Chapter 5

Trade-credit in an economic production quantity model*

5.1 Introduction

In the modern marketing strategy, supplier provides a fixed time period to retailer for settling the purchasing amount. This fixed time-period is known as trade-credit-period. Additionally, to raise their market shares, retailer also provides trade-credit to their consumers. As result, retailer can hold their some payments within the duration of permissible period given by the supplier. Customer must pay the purchasing amount instantly at the time of having products from the retailer. Hence the retailer can enhance their profit due to obtain interests from their consumers.

In this direction, Arcelus *et al.* (2003) made an inventory system for deteriorating items and trade-credit policy. Chang *et al.* (2003) investigated an ordering inventory with deterioration, permissible delay-in-payments. Ouyang *et al.* (2006) studied a non-instantaneous deteriorating

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inventory model for delay-in-payments. Mahata (2012) obtained an EPQ model with two-level trade-credit policy. Soni and Patel (2012) made an integrated-inventory system with variable production rate, price-dependent demand, and two-level trade-credit strategy. Ouyang *et al.* (2013) deduced an ordering inventory model with some assumptions (i) interest rates applied by the supplier are not greater than the interest rate received by the wholesaler, and (ii) independent delay-period provided by the supplier of the permissible delay-period allowed by the wholesaler. Chen *et al.* (2014) derived ordering inventory system with permissible delay-in-payments. Sarkar *et al.* (2014) extended earlier research articles by adding some inspection process throughout production. Shah *et al.* (2014) depicted a price-sensitive inventory model and time varying demand for two-level trade-credit.

Taft (1918) first considered an EPQ model in the inventory literature. Szendrovits and Goyal (1981) discussed a production-inventory model for developing two items produced through different stages. Earlier, it is believed that in any EPQ model, all produced products are non-defective. Cheng (1989) formed a production-inventory model b mentioning defective items. Teng *et al.* (2005) derived a deterministic EPQ model with time-varying demand. Chiu *et al.* (2011) extended previous works with shipment and quality assurance matters. Pal *et al.* (2013) represented a production-inventory model with stochastic demand and defective production process. Chang (2013) developed an EPQ model defective quality products. Wee *et al.* (2013) deduced a production-inventory system with shortages.

It can be observed in any production factory that almost all physical products deteriorated during time. Most of the earlier research works are formulated with the assumption with constant deterioration. It can be concluded deterioration of products as time-dependent if one can measure practical situation. For instance, fruits and vegetables are deteriorated throughout all the time.

Therefore, to relate with real life observations deterioration of items are considered as variable. Chakrabarty *et al.* (1998) considered an ordering quantity model with probabilistic deterioration, backlogging, and trended demand. Chu and Chen (2002) invented few inventory replenishment techniques by highlighting exponentially declining deterioration rate of items. Chung and Wee (2011) determined a deteriorating inventory system in which deterioration is assumed for short life-cycle of products.

Covert and Philip (1973) surveyed an EOQ model for weibull distribution deteriorating items. Hariga (1996) derived a deteriorating ordering model with time-varying demand. Goyal and Giri (2003) depicted an inventory system for time-varying demand and deterioration of products. Mukhopadhyay *et al.* (2004) formulated a pricing-ordering inventory model with deteriorating items. Skouri *et al.* (2009) addressed some inventory models in which ramp type demand, weibull distributed deterioration along with partial backlogging. Sana (2010) obtained an inventory system for time-dependent deterioration as well as partial backlogging. Sett *et al.* (2012) provided a two-warehouse inventory model that provides time-dependent deterioration and increasing demand. Sarkar (2012b) discussed an ordering inventory system where delay-in-payments and time-dependent deterioration are assumed. Chung and Cárdenas-Barrón (2013) made a deteriorating inventory system for stock-dependent demand along with two-level trade-credit technique. Sarkar *et al.* (2013) surveyed a deteriorating inventory model by providing that both component cost and selling-price are increases with time. Later, Sarkar and Sarkar (2013b) extended previous research works by assuming time-varying deterioration. Sarkar and Sarkar (2013) deduced an EMQ model by assuming probabilistic deterioration. Sarkar (2013) derived a production model for probabilistic deterioration. Chung *et al.* (2014) a deteriorating inventory system with the consideration of two-level trade-credit policy with continuous deterioration.

Wu *et al.* (2014) invented an Economic Order Quantity (EOQ) model on the basis such assumptions which are (i) supplier offers an up-stream trade-credit policy but the retailer provides a down-stream trade-credit strategy, (ii) retailers down-stream trade-credit makes an opportunity to raise sales, opportunity cost, and profit, and (iii) deteriorating products are decayed with time and has its expiration dates. Sarkar *et al.* (2014) extended the model of Mahata (2012) by adding the concept of time-dependent deterioration for fixed lifetime items with two-level trade-credit policy. De and Sana (2015) observed an inventory system in which both shortages and demand are dependent on promotional impact and selling price. See Table 5.1 for contribution of several authors.

Table 5.1: Contribution of the several authors

Author(s)	**EPQ**	**Two-level trade-credit policy**	**Constant deterioration**	**Variable deterioration**
Taft (1918)	√			
Covert and Philip (1973)				√
Szendrovits and Goyal (1981)	√			
Cheng (1989)	√			
Hariga (1996)			√	
Chakrabarty *et al.* (1998)				√
Chu and Chen (2002)				√
Goyal and Giri (2003)			√	
Arcelus *et al.* (2003)			√	

Author(s)	**EPQ**	**Two-level trade-credit policy**	**Constant deterioration**	**Variable deterioration**
Chang *et al.* (2003)			√	
Mukhopadhyay *et al.* (2004)				√
Teng *et al.* (2005)	√			
Ouyang *et al.* (2006)			√	
Skouri *et al.* (2009)				√
Sana (2010)				√
Chiu *et al.* (2011)	√			
Chung and Wee (2011)			√	
Sett *et al.* (2012)				√
Mahata (2012)		√	√	
Soni and Patel (2012)		√		
Sarkar (2012b)		√	√	
Sarkar (2013)				√
Sarkar *et al.* (2013)			√	
Sarkar and Sarkar (2013b)				√
Chung and Cárdenas-Barrón (2013)		√	√	

Author(s)	**EPQ**	**Two-level trade-credit policy**	**Constant deterioration**	**Variable deterioration**
Sarkar *et al.* (2014)		$\surd$		
Shah *et al.* (2014)		$\surd$		
Wu *et al.* (2014)		$\surd$	$\surd$	
This chapter	$\surd$	$\surd$		$\surd$

This chapter mentioning the technique that supplier provides their retailer a full trade-credit-period. On the other situation, retailer provides their consumers a partial trade-credit-period. Deterioration of products is assumed as exponential related to time in this chapter. This chapter is presented to minimized retailer's optimal cost function by calculating the cycle length. In final section of this chapter, numerical examples and also corresponding sensitivity analysis are discussed for seven cases.

