Chapter 4

Analysis of prey-predator three species fishery model with harvesting including prey refuge and migration*

In this chapter, a prey-predator system with Holling type II functional response for the predator population including prey refuge region has been analyzed. Also a harvesting effort has been considered on predator population. The density-dependent mortality rate for the prey, predator and specialist predator has been considered. The equilibria of the proposed system have been determined. Local and global stabilities for the system have been discussed. The analytic approach have been used to derive the global asymptotic stabilities of the system. The maximal predator per capita consumption rate has been considered as a bifurcation parameter to evaluate Hopf bifurcation in the neighborhood of interior equilibrium point. Also, fishing effort have been used to harvest predator population of the system as a control to develop a dynamic framework to investigate the optimal utilization of the resource, sustainability properties of the stock and the resource rent is earned from the resource. Finally, some numerical simulations have been presented to verify the analytic results and the system has been analyzed through graphical illustrations.

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4.1 Introduction

The relationship between prey and predator is natural phenomena for universal existence in our ecological system. There are different types of ecological systems. Interactions of prey and their predators are one of the common and well known ecological systems. This ecological system is one of the important field in the study of mathematical ecology.

Most of the researchers have concentrated their attention on two species with prey-predator including prey refuge with Holling type II functional response; but we consider three species prey-predator-specialist predator interactions with prey refuge region with Holling type II functional response and also we introduce the predator harvesting effort in this chapter which is more realistic to analyze the whole system. Thus it can be concluded that prey refuge is a natural phenomenon of prey population as they always want to escape from predator. So, this phenomenon has been considered in this present chapter too. These are the main motivations of this chapter.

A real-life example of our proposed problem has been considered to show the feasibility and effectiveness of this chapter.

A good example is an ecological system of the lake where fishes (*Puntiun ticto, Amhypharyngodon mola* etc) are living on zooplankton (*Mesocyclops leuckrti, Daphnia hyalina* etc). Also snakes(*Bungarus fasciatus, Xenochrophis piscator*etc) eat these fishes. So in such a system zooplankton can be considered as prey, fish can be considered as predator and snakes can be considered as specialist predator. Fishes are also harvested, so the density of the predator population decreases. There are some areas where the fishes cannot enter. So, the zooplanktons are safe from that region, which may be considered as refuge region. So such system is a good example for the proposed system.

4.2 Notation

Table-4.2.1: Description of the parameters.

Parameter	Description of the parameters
x	Population of prey in refuge region at time t
y	Population of prey in predatory region at time t
z	Population of predator at time t
u	Population of specialist predator at time t
r	Intrinsic birth rate of prey in refuge region
K	Environmental carrying capacity of the prey in refuge region
s	Intrinsic birth rate of prey in predatory region
L	Environmental carrying capacity of the prey in predatory region
a, b	Half saturation constants
d_1	Natural death rate of prey
d_2	Natural death rate of predator
d_3	Natural death rate of specialist predator
n	Predator's consumption rate on prey
n_1	specialist predator 's consumption rate on predator
σ_1	Per unit migration of the prey population in refuge region
σ_2	Per unit emigration of the prey population in refuge region
q	Catchability co-efficient
E	Fishing effort for harvesting the specialist predator population

4.3 Formulation of the model

A prey-predator model has been considered with prey refuge and it is assumed that only predator population is harvested. Generally, the birth rate of a prey in the refuge region and predatory region will be different due to available of food sources and other considerable factors. For this reason, different birth rates have been considered of the prey in two different regions. The presence of generalist predator has been included in this model. Most of the researchers have studied either on a fixed number of prey population in refuge region or on a proportion of prey population in the refuge region. But the carrying capacity of the refuge and predatory regions are different. Considering all these factors, total prey populations are broken into two parts: first is refuge region with density x at time t and second is predatory region with density y at time t. Again we assume that, at time t, the predator and generalist predator populations are denoted by z and u respectively. We consider in the proposed model that, the predator population consumes the prey population with Holling type-II functional response or Michaelis-Menten functional responce which is $\frac{m}{a+y}$ where m denotes the maximal predator per capita consumption rate, i.e., the maximum number of prey population can be eaten by a predator in each time unit and the half capturing saturation constant denoted by a i.e., the number of prey necessary to obtain one-half of the maximum rate m. Also, the specialist predator population consumes the predator population with Holling type-II functional response which is $\frac{m_1}{b+z}$ where m_1 denotes the maximal generalist predator per capita consumption rate, i.e., the maximum number of predator population can be eaten by a specialist predator in each time unit and the half capturing saturation constant denoted by b i.e., the number of predator necessary to obtain one-half of the maximum rate m_1. Considering that, in predatory region, interaction may occur to prey population in predatory region and the specialist predator population can interact with predator population. Again, considering that intra-specific competitions are occurred between the predator as well as specialist predator also, for their existence. Based on the assumptions that, the system of differential equation is described as follows:

$$\left.\begin{aligned}
\frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\sigma_1 x+\sigma_2 y-d_1 x\\
\frac{dy}{dt} &= sy\left(1-\frac{y}{L}\right)+\sigma_1 x-\sigma_2 y-d_1 y-\frac{myz}{a+y}\\
\frac{dz}{dt} &= \frac{nyz}{a+y}-d_2 z-\gamma z^2-\frac{m_1 zu}{b+z}-h(t)\\
\frac{du}{dt} &= \frac{n_1 zu}{b+z}-d_3 u-\delta u^2
\end{aligned}\right\} \tag{4.1}$$

with initial conditions $x(0)\geq 0, y(0)\geq 0, z(0)\geq 0, u(0)\geq 0$, where natural death rate of prey, predator and specialist predator population denoted as d_1, d_2 and d_3 respectively and intra-specific competition coefficient of predator and specialist predator denoted as γ and δ respectively. Again the predator population consumes prey at the rate n $(0 < n \leq m)$ and the specialist predator population consumes predator at the rate n_1 $(0 < n_1 \leq m_1)$.

Considering the harvest rate as $h(t) = qEz$, the system (4.1) becomes

$$\left.\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - d_1 x\\ \frac{dy}{dt} &= sy\left(1-\frac{y}{L}\right) + \sigma_1 x - \sigma_2 y - d_1 y - \frac{myz}{a+y}\\ \frac{dz}{dt} &= \frac{nyz}{a+y} - d_2 z - \gamma z^2 - \frac{m_1 zu}{b+z} - qEz\\ \frac{du}{dt} &= \frac{n_1 zu}{b+z} - d_3 u - \delta u^2\end{aligned}\right\} \tag{4.2}$$

with initial conditions $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0, u(0) \geq 0$.

4.4 Equilibria and their existence criteria

Different equilibrium points of the system have been analyzed. It is seen that the system has four possible equilibria. Now these equilibria are given as follows:

(i) The equilibrium point $B_0(0,0,0,0)$ which is trivial.

(ii) The equilibrium point $B_1(x_1, y_1, 0, 0)$ without any effect of the predator and specialist predator, where x_1 is the positive root of the following equation:

$$\frac{sr^2}{LK^2\sigma_2^2}x_1^3 + \left[\frac{s}{L\sigma_2^2}(\sigma_1+d_1-r)^2 + \frac{r}{\sigma_2 K}(\sigma_2+d_1-s)\right]x_1 + \frac{1}{\sigma_2}\{(\sigma_1 + d_1 - r)(\sigma_2+d_1-s) - \sigma_1\sigma_2\} + \frac{2sr}{LK\sigma_2^2}(\sigma_1+d_1-r)x_1^2 = 0 \tag{4.3}$$

and

$$y_1 = \frac{x_1}{\sigma_2}[(\sigma_1+d_1-r) + \frac{r}{K}x_1] \tag{4.4}$$

$$x_1 > 0, y_1 > 0$$

(iii) The equilibrium point $B_2(x^*, y^*, z^*, 0)$ without any effect on specialist predator, where x^* is the positive root of the following equation:

$$R_8 + R_7 x + R_6 x^2 + R_5 x^3 + R_4 x^4 + R_3 x^5 + R_2 x^6 - R_1 x^7 = 0$$

and

$$y^* = \frac{x^*}{\sigma_2}[d_1 + \sigma_1 - r + \frac{r}{K}x^*]$$

$$z^* = \frac{1}{\gamma}\left\{\frac{ny^*}{a+y^*} - d_2 - qE\right\}$$

So, the sufficient conditions for the system (4.2) has a specialist predator free equilibrium point are $\frac{r}{K}x^* > r - (d_1+\sigma_1)$, $ny^* > (d_2+qE)(a+y^*)$ and all

$R_i > 0$, $(i = 1, 2, ..., 8)$. For detailed analysis, we refer to Appendix.
(iv) The interior equilibrium point $B_3(\overline{x}, \overline{y}, \overline{z}, \overline{u})$ where $\overline{x}, \overline{y}, \overline{z}$ and $\overline{u}$ are the positive roots of the equation $\dot{x} = \dot{y} = \dot{z} = \dot{u} = 0$.
It may be noted that

$$\overline{u} = \frac{1}{\delta}[\frac{n_1\overline{z}}{b+\overline{z}} - d_3] \tag{4.5}$$

$$\overline{z} = \frac{a+\overline{y}}{a\overline{y}}\left[s\overline{y}\left(1-\frac{\overline{y}}{L}\right) + \sigma_1\overline{x} - \sigma_2\overline{y} - d_1\overline{y}\right] \tag{4.6}$$

$$\overline{y} = \frac{\overline{x}}{\sigma_2}[d_1 + \sigma_1 - r + \frac{r}{K}\overline{x}] \tag{4.7}$$

and $\overline{x}$ is the positive root of the following equation

$$T_{18} + T_{17}x + T_{16}x^2 + T_{15}x^3 + T_{14}x^4 + T_{13}x^5 + T_{12}x^6 + T_{11}x^7 + T_{10}x^8 + T_9x^9$$
$$+T_8x^{10} + T_7x^{11} + T_6x^{12} + T_5x^{13} + T_4x^{14} + T_3x^{15} + T_2x^{16} - T_1x^{17} = 0 \tag{4.8}$$

So, $(n_1 - d_3)\overline{z} > d_3b$, $s\overline{y} + \sigma_1\overline{x} > (\sigma_2 + d_1 + \frac{s\overline{y}}{L})\overline{y}$, $d_1 + \sigma_1 + \frac{r\overline{x}}{K} > r$ and all $T_i > 0$, $(i = 1, 2, 3, ..., 18)$ are the sufficient conditions for the system with a positive interior equilibrium point. For detailed analysis, we refer to Appendix.

