Chapter 2

Analysis of prey-predator three species models with vertebral and invertebral predators*

In this chapter, a mathematical model has been considered involving three species namely prey, predator and generalist predator. Different types of functional responses have been considered to formulate the mathematical model for predator and specialist predator. Main intention of this study is to establish the local and global stabilities for the proposed model around its interior equilibrium point. A numerical example is considered to illustrate the proposed system of this chapter. The stability of the system has been analyzed using some graphical representations.

2.1 Introduction

In ecological systems, the interaction of predator and prey is a common phenomena for universal existence. The existence of ecological system is one of the important fields in the study of mathematical ecology. The problem of the ecological system can be solved by simple mathematics at first sight, but they are, in fact very challenging and complicated. There are different kinds of predator-prey models based on some difficulties. Generally, a specialist

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predator-predator-prey system may be defined in the form as follows:

$$\left.\begin{aligned} \frac{dx}{dt} &= rx\big(1-\tfrac{x}{K}\big) - yp(x) \\ \frac{dy}{dt} &= \mu_1 yp(x) - zq(y) - d_1 y \\ \frac{dz}{dt} &= \mu_2 zq(y) - \gamma z^2 - d_2 z \end{aligned}\right\} \tag{2.1}$$

with initial conditions $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$. Where $x(t), y(t)$ and $z(t)$ denote the population of prey, predator and specialist predator respectively at time t. $p(x)$ and $q(y)$ denote the functional responses of predator and specialist predator respectively.

Most of the researchers have formulated their models on three species prey-predator-specialist predator interactions with same functional response for predator and specialist predator but in this chapter, three species prey-predator-specialist predator interactions with different functional responses have been considered, which is more realistic to analyze the whole system. Holling type II functional response usually is suited for the invertebral predators that has been used to formulate the mathematical model in this chapter. For the vertebral predators, Holling type III functional response has been used which is more fitted to describe the relationship between predator and prey. These are the main motivations of this chapter.

Here a real-life example is described to understand the phenomena. The pond ecology has been considered to formulate the model, where Diatom (*Phylum bacillariophyta*) is prey, Daphnia (*Daphnia pulex*) is predator and Channa (*Channa amphibeus*) is the specialist predator. Since zooplankton are invertebral and fishes are vertebral. So, the predators give different responses generalist on the prey. For this reason, Holling type II functional response has been used to describe the relationship between the predator and the prey and Holling type III functional response to describe the relationship between the generalist-predator and the predator. Based on this consideration, the system of differential equations has been developed. Assuming that, $p(x) = \frac{\alpha x}{a+x}$ and

$q(y) = \frac{my^2}{b+y^2}$. Then the system (2.1) becomes

$$\left.\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a+x} \\ \frac{dy}{dt} &= \mu_1 \frac{\alpha xy}{a+x} - d_1 y - \frac{my^2 z}{b+y^2} \\ \frac{dz}{dt} &= \mu_2 \frac{my^2 z}{b+y^2} - d_2 z - \gamma z^2 \end{aligned}\right\} \tag{2.2}$$

with same initial conditions $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$. By choosing the dimensionless variables as $\beta = \mu_1 \alpha$, $n = \mu_2 m$, the system (2.2) becomes as:

$$\left.\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a+x} \\ \frac{dy}{dt} &= \frac{\beta xy}{a+x} - d_1 y - \frac{my^2 z}{b+y^2} \\ \frac{dz}{dt} &= \frac{ny^2 z}{b+y^2} - d_2 z - \gamma z^2 \end{aligned}\right\} \tag{2.3}$$

with same initial conditions $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$.

2.2 Notations

Table-2.2.1: Description of the parameters.

Parameter	Description of the parameters
x	Population of prey at time t
y	Population of predator at time t
z	Population of specialist predator at time t
r	Intrinsic growth rate of prey
K	Environmental carrying capacity of the prey
α	Capture rate of the predator to prey
m	Capture rate of the specialist predator to predator
a, b	Half saturation constants
d_1	Natural death rate of predator
d_2	Natural death rate of specialist predator
β	Predator's consumption rate on prey
n	Specialist predator's consumption rate on predator
γ	Intra-specific competition coefficient of generalist predator

