Chapter 2

Two-person zero-sum game in triangular intuitionistic fuzzy environment*

The main intention of this chapter is to formulate a *two-person zero-sum game*, sometimes called as matrix game, in triangular intuitionistic fuzzy environment and to solve it by ranking function approach. Triangular intuitionistic fuzzy numbers are used as payoff elements and robust ranking technique is used as ranking approach. We analyze numerical examples to validate the proposed technique.

2.1 Motivation

Motivated by the uncertainty of real-life problems, triangular intuitionistic fuzzy numbers are treated on considering payoff elements. A new robust ranking technique is used to rank the triangular intuitionistic fuzzy numbers and to solve the considered fuzzy game having real-life applications.

2.2 Introduction

In reality, we are often faced with conflicting, cooperative and co-parallel game like problems or situations, some of which may be solved by classical game theory. But the uncertainties existing in various forms restrains the 'crisp' data to tackle most of such critical problems which are being managed intelligently by 'fuzzy set' representing an uncertainty of payoffs of games. In solving real-life conflicts and cooperative problems, fuzzy game theory is being applied effectively. A player opting for a pure strategy may select at random a row or a column followed by some probability processes. These probability based strategies are known as mixed strategies in which the payoff may be calculated only in expected sense so that (if the game is played several times) when each player what expects to receive and the player actually receives on average is represented by the payoff. While the members are considered as crisp in nature, duality in Linear Programming Problem (LPP) and matrix game play the role of twin sisters. But in real-life situations, ambigu-

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ity in the judgement of decision makers, imprecision and uncertainty occurred in the system, etc., lead to the fuzzy numbers which assume different forms starting from triangular fuzzy numbers to diamond fuzzy numbers incorporated with reverse order fuzzy numbers and some others.
The Intuitionistic Fuzzy Number (IFN) plays a vital role among these considering the degree of belongingness and that of non-belongingness which are complementary to each other (complement to 1). An Intuitionistic Fuzzy Set (IFS) includes both the membership degrees as well as non-membership degrees and the degree of hesitation is accommodated by equalling to 1 minus sum of both these two degrees. The IFS, in today's uncertainty situations, describes information more comfortably. Here the problems of matrix games have been considered in Triangular Intuitionistic Fuzzy Number (TIFN) based environment where focus has been made on robust ranking technique to solve matrix game problems. Literally the word 'robust' is synonymous to 'able-bodied', 'fit', 'sturdy' or 'well-conditioned'. For a perfect ranking of fuzzy numbers this sturdy type nature of robust ranking technique contributes a lot.

2.3 Basic Concepts

Definition 2.3.1 *[4] Let X denote a universe of discourse, then an IFS A_{IF} in X is given by a set of ordered triplet as described below:*

$$A_{IF} = \{\langle x, \mu_{A_{IF}}(x), \gamma_{A_{IF}}(x)\rangle : x \in X\}, \tag{2.1}$$

where $\mu_{A_{IF}}(x),\ \gamma_{A_{IF}}(x)\ :\ X \to [0,1]$ are functions such that $0 \leq \mu_{A_{IF}}(x)+\gamma_{A_{IF}}(x)\ \leq 1,\ x \in X$. For each x, $\mu_{A_{IF}}(x)$ and $\gamma_{A_{IF}}(x)$ from Eq.(2.1) represent the degree of membership and degree of non-membership functions respectively. Again the function $\pi_{A_{IF}}(x) = 1 - \mu_{A_{IF}}(x) - \gamma_{A_{IF}}(x)$ is called "degree of hesitation" of the element x in the set A_{IF}. If $\pi_{A_{IF}}(x) = 0,\ x \in X$, then the IFS becomes a fuzzy set.

Definition 2.3.2 *An intuitionistic fuzzy number (IFN) A_{IF} is an IF subset of real numbers $\mathbb{R}$ if:*

(i) A_{IF} *is normal.*

(ii) A_{IF} *is convex for the membership function* $\mu_{A_{IF}}(x)$,
i.e., $\mu_{A_{IF}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{A_{IF}}(x_1), \mu_{A_{IF}}(x_2)\}$ *for* $x_1, x_2 \in \mathbb{R}; \lambda \in [0,1]$.

(iii) A_{IF} *is concave for the non-membership function* $\gamma_{A_{IF}}(x)$,
i.e., $\gamma_{A_{IF}}(\lambda x_1 + (1-\lambda)x_2) \leq \max\{\gamma_{A_{IF}}(x_1), \gamma_{A_{IF}}(x_2)\}$ *for* $x_1, x_2 \in \mathbb{R}; \lambda \in [0,1]$.

