Chapter 3

Some properties of fuzzy soft groups

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3.1 Introduction

Molodtsov's soft set theory [118] has achieved much more observations to the researchers from the beginning of its introduction because, in a soft set, to define an alternative at the initial stage, we do not need the exact description of an alternative rather, we can define an alternative through some parameters based on human cognitive process. Therefore, experiments on soft sets have been very efficacious and several exigent emanations have been done by the researchers. After the introduction of soft set theory [118], Maji et al. [106, 107] worked on further development of soft set theory. They proposed some basic operational laws, properties and theorems on soft sets. Further, they [104] also used soft set theory in the field of decision-making. Moreover, Roy and Maji [139] defined the concept of fuzzy soft set theory by integrating soft set theory theory with fuzzy set theory and then provided its application in decision-making [139]. After Roy and Maji's [139] work, many researchers [22, 61, 91–93] have used fuzzy soft set theory for solving many real-life problems in decision-making, disease diagnosis, supplier selection, etc. Furthermore, Çagman and Enginoğlu [33, 36] introduced fuzzy soft set through matrix form and proposed the idea of fuzzy soft set. Further, they used this fuzzy soft matrix in solving decision-making problems.

Group theory is one of the basic algebraic structures on classical mathematics. In 1971, Rosenfeld [138] initiated the notion group theory on fuzzy environment and proposed the

[1]This chapter has been published in **Annals of Fuzzy Mathematics and Informatics, 13 (1) (2017): 47-61**.

idea of fuzzy group. Afterwards, this idea has been developed through soft set theory. The researchers Aktas and Cagman [16] introduced the notion of soft group and some of its related theorems and properties. They also showed that, fuzzy group is a special case of a soft group. Basically, a soft group is a parameterized family of subgroups over a universal set which also forms a group under some binary operations. Then, several articles have been published to develop the idea of soft group such as, soft cyclic group [17], normalistic soft groups [147], soft function [110], etc. Meanwhile, Aygünoğlu and Aygün [20] combined soft group with fuzzy group and initiated the idea of fuzzy soft group. Moreover, they also defined fuzzy soft homomorphism, fuzzy soft function and normal fuzzy soft group over fuzzy soft group. Further, several researchers have worked [102, 113, 141] on developing fuzzy soft group in different directions such as, fuzzy soft normal subgroup, homomorphic image, homomorphic pre-image, etc.

Now, from the existing literature, it has been observed that, there exist several works on fuzzy soft group including, fuzzy soft homomorphism, fuzzy soft function, homomorphic image, homomorphic pre-image, fuzzy soft normal subgroup, etc. But, the concept of order of an element of a fuzzy soft group has not yet been introduced. Moreover, though, Aktas and Qzlu [17] illustrated cyclic group through soft set but, till now no one is introduce the notion of cyclic group through fuzzy soft set. Therefore, to fill up this research gap, in this chapter, firstly, we have introduced the idea of order of an element of a fuzzy soft group. Then, we have proposed the idea of fuzzy soft cyclic group as a generalization of soft cyclic group [17]. It has been shown that, intersection of two fuzzy soft cyclic groups is a fuzzy soft cyclic group and for the case of union of two fuzzy soft cyclic groups, it has been concluded that, with some certain conditions, union of two fuzzy soft cyclic groups will from a fuzzy soft cyclic group. Some examples have also been discussed in this chapter to justify the existence of these proposed definitions and theorems.

Now, the organization of this chapter is as following. In section 3.2, some basic preliminaries including, soft group, fuzzy soft group, fuzzy soft abelian group, etc. have been illustrated. In section 3.3, we have introduced the order of an element of a fuzzy soft group. Then, in Section 3.4, we have proposed the idea of fuzzy soft cyclic group. In Sections 3.3 and 3.4, some related properties and theorems have also been developed. Section 3.5 contains some conclusions.

3.2 Some basic relevant notions

In this section, some existing ideas have been recalled based on which our subsequent discussions have been proceeded. In this chapter, $I = [0, 1]$ has been considered as the unit closed interval.

(i) **Union of two FSSs [139].**

Let us consider two fuzzy soft sets $(\tilde{f}, A)$ and $(\tilde{g}, B)$ over an initial universe X where, A, B are two subsets of the parameter set E. Then, the union of $(\tilde{f}, A)$ and $(\tilde{g}, B)$ is also a fuzzy soft set. Mathematically, it can be denoted by, $(\tilde{f}, A)\tilde{\cup}(\tilde{g}, B) = (\tilde{h}, C)$, where $C = A \cup B$ and is defined as follows: $\forall c \in C$,

$$\tilde{h}_c(x) = \begin{cases} \tilde{f}_c(x) & , if\ c \in A - B \\ \tilde{g}_c(x) & , if\ c \in B - A \\ \tilde{f}_c(x)\bar{\cup}\tilde{g}_c(x) & , if\ c \in A \cap B \end{cases}$$

where, $\bar{\cup}$ is the fuzzy union.