2.3 Analysis of the model at its interior equilibrium

The analysis of the model has been taken around its interior equilibrium point $\overline{B}(\bar{x}, \bar{y}, \bar{z})$, where $\overline{B}(\bar{x}, \bar{y}, \bar{z})$ are the positive roots of the equation $\dot{x} = \dot{y} = \dot{z} = 0$. Then we have,

$$\bar{z} = \frac{1}{\gamma}\left[\frac{n\bar{y}^2}{b+\bar{y}^2} - d_2\right] \tag{2.4}$$

$$\bar{y} = \frac{r}{\alpha}(a+\bar{x})\left(1-\frac{\bar{x}}{K}\right) \tag{2.5}$$

and $\bar{x}$ is the positive root of the following equation

$$T_9x^9 - T_8x^8 - T_7x^7 - T_6x^6 - T_5x^5 - T_4x^4 - T_3x^3 - T_2x^2 - T_1x - T_0 = 0 \tag{2.6}$$

where

$$\begin{aligned}
T_9 &= c_3^4(\beta - d_1),\\
T_8 &= 4c_2c_3^3(d_1-\beta) + d_1ac_3^4,\\
T_7 &= (4c_1c_3^3 + 6c_2^2c_3^2)(d_1-\beta) + 4d_1ac_2c_3^3 + \frac{m}{\gamma}(n-d_2)c_3^3,\\
T_6 &= (12c_1c_2c_3^2 + 4c_2^3c_3)(d_1-\beta) + ad_1(4c_1c_3^3 + 6c_2^2c_3^2) + 3\frac{m}{\gamma}(n-d_2)c_2c_3^2\\
&\quad + \frac{amc_3^3}{\gamma}(n-d_2),\\
T_5 &= (12c_1c_2^2c_3 + c_2^4 + 6c_1^2c_3^2 + 2bc_3^2)(d_1-\beta) + ad_1(12c_1c_2c_3^2 + 4c_2^3c_3)\\
&\quad + 3\frac{m}{\gamma}(c_1c_3^2 + c_2^2c_3)(n-d_2) + 3\frac{am}{\gamma}c_2c_3^2(n-d_2),\\
T_4 &= 4(c_1c_2^3 + 3c_1^2c_2c_3 + bc_2c_3)(d_1-\beta) + \frac{m}{\gamma}(n-d_2)\{6c_1c_2c_3 + c_2^3\\
&\quad + 3a(c_1c_3^2 + c_2^2c_3)\} + ad_1(12c_1c_2^2c_3 + c_2^4 + 6c_1^2c_3^2 + 2bc_3^2),\\
T_3 &= (d_1-\beta)\{6c_1^2c_2^2 + 4c_1^3c_3 + 2b(c_2^2 + 2c_1c_3)\} + \frac{m}{\gamma}(n-d_2)(3c_1c_2^2 + 3c_1^2c_3\\
&\quad + 6ac_1c_2c_3 + ac_2^3) - \frac{m}{\gamma}bd_2c_3 + 4ad_1(c_1c_2^3 + 3c_1^2c_2c_3 + bc_2c_3),\\
T_2 &= 4(d_1-\beta)(c_1^3c_2 + bc_1c_2) + 3\frac{m}{\gamma}(n-d_2)(c_1^2c_2 + ac_1c_2^2 + ac_1^2c_3) + ad_1(6c_1^2c_2^2\\
&\quad + 4c_1^3c_3 + 2bc_2^2 + 4bc_1c_3) - \frac{m}{\gamma}bd_2(c_2 + ac_3),
\end{aligned}$$

$$
\begin{aligned}
T_1 &= (d_1-\beta)(c_1^4+2bc_1^2+b^2)+4ad_1c_1(c_1^2c_2+bc_2)+\frac{m}{\gamma}(n-d_2)(c_1^3+3ac_1^2c_2) \\
&\quad -\frac{m}{\gamma}bd_2(c_1+ac_2), \\
T_0 &= ad_1(c_1^4+2bc_1^2+b^2)+\frac{am}{\gamma}\{(n-d_2)c_1^3-bd_2c_1\}, c_1=ar, c_2=r\left(1-\frac{a}{K}\right), \\
c_3 &= -\frac{r}{K}.
\end{aligned}
$$

Hence, the sufficient condition for the system (2.3) has a positive interior equilibrium point if $T_i, (i=0,1,2,...,9)$ are all positive with $K>\bar{x}$ and $\bar{y}^2>\frac{bd_2}{n-d_2}$.