4.5 Boundedness

Theorem 4.5.1. *Solutions of the system* (4.2) *are bounded.*

Proof: From the first two equations of system (4.2), it is seen that carrying capacity of total prey population is $K+L$. For $\epsilon > 0$, we have $x+y \leq K+L+\epsilon$ as $t \to \infty$. Thus, we consider $x \leq K+\epsilon_1 \;\; as \;\; t \to \infty$ and $y \leq L+\epsilon_2 \;\; as \;\; t \to \infty$ where $\epsilon_1 > 0$ and $\epsilon_2 > 0$.
Now, let $P = x + y + \frac{m}{n}z + \frac{mm_1}{nn_1}u$
Then we get,

$$\frac{dP}{dt} \leq rx + sy - \frac{\rho m}{n}z - \frac{\rho mm_1}{nn_1}u, \qquad \text{where} \quad \rho = \min\{d_2 + qE, d_3\}$$

$$i.e, \quad \frac{dP}{dt} \leq -\rho P + (r+\rho)x + (s+\rho)y$$

$$i.e, \quad \frac{dP}{dt} \leq -\rho P + (r+\rho)(K+\epsilon_1) + (s+\rho)(L+\epsilon_2)$$

$$i.e, \quad \frac{dP}{dt} + \rho P \leq I, \quad where \quad I = (r+\rho)(K+\epsilon_1) + (s+\rho)(L+\epsilon_2)$$

On integrating both sides of above equation and applying the theorem (9), we obtain:
$0 < P \leq \frac{I}{\rho}(1 - e^{-\rho t}) + P[x(0), y(0), z(0), u(0)]$ as $t \to \infty$. Also, we have $0 < P \leq \frac{I}{\rho} + P(0)$. From above analysis, we conclude that the solution space (x, y, z, u) is bounded in the specified region. Thus, the theorem holds.

4.6 Local Stability

In this section, the stability of the system (4.2) has been analyzed at trivial, semi-trivial and interior equilibrium points.

Theorem 4.6.1. *The system* (4.2) *is locally asymptotically stable at the trivial equilibrium point* $B_0(0, 0, 0, 0)$ *if* $r+s < (\sigma_1+\sigma_2)+2d_1$ *and* $rs > \sigma_2 r+\sigma_1 s+d_1^2$.

Proof: The characteristic equation of the system (4.2) around its trivial equilibrium point is given by:
$(\lambda + d_3)(\lambda + d_2 + \gamma + qE)[\lambda^2 - (M + N)\lambda + MN - \sigma_1\sigma_2] = 0$
where $M = r - \sigma_1 - d_1$, $N = s - \sigma_2 - d_1$
Clearly, for $M + N < 0$ and $MN - \sigma_1\sigma_2 > 0$ i.e, $r + s < \sigma_1 + \sigma_2 + 2d_1$ and $rs > \sigma_2 r + \sigma_1 s + d_1^2$, all the eigen values of the system become negative at $B_0(0, 0, 0, 0)$, thus the system is locally asymptotically stable around $B_0(0, 0, 0, 0)$.
From the above conditions, it is seen that the trivial equilibrium is locally asymptotically stable if the birth rates of prey is less than the sum of death rates and migration rates of prey, i.e, all the species are going to extinct forever, if the birth rates of prey are less than the sum of death rate and migration rates of prey.

Theorem 4.6.2. *The system* (4.2) *is locally asymptotically stable at the predator and specialist predator free equilibrium point* $B_1(x_1, y_1, 0, 0)$ *if* $\frac{ny_1}{a+y_1} < d_2 + \gamma + qE$, $\quad (r + s) < 2d_1 + \sigma_1 + \sigma_2 + 2(\frac{r}{K}x_1 + \frac{s}{L}y_1)$ *and* $(r - \frac{2r}{K}x_1 - \sigma_1 - d_1)(s - \frac{2s}{L}y_1 - \sigma_2 - d_1) > \sigma_1\sigma_2$

Proof: The characteristic equation of the system (4.2) at $B_1(x_1, y_1, 0, 0)$ is given by:

$(\lambda + d_3)(\lambda - \frac{ny_1}{a+y_1} + d_2 + \gamma + qE)[\lambda^2 - (M_1 + N_1)\lambda + M_1N_1 - \sigma_1\sigma_2] = 0$
where $M_1 = r - \frac{2r}{K}x_1 - \sigma_1 - d_1$, $N_1 = s - \frac{2s}{L}y_1 - \sigma_2 - d_1$
Clearly, $\frac{ny_1}{a+y_1} < d_2 + \gamma + qE$
for $M_1 + N_1 < 0$ i.e, $(r + s) < 2d_1 + \sigma_1 + \sigma_2 + 2(\frac{r}{K}x_1 + \frac{s}{L}y_1)$
and $M_1N_1 - \sigma_1\sigma_2 > 0$ i.e, $(r - \frac{2r}{K}x_1 - \sigma_1 - d_1)(s - \frac{2s}{L}y_1 - \sigma_2 - d_1) > \sigma_1\sigma_2$, all the eigen values of the system become negative at $B_1(x_1, y_1, 0, 0)$, thus the system is locally asymptotically stable around $B_1(x_1, y_1, 0, 0)$.
From the above conditions, it is seen that the predator and specialist predator free equilibrium is locally asymptotically stable if the birth rates of prey is less than the sum of death rates, migration rates of prey and twice times ratio of birth rates all over prey populations and carrying capacity of the system, i.e, all the species are going to extinct forever, if the birth rates of prey are less than the sum of death rates, migration rates of prey and twice times ratio of birth rates all over prey populations and carrying capacity of the system.
Now the characteristic equation of system (4.2) around its interior equilibrium reduces to
$\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D = 0$
where

$$A = \frac{r}{K}\bar{x} + \frac{s}{L}\bar{y} + \gamma\bar{z} + \delta\bar{u} + \sigma_1\frac{\bar{x}}{\bar{y}} + \sigma_2\frac{\bar{y}}{\bar{x}} - \frac{m\bar{y}\bar{z}}{(a+\bar{y})^2} - \frac{m_1\bar{z}\bar{u}}{(b+\bar{z})^2}$$

$$B = -\sigma_1\sigma_2 + \frac{m_1n_1b\bar{z}\bar{u}}{(b+\bar{z})^3} + \frac{mna\bar{y}\bar{z}}{(a+\bar{y})^3} + (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4)$$

$$C = -(r_1r_2r_3 + r_1r_3r_4 + r_1r_2r_4 + r_2r_3r_4) + \sigma_1\sigma_2(r_3 + r_4) - \frac{m_1n_1b\bar{z}\bar{u}}{(b+\bar{z})^3}(r_1 + r_2)$$
$$-\frac{mna\bar{y}\bar{z}}{(a+\bar{y})^3}(r_1 + r_4)$$

$$D = r_1r_2r_3r_4 - \sigma_1\sigma_2r_3r_4 + \frac{m_1n_1b\bar{z}\bar{u}}{(b+\bar{z})^3}(r_1r_2 + \sigma_1\sigma_2) + \frac{amn\bar{y}\bar{z}}{(a+\bar{y})^3}r_1r_4$$

$$r_1 = -\frac{r}{K}\bar{x} - \sigma_2\frac{\bar{y}}{\bar{x}}, \quad r_2 = -\frac{s}{L}\bar{y} - \sigma_1\frac{\bar{x}}{\bar{y}} + \frac{m\bar{y}\bar{z}}{(a+\bar{y})^2}$$

$$r_3 = -\gamma\bar{z} + \frac{m_1\bar{z}\bar{u}}{(b+\bar{z})^2}, \quad r_4 = -\delta\bar{u}$$

Let us consider, $A = c_1 - nc_2, B = c_3 - nc_4, C = c_5 - nc_6, D = c_7 - nc_8$. Then we have $BC - AD = Q_1n^2 + Q_2n + Q_3 = \varphi(n)$ (say), where $Q_1 = c_4c_6 -$

$c_2c_8, Q_2 = c_2c_7 + c_1c_8 - c_5c_4 - c_3c_6, Q_3 = c_3c_5 - c_1c_7$ and $det \begin{pmatrix} C & D & 0 \\ A & B & C \\ 0 & 1 & A \end{pmatrix} =$ $Q_4n^3 + Q_5n^2 + Q_6n + Q_7 = \psi(n)$ (say), where $Q_4 = c_2^2c_8 - c_2c_4c_6,\ Q_5 = c_2c_4c_5 + c_2c_3c_6 + c_1c_4c_6 - c_2^2c_7 - 2c_1c_2c_8 - c_6^2,\ Q_6 = c_1^2c_8 + 2c_1c_2c_7 + 2c_5c_6 - c_2c_3c_5 - c_1c_4c_5 - c_1c_3c_6,\ Q_7 = c_1c_3c_5 - c_1^2c_7 - c_5^2.$

Here all Q_j, $(j = 1 \ \ to \ \ 7)$ are functions of n since the interior equilibrium depends on n. But for a known parameter set it is possible to find all the values of Q_j in terms of n. In that case, we assume that $\overline{n}$ is the common positive root of $\varphi(n) = 0$ and $\psi(n) = 0$. Now using Routh-Hurwitz criteria around the interior equilibrium point, we can state and prove the following theorem for the local asymptotic stability of the system (4.2).