(ii) **Intersection of two FSSs [139].**

The intersection of $(\tilde{f}, A)$ and $(\tilde{g}, B)$ over the universe X is also a fuzzy soft set. mathematically, it can be denoted by, $(\tilde{f}, A)\tilde{\cap}(\tilde{g}, B) = (\tilde{h}, C)$ where, $C = A \cap B$ and is defined as follows: $\forall c \in C$,

$$\tilde{h}_c(x) = \tilde{f}_c(x)\bar{\cap}\tilde{g}_c(x).$$

where, $\bar{\cap}$ is the fuzzy intersection.

(iii) **Soft group [16].**

Let, the universal set X forms a group under some binary compositions i.e., let $(X, .)$ be a group and E be a non-empty parameterized set. R is an arbitrary binary relation between the elements of the set E and the elements of the set X. Now, let us consider a soft set (f, E) over the set X where, f is given by, $f : E \to P(X)$ such that,

$f(a)$=$\{y \in X$: $(a, y) \in R$, $a \in E$ and $y \in X\}$ and $R = \{(a, y) \in E \times X : y \in f(a)\}$.

Then, the soft set (f, E) is said to be a soft group over X if, $\forall a \in E$, $f(a) < X$, i.e., for each $a \in E$, $f(a)$ provides a collection of subgroups of the set X.

(iv) **Fuzzy soft group [20].**

Let us consider that, $(X, .)$ be a group and $(\tilde{f}, E)$ be a fuzzy soft set over X where, E be a parameterized set. Then, a fuzzy soft set $(\tilde{f}, E)$ is said to be a fuzzy soft group over the universal set X if, $\forall a \in E$ and $x, y \in X$,

(i) $f_a(x.y) \geq min\{f_a(x), f_a(y)\}$,

(ii) $f_a(x^{-1}) \geq f_a(x)$.

i.e., in other words, a fuzzy soft set $(\tilde{f}, E)$ is called a fuzzy soft group over X if and only if, $\forall a \in E$, $f_a\ (f(a))$ forms a fuzzy group over X (Rosenfeld's sense).

Example 3.1. Now, consider an universal set as, $X = \{1, -1, i, -i\}$ which forms a group under multiplication and let, $E = \{e_1, e_2\}$ be a set of parameters where,
e_1 indicates the 'square elements of X' and e_2 indicates the 'cubic elements of X'.
It is noted that we do not take any element more than one time.
Now, let us consider a mapping $\tilde{f} : E \to I^X$ such that $\tilde{f}_a : X \to [0, 1]$ where,

$$\tilde{f}_a(x) = \{\tfrac{1}{n} : x^n = 1, a \in E, x \in X\}.$$

Then,

$$(\tilde{f}, E) = \{(e_1, ((1, 1), (-1, 0.5))), (e_2, ((1, 1), (-1, 0.5), (i, 0.25), (-i, 0.25)))\}$$

is a fuzzy soft group over the universe X.

(v) **Fuzzy soft abelian group [20].**

Assume that, $(X, .)$ be a group and $(\tilde{f}, E)$ be a fuzzy soft group over X, where E be the parameterized set. Then, the fuzzy soft group $(\tilde{f}, E)$ is said to be a fuzzy soft abelian group over X, if $\forall a \in E$, $\tilde{f}_a(x.y) = \tilde{f}_a(y.x), \forall x, y \in X$.

Theorem 3.1. [20]
Let $(X, .)$ be group and $(\tilde{f}, E)$ be a fuzzy soft group over X where, E is the set of parameters. Then for each $a \in E$ and $x \in X$, the following conditions satisfies truly:
(1) $\tilde{f}_a(x^{-1}) = \tilde{f}_a(x)$,
(2) $\tilde{f}_a(e) \geq \tilde{f}_a(x)$.

Theorem 3.2. [20](*Necessary and Sufficient Condition of a fuzzy soft group.*)
Now, consider that, $(\tilde{f}, E)$ be a fuzzy soft set over X where, $(X, .)$ is also forms a group. Then $(\tilde{f}, E)$ will be a fuzzy soft group over X if and only if $\forall a \in E$, $\tilde{f}_a(x.y^{-1}) \geq min\{\tilde{f}_a(x), \tilde{f}_a(y)\}$ where, $x, y \in X$.