2.4 Boundedness

Theorem 2.4.1. *All the solutions of the system* (2.3) *are always bounded.*

Proof: Since K is the carrying capacity of the prey population then from first equation of the system (2.3), we have $x \le K+\epsilon$ as $t\to\infty$ for $\epsilon>0$. Now, let $w = x+\frac{\alpha}{\beta}y+\frac{\alpha m}{\beta n}z$,
Then we have

$$
\begin{aligned}
&\frac{dw}{dt} = \frac{dx}{dt}+\frac{\alpha}{\beta}\frac{dy}{dt}+\frac{\alpha m}{\beta n}\frac{dz}{dt} \\
&\frac{dw}{dt} \le rx-\frac{\rho\alpha}{\beta}y-\frac{\rho\alpha m}{\beta n}z, \qquad \text{where} \quad \rho=\min\{d_1,d_2\} \\
i.e, \qquad &\frac{dw}{dt} \le -\rho w+(r+\rho)x \\
i.e, \qquad &\frac{dw}{dt} \le -\rho w+(r+\rho)(K+\epsilon) \\
i.e, \qquad &\frac{dw}{dt}+\rho w \le I, \quad where \qquad I=(r+\rho)(K+\epsilon)
\end{aligned}
$$

Integrating both sides of above equation and applying the theorem of differential inequality, we obtain $0<w\le \frac{I}{\rho}(1-e^{-\rho t})+w[x(0),y(0),z(0)]$ for $t\to\infty$. Again, $0<w\le\frac{I}{\rho}+w(0)$ for $\epsilon>0$. Hence from above expression, we may conclude that the solution space (x,y,z) is bounded in the region R_+^3. Hence the theorem follows:

2.5 Local Stability

In this section, we analyze the stability of the system (2.3) at interior equilibrium point.

Now the characteristic equation of the system (2.3) around its interior equilibrium reduces to

$$\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 = 0 \tag{2.7}$$

where

$$\begin{aligned}
h_1 &= \frac{r}{K}\bar{x} - \frac{\alpha\bar{x}\bar{y}}{(a+\bar{x})^2} - \frac{m\bar{y}\bar{z}(\bar{y}^2-b)}{(b+\bar{y}^2)^2} + \gamma\bar{z} \\
h_2 &= \frac{r\gamma}{K}\bar{x}\bar{z} - \frac{\alpha\gamma\bar{x}\bar{y}\bar{z}}{(a+\bar{x})^2} + \frac{\alpha\beta a\bar{x}\bar{y}}{(a+\bar{x})^3} - \frac{m\gamma\bar{y}\bar{z}^2(\bar{y}^2-b)}{(b+\bar{y}^2)^2} - \frac{rm}{K}\frac{\bar{x}\bar{y}\bar{z}(\bar{y}^2-b)}{(b+\bar{y}^2)^2} \\
&\quad + \frac{m\alpha\bar{x}\bar{y}^2\bar{z}(\bar{y}^2-b)}{(a+\bar{x})^2(b+\bar{y}^2)^2} + 2\frac{bnm\bar{y}^3\bar{z}}{(b+\bar{y}^2)^3} \\
h_3 &= \frac{a\alpha\beta\gamma\bar{x}\bar{y}\bar{z}}{(a+\bar{x})^3} - \frac{r}{K}\gamma\frac{m\bar{x}\bar{y}\bar{z}^2(\bar{y}^2-b)}{(b+\bar{y}^2)^2} + \frac{m\alpha\gamma\bar{x}\bar{y}^2\bar{z}^2(\bar{y}^2-b)}{(a+\bar{x})^2(b+\bar{y}^2)^2} + 2\frac{bnmr\bar{x}\bar{y}^3\bar{z}}{K(b+\bar{y}^2)^3} \\
&\quad - 2\frac{bnm\alpha\bar{x}\bar{y}^4\bar{z}}{(a+\bar{x})^2(b+\bar{y}^2)^3}
\end{aligned}$$