5.2 Mathematical model

The following notation are applied to derive this model.

P_2 production or replenishment rate (unit/year)

A_2 retailer's ordering cost (\$/order)

D rate of demand (unit/year)

c cost of purchasing product (\$/unit)

h holding cost excluding interest charges (\$/unit/year)

s	selling-price, $s \geq c$, (\$/unit)
α_2	customers fraction of the total payment owed to retailer (\$/unit)
M_2	retailer's trade-credit-period which they allowed by the supplier (years)
N_2	customers trade-credit-period given by the retailer (years)
I_{e2}	interest received from customers by retailer (\$/year)
I_{c2}	interest paid by retailer which is charged by the supplier (\$/year)
$\theta_2(t)$	time-dependent deterioration rate, $0 < \theta_2(t) < 1$
t_1	production time length in a cycle (years)
T	length of the cycle time (years)
$TRC(T)$	retailer's annual total cost (\$/year)
T^*	optimal cycle length (years)

Some assumptions are prepared to produce this model which are given as follow:

1. The model assumes both production rate and demand rate are constant.
2. Supplier allowed a full trade-credit policy to their retailer. In addition, retailer gives a partial trade-credit policy to their consumers.
3. By making a partial payment, retailer can obtain more interest at a rate I_{e2} from their consumers.
4. While $T \geq M_2$, account is adjusted at $T = M_2$. Then the retailer pays the interest charges on products in stock at a rate I_{c2}.

5. While $T \leq M_2$, account is adjusted at $T = M_2$. Then the retailer does not required to pay any interest to supplier.

6. Deterioration rate is exponential over time.

7. Time horizon is considered as infinite.

8. Lead time is taken as negligible and shortages are not considered in this model.

This model assumes a production-inventory system. At time $t = 0$, production increases and raises to the time $t = t_1$. During the time $[0, t_1]$ level of the inventory is increased by production, demand, as well as deterioration. At the time $t = t_1$, production terminates and then the inventory level decreases to $t = T$ for deterioration and consumption. Figure 5.1 illustrates the graphical formation of the inventory system.

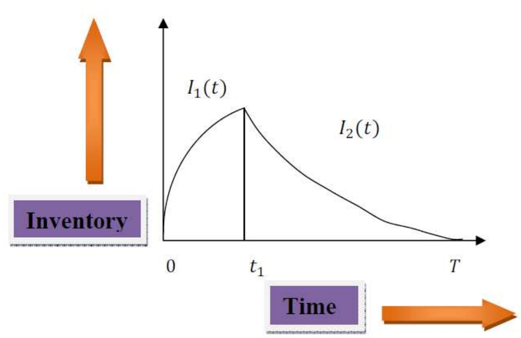

Figure 5.1: Graphical representation of the inventory model

The differential equation of the inventory model in time interval $[0,\, t_1]$is

$$\frac{dI_1(t)}{dt} + e^{-\theta_2 t} I_1(t) = P_2 - D, \quad 0 \leq t \leq t_1$$

where the initial condition is $I_1(0) = 0$.

In the time interval $[t_1, T]$, the differential equation of the inventory model is

$$\frac{dI_2(t)}{dt} + e^{-\theta_2 t} I_2(t) = -D, \quad t_1 \le t \le T$$

where boundary condition is $I_2(T) = 0$.

One can determine solutions of these equations are

$$I_1(t) = (P_2 - D)\left(t + \frac{t^2}{2}\right) e^{-\left(t - \frac{\theta_2 t^2}{2}\right)}.$$

and

$$I_2(t) = D\left((T - t) + \frac{(T^2 - t^2)}{2}\right) e^{-\left(t - \frac{\theta_2 t^2}{2}\right)}.$$

[In this case, Taylor series expansion of this exponential function is taken to second order as θ_2 is small.]

Using the continuity condition during $t = t_1$, one has $I_1(t_1) = I_2(t_1)$.

Then, $t_1 = \sqrt{1 + \frac{2D}{P_2}\left(T + \frac{T^2}{2}\right)} - 1$.

There are two cases for annual total cost which are $M_2 \ge N_2$ and $M_2 < N_2$.

A. Annual total cost while $M_2 \ge N_2$

This section calculated the annual total cost when $M_2 \ge N_2$.

Retailer's annual ordering cost as $= \frac{A_2}{T}$.

Annual stock holding cost where interest charges is not considering given by

$$\begin{aligned} &= \frac{h}{T}\left[\int_0^{t_1} I_1(t)dt + \int_{t_1}^{T} I_2(t)dt\right] \\ &= \frac{h(P_2 - D){t_1}^2}{2T} + \frac{hD}{T}\left(\frac{T^2}{2} - Tt_1 + \frac{{t_1}^2}{2} + \frac{T{t_1}^2}{2} - \frac{T^2 t_1}{2} + \frac{T^2{t_1}^2}{4}\right). \end{aligned}$$

Cost for deterioration is $= \frac{c(P_2 t_1 - DT)}{T}$.

There are four cases for interest charged which are as

Case A.(i) $M_2 \leq t_1$ **i.e.,** $M_2 \leq t_{M2} \leq T$

See Figure 5.2 for the inventory position.

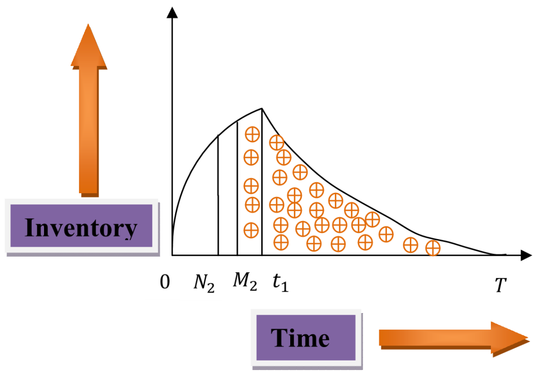

Figure 5.2: Total interest payable while $T \geq t_{M2}$

Annual interest payable is

$$
\begin{aligned}
&= \frac{cI_{c2}}{T}\left[\int_{M_2}^{t_1} I_1(t)dt + \int_{t_1}^{T} I_2(t)dt\right] \\
&= \frac{cI_{c2}(P_2 - D)({t_1}^2 - {M_2}^2)}{2T} + \frac{cI_{c2}D}{T}\left(\frac{T^2}{2} - Tt_1 + \frac{{t_1}^2}{2} + \frac{T{t_1}^2}{2} - \frac{T^2 t_1}{2} + \frac{T^2 {t_1}^2}{4}\right).
\end{aligned}
$$

Case A.(ii) $t_1 \leq M_2 \leq T$ **i.e.,** $M_2 \leq T \leq t_{M2}$

See Figure 5.3 for the present inventory position.