Theorem 3.3. [20]
Let $(X, .)$ be a group and $(\tilde{f}, A)$, $(\tilde{g}, B)$ be two fuzzy soft groups over X where, $A, B \subset E$. Then the intersection of $(\tilde{f}, A)$ and $(\tilde{g}, B)$ is also forms a fuzzy soft group over the universe X. Mathematically, it is denoted by, $(f, A)\tilde{\cap}(g, B)$.

Theorem 3.4. [20]
Let $(X, .)$ be a group and $(\tilde{f}, A)$, $(\tilde{g}, B)$ be two fuzzy soft groups over X where, $A, B \subset E$. Now if, $A \cap B = \phi$, then the union of $(\tilde{f}, A)$ and $(\tilde{g}, B)$ is also forms a fuzzy soft group over the universe X. Mathematically, it is denoted by, $(\tilde{f}, A)\tilde{\cup}(\tilde{g}, B)$.

3.3 Order of an element of a fuzzy soft group

Definition 3.1. Let, $(X, .)$ be a group and $(\tilde{f}, E)$ be a fuzzy soft group over X which is defined as follows:

$$(\tilde{f}, E) = \{(a, \tilde{f}(a)) | \forall a \in E\} = \{(a, (x, \tilde{f}_a(x))) | \forall a \in E, x \in X\}$$

where, $\tilde{f}_a(x)$ is the fuzzy evaluation of an element $x \in X$ with respect to a parameter $a \in E$. Then, $(x^n, \tilde{f}_a(x^n))$ is called the n^{th} power of an element x with respect to a parameter $a \in E$ where, $n \in Z$ (the set of integers).

Example 3.2. Consider the fuzzy soft group $(\tilde{f}, E)$ of Example 3.1 as follows:

$$(\tilde{f}, E) = \{(e_1, ((1, 1), (-1, 0.5))), (e_2, ((1, 1), (-1, 0.5), (i, 0.25), (-i, 0.25)))\}$$

Then, with respect to the parameter e_2, the third power of the element i in X is,
$(i^3, \tilde{f}_{e_2}(i^3)) = (-i, \tilde{f}_{e_2}(-i)) = (-i, 0.25)$.
Note: If $(X, .)$ be an additive group, then the n^{th} power of an element $x \in X$ with respect to the parameter $a \in E$ is denoted by, $\{(nx, \tilde{f}_a(nx)) : a \in E,\ x \in X\}$, $n \in Z$ (the set of integers).

Definition 3.2. Let, $(\tilde{f}, E)$ be a fuzzy soft group over the universe X where, $(X, .)$ is also a group. Then $\forall a \in E$, $\{\tilde{f}(a)\}^n = \{x^n/\tilde{f}_a(x^n) : \forall x \in X\}$ is called the n^{th} power of the $a - approximation$ $\tilde{f}(a)$ of the fuzzy soft group $(\tilde{f}, E))$ over X where, n is an integer.

Example 3.3. Let us consider the set of all permutations on the set $\{1, 2, 3\}$ as a universal set i.e., $X = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$, which forms a group under multiplication. Again consider that, $E = X = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ be a parameterized set.
Now, consider a fuzzy soft set $(\tilde{f}, E)$ over X as follows:
$(\tilde{f}, E) = \{(\rho_0, \{(\rho_0, 1)\}), (\rho_1, \{(\rho_0, 1), (\rho_1, 0.33), (\rho_2, 0.33)\}), (\rho_2, \{(\rho_0, 1), (\rho_1, 0.33), (\rho_2, 0.33)\}), (\rho_3, \{(\rho_0, 1), (\rho_3, 0.5)\}), (\rho_4, \{(\rho_0, 1), (\rho_4, 0.5)\}), (\rho_5, \{(\rho_0, 1), (\rho_5, 0.5)\})\}$.
This fuzzy soft set satisfies all the conditions of fuzzy soft group. Thus, $(\tilde{f}, E)$ is a fuzzy soft group over X.
Now, $\{\tilde{f}(\rho_1)\}^2 = \{((\rho_0)^2, \tilde{f}_{\rho_1}({\rho_0}^2)), ((\rho_1)^2, \tilde{f}_{\rho_1}({\rho_1}^2)), ((\rho_2)^2, \tilde{f}_{\rho_1}({\rho_2}^2))\} =$
$\{(\rho_0, f_{\rho_1}(\rho_0)), (\rho_2, f_{\rho_1}(\rho_2)), (\rho_1, f_{\rho_1}(\rho_1))\} = \{(\rho_0, 1), (\rho_2, 0.33), (\rho_1, 0.33)\}$.
So the 2nd-power of $\tilde{f}(\rho_1) = \{(\rho_0, 1), (\rho_2, 0.33), (\rho_1, 0.33)\}$.

