Chapter 3

Effects on prey-predator with different functional responses*

In this chapter, the effects on prey of two predators which are also related in terms of prey-predator relationship has been investigated. Different type of functional responses are considered to formulate the mathematical model for predator and generalist predator of the proposed model. Harvesting effort for the generalist predator is considered and the density dependent mortality rate for predator and generalist predator are incorporated. Local stability as well as global stability for the system are discussed. The different bifurcation parameters have been analyzed to evaluate Hopf bifurcation in the neighborhood of interior equilibrium point. Finally, some numerical simulations and graphical figures are provided to verify our analytical results with the help of different sets of parameters.

3.1 Introduction

The interaction between prey and predator is one of basic interspecies relationship in the biology and ecology. It is also the basic problem of the complicated food chain, food web and biochemical network structure. In the study of interacting population dynamics, a functional response of predator to prey density refers to the change in the density of prey per unit time per predator as a

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function of the prey density. There are mainly three types of functional responses namely Holling types I, II and III. Type I occurs, when there is a linear situation to a maximum in the number of prey eaten per predator as prey density decreases. Again type II occurs, when the response arises at a decreasing rate towards a maximum value. Finally, type III occurs, when the response is sigmoid, again approaching an upper asymptote. The subject of harvesting in prey-predator systems is described as a multi-disciplinary area of research which considered by economists and ecologists. In many earlier studies, it is shown that harvesting has a strong impact on population dynamics, ranging from rapid depletion to complete preservation of biological populations.

They considered only a simple food chain (the number of prey and predator is only one in this food chain) in their model, but normally in an ecological system, there are so many species which are interacted to each other. For example, in an aquatic ecosystem, so many micro-organisms and fishes are lived. Fishes live on phytoplankton and zooplankton. Also phytoplankton is eaten by zooplankton. So, there are more than one prey and their food resources are different. Again micro-organisms and fishes are belonged in different classes and functional responses of predator to the prey population are different.

The main motivations of the this chapter are as follows: It is considered that one species consumes more than one species which are also related in prey-predator relationship. Also different functional responses are considered as per class of the different species. In addition to these in the chapter, harvesting effort has been introduced on generalist predator which is more realistic to analyze the whole system.

3.2 Notation

Table-3.2.1: Description of the parameters.

Parameter	Description of the parameters
x	Population of prey at time t
y	Population of predator at time t
z	Population of generalist predator at time t

r	Intrinsic growth rate of prey
K	Environmental carrying capacity of the prey
α	Capture rate of the predator to prey
m	Capture rate of the generalist predator to prey
n	Capture rate of the generalist predator to predator
a, b_1, b_2	Half saturation constants
d_1	Natural death rate of predator
d_2	Natural death rate of generalist predator
β	Predator's consumption rate on prey
n_1	Generalist predator's consumption rate on predator
m_1	Generalist predator's consumption rate on prey

3.3 Mathematical model

Assume that $x(t), y(t)$ and $z(t)$ denote the population of prey, predator and generalist predator respectively at time t. Fishes live on phytoplankton and zooplankton. Also, phytoplankton is eaten by zooplankton. Again, fishes are the member of vertebrata and planktons are member of invertebrates. So the predator and the generalist predator give different responses on the prey. For this reason, it has been consider that, the predator population consumes the prey population with Holling type-II functional response or Michaelis-Menten functional response which is denoted by $\frac{x}{a+x}$; and the generalist-predator population consumes the predator and prey population with Holling type-III functional response, denoted by $\frac{y^2}{b_2+y^2}$ and $\frac{x^2}{b_1+x^2}$ respectively. Then the system of equations in reserve region becomes as follows:

$$\left.\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\frac{\alpha xy}{a+x}-m\frac{x^2z}{b_1+x^2} \\ \frac{dy}{dt} &= \frac{\beta xy}{a+x}-d_1y-\frac{ny^2z}{b_2+y^2} \\ \frac{dz}{dt} &= n_1\frac{y^2z}{b_2+y^2}+m_1\frac{x^2z}{b_1+x^2}-d_2z \end{aligned}\right\} \tag{3.1}$$

with initial conditions $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$. Here, we consider b_1 and b_2 as saturation constants in functional response of generalist predator in prey and predator respectively. For the simplification of calculation, we consider

$b_1 = b_2 = b$. Then the system of differential equations (3.1) reduces as follows:

$$\left.\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a+x} - m\frac{x^2 z}{b+x^2} \\ \frac{dy}{dt} &= \frac{\beta xy}{a+x} - d_1 y - \frac{ny^2 z}{b+y^2} \\ \frac{dz}{dt} &= n_1\frac{y^2 z}{b+y^2} + m_1\frac{x^2 z}{b+x^2} - d_2 z \end{aligned}\right\} \tag{3.2}$$

with initial conditions $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$. The growing of human needs for more food and more energy have led to increase the exploitation of these resources. Since fishes are harvested for human needs, for this reason harvesting of fisheries have been drawing more attention in this model. The harvest rate is denoted as $h(t)$ and it is considered as $h(t) = qEz$ where q denotes as catchability co-efficient; E denotes the fishing effort used to harvest of predator population. Thus the system (3.2) rewrites as follows:

$$\left.\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a+x} - m\frac{x^2 z}{b+x^2} \\ \frac{dy}{dt} &= \frac{\beta xy}{a+x} - d_1 y - \frac{ny^2 z}{b+y^2} \\ \frac{dz}{dt} &= n_1\frac{y^2 z}{b+y^2} + m_1\frac{x^2 z}{b+x^2} - d_2 z - qEz \end{aligned}\right\} \tag{3.3}$$

with initial conditions $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$.

3.4 Local stability

In this section, we analyze the stability of the system (3.3) at an interior equilibrium point $\bar{B}(\bar{x}, \bar{y}, \bar{z})$.
Now the characteristic equation of the system (3.3) around its interior equilibrium $\bar{B}(\bar{x}, \bar{y}, \bar{z})$ is calculated as follows:

$$\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 = 0, \tag{3.4}$$

where

$$\begin{aligned}
h_1 &= \frac{r}{K}\bar{x} + m\bar{x}\bar{z}\frac{b-\bar{x}^2}{(b+\bar{x}^2)^2} + n\bar{y}\bar{z}\frac{b-\bar{y}^2}{(b+\bar{y}^2)^2} - \frac{\alpha\bar{x}\bar{y}}{(a+\bar{x})^2},\\
h_2 &= n\bar{y}\bar{z}\frac{\bar{y}^2-b}{(b+\bar{y}^2)^2}\left\{\frac{\alpha\bar{x}\bar{y}}{(a+\bar{x})^2} + m\bar{x}\bar{z}\frac{\bar{x}^2-b}{(b+\bar{x}^2)^2} - \frac{r}{K}\bar{x}\right\} + a\alpha\beta\frac{\bar{x}\bar{y}}{(a+\bar{x})^3}\\
&\quad +2mm_1b\frac{\bar{x}^3\bar{z}}{(b+\bar{x}^2)^3},\\
h_3 &= -2nm_1b\alpha\frac{\bar{x}^2\bar{y}^2\bar{z}}{(a+\bar{x})(b+\bar{y}^2)(b+\bar{x}^2)^2} + 2abmn_1\beta\frac{\bar{x}^2\bar{y}^2\bar{z}}{(b+\bar{x}^2)(a+\bar{x})^2(b+\bar{y}^2)^2},\\
&\quad -2nmm_1b\frac{\bar{x}^3\bar{y}\bar{z}^2(\bar{y}^2-b)}{(b+\bar{x}^2)^3(b+\bar{y}^2)^2}.
\end{aligned}$$

