Chapter 5

Prey-predator model with migration and harvesting*

In this chapter, we consider a prey-predator model with a reserve region of predator where generalist predator cannot enter. Based on the intake capacity of food and other factors, we introduce the predator population which consumes the prey population with Holling type-II functional response; and generalist predator population consumes the predator population with Beddington-DeAngelis functional response. The density-dependent mortality rate for prey and generalist predator are considered. The equilibria of proposed system are determined. Local stability for the system is discussed. The environmental carrying capacity is considered as a bifurcation parameter to evaluate Hopf bifurcation in the neighborhood of interior equilibrium point. Here the fishing effort is used as a control parameter to harvest the generalist predator population of the system. With the help of this control parameter, a dynamic framework is developed to investigate the optimal utilization of resources, sustainability properties of the stock and the resource rent. Finally, we present a numerical simulation to verify the analytical results and the system is analyzed through graphical illustrations.

*A part of this chapter has been communicated to the International Journal

5.1 Introduction

Predation is an important factor in a food chain. In predation, predators may or may not kill their prey prior to feeding on them, but the act of predation often results in the death of its prey through consumption. Moreover, predation is an interaction between the species in which one species uses another species as food. Generally, successful predation leads to increase in a population size of predator and to decrease in a population size of prey. In this way, predation controls a food chain as well as an ecological system.

Again harvesting is the process of catching or killing those substances which are used to prepare foods. To fill up the demand of human needs and balance the species for future harvesting, harvesting effort is very much essential to optimize economically.

A nature reserve is a protected area which is very importance for an ecological system. It may be designated by government institutions in some countries or by private land owners. In natural reserve region, limited number of species lived, all species cannot enter. They considered only one species i.e., prey lives in a reserve region in their works. But, many cases, there are more than one species lives in a reserve region. They are also related in prey-predator relationship.

We consider a real-life example of our proposed problem for showing the feasibility and effectiveness of this work. In geomorphology, drainage systems are the patterns formed by the streams, rivers, and lakes in a particular drainage basin. In drainage system, mosquito larvas are grown usually. From which different types of mosquitoes are born. They are the carrier of different types of harmful diseases like Dengue, Malaria, Meningitis etc. Nowadays, harmful diseases are attacking to human population. A large number of people are affected by these harmful diseases. Controlling of mosquito population is an essential measure to prevent the harmful diseases. Mosquito fishes (Gambezi) live and also cultivate in a drainage system to prevent the mosquito populations. Here, Gambezi is considered as predator; and mosquito larva is treated as prey. Again, a drainage basin is an area of land where all surface water from drainage system usually falls into the basin, as well as the water connects another body of water, such as a river, lake, reservoir, estuary, wetland, sea or ocean. Fishes like catfish lives in the other body of water like river, lake, reservoir etc. Catfish is considered here as generalist predator, which consumes mosquito fishes for their diet but catfish does not enter in drainage system. For this reason, drainage system is here the reserve region for the mosquito fish. Many researchers have paid their attention on different regions such as

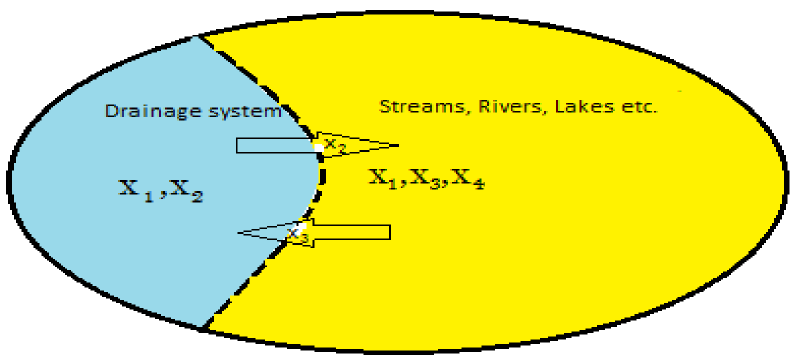

Figure 5.1: Pictorial representation of the model.

refuge region and predatory region and shown their impacts on prey-predator in these regions. But, to the best of our knowledge, for the first time, we consider the interaction among the generalist predator, predator and prey in reserve region. This consideration is shown mathematically and analyzed the results. In addition to the above, we introduce the generalist predator harvesting effort which is more realistic to analyze the whole system. Based on the aforementioned facts, we motivate to design this chapter.

5.2 Notation

Table-5.2.1: Description of the parameters.

Parameter	Description of the parameters
x_1	Population of prey at time t
x_2	Population of predator in reserve region at time t
x_3	Population of predator in other region at time t
x_4	Population of generalist predator at time t
r	Intrinsic growth rate of prey
K	Environmental carrying capacity of the prey
γ	Maximal predator per capita consumption rate
a_1	Half saturation constant
d_1	Natural death rate of prey
d_3	Natural death rate of generalist predator
s_1, s_2	Predator's consumption rate on prey
β_1	Generalist predator's consumption rate on predator
m_1	Per unit emigration of the predator population in reserve region
m_2	Per unit migration of the predator population in reserve region
q	Catchability co-efficient
E	Fishing effort for harvesting the generalist predator population