Definition 3.3. Let $(X, .)$ be a group with an identity element e and $(\tilde{f}, E)$ be a fuzzy soft group over X. If there exists a least positive integer n such that, $\{\tilde{f}_a(x^n) = \tilde{f}_a(e) : a \in E, x \in X\}$ then, the least positive integer n is called the order of the element $x \in X$ over the fuzzy soft group $(\tilde{f}, E)$ with respect to a parameter $a \in E$.

Mathematically, it is denoted by $|\tilde{f}_a(x)|$ or $O(\tilde{f}_a(x))$.

If no such n exists then, x is said to have an infinite order corresponding to the parameter $a \in E$ over the fuzzy soft group $(\tilde{f}, E)$.

Example 3.4. Consider the fuzzy soft group $(\tilde{f}, E)$ as given in Example 3.1.
$(\tilde{f}, E) = \{(e_1, ((1,1), (-1, 0.5))), (e_2, ((1,1), (-1, 0.5), (i, 0.25), (-i, 0.25)))\}$.
Now, $\tilde{f}_{e_1}((-1)^2) = \tilde{f}_{e_1}(1)$. So, corresponding to the parameter e_1, order of -1 is 2 i.e., $O(\tilde{f}_{e_1}(-1)) = 2$.
Again $O(\tilde{f}_{e_2}(-i)^4) = \tilde{f}_{e_2}(1)$. So, corresponding to the parameter e_2, order of $-i$ is 4 i.e., $O(\tilde{f}_{e_2}(-i)) = 4$.

Definition 3.4. Let $(X, .)$ be a group of finite order and $(\tilde{f}, E)$ be a fuzzy soft group over X. Then $\forall a \in E$, the order of $\tilde{f}(a)$ or $\tilde{f}_a$ is defined as follows:

$$O(\tilde{f}(a)) = lcm_{x \in X}\{O(\tilde{f}_a(x))\}.$$

Example 3.5. Now, consider the fuzzy soft group $(\tilde{f}, E)$ from Example 3.1 as follows:

$$(\tilde{f}, E) = \{(e_1, ((1,1), (-1, 0.5))), (e_2, ((1,1), (-1, 0.5), (i, 0.25), (-i, 0.25)))\}$$

.
Then, $O(f_{e_2}(1)) = 1$, $O(f_{e_2}(-1)) = 2$, $O(f_{e_2}(i)) = 4$, $O(f_{e_2}(-i)) = 4$.
So, $O(f(e_2)) = lcm\{O(f_{e_2}(1)), O(f_{e_2}(-1)), O(f_{e_2}(i)), O(f_{e_2}(-i))\} = lcm\{1, 2, 4, 4\} = 4$.

Theorem 3.5. Let $(X, .)$ be a group and $(\tilde{f}, E)$ be a fuzzy soft group over X. Then with respect to a parameter $a \in E$, $O(f_a(x)) = O(f_a(x^{-1}))$ where, $x \in X$.

Proof: Case-I: Suppose $O(\tilde{f}_a(x)) = n$ (finite).
Then, $\tilde{f}_a(x^n) = \tilde{f}_a(e)$ which implies that,

$$\tilde{f}_a(x^{-1})^n = \tilde{f}_a((x^n)^{-1}) = \tilde{f}_a(x^n) = \tilde{f}_a(e).$$

Thus $\tilde{f}_a(x^{-1})^n = \tilde{f}_a(e)$. So, $O(\tilde{f}_a(x)) = O(\tilde{f}_a(x^{-1}))$.

Case-II: Suppose, $O(\tilde{f}_a(x))$ is infinite.
Then, we will prove that, $O(\tilde{f}_a(x^{-1}))$ is infinite.
For this, assume that, $O(\tilde{f}_a(x^{-1}))$ is finite, say $O(\tilde{f}_a(x^{-1})) = m$ (a finite number). Then, $\tilde{f}_a((x^{-1})^m) = \tilde{f}_a(e)$. Thus $\tilde{f}_a((x^m)^{-1}) = \tilde{f}_a(e)$. Then, from Theorem 3.1 we get that, $\tilde{f}_a(x^m) = \tilde{f}_a(e)$ which implies that, $O(\tilde{f}_a(x)) = m$, a finite number, which contradicts our assumption. Hence, obviously $O(\tilde{f}_a(x^{-1}))$ is infinite.
Thus, we have seen that, the theorem is true for both of the cases.