Now we consider $h_i \quad (i=1,2,3)$ as $h_1 = k_1 - k_2\alpha, \quad h_2 = k_3 - k_4\alpha, \quad h_3 = k_5 - k_6\alpha$ with respect to the parameter α, where

$$\begin{aligned}
k_1 &= \frac{r}{K}\bar{x} + m\bar{x}\bar{z}\frac{b-\bar{x}^2}{(b+\bar{x}^2)^2} + n\bar{y}\bar{z}\frac{b-\bar{y}^2}{(b+\bar{y}^2)^2},\\
k_2 &= \frac{\bar{x}\bar{y}}{(a+\bar{x})^2},\\
k_3 &= n\bar{y}\bar{z}\frac{\bar{y}^2-b}{(b+\bar{y}^2)^2}\left\{m\bar{x}\bar{z}\frac{\bar{x}^2-b}{(b+\bar{x}^2)^2} - \frac{r}{K}\bar{x}\right\} + 2mm_1b\frac{\bar{x}^3\bar{z}}{(b+\bar{x}^2)^3},\\
k_4 &= n\bar{x}\bar{y}^2\bar{z}\frac{(b-\bar{y}^2)}{(a+\bar{x})^2(b+\bar{y}^2)^2} - a\beta\frac{\bar{x}\bar{y}}{(a+\bar{x})^3},\\
k_5 &= 2abmn_1\beta\frac{\bar{x}^2\bar{y}^2\bar{z}}{(b+\bar{x}^2)(a+\bar{x})^2(b+\bar{y}^2)^2} - 2nmm_1b\frac{\bar{x}^3\bar{y}\bar{z}^2(\bar{y}^2-b)}{(b+\bar{x}^2)^3(b+\bar{y}^2)^2},\\
k_6 &= 2nm_1b\frac{\bar{x}^2\bar{y}^2\bar{z}}{(a+\bar{x})(b+\bar{y}^2)(b+\bar{x}^2)^2}.
\end{aligned}$$

Now using Routh-Hurwitz criteria around the interior equilibrium point, we state and prove the following theorem for the local asymptotic stability of the system (3.3).

Theorem 3.4.1. *The system* (3.3) *will be locally asymptotically stable around its interior equilibrium point, if* $min\{\frac{k_1}{k_2}, \frac{k_3}{k_4}, \frac{k_5}{k_6}\} > \alpha > \alpha^*$, *where* α^* *is the largest root of the equation* $\psi(\alpha) = k_2k_4\alpha^2 + (k_6 - k_2k_3 - k_1k_4)\alpha + (k_1k_3 - k_5) = 0$.

Proof: The system will be locally asymptotically stable at the interior equilibrium point $\overline{B}(\bar{x}, \bar{y}, \bar{z})$, if Routh-Hurwitz criteria around the interior equilibrium

point holds.
Using Routh-Hurwitz criteria, we conclude that all the eigen values of the system (3.3) contain the negative real part at $\bar{B}$. i.e., all the roots of the equation (3.4) have negative real part, i.e., $h_1, h_3 > 0$ and $h_1 h_2 > h_3$.
Again $h_1, h_3 > 0$ and $h_1 h_2 > h_3$ when $\min\{\frac{k_1}{k_2}, \frac{k_3}{k_4}, \frac{k_5}{k_6}\} > \alpha > \alpha^*$.
Hence, the system is locally asymptotically stable at the interior equilibrium point.
This completes the proof of the theorem.
For more analysis from Theorem 3.4.1, we state the following lemma.

Lemma 1. *From the above theorem, we can conclude that the system* (3.3) *will be locally asymptotically stable for $\alpha > \alpha^*$ and unstable for $\alpha < \alpha^*$.*

Theorem 3.4.2. *The system* (3.3) *undergoes through a Hopf bifurcation at its interior equilibrium for $\alpha = \alpha^*$.*

Proof: For $\alpha = \alpha^*$, we have $h_1 h_2 - h_3 = 0$ and then the eigenvalues of the system at $\bar{B}$ can be represented as $\lambda_1 = -h_1$ and $\lambda_{2,3} = \pm i\sqrt{h_2}$. Considering $\lambda_1 = \phi_1(\alpha)$ and $\lambda_{2,3} = \phi_2(\alpha) \pm i\phi_3(\alpha)$. Now it is clearly to show that $\frac{d\phi}{d\alpha} = 0$ at the point $\alpha = \alpha^*$. Again we have $\phi(\alpha) = 0$. Therefore, it is obvious to show that our system (3.3) follows a Hopf bifurcation at its interior equilibrium for the critical value of α, i.e., for $\alpha = \alpha^*$, with the help of given conditions (Venkatsubramanian et al. (103)).
So, this suggests the proof of the theorem.

The constant parameters involving in the prey-predator model are generally described for approaching the steady state where the species coexists in equilibrium. The dynamical behavior of the model may vary if the parameters involved in the system are changed and then the values of the critical parameters at which such effects happened are known as bifurcation points. The main aim of this study is to determine the stability behavior of the model due to presence of various density-dependent factors to the prey-predator interactions. To study the transition of the system with respect to the small changes in the density dependent factors, we consider α, m and n as bifurcation parameters and α^*, m^* and n^* denote the critical values or the bifurcating values of the concerned bifurcation parameters.

Now we can also choose h_1, h_2 and h_3 with respect to the parameter n, in the form as

$h_1 = l_1 + l_2 n$, $h_2 = l_3 + l_4 n$ and $h_3 = l_5 - l_6 n$, where

$$
\begin{aligned}
l_1 &= \frac{r}{K}\bar{x} + m\bar{x}\bar{z}\frac{b-\bar{x}^2}{(b+\bar{x}^2)^2} - \frac{\alpha\bar{x}\bar{y}}{(a+\bar{x})^2},\\
l_2 &= \bar{y}\bar{z}\frac{b-\bar{y}^2}{(b+\bar{y}^2)^2},\\
l_3 &= a\alpha\beta\frac{\bar{x}\bar{y}}{(a+\bar{x})^3} + 2mm_1 b\frac{\bar{x}^3\bar{z}}{(b+\bar{x}^2)^3},\\
l_4 &= \bar{y}\bar{z}\frac{\bar{y}^2-b}{(b+\bar{y}^2)^2}\left\{\frac{\alpha\bar{x}\bar{y}}{(a+\bar{x})^2} + m\bar{x}\bar{z}\frac{\bar{x}^2-b}{(b+\bar{x}^2)^2} - \frac{r}{K}\bar{x}\right\},\\
l_5 &= 2abmn_1\beta\frac{\bar{x}^2\bar{y}^2\bar{z}}{(b+\bar{x}^2)(a+\bar{x})^2(b+\bar{y}^2)^2},\\
l_6 &= 2m_1 b\alpha\frac{\bar{x}^2\bar{y}^2\bar{z}}{(a+\bar{x})(b+\bar{y}^2)(b+\bar{x}^2)^2} + 2mm_1 b\frac{\bar{x}^3\bar{y}\bar{z}^2(\bar{y}^2-b)}{(b+\bar{x}^2)^3(b+\bar{y}^2)^2}.
\end{aligned}
$$

Theorem 3.4.3. *The system* (3.3) *undergoes through a bifurcation at its interior equilibrium for $n = n^*$ where $n^* = \frac{l_5}{l_6}$ provided that $n^* > \max\{0, -\frac{l_1}{l_2}, -\frac{l_3}{l_4}\}$.*

Proof: For $n = n^*$, we have $h_3 = 0$ (where $h_3 = l_5 - l_6 n$). Then the characteristic equation is $\lambda^3 + h_1\lambda^2 + h_2\lambda = 0$. It can be concluded that from the equation (3.1), both h_1 and h_2 are positive since $n^* > \max\{0, -l_1/l_2, -l_3/l_4\}$. Consequently, the characteristic equation has a simple zero eigen value around its interior equilibrium for $n = n^*$. Hence, the system (3.3) passes through a bifurcation at $n = n^*$ around its interior equilibrium point $\bar{B}$.