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5.3 Mathematical model

Assume that, at any time t, x_1 and x_2 denote the mosquito larva population (prey) and gambezi population (predator) respectively. Since the mosquito larva is considered as prey and gambezi as predator, so the predator population consumes the prey population with Holling type-II functional response or Michaelis-Menten functional response which is considered as $\frac{x_1}{a_1+x_1}$. Then the system of equations in reserve region becomes as follows:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= rx_1\left(1-\frac{x_1}{K}\right)-\gamma\frac{x_1x_2}{a_1+x_1}-d_1x_1 \\ \frac{dx_2}{dt} &= s_1\frac{x_1x_2}{a_1+x_1} \end{aligned}\right\} \tag{5.1}$$

with initial conditions $x_1(0)\geq 0, x_2(0)\geq 0$, where d_1 denotes as death rate of prey population and γ denotes the maximal predator per capita consumption rate, i.e., the maximum number of prey population can be eaten by the predator in each time unit and the half capturing saturation constant denoted by a_1 i.e., the number of prey necessary to obtain one-half of the maximum rate α_1. To analyze the situation more practically, here drainage basin has been chosen in which one part is formed by streams, rivers and lakes as shown in Figure 5.1. The drainage system is opened in a certain place like as river and ocean. There are so many species which do not enter in drainage system like catfishes (generalist predator) which consume mosquito fish(Gambezi). Here, for mosquito fish, drainage system is a reserve region. Again consider that, at any time t, x_3 and x_4 denote the gambezi population and catfish population in other region respectively. Generally, the growth rate of a predator in the reserve region is different from other areas. For this reason, different growth rates are considered for the predator in two different regions. Assume that, s_2 is the growth rate of predator population in other region. Again, generalist predator population i.e., fish consumes the gambezi, so predator population with Beddington-DeAngelis functional response is treated as $\beta\frac{x_3}{a+bx_3+mx_4}$, where β, a, b and m are positive integers. Then the system described by the following system of equations as:

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= rx_1\left(1-\frac{x_1}{K}\right) - \gamma\frac{x_1x_3}{a_1+x_1} - d_1x_1 \\
\frac{dx_3}{dt} &= s_2\frac{x_1x_3}{a_1+x_1} - \beta\frac{x_3x_4}{a+bx_3+mx_4} \\
\frac{dx_4}{dt} &= \beta_1\frac{x_3x_4}{a+bx_3+mx_4} - d_3x_4
\end{aligned}\right\} \tag{5.2}$$

with initial conditions $x_1(0) \geq 0, x_3(0) \geq 0, x_4(0) \geq 0$, where d_3 and β_1 denotes as death rate of generalist predator population and generalist predator's consumption rate on predator respectively. Combining the systems (5.1) and (5.2), the dynamics of the reserved region is designed by the following differential equations as:

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= rx_1\left(1-\frac{x_1}{K}\right) - \gamma\frac{x_1x_2}{a_1+x_1} - \gamma\frac{x_1x_3}{a_1+x_1} - d_1x_1 \\
\frac{dx_2}{dt} &= s_1\frac{x_1x_2}{a_1+x_1} \\
\frac{dx_3}{dt} &= s_2\frac{x_1x_3}{a_1+x_1} - \beta\frac{x_3x_4}{a+bx_3+mx_4} \\
\frac{dx_4}{dt} &= \beta_1\frac{x_3x_4}{a+bx_3+mx_4} - d_3x_4
\end{aligned}\right\} \tag{5.3}$$

with initial conditions $x_1(0) \geq 0, x_2(0) \geq 0, x_3(0) \geq 0$ and $x_4(0) \geq 0$.
In a reserved region, we consider two regions which are connected to each other. So, migration as well as emigration should occur between these regions. Again the predator population can interact with prey population in all regions but the generalist predator population cannot interact with predator population in the reserve region. Therefore, the system of equations (5.3) becomes as follows:

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= rx_1\left(1 - \frac{x_1}{K}\right) - \gamma\frac{x_1x_2}{a_1 + x_1} - \gamma\frac{x_1x_3}{a_1 + x_1} - d_1x_1\\
\frac{dx_2}{dt} &= s_1\frac{x_1x_2}{a_1 + x_1} + m_1x_3 - m_2x_2\\
\frac{dx_3}{dt} &= s_2\frac{x_1x_3}{a_1 + x_1} - \beta\frac{x_3x_4}{a + bx_3 + mx_4} - m_1x_3 + m_2x_2\\
\frac{dx_4}{dt} &= \beta_1\frac{x_3x_4}{a + bx_3 + mx_4} - d_3x_4
\end{aligned}\right\} \tag{5.4}$$

with initial conditions $x_1(0) \geq 0, x_2(0) \geq 0, x_3(0) \geq 0$ and $x_4(0) \geq 0$. Again, generalist predators i.e., catfishes are harvested for human needs. Without loss of generality, at any time t, the harvest rate is denoted by $h(t)$ and is defined by $h(t) = qEx_4$. Then the system of equations (5.4) reduces as follows:

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= rx_1\left(1 - \frac{x_1}{K}\right) - \gamma\frac{x_1x_2}{a_1 + x_1} - \gamma\frac{x_1x_3}{a_1 + x_1} - d_1x_1\\
\frac{dx_2}{dt} &= s_1\frac{x_1x_2}{a_1 + x_1} + m_1x_3 - m_2x_2\\
\frac{dx_3}{dt} &= s_2\frac{x_1x_3}{a_1 + x_1} - \beta\frac{x_3x_4}{a + bx_3 + mx_4} - m_1x_3 + m_2x_2\\
\frac{dx_4}{dt} &= \beta_1\frac{x_3x_4}{a + bx_3 + mx_4} - d_3x_4 - qEx_4
\end{aligned}\right\} \tag{5.5}$$

