Appendix A

Some derivations

A.1 Iterative formula:

Here, we describe the iteration formula used at the *solution methodology* section in Chapter 2. Now, we refer to

$$Z(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{p} \alpha_i w_{ij}^B \phi(c_i, d_i; x_j, y_j),$$

where $\phi(c_i, d_i; x_j, y_j) = [(c_i - x_j)^2 + (d_i - y_j)^2]^{1/2}$ and the terms w_{ij}^B are constants. Differentiating Z with respect to (x_j, y_j) and equating with 0, we get

$$\sum_{i=1}^{m} \frac{\alpha_i w_{ij}^B (c_i - x_j)}{\phi(c_i, d_i; x_j, y_j)} = 0 \quad (j = 1, 2, \ldots, p), \tag{A.1}$$

$$\sum_{i=1}^{m} \frac{\alpha_i w_{ij}^B (d_i - y_j)}{\phi(c_i, d_i; x_j, y_j)} = 0 \quad (j = 1, 2, \ldots, p). \tag{A.2}$$

Now, from Eqs. (A.1)-(A.2) we obtain

$$\sum_{i=1}^{m} \frac{\alpha_i w_{ij}^B c_i}{\phi(c_i, d_i; x_j, y_j)} - x_j \sum_{i=1}^{m} \frac{\alpha_i w_{ij}^B}{\phi(c_i, d_i; x_j, y_j)} = 0 \quad (j = 1, 2, \ldots, p),$$

$$\sum_{i=1}^{m} \frac{\alpha_i w_{ij}^B d_i}{\phi(c_i, d_i; x_j, y_j)} - y_j \sum_{i=1}^{m} \frac{\alpha_i w_{ij}^B}{\phi(c_i, d_i; x_j, y_j)} = 0 \quad (j = 1, 2, \ldots, p).$$

Then,

$$x_j = \frac{\sum_{i=1}^{m} \frac{\alpha_i w_{ij}^B c_i}{\phi(c_i, d_i; x_j, y_j)}}{\sum_{i=1}^{m} \frac{\alpha_i w_{ij}^B}{\phi(c_i, d_i; x_j, y_j)}} \quad (j = 1, 2, \ldots, p),$$

$$y_j = \frac{\sum_{i=1}^{m} \frac{\alpha_i w_{ij}^B d_i}{\phi(c_i, d_i; x_j, y_j)}}{\sum_{i=1}^{m} \frac{\alpha_i w_{ij}^B}{\phi(c_i, d_i; x_j, y_j)}} \quad (j = 1, 2, \ldots, p).$$

These equations are solved iteratively (motivated by the concept of Cooper [30]). The iteration equations for (x_j, y_j) are as follows:

$$x_j^{k+1} = \frac{\sum_{i=1}^{m} \frac{\alpha_i w_{ij}^B c_i}{\phi(c_i,d_i;x_j^k,y_j^k)}}{\sum_{i=1}^{m} \frac{\alpha_i w_{ij}^B}{\phi(c_i,d_i;x_j^k,y_j^k)}} \quad (j = 1,2,\ldots,p;\ k \in \mathbb{N}), \tag{A.3}$$

$$y_j^{k+1} = \frac{\sum_{i=1}^{m} \frac{\alpha_i w_{ij}^B d_i}{\phi(c_i,d_i;x_j^k,y_j^k)}}{\sum_{i=1}^{m} \frac{\alpha_i w_{ij}^B}{\phi(c_i,d_i;x_j^k,y_j^k)}} \quad (j = 1,2,\ldots,p;\ k \in \mathbb{N}), \tag{A.4}$$

where $\phi(c_i,d_i;x_j^k,y_j^k) = [(c_i - x_j^k)^2 + (d_i - y_j^k)^2]^{1/2}$. The initial estimates of (x_j, y_j) are simply chosen by the weighted mean coordinates:

$$x_j^0 = \frac{\sum_{i=1}^{m} \alpha_i w_{ij}^B c_i}{\sum_{i=1}^{m} \alpha_i w_{ij}^B} \quad (j = 1,2,\ldots,p), \tag{A.5}$$

$$y_j^0 = \frac{\sum_{i=1}^{m} \alpha_i w_{ij}^B d_i}{\sum_{i=1}^{m} \alpha_i w_{ij}^B} \quad (j = 1,2,\ldots,p). \tag{A.6}$$

A.2 Optimality condition:

Here, the criterion of local minimum used at the 'methodology' Section in Chapter 3 is displayed.