(iv) $\mu_{A_{IF}}$ *is piecewise continuous.*

Definition 2.3.3 *A TIFN $\hat{A} = \langle(\zeta_1, \zeta_2, \zeta_3), (\overline{\zeta}_1, \zeta_2, \overline{\zeta}_3)\rangle$ where $\overline{\zeta}_1 \leq \zeta_1 \leq \zeta_2 \leq \zeta_3 \leq \overline{\zeta}_3$, is defined by its membership and non-membership functions respectively, given as follows:*

$$\mu_{\hat{A}}(x) = \begin{cases} \frac{x-\zeta_1}{\zeta_2-\zeta_1}, & \text{if } \zeta_1 \leq x \leq \zeta_2, \\ \frac{\zeta_3-x}{\zeta_3-\zeta_2}, & \text{if } \zeta_2 \leq x \leq \zeta_3, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \gamma_{\hat{A}}(x) = \begin{cases} \frac{\zeta_2-x}{\zeta_2-\overline{\zeta}_1}, & \text{if } \overline{\zeta}_1 \leq x \leq \zeta_2, \\ \frac{x-\zeta_2}{\overline{\zeta}_3-\zeta_2}, & \text{if } \zeta_2 \leq x \leq \overline{\zeta}_3, \\ 1, & \text{otherwise.} \end{cases}$$

Arithmetic operations on TIFNs : Let $\hat{A} = \langle(\zeta_1, \zeta_2, \zeta_3), (\overline{\zeta}_1, \zeta_2, \overline{\zeta}_3)\rangle$, $\hat{B} = \langle(\tau_1, \tau_2, \tau_3), (\overline{\tau}_1, \tau_2, \overline{\tau}_3)\rangle$ represent two TIFNs, then the addition, substraction, and scalar multiplication of the numbers are stated [77; 79] as below:

$$\text{Addition}: \quad \hat{A} + \hat{B} = \langle(\zeta_1+\tau_1, \zeta_2+\tau_2, \zeta_3+\tau_3), (\overline{\zeta}_1+\overline{\tau}_1, \zeta_2+\tau_2, \overline{\zeta}_3+\overline{\tau}_3)\rangle.$$

$$\text{Substraction}: \quad \hat{A} - \hat{B} = \langle(\zeta_1-\tau_3, \zeta_2-\tau_2, \zeta_3-\tau_1), (\overline{\zeta}_1-\overline{\tau}_3, \zeta_2-\tau_2, \overline{\zeta}_3-\overline{\tau}_1)\rangle.$$

Scalar multiplication: For any real number k,

$$k\hat{A} = \begin{cases} \langle (k\zeta_1, k\zeta_2, k\zeta_3), (k\overline{\zeta}_1, k\zeta_2, k\overline{\zeta}_3) \rangle, & \text{if } k \geq 0, \\ \langle (k\zeta_3, k\zeta_2, k\zeta_1), (k\overline{\zeta}_3, k\zeta_2, k\overline{\zeta}_1) \rangle, & \text{if } k < 0. \end{cases}$$

Definition 2.3.4 *α-cut of a TIFN $\hat{A} = \langle (\zeta_1, \zeta_2, \zeta_3), (\overline{\zeta}_1, \zeta_2, \overline{\zeta}_3) \rangle$ is the set of all x, whose degrees of membership are greater than or equal to α, i.e., $\hat{A}_\alpha = \{x : \mu_{\hat{A}}(x) \geq \alpha,\ \alpha \in (0, 1],\ x \in X\}$.*

Now, $\mu_{\hat{A}}(x) \geq \alpha \ \Rightarrow \frac{x-\zeta_1}{\zeta_2-\zeta_1} \geq \alpha$ and $\frac{\zeta_3-x}{\zeta_3-\zeta_2} \geq \alpha$, or $x \geq \zeta_1 + \alpha(\zeta_2 - \zeta_1)$ and $x \leq \zeta_3 - \alpha(\zeta_3 - \zeta_2)$. Therefore, $[\zeta_1 + \alpha(\zeta_2 - \zeta_1), \zeta_3 - \alpha(\zeta_3 - \zeta_2)]$ is the α-cut interval of $\hat{A} = \langle (\zeta_1, \zeta_2, \zeta_3), (\overline{\zeta}_1, \zeta_2, \overline{\zeta}_3) \rangle$.

Definition 2.3.5 *Let $\mathbb{R}$ be the set of all real numbers, then the interval number C is a closed interval denoted by $C = [c^l, c^u]$ and is considered as follows:*

$$C = [c^l, c^u] = \{x : c^l \leq x \leq c^u;\ c^l, c^u \in \mathbb{R}\},$$

where c^l and c^u are respectively lower and upper bounds of the interval C. If $c^l = c^u$ then C reduces to a real number. An interval C can be also denoted by

$$C =< c_c, c_w >= \{x : c_c - c_w \leq x \leq c_c + c_w;\ x \in \mathbb{R}\},$$

where c_c and c_w are respectively the center and the width of the interval C and $c_c = \frac{c^l+c^u}{2}$ and $c_w = \frac{c^u-c^l}{2}$.