Theorem 4.6.3. *Assuming all Q_j $(j = 1 \ \ to \ \ 7)$, C, $BC-AD$ and $\begin{vmatrix} C & D & 0 \\ A & B & C \\ 0 & 1 & A \end{vmatrix}$ be positive. Then the equilibrium point $B_3(\bar{x}, \bar{y}, \bar{z}, \bar{u})$ of the system* (4.2) *is locally asymptotically stable.*

Proof: Using Routh-Hurwitz criterion the conclusion becomes all eigenvalues of the system (4.2) around its interior equilibrium point $B_3(\bar{x}, \bar{y}, \bar{z}, \bar{u})$ have negative real parts. Consequently the system will be locally asymptotically stable. Hence the theorem.

Lemma 2. *In the above theorem, we have proved that for a known parameter set we can find all Q_j in terms of n. In this case if $\bar{n}$ is the only common positive root. Then for $n > \bar{n}$, $BC - AD$ and det $\begin{pmatrix} C & D & 0 \\ A & B & C \\ 0 & 1 & A \end{pmatrix}$ are positive. Again if $BC - AD$ and det $\begin{pmatrix} C & D & 0 \\ A & B & C \\ 0 & 1 & A \end{pmatrix}$ are negative then the system* (4.2) *must be unstable around $B_3(\bar{x}, \bar{y}, \bar{z}, \bar{u})$. Also if det $\begin{pmatrix} C & D & 0 \\ A & B & C \\ 0 & 1 & A \end{pmatrix} = 0$ then the system* (4.2) *undergoes through a bifurcation. In next theorem, we describe about Hopf bifurcation.*

Theorem 4.6.4. *The system* (4.2) *follows Hopf bifurcation about the point* $B_3(\bar{x}, \bar{y}, \bar{z}, \bar{u})$ *for* $n = \bar{n}$.

Proof: For $n = \bar{n}$,
we have $det \begin{pmatrix} C & D & 0 \\ A & B & C \\ 0 & 1 & A \end{pmatrix} = 0$ and then the eigenvalues of the system at $B_3(\bar{x}, \bar{y}, \bar{z}, \bar{u})$ can be represented as $\lambda_{1,2} = \pm i\sqrt{C_1}$ and $\lambda_{3,4} = \pm i\sqrt{C_2}$.
Considering $\lambda_{1,2} = \phi_1(n) \pm i\phi_2(n)$ and $\lambda_{3,4} = \phi_3(n) \pm i\phi_4(n)$. Now it is obvious to show that $\frac{d\phi}{dn} \neq 0$ at the point $n = \bar{n}$ where ϕ represents ϕ_1 and ϕ_3. Again, we have $\phi(\bar{n}) = 0$. Therefore, it is obvious to show that our system (4.2) follows a Hopf bifurcation at its interior equilibrium for the critical value of n, i.e, for $n = \bar{n}$, with the help of given conditions (103). So, the theorem is obvious.
Prey-predator models with constant parameters are often found to approach a steady state in which the species coexist in equilibrium. But if parameters used in the model are changed, other types of dynamical behavior may occur and the critical parameter values at which such transitions happen are called bifurcation points. The purpose of this study is to determine the stability behavior of the system in presence of different density-dependent factors of the prey-predator interactions. To study the transition of the system with respect to the small changes in the density dependent factors, we consider, n as bifurcation parameter and n represent the critical value or the bifurcating value of the concerned bifurcation parameter.

4.7 Global Stability

We now state and prove the globally asymptotically stability at an interior equilibrium point with the help of Lyapunov function.

Theorem 4.7.1. *The system* (4.2) *will be globally asymptotically stable at an interior equilibrium point* $B_3(\bar{x}, \bar{y}, \bar{z}, \bar{u})$*, if the sufficient conditions that no trajectory of the solution path meets the coordinate axes and* $R_1(0) > 0, R_2(0) > 0$ *where* $R_1(y) = \frac{s}{L} - \frac{1}{2}(\frac{\sigma_2}{\bar{x}} + \frac{\sigma_1}{\bar{y}}) - \frac{m\bar{z}}{(a+y)(a+\bar{y})}$ *and* $R_2(z) = \gamma - \frac{m_1\bar{u}}{(b+z)(b+\bar{z})}$,

$r \geq \frac{K}{2}(\frac{\sigma_2}{\bar{x}} + \frac{\sigma_1}{\bar{y}})$ *hold simultaneously.*

Proof: Let us choose a Lyapunov function which is defined as follows
$V(x,y,z,u) = \int_{\bar{x}}^{x} \frac{x-\bar{x}}{x}dx + p_1\int_{\bar{y}}^{y} \frac{y-\bar{y}}{y}dy + p_2\int_{\bar{z}}^{z} \frac{z-\bar{z}}{z}dz + p_3\int_{\bar{u}}^{u} \frac{u-\bar{u}}{u}du$
where $p_i(i = 1,2,3)$ are suitable positive constants to be determined in the following subsequent steps.
Time derivative of the equation along the solutions of the system(4.2) is given by

$$
\begin{aligned}
\frac{dV}{dt} = & \quad \frac{x-\bar{x}}{x}\frac{dx}{dt} + p_1\frac{y-\bar{y}}{y}\frac{dy}{dt} + p_2\frac{z-\bar{z}}{z}\frac{dz}{dt} + p_3\frac{u-\bar{u}}{u}\frac{du}{dt} \\
= & \quad (x-\bar{x})\left\{r\left(1-\frac{x}{K}\right) - \sigma_1 + \sigma_2\frac{y}{x} - d_1\right\} \\
& +p_1(y-\bar{y})\left\{s\left(1-\frac{y}{L}\right) + \sigma_1\frac{x}{y} - \sigma_2 - d_1 - \frac{mz}{a+y}\right\} \\
& +p_2(z-\bar{z})\left\{\frac{ny}{a+y} - d_2 - \gamma z - \frac{m_1 u}{b+z} - qE\right\} \\
& +p_3(u-\bar{u})\left\{\frac{n_1 z}{b+z} - d_3 - \delta u\right\}
\end{aligned}
$$

Again at the interior equilibrium point $B_3(\bar{x},\bar{y},\bar{z},\bar{u})$, $\dot{x} = \dot{y} = \dot{z} = \dot{u} = 0$ i.e, $r - \sigma_1 - d_1 = \frac{r}{K}\bar{x} - \frac{\bar{y}}{\bar{x}}\sigma_2$, $s - \sigma_2 - d_1 = \frac{s}{L}\bar{y} - \sigma_1\frac{\bar{x}}{\bar{y}} + m\frac{\bar{z}}{a+\bar{y}}$, $-d_2 - qE = \gamma\bar{z} + m_1\frac{\bar{u}}{b+\bar{z}} - n\frac{\bar{y}}{a+\bar{y}}$, $-d_3 = \delta\bar{u} - n_1\frac{\bar{z}}{b+\bar{z}}$. Substituting these, we have

$$
\begin{aligned}
\frac{dV}{dt} = & \quad (x-\bar{x})\left\{-\frac{r}{K}(x-\bar{x}) + \sigma_2\left(\frac{y}{x} - \frac{\bar{y}}{\bar{x}}\right)\right\} \\
& +p_1(y-\bar{y})\left\{-\frac{s}{L}(y-\bar{y}) + \sigma_1\left(\frac{x}{y} - \frac{\bar{x}}{\bar{y}}\right) - m\left(\frac{z}{a+y} - \frac{\bar{z}}{a+\bar{y}}\right)\right\} \\
& +p_2(z-\bar{z})\left\{-\gamma(z-\bar{z}) + n\left(\frac{y}{a+y} - \frac{\bar{y}}{a+\bar{y}}\right) - m_1\left(\frac{u}{b+z} - \frac{\bar{u}}{b+\bar{z}}\right)\right\} \\
& +p_3(u-\bar{u})\left\{-\delta(u-\bar{u}) + n_1\left(\frac{z}{b+z} - \frac{\bar{z}}{b+\bar{z}}\right)\right\} \\
= & \quad -\left(\frac{r}{K} + \frac{\sigma_2 y}{x\bar{x}}\right)(x-\bar{x})^2 - p_1\left(\frac{s}{L} + \frac{\sigma_1 x}{y\bar{y}} - \frac{m\bar{z}}{(a+y)(a+\bar{y})}\right)(y-\bar{y})^2 \\
& -p_2\left(\gamma - \frac{m_1\bar{u}}{(b+z)(b+\bar{z})}\right)(z-\bar{z})^2 + \left(p_2\frac{an}{(a+\bar{y})} - p_1 m\right)\frac{(y-\bar{y})(z-\bar{z})}{(a+y)} \\
& + \left(p_3\frac{bn_1}{b+\bar{z}} - m_1 p_2\right)\frac{(z-\bar{z})(u-\bar{u})}{b+z} - p_3\delta(u-\bar{u})^2 \\
& + \left(\frac{\sigma_2}{\bar{x}} + p_1\frac{\sigma_1}{\bar{y}}\right)(x-\bar{x})(y-\bar{y})
\end{aligned}
$$