Now,

$$Z = \sum_{i=1}^{m} \sum_{j=1}^{p} \left[\alpha_i w_{ij} \phi(a_i,b_i;x_j,y_j) + f_{ij} u_{ij} \right], \tag{A.7}$$

$$\text{where } \ \phi(a_i,b_i;x_j,y_j) = \sqrt{(a_i - x_j)^2 + (b_i - y_j)^2}.$$

Differentiating Z with respect to (x_j, y_j) to find the minimum yields. Then, we get

$$\sum_{i=1}^{m} - \frac{\alpha_i w_{ij}(a_i - x_j)}{\phi(a_i,b_i;x_j,y_j)} = 0 \quad (j = 1,2,\ldots,p), \tag{A.8}$$

$$\text{and} \quad \sum_{i=1}^{m} - \frac{\alpha_i w_{ij}(b_i - y_j)}{\phi(a_i,b_i;x_j,y_j)} = 0 \quad (j = 1,2,\ldots,p). \tag{A.9}$$

It must be shown that the solution of equations (A.8) to (A.9) yield a minimum. The conditions for a minimum are as follows:

$$\frac{\partial^2 Z}{\partial x_j^2} > 0,$$

$$\text{and} \quad \frac{\partial^2 Z}{\partial x_j^2} \frac{\partial^2 Z}{\partial y_j^2} - \left(\frac{\partial^2 Z}{\partial x_j \partial y_j} \right)^2 > 0.$$

Substituting the expressions we have,

$$\sum_{i=1}^{m} \frac{\alpha_i w_{ij}[(a_i - x_j)^2 + 1]}{[(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2}} > 0 \quad (j = 1,2,\ldots,p), \ \text{and} \tag{A.10}$$

$$\sum_{i=1}^{m} \frac{\alpha_i w_{ij}[(a_i - x_j)^2 + 1]}{[(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2}} \sum_{i=1}^{m} \frac{\alpha_i w_{ij}[(b_i - y_j)^2 + 1]}{[(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2}} - \left(\sum_{i=1}^{m} \frac{\alpha_i w_{ij}(a_i - x_j)(b_i - y_j)}{[(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2}} \right)^2 > 0. \tag{A.11}$$

It is easily shown that equations (A.10) to (A.11) are true for all values of x_j and y_j. This establishes that the solution of equation (A.7) leads to a minimum.

A.3 Find (x_j, y_j):

Here, the equations are derived iteratively in similar way of A.1, which are used at the 'Methodology' Section 5.3 in Chapter 5.

(i) Here, the iterative formula of $Z_{1(x,y)} = \sum_{i=1}^{m} \sum_{j=1}^{p} e_i w_{ij}^B \phi(u_i, v_i; x_j, y_j)$ are presented as below:

$$x_j^0 = \frac{\sum_{i=1}^{m} e_i w_{ij}^B u_i}{\sum_{i=1}^{m} e_i w_{ij}^B} \quad (j = 1, 2, \ldots, p),$$

$$y_j^0 = \frac{\sum_{i=1}^{m} e_i w_{ij}^B v_i}{\sum_{i=1}^{m} e_i w_{ij}^B} \quad (j = 1, 2, \ldots, p),$$

$$x_j^{r+1} = \frac{\sum_{i=1}^{m} \frac{e_i w_{ij}^B u_i}{\phi(u_i, u_i; x_j^r, y_j^r)}}{\sum_{i=1}^{m} \frac{e_i w_{ij}^B}{\phi(u_i, v_i; x_j^r, y_j^r)}} \quad (j = 1, 2, \ldots, p;\ r \in \mathbb{N}),$$

$$y_j^{r+1} = \frac{\sum_{i=1}^{m} \frac{e_i w_{ij}^B v_i}{\phi(u_i, v_i; x_j^r, y_j^r)}}{\sum_{i=1}^{m} \frac{e_i w_{ij}^B}{\phi(u_i, v_i; x_j^r, y_j^r)}} \quad (j = 1, 2, \ldots, p;\ r \in \mathbb{N}),$$

where $\phi(u_i, v_i; x_j^r, y_j^r) = [(u_i - x_j^r)^2 + (v_i - y_j^r)^2 + \delta_{ij}]^{1/2}$.