Now we consider h_i, $(i = 1, 2, 3)$ as $h_1 = k_1 - k_2 m$, $h_2 = k_3 - k_4 m$, $h_3 = k_5 - k_6 m$, where

$$\begin{aligned}
k_1 &= \frac{r}{K}\bar{x} - \frac{\alpha\bar{x}\bar{y}}{(a+\bar{x})^2} + \gamma\bar{z} \\
k_2 &= \frac{\bar{y}\bar{z}(\bar{y}^2-b)}{(b+\bar{y}^2)^2} \\
k_3 &= \frac{r\gamma}{K}\bar{x}\bar{z} - \frac{\alpha\gamma\bar{x}\bar{y}\bar{z}}{(a+\bar{x})^2} + \frac{\alpha\beta a\bar{x}\bar{y}}{(a+\bar{x})^3} \\
k_4 &= \frac{\gamma\bar{y}\bar{z}^2(\bar{y}^2-b)}{(b+\bar{y}^2)^2} + \frac{r}{K}\frac{\bar{x}\bar{y}\bar{z}(\bar{y}^2-b)}{(b+\bar{y}^2)^2} - \frac{\alpha\bar{x}\bar{y}^2\bar{z}(\bar{y}^2-b)}{(a+\bar{x})^2(b+\bar{y}^2)^2} - 2\frac{bn\bar{y}^3\bar{z}}{(b+\bar{y}^2)^3} \\
k_5 &= \frac{a\alpha\beta\gamma\bar{x}\bar{y}\bar{z}}{(a+\bar{x})^3} \\
k_6 &= \gamma\frac{r\bar{x}\bar{y}\bar{z}^2(\bar{y}^2-b)}{K(b+\bar{y}^2)^2} - \frac{\alpha\gamma\bar{x}\bar{y}^2\bar{z}^2(\bar{y}^2-b)}{(a+\bar{x})^2(b+\bar{y}^2)^2} - \frac{2bnr\bar{x}\bar{y}^3\bar{z}}{K(b+\bar{y}^2)^3} + \frac{2bn\alpha\bar{x}\bar{y}^4\bar{z}}{(a+\bar{x})^2(b+\bar{y}^2)^3}
\end{aligned}$$

Now using Routh-Hurwitz criteria around the interior equilibrium point, we can state and prove the following theorem for the local asymptotic stability of the system (2.3).

Theorem 2.5.1. *The system* (2.3) *will be locally asymptotically stable around its interior equilibrium point, if* $min\{\frac{k_2}{k_1}, \frac{k_6}{k_5}\} > m > m^*$, *where* m^* *is the biggest root of the equation* $\psi(m) = k_2k_4m^2 + (k_6 - k_2k_3 - k_1k_4)m + (k_1k_3 - k_5) = 0$, *if those conditions hold* $(i)\frac{r}{K} \geq \frac{\alpha\bar{y}}{(a+\bar{x})^2}$ *and* $(ii)(\bar{y}^4 - b^2)\gamma \geq 2bn\bar{y}^2$

Proof: The system will be locally asymptotically stable at the interior equilibrium point $\overline{B}(\bar{x}, \bar{y}, \bar{z})$, if Routh-Hurwitz criteria around the interior equilibrium point holds.
Using Routh-Hurwitz criteria, we conclude that all the eigen values of the system (2.3) contain the negative real part at $\bar{B}$. i.e., all the roots of the equation (2.7) have negative real part
i.e., $h_1, h_3 > 0$ and $h_1h_2 > h_3$.
Again $h_1, h_3 > 0$ and $h_1h_2 > h_3$ when $min\{\frac{k_2}{k_1}, \frac{k_6}{k_5}\} > m > m^*$
i.e. $\frac{r}{K} \geq \frac{\alpha\bar{y}}{(a+\bar{x})^2}$ and $(\bar{y}^4 - b^2)\gamma \geq 2bn\bar{y}^2$
Hence the system is locally asymptotically stable at the interior equilibrium point.

Theorem 2.5.2. *The system* (2.3) *undergoes through a Hopf bifurcation at its interior equilibrium for* $m = m^*$.

Proof: For $m = m^*$, we have $h_1h_2 - h_3 = 0$ and then the eigenvalues of the system at $\bar{B}$ can be represented as $\lambda_1 = -h_1$ and $\lambda_{2,3} = \pm i\sqrt{h_2}$ Considering $\lambda_1 = \phi_1(m)$ and $\lambda_{2,3} = \phi_2(m) \pm i\phi_3(m)$. Now it is obvious to show that $\frac{d\phi_2}{dm}$ is non zero at the point $m = m^*$. Again, we have $\phi(m^*) = 0$. Therefore, it is obvious to show that our system (2.3) follows a Hopf bifurcation at its interior equilibrium for the critical value of m, i.e, for $m = m^*$, with the help of given conditions (103). So, the theorem is obvious.