Assuming that, $p_1 = 1,\ \ p_2 = \frac{m(a+\bar{y})}{an},\ \ p_3 = \frac{mm_1(a+\bar{y})(b+\bar{z})}{abnn_1}$
Then the above expression becomes as follows:

$$\begin{aligned}\frac{dV}{dt} = \quad & -\left(\frac{r}{K}+\frac{\sigma_2 y}{x\bar{x}}\right)(x-\bar{x})^2-\left(\frac{s}{L}+\frac{\sigma_1 x}{y\bar{y}}-\frac{m\bar{z}}{(a+y)(a+\bar{y})}\right)(y-\bar{y})^2\\ & -\frac{m(a+\bar{y})}{an}\left(\gamma-\frac{m_1\bar{u}}{(b+z)(b+\bar{z})}\right)(z-\bar{z})^2+\left(\frac{\sigma_2}{\bar{x}}+\frac{\sigma_1}{\bar{y}}\right)(x-\bar{x})(y-\bar{y})\\ & -\frac{mm_1(a+\bar{y})(b+\bar{z})}{abnn_1}\delta(u-\bar{u})^2\end{aligned}$$

Now, if no trajectory of the solution path meets the coordinate axes, then we always have, x/y and y/x are positive. Thus, we have

$$\begin{aligned}\frac{dV}{dt} \leq \quad & -\frac{r}{K}(x-\bar{x})^2-\left(\frac{s}{L}-\frac{m\bar{z}}{(a+y)(a+\bar{y})}\right)(y-\bar{y})^2\\ & -\frac{m(a+\bar{y})}{an}\left(\gamma-\frac{m_1\bar{u}}{(b+z)(b+\bar{z})}\right)(z-\bar{z})^2\\ & +\left(\frac{\sigma_2}{\bar{x}}+\frac{\sigma_1}{\bar{y}}\right)(x-\bar{x})(y-\bar{y})\\ \leq \quad & -\left[\sqrt{\frac{1}{2}\left(\frac{\sigma_2}{\bar{x}}+\frac{\sigma_1}{\bar{y}}\right)}\{(x-\bar{x})-(y-\bar{y})\}\right]^2\\ & -\left\{\frac{r}{K}-\frac{1}{2}\left(\frac{\sigma_2}{\bar{x}}+\frac{\sigma_1}{\bar{y}}\right)\right\}(x-\bar{x})^2\\ & -\left\{\frac{s}{L}-\frac{1}{2}\left(\frac{\sigma_2}{\bar{x}}+\frac{\sigma_1}{\bar{y}}\right)-\frac{m\bar{z}}{(a+y)(a+\bar{y})}\right\}(y-\bar{y})^2\\ & -\frac{m(a+\bar{y})}{an}\left(\gamma-\frac{m_1\bar{u}}{(b+z)(b+\bar{z})}\right)(z-\bar{z})^2\end{aligned}$$

Since, $R_1(0) > 0,\quad R_2(0) > 0\ ,\ r \geq \frac{K}{2}\left(\frac{\sigma_2}{\bar{x}}+\frac{\sigma_1}{\bar{y}}\right)$, then from the above expression, we conclude that $\frac{dV}{dt} \leq 0$. So, the theorem is holds.

Note 4.1: ‘No trajectory meet the co-ordinate axes’ means that the isoclines would remain always in the positive quadrant and it never goes to any other quadrant for positive initial conditions and this is an essential criteria for each and every ecological system.

4.8 Optimal Control

In economic ground, the fundamental problem regarding the commercial exploitation of renewable resources is to determine the optimal trade-off between present and future harvesting. Study of this section emphasises the better profit of the fisheries. It is thoroughly a study of the optimal harvesting policy and the profit earned by harvesting. Then we focus on quadratic cost and conservation of fish population by constraining the latter to stay always above a critical threshold. Specially, in this chapter, we study on quadratic cost of harvesting, usually it is taken as linear. The main reason for using quadratic cost is to obtain an analytical expression for the optimal harvesting. It is assumed that the price function and biomass are inversely proportional. Thus, to maximize the total discounted net revenues from the fishery, the optimal control problem can be designed as follows:

$$J(E) = \int_{t_0}^{t_1} e^{-\eta t}[(p - \omega qEz)qEz - cE]dt \tag{4.9}$$

where $[t_0, t_1]$ is the time interval of the observation, p is the constant price per unit biomass, c denotes the constant cost of harvesting effort, ω denotes the economic constant and the instantaneous annual discount rate denoted by η. The problem (4.9), subject to the population of the system (4.2) and control constraint $0 \leq E \leq E_{max}$, can be solved by applying Pontryagin's maximum principle (80). The convexity of the objective function with respect to E, the linearity of the differential equations in the control variable and the compactness of the range values of the state variables can be combined to give the existence of the optimal control.

Considering E_δ to be an optimal control with corresponding states x_η, y_η, z_η and u_η, we take $A_\eta(x_\eta, y_\eta, z_\eta, u_\eta)$ as optimal equilibrium point. Here, it is interested to derive an optimal control E_η such that

$J(E_\eta) = max\{J(E) : E \in U\}$,

where U is the control set defined as follows:

$U = \{E : [t_0, t_1] \rightarrow [0, E_{max}]$, E is the Lebesgue measurable$\}$

Now the Hamiltonian of this optimal control problem is as follows:

$$\begin{aligned} H = \quad & (p - \omega qEz)qEz - cE + \mu_1 \left\{ rx\left(1 - \frac{x}{k}\right) - \sigma_1 x + \sigma_2 y - d_1 x \right\} \\ & + \mu_2 \left\{ sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y - d_1 y - \frac{myz}{a+y} \right\} \\ & + \mu_3 \left\{ \frac{nyz}{a+y} - d_2 z - \gamma z^2 - \frac{m_1 zu}{b+z} - qEz \right\} \\ & + \mu_4 \left\{ \frac{n_1 zu}{b+z} - d_3 u - \delta u^2 \right\} \end{aligned} \tag{4.10}$$

where μ_1, μ_2, μ_3 and μ_4 are adjoint variables. Here, the transversality conditions give $\mu_i(t_1) = 0, i = 1, 2, 3, 4$.

Now, it is possible to find the characterization of the optimal control E_η, on the set $\{t : \quad 0 < E_\eta(t) < E_{max}\}$.

We have $\frac{\partial H}{\partial E} = pqz - 2\omega q^2 z^2 E - c - \mu_3 qz$

Thus at $A_\eta(x_\eta, y_\eta, z_\eta, u_\eta)$, $\;E = E_\eta(t)$ and $\frac{\partial H}{\partial E} = pqz_\eta - 2\omega q^2 z_\eta^2 E - c - \mu_3 qz_\eta = 0$.

This implies that,

$E_\eta = \frac{pqz_\eta - c - \mu_3 qz_\eta}{2\omega q^2 z_\eta^2}$

Now the adjoint equations at the point $A_\eta(x_\eta, y_\eta, z_\eta, u_\eta)$ are

$$\frac{d\mu_1}{dt} = \eta\mu_1 - \frac{\partial H}{\partial x}|_{A_\eta} = \quad \eta\mu_1 - \left[\mu_1\left(r - \frac{2rx_\eta}{K} - \sigma_1 - d_1\right) + \mu_2\sigma_1\right] \tag{4.11}$$

$$\begin{aligned} \frac{d\mu_2}{dt} = \eta\mu_2 - \frac{\partial H}{\partial y}|_{A_\eta} = \quad & \eta\mu_2 - \left[\mu_1\sigma_2 + \mu_2\left\{s - \frac{2sy_\eta}{L} - \sigma_2 - d_1 - \frac{maz_\eta}{(a+y_\eta)^2}\right\}\right. \\ & \left. + \mu_3 \frac{naz_\eta}{(a+y_\eta)^2}\right] \end{aligned} \tag{4.12}$$

$$\begin{aligned} \frac{d\mu_3}{dt} = \eta\mu_3 - \frac{\partial H}{\partial z}|_{A_\eta} = \quad & \eta\mu_3 - \left[\mu_3\left\{\frac{ny_\eta}{a+y_\eta} - d_2 - 2\gamma z_\eta - \frac{m_1 bu_\eta}{(b+z_\eta)^2} - qE\right\}\right. \\ & \left. + 2\omega q^2 E^2 z_\eta - \frac{m\mu_2 y_\eta}{a+y_\eta} + \mu_4 \frac{n_1 bu_\eta}{(b+z_\eta)^2}\right] - pqE \end{aligned} \tag{4.13}$$

$$\frac{d\mu_4}{dt} = \eta\mu_4 - \frac{\partial H}{\partial u}|_{A_\eta} = \quad \eta\mu_4 - \left[\mu_4\left\{\frac{n_1 z_\eta}{b+z_\eta} - d_3 - 2\delta u_\eta\right\} - \mu_3 \frac{m_1 z_\eta}{b+z_\eta}\right] \tag{4.14}$$

Equations (4.11) to (4.14) are first order system of simultaneous differential equations and the analytical solution of the equations with the help of initial conditions $\mu_i(t_1) = 0, i = 1, 2, 3, 4$ is easily obtained. After that, it is formulated the optimal control problem through considering fishing effort as control parameter and the optimal control problem will be numerically solved using a forward-backward sweep technique of 4th order Runge-Kutta method to pursue numerical simulations in the later. Based on the above analysis, we describe the following lemma.

Lemma 3. *There exists an optimal control E_η and corresponding solutions to the system* (4.2) x_η, y_η, z_η *and* u_η *that maximizes* $J(E)$ *over* U. *Also, there exist adjoint functions* μ_1, μ_2, μ_3 *and* μ_4 *which satisfy equations (4.11) to (4.14) respectively, with transversality conditions* $\mu_i(t_1) = 0, i = 1, 2, 3, 4$. *So, the optimal control is given by* $E_\eta = \frac{pqz_\eta - c - \mu_3 qz_\eta}{2\omega q^2 z_\eta^2}$.