Annual interest payable is

$$
\begin{aligned}
&= \frac{cI_{c2}}{T}\left[\int_{M_2}^{T} I_2(t)dt\right] \\
&= \frac{cI_{c2}D}{T}\left(\frac{T^2}{2} - TM_2 + \frac{{M_2}^2}{2} + \frac{T{M_2}^2}{2} - \frac{T^2 M_2}{2} + \frac{T^2 {M_2}^2}{4}\right).
\end{aligned}
$$

Case A.(iii) $N_2 \leq T \leq M_2$

The annual interest payable as 0.

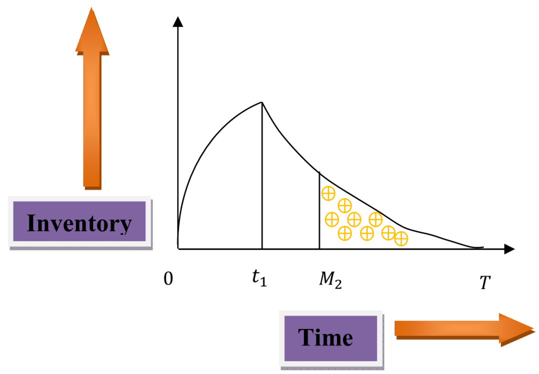

Figure 5.3: Total interest payable while $M_2 \leq T \leq t_{M2}$ and $t_1 \leq N_2 \leq M_2$

Case A.(iv) $0 < T \leq N_2$

The annual interest payable of retailer to supplier is 0.

There arises four cases for interest earned per year of retailer from their consumers.

Case A.(a) $M_2 \leq t_1$ **i.e.,** $M_2 \leq t_{M2} \leq T$

See Figure 5.4 for the present inventory position.

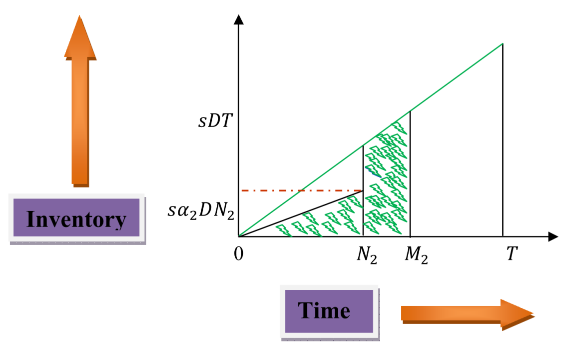

Figure 5.4: Total interest earned while $M_2 \leq T$

Annual interest earned is

$$\begin{aligned} &= \frac{sI_{e2}}{T}\left[\frac{DN_2{}^2\alpha_2}{2} + \frac{(DN_2 + DM_2)(M_2 - N_2)}{2}\right] \\ &= \frac{sI_{e2}D}{2T}\left[M_2{}^2 - (1-\alpha_2)N_2{}^2\right]. \end{aligned}$$

Case A.(b) $t_1 \leq M_2 \leq T$ **i.e.,** $M_2 \leq T \leq t_{M2}$

Retailer's annual interest earned from consumers is $= \frac{sI_{e2}D}{2T}[{M_2}^2 - (1-\alpha_2){N_2}^2]$.

which is same as Case A.(a)

Case A.(c) $N_2 \leq T \leq M_2$

See Figure 5.5 for the present inventory position.

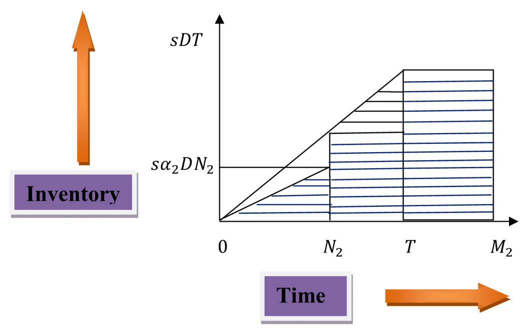

Figure 5.5: Total interest earned while $N_2 \leq T \leq M_2$

Annual interest earned is

$$\begin{aligned} &= \frac{sI_{e2}}{T}\left[\frac{D{N_2}^2\alpha_2}{2} + \frac{(DT+DN_2)(T-N_2)}{2} + (M_2-T)DT\right] \\ &= \frac{sI_{e2}D}{2T}\left[2M_2T - (1-\alpha_2){N_2}^2 - T^2\right]. \end{aligned}$$

Case A.(d) $0 < T \leq N_2$

See Figure 5.6 for the present inventory position.

Annual interest earned is

$$\begin{aligned} &= \frac{sI_{e2}}{T}\left[\frac{DT^2\alpha_2}{2} + \alpha_2 DT(N_2-T) + (M_2-N_2)DT\right] \\ &= sI_{e2}D\left[M_2 - (1-\alpha_2)N_2 - \frac{\alpha_2 T}{2}\right]. \end{aligned}$$

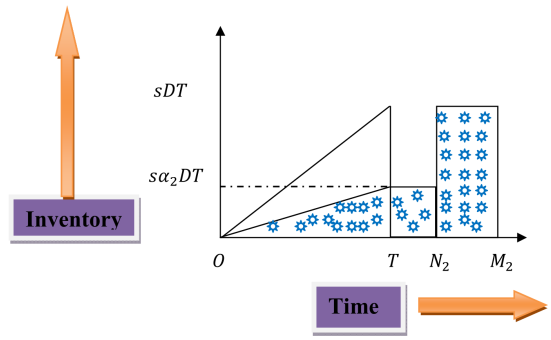

Figure 5.6: Total interest earned while $0 < T \leq N_2$

Retailer's annual total cost is

$TRC(T)$ = ordering cost + holding charge + deterioration or decay cost + interest payable − interest earned.