(ii) As similar the iterations for $Z_{2(x,y)} = \sum_{i=1}^{m} \sum_{j=1}^{p} e_i w_{ij}^B \psi(u_i, v_i; x_j, y_j)$ are

$$x_j^0 = \frac{\sum_{i=1}^{m} e_i w_{ij}^B u_i}{\sum_{i=1}^{m} e_i w_{ij}^B} \quad (j = 1, 2, \ldots, p),$$

$$y_j^0 = \frac{\sum_{i=1}^{m} e_i w_{ij}^B v_i}{\sum_{i=1}^{m} e_i w_{ij}^B} \quad (j = 1, 2, \ldots, p),$$

$$x_j^{r+1} = \frac{\sum_{i=1}^{m} \frac{e_i w_{ij}^B u_i}{\psi(u_i, u_i; x_j^r, y_j^r)}}{\sum_{i=1}^{m} \frac{e_i w_{ij}^B}{\psi(u_i, v_i; x_j^r, y_j^r)}} \quad (j = 1, 2, \ldots, p;\ r \in \mathbb{N}),$$

$$y_j^{r+1} = \frac{\sum_{i=1}^{m} \frac{e_i w_{ij}^B v_i}{\psi(u_i, v_i; x_j^r, y_j^r)}}{\sum_{i=1}^{m} \frac{e_i w_{ij}^B}{\psi(u_i, v_i; x_j^r, y_j^r)}} \quad (j = 1, 2, \ldots, p;\ r \in \mathbb{N}),$$

where $\psi(u_i, v_i; x_j^r, y_j^r) = [(u_i - x_j^r)^2 + (v_i - y_j^r)^2 + t_{ij}]^{1/2}$.

(iii) Again the iterations for Z_3 are as follows:

Case 3.1: The iterations for $Z_{3(x,y)} = \sum_{i=1}^{m}\sum_{j=1}^{p}(\alpha+\gamma)w_{ij}^{B}\varphi(u_i,v_i;x_j,y_j) - \gamma C$ are as:

$$x_j^0 = \frac{\sum_{i=1}^{m}(\alpha+\gamma)w_{ij}^{B}u_i}{\sum_{i=1}^{m}(\alpha+\gamma)w_{ij}^{B}} \quad (j=1,2,\ldots,p),$$

$$y_j^0 = \frac{\sum_{i=1}^{m}(\alpha+\gamma)w_{ij}^{B}v_i}{\sum_{i=1}^{m}(\alpha+\gamma)w_{ij}^{B}} \quad (j=1,2,\ldots,p),$$

$$x_j^{r+1} = \frac{\sum_{i=1}^{m}\frac{(\alpha+\gamma)w_{ij}^{B}u_i}{\varphi(u_i,u_i;x_j^r,y_j^r)}}{\sum_{i=1}^{m}\frac{(\alpha+\gamma)w_{ij}^{B}}{\varphi(u_i,v_i;x_j^r,y_j^r)}} \quad (j=1,2,\ldots,p;\ r\in\mathbb{N}),$$

$$y_j^{r+1} = \frac{\sum_{i=1}^{m}\frac{(\alpha+\gamma)w_{ij}^{B}v_i}{\varphi(u_i,v_i;x_j^r,y_j^r)}}{\sum_{i=1}^{m}\frac{(\alpha+\gamma)w_{ij}^{B}}{\varphi(u_i,v_i;x_j^r,y_j^r)}} \quad (j=1,2,\ldots,p;\ r\in\mathbb{N}),$$

where $\varphi(u_i,v_i;x_j^r,y_j^r) = [(u_i-x_j^r)^2+(v_i-y_j^r)^2+\delta_{ij}]^{1/2}$.