Theorem 3.4.4. *The system* (3.3) *also undergoes through a bifurcation at its interior equilibrium for $m = m^*$ where m^* can be obtained by solving $h_3 = 0$ provided m^* is positive; h_1 and h_2 are also positive.*

Proof: Straight forward.

3.5 Global stability

Here we consider the general method (Li and Muldowney (58)) to show an n-dimensional autonomous dynamical system $f : D \to R^n$, $D \subset R^n$, an open and simply connected set and $f \in C^1(D)$, where the dynamical system is as follows:

$$\frac{dx}{dt} = f(x) \tag{3.5}$$

which is globally stable under certain parametric conditions. We refer to the works of Haque et al. (39), Bunomo et al. (10), Kar and Mondal (50) for detailed discussion.

Now we consider the conditions as stated in below:

(1) The autonomous dynamical system (3.5) has a unique interior equilibrium point $\bar{x}$ in D.

(2) The domain D is simply connected.

(3) There is a compact absorbing set $\Omega \subset D$.

The unique interior equilibrium point $\bar{x}$ in D of the system (3.5) is globally asymptotically stable if the system is locally asymptotically stable and all the trajectories in D converges to its interior equilibrium point.

Theorem 3.5.1. *The system* (3.3) *is globally asymptotically stable around its interior equilibrium point if* $d_2 + qE < \mu_2$, *where* $\mu_2 = n_1\frac{y^2}{b+y^2} + m_1\frac{x^2}{b+x^2} + min\{mxz\frac{b-x^2}{(b+x^2)^2} + nyz\frac{b-y^2}{(b+y^2)^2} - \frac{\alpha xy}{(a+x)^2} + \frac{r}{K}x - m\frac{x^2}{b+x^2},\ \ mxz\frac{b-x^2}{(b+x^2)^2} - \frac{\alpha xy}{(a+x)^2} + \frac{r}{K}x - \frac{\beta ay}{(a+x)^2} - \frac{2bn_1yz}{(b+y^2)^2}, \frac{\alpha x}{a+x} + nyz\frac{b-y^2}{(b+y^2)^2} - \frac{2bn_1yz}{(b+y^2)^2}\}$.

Proof: Let $J^{[2]}$ be the second additive compound matrix with order ${}^3C_2 \times {}^3C_2$. Hence, we have,

$$J^{[2]} = \frac{\partial f^{[2]}}{\partial x} = \begin{pmatrix} J_{11} + J_{22} & J_{23} & -J_{13} \\ J_{32} & J_{11} + J_{33} & J_{12} \\ -J_{31} & J_{21} & J_{22} + J_{33} \end{pmatrix}$$

Now we introduce the above expression in our system to show that our system (3.3) will be globally stable around its interior equilibrium point $\bar{B}$. The system of equation (3.3) can be described as below:

$$\frac{dX}{dt} = f(X)$$

where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $f(X) = \begin{pmatrix} rx\left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a+x} - m\frac{x^2 z}{b+x^2} \\ \frac{\beta xy}{a+x} - d_1 y - \frac{ny^2 z}{b+y^2} \\ n_1\frac{y^2 z}{b+y^2} + m_1\frac{x^2 z}{b+x^2} - d_2 z - qEz \end{pmatrix}$.

Then

$$V = \frac{\partial f}{\partial X} = \begin{pmatrix} mxz\frac{x^2-b}{(b+x^2)^2} + \frac{\alpha xy}{(a+x)^2} - \frac{r}{K}x & -\frac{\alpha x}{a+x} & -m\frac{x^2}{b+x^2} \\ \frac{\beta ay}{(a+x)^2} & nyz\frac{y^2-b}{(b+y^2)^2} & -n\frac{y^2}{b+y^2} \\ 2m_1 b\frac{xz}{(b+x^2)^2} & 2bn_1\frac{yz}{(b+y^2)^2} & 0 \end{pmatrix} \tag{3.6}$$

where $V(x, y, z) = (J_{ij})_3$ be the Jacobian matrix of the system (3.3) at its interior equilibrium point.

$$J^{[2]} = \begin{pmatrix} \frac{mxz(x^2-b)}{(b+x^2)^2} + \frac{nyz(y^2-b)}{(b+y^2)^2} + \frac{\alpha xy}{(a+x)^2} - \frac{r}{K}x & -\frac{ny^2}{b+y^2} & \frac{mx^2}{b+x^2} \\ \frac{2bn_1 yz}{(b+y^2)^2} & \frac{mxz(x^2-b)}{(b+x^2)^2} + \frac{\alpha xy}{(a+x)^2} - \frac{r}{K}x & -\frac{\alpha x}{a+x} \\ -\frac{2m_1 bxz}{(b+x^2)^2} & \frac{\beta ay}{(a+x)^2} & \frac{nyz(y^2-b)}{(b+y^2)^2} \end{pmatrix}$$

We consider $M(X) \in C^1(D)$ in such a way that $M = diag\{x/z, x/z, x/z\}$. Then $M_f M^{-1} = diag\{\dot{x}/x - \dot{z}/z, \dot{x}/x - \dot{z}/z, \dot{x}/x - \dot{z}/z\}$ and $MJ^{[2]}M^{-1} = J^{[2]}$, where matrix M_f is obtained by replacing each entity M_{ij} of M by its derivative in the direction of solution (3.3). In addition, we have

$$B = M_f M^{-1} + MJ^{[2]}M^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where M_f is represented by

$$(M_{ij}(X))_f = \left(\frac{\partial M_{ij}}{\partial x}\right)^t f(x) = \nabla M_{ij}.f(x), \tag{3.7}$$

and $B_{11} = \dot{x}/x - \dot{z}/z + mxz\frac{x^2-b}{(b+x^2)^2} + nyz\frac{y^2-b}{(b+y^2)^2} + \frac{\alpha xy}{(a+x)^2} - \frac{r}{K}x$,

$B_{12} = \begin{pmatrix} -n\frac{y^2}{b+y^2} & m\frac{x^2}{b+x^2} \end{pmatrix}$, $\quad B_{21} = \begin{pmatrix} 2bn_1\frac{yz}{(b+y^2)^2} \\ -2m_1 b\frac{xz}{(b+x^2)^2} \end{pmatrix}$,

$B_{22} = \begin{pmatrix} \dot{x}/x - \dot{z}/z + mxz\frac{x^2-b}{(b+x^2)^2} + \frac{\alpha xy}{(a+x)^2} - \frac{r}{K}x & -\frac{\alpha x}{a+x} \\ \frac{\beta ay}{(a+x)^2} & \dot{x}/x - \dot{z}/z + nyz\frac{y^2-b}{(b+y^2)^2} \end{pmatrix}$.