Definition 2.3.6 *Robust Ranking Technique [102] which satisfies compensation, costs, linearity, and additive properties and provides results associated with practical human intuition. If $\hat{z}$ is an TIFN, whatever be its representation as membership function, its Robust Ranking is defined by*

$$R_1(\hat{z}) = \int_{\alpha=0}^{1} (0.5) \times (z_\alpha^L + z_\alpha^U) d\alpha, \tag{2.2}$$

where z_α^L and z_α^U of Eq.(2.2) are the lower and upper bounds of $[z_\alpha^L, z_\alpha^U]$ obtained from the α-cut interval of $\hat{z}$. Here $R_1(\hat{z})$ gives the representative value of $\hat{z}$.

2.4 Mathematical Model

In this section, two-person zero-sum game is discussed under proposed ranking technique, methodologically.

2.4.1 Robust ranking technique

Here we consider another form of robust ranking $R(\hat{z})$ of $\hat{z}$, slightly differed from $R_1(\hat{z})$ which is defined by

$$R(\tilde{z}) = \int_{\alpha=0}^{1} \left(\frac{\sqrt{3}}{3}\right) \times (z_\alpha^L + z_\alpha^U) d\alpha. \tag{2.3}$$

In Eq.(2.3), $[z_\alpha^L, z_\alpha^U]$ is the α-cut interval of the TIFN $\hat{z}$. The ranking of the TIFN is slightly changed satisfied with compensation, costs, linearity, and additive properties. The results obtained from practical human intuition are also incorporated in it.

2.4.2 Two-person zero-sum game in crisp environment

A two-person zero-sum game in matrix form means that there is a matrix $A = (a_{ij}), (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$ of real numbers, called payoff matrix, so that if player I, the row player, chooses to play row i and player II, the column player, chooses to play column j, then the payoff to player I is a_{ij} and that of player II is $-a_{ij}$. Both players want to choose strategies that will benefit their individual payoffs.
Considering the matrix game with the set of pure strategies $S_1(= \{\alpha_1, \alpha_2, \ldots, \alpha_p\})$ and $S_2(= \{\beta_1, \beta_2, \ldots, \beta_q\})$ and that of mixed strategies Y and Z for two players I and II respectively are defined as: $Y = \{y = (y_1, y_2, \ldots, y_p)^T : \sum_{i=1}^{p} y_i = 1,\ y_i \geq 0,\ i = 1, 2, \ldots, p\}$, and $Z = \{z = (z_1, z_2, \ldots, z_q)^T : \sum_{j=1}^{q} z_j = 1,\ z_j \geq 0,\ j = 1, 2, \ldots, q\}$. Here $y_i\ (i = 1, 2, \ldots, p)$ and $z_j\ (j = 1, 2, \ldots, q)$ are probabilities in which the players I and II choose their pure strategies $\alpha_i \in S_1\ (i = 1, 2, \ldots, p)$ and $\beta_j \in S_2\ (j = 1, 2, \ldots, q)$ respectively. Then the game is defined as $G \equiv (Y, Z, A)$.

2.4.3 Two-person zero-sum game in triangular intuitionistic fuzzy environment

Game problems, depicted in real World situation, generally have the payoff elements in imprecise form and if the corresponding payoff matrix is formed with TIFNs then the payoff matrix can be considered as triangular intuitionistic fuzzy payoff matrix $\hat{A}$, where $\hat{A}$= $(\hat{a}_{ij})$, and then the game is considered as $G \equiv (Y, Z, \hat{A})$. While using the best strategies by both the players, the maximum guaranteed profit/gain to the maximizing player I or the minimum possible loss to the minimizing player II is assigned to the value of the game. In maximin or minimax principle, a strategy corresponding to the best of the worst outcomes enlisted out of one's all potential strategies is choosen as the most suitable strategy for him/her. The game is said to reach to a saddle point when maximin for player I becomes equal to the minimax of player II. Classically the term 'saddle point' is due to von Neumann and Oskar Morgenstern [110].
Let (m, n)-th position of $\hat{A} = (\hat{a}_{ij})_{p\times q}$, the payoff matrix, is a saddle point, then,