2.6 Global Stability

Now we discuss the general method (58) to show an n-dimensional autonomous dynamical system $f : D \to R^n$, $D \subset R^n$, an open and simply connected set and $f \in C^1(D)$, where the dynamical system is as follows:

$$\frac{dx}{dt} = f(x) \tag{2.8}$$

which is globally stable under certain parametric conditions. We refer to the works of Haque et al. (39), Bunomo et al. (10), Kar and Mondal (50) for detailed discussion. Now we consider the following conditions.
(i) The autonomous dynamical system (2.8) has a unique interior equilibrium point $\bar{x}$ in D.
(ii) The domain D is simply connected.
(iii) There is a compact absorbing set $\Omega \subset D$.
The unique interior equilibrium point $\bar{x}$ in D of the system (2.8) is globally asymptotically stable if the system is locally asymptotically stable and all the trajectories in D converges to its interior equilibrium point.

Theorem 2.6.1. *The system* (2.3) *is globally asymptotically stable around its interior equilibrium if* $d_2 < g_2$, *where* $g_2 = \frac{ng_1^2}{b+g_1^2} + min\{-\gamma g_1 - \frac{mg_1^2(g_1^2-b)}{(b+g_1^2)^2} + \frac{r}{K}g_1 - \frac{\alpha g_1^2}{(a+g_1)^2} - \frac{mg_1^2}{b+g_1^2}, \frac{r}{K}g_1 - \frac{\alpha g_1^2}{(a+g_1)^2} - \frac{\beta a g_1}{(a+g_1)^2}, \frac{\alpha g_1}{a+g_1} - \frac{mg_1^2(g_1^2-b)}{(b+g_1^2)^2}\}$ *with* $g_1 \in R^+$ *such that for* $t_1 > 0$ *we have* $g_1 = inf\{x(t), y(t), z(t)\}$ *whenever* $t > t_1$.

Proof: Let $J^{|2|}$ be the second additive compound matrix with order ${}^3C_2 \times {}^3C_2$. Hence,

$$J^{|2|} = \frac{\partial f^{|2|}}{\partial x} = \begin{pmatrix} J_{11}+J_{22} & J_{23} & -J_{13} \\ J_{32} & J_{11}+J_{33} & J_{12} \\ -J_{31} & J_{21} & J_{22}+J_{33} \end{pmatrix}$$

where $J = (J_{ij})_3$ is the variational matrix of the system (2.3). Then from the system equation we have

$$J^{|2|} = \begin{pmatrix} \frac{myz(y^2-b)}{(b+y^2)^2} - \frac{r}{K}x + \frac{\alpha xy}{(a+x)^2} & -\frac{my^2}{b+y^2} & 0 \\ \frac{2bnyz}{(b+y^2)^2} & -\frac{r}{K}x + \frac{\alpha xy}{(a+x)^2} - \gamma z & -\frac{\alpha x}{a+x} \\ 0 & \frac{\beta a y}{(a+x)^2} & \frac{myz(y^2-b)}{(b+y^2)^2} - \gamma z \end{pmatrix}$$

We consider $M(X) \in C^1(D)$ in such a way that $M = diag\{x/z, x/z, x/z\}$. Then $M_f M^{-1} = diag\{\dot{x}/x - \dot{z}/z, \dot{x}/x - \dot{z}/z, \dot{x}/x - \dot{z}/z\}$ and $MJ^{|2|}M^{-1} = J^{|2|}$, where matrix M_f is obtained by replacing each entity M_{ij} of M by its derivative in the direction of solution (2.3). In addition, we have

$$B = M_f M^{-1} + MJ^{|2|}M^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

where $B_{11} = \frac{\dot{x}}{x} - \frac{\dot{z}}{z} + \frac{myz(y^2-b)}{(b+y^2)^2} - \frac{r}{K}x + \frac{\alpha xy}{(a+x)^2}$, $B_{12} = \begin{pmatrix} -\frac{my^2}{b+y^2} & 0 \end{pmatrix}$, $B_{21} = \begin{pmatrix} \frac{2bnyz}{(b+y^2)^2} \\ 0 \end{pmatrix}$, $B_{22} = \begin{pmatrix} \frac{\dot{x}}{x} - \frac{\dot{z}}{z} - \frac{r}{K}x + \frac{\alpha xy}{(a+x)^2} - \gamma z & -\frac{\alpha x}{a+x} \\ \frac{\beta ay}{(a+x)^2} & \frac{\dot{x}}{x} - \frac{\dot{z}}{z} + \frac{myz(y^2-b)}{(b+y^2)^2} - \gamma z \end{pmatrix}$.