4.9 Numerical Simulation

Some arbitrary data have been assumed for describing the analytical results. Using the MATLAB 7.10 software, we analyze the sensitivity analysis of the experiments. Again, we see that the parameters introduced in the system are not taken into consideration from real-life problems, so the prime characteristics are analyzed by the simulations presented here should be treated from a qualitative, rather than a quantitative point of view. However, numerous scenarios covering the breadth of the biological feasible parameter space have been conducted and the results shown above display the gamut of dynamical results collected from all the scenarios tested. Assuming that parameter set is taken as $P_1 = \{r, s, K, L, \sigma_1, \sigma_2, d_1, m, a, n, d_2, \gamma, m_1, n_1, b, q, E, d_3, \delta\}$ $= \{1.1, 1.2, 30, 40, 0.27, 0.15, 0.3, 0.9, 3.8, 0.8, 0.2, 0.01, 0.3, 0.12, 2.9, 0.3, 0.76, 0.02, 0.02\}$ and initial point is also taken as $B(11, 10, 7, 3)$. The phase portrait of the system (4.2) with the parameter set P_1 with respect to x, y, z and y, z, u are shown in the Figures 4.1 and 4.2 respectively. It is seen from the Figures 4.3 and 4.4 that if we consider the system (4.2) with harvesting and migration, then the system will be stable faster than the system which is considered without harvesting. From the Figures 4.3 and 4.5, it is seen that when migration is considered, then the system is stable where as it is unstable when migration is not considered. So, the proposed model is more realistic than the model without migration as well as harvesting. For the existence of the system, the intrinsic birth rate of prey population in the refuge region and also predatory region has an important role. From Figures 4.6 and 4.7, we see that, the density of the prey populations in the refuge region as well as predatory region are directly proportional to the intrinsic birth rate of prey population in that region. Functional response is most important concept to describe the prey-predator interaction. Figures 4.8 and 4.9 illustrate the sensitivity of prey-predator interaction. From the Figures 4.8 and 4.9, it is seen that, m and a are directly proportional to the density of all the four populations. The Figure 4.10 shows that the change of n is directly proportional to the density of predator population and same result for the specialist predator whereas the change of n is inversely proportional to the density of prey population in the refuge region and predatory region. Environmental carrying capacity has an important role for the existence of any population. The sensitivity of environmental carrying capacity of prey population in refuge region is described in Figure 4.11. From Figure 4.11, it is seen that the change of K is directly proportional to the density of the population. Again, due to the migration of prey populations between two regions, the increase in the density of prey in refuge region will be the cause for the decrease of prey population in the predatory region. Figures 4.12 and 4.13, we see the populations for the change of the migration and emigration parameters. It is seen in Figure 4.12, the change of the migration parameter σ_1 is directly proportional to the density of the prey population in the refuge region where as it is inversely proportional to the density of prey population in the predatory region. Similarly, Figure 4.13 is seen that the change of σ_2 is directly proportional to the density of prey population in the predatory region where as it is inversely proportional to the density of the prey population in the refuge region. Figure 4.14 shows that, the natural death rate of predator population d_2 is inversely proportional to the density of the predator and specialist predator populations and directly proportional to the change of the density of both the prey populations.

4.10 Chapter Summery

An effect of specialist predator in a prey-predator model with Holling type II functional response has been introduced and a prey refuge region with harvesting effort on predator has been considered. Again a density dependent mortality rate has been considered for prey, predator and specialist predator. Also, the different birth rates and different carrying capacities for the prey populations in the refuge region have been described. In this context, this research work is significantly different in compare to other works in this area. In addition, migrations of the prey populations has been included between two regions. The local as well as global stability around the equilibria have been discussed. The problem has been illustrated with a numerical example. Also the proposed model has been analyzed with some geometrical figures. Global stability of the system is shown by using a suitable analytical approach. It has been observed that three possible equilibria exist, one is trivial equilibrium point, one as predator and specialist predator free and the most important one is the interior equilibrium point. From this study, it has been concluded that the obtained results are not only feasible to analyze the biological, social and economic impacts of existing resource, but also provide appropriate measures to maintain long-run sustainability of the resource.

4.11 Appendix

$$
\begin{aligned}
R_1 = &\quad \frac{sr^4\gamma}{LK^4\sigma_2^4}\\
R_2 = &\quad \gamma(b_1c_5 + a_1c_6)\\
R_3 = &\quad \gamma(ac_6 + a_1c_5 + b_1c_4)\\
R_4 = &\quad \gamma(ac_5 + a_1c_4 + b_1c_3)\\
R_5 = &\quad \gamma(ac_4 + a_1c_3 + c_2b_1) - nb_1^2 + (d_1 + qE)b_1^2\\
R_6 = &\quad \gamma(ac_3 + a_1c_2 + b_1c_1) - 2a_1b_1n + 2a_1b_1(d_1 + qE)\\
R_7 = &\quad \gamma(ac_2 + a_1c_1) - na_1^2 + (d_1 + qE)(ab_1 + a_1^2)\\
R_8 = &\quad \gamma ac_1 + aa_1(d_1 + qE)\\
T_1 = &\quad \frac{s\gamma r^4}{L\alpha_1K^4\sigma_2^4}\\
T_2 = &\quad (a_1c_6^3 + 3b_1c_5c_6^2)c_{21}\\
T_3 = &\quad c_{21}[b_1(3c_5^2c_6 + 3c_4c_6^2) + 3a_1c_5c_6^2] + c_{20}c_6^3\\
T_4 = &\quad 3c_{20}c_5c_6^2 + c_{21}[a_1(3c_5^2c_6 + 3c_4c_6^2) + b_1(c_5^3 + 6c_4c_3c_6 + 3c_3c_6^2)]\\
T_5 = &\quad c_{21}[a_1(c_5^3 + 6c_3c_4c_6 + 3c_3c_6^2) + b_1(3c_4c_5^2 + 3c_4^2c_6 + 3c_2c_6^2 + 6c_3c_5c_6)]\\
&\quad +c_{20}(3c_5^2c_6 + 3c_4c_6^2) + c_{18}b_1^2c_6^2\\
T_6 = &\quad c_{18}[2b_1^2c_5c_6 + 2a_1b_1c_6^2] + c_{20}(c_5^2 + 6c_4c_3c_6 + 3c_3c_6^2)] + c_{21}[a_1(3c_4c_5^2\\
&\quad +3c_4^2c_6 + 3c_2c_6^2 + 6c_3c_5c_6) + b_1(3c_4^2c_5 + 3c_1c_6^2\\
&\quad +3c_3c_5^2 + 6c_2c_5c_6 + 6c_3c_4c_6)]\\
T_7 = &\quad c_{18}[a_1^2c_6^2 + 4a_1b_1c_6c_5 + b_1^2(2c_4c_6 + c_5^2)] + c_{19}b_1c_6^2 + c_{20}(3c_4c_5^2 + 3c_4^2c_6\\
&\quad +3c_2c_6^2 + 6c_3c_5c_6) + c_{21}[a_1(3c_4^2c_5 + 3c_1c_6^2 + 6c_2c_5c_6 + 6c_3c_4c_6 + 3c_3c_5^2)\\
&\quad +b_1(c_4^3 + 3c_2c_5^2 + 3c_3^2c_6 + 6c_1c_5c_6 + 6c_2c_4c_6 + 6c_3c_4c_5)]\\
T_8 = &\quad c_{18}[2a_1^2c_5c_6 + 2a_1b_1(2c_4c_6 + c_5^2) + b_1^2(2c_4c_5 + 2c_3c_6)] + c_{19}(a_1c_6^2\\
&\quad +2b_1c_5c_6) + c_{20}(3c_4^2c_5 + 3c_1c_6^2 + 6c_2c_5c_6 + 6c_3c_4c_6 + 3c_3c_5^2)\\
&\quad +c_{21}[a_1(c_4^3 + 3c_2c_5^2 + 3c_3^2c_6 + 6c_1c_5c_6 + 6c_2c_4c_6 + 6c_3c_4c_5)\\
&\quad +b_1(3c_1c_5^2 + 3c_3c_4^2 + 3c_3^2c_5 + 6c_2c_4c_5 + 6c_1c_4c_6 + 6c_2c_3c_6)]
\end{aligned}
$$