with initial conditions $x_1(0) \geq 0, x_2(0) \geq 0, x_3(0) \geq 0$ and $x_4(0) \geq 0$. Since x_2 and x_3 are both predator populations, one is in the reserve region and another is in other region respectively. To simplicity of the calculation and the predators x_2 and x_3 are the same species, the term of functional response of predator on prey in reserve region and other region is treated as same. So, the system of equations (5.5) is described as follows:

$$\left.\begin{aligned}\frac{dx_1}{dt}&=rx_1\left(1-\frac{x_1}{K}\right)-\alpha\frac{x_1x_2}{a_1+x_1}-d_1x_1\\\frac{dx_2}{dt}&=s_1\frac{x_1x_2}{a_1+x_1}+m_1x_3-m_2x_2\\\frac{dx_3}{dt}&=s_2\frac{x_1x_3}{a_1+x_1}-\beta\frac{x_3x_4}{a+bx_3+mx_4}-m_1x_3+m_2x_2\\\frac{dx_4}{dt}&=\beta_1\frac{x_3x_4}{a+bx_3+mx_4}-d_3x_4-qEx_4\end{aligned}\right\}\tag{5.6}$$

with initial conditions $x_1(0)\geq 0, x_2(0)\geq 0, x_3(0)\geq 0, x_4(0)\geq 0$ and $\alpha=2\gamma$.

5.4 Equilibria and their existence criteria

The steady state of system of equations (5.6) is calculated by solving the equations. Let the interior equilibrium point is denoted by $P(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4)$ where $\bar{x}_1,\bar{x}_2,\bar{x}_3$ and $\bar{x}_4$ are the positive roots of the equation $\dot{x_1}=\dot{x_2}=\dot{x_3}=\dot{x_4}=0$, where $\dot{x}_i=\frac{dx_i}{dt}$, $i=1,2,3$ and 4.

It may be noted that

$$\bar{x}_4=\quad\frac{1}{m}\left\{\left(\frac{\beta_1}{d_3+qE}-b\right)\bar{x}_3-a\right\}\tag{5.7}$$

$$\bar{x}_3=\quad\frac{1}{m_1}\left\{m_2\bar{x}_2-s_1\frac{\bar{x}_1\bar{x}_2}{a_1+\bar{x}_1}\right\}\tag{5.8}$$

$$\bar{x}_2=\quad\frac{a_1+\bar{x}_1}{\alpha}\left\{r\left(1-\frac{\bar{x}_1}{K}\right)-d_1\right\}\tag{5.9}$$

and $\bar{x}_1$ is the positive root of the following equation

$$B_5x_1^5+B_4x_1^4+B_3x_1^3+B_2x_1^2+B_1x_1-B_0=0,\tag{5.10}$$

where

$$\begin{aligned}B_0=&\quad\beta a_1C_8(C_1C_8+C_2),\\B_1=&\quad C_8^2\{s_2(b+C_1m)-\beta C_1\}+C_8\{s_2(a+C_2m)+C_5s_1(b+C_1m)-\beta C_2\}\\&\quad-\beta a_1C_7(2C_1C_8+C_2)+C_5s_1(a+mC_2),\\B_2=&\quad-2\beta a_1C_1C_7C_8(C_7^2+2C_6C_8)+2C_7C_8\{s_2(b+C_1m)-\beta C_1\}-\beta a_1C_2C_6\\&\quad+C_7\{s_2(a+C_2m)+C_5s_1(b+C_1m)-\beta C_2\}+s_1C_4(a+mC_2),\\B_3=&\quad(C_7^2+2C_6C_8)\{s_2(b+C_1m)-\beta C_1\}+C_6\{s_2(a+C_2m)+C_5s_1(b+C_1m)-\beta C_2\}\\&\quad+C_7C_4s_1(b+C_1m)+s_1C_3(a+mC_2)-2\beta a_1C_1C_6C_7,\\B_4=&\quad 2C_6C_7\{s_2(b+C_1m)-\beta C_1\}+C_6C_4s_1(b+C_1m)+C_7C_3s_1(b+C_1m)-\beta a_1C_1C_6^2,\\B_5=&\quad C_6^2\{s_2(b+C_1m)-\beta C_1\}+C_6C_3s_1(b+C_1m),\end{aligned}$$

and

$$
\begin{aligned}
C_1 &= \frac{1}{m}\left(\frac{\beta_1}{d_3+qE}-b\right), & C_5 &= \frac{a_1}{\alpha}(r-d_1),\\
C_2 &= -\frac{a}{m}, & C_6 &= \frac{r}{m_1\alpha K}(s_1-m_2),\\
C_3 &= -\frac{r}{\alpha K}, & C_7 &= \frac{1}{\alpha m_1}\{(m_2-s_1)(r-d_1)-a_1m_2\frac{r}{K}\},\\
C_4 &= \frac{1}{\alpha}(r-d_1-a_1\frac{r}{K}), & C_8 &= \frac{a_1m_2}{\alpha m_1}(r-d_1).
\end{aligned}
$$

So, the sufficient conditions for the system with a positive interior equilibrium point are $r > d_1$, $\beta_1 > b(d_3+qE)$ and for all B_i $(i=0,1,2,3,4,5)$ are positive.