$$\begin{aligned} \langle (a_{mn}, \underset{\sim}{a}_{mn}, \tilde{a}_{mn}); \mu_{\hat{a}_{mn}}, \gamma_{\hat{a}_{mn}} \rangle &= \max_i \{ \min_j \langle (a_{ij}, \underset{\sim}{a}_{ij}, \tilde{a}_{ij}); \mu_{\hat{a}_{ij}}, \gamma_{\hat{a}_{ij}} \rangle \} \\ &= \min_j \{ \max_i \langle (a_{ij}, \underset{\sim}{a}_{ij}, \tilde{a}_{ij}); \mu_{\hat{a}_{ij}}, \gamma_{\hat{a}_{ij}} \rangle \}, \end{aligned}$$

and this implies that the saddle point entry is, (m, n)-th, value of the game, where $\mu_{\hat{a}_{ij}}$ and $\gamma_{\hat{a}_{ij}}$ respectively denote the degree of acceptance and degree of non-acceptance of $(a_{ij},\ \underset{\sim}{a}_{ij},\ \tilde{a}_{ij})$.
Assuming that there exist $\hat{v}^* \in V$ and $\hat{w}^* \in W$ and if there are no other $\hat{v}$ and $\hat{w}$ such that $\hat{v}^* \tilde{\leq} \hat{v}$ and $\hat{w}^* \tilde{\geq} \hat{w}$, where, V and W finger the sets of all reasonable game values $\hat{v}$ and $\hat{w}$ for players I and II respectively, then $(y^*, z^*, \hat{v}^*, \hat{w}^*)$ is called a solution of the TIFMG. And y^* and z^* are called maximin and minimax strategies for players I and II respectively and $\hat{v}^*$ and $\hat{w}^*$ are called respectively player I's gain-floor and player II's loss-ceiling. Let $\hat{\nu}^* = \hat{v}^* \wedge \hat{w}^*$ with the membership function $\mu_{\hat{\nu}^*}(x) = \min\{\mu_{\hat{v}^*}(x), \mu_{\hat{w}^*}(x)\}$. Then $\hat{\nu}^*$ is called a fuzzy value of TIFMG. Also, y^* and z^*, maximin and minimax strategies for players I and II respectively are obtained by solving the following fuzzy mathematical programming problems, given below:

$$\begin{aligned}&\text{maximize } \hat{v}\\&\text{subject to} \begin{cases} y^T\hat{A}z \tilde{\geq} \hat{v},\ z\in Z,\\ y\in Y,\ \ \hat{v}\in TIFN(\mathbb{R}),\end{cases}\end{aligned}$$

and

$$\begin{aligned}&\text{minimize } \hat{w}\\&\text{subject to} \begin{cases} y^T\hat{A}z \tilde{\leq} \hat{w},\ y\in Y,\\ z\in Z,\ \ \hat{w}\in TIFN(\mathbb{R}).\end{cases}\end{aligned}$$

where, '$\tilde{\geq}$' and '$\tilde{\leq}$' denote intuitionistic fuzzy inequalities and 'TIFN($\mathbb{R}$)' is the chewed form of Triangular Intuitionistic Fuzzy Numbers whose entries are real numbers.

$$\begin{aligned}E(\hat{A}) &= y^T\hat{A}z\\&= \sum_{i=1}^{p}\sum_{j=1}^{q}\hat{a}_{ij}y_iz_j\\&= \sum_{i=1}^{p}\sum_{j=1}^{q}\langle(a_{ij},\ \underset{\sim}{a}_{ij},\ \tilde{a}_{ij});\ \mu_{\hat{a}_{ij}},\ \gamma_{\hat{a}_{ij}}\rangle y_iz_j\\&= \left\langle\left(\sum_{i=1}^{p}\sum_{j=1}^{q}a_{ij}y_iz_j, \sum_{i=1}^{p}\sum_{j=1}^{q}\underset{\sim}{a}_{ij}y_iz_j, \sum_{i=1}^{p}\sum_{j=1}^{q}\tilde{a}_{ij}y_iz_j\right); \mu_{\hat{a}_{ij}}, \gamma_{\hat{a}_{ij}}\right\rangle,\\ \text{and } E(-\hat{A}) &= y^T(-\hat{A})z\\&= \sum_{i=1}^{p}\sum_{j=1}^{q}(-\hat{a}_{ij})y_iz_j\\&= \sum_{i=1}^{p}\sum_{j=1}^{q}\langle-(a_{ij},\ \underset{\sim}{a}_{ij},\ \tilde{a}_{ij});\ \mu_{\hat{a}_{ij}},\ \gamma_{\hat{a}_{ij}}\rangle y_iz_j\\&= \left\langle-\left(\sum_{i=1}^{p}\sum_{j=1}^{q}a_{ij}y_iz_j, \sum_{i=1}^{p}\sum_{j=1}^{q}\tilde{a}_{ij}y_iz_j, \sum_{i=1}^{p}\sum_{j=1}^{q}\underset{\sim}{a}_{ij}y_iz_j\right); \mu_{\hat{a}_{ij}}, \gamma_{\hat{a}_{ij}}\right\rangle.\end{aligned}$$

Both $E(\hat{A})$ and $E(-\hat{A})$ are TIFNs.