$$
\begin{aligned}
T_9 = \quad & -c_{17}c_6b_1^3 + c_{18}[a_1^2(2c_4c_6 + c_5^2) + 2a_1b_1(2c_4c_5 + 2c_3c_6) + b_1^2(c_4^2 + 2c_2c_6 \\
& +2c_3c_5)] + c_{19}[2a_1c_5c_6 + b_1(2c_4c_6 + c_5^2)] + c_{20}(c_4^3 + 3c_2c_5^2 + 3c_3^2c_6 \\
& +6c_1c_5c_6 + 6c_2c_4c_6 + 6c_3c_4c_5) + c_{21}[a_1(3c_1c_5^2 + 3c_3c_4^2 + 3c_3^2c_5 + 6c_2c_4c_5 \\
& +6c_1c_4c_6 + 6c_2c_3c_6) + b_1(3c_2c_4^2 + 3c_2^2c_6 + 3c_3^2c_4 + 6c_1c_4c_6 + 6c_1c_3c_6 \\
& +6c_2c_3c_5)] \\
T_{10} = \quad & -c_{17}(3c_6a_1b_1^2 + c_5b_1^3) + c_{18}[a_1^2(2c_4c_5 + 2c_3c_6) + 2a_1b_1(c_4^2 + 2c_2c_6 + 2c_3c_5) \\
& +b_1^2(2c_1c_6 + 2c_2c_5 + 2c_3c_4)] + c_{19}[a_1(2c_4c_6 + c_5^2) + b_1(2c_4c_5 + 2c_3c_6)] \\
& +c_{20}(3c_1c_5^2 + 3c_3c_4^2 + 3c_3^2c_5 + 6c_2c_4c_5 + 6c_1c_4c_6 + 6c_2c_3c_6) \\
& +c_{21}[a_1(3c_2c_4^2 + 3c_2^2c_6 + 3c_3^2 + 6c_1c_4c_5 + 6c_1c_3c_6 + 6c_2c_3c_5) \\
& +b_1(c_3^3 + 3c_1c_4^2 + 3c_2^2c_5 + 6c_1c_3c_5 + 6c_1c_2c_6 + 6c_2c_3c_4)] \\
T_{11} = \quad & -c_{17}(b_1^3c_4 + 3c_5a_1b_1^2 + 3c_6a_1^1b_1) - c_{16}c_6b_1^3 + c_{18}[a_1^2(c_4^2 + 2c_2c_6 + 2c_3c_5) \\
& +2a_1b_1(2c_1c_6 + 2c_2c_5 + 2c_3c_4) + b_1^2(c_3^2 + 2c_1c_5 + 2c_2c_4)] + c_{19}[a_1(2c_4c_5 \\
& +2c_3c_6) + b_1(c_4^2 + 2c_2c_6 + 2c_3c_5)] + c_{20}(3c_2c_4^2 + 3c_2^2c_6 + 3c_3^2c_4 \\
& +6c_1c_4c_5 + 6c_1c_3c_6 + 6c_2c_3c_5) + c_{21}[a_1(c_3^3 + 3c_1c_4^2 + 3c_2^2c_5 + 6c_1c_3c_5 \\
& +6c_1c_2c_6 + 6c_2c_3c_4) + b_1(3c_1c_3^2 + 3c_1^2c_6 + 3c_2^2c_4 + 6c_1c_3c_4 + 6c_1c_2c_5)] \\
T_{12} = \quad & -c_{16}(c_5b_1^3 + 2b_1a_1c_6) - c_{17}(c_3b_1^3 + 3a_1b_1^2c_4 + 3c_5a_1^2b_1 + c_6a_1^3) \\
& +c_{18}[a_1^2(2c_1c_6 + 2c_2c_5 + 2c_3c_4) + 2a_1b_1(c_3^2 + 2c_1c_5 + 2c_3c_4) + b_1^2(2c_1c_4 + 2c_2c_3)] \\
& +c_{19}[a_1(c_4^2 + 2c_2c_6 + 2c_3c_5) + b_1(2c_1c_6 + 2c_2c_5 + 2c_3c_4) + c_{20}(c_3^3 + 3c_1c_4^2 \\
& +3c_2^2c_5 + 6c_1c_3c_5 + 6c_1c_2c_6 + 6c_2c_3c_4) + c_{21}[a_1(3c_1c_3^2 + 3c_1^2c_6 + 3c_2^2c_4 \\
& +6c_1c_3c_4 + 6c_1c_2c_5) + b_1(3c_2^2c_3 + 3c_1c_3^2 + 3c_1^2c_5 + 6c_1c_2c_4)] \\
T_{13} = \quad & -c_{15}b_1^4 - c_{16}(a_1^2c_6 + 2a_1b_1c_5 + b_1^3c_4) - c_{17}(c_2b_1^3 + 3c_3a_1b_1^2 \\
& +3c_4a_1^2b_1 + c_5a_1^3) + c_{18}[a_1^2(c_3^2 + 2c_1c_5 + 2c_2c_4) + 2a_1b_1(2c_1c_4 + 2c_2c_3) \\
& +b_1^2(c_2^2 + 2c_1c_3)] + c_{19}[a_1(2c_1c_6 + 2c_2c_5 + 2c_3c_4) + b_1(c_3^2 + 2c_1c_5 + 2c_2c_4)] \\
& +c_{20}(3c_1c_3^2 + 3c_1^2c_6 + 3c_2^2c_4 + 6c_1c_3c_4 + 6c_1c_2c_5) + c_{21}[a_1(3c_2^2c_3 \\
& +3c_1c_3^2 + 3c_1^2c_5 + 6c_1c_2c_4) + b_1(c_2^3 + 6c_1c_2c_3 + 3c_1^2c_4)]
\end{aligned}
$$

$$
\begin{aligned}
T_{14} = \quad & -4c_{14}b_1^3 - 6c_{15}a_1^2b_1^2 - c_{16}(c_4a_1^2 + 2c_3a_1b_1 + c_2b_1^3) - c_{17}(c_3a_1^3 \\
& +3c_2a_1^2b_1 + 3c_1a_1b_1^2) + c_{18}[a_1^2(c_2^2 + 2c_1c_3) + 4a_1b_1c_1c_2 + c_1^2b_1^2] \\
& +c_{19}[a_1(2c_1c_4 + 2_2c_3) + b_1(c_2^2 + 2c_1c_3)] + c_{20}(c_2^3 + 6c_1c_2c_3 + 3c_1^2c_4) \\
& +c_{21}[a_1(3c_1c_2^2 + 3c_1^2c_3) + 3b_1c_1^2c_4] \\
T_{15} = \quad & -c_{14}b_1^3 - 6c_{15}a_1^2b_1^2 - c_{16}(c_4a_1^2 + 2c_3a_1b_1 + c_2b_1^3) - c_{17}(c_3a_1^3 \\
& +3c_2a_1^2b_1 + 3c_1a_1b_1^2) + c_{18}[a_1^2(c_2^2 + 2c_1c_3) + 4a_1b_1c_1c_2 + c_1^2b_1^2] \\
& +c_{19}[a_1(2c_1c_4 + 2c_2c_3)b_1(c_2^2 + 2c_1c_3)] + c_{20}(c_2^3 + 6c_1c_2c_3 + 3c_1^2c_4) \\
& +c_{21}[a_1(3c_1c_2^2 + 3c_1^2c_3) + 3b_1c_1^2c_4] \\
T_{16} = \quad & -3c_{14}a_1b_1^2 - 4c_{15}a_1^3b_1 - c_{16}c_1b_1^3 - c_{17}(3c_1a_1^2b_1 + c_2a_1^3) \\
& +c_{18}(2a_1^2c_1c_2 + 2a_1b_1c_1^2) + c_{19}[a_1(c_2^2 + 2c_1c_3) + 2b_1c_1c_2] + c_{20}(3c_1c_2^2 \\
& +3c_1^2c_3) + c_{21}(3a_1c_1^2c_4 + b_1c_1^3) \\
T_{17} = \quad & -3c_{14}a_1b_1 - c_{15}a_1^4 - c_{16}(2a_1b_1c_1 + c_2a_1^2) - c_{17}c_1a_1^3 + c_{18}a_1^2c_1^2 \\
& +c_{19}(2a_1c_1c_2 + b_1c_1^2) + 3c_{20}c_1^2c_4 + c_{21}a_1c_1^3 \\
T_{18} = \quad & -c_{14}a_1^3 - c_{16}c_1a_1^2 + c_{19}c_1^2a_1 + c_{20}c_1^3
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 = \quad & \frac{r - \sigma_1 - d_1}{\sigma_2}, b_1 = -\frac{r}{K\sigma_2}, c_1 = \frac{1}{m}[a\sigma_1 + \frac{a}{\sigma_2}(r - \sigma_1 - d_1)(s - \sigma_2 - d_1)] \\
c_2 = \quad & \frac{1}{m}[\frac{\sigma_1}{\sigma_2}(r - \sigma_1 - d_1) - \frac{ar}{K\sigma_2}(s - \sigma_2 - d_1) + (s - \sigma_2 - d_1)(\frac{r - \sigma_1 - d_1}{\sigma_2})^2] \\
c_3 = \quad & -\frac{r\sigma_1}{K\sigma_2} - 2\frac{r}{K\sigma_2^2}(r - \sigma_1 - d_1)(s - \sigma_2 - d_1) - \frac{s}{L}(\frac{r - \sigma_1 - d_1}{\sigma_2})^3 \\
c_4 = \quad & \frac{r^2}{K^2\sigma_2^2}(s - \sigma_2 - d_1) + 3\frac{sr}{LK}\frac{(r - \sigma_1 - d_1)^2}{\sigma_2^3} \\
c_5 = \quad & -3\frac{sr^2}{LK^2}\frac{r - \sigma_1 - d_1}{\sigma_2^3}, \ c_6 = \frac{sr^3}{LK^3\sigma_2^3} \\
c_{14} = \quad & -\frac{ab^2}{m_1}(d_2 + qE) - \frac{abd_3}{\delta}, \ \ c_{15} = \frac{1}{m_1}(b^2n - b^2d_2 - b^2qE) - \frac{bd_3}{\delta} \\
c_{16} = \quad & -\frac{1}{m_1}(2abd_2 + 2abqE + \gamma ab^2), \\
& c_{17} = \frac{d_1 - n_1}{\delta} + \frac{1}{m_1}(2bn - 2bd_2 - 2bqE - \gamma b^2), \ \ c_{18} = \frac{1}{m_1}(d_2 + qE - 2b\gamma - n),
\end{aligned}
$$

$$c_{19} = \frac{1}{m_1}(ad_2 + aqE - 2ab\gamma), \;\; c_{20} = \frac{a\gamma}{m_1}, \;\; c_{21} = \frac{\gamma}{m_1}$$

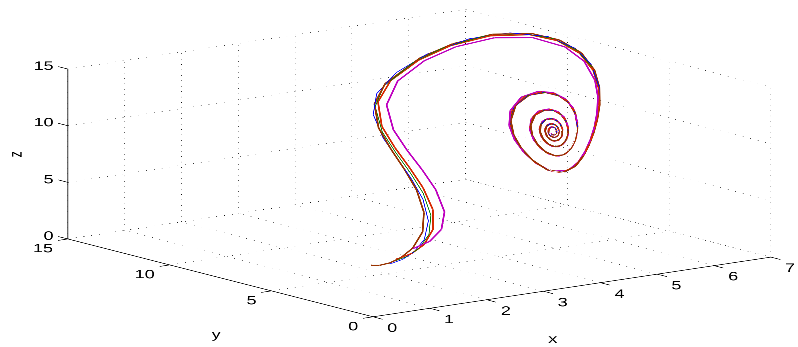

Figure 4.1: Phase space diagram of the system (4.2) with the parameter set P_1 with respect to x,y,z.