Let (u_1, u_2, u_3) denote the vector in R^3, choose a norm in R^3 as $|u_1, u_2, u_3| =$

$max\{|u_1|, |u_2|+|u_3|\}$ and let Γ be the Lozinskii measure (68) of B with respect to with a vector norm in R^N, $N = ^n C_2$, then we get

$$\Gamma(B) = \lim_{h \to 0^+} \frac{|l + hB| - 1}{h}. \tag{3.8}$$

If the conditions (1), (2) and (3) hold then we can write the following inequality (Li and Muldowney (58)) as:

$$\lim\sup\sup \frac{1}{t} \int_0^t \Gamma(B(x(s, x_0)))ds < 0. \tag{3.9}$$

The condition (3.9) ensures that there are no orbits (i.e., homoclinic orbits, heteroclinic cycles and periodic orbits) which give rise to a simple closed rectifiable curve in D, invariant for the system (3.5). It is also a robust Bendixson criterion. Now, based on the above discussion, we are to show that our system (3.3) is globally stable around its interior equilibrium. Then, we have the following estimate (Li and Muldowney (58)):

$$\Gamma(B) \leq \sup\{b_1, b_2\}, \tag{3.10}$$

where $b_1 = \Gamma_1(B_{11}) + |B_{12}|$, $b_2 = |B_{21}| + \Gamma_1(B_{22})$ and Γ_1 denotes the Lozinskii measure with respect to l_1 vector norm, $|B_{12}|$ and $|B_{21}|$ are matrix norms with respect to l_1 norm. Then we get

$$\begin{aligned}
\Gamma_1(B_{11}) &= \dot{x}/x - \dot{z}/z + mxz\frac{x^2 - b}{(b + x^2)^2} + nyz\frac{y^2 - b}{(b + y^2)^2} + \frac{\alpha xy}{(a + x)^2} - \frac{r}{K}x, \\
|B_{12}| &= m\frac{x^2}{b + x^2}, \\
|B_{21}| &= \frac{2bn_1yz}{(b + y^2)^2}, \\
\Gamma_1(B_{22}) &= \frac{\dot{x}}{x} - \frac{\dot{z}}{z} + max\Big\{mxz\frac{x^2 - b}{(b + x^2)^2} + \frac{\alpha xy}{(a + x)^2} - \frac{r}{K}x + \frac{\beta ay}{(a + x)^2}, \\
& \quad nyz\frac{y^2 - b}{(b + y^2)^2} - \frac{\alpha x}{a + x}\Big\},
\end{aligned}$$

Hence

$$b_1 = \dot{x}/x - \dot{z}/z + mxz\frac{x^2 - b}{(b + x^2)^2} + nyz\frac{y^2 - b}{(b + y^2)^2} + \frac{\alpha xy}{(a + x)^2} - \frac{r}{K}x + m\frac{x^2}{b + x^2},$$

and

$$\begin{aligned}
b_2 &= \frac{\dot{x}}{x} - \frac{\dot{z}}{z} + \frac{2bn_1yz}{(b+y^2)^2} + max\Big\{mxz\frac{x^2-b}{(b+x^2)^2} + \frac{\alpha xy}{(a+x)^2} - \frac{r}{K}x + \frac{\beta ay}{(a+x)^2}, \\
&\quad nyz\frac{y^2-b}{(b+y^2)^2} - \frac{\alpha x}{a+x}\Big\}.
\end{aligned}$$

Now, using $\frac{\dot{z}}{z} = n_1\frac{y^2}{b+y^2} + m_1\frac{x^2}{b+x^2} - d_2 - qE$ from the system (3.3), the expression becomes,

$$\begin{aligned}
b_1 &= \frac{\dot{x}}{x} - n_1\frac{y^2}{b+y^2} - m_1\frac{x^2}{b+x^2} + d_2 + qE + mxz\frac{x^2-b}{(b+x^2)^2} + nyz\frac{y^2-b}{(b+y^2)^2} \\
&\quad + \frac{\alpha xy}{(a+x)^2} - \frac{r}{K}x + m\frac{x^2}{b+x^2},
\end{aligned}$$

and

$$\begin{aligned}
b_2 &= \frac{\dot{x}}{x} - n_1\frac{y^2}{b+y^2} - m_1\frac{x^2}{b+x^2} + d_2 + qE + \frac{2bn_1yz}{(b+y^2)^2} + max\Big\{mxz\frac{x^2-b}{(b+x^2)^2} \\
&\quad + \frac{\alpha xy}{(a+x)^2} - \frac{r}{K}x + \frac{\beta ay}{(a+x)^2}, \quad nyz\frac{y^2-b}{(b+y^2)^2} - \frac{\alpha x}{a+x}\Big\} \\
&= \frac{\dot{x}}{x} - n_1\frac{y^2}{b+y^2} - m_1\frac{x^2}{b+x^2} + d_2 + qE + \frac{2bn_1yz}{(b+y^2)^2} - min\Big\{mxz\frac{b-x^2}{(b+x^2)^2} \\
&\quad - \frac{\alpha xy}{(a+x)^2} + \frac{r}{K}x - \frac{\beta ay}{(a+x)^2}, \quad \frac{\alpha x}{a+x} + nyz\frac{b-y^2}{(b+y^2)^2}\Big\}.
\end{aligned}$$

Now, from (3.10) we get

$$\begin{aligned}
\Gamma(B) &\leq \frac{\dot{x}}{x} - n_1\frac{y^2}{b+y^2} - m_1\frac{x^2}{b+x^2} + d_2 + qE - min\Big\{mxz\frac{b-x^2}{(b+x^2)^2} \\
&\quad + nyz\frac{b-y^2}{(b+y^2)^2} - \frac{\alpha xy}{(a+x)^2} + \frac{r}{K}x - m\frac{x^2}{b+x^2}, \quad mxz\frac{b-x^2}{(b+x^2)^2} \\
&\quad - \frac{\alpha xy}{(a+x)^2} + \frac{r}{K}x - \frac{\beta ay}{(a+x)^2} - \frac{2bn_1yz}{(b+y^2)^2}, \frac{\alpha x}{a+x} + nyz\frac{b-y^2}{(b+y^2)^2} \\
&\quad - \frac{2bn_1yz}{(b+y^2)^2}\Big\}, \\
i.e., \Gamma(B) &\leq \frac{\dot{x}}{x} + d_2 + qE - \mu_2,
\end{aligned}$$

where, $\mu_2 = n_1\frac{y^2}{b+y^2} + m_1\frac{x^2}{b+x^2} + min\{mxz\frac{b-x^2}{(b+x^2)^2} + nyz\frac{b-y^2}{(b+y^2)^2} - \frac{\alpha xy}{(a+x)^2} + \frac{r}{K}x - m\frac{x^2}{b+x^2}, \quad mxz\frac{b-x^2}{(b+x^2)^2} - \frac{\alpha xy}{(a+x)^2} + \frac{r}{K}x - \frac{\beta ay}{(a+x)^2} - \frac{2bn_1yz}{(b+y^2)^2}, \frac{\alpha x}{a+x} + nyz\frac{b-y^2}{(b+y^2)^2} - \frac{2bn_1yz}{(b+y^2)^2}\}$.

i.e.,

$$\frac{1}{t}\int_0^t \Gamma(B)ds \leq \frac{1}{t}log\frac{x(t)}{x(0)} - (g_2 - d_2).$$

Therefore,

$$\lim_{t\to\infty} sup \sup \frac{1}{t}\int_0^t \Gamma(B(s,x_0))ds < -(\mu_2 - d_2 - qE) < 0, i.e, d_2 + qE < \mu_2.$$

This shows the proof of the theorem.