Theorem 3.6. Let $(X, .)$ be a group with an identity element e and let $(\tilde{f}, E)$ be a fuzzy soft group over X. If corresponding to some element $a \in E$, there exists an element $x \in X$ such that, $\tilde{f}_a(x^m) = \tilde{f}_a(e)$. Then, the order of $\tilde{f}_a(x)$ is a divisor of m. i.e., $O(\tilde{f}_a(x)) \mid m$.

Proof: Consider that, $O(\tilde{f}_a(x)) = n$ then, $\tilde{f}_a(x^n) = \tilde{f}_a(e)$.
Now, let us assume that, n does not divide m. Then, by using divisor algorithm, there exists some integers s and t where, $m = ns + t$ and $0 \leq t < n$.
On one hand,

$$\begin{aligned} \tilde{f}_a(x^t) &= \tilde{f}_a(x^{m-ns}) = \tilde{f}_a(x^m.x^{-ns}) \\ &\geq min\{\tilde{f}_a(x^m), \tilde{f}_a(x^{-ns})\}[by\ using\ the\ definition\ of\ fuzzy\ soft\ group] \\ &= min\{\tilde{f}_a(e), \tilde{f}_a(x^{ns})^{-1}\}[by\ the\ given\ condition] \\ &= \tilde{f}_a(x^{ns})^{-1}[by\ using\ Theorem\ 3.1] \\ &= \tilde{f}_a(x^{ns}) = \tilde{f}_a(x^n)^s \\ &\geq \tilde{f}_a(x^n) = \tilde{f}_a(e) \end{aligned}$$

Thus we get that, $\tilde{f}_a(x^t) \geq \tilde{f}_a(e)$.
Again it is obvious that, $\tilde{f}_a(x^t) \leq \tilde{f}_a(e)$.
So, from these two inequalities it is obtained that, $\tilde{f}_a(x^t) = \tilde{f}_a(e)$ which contradicts the minimality of n.
So, we can conclude that, n is a divisor of m, i.e., $O(\tilde{f}_a(x)) \mid m$ holds truly.

Theorem 3.7. let $(X, .)$ be a group of finite order and $(\tilde{f}, E)$ be a fuzzy soft group over X. Now, if $O(\tilde{f}_a(x)) = n$ for some $a \in E$ and $x \in X$ then, for a positive integer m, $O(\tilde{f}_a(x^m)) = \frac{n}{gcd(m,n)}$.

Proof: Let $gcd(m, n) = d$. Then, there exist some elements u, v such that, $d = mu + nv$.
Now let, $O(\tilde{f}_a(x^m)) = t$. Then, for some $a \in E$ and $x \in X$ we have,

$$\tilde{f}_a(x^{mt}) = \tilde{f}_a(e).$$

Again since, $O(\tilde{f}_a(x)) = n$ then, $\tilde{f}_a(x^n) = \tilde{f}_a(e)$.
Now,

$$\begin{aligned} \tilde{f}_a(x^m)^{\frac{n}{d}} &= \tilde{f}_a(x^n)^{\frac{m}{d}} = \tilde{f}_a((x^n)^k)(we\ consider\ that,\ \frac{m}{d} = k) \\ &\geq \tilde{f}_a(x^n) \\ &= \tilde{f}_a(e) \end{aligned} \tag{3.1}$$

Moreover, it is also obvious that, $\tilde{f}_a(x^m)^{\frac{n}{d}} \leq \tilde{f}_a(e)$ (by using Theorem 3.1).
Thus we get that, $\tilde{f}_a(x^m)^{\frac{n}{d}} = \tilde{f}_a(e)$ which concludes that,

$$t \mid (n/d) \tag{3.2}$$

On the other hand,

$$\begin{aligned}
\tilde{f}_a(x^{td}) \quad &= \tilde{f}_a(x^{t(mu+nv)}) = \tilde{f}_a(x^{tmu}.x^{tnv}) \\
&\geq min\{\tilde{f}_a(x^{tmu}), \tilde{f}_a(x^{tnv})\} \\
&= min\{\tilde{f}_a((x^{mt})^u), \tilde{f}_a((x^n)^{vt})\} \\
&\geq min\{\tilde{f}_a(x^{mt}), \tilde{f}_a(x^n)\} \\
&= min\{\tilde{f}_a(e), \tilde{f}_a(e)\}
\end{aligned}$$

Then, $\tilde{f}_a(x^{td}) \geq \tilde{f}_a(e)$. Moreover, it is obvious that, $\tilde{f}_a(e) \geq \tilde{f}_a(x^{td})$.
So, then we get, $\tilde{f}_a(x^{td}) = \tilde{f}_a(e)$ which implies that, $n \mid td$. Therefore, we can conclude that,

$$(n/d) \mid t \tag{3.3}$$

Hence, from Equations 3.2 and 3.3 we get that, $t = \frac{n}{d}$ which proves this theorem.