$$TRC(T) = \begin{cases} TRC_1(T), \text{ if } T \geq t_{M2} \\ TRC_2(T), \text{ if } M_2 \leq T \leq t_{M2} \\ TRC_3(T), \text{ if } N_2 \leq T \leq M_2 \\ TRC_4(T), \text{ if } 0 < T \leq N_2 \end{cases}$$

where the cost expressions are given by

$$\begin{aligned} TRC_1(T) &= \frac{A_2}{T} + \frac{h(P_2 - D){t_1}^2}{2T} + \frac{hD}{T}\left(\frac{T^2}{2} - Tt_1 + \frac{{t_1}^2}{2} + \frac{T{t_1}^2}{2} - \frac{T^2t_1}{2} + \frac{T^2{t_1}^2}{4}\right) \\ &+ \frac{cI_{c2}(P_2 - D)({t_1}^2 - {M_2}^2)}{2T} + \frac{cI_{c2}D}{T}\Big(\frac{T^2}{2} - Tt_1 + \frac{{t_1}^2}{2} + \frac{T{t_1}^2}{2} - \frac{T^2t_1}{2} \\ &+ \frac{T^2{t_1}^2}{4}\Big) + \frac{c(P_2t_1 - DT)}{T} - \frac{sI_{e2}D}{2T}[{M_2}^2 - (1 - \alpha_2){N_2}^2], \end{aligned}$$

$$\begin{aligned} TRC_2(T) &= \frac{A_2}{T} + \frac{h(P_2 - D){t_1}^2}{2T} + \frac{hD}{T}\left(\frac{T^2}{2} - Tt_1 + \frac{{t_1}^2}{2} + \frac{T{t_1}^2}{2} - \frac{T^2t_1}{2} + \frac{T^2{t_1}^2}{4}\right) \\ &+ \frac{cI_{c2}D}{T}\left(\frac{T^2}{2} - TM_2 + \frac{{M_2}^2}{2} + \frac{T{M_2}^2}{2} - \frac{T^2M_2}{2} + \frac{T^2{M_2}^2}{4}\right) \\ &+ \frac{c(P_2t_1 - DT)}{T} - \frac{sI_{e2}D}{2T}[{M_2}^2 - (1 - \alpha_2){N_2}^2], \end{aligned}$$

$$\begin{aligned} TRC_3(T) &= \frac{A_2}{T} + \frac{c(P_2 t_1 - DT)}{T} - \frac{sI_{e2}D}{2T}[2M_2T - (1-\alpha_2){N_2}^2 - T^2] + \frac{h(P_2 - D){t_1}^2}{2T} \\ &+ \frac{hD}{T}\left(\frac{T^2}{2} - Tt_1 + \frac{{t_1}^2}{2} + \frac{T{t_1}^2}{2} - \frac{T^2 t_1}{2} + \frac{T^2{t_1}^2}{4}\right), \end{aligned}$$

and

$$\begin{aligned} TRC_4(T) &= \frac{A_2}{T} + \frac{h(P_2 - D){t_1}^2}{2T} + \frac{hD}{T}\left(\frac{T^2}{2} - Tt_1 + \frac{{t_1}^2}{2} + \frac{T{t_1}^2}{2} - \frac{T^2 t_1}{2} + \frac{T^2{t_1}^2}{4}\right) \\ &+ \frac{c(P_2 t_1 - DT)}{T} - sI_{e2}D\Big[M_2 - (1-\alpha_2)N_2 - \frac{\alpha_2 T}{2}\Big]. \end{aligned}$$

As, $t_{M2} = \sqrt{1 + \frac{2P_2}{D}\left(M_2 + \frac{{M_2}^2}{2}\right)} - 1$ and the continuity condition at t_{M2} states that

$TRC_1(t_{M2}) = TRC_2(t_{M2})$, $TRC_2(t_{M2}) = TRC_3(t_{M2})$, $TRC_3(t_{M2}) = TRC_4(t_{M2})$.

Therefore $TRC(T)$, $TRC_1(T)$, $TRC_2(T)$, $TRC_3(T)$, and $TRC_4(T)$ all are well defined while $T > 0$.

B. Annual total cost while $M_2 < N_2$

This case calculated retailer's annual total cost while $M_2 < N_2$.

Annual ordering cost is $= \frac{A_2}{T}$.

Annual stock holding cost without interest charges is

$$\begin{aligned} &= \frac{h}{T}\left[\int_0^{t_1} I_1(t)dt + \int_{t_1}^{T} I_2(t)dt\right] \\ &= \frac{h(P_2 - D){t_1}^2}{2T} + \frac{hD}{T}\left(\frac{T^2}{2} - Tt_1 + \frac{{t_1}^2}{2} + \frac{T{t_1}^2}{2} - \frac{T^2 t_1}{2} + \frac{T^2{t_1}^2}{4}\right). \end{aligned}$$

Deterioration cost is $= \frac{c(P_2 t_1 - DT)}{T}$.

There arises three cases due to interest charged which are

Case B.(i) $t_{M2} \leq T$

See Figure 5.2 for the present position of inventory.

Annual interest payable is

$$\begin{aligned} &= \frac{cI_{c2}}{T}\left[\int_{M_2}^{t_1} I_1(t)dt + \int_{t_1}^{T} I_2(t)dt\right] \\ &= \frac{cI_{c2}(P_2 - D)({t_1}^2 - {M_2}^2)}{2T} + \frac{cI_{c2}D}{T}\left(\frac{T^2}{2} - Tt_1 + \frac{{t_1}^2}{2} + \frac{T{t_1}^2}{2} - \frac{T^2 t_1}{2} + \frac{T^2{t_1}^2}{4}\right). \end{aligned}$$

Case B.(ii) $M_2 \leq T \leq t_{M2}$

See Figure 5.3 for the present position of inventory.

Annual interest payable is

$$\begin{aligned} &= \frac{cI_{c2}}{T}\left[\int_{M_2}^{T} I_2(t)dt\right] \\ &= \frac{cI_{c2}D}{T}\left(\frac{T^2}{2} - TM_2 + \frac{{M_2}^2}{2} + \frac{T{M_2}^2}{2} - \frac{T^2M_2}{2} + \frac{T^2{M_2}^2}{4}\right). \end{aligned}$$

Case B.(iii) $0 < T \leq M_2$

Retailer's annual interest payable is 0.

Three cases obtains for interest earned per year which are

Case B.(a) $t_{M2} \leq T$

See Figure 5.7 for the present position of inventory.