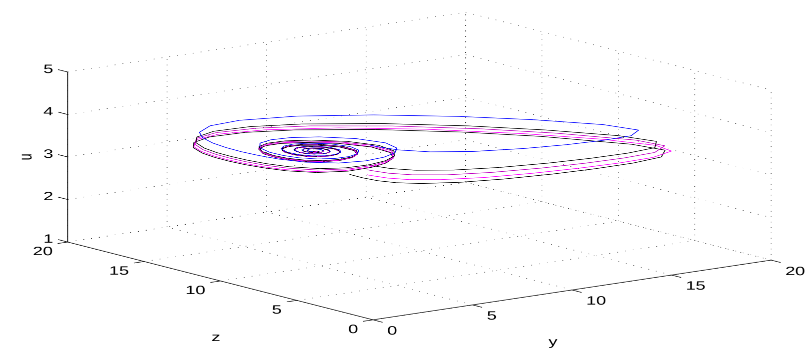

Figure 4.2: Phase space diagram of the system (4.2) with the parameter set P_1 with respect to y,z,u.

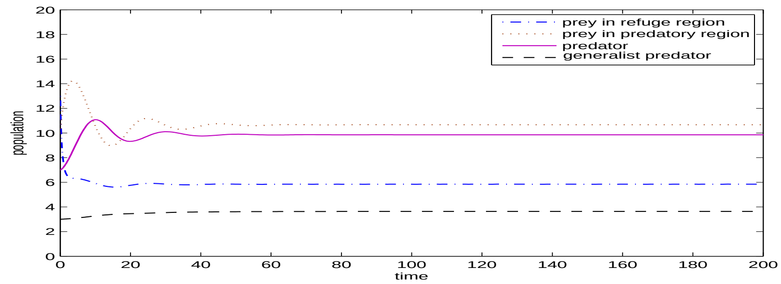

Figure 4.3: Graphical representation of the system (4.2) with migration and harvesting.

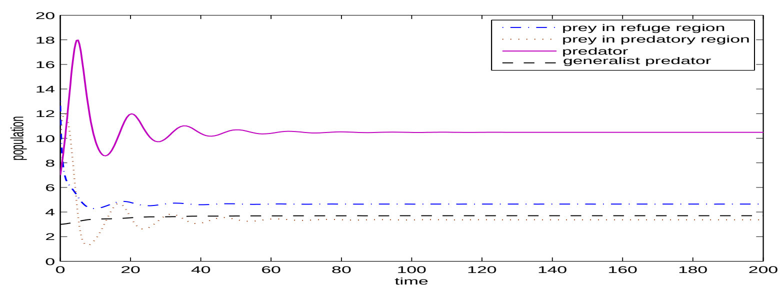

Figure 4.4: Graphical representation of the system (4.2) without harvesting.

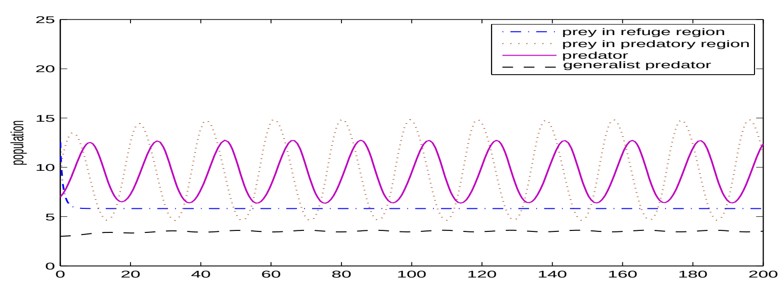

Figure 4.5: Graphical representation of the system (4.2) without migration.

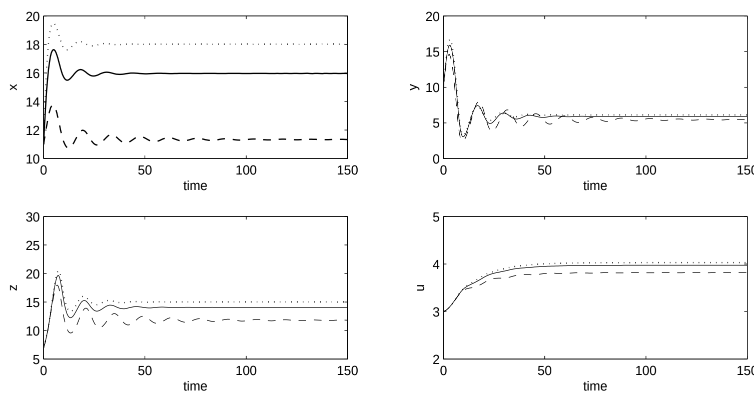

Figure 4.6: *Change of x, y, z and u of the system (4.2) with respect to change of intrinsic birth rate of prey population in refuge region with parameter set* $\{s, K, L, \sigma_1, \sigma_2, d_1, m, a, n, d_2, \gamma, m_1, n_1, b, q, E, d_3, \delta\} = \{1.2, 30, 40, 0.27, 0.15, 0.3, 0.9, 3.8, 0.8, 0.2, 0.01, 0.3, 0.12, 2.9, 0.3, 0.76, 0.02, 0.02\}$. *Here (- - -) line corresponds to $r = 0.8$, (—) line to $r = 1.1$ and ($\cdots$) line to $r = 1.3$.*

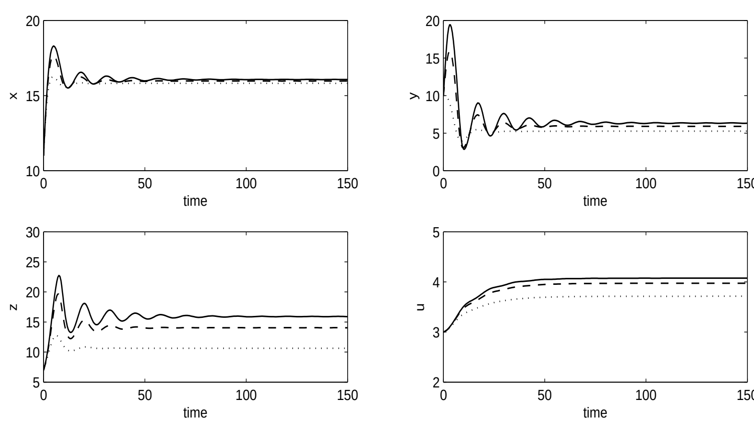

Figure 4.7: *Change of x, y, z and u of the system (4.2) with respect to change of intrinsic birth rate of prey population in predatory region with parameter set* $\{r, K, L, \sigma_1, \sigma_2, d_1, m, a, n, d_2, \gamma, m_1, n_1, b, q, E, d_3, \delta\} = \{1.1, 30, 40, 0.27, 0.15, 0.3, 0.9, 3.8, 0.8, 0.2, 0.01, 0.3, 0.12, 2.9, 0.3, 0.76, 0.02, 0.02\}$. *Here ($\cdots$) line corresponds to $s = 0.8$, (- - -) line to $s = 1.2$ and (—) line to $s = 1.4$.*

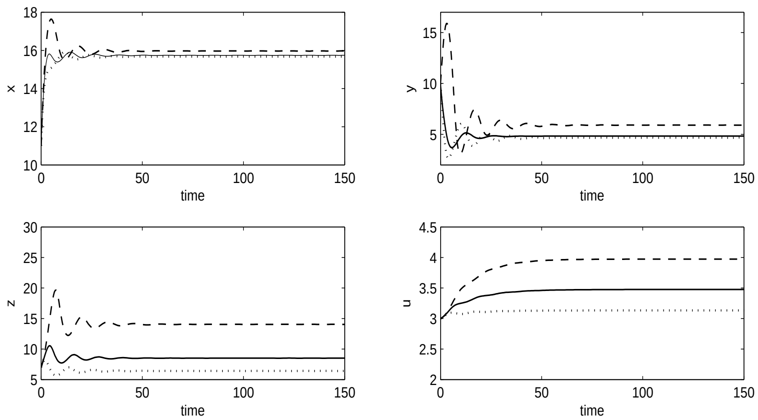

Figure 4.8: *Change of x, y, z and u of the system (4.2) with respect to change of m with parameter set* $\{r, s, K, L, \sigma_1, \sigma_2, d_1, a, n, d_2, \gamma, m_1, n_1, b, q, E, d_3, \delta\}$ $=\{1.1, 1.2, 30, 40, 0.27, 0.15, 0.3, 3.8, 0.8, 0.2, 0.01, 0.3, 0.12, 2.9, 0.3, 0.76, 0.02, 0.02\}$. *Here (- - -) line corresponds to $m = 0.9$, (—) line to $m = 1.5$ and ($\cdots$) line to $m = 2$.*