5.5 Local stability

The stability criterion of the system is analyzed at the interior equilibrium point. The Jacobian matrix of the system (5.6) at the interior equilibrium point P is denoted by V and is defined as follows:

$$
V = \begin{pmatrix}
-\frac{r}{K}\bar{x}_1+\alpha\frac{\bar{x}_1\bar{x}_2}{(a_1+\bar{x}_1)^2} & -\alpha\frac{\bar{x}_1}{a_1+\bar{x}_1} & 0 & 0\\
\frac{a_1s_1\bar{x}_2}{(a_1+\bar{x}_1)^2} & \frac{s_1\bar{x}_1}{a_1+\bar{x}_1}-m_2 & m_1 & 0\\
\frac{a_1s_1\bar{x}_3}{(a_1+\bar{x}_1)^2} & m_2 & \frac{s_2\bar{x}_1}{a_1+\bar{x}_1}-m_1-\beta\bar{x}_4\frac{a+b\bar{x}_4}{(a+b\bar{x}_3+m\bar{x}_4)^2} & \frac{\beta\bar{x}_3(a+b\bar{x}_3)}{(a+b\bar{x}_3+m\bar{x}_4)^2}\\
0 & 0 & \frac{\beta_1\bar{x}_4(a+m\bar{x}_4)}{(a+b\bar{x}_3+m\bar{x}_4)^2} & -\frac{m\beta_1\bar{x}_3\bar{x}_4}{(a+b\bar{x}_3+m\bar{x}_4)^2}
\end{pmatrix}
$$

.

Now the characteristic equation of system (5.6) around its interior equilibrium is

$det(V-\lambda I)=0$, i.e., $\lambda^4+h_1\lambda^3+h_2\lambda^2+h_3\lambda+h_4=0$.

where I represents an identity matrix of order 4 and

$$
\begin{aligned}
h_1 &= \frac{r}{K}\bar{x}_1-\alpha\frac{\bar{x}_1\bar{x}_2}{(a_1+\bar{x}_1)^2}-\frac{s_1\bar{x}_1+s_2\bar{x}_1}{a_1+\bar{x}_1}+m_2+m_1+\\
&\quad \bar{x}_4\frac{\{\beta(a+b\bar{x}_4)+\beta_1m\bar{x}_3\}}{(a+b\bar{x}_3+m\bar{x}_4)^2},\\
h_2 &= r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4-m_1m_2+\\
&\quad \frac{\beta\beta_1\bar{x}_3\bar{x}_4(a+b\bar{x}_3)(a+m\bar{x}_4)}{(a+b\bar{x}_3+m\bar{x}_4)^4}+\frac{\alpha a_1s_1\bar{x}_1\bar{x}_2}{(a_1+\bar{x}_1)^3},
\end{aligned}
$$

$$
\begin{aligned}
h_3 &= -(r_1r_2r_3 + r_1r_3r_4 + r_1r_2r_4 + r_2r_3r_4) + m_1m_2(r_1 + r_4) \\
&\quad -\frac{\beta\beta_1\bar{x}_3\bar{x}_4(a + b\bar{x}_3)(a + m\bar{x}_4)}{(a + b\bar{x}_3 + m\bar{x}_4)^4}(r_1 + r_2) - \frac{\alpha a_1 s_1\bar{x}_1\bar{x}_2}{(a_1 + \bar{x}_1)^3}(r_3 + r_4) \\
&\quad +m_1\alpha a_1 s_1\frac{\bar{x}_1\bar{x}_3}{(a_1 + \bar{x}_1)^3}, \\
h_4 &= \left\{r_1r_2 + \frac{\alpha a_1 s_1\bar{x}_1\bar{x}_2}{(a_1 + \bar{x}_1)^3}\right\}\left\{\frac{\beta\beta_1\bar{x}_3\bar{x}_4(a + b\bar{x}_3)(a + m\bar{x}_4)}{(a + b\bar{x}_3 + m\bar{x}_4)^4} + r_3r_4\right\} \\
&\quad -m_1m_2r_1r_4 - \alpha m_1 a_1 s_1\frac{\bar{x}_1\bar{x}_3 r_4}{(a_1 + \bar{x}_1)^3}, \\
r_1 &= -\frac{r}{K}\bar{x}_1 + \alpha\frac{\bar{x}_1\bar{x}_2}{(a_1 + \bar{x}_1)^2}, \quad r_2 = \frac{s_1\bar{x}_1}{a_1 + \bar{x}_1} - m_2, \\
r_3 &= \frac{s_2\bar{x}_1}{a_1 + \bar{x}_1} - m_1 - \beta\bar{x}_4\frac{a + b\bar{x}_4}{(a + b\bar{x}_3 + m\bar{x}_4)^2}, \quad r_4 = -\frac{m\beta_1\bar{x}_3\bar{x}_4}{(a + b\bar{x}_3 + m\bar{x}_4)^2}.
\end{aligned}
$$

Assume that, $h_1 = \frac{c_1}{K} - c_2, h_2 = \frac{c_3}{K} - c_4, h_3 = \frac{c_5}{K} - c_6, h_4 = \frac{c_7}{K} - c_8$. If we consider $\frac{1}{K} = K_1$, then, we have $h_2h_3 - h_1h_4 = Q_1 + Q_2K_1 + Q_3K_1^2 = \varphi(K_1)$ (say), where $Q_1 = c_4c_6 - c_2c_8$,

$Q_2 = c_2c_7 + c_1c_8 - c_5c_4 - c_3c_6, Q_3 = c_3c_5 - c_1c_7$ and $\begin{vmatrix} h_3 & h_4 & 0 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = Q_4 + Q_5K_1 + Q_6K_1^2 + Q_7K_1^3 = \psi(K_1)$ (say), where $Q_4 = c_2^2c_8 - c_6^2 - c_2c_4c_6$, $Q_5 = c_2c_4c_5 + c_2c_3c_6 + c_1c_4c_6 - c_2^2c_7 - 2c_1c_2c_8 + 2c_5c_6$, $Q_6 = c_1^2c_8 + 2c_1c_2c_7 - c_2c_3c_5 - c_1c_4c_5 - c_1c_3c_6 - c_5^2$, $Q_7 = c_1c_3c_5 - c_1^2c_7$.