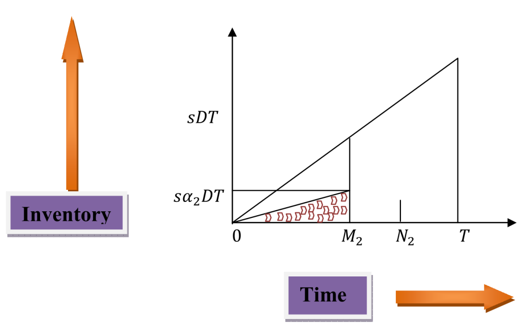

Figure 5.7: Total interest earned while $t_{M2} \leq T$

Annual interest earned $= \frac{sI_{e2}D{M_2}^2\alpha_2}{2T}$.

Case B.(b) $M_2 \leq T \leq t_{M2}$

Retailer's annual interest earned from consumers is $= \frac{sI_{e2}D{M_2}^2\alpha_2}{2T}$.

Which is same as Case B.(a).

Case B.(c) $0 < T \leq M_2$

See Figure 5.8 for the present position of inventory.

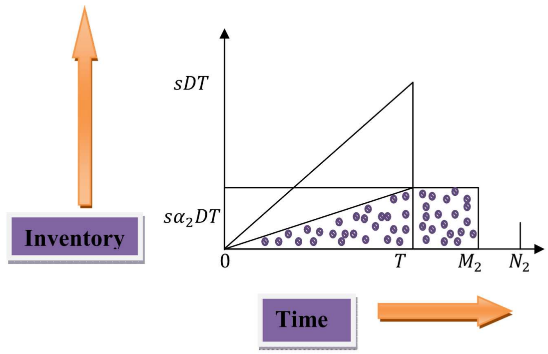

Figure 5.8: Total interest earned while $0 < T \leq M_2$

Annual interest earned is

$$= \frac{sI_{e2}}{T}\left[\frac{DT^2\alpha_2}{2} + \alpha_2 DT(M_2 - T)\right] = sI_{e2}D\alpha_2\left[M_2 - \frac{T}{2}\right].$$

Retailer's annual total cost is

$TRC(T)$ = ordering cost + holding or carrying cost + deterioration or decay charge + interest payable − interest earned.

$$TRC(T) = \begin{cases} TRC_5(T); \text{ if } T \geq t_{M2} \\ TRC_6(T); \text{ if } M_2 \leq T \leq t_{M2} \\ TRC_7(T); \text{ if } 0 < T \leq M_2 \end{cases}$$

where the costs are given by

$$\begin{aligned} TRC_5(T) &= \frac{A_2}{T} + \frac{h(P_2 - D){t_1}^2}{2T} + \frac{hD}{T}\left(\frac{T^2}{2} - Tt_1 + \frac{{t_1}^2}{2} + \frac{T{t_1}^2}{2} - \frac{T^2 t_1}{2} + \frac{T^2{t_1}^2}{4}\right) \\ &+ \frac{cI_{c2}(P_2 - D)({t_1}^2 - {M_2}^2)}{2T} + \frac{cI_{c2}D}{T}\left(\frac{T^2}{2} - Tt_1 + \frac{{t_1}^2}{2} + \frac{T{t_1}^2}{2} - \frac{T^2 t_1}{2}\right. \\ &+ \left.\frac{T^2{t_1}^2}{4}\right) - \frac{sI_{e2}D{M_2}^2\alpha_2}{2T} + \frac{c(P_2 t_1 - DT)}{T}, \end{aligned}$$

$$
\begin{aligned}
TRC_6(T) &= \frac{A_2}{T} + \frac{h(P_2 - D){t_1}^2}{2T} + \frac{hD}{T}\left(\frac{T^2}{2} - Tt_1 + \frac{{t_1}^2}{2} + \frac{T{t_1}^2}{2} - \frac{T^2 t_1}{2} + \frac{T^2{t_1}^2}{4}\right) \\
&+ \frac{cI_{c2}D}{T}\Big(\frac{T^2}{2} - TM_2 + \frac{{M_2}^2}{2} + \frac{T{M_2}^2}{2} - \frac{T^2M_2}{2} + \frac{T^2{M_2}^2}{4}\Big) \\
&- \frac{sI_{e2}D{M_2}^2\alpha_2}{2T} + \frac{c(P_2t_1 - DT)}{T},
\end{aligned}
$$

and

$$
\begin{aligned}
TRC_7(T) &= \frac{A_2}{T} + \frac{h(P_2 - D){t_1}^2}{2T} + \frac{hD}{T}\left(\frac{T^2}{2} - Tt_1 + \frac{{t_1}^2}{2} + \frac{T{t_1}^2}{2} - \frac{T^2 t_1}{2} + \frac{T^2{t_1}^2}{4}\right) \\
&+ \frac{c(P_2t_1 - DT)}{T} - sI_{e2}D\alpha_2\left[M_2 - \frac{T}{2}\right].
\end{aligned}
$$

At t_{M2}, $TRC_5(t_{M2}) = TRC_6(t_{M2})$, $TRC_6(t_{M2}) = TRC_7(t_{M2})$, and $TRC(T)$, $TRC_5(T)$, $TRC_6(T)$, and $TRC_7(T)$ are well defined while $T > 0$.

Lemma

If $j(t)$ is a continuous function on (a,b) and $\frac{dj(t)}{dt} = 0$, then $j(t)$ will be convex.

Proof: To establish the statement of the above Lemma, two cases arises which are $M_2 \geq N_2$ and $M_2 < N_2$.

Case A $M_2 \geq N_2$

For this case, four subcases are appeared which are as follows:

Case A.(1) $T \geq t_{M2}$

Case A.(2) $M_2 \leq T \leq t_{M2}$

Case A.(3) $N_2 \leq T \leq M_2$

Case A.(4) $0 < T \leq N_2$

Case B $M_2 < N_2$

Three subcases are discussed for this case which are given below:

Case B.(1) $T \geq t_{M2}$

Case B.(2) $M_2 \leq T \leq t_{M2}$

Case B.(3) $0 < T \leq M_2$

The convexity of each cost functions $TRC_1(T), TRC_2(T), TRC_3(T), TRC_4(T), TRC_5(T), TRC_6(T)$, and $TRC_7(T)$ are illustrated in Appendix A3.

5.3 Numerical examples

Some numerical examples are presented in this chapter.

Example 1(a)

Let A_2 = \$200/order, P_2 = 3000 units/year, s = \$75/unit, D = 2500 units/year, c = \$50/unit, I_{c2} = \$0.15/\$/year, h = \$15/unit/year, I_{e2} = \$0.1/\$/year, $M_2 = 0.1$ year, $\alpha_2 = 0.05$, $N_2 = 0.05$ year, $\theta_2 = 0.05$, then the retailer's annual total cost is $TRC_1(T)$ = \$2499.87 and length of cycle time $T = 0.1$ year. Figure 5.9 shows the minimum of the annual total cost $TRC_1(T)$ at optimal cycle time (T).