Theorem 3.8. Let $(X, .)$ be a group and $(\tilde{f}, E)$ be a fuzzy soft group over X. If $(\tilde{f}, E)$ be a fuzzy soft abelian group over X and for some $a \in E$ and $x, y \in X$, $gcd(O(\tilde{f}_a(x)), O(\tilde{f}_a(y))) = 1$ then, $O(\tilde{f}_a(xy)) = O(\tilde{f}_a(x))O(\tilde{f}_a(y))$.

Proof: Let $O(\tilde{f}_a(x)) = m$, $O(\tilde{f}_a(y)) = n$ and $O(\tilde{f}_a(xy)) = k$ where, $gcd(m, n) = 1$.
Now, we will prove that, $k = mn$.
Since, $O(\tilde{f}_a(x)) = m$ then,

$$\tilde{f}_a(x^m) = \tilde{f}_a(e) \tag{3.4}$$

Similarly, $O(\tilde{f}_a(y)) = n$ implies that,

$$\tilde{f}_a(y^n) = \tilde{f}_a(e) \tag{3.5}$$

Moreover, $O(\tilde{f}_a(xy)) = k$ implies that,

$$\tilde{f}_a((xy)^k) = \tilde{f}_a(e). \tag{3.6}$$

Now,

$$\begin{aligned}
\tilde{f}_a((xy)^{mn}) \quad &= \tilde{f}_a(x^{mn}.y^{mn}) \\
&\geq min\{\tilde{f}_a(x^{mn}), \tilde{f}_a(y^{mn})\} \\
&= min\{\tilde{f}_a((x^m)^n), \tilde{f}_a((y^n)^m)\} \\
&\geq min\{\tilde{f}_a(x^m), \tilde{f}_a(y^n)\} \\
&= min\{\tilde{f}_a(e), \tilde{f}_a(e)\} = \tilde{f}_a(e)
\end{aligned}$$

Again, obviously, $\tilde{f}_a(e) \geq \tilde{f}_a((xy)^{mn})$ which concludes that, $\tilde{f}_a((xy)^{mn}) = \tilde{f}_a(e)$.
Then, by using Theorem 3.6 we get that,

$$k \mid mn \tag{3.7}$$

Now, we will only prove that, $mn \mid k$.
Clearly,

$$\begin{aligned}\tilde{f}_a(x^k) &= \tilde{f}_a(x^k.y^k.y^{-k}) = \tilde{f}_a((xy)^k.y^{-k}) \\ &\geq min\{\tilde{f}_a(xy)^k, \tilde{f}_a(y^{-k})\} \\ &\geq min\{\tilde{f}_a(xy)^k, \tilde{f}_a(y)^k\} \\ &= min\{\tilde{f}_a(e), \tilde{f}_a(y^k)\}(by\ using\ Equation\ 3.6) \\ &= \tilde{f}_a(y^k)\ (by\ using\ Theorem\ 3.1)\end{aligned}$$

Similarly, $\tilde{f}_a(y^k) \geq \tilde{f}_a(x^k)$.
So, $\tilde{f}_a(x^k) = \tilde{f}_a(y^k)$ and obviously, $\tilde{f}_a(x^{nk}) = \tilde{f}_a(y^{nk}) = \tilde{f}_a(e)$.
Then, from Equation 3.4, we have, $m \mid nk$. But, since $(m, n) = 1$ then we get that,

$$m \mid k. \tag{3.8}$$

In the similar way we can also prove that,

$$n \mid k. \tag{3.9}$$

So from Equations 3.8 and 3.9, we get that,

$$mn \mid k. \tag{3.10}$$

Hence, form Equations 3.7 and 3.10, it is obtained that, $k = mn$.

Theorem 3.9. Let us consider that, $O(\tilde{f}_a(x)) = n$. Then, $O(\tilde{f}_a(x^p)) = n$ if and only if, p is prime to n.