Let (u_1, u_2, u_3) denote the vector in R^3, choose a norm in R^3 as $|u_1, u_2, u_3| = max\{|u_1|, |u_2| + |u_3|\}$ and let Γ be the Lozinskii measure with respect to this norm. Then, we have the following estimate (58):

$$\Gamma(B) \leq \{b_1, b_2\} \tag{2.9}$$

Where $b_1 = \Gamma_1(B_{11}) + |B_{12}|$,$b_2 = |B_{21}| + \Gamma_1(B_{22})$ and Γ_1 denotes the Lozinskii measure with respect to l_1 vector norm, $|B_{12}|$ and $|B_{21}|$ are matrix norms with respect to l_1 norm. Then we get

$$\begin{aligned}
\Gamma_1(B_{11}) &= \dot{x}/x - \dot{z}/z - \frac{r}{K}x + \frac{\alpha xy}{(a+x)^2} - \gamma z \\
|B_{12}| &= \frac{my^2}{b+y^2} \\
|B_{21}| &= \frac{2bnyz}{(b+y^2)^2} \\
\Gamma_1(B_{22}) &= \frac{\dot{x}}{x} - \frac{\dot{z}}{z} - \gamma z + max\Big\{\frac{\alpha xy}{(a+x)^2} - \frac{r}{K}x + \frac{\beta ay}{(a+x)^2}, \frac{myz(y^2-b)}{(b+y^2)^2} \\
&\quad - \frac{\alpha x}{a+x}\Big\}
\end{aligned}$$

Hence

$$b_1 = \frac{\dot{x}}{x} - \frac{\dot{z}}{z} + \frac{myz(y^2-b)}{(b+y^2)^2} - \frac{r}{K}x + \frac{\alpha xy}{(a+x)^2} + \frac{my^2}{b+y^2}$$

and

$$\begin{aligned}
b_2 &= \frac{\dot{x}}{x} - \frac{\dot{z}}{z} - \gamma z + \frac{2bnyz}{(b+y^2)^2} + max\Big\{ - \frac{r}{K}x + \frac{\alpha xy}{(a+x)^2} + \frac{\beta ay}{(a+x)^2}, \ -\frac{\alpha x}{a+x} \\
&\quad + \frac{myz(y^2-b)}{(b+y^2)^2}\Big\}
\end{aligned}$$

Now using $\frac{\dot{z}}{z} = \frac{ny^2}{b+y^2} - \gamma z - d_2$ from the system (2.3), the expression becomes,

$$b_1 = \frac{\dot{x}}{x} - \frac{ny^2}{b+y^2} + \gamma z + d_2 + \frac{myz(y^2-b)}{(b+y^2)^2} - \frac{r}{K}x + \frac{\alpha xy}{(a+x)^2} + \frac{my^2}{b+y^2}$$

and

$$b_2 = \frac{\dot{x}}{x} - \frac{ny^2}{b+y^2} + d_2 + \frac{2bnyz}{(b+y^2)^2} - min\Big\{\frac{r}{K}x - \frac{\alpha xy}{(a+x)^2} - \frac{\beta ay}{(a+x)^2}, \frac{\alpha x}{a+x} - \frac{myz(y^2-b)}{(b+y^2)^2}\Big\}$$

Now from (2.9) we get

$$\Gamma(B) \le \frac{\dot{x}}{x} - \frac{ny^2}{b+y^2} + d_2 - min\Big\{-\gamma z - \frac{myz(y^2-b)}{(b+y^2)^2} + \frac{r}{K}x - \frac{\alpha xy}{(a+x)^2} - \frac{my^2}{b+y^2}, \frac{r}{K}x - \frac{\alpha xy}{(a+x)^2} - \frac{\beta ay}{(a+x)^2}, \frac{\alpha x}{a+x} - \frac{myz(y^2-b)}{(b+y^2)^2}\Big\}$$