2.4.4 Reasonable solutions and strategies

Assume $\hat{v} = \langle(v, \underset{\sim}{v}, \tilde{v}); \mu_{\hat{v}}, \gamma_{\hat{v}}\rangle \in TIFN(\mathbb{R}), \hat{w} = \langle(w, \underset{\sim}{w}, \tilde{w}); \mu_{\hat{w}}, \gamma_{\breve{w}}\rangle \in TIFN(\mathbb{R})$ be two triangular IF numbers. Suppose that there exist $y^* \in Y,\ z^* \in Z$. Then $(y^*, z^*, \hat{v}, \hat{w})$ is called a reasonable (highly acceptable than others in the correspondence situation) solution of the matrix game and for any $y^* \in Y$ and $z^* \in Z$, we have $y^{*T}\hat{A}z\tilde{\geq}\hat{v}$ and $y^T\hat{A}z^*\tilde{\leq}\hat{w}$. If $(y^*, z^*, \hat{v}, \hat{w})$ is a reasonable solution of the IFMG then $\hat{v}$ and $\hat{w}$ are called reasonable values for players I and II respectively. Similarly y^* and z^* are called reasonable strategies for players I and II respectively.

Theorem 2.4.1 *If a payoff matrix with TIFN as payoff has at (m, n)-th position, the value of the game as $\hat{a}_{mn}$, then after defuzzification with the help of robust ranking technique, the value of the game is $R(\hat{a}_{mn})$ at (m, n)-th position.*

Proof: If (m,n)-th be the saddle point of the payoff matrix and $\hat{a}_{mn}$ be the value of the game, then

$$\begin{aligned}\hat{a}_{mn} &= \max_{1\leq i\leq p}\{\min_{1\leq j\leq q}\{(\hat{a}_{ij})\}\} \\ &= \min_{1\leq j\leq q}\{\max_{1\leq i\leq p}\{(\hat{a}_{ij})\}\}. \end{aligned} \tag{2.4}$$

Now,

$$\begin{aligned}R(\hat{a}_{mn}) &= R(\max_{1\leq i\leq p}\{\min_{1\leq j\leq q}\{(\hat{a}_{ij})\}\}) \\ &= R(\min_{1\leq j\leq q}\{\max_{1\leq i\leq p}\{(\hat{a}_{ij})\}\}). \end{aligned} \tag{2.5}$$

Therefore,

$$\begin{aligned}R(\hat{a}_{mn}) &= \max_{1\leq i\leq p} R(\{\min_{1\leq j\leq q}\{(\hat{a}_{ij})\}\}) \\ &= \min_{1\leq j\leq q} R(\{\max_{1\leq i\leq p}\{(\hat{a}_{ij})\}\}). \end{aligned} \tag{2.6}$$

Therefore, $R(\hat{a}_{mn}) = \max_{1\leq i\leq p}\{\min_{1\leq j\leq q}\{R(\hat{a}_{ij})\}\} = \min_{1\leq j\leq q}\{\max_{1\leq i\leq p}\{R(\hat{a}_{ij})\}\}$.
Thus the theorem is proved. □

Theorem 2.4.2 *If (y^*, z^*) be the solution of the payoff matrix with mixed strategies, then (y^*, z^*) is also the solution of the payoff matrix after defuzzification by robust ranking $R(\hat{a}_{mn})$.*

Proof: Let (y^*, z^*) be the solution of the payoff matrix. Then,

$$\begin{aligned}\min_z\{\max_y\{E(y,z)\}\} &= E(y^*, z^*) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{q}\hat{a}_{ij}y_i^* z_j^* \\ &= \max_y\{\min_z\{E(y,z)\}\}. \end{aligned} \tag{2.7}$$

Therefore, from Eq.(2.7), we get

$$\begin{aligned}R\{\min_z\{\max_y\{E(y,z)\}\}\} &= R\{E(y^*, z^*)\} \\ &= R\{\sum_{i=1}^{p}\sum_{j=1}^{q}\hat{a}_{ij}y_i^* z_j^*\} \\ &= R\{\max_y\{\min_z\{E(y,z)\}\}\}. \end{aligned} \tag{2.8}$$

And

$$\begin{aligned}\min_z\{\max_y R\{E(y,z)\}\} &= R\{E(y^*, z^*)\} \\ &= \sum_{i=1}^{p}\sum_{j=1}^{q}R(\hat{a}_{ij})y_{ij}^* z_{ij}^* \\ &= E(R(\hat{a}_{ij})) \\ &= \max_y\{\min_z R\{E(y,z)\}\}. \end{aligned} \tag{2.9}$$