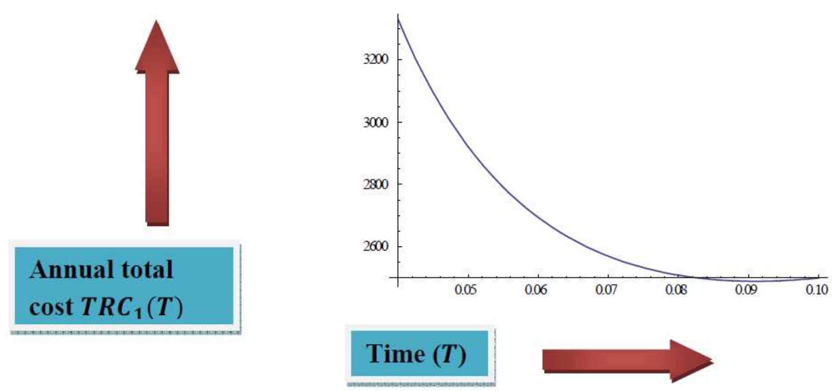

Figure 5.9: Annual total cost $TRC_1(T)$ versus time (T)

Example 2(a)

Let A_2 = \$150/order, P_2 = 3000 units/year, s = \$75/unit, D = 2500 units/year, I_{e2} = \$0.1/\$/year, c = \$50/unit, h = \$15/unit/year, I_{c2} = \$0.15/\$/year, M_2 = 0.1 year, $\alpha_2 = 0.05$, N_2 = 0.08 year, $\theta_2 = 0.05$, then the retailer's annual total cost is $TRC_2(T)$ = \$2393.07 and length of cycle time T = 0.1 year. Figure 5.10 shows the minimum of the annual total cost $TRC_2(T)$ at optimal cycle time (T).

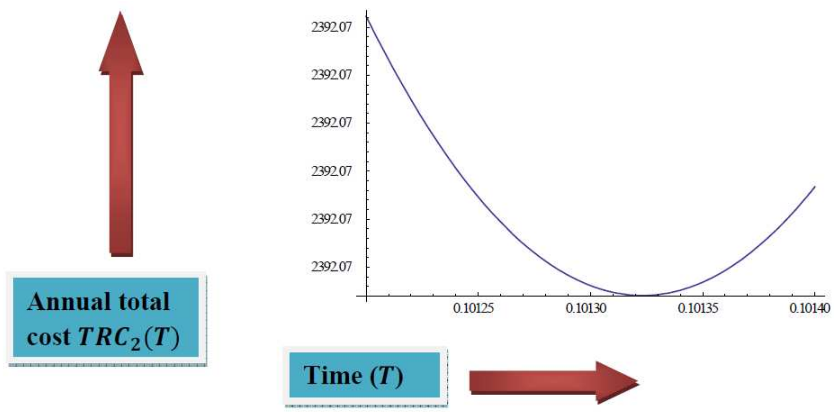

Figure 5.10: Annual total cost $TRC_2(T)$ versus time (T)

Example 3(a)

Let A_2 = \$100/order, P_2 = 4000 units/year, D = 2500 units/year, s = \$75/unit, N_2 = 0.05 year, c = \$50/unit, h = \$15/unit/year, I_{c2} = \$0.15/\$/year, M_2 = 0.1 year, $\alpha_2 = 0.05$, I_{e2} = \$0.1/\$/year, $\theta_2 = 0.05$, then the retailer's annual total cost is $TRC_3(T)$ = \$2481.5 and length of cycle time T = 0.05 year. Figure 5.11 shows the minimum of the annual total cost $TRC_3(T)$ at optimal cycle time (T).

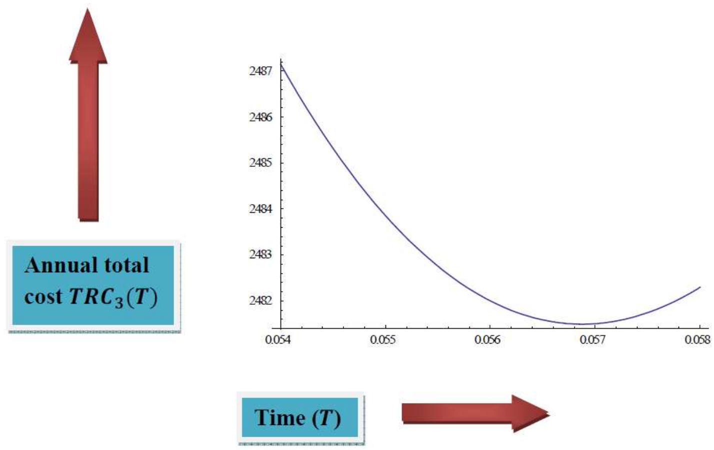

Figure 5.11: Annual total cost $TRC_3(T)$ versus time (T)

Example 4(a)

Let $A_2 = \$50$/order, $P_2 = 4000$ units/year, $D = 2500$ units/year, $I_{c2} = \$0.15/\$/\text{year}$, $s = \$100$/unit, $c = \$50$/unit, $N_2 = 0.08$ year, $\alpha_2 = 0.05$, $h = \$15$/unit/year, $I_{e2} = \$0.12/\$/\text{year}$, $M_2 = 0.12$ year, $\theta_2 = 0.05$, then the retailer's annual total cost is $TRC_4(T) = \$1148.32$ and length of cycle time $T = 0.04$ year. Figure 5.12 shows the minimum of the annual total cost $TRC_4(T)$ at optimal cycle time (T).

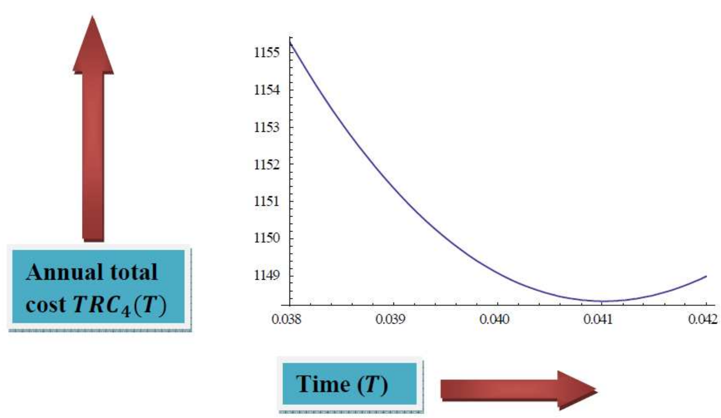

Figure 5.12: Annual total cost $TRC_4(T)$ versus time (T)

Example 5(a)

Let $A_2 = \$150$/order, $I_{c2} = \$0.15/\$$/year, $P_2 = 3000$ units/year, $M_2 = 0.02$ year, $s = \$75$/unit, $D = 2500$ units/year, $c = \$50$/unit, $h = \$15$/unit/year, $I_{e2} = \$0.1/\$$/year, $N_2 = 0.05$ year, $\alpha_2 = 0.05$, $\theta_2 = 0.05$, then the retailer's annual total cost is $TRC_5(T) = \$2886.75$ and length of cycle time $T = 0.1$ year. Figure 5.13 shows the minimum of the annual total cost $TRC_5(T)$ at optimal cycle time (T).