$$i.e., \Gamma(B) \le \frac{\dot{x}}{x} + d_2 - g_2$$

where, $g_2 = \frac{ng_1^2}{b+g_1^2} + min\{-\gamma g_1 - \frac{mg_1^2(g_1^2-b)}{(b+g_1^2)^2} + \frac{r}{K}g_1 - \frac{\alpha g_1^2}{(a+g_1)^2} - \frac{mg_1^2}{b+g_1^2}, \frac{r}{K}g_1 - \frac{\alpha g_1^2}{(a+g_1)^2} - \frac{\beta a g_1}{(a+g_1)^2},$
$\frac{\alpha g_1}{a+g_1} - \frac{mg_1^2(g_1^2-b)}{(b+g_1^2)^2}\}$ and $g_1 = inf\{x(t), y(t), z(t)\} \in R$.
i.e.,

$$\frac{1}{t}\int_0^t \Gamma(B)ds \le \frac{1}{t}log\frac{x(t)}{x(0)} - (g_2 - d_2)$$

Therefore,

$$\lim_{t\to\infty} sup \sup \frac{1}{t}\int_0^t \Gamma(B(s,x_0))ds < -(g_2 - d_2) < 0$$

Hence the theorem is proved.

2.7 Numerical Simulation

Some arbitrarily data have been assumed for describing the analytical results. Using the MATLAB 7.10 software, we have analyzed the sensitivity analysis of the experiment. Assuming that, $r = 1.1, K = 9.8, \alpha = 1.3, a = 6, \beta = 1.2, b = 7, \gamma = 0.39, m = 1.2, n = 0.9, d_1 = 0.3$ and $d_2 = 0.12$ and also that, initially, the prey population is 8, the predator population is 7 and the specialist predator population is 1. In Figures 2.1 and 2.2, we show the stability in the form of phase space diagram and solution curve for $m = 0.5$. In Figures 2.3 and 2.4, we have shown instability in the form of phase space diagram and solution curve for $m = 0.4$ of the system (2.3). Again from Figure 2.5, we have seen that, after certain time the system reaches equilibrium position. So the proposed model obeys the law of universal existence in the presence of so many interactions.

2.8 Chapter Summary

Different functional responses have been introduced for specialist predator in predator and predator in prey in this chapter. Again a density dependent mortality rate for predator and specialist predator have been introduced and also intra-specific competition for specialist predator have been considered. In this context, this chapter is significantly differ from other works in this area. The local as well as global stability around its interior equilibrium point has been discussed. The study has been illustrated with a numerical example. Also the proposed model has been analyzed with some geometrical representations. Finally, it is remarked that, the derived results are not only feasible but also have great impacts on ecological systems from the biological and social points of view.

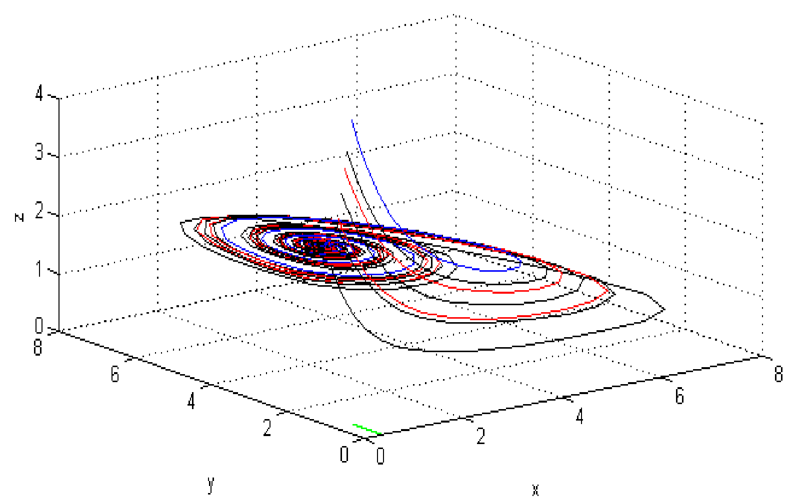

Figure 2.1: *Phase space diagram of the system* (2.3) *for* $m(= 0.5) > m^*$.