3.6 Optimal Control

Here, a harvesting effort is applied to the generalist predator population. Now, the aim is to calculate the optimal profit in bionomic equilibrium state. The bionomic equilibrium is a concept of economic equilibrium as well as biological equilibrium. The net economic revenue obtained from the fishery is $p(x, y, z, E, t)$ = The total revenue obtained by selling the harvested biomass − the total cost for the effort devoted to harvesting $= pqEz - cE$, where p is the constant price per unit biomass of the generalist predator and c is the constant cost per unit effort, then we consider the present value J of a continuous time-stream of revenues as

$$J = \int_0^\infty e^{-\delta t} p(x, y, z, E, t)dt, \tag{3.11}$$

where δ denotes the instantaneous annual rate of discount (82). Our problem is to maximize J subject to the state equations (3.3) using *Pontryagin's maximum principle* (80). The control variable $E(t)$ is subject to the constraint set $0 \leq E \leq E_{max}$. At first, we construct the corresponding Hamiltonian function as follows:

$$\begin{aligned} H = \quad & e^{-\delta t}(pqz - c)E + \lambda_1 \left\{ rx\left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a + x} - m\frac{x^2 z}{b + x^2} \right\} + \lambda_2 \Big(\frac{\beta xy}{a + x} \\ & -d_1 y - \frac{ny^2 z}{b + y^2}\Big) + \lambda_3 \left(\frac{n_1 y^2 z}{b + y^2} + \frac{m_1 x^2 z}{b + x^2} - d_2 z - qEz \right) \end{aligned} \tag{3.12}$$

where λ_i $(i = 1, 2, 3)$ are called the adjoint variables.

By Pontryagin's maximum principle, the adjoint equations are as follows:

$$\begin{aligned}
\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x} &= -\lambda_1 \left\{ r - 2\frac{r}{K}x - \frac{a\alpha y}{(a+x)^2} - 2mb\frac{xz}{(b+x^2)^2} \right\} \\
&\quad -\lambda_2 a\beta \frac{y}{(a+x)^2} - 2\lambda_3 m_1 b \frac{xz}{(b+x^2)^2}, \qquad (3.13) \\
\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y} &= \lambda_1 \frac{\alpha x}{a+x} + \lambda_2 \left\{ 2nb\frac{yz}{(b+y^2)^2} - \beta\frac{x}{a+x} + d_1 \right\} \\
&\quad -2\lambda_3 n_1 b \frac{yz}{(b+y^2)^2}, \qquad (3.14) \\
\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial z} &= -pqEe^{-\delta t} - \lambda_3 \left\{ n_1\frac{y^2}{b+y^2} + m_1\frac{x^2}{b+x^2} - d_2 - qE \right\} \\
&\quad +\lambda_1 m \frac{x^2}{b+x^2} + \lambda_2 n \frac{y^2}{b+y^2}. \qquad (3.15)
\end{aligned}$$

Now we derive an optimal equilibrium solution of the problem at the interior equilibrium $E^*(x^*, y^*, z^*)$. Then from equations (3.13), (3.14) and (3.15), we get

$$\frac{d\lambda_1}{dt} = \lambda_1 A_1 - \lambda_2 A_2 - \lambda_3 A_3, \qquad (3.16)$$

$$\frac{d\lambda_2}{dt} = \lambda_1 A_4 + \lambda_2 A_5 - \lambda_3 A_6, \qquad (3.17)$$

$$\frac{d\lambda_3}{dt} = -pqEe^{-\delta t} + \lambda_1 A_7 + \lambda_2 A_8 - \lambda_3 A_9. \qquad (3.18)$$

where $A_1 = 2\frac{r}{K}x^* + \frac{a\alpha y^*}{(a+x^*)^2} + 2mb\frac{x^*z^*}{(b+x^{*2})^2} - r$, $A_2 = a\beta\frac{y^*}{(a+x^*)^2}$, $A_3 = 2m_1 b\frac{x^*z^*}{(b+x^{*2})^2}$, $A_4 = \alpha\frac{x^*}{a+x^*}$, $A_5 = n\frac{(b-y^{*2})y^*z^*}{(b+y^{*2})^2}$, $A_6 = 2n_1 b\frac{y^*z^*}{(b+y^{*2})^2}$, $A_7 = m\frac{x^{*2}}{b+x^{*2}}$, $A_8 = n\frac{y^{*2}}{b+y^{*2}}$, $A_9 = 0$. Solving these three equations (3.16), (3.17) and (3.18), we get a third order differential equation in λ_1. i.e.,

$$(D^3 + H_1 D^2 + H_2 D + H_3)\lambda_1 = H_4 pqEe^{-\delta t} \qquad (3.19)$$

where

$$\begin{aligned}
D &\equiv \frac{d}{dt} \\
H_1 &= A_9 - A_1 - A_5 \\
H_2 &= A_3A_7 + A_1A_5 + A_2A_4 - (A_1A_9 + A_5A_9 + A_6A_8) \\
H_3 &= A_1A_5A_9 + A_1A_6A_8 + A_2A_4A_9 - (A_2A_6A_7 + A_3A_5A_7 + A_3A_4A_8) \\
H_4 &= -(A_2A_6 + A_3A_5 + A_3\delta)
\end{aligned}$$

Using Laplace transform and denoting $L(\lambda_1) = \bar{\lambda_1}$, we solve the equation (3.19) and obtain the following result as:

$$\begin{aligned}\bar{\lambda_1} &= \frac{H_4pqE}{(s+\delta)(s^3+H_1s^2+H_2s+H_3)} + \frac{C_1(s^2+H_1s+H_2)}{s^3+H_1s^2+H_2s+H_3} \\ &\quad + \frac{(s+H_1)C_2}{s^3+H_1s^2+H_2s+H_3} + \frac{C_3}{s^3+H_1s^2+H_2s+H_3}\end{aligned}$$

Where $C_1 = \lambda_1(0)$, $C_2 = \lambda_1'(0)$ and $C_3 = \lambda_1''(0)$.

Now we consider $s^3 + H_1s^2 + H_2s + H_3 = (s+\alpha)(s+\beta)(s+\gamma)$ where $\sum \alpha = H_1$, $\sum \alpha\beta = H_2$ and $\alpha\beta\gamma = H_3$. Then by inverse Laplace transform we get,

$$\begin{aligned}\lambda_1 &= H_4pqE\{-\frac{e^{-\alpha t}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{e^{-\beta t}}{(\alpha-\beta)(\beta-\gamma)(\beta-\delta)} \\ &\quad + \frac{e^{-\gamma t}}{(\alpha-\gamma)(-\beta+\gamma)(\gamma-\delta)} + \frac{e^{-\delta t}}{(\gamma-\delta)(\delta-\alpha)(\delta-\beta)}\} \\ &\quad + \frac{e^{-\alpha t}}{(\alpha-\beta)(\alpha-\gamma)}(C_3 - H_3C_1 - C_2\alpha + C_2H_1) \\ &\quad + \frac{e^{-\beta t}}{(\alpha-\beta)(\beta-\gamma)}(-C_3 + H_3C_1 + C_2\beta - C_2H_1) \\ &\quad + \frac{e^{-\gamma t}}{(\alpha-\gamma)(\beta-\gamma)}(C_3 - H_3C_1 - C_2\gamma + C_2H_1) + C_1\end{aligned}$$

In similar way we can derive the values of λ_2 and λ_3.

Using the values of λ_1, λ_2 and λ_3, we get Hamiltonian function from (3.12) by which we can determine the optimality using Pontryagin's maximum principle. The numerical illustrations of this system are discussed in the next section.