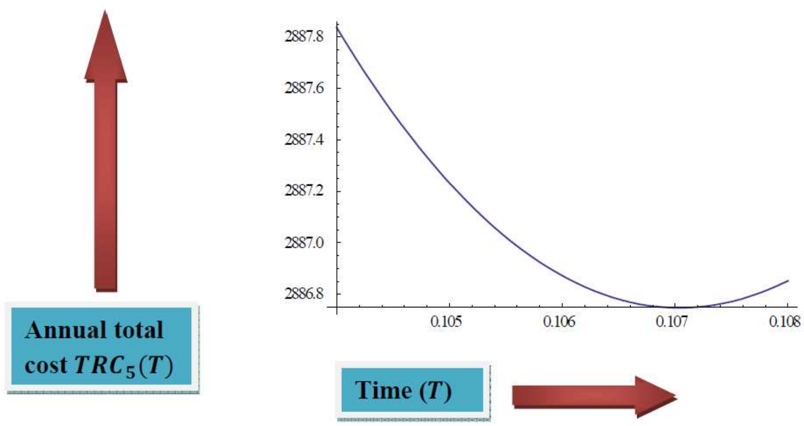

Figure 5.13: Annual total cost $TRC_5(T)$ versus time (T)

Example 6(a)

Let $D = 2500$ units/year, $A_2 = \$100$/order, $P_2 = 3500$ units/year, $s = \$75$/unit, $c = \$50$/unit, $I_{e2} = \$0.1/\$$/year, $h = \$15$/unit/year, $N_2 = 0.5$ year, $M_2 = 0.14$ year, $I_{c2} = \$0.15/\$$/year, $\alpha_2 = 0.05$, $\theta_2 = 0.05$, then the retailer's annual total cost is $TRC_6(T) = \$3289.12$ and length of cycle time $T = 0.1$ year. Figure 5.14 shows the minimum of the annual total cost $TRC_6(T)$ at optimal cycle time (T).

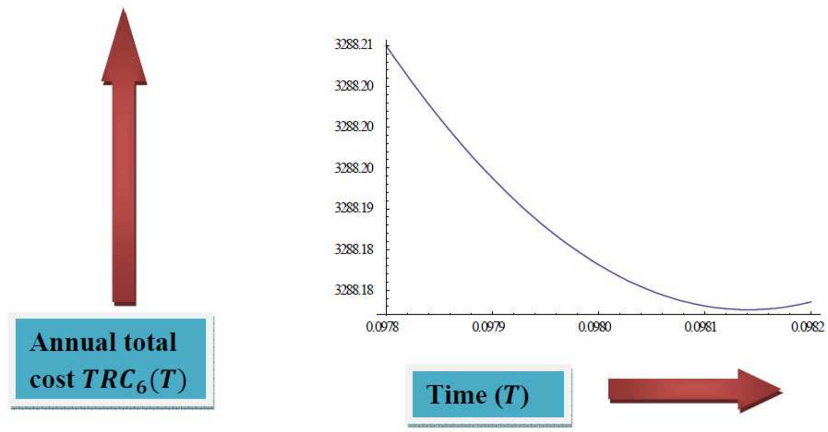

Figure 5.14: Annual total cost $TRC_6(T)$ versus time (T)

Example 7(a)

Let s = \$100/unit, A_2 = \$50/order, D = 2500 units/year , c = \$50/unit, I_{c2} = \$0.15/\$/year, P_2 = 4000 units/year, I_{e2} = \$0.1/\$/year, $N_2 = 0.2$ year, $\alpha_2 = 0.05$, h = \$15/unit/year, $M_2 = 0.1$ year, $\theta_2 = 0.05$, then the retailer's annual total cost is $TRC_7(T)$ = \$2338.19 and length of cycle time $T = 0.04$ year. Figure 5.15 shows the minimum of the annual total cost $TRC_7(T)$ at optimal cycle time (T).

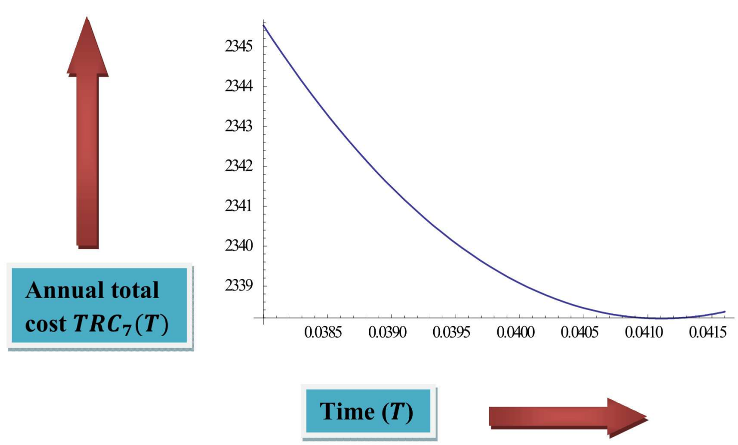

Figure 5.15: Annual total cost $TRC_7(T)$ versus time (T)

Case Study

The concept of exponentially deteriorated products is highlighted. Also, retailers offer partial trade credit policy to their customers for enhancing business strategy. In this proposed model, major factor is deterioration rate of products is exponential over time. A half cut apple is the perfect example of this concept i.e. exponentially deteriorating items. If we cut an apple in half, it can be easily notice that half cut apple changes its colour. At first, the colour of that half cut apple was white. After sometime its colour changes to light brown. Finally, that half cut apple turned into dark brown colour. It is mainly because of the deterioration of products.In first time interval i.e. until that apple turns into light brown colour, the deterioration rate of that apple was minor with time. After that time interval i.e. the deterioration rate of that apple increases rapidly with time. The dark brown colour of half cut apple indicates that the apple becomes totally deteriorated or spoiled.