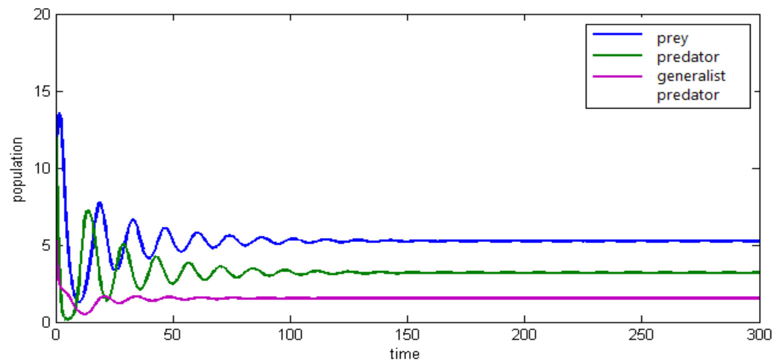

Figure 2.2: *Solution curve of the system* (2.3) *for* $m(= 0.5) < m^*$.

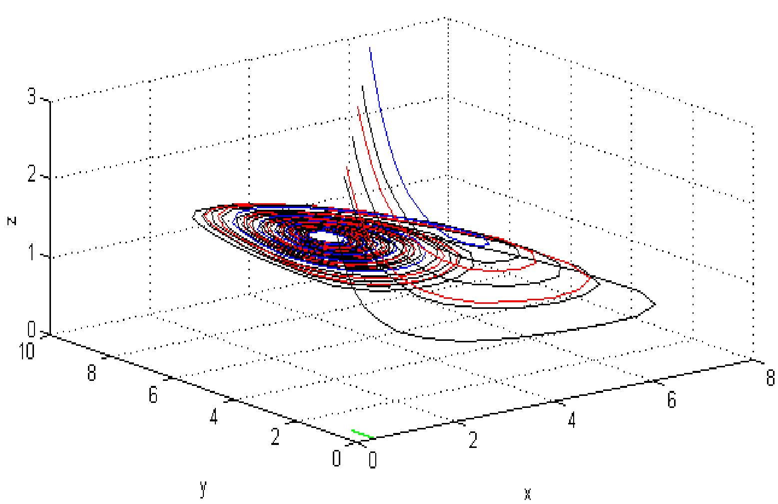

Figure 2.3: *Phase space diagram of the system* (2.3) *for* $m(= 0.4) < m^*$.

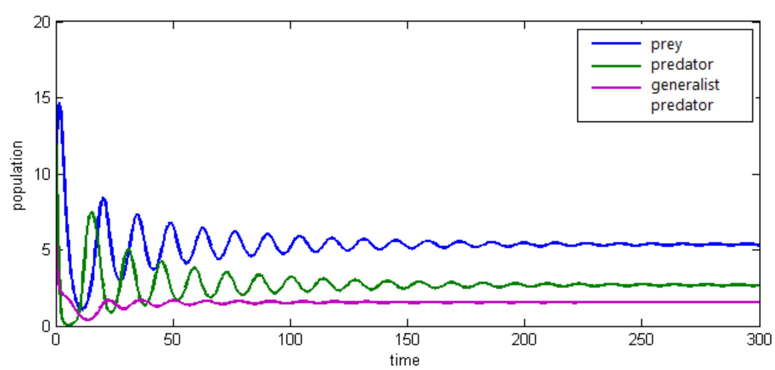

Figure 2.4: *Solution curve of the system* (2.3) *for* $m(= 0.4) < m^*$.

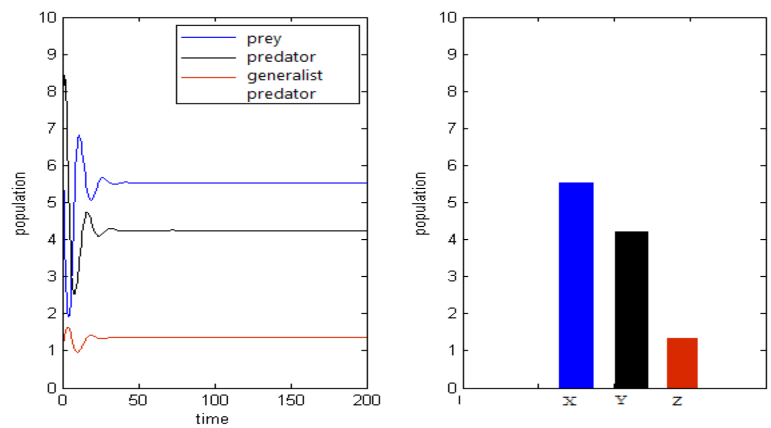

Figure 2.5: *Graphical representation of the system and the bar diagram of populations at stable state.*