3.7 Numerical simulation

Some arbitrary data are assumed for describing the analytical results. Using the MATLAB 7.10 software, we analyze the sensitivity analysis of the experiments. Again, we observe that the parameters involved in the system are not taken into consideration from real-life problems, so the prime characteristics are analyzed by the simulations described here should be treated from a qualitative rather than a quantitative point of view. However, numerous scenarios covering the breadth of the biological feasible parameter space are conducted

and the results shown above display the gamut of dynamical results collected from all the scenarios tested. Assume that parameter set is taken as
$P_1 = \{r = 1.1; K = 18; a = 6; \beta = 1.2; b = 7; n_1 = 0.9; d_1 = 0.3; d_2 = 0.12; m_1 = 0.2; m = 0.4; q = 0.2; E = 0.76; n = 1.3;\}$ and initial point taken as $B(13, 10, 7)$. According to our theoretical result, the system is locally asymptotically stable for $\alpha > \alpha^*$ and unstable for $\alpha < \alpha^*$. For the set of parametric value P_1, from Figures 3.1 and 3.2, we see that for the value of $\alpha(= 4.2) > \alpha^*$ the system will be stable. Again from Figures 3.3 and 3.4, we observe that for the value of $\alpha(= 3.9) < \alpha^*$ the system will be unstable.

It is well known that fish (generalist predator) has an economical demand in marketing management. Due to this fact, we include the harvesting for generalist predator population in the dynamical system. From Figures 3.1 and 3.2 with the parameter set $P_2 = \{r = 1.1; K = 18; a = 6; \beta = 1.2; b = 7; n_1 = 0.9; d_1 = 0.3; d_2 = 0.12; m_1 = 0.2; m = 0.4; q = 0.2; E = 0.76; \alpha = 4.2; n = 1.3\}$, here we observe that system with harvesting is stable after certain time. But when we consider same parameter set P_2 in absence of harvesting i.e., $q = 0; E = 0$, then we note that the system is unstable from Figures 3.5 and 3.6. Therefore from Figures 3.1 and 3.2, we conclude that the system is stable with harvesting whereas the system is unstable at the same time without harvesting with the help of Figures 3.5 and 3.6.

For the existence of the system, the intrinsic growth rate of prey population has an important role. From Figure 3.7, we see that the density of the prey population is directly proportional to the intrinsic growth rate of prey population. The sensitivity of environmental carrying capacity of prey population is described in Figure 3.8. From Figure 3.8, it is seen that the change of K is directly proportional to the density of the population. Functional response is most important concept to describe the prey-predator interaction. Figures 3.9 and 3.10 illustrate the sensitivity of prey-predator interaction. From Figures 3.9 and 3.10, it is seen that α and m are inversely proportional to the density of three populations. Figure 3.11 shows that the change of β is directly proportional to the density of predator population and same result for the generalist

predator whereas the change of β is inversely proportional to the density of prey population. Again from Figures 3.12 and 3.13, it is observed that the change of n and n_1 are directly proportional to the density of prey population whereas the change of n and n_1 are inversely proportional to the density of predator population and same result for the generalist predator.
Figure 3.14 shows that, the natural death rate of predator population d_1 is inversely proportional to the density of the predator and generalist predator population and directly proportional to the change of the density of prey population.

3.8 Chapter Summary

Prey-predator model with three species has been described. Here the effects on a prey of two predators which are also related in a prey-predator relationship have been considered. Also, different types of functional responses have been considered for predator and generalist predator. The harvesting effort has been applied only for the generalist predator. The density-dependent mortality rates for the predator and generalist predator have been considered. The local stability as well as global stability for the system at the interior equilibrium point have been discussed. Different parameters have been considered as a bifurcation parameter to evaluate Hopf bifurcation in the neighborhood of interior equilibrium point. With different set of parameters, the model has been verified through numerical simulations and graphical Figures.

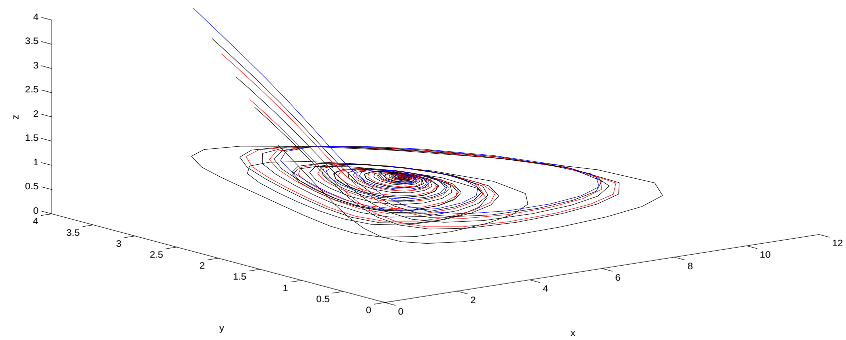

Figure 3.1: Phase space diagram of the system (3.3) with the parameter set $\{r = 1.1; K = 18; a = 6; \beta = 1.2; b = 7; n_1 = 0.9; d_1 = 0.3; d_2 = 0.12; m_1 = 0.2; m = 0.4; q = 0.2; E = 0.76; n = 1.3\}$ and $\alpha = 4.2$ with respect to x,y and z.

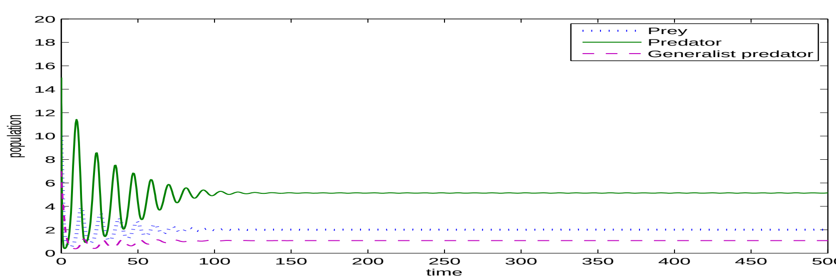

Figure 3.2: Graphical representation of the system (3.3) with parameter set $\{r = 1.1; K = 18; a = 6; \beta = 1.2; b = 7; n_1 = 0.9; d_1 = 0.3; d_2 = 0.12; m_1 = 0.2; m = 0.4; q = 0.2; E = 0.76; n = 1.3\}$ and $\alpha = 4.2$.

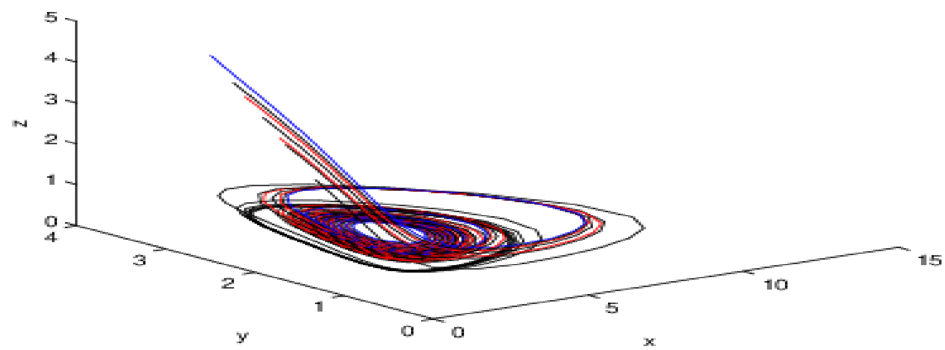

Figure 3.3: Phase space diagram of the system (3.3) with the parameter set $\{r = 1.1; K = 18; a = 6; \beta = 1.2; b = 7; n_1 = 0.9; d_1 = 0.3; d_2 = 0.12; m_1 = 0.2; m = 0.4; q = 0.2; E = 0.76; n = 1.3\}$ and $\alpha = 3.9$ with respect to *x,y,z*.