Here all Q_j $(j = 1, 2, \ldots, 7)$ are functions of K_1, as the interior equilibrium point depends on K_1. But for a known parameter set, it is possible to find all the values of Q_j in terms of K_1. In that case, we assume that $\bar{K_1}$ is the common positive root of $\varphi(K_1) = 0$ and $\psi(K_1) = 0$. Now using Routh-Hurwitz criteria around at the interior equilibrium point, we can now state and prove the following theorems for analyzing the local stability of the system (5.6).

Theorem 5.5.1. *Consider that all* Q_j $(j = 1, 2, \ldots, 7)$, C, $h_2h_3 - h_1h_4$ *and* $\begin{vmatrix} h_3 & h_4 & 0 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix}$ *are positive. Then at the equilibrium point* $P(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ *of the system* (5.6) *is locally asymptotically stable.*

Proof: Using Routh-Hurwitz criterion, the conclusion becomes all eigenvalues of the system (5.6) around its interior equilibrium point $P(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ have negative real parts. Consequently the system will be locally asymptotically stable. This completes the proof of the theorem.

Lemma 4. *In Theorem 1, we have proved that for a known parameter set, we can find all Q_j in terms of K_1. In this case if $\bar{K}_1$ is the only common positive root. Then for $K_1 > \bar{K}_1$, $h_2h_3 - h_1h_4$ and $\begin{vmatrix} h_3 & h_4 & 0 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix}$ are positive. Again if $h_2h_3 - h_1h_4$ and $\begin{vmatrix} h_3 & h_4 & 0 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix}$ are negative then the system (5.6) will be unstable around $\bar{P}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$.*

Also if $\det \begin{pmatrix} h_3 & h_4 & 0 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{pmatrix} = 0$, then the system (5.6) undergoes through a bifurcation.

Again $K_1 = \frac{1}{K}$, thus we have, for $K < \bar{K}$, $h_2h_3 - h_1h_4$ and $\det \begin{pmatrix} h_3 & h_4 & 0 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{pmatrix}$ are positive. Again if $h_2h_3 - h_1h_4$ and $\det \begin{pmatrix} h_3 & h_4 & 0 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{pmatrix}$ are negative then the system (5.6) must be unstable around $\bar{P}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$. Also if $\begin{vmatrix} h_3 & h_4 & 0 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = 0$, then the system (5.6) undergoes through a bifurcation.

Theorem 5.5.2. *System (5.6) undergoes through Hopf bifurcation around $\bar{P}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ for $K_1 = \bar{K}_1$.*

Proof: For $K_1 = \bar{K}_1$, we have *det* $\begin{pmatrix} h_3 & h_4 & 0 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{pmatrix} = 0$ and then the eigenvalues of the system at $\bar{P}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ which can be expressed as $\lambda_{1,2} = \phi_1(K_1) \pm i\phi_2(K_1)$ and $\lambda_{3,4} = \phi_3(K_1) \pm i\phi_4(K_1)$. Now it is easy to show that $\frac{d\phi}{dK_1}$ is non-zero at the point $K_1 = \bar{K}_1$. Also we see $\phi(\bar{K}_1) = 0$. Hence, using the given conditions in Venkatsubramanian et al. (103), it is clear to show that our system (5.6) undergoes a Hopf bifurcation at its interior equilibrium point for the critical value of K_1, i.e., for $K_1 = \bar{K_1}$. This completes the proof of the theorem.

5.6 Optimal control

For a better economical point of the fishery as well as the economic development of human life, we optimize the profit in bionomic equilibrium state. The bionomic equilibrium is a combination of economic equilibrium as well as biological equilibrium. Assuming that p is a constant price per unit biomass, c is a constant cost of harvesting effort. Then the net economic revenue calculated from the fishery is $P(x_1, x_2, x_3, x_4, E, t)$ = the total revenue obtained by selling the harvested biomass − the total cost for effort devoted to harvesting $= pqEx_4 - cE$, then we consider an integral J which is a continuous time-stream of revenues as follows:

$$J = \int_0^\infty e^{-\delta t} P(x_1, x_2, x_3, x_4, E, t)dt, \tag{5.11}$$

where δ denotes the instantaneous annual rate of discount (82). Our problem is to maximize J subject to the state equations (5.6) using *Pontryagin's maximum principle* (80). The control variable $E(t)$ is subject to the constraint set, $0 \le E \le E_{max}$. At first, we construct the corresponding Hamiltonian function of this optimal control problem as stated below:

$$\begin{aligned} H = \quad & e^{-\delta t}(pqz - c)E + \lambda_1 \left\{ rx_1 \left(1 - \frac{x_1}{K}\right) - \alpha \frac{x_1 x_2}{a_1 + x_1} - d_1 x_1 \right\} \\ & + \lambda_2 \left(s_1 \frac{x_1 x_2}{a_1 + x_1} + m_1 x_3 - m_2 x_2 \right) \\ & + \lambda_3 \left(s_2 \frac{x_1 x_3}{a_1 + x_1} - m_1 x_3 + m_2 x_2 - \beta \frac{x_3 x_4}{a + bx_3 + mx_4} \right) \\ & + \lambda_4 \left(\beta_1 \frac{x_3 x_4}{a + bx_3 + mx_4} - d_3 x_4 - qEx_4 \right), \end{aligned} \tag{5.12}$$

where $\lambda_i \;\; (i = 1, 2, 3, 4)$ are called the adjoint variables.