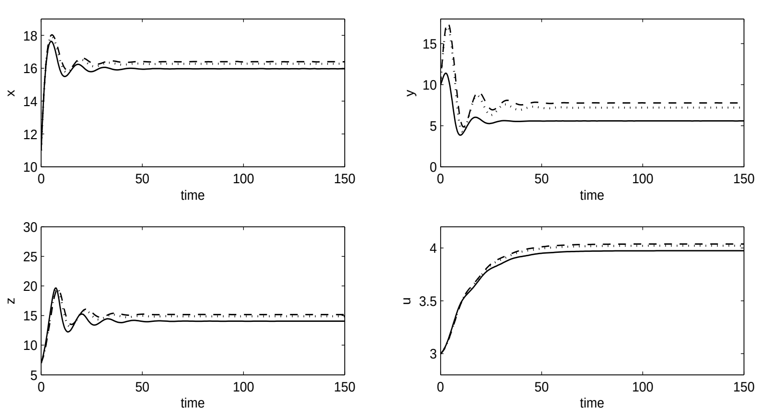

Figure 4.9: *Change of x, y, z and u with respect to change of a with parameter set* $\{r, s, K, L, \sigma_1, \sigma_2, d_1, m, n, d_2, \gamma, m_1, n_1, b, q, E, d_3, \delta\}$ $=$ $\{1.1, 1.2, 30, 40, 0.27, 0.15, 0.3, 0.9, 0.8, 0.2, 0.01, 0.3, 0.12, 2.9, 0.3, 0.76, 0.02, 0.02\}$. *Here (—) line corresponds to $a = 3.8$, ($\cdots$) line to $a = 4.5$ and (- - -) line to $a = 4.8$.*

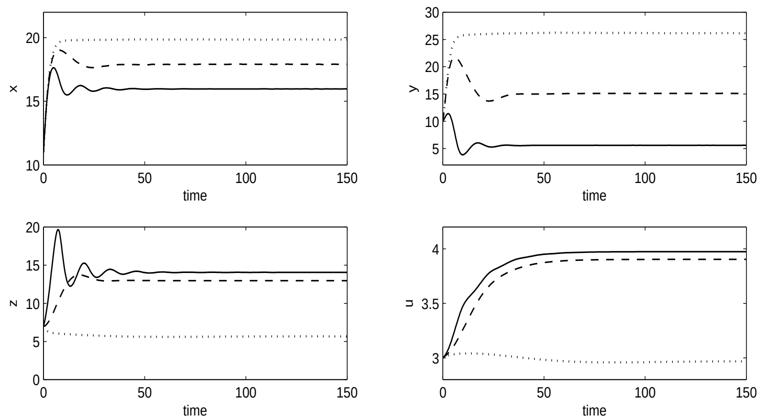

Figure 4.10: *Change of x, y, z and u with respect to change of n with parameter set* $\{r, s, K, L, \sigma_1, \sigma_2, d_1, m, a, d_2, \gamma, m_1, n_1, b, q, E, d_3, \delta\} = \{1.1, 1.2, 30, 40, 0.27, 0.15, 0.3, 0.9, 3.8, 0.2, 0.01, 0.3, 0.12, 2.9, 0.3, 0.76, 0.02, 0.02\}$. *Here ($\cdots$) line corresponds to $n = 0.5$, (- - -) line to $n = 0.6$ and (—) line to $n = 0.8$.*

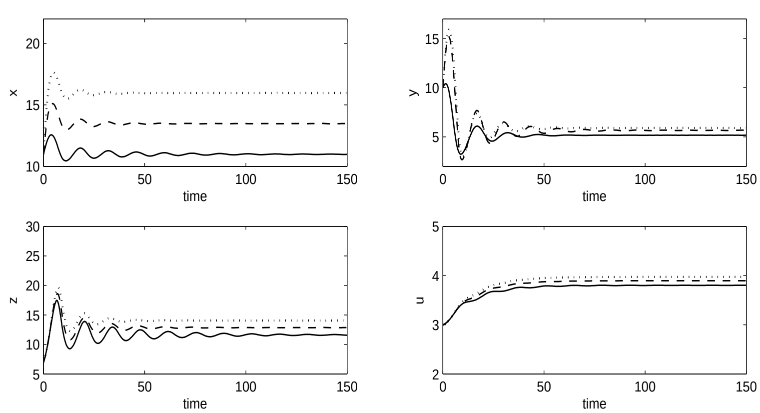

Figure 4.11: *Change of x, y, z and u with respect to change of K with parameter set* $\{r, s, L, \sigma_1, \sigma_2, d_1, m, a, n, d_2, \gamma, m_1, n_1, b, q, E, d_3, \delta\} = \{1.1, 1.2, 40, 0.27, 0.15, 0.3, 0.9, 3.8, 0.8, 0.2, 0.01, 0.3, 0.12, 2.9, 0.3, 0.76, 0.02, 0.02\}$. *Here (—) line corresponds to $K = 20$, (- - -) line to $K = 25$ and ($\cdots$) line to $K = 30$.*

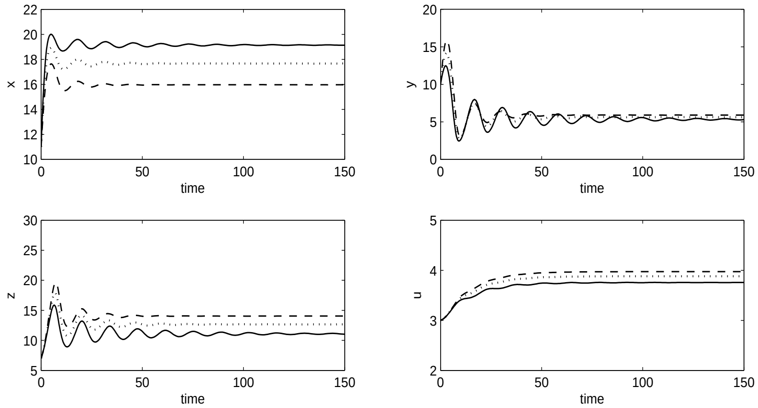

Figure 4.12: *Change of* x, y, z *and* u *with respect to change of* σ_1 *with parameter set* $\{r, s, K, L, \sigma_2, d_1, m, a, n, d_2, \ \gamma, m_1, n_1, b, q, E, d_3, \delta\} = \{1.1, 1.2, 30, 40, 0.15, 0.3, 0.9, 3.8, 0.8, 0.2, 0.01, 0.3, 0.12, 2.9, 0.3, 0.76, 0.02, 0.02\}$. *Here (—) line corresponds to* $\sigma_1 = 0.14$*, (···) line to* $\sigma_1 = 0.2$ *and (- - -) line to* $\sigma_1 = 0.27$.

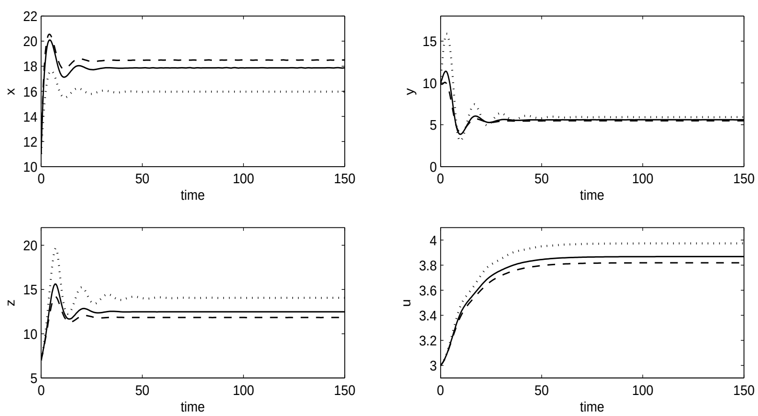

Figure 4.13: *Change of* x, y, z *and* u *with respect to change of* σ_2 *with parameter set* $\{r, s, K, L, \sigma_1, d_1, m, a, n, d_2, \ \gamma, m_1, n_1, b, q, E, d_3, \delta\} = \{1.1, 1.2, 30, 40, 0.27, 0.3, 0.9, 3.8, 0.8, 0.2, 0.01, 0.3, 0.12, 2.9, 0.3, 0.76, 0.02, 0.02\}$. *Here (···) line corresponds to* $\sigma_2 = 0.15$*, (—) line to* $\sigma_2 = 0.4$ *and (- - -) line to* $\sigma_2 = 0.5$.

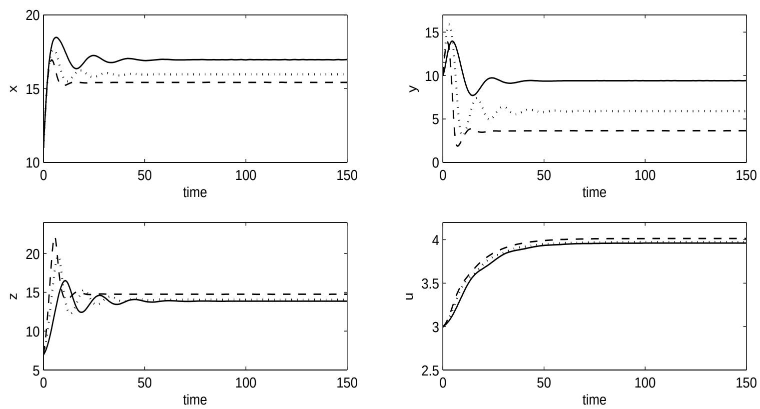

Figure 4.14: *Change of x, y, z and u with respect to change of d_2 with parameter set* $\{r, s, K, L, \sigma_1, \sigma_2, d_1, m, a, n, \gamma, m_1, n_1, b, q, E, d_3, \delta\} = \{1.1, 1.2, 30, 40, 0.27, 0.15, 0.3, 0.9, 3.8, 0.8, 0.01, 0.3, 0.12, 2.9, 0.3, 0.76, 0.02, 0.02\}$. *Here (- - -) line corresponds to $d_2 = 0.1$, ($\cdots$) line to $d_2 = 0.2$ and (—) line to $d_2 = 0.3$.*