Proof: Let, $O(\tilde{f}_a(x^p)) = n$. Now, we will prove that, $gcd(p, n) = 1$.
Suppose, $O(\tilde{f}_a(x)) = n$. Then, $\tilde{f}_a(x^n) = \tilde{f}_a(e)$. Now, from Theorem 3.7 we have, $O(\tilde{f}_a(x^p)) = \frac{n}{gcd(p,n)}$. Thus $gcd(p, n) = \frac{n}{O(\tilde{f}_a(x^p))}$. So $gcd(p, n) = \frac{n}{n} = 1$.
The proof of the converse part of this theorem is similar to the Theorem 3.7.

Definition 3.5. Let $(X, .)$ be a group and let $(\tilde{f}, E)$ be a fuzzy soft group over X.Then for each $a \in E$, the set $Z(\tilde{f}(a)) = \{x \in X : \tilde{f}_a(xy) = \tilde{f}_a(yx), \forall y \in X\}$ is called the center of the $a - approximation$ $\tilde{f}(a)$ of the fuzzy soft group $(\tilde{f}, E)$ over X.

Definition 3.6. Let $(X, .)$ be a group and $(\tilde{f}, E)$ be a fuzzy soft group over X. Then the set, $C(\tilde{f}(a))$=$\{x \in X$: $y, z \in X$, $\tilde{f}_a(xy) = \tilde{f}_a(yx)$ and $\tilde{f}_a(xyz) = \tilde{f}_a(yxz)$, $a \in E\}$ is called the centralizer of $\tilde{f}(a)$ of the fuzzy soft group $(\tilde{f}, E)$ over X.

Theorem 3.10. Let $(X, .)$ be a group and $(\tilde{f}, E)$ be a fuzzy soft group over X, then $x \in C(\tilde{f}(a))$ if and only if for some $a \in E$,

$\tilde{f}_a(xy_1y_2...y_n) = \tilde{f}_a(y_1xy_2...y_n) = ... = f_a(y_1y_2....y_nx)$ where, all $y_1, y_2, ..., y_n$ are in the set X.

Proof: The proof is straightforward.

3.4 Fuzzy soft cyclic group

Definition 3.7. Let $(X, .)$ be a group and let $(\tilde{f}, E)$ be a fuzzy soft group over X. If $\forall a \in E$, there exists an element $x \in X$ such that, $\tilde{f}(a) =< (x, \tilde{f}_a(x)) >$, i.e., $\tilde{f}(a) = \{(x^n, \tilde{f}_a(x^n)) : n \in Z\}$ then, $(\tilde{f}, E)$ is said to be a fuzzy soft cyclic group over X. Here, $(x, \tilde{f}_a(x))$ is called the fuzzy soft generator of $\tilde{f}(a)$ with respect to the element $a \in E$ over the fuzzy soft cyclic group $(\tilde{f}, E)$.
Note. Each $\tilde{f}(a)$ be a fuzzy cyclic group over X.

Example 3.6. Let $(X, .)$ be a cyclic group and let $(\tilde{f}, E)$ be a soft cyclic group over X, then $(\tilde{f}, E)$ is also a fuzzy soft cyclic group over X. Since every subgroup of a cyclic group is cyclic and each soft group is consider as a fuzzy soft group.

Example 3.7. Let $X = \{1, -1, i, -i\}$ be a set which forms a group under multiplication and let $E = \{e_1, e_2\}$ be the parameterized set where,e_1 stands for the parameter 'all real roots' and e_2 stands for the parameter 'all roots'.

Now, let consider a mapping as, $\tilde{f} : E \to I^X$ such that, $\tilde{f}_a : X \to [0, 1]$ where,

$$\tilde{f}_a(x) = \{\tfrac{1}{n} : x^n = 1\}.$$

Then, the fuzzy soft set

$$(\tilde{f}, E) = \{(e_1, ((1, 1), (-1, 0.5))), (e_2, ((1, 1), (-1, 0.5), (i, 0.25), (-i, 0.25)))\}$$

satisfies all the conditions of a fuzzy soft group. So, $(\tilde{f}, E)$ is a fuzzy soft group over X.
Now, with respect to the parameter e_1, $< (1, 1) >= \{(1, 1)\}$ and $< (-1, 0.5) >= \{(1, 1), (-1, 0.5)\}$.
Then $< (-1, 0.5) >$ is the generator of $\tilde{f}(e_1)$.
In the similar way, $< (i, 0.25) >$ and $< (-i, 0.25) >$ are also two generators of $f(e_2)$. So $(\tilde{f}, E)$ be a fuzzy soft cyclic group over X.