By Pontryagin's maximum principle, the adjoint equations are described as follows:

$$\begin{aligned}
\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x_1} &= -\lambda_1\left\{r - 2\frac{r}{K}x_1 - \frac{a_1\alpha x_2}{(a_1+x_1)^2} - d_1\right\} - \lambda_2 a_1 s_1\frac{x_2}{(a_1+x_1)^2} \\
&\quad -\lambda_3\frac{a_1 s_2 x_3}{(a_1+x_1)^2}, \qquad (5.13)\\
\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial x_2} &= \lambda_1\frac{\alpha x_1}{a_1+x_1} - \lambda_2\frac{s_1 x_1}{a_1+x_1} + \lambda_2 m_2 - \lambda_3 m_2, \qquad (5.14)\\
\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial x_3} &= -\lambda_2 m_1 + \lambda_3\left(m_1 + \beta x_4\frac{(a+mx_4)}{(a+bx_3+mx_4)^2} - \frac{s_2 x_1}{a_1+x_1}\right)\\
&\quad -\lambda_4\beta_1 x_4\frac{(a+mx_4)}{(a+bx_3+mx_4)^2}, \qquad (5.15)\\
\frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial x_4} &= -pqe^{-\delta t} + \lambda_3\beta x_3\frac{(a+bx_3)}{(a+bx_3+mx_4)^2} - \lambda_4\beta_1 x_3\frac{(a+bx_3)}{(a+bx_3+mx_4)^2}\\
&\quad +\lambda_4(d_3+qE). \qquad (5.16)
\end{aligned}$$

Now an optimal equilibrium solution has been derived of the problem at the interior equilibrium point, $\bar{P}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$. Then from equations (5.13) to (5.16), we get

$$\frac{d\lambda_1}{dt} = \lambda_1 A_1 - \lambda_2 A_2 - \lambda_3 A_3, \qquad (5.17)$$

$$\frac{d\lambda_2}{dt} = \lambda_1 A_4 - \lambda_2 A_5 - \lambda_3 A_6, \qquad (5.18)$$

$$\frac{d\lambda_3}{dt} = -\lambda_2 A_7 + \lambda_3 A_8 - \lambda_4 A_9, \qquad (5.19)$$

$$\frac{d\lambda_4}{dt} = -pqEe^{-\delta t} + \lambda_3 A_{10} - \lambda_4 A_{11}. \qquad (5.20)$$

where $A_1 = 2\frac{r}{K}\bar{x}_1 + \frac{a_1\alpha\bar{x}_2}{(a_1+\bar{x}_1)^2} + d_1 - r$, $A_2 = a_1 s_1\frac{\bar{x}_2}{(a_1+\bar{x}_1)^2}$, $A_3 = a_1 s_2\frac{\bar{x}_3}{(a_1+\bar{x}_1)^2}$, $A_4 = \alpha\frac{\bar{x}_1}{a_1+\bar{x}_1}$, $A_5 = \frac{s_1\bar{x}_1}{a_1+\bar{x}_1} - m_2$, $A_6 = m_2$, $A_7 = m_1$, $A_8 = m_1 + \beta\bar{x}_4\frac{(a+m\bar{x}_4)}{(a+b\bar{x}_3+m\bar{x}_4)^2} - \frac{s_2\bar{x}_1}{a_1+\bar{x}_1}$, $A_9 = \beta_1\bar{x}_4\frac{(a+m\bar{x}_4)}{(a+b\bar{x}_3+m\bar{x}_4)^2}$, $A_{10} = \beta\bar{x}_3\frac{(a+b\bar{x}_3)}{(a+b\bar{x}_3+m\bar{x}_4)^2}$, $A_{11} = \beta_1\bar{x}_3\frac{(a+b\bar{x}_3)}{(a+b\bar{x}_3+m\bar{x}_4)^2} + \lambda_4(d_3+qE)$.
Using the values of λ_1, λ_2 and λ_3 in equations (5.17) to (5.20), we get Hamiltonian function by which we can calculate the optimality using Pontryagin's maximum principle. The numerical illustration of this system is described in the next Section i.e., Section 5.7.

5.7 Numerical simulation

Consider the discussed phenomenon with their relative factors, we justify the analytical results of the proposed system with the hypothetical data set. The results of the simulation experiments are taken using the MATLAB 7.10. Choose the parameter set arbitrarily as $P_1 = \{r = 2.5, \alpha = 1.3, a_1 = 5, d_1 = 0.03, m_1 = 0.2, m_2 = 0.5, s_1 = 0.6, s_2 = 0.6, \beta = 0.8, \beta_1 = 0.2, d_3 = 0.2, a = 5, b = 0.1, m = 0.05, q = 0.2, E = 0.76\}$. Now K can be taken as two cases:
(i) $K(= 14) < \bar{K}$ and (ii) $K(= 16) > \bar{K}$.
Two cases has been considered as follows:
Case (i), the Figures 5.2, 5.3, 5.4 and 5.5 have been drawn by choosing the parameter set P_1. From Figure 5.2, we see that the proposed system is stable after certain time. Figure 5.3 shows the phase space diagram of the system with respect to x_1, x_2 and x_3. Similarly, Figures 5.4 and 5.5 show the phase diagram of the system (5.6) with respect to x_2, x_3, x_4 and x_1, x_3, x_4 respectively.
Case (ii), the Figures 5.6, 5.7, 5.8 and 5.9 have been drawn by taking the parameter set P_1. From Figure 5.6, we observe that the proposed system is unstable. Figure 5.7 shows the phase space diagram of the system with respect to x_1, x_2 and x_3. Similarly, Figures 5.8 and 5.9 show the phase diagram of the system (5.6) with respect to x_2, x_3, x_4 and x_1, x_3, x_4 respectively.

