > u_1 > u_2$. So, the alternative u_3 is the optimal solution.

Hence, the place u_3 is the best location for constructing a manufacturing plant of the company.

4.7 Conclusion

In this chapter, we have revisited Sun's [153] linguistic valued soft set based group decision-making approach. We have found out that, in their algorithm, the formula for deriving the weight of a decision maker is not right. At the same time we have pointed out that, linguistic scale function based operational laws is more suitable than the index-based operational laws as discussed in Section 4.5. Then, we have proposed a new approach for solving linguistic valued soft set based group decision-making problems through the notion of linguistic scale function.
The main contributions in this chapter are as follows:

- A new axiomatic definition of similarity measure for linguistic valued sets has been introduced.

- The definition of ranking function of an alternative in a linguistic valued soft set has been redefined through the notion of linguistic scale function.
- The importance of the satisfaction level of a decision maker in a linguistic valued soft set based group decision-making problem has been illustrated and determined.
- A novel linguistic valued soft approach in multi-criteria group decision-making through the notion of linguistic scale function has been proposed.
- Furthermore, we have applied our proposed approach in selecting the location for constructing a manufacturing plant of a company.

In further research, one can extend this proposed approach to other uncertain environments such as, probabilistic soft set, probabilistic linguistic soft set, type-2 fuzzy soft set, etc. Moreover, in this chapter, we have handled linguistic valued soft set based decision making problems through linguistic scale function. So, as a further research, one can give attention on solving linguistic valued soft set based problems by using other computational techniques such as, 2-tuple linguistic representation model, transformation of linguistic variables into some fuzzy numbers and granulation of linguistic terms.