Therefore, (y^*, z^*) is also a solution of the defuzzified payoff matrix and the value of the game is $\hat{V}(y^*, z^*) = \sum_{i=1}^{p}\sum_{j=1}^{q}R(\hat{a}_{ij})y_{ij}^* z_{ij}^*$. □

2.4.5 Algorithm for solving TIFMG

Li's [78] ratio ranking method of TIFNs is unable to rank between fuzzy numbers. For example, consider two TIFNs $\langle(-6, 1, 2); 0.6, 0.3\rangle$ and $\langle(-6, 1, 2); 0.5, 0.4\rangle$. These two fuzzy numbers having different membership and non-membership degrees are not comparable by ratio ranking method [78] because both of their ratio ranking results are zero.
If we consider the score and accuracy function approaches of Jianqiang and Zhong [63], i.e., for TIFN A_i=$\langle(a_i, b_i, c_i); \mu_i, \gamma_i\rangle$, and for $i \neq j$, if $S(A_i)$>$S(A_j)$, then A_i>A_j and if $S(A_i) = S(A_j)$, then A_i>A_j if $H(A_i)$>$H(A_j)$, where $S(A_i) = I(A_i) \times (\mu_i - \gamma_i)$, $H(A_i) = I(A_i) \times (\mu_i + \gamma_i)$ and $I(A_i) = \frac{(a_i+b_i+c_i)\times(1+\mu_i-\gamma_i)}{8}$. As for example, we consider A_1 and A_2, given as: $A_1 = \langle(0.56, 1.54, 0.90); 0.5, 0.5\rangle$, $A_2 = \langle(0.50, 1.55, 0.95); 0.5, 0.5\rangle$. Then we get $S(A_1) = 0$, $S(A_2) = 0$, and $H(A_1) = 0.3750$, $H(A_2) = 0.3750$, i.e., A_1 and A_2 are not comparable.
If we use the ranking method acquired by Rezvani [117], then, for TIFN A_i=$\langle(a_i, b_i, c_i); \mu_i, \gamma_i\rangle$, and for $i \neq j$, if $V(A_i) \geq V(A_j)$, then $A_i \geq A_j$, where $V(A_i) = \frac{a_i+2b_i+c_i}{6}$.
But if $A_1 = \langle(0.55, 1.30, 0.75); 0.4, 0.3\rangle$, $A_2 = \langle(0.45, 1.35, 0.75); 0.1, 0.8\rangle$ then A_1 and A_2 are incomparable. Using accuracy function defined by Xu [155], we define $h(A_i) = \frac{(a_i+b_i+c_i)}{2}$, where we have $A_i = \langle(a_i, b_i, c_i); \mu_i, \gamma_i\rangle$ and if we take two TIFNs as $A_1 = \langle(1, 2, 3); 0.4, 0.3\rangle$ and $A_2 = \langle(1.5, 2, 2.5); 0.3, 0.2\rangle$, we notice that A_1 and A_2 are still incomparable.
Thus, we notice that Li's [78] ratio ranking method, the score and accuracy function approaches of Jianqiang and Zhong [63], Rezvani's [117] ranking method, accuracy function of Xu [155], etc., sometimes fail to make a comparison among triangular intuitionistic fuzzy numbers.
In this chapter, we solve TIFMGs through the algorithmic steps of **Algorithm 1**.

Algorithm 1: Algorithm for solving TIFN payoff two-person zero sum game.

Input: Matrix game with triangular IF type payoffs
Output: Optimal solution
1 Calculate defuzzified payoff values of corresponding payoff matrix
2 Check Saddle point existence:
(I) if saddle point exists using maximin-minimax principle then the optimal solution achieved
(II) if saddle point does not exist, then use mixed strategy method is used
3 The defuzzified solutions of the game arise
4 Using given original payoff matrix, required solutions are obtained

2.5 Numerical Illustrations

To illustrate our proposed method in this chapter, we solve two problems from real-life experiences, given below.

2.5.1 Strike policy problem

Under unexplained circumstances a 'strike' is called by some political parties against ruler. Generally, people of the state feel uneasy. Parties or agencies want to make the strike a success

whereas the ruling systems, say government wishes to enforce it into a false. This situation occurs a real-life game problem. Player I, i.e., strike supporter wants to maximize their fulfillment of agenda through strike by drawing attention of people of state to their demands, while the player II, i.e., the government wants to minimize demands.
In this example, we demonstrate the situation by a 2×2 matrix game with TIFNs as payoffs. We consider player I has alternatives 1 and 2, as:

1. Shutting down the institutions (academic, professional, commercial).
2. Campaigning for the issues of the strike against the ruling system.