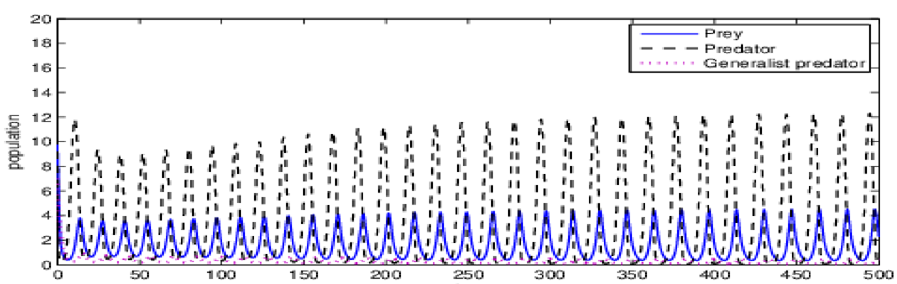

Figure 3.4: Graphical representation of the system (3.3) with parameter set $\{r = 1.1; K = 18; a = 6; \beta = 1.2; b = 7; n_1 = 0.9; d_1 = 0.3; d_2 = 0.12; m_1 = 0.2; m = 0.4; q = 0.2; E = 0.76; n = 1.3\}$ and $\alpha = 3.9$.

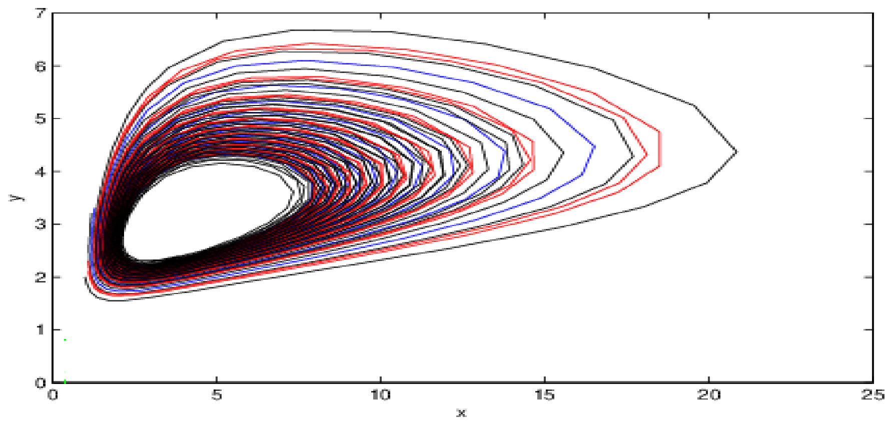

Figure 3.5: Phase space diagram of the system (3.3) in absence of harvesting.

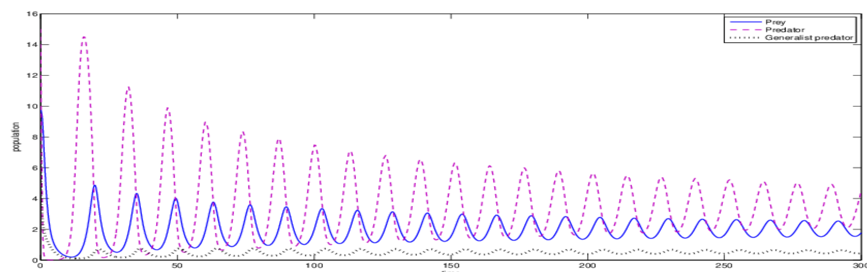

Figure 3.6: Solution curve of the system (3.3) in absence of harvesting.

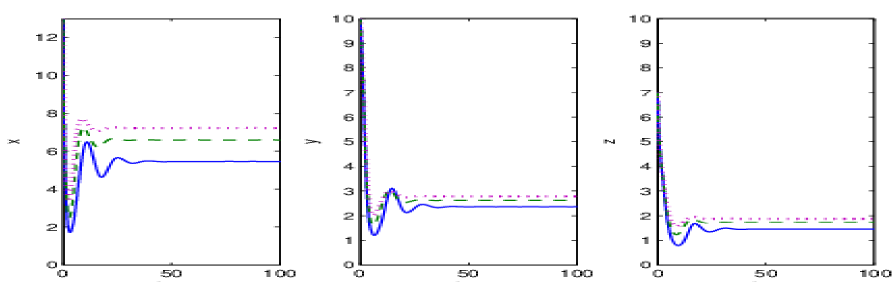

Figure 3.7: *Change of* x, y *and* z *of the system* (3.3) *with respect to change of intrinsic growth rate of prey population with parameter set* $\{K = 9.8; a = 6; \beta = 1.2; b = 7; n = 1; n_1 = 0.9; d_1 = 0.3; d_2 = 0.12; m_1 = 0.2; m = 0.4; q = 0.2; E = 0.76; \alpha = 1.3;\}$. *Here (—) line corresponds to* $r = 0.8$, *(- - -) line to* $r = 1.1$ *and* $(\cdots)$ *line to* $r = 1.4$.

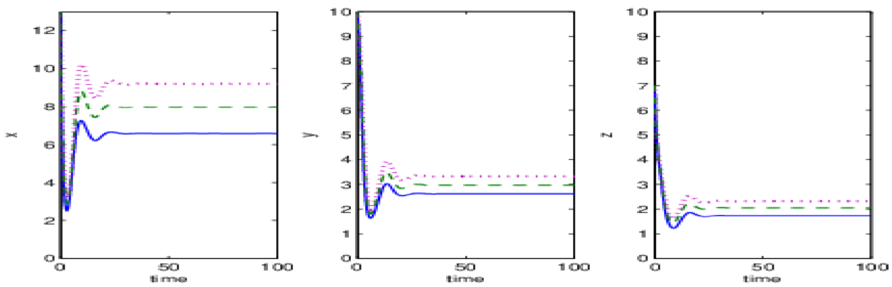

Figure 3.8: *Change of x, y and z of the system* (3.3) *with respect to change of carrying capacity of prey population K with parameter set* $\{r = 1.1; a = 6; \beta = 1.2; b = 7; n = 1; n_1 = 0.9; d_1 = 0.3; d_2 = 0.12; m_1 = 0.2; m = 0.4; q = 0.2; E = 0.76; \alpha = 1.3;\}$. *Here (—) line corresponds to $K = 9.8$, (- - -) line to $K = 12$ and ($\cdots$) line to $K = 14$.*

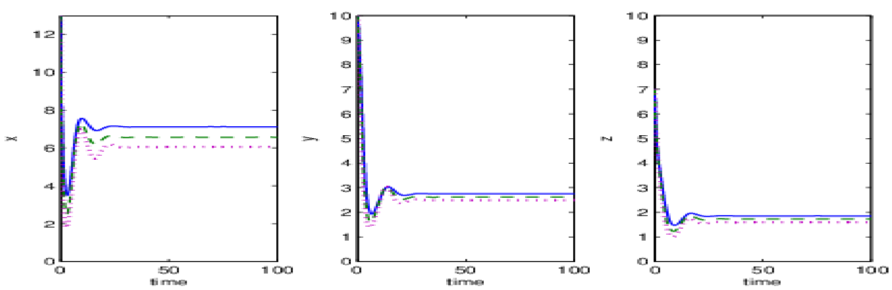

Figure 3.9: *Change of x, y and z of the system* (3.3) *with respect to change of α with parameter set* $\{r = 1.1; K = 9.8; a = 6; \beta = 1.2; b = 7; n = 1; n_1 = 0.9; d_1 = 0.3; d_2 = 0.12; m_1 = 0.2; m = 0.4; q = 0.2; E = 0.76;\}$. *Here (—) line corresponds to $\alpha = 1$, (- - -) line to $\alpha = 1.3$ and ($\cdots$) line to $\alpha = 1.6$.*