Example 3.8. Let $X = \{e, a, b, c\}$ be the initial universal set which is not a cyclic group and let $E = \{e, a, b, c\}$ be the parameterized set. We now construct a mapping $\tilde{f} : E \to I^X$ such that, $\tilde{f}(a) = \tilde{f}_a : X \to [0.1]$ which is defined as:

$$\tilde{f}_a(x) = \{\frac{1}{n} : a^n = x, \forall a \in E \text{ and } x \in X\}.$$

Then the fuzzy soft set $(\tilde{f}, E)$ is defined as follows:

$$(\tilde{f}, E) = \{(e, \{(e, 1)\}), (a, \{(e, 1), (a, 0.5)\}), (b, \{(e, 1), (b, 0.5)\}), (c, \{(e, 1), (c, 0.5)\})\}.$$

Clearly, this fuzzy soft set satisfies the conditions of a fuzzy soft group. Thus $(\tilde{f}, E)$ is a fuzzy soft group over X.
Now, we have seen that, $< (e, 1) >, < (a, 0.5) >, < (b, 0.5) >, < (c, 0.5) >$ are the fuzzy soft generators with respect to the parameter e, a, b, c, respectively. So $(\tilde{f}, E)$ is a fuzzy soft cyclic group over X. But here X is not a cyclic group.

Theorem 3.11. Let $(X, .)$ be a group and $(\tilde{f}, A)$ and $(\tilde{g}, B)$ be two fuzzy soft cyclic groups over X where, $A, B \subseteq E$. If $A \cap B = \phi$, then $(\tilde{f}, A)\tilde{\cup}(\tilde{g}, B)$ is also a fuzzy soft cyclic group over X.

Proof: Let $(\tilde{f}, A)\tilde{\cup}(\tilde{g}, B) = (\tilde{h}, C)$ where, $C = A \cup B$. Then, since $A \cap B = \phi$ so, $\forall c \in C$, $\tilde{h}(c) = \tilde{f}_c$ if, $c \in A - B$ and $\tilde{h}(c) = \tilde{g}_c$ if, $c \in B - A$. Since both the $(\tilde{f}, A)$ and $(\tilde{g}, B)$ are both fuzzy soft cyclic groups over X so then, $(\tilde{f}, A)\tilde{\cup}(\tilde{g}, B) = (\tilde{h}, C)$ is also a fuzzy soft cyclic group over X.

Theorem 3.12. Let $(X, .)$ be a group and $(\tilde{f}, A)$ and $(\tilde{g}, B)$ be two fuzzy soft cyclic groups over X where, $A, B \subseteq E$. Then their intersection $(\tilde{f}, A)\tilde{\cap}(\tilde{g}, B)$ is also a fuzzy soft cyclic group over X.

Proof: It is trivial.

Theorem 3.13. In a finite fuzzy soft cyclic group, the inverse element of a generator with respect to a parameter is also a generator with respect to the same parameter.

Proof: Let $(\tilde{f}, E)$ be a finite fuzzy soft cyclic group over a finite group $(X, .)$ where, A is the parameterized set. Then, for some $a \in E$, there exists an element$x \in X$ such that $f(a) =< (x, \tilde{f}_a(x)) >$, i.e., $\tilde{f}(a) = \{(x^n, \tilde{f}_a(x^n)) : n \text{ is an integer}\}$. Now, let us considered that, $(x^i, \tilde{f}_a(x^i)) = (u, \tilde{f}_a(u))$, where i is an integer and $u \in X$. Then, we can write that, $(u, \tilde{f}_a(u)) = (x^i, \tilde{f}_a(x^i) = ((x^{-1})^{-i}, \tilde{f}_a(x^{-1})^{-i})$. Now, since $-i$ is also an integer so, $(u, \tilde{f}_a(u)) = ((x^{-1})^m, \tilde{f}_a(x^{-1})^m)$, $m \in Z$. So, we have seen that, $< (x^{-1})/\tilde{f}_a(x^{-1}) >$ is also a fuzzy soft generator of $\tilde{f}(a)$ over the fuzzy soft cyclic group $(\tilde{f}, E)$.

3.5 Conclusion

In this chapter we have developed some properties of classical group theory through fuzzy soft sets. In this regard, firstly, we have defined the order of an element of a fuzzy soft group with respect to a parameter. Then, we have defined the notion of fuzzy soft cyclic group. Some examples and related theorems have also been discussed to illustrate these ideas. From, these theorems, it can be concluded that, the union of two fuzzy soft cyclic groups can form a fuzzy soft cyclic group if, the corresponding set of parameters have a null intersection.

As a further research, one can study the other classical properties like, ring theory, filed theory, etc. through fuzzy soft sets. Moreover, one can also study the further generalizations of fuzzy soft sets like, intuitionistic fuzzy soft sets, linguistic valued soft sets, neutrosophic soft sets, etc.