Similarly, player II may opt for the following two alternatives.

1. Circulating notices to work at workplace, mandatory.
2. Campaigning against the demands of strike supporters and the strike.

In such a situation we may consider a TIFMG and the matrix is stated in Table 2.1. Here, the cell

Table 2.1: Payoff Matrix for TIFMG.

	1	2
1	$\langle(40, 60, 90); 0.5, 0.4\rangle$	$\langle(50, 70, 80); 0.6, 0.2\rangle$
2	$\langle(40, 70, 80); 0.4, 0.3\rangle$	$\langle(30, 50, 60); 0.5, 0.3\rangle$

position $(1, 1)$ of Table 2.1 shows payoff $\langle(40, 60, 90); 0.5, 0.4\rangle$ which indicates that when the players I and II use alternatives 1 and 1 respectively, then we say that after government's mandatory circulation to work at workplace, strike supporters are able to shut-down the institutions 60 percent in number with minimum 40 percent and maximum 90 percent with membership degree 0.5 and non-membership degree 0.4.
Now we derive α-cut of $\langle(40, 60, 90)\rangle$ as $[40 + 20\alpha, 90 - 30\alpha]$ and the proposed robust ranking index is calculated as:

$$\begin{aligned} R(\langle(40, 60, 90)\rangle) &= \int_{\alpha=0}^{1} \left(\frac{\sqrt{3}}{3}\right) \times (40 + 20\alpha + 90 - 30\alpha) d\alpha \\ &= 72.1687836 \end{aligned}$$

Similarly, other indices are: $R(\langle(50, 70, 80)\rangle) = 77.9422863$, $R(\langle(40, 70, 80)\rangle) = 75.0555349$, $R(\langle(30, 50, 60)\rangle) = 54.8482755$ and then the defuzzified matrix is described in Table 2.2.

Table 2.2: Defuzzified Payoff Matrix for TIFMG.

	1	2
1	72.1687836	77.9422863
2	75.0555349	54.8482755

Since maximum of row minimum exists at the position $(1,1)$ and minimum of column maximum occurs at the position $(2,1)$ of Table 2.2, saddle point does not exist. So we use mixed strategy method by solving strategically the matrix game and we get, using LINGO 14.0 software with a 32-bit machine, from the following pair of linear programming problems where we search for player I, the strategies $(x_1, x_2),\ x_i \geq 0, \sum x_i = 1, x_i = p_i v, v = \frac{1}{\sum p_i}\ (i = 1,2)$.

$$\text{minimize} \quad p_1 + p_2$$
$$\text{subject to} \begin{cases} 72.1687836p_1 + 75.0555349p_2 \geq 1, \\ 77.9422863p_1 + 54.8482755p_2 \geq 1, \\ p_1, p_2 \geq 0. \end{cases}$$

And for player II, the strategies $(y_1, y_2),\ y_j \geq 0, \sum y_j = 1, y_j = q_j w,\ w = \frac{1}{\sum q_j}\ (j = 1,2)$.

$$\text{maximize} \quad q_1 + q_2$$
$$\text{subject to} \begin{cases} 72.1687836q_1 + 77.9422863q_2 \leq 1, \\ 75.0555349q_1 + 54.8482755q_2 \leq 1, \\ q_1, q_2 \geq 0. \end{cases}$$

Consequently, we get the optimal strategies as $(x_1^*, x_2^*) = (0.7777778, 0.2222222)$ and $(y_1^*, y_2^*) = (0.8888892, 0.1111108)$. The defuzzified value of the game is $v = 72.8103029 = w$ and the value of the game as TIFN is: $(0.7777778) \times \langle (40, 60, 90); 0.5, 0.4 \rangle + (0.2222222) \times \langle (40, 70, 80); 0.4, 0.3 \rangle$ $= \langle (40.0000055,\ 62.2222310,\ 87.7777896); 0.4777778,\ 0.3777778 \rangle$.
and this indicates that the expected optimization to the demands of the strike supporters against the government is in percentage 62.2222310 with the membership degree 0.4777778 and the non-membership degree 0.3777778.

2.5.2 Online shopping-marketing problem

There are a lot of online-marketing houses throughout the World, namely, 'Amazon', 'Flipkart', 'Snapdeal', 'Homeshop18', 'Ebay', 'Shopclues', 'Paytm', etc,. Among them we consider two companies, say C_1 and C_2, whose targeted aims are to increase their market shares under increasing demand of products in market. The two companies consider two strategies to increase their sales by rebating on prices of commodities in two ways, i.e., Strategy 1: COD or Cash On Delivery and Strategy 2: Net-Banking payment or Debit-Card payment. We consider these two companies as players I and II respectively. Companies estimate their sales-amount but a hesitation arises on the exactness of the sales-amount. Here we use TIFNs to express such ambiguity of the data-character. The payoff matrix of the above problem is given in Table 2.3. In

Table 2.3: Payoff matrix-1 for TIFMG.