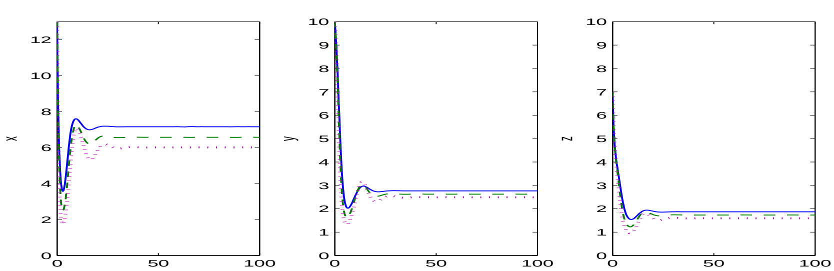

Figure 3.10: *Change of x, y and z of the system* (3.3) *with respect to change of m with parameter set* $\{r = 1.1; K = 9.8; a = 6; \beta = 1.2; b = 7; n = 1; n_1 = 0.9; d_1 = 0.3; d_2 = 0.12; m_1 = 0.2; q = 0.2; E = 0.76; \alpha = 1.3;\}$. *Here (—) line corresponds to $m = 0.1$, (- - -) line to $m = 0.4$ and ($\cdots$) line to $m = 0.7$.*

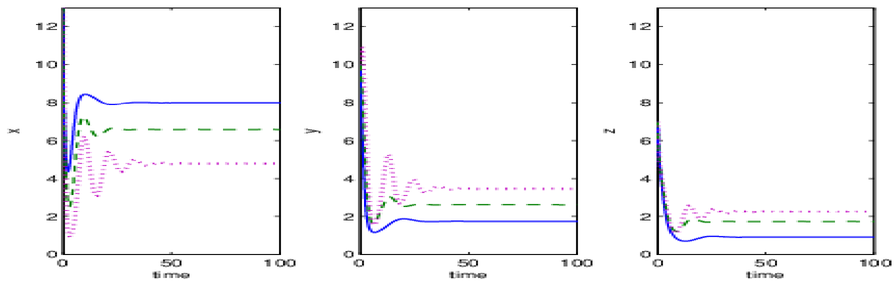

Figure 3.11: *Change of x, y and z of the system* (3.3) *with respect to change of β with parameter set* $\{r = 1.1; K = 9.8; a = 6; b = 7; n = 1; n_1 = 0.9; d_1 = 0.3; d_2 = 0.12; m_1 = 0.2; m = 0.4; q = 0.2; E = 0.76; \alpha = 1.3; \}$. *Here (—) line corresponds to $\beta = 0.8$, (- - -) line to $\beta = 1.2$ and ($\cdots$) line to $\beta = 1.6$.*

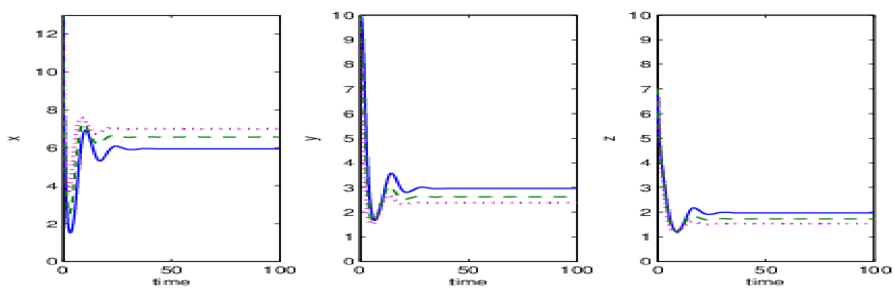

Figure 3.12: *Change of x, y and z of the system* (3.3) *with respect to change of n with parameter set* $\{r = 1.1; K = 9.8; a = 6; \beta = 1.2; b = 7; n_1 = 0.9; d_1 = 0.3; d_2 = 0.12; m_1 = 0.2; m = 0.4; q = 0.2; E = 0.76; \alpha = 1.3; \}$. *Here (—) line corresponds to $n = 0.8$, (- - -) line to $n = 1$ and ($\cdots$) line to $n = 1.2$.*

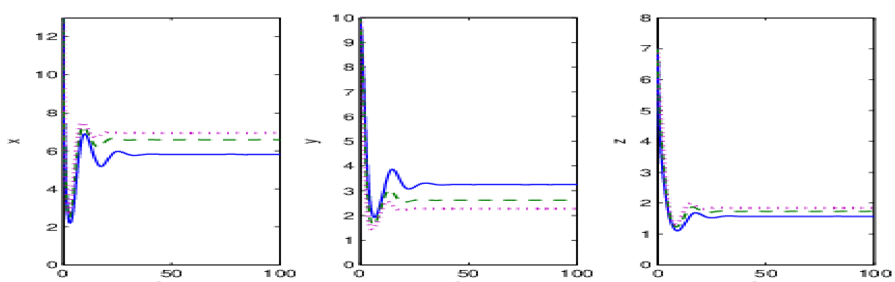

Figure 3.13: *Change of x, y and z of the system* (3.3) *with respect to change of n_1 with parameter set* $\{r = 1.1; K = 9.8; a = 6; \beta = 1.2; b = 7; n = 1; n_1 = 0.9; d_1 = 0.3; d_2 = 0.12; m_1 = 0.2; m = 0.4; q = 0.2; E = 0.76; \alpha = 1.3; \}$. *Here (—) line corresponds to $n_1 = 0.7$, (- - -) line to $n_1 = 0.9$ and ($\cdots$) line to $n_1 = 1.1$.*

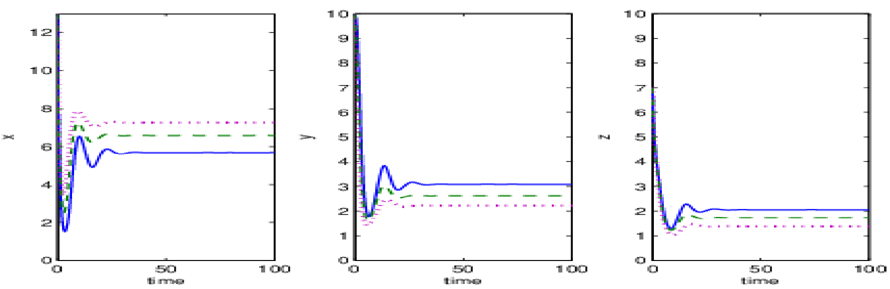

Figure 3.14: *Change of* x, y *and* z *of the system* (3.3) *with respect to change of* d_1 *with parameter set* $\{r = 1.1; K = 9.8; a = 6; \beta = 1.2; b = 7; n = 1; n_1 = 0.9; d_2 = 0.12; m_1 = 0.2; m = 0.4; q = 0.2; E = 0.76; \alpha = 1.3;\}$. *Here (—) line corresponds to* $d_1 = 0.2$, *(- - -) line to* $d_1 = 0.3$ *and* $(\cdots)$ *line to* $d_1 = 0.4$.