5.8 Chapter Summary

In this chapter, the mathematical model has been formulated on prey-predator relationship in a reserved region. To the best of the knowledge, this work is the first attempt to consider reserve region for the predator population. Based on the discussed phenomena, a food chain has been considered with a reserve region of predator. A study of potential effects of generalist predator in a predator and predator in a prey has been introduced with Beddington-DeAngelis functional response and Holling type II functional response respectively. The density-dependent mortality rates for prey and generalist predator have been assumed. Depending on different environments, we have considered different consumption rates of predator populations. The equilibria of the proposed system have been determined with the discussions on local stability for the system. The proposed model has been illustrated with a numerical simulation and analyzed through the geometrical figures.

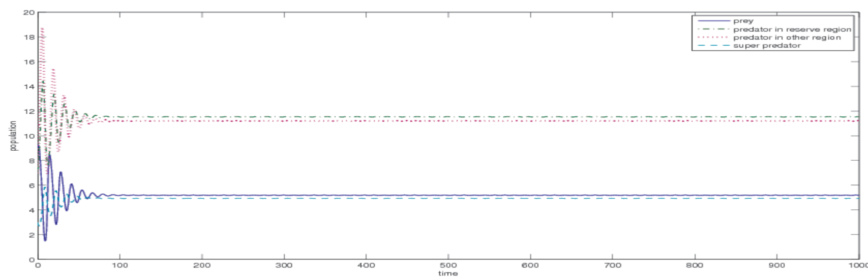

Figure 5.2: Graphical representation of the system (5.6) for $K(=14) < \bar{K}$ i.e., $K_1 = \frac{1}{14} > \bar{K}_1$.

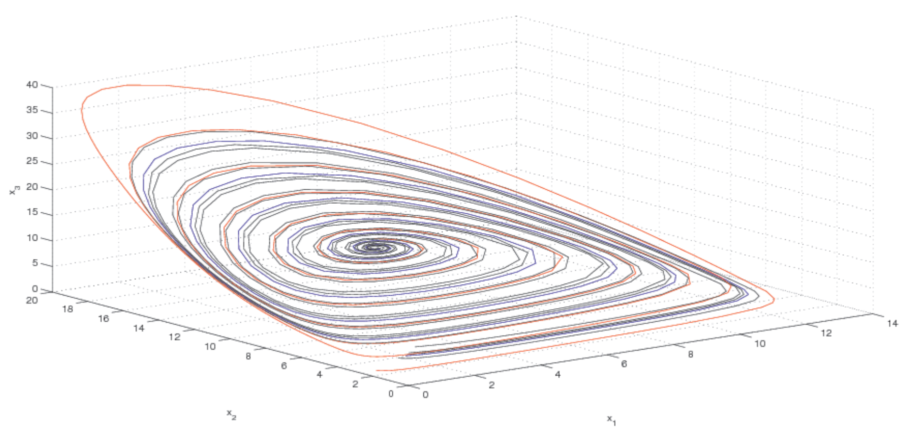

Figure 5.3: Phase space diagram of the system (5.6) with the parameter set P_1 and $K(=14) < \bar{K}$ i.e., $K_1 = \frac{1}{14} > \bar{K}_1$ with respect to x_1, x_2 and x_3.

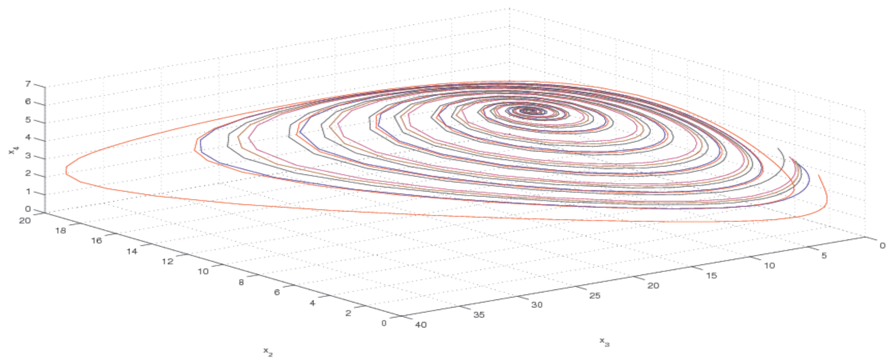

Figure 5.4: Phase space diagram of the system (5.6) with the parameter set P_1 and $K(=14) < \bar{K}$ i.e., $K_1 = \frac{1}{14} > \bar{K}_1$ with respect to x_2, x_3 and x_4.