	Strategy 1	Strategy 2
Strategy 1	$\langle (175, 180, 190); 0.6, 0.2 \rangle$	$\langle (150, 156, 158); 0.6, 0.1 \rangle$
Strategy 2	$\langle (80, 90, 100); 0.9, 0.1 \rangle$	$\langle (175, 180, 190); 0.6, 0.2 \rangle$

Table 2.3, the first element of the payoff matrix $\langle (175, 180, 190); 0.6, 0.2 \rangle$ indicates that the sales amount of company C_1 is about 180 with lower and upper bounds respectively 175 and 190 with membership degree 0.6 and non-membership degree 0.2, when both C_1 and C_2 use Strategy 1 simultaneously. We can explain the other elements of payoff matrix similarly.

Here, we first convert the payoff matrix into a defuzzified matrix using our proposed robust ranking technique and achieve in Table 2.4. Using the same mixed strategy method as previous

Table 2.4: Achieved Payoff matrix-2 for TIFMG.

	Strategy 1	Strategy 2
Strategy 1	209.2894725	178.9785834
Strategy 2	103.9230485	209.2894725

one, we get the optimal strategies as $(x_1^*, x_2^*) = (0.7765944, 0.2234056)$ and the defuzzified value of the game is 185.7496578. We obtain the value of the game as a TIFN:
$(0.7765944) \times \langle(175, 180, 190); 0.6, 0.2\rangle + (0.2234056) \times \langle(80, 90, 100); 0.9, 0.1\rangle$
$= \langle(153.7764706, 159.8934984, 169.8934984); 0.6670216, 0.1776594\rangle$.
From the above, we conclude that the sales-amount of company C_1 is 159.8934984 with lower bound 153.7764706 and upper bound 169.8934984, when C_1 and C_2 both choose Strategy 1 simultaneously with membership degree 0.6670216 and non-membership degree 0.1776594.

2.6 Result and Discussion

Comparison of fuzzy numbers can be explained by many ranking methods. The parametric method of comparing fuzzy numbers, mainly in fuzzy decision making problem, is more efficient than non-parametric method.

- In the first example, we have used ranking function approach to solve TIFMGs. In percentage, we have calculated the expected rate of strike-fulfillment as 62.2222310. We have derived the defuzzified value of the game as 72.8103029. Even if we use robust ranking technique [102] defined in Definition 2.3.6, we get the value of the game 63.8636485. When we have discussed from government's corner, i.e., considering player II, we have seen that the government wishes to minimize the demand of strike supporters with 61.1111888 percentage (using our proposed ranking technique).

- In the second example, we have derived the value of the game 159.8934984 which is more efficient than the results obtained previous by others [103; 104].

Table 2.5: Comparison among our derived solution and others.

Comparison with respect to Example 2.5.2	
Articles	Results
Nan et al. [103]	$\langle(152.37, 158.44, 165.18); 0.6, 0.2\rangle$
Nan et al. [104]	$\langle(148, 155, 162); 0.6, 0.2\rangle$
Robust ranking [102]	$\langle(153.776596, 159.893618, 169.893617); 0.6670212, 0.1776595\rangle$
Our proposed robust ranking [12]	$\langle(153.7764706, 159.8934984, 169.8934984); 0.6670216, 0.1776594\rangle$

From Table 2.5, the ranking results gained by the proposed technique are akin with the results obtained by [102]. This manifests the effectiveness of the proposed technique. Though the value of the game (upto 7 decimal places) slightly varies as 0.0001196, the lower and upper levels of the game value with membership and non-membership degrees are significantly changed and we get larger membership degree and smaller non-membership degree in comparison with [102]. Therefore, the proposed technique is suitable for utilization and gratification.

2.7 Conclusion

In this chapter, we have analyzed the matrix game under the IF environment. We have used defuzzification technique using robust ranking technique to the payoff matrix and then solved the triangular IF matrix game to obtain the value of the game. Here we have achieved a better result than the others. To solve the formulated problem, we have constructed an algorithm and solved the matrix game. Robust ranking technique, defined in the section 'Basic Concepts', has a multiplication coefficient 0.5 whereas we have changed it by 0.57735 and obtained a better result. Robust ranking technique is more effective than the different score functions and the accuracy functions [144; 153]. Using our robust ranking technique we have shown that the government is more aggressive to minimize the demands of strike supporters. Again using the same technique, with great interest, we have shown that in Online Shopping-Marketing Problem, Cash On Delivery is more effective way to increase the sales-amount than Net-Banking or Debit-Card payment options from company's view-point.