Case 3.2: The iterations for $Z_{3(x,y)} = \sum_{i=1}^{m}\sum_{j=1}^{p}(\alpha+P_c\beta)w_{ij}^{B}\varphi(u_i,v_i;x_j,y_j) - P_c\beta C$ are as:

$$x_j^0 = \frac{\sum_{i=1}^{m}(\alpha+P_c\beta)w_{ij}^{B}u_i}{\sum_{i=1}^{m}(\alpha+P_c\beta)w_{ij}^{B}} \quad (j=1,2,\ldots,p),$$

$$y_j^0 = \frac{\sum_{i=1}^{m}(\alpha+P_c\beta)w_{ij}^{B}v_i}{\sum_{i=1}^{m}(\alpha+P_c\beta)w_{ij}^{B}} \quad (j=1,2,\ldots,p),$$

$$x_j^{r+1} = \frac{\sum_{i=1}^{m}\frac{(\alpha+P_c\beta)w_{ij}^{B}u_i}{\varphi(u_i,u_i;x_j^r,y_j^r)}}{\sum_{i=1}^{m}\frac{(\alpha+P_c\beta)w_{ij}^{B}}{\varphi(u_i,v_i;x_j^r,y_j^r)}} \quad (j=1,2,\ldots,p;\ r\in\mathbb{N}),$$

$$y_j^{r+1} = \frac{\sum_{i=1}^{m}\frac{(\alpha+P_c\beta)w_{ij}^{B}v_i}{\varphi(u_i,v_i;x_j^r,y_j^r)}}{\sum_{i=1}^{m}\frac{(\alpha+P_c\beta)w_{ij}^{B}}{\varphi(u_i,v_i;x_j^r,y_j^r)}} \quad (j=1,2,\ldots,p;\ r\in\mathbb{N}),$$

where $\varphi(u_i,v_i;x_j^r,y_j^r) = [(u_i-x_j^r)^2+(v_i-y_j^r)^2+\delta_{ij}]^{1/2}$.

A.4 Iterations for (x_j, y_j):

Here, the equations are derived iteratively in similar way of A.1, which are used at the 'Solution methodology' Section 6.3 in Chapter 6.

1. Herein, the iterations of

$$Z_{1(x,y)} = \sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\sum_{l\in L}\left(\alpha_i\phi_l(u_i,v_i;r_k,s_k) + \alpha_k'\phi_l(r_k,s_k;x_j,y_j)\right)w_{ikj}^{lB}$$

is presented.

$$x_j^0 = \frac{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\beta_k'\varepsilon_l' w_{ikj}^{lB} r_k}{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\beta_k'\varepsilon' w_{ikj}^{lB}} \quad (j = 1,2,\ldots,p),$$

$$y_j^0 = \frac{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\beta_k'\varepsilon' w_{ikj}^{lB} s_k}{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\beta_k'\varepsilon' w_{ikj}^{lB}} \quad (j = 1,2,\ldots,p),$$

$$x_j^{k'+1} = \frac{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\frac{\alpha_k'\varepsilon_l w_{ikj}^{lB} r_k}{\varphi(r_k,s_k;x_j^{k'},y_j^{k'})}}{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\frac{\alpha_k'\varepsilon_l w_{ikj}^{lB}}{\varphi(r_k,s_k;x_j^{k'},y_j^{k'})}} \quad (j = 1,2,\ldots,p;\ k' \in \mathbb{N}),$$

$$y_j^{k'+1} = \frac{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\frac{\alpha_k'\varepsilon_l w_{ikj}^{lB} s_k}{\varphi(r_k,s_k;x_j^{k'},y_j^{k'})}}{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\frac{\alpha_k'\varepsilon_l w_{ikj}^{lB}}{\varphi(r_k,s_k;x_j^{k'},y_j^{k'})}} \quad (j = 1,2,\ldots,p;\ k' \in \mathbb{N}),$$

where $\varphi(r_k,s_k;x_j^{k'},y_j^{k'}) = [(r_k - x_j^{k'})^2 + (s_k - y_j^{k'})^2 + \delta_{kj}]^{1/2}$. The initial iteration of (x_j, y_j) is the weighted mean coordinate:

2. Similarly, the iterations for

$$Z_{2(x,y)} = \sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\sum_{l\in L}\big(\beta_i\psi_l(u_i,v_i;r_k,s_k) + \beta_k'\psi_l(r_k,s_k;x_j,y_j)\big) w_{ikj}^{lB}$$

are

$$x_j^0 = \frac{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\beta_k'\varepsilon_l' w_{ikj}^{lB} r_k}{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\beta_k'\varepsilon_l' w_{ikj}^{lB}} \quad (j = 1,2,\ldots,p),$$

$$y_j^0 = \frac{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\beta_k'\varepsilon_l' w_{ikj}^{lB} s_k}{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\beta_k'\varepsilon_l' w_{ikj}^{lB}} \quad (j = 1,2,\ldots,p),$$

$$x_j^{k'+1} = \frac{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\frac{\beta_k'\varepsilon_l' w_{ikj}^{lB} r_k}{\tau(r_k,s_k;x_j^{k'},y_j^{k'})}}{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\frac{\beta_k'\varepsilon_l' w_{ikj}^{lB}}{\tau(r_k,s_k;x_j^{k'},y_j^{k'})}} \quad (j = 1,2,\ldots,p;\ k' \in \mathbb{N}),$$

$$y_j^{k'+1} = \frac{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\frac{\beta_k'\varepsilon_l' w_{ikj}^{lB} s_k}{\tau(r_k,s_k;x_j^{k'},y_j^{k'})}}{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\frac{\beta_k'\varepsilon_l' w_{ikj}^{lB}}{\tau(r_k,s_k;x_j^{k'},y_j^{k'})}} \quad (j = 1,2,\ldots,p;\ k' \in \mathbb{N}),$$

where $\tau(r_k,s_k;x_j^{k'},y_j^{k'}) = [(r_k - x_j^{k'})^2 + (s_k - y_j^{k'})^2 + t_{kj} + \delta_{kj}]^{1/2}$.

3. Furthermore, the iterative formula for Z_3 are:

The iterations for

$$\begin{aligned} Z_{3(x,y)} = &\sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\sum_{l\in L}(d_{2k}A_{ik} + d_{1j}G_{kj}) w_{ikj}^{lB} + \sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\sum_{l\in L} H_k w_{ikj}^{lB} \\ &+ \sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\sum_{l\in L} D_k w_{ikj}^{lB} + \sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\sum_{l\in L} B_i w_{ikj}^{lB} + \sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\sum_{l\in L} g_k w_{ikj}^{lB} \\ &+ \sum_{k\in K} f_k + \gamma\sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\sum_{l\in L}\big(\rho_l(u_i,v_i;r_k,s_k) + \rho_l(r_k,s_k;x_j,y_j)\big) w_{ikj}^{lB} \end{aligned}$$

are stated subsequently:

$$x_j^0 = \frac{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\varepsilon_l'' w_{ikj}^{lB} r_k}{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\varepsilon_l'' w_{ikj}^{lB}} \quad (j=1,2,\ldots,p),$$

$$y_j^0 = \frac{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\varepsilon_l'' w_{ikj}^{lB} s_k}{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\varepsilon_l'' w_{ikj}^{lB}} \quad (j=1,2,\ldots,p),$$

$$x_j^{k'+1} = \frac{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\frac{\varepsilon_l'' w_{ikj}^{lB} r_k}{\rho(r_k,s_k;x_j^{k'},y_j^{k'})}}{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\frac{\varepsilon_l'' w_{ikj}^{lB}}{\rho(r_k,s_k;x_j^{k'},y_j^{k'})}} \quad (j=1,2,\ldots,p;\ k'\in\mathbb{N}),$$

$$y_j^{k'+1} = \frac{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\frac{\varepsilon_l'' w_{ikj}^{lB} s_k}{\rho(r_k,s_k;x_j^{k'},y_j^{k'})}}{\sum_{i\in I}\sum_{k\in K}\sum_{l\in L}\frac{\varepsilon_l'' w_{ikj}^{lB}}{\rho(r_k,s_k;x_j^{k'},y_j^{k'})}} \quad (j=1,2,\ldots,p;\ k'\in\mathbb{N}),$$

where $\rho(r_k,s_k;x_j^{k'},y_j^{k'}) = [(r_k-x_j^{k'})^2+(s_k-y_j^{k'})^2+\delta_{kj}]^{1/2}$.