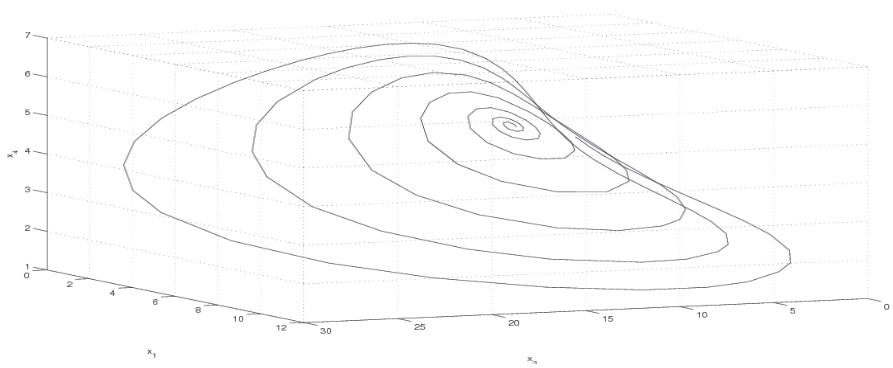

Figure 5.5: Phase space diagram of the system (5.6) with the parameter set P_1 and $K(=14) < \bar{K}$ i.e., $K_1 = \frac{1}{14} > \bar{K}_1$ with respect to x_1, x_3 and x_4.

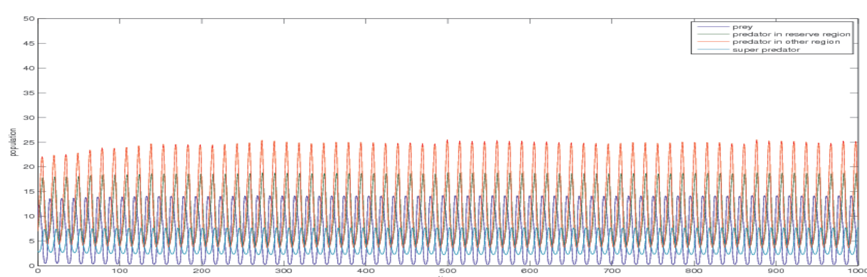

Figure 5.6: Graphical representation of the system (5.6) for $K(=16) > \bar{K}$ i.e., $K_1 = \frac{1}{16} < \bar{K}_1$.

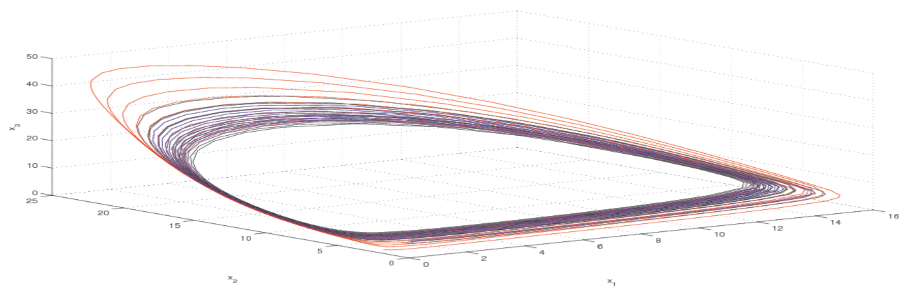

Figure 5.7: Phase space diagram of the system (5.6) with the parameter set P_1 and $K(=16) > \bar{K}$ i.e., $K_1 = \frac{1}{16} < \bar{K}_1$ with respect to x_1, x_2 and x_3.

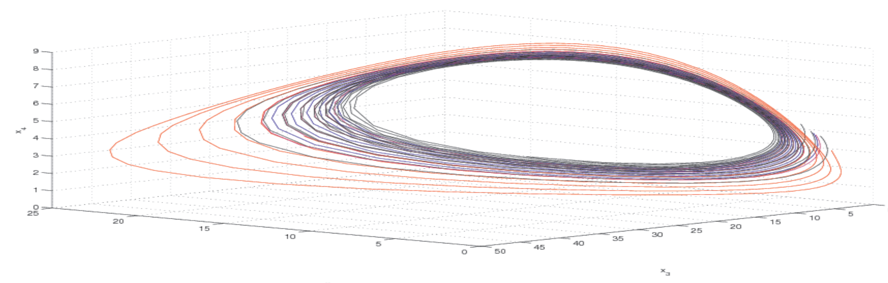

Figure 5.8: Phase space diagram of the system (5.6) with the parameter set P_1 and $K(=16) > \bar{K}$ i.e., $K_1 = \frac{1}{16} < \bar{K}_1$ with respect to x_2, x_3 and x_4.

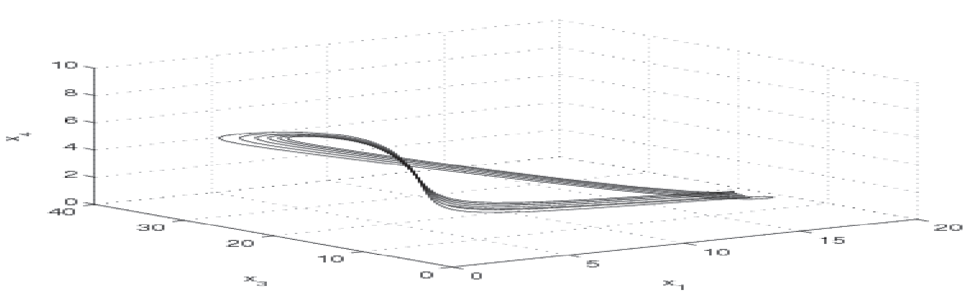

Figure 5.9: Phase space diagram of the system (5.6) with the parameter set P_1 and $K(=16) > \bar{K}$ i.e., $K_1 = \frac{1}{16} < \bar{K}_1$ with respect to x_1, x_3 and x_4.