Numerical examples

Example 1(b)

Let $A_2 = \$210$/order, $P_2 = 3010$ units/year, $s = \$76$/unit, $D = 2400$ units/year, $c = \$50.5$/unit, $I_{c2} = \$0.14/\$/\text{year}$, $h = \$16$/unit/year, $I_{e2} = \$0.11/\$/\text{year}$, $M_2 = 0.09$ year, $\alpha_2 = 0.04$, $N_2 = 0.06$ year, $\theta_2 = 0.04$, then the retailer's annual total cost is $TRC_1(T) = \$3123.18$ and length of cycle time $T = 0.1$ year. Figure 5.16 shows the minimum of the annual total cost $TRC_1(T)$ at optimal cycle time (T).

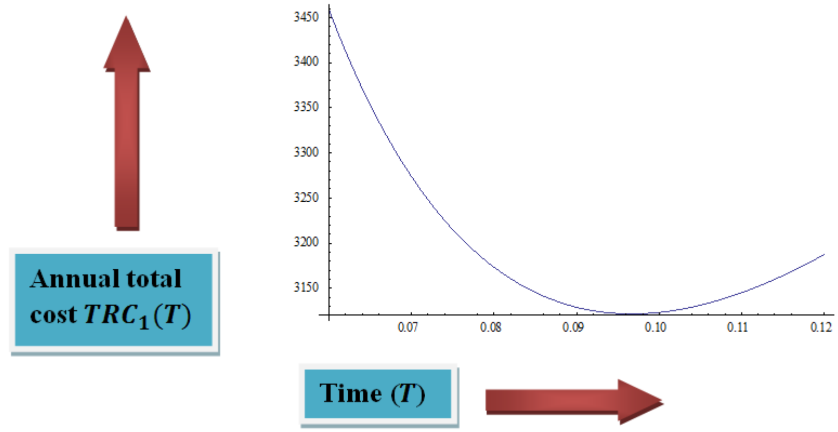

Figure 5.16: Annual total cost $TRC_1(T)$ versus time (T)

Example 2(b)

Let $A_2 = \$160$/order, $P_2 = 4000$ units/year, $s = \$100$/unit, $D = 2600$ units/year, $I_{e2} = \$0.12/\$$/year, $c = \$30$/unit, $h = \$5$/unit/year, $I_{c2} = \$0.4/\$$/year, $M_2 = 0.09$ year, $\alpha_2 = 0.01$, $N_2 = 0.07$ year, $\theta_2 = 0.01$, then the retailer's annual total cost is $TRC_2(T) = \$2574.17$ and length of cycle time $T = 0.09$ year. Figure 5.17 shows the minimum of the annual total cost $TRC_2(T)$ at optimal cycle time (T).

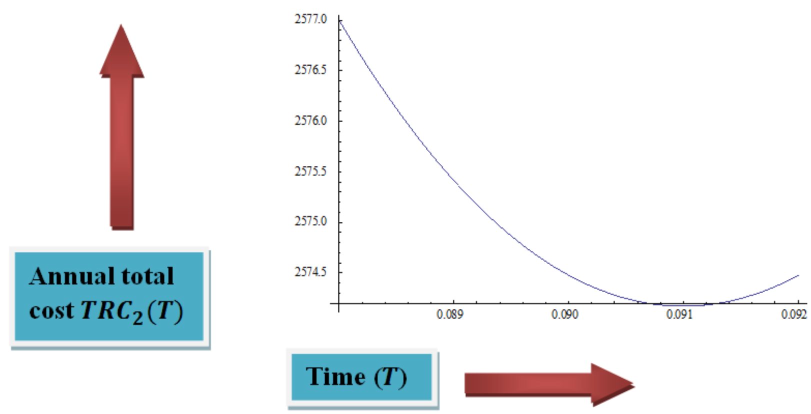

Figure 5.17: Annual total cost $TRC_2(T)$ versus time (T)

Example 3(b)

Let $A_2 = \$97$/order, $P_2 = 4500$ units/year, $D = 1600$ units/year, $s = \$85$/unit, $N_2 = 0.04$ year, $c = \$30$/unit, $h = \$25$/unit/year, $I_{c2} = \$0.3/\$/\text{year}$, $M_2 = 0.09$ year, $\alpha_2 = 0.04$, $I_{e2} = \$0.09/\$/\text{year}$, $\theta_2 = 0.01$, then the retailer's annual total cost is $TRC_3(T) = \$2697.72$ and length of cycle time $T = 0.05$ year. Figure 5.18 shows the minimum of the annual total cost $TRC_3(T)$ at optimal cycle time (T).

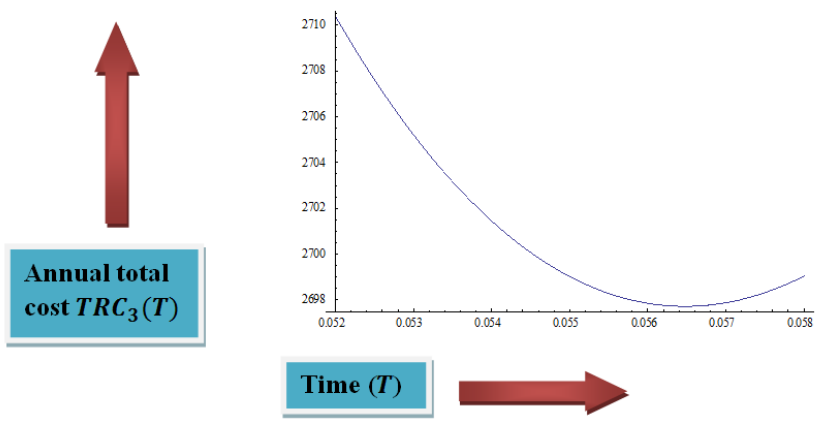

Figure 5.18: Annual total cost $TRC_3(T)$ versus time (T)

Example 4(b)

Let $A_2 = \$100$/order, $P_2 = 4500$ units/year, $D = 1000$ units/year, $I_{c2} = \$0.3/\$/\text{year}$, $s = \$90$/unit, $c = \$30$/unit, $N_2 = 0.2$ year, $\alpha_2 = 0.01$, $h = \$10$/unit/year, $I_{e2} = \$0.1/\$/\text{year}$, $M_2 = 0.3$ year, $\theta_2 = 0.01$, then the retailer's annual total cost is $TRC_4(T) = \$1557.56$ and length of cycle time $T = 0.08$ year. Figure 5.19 shows the minimum of the annual total cost $TRC_4(T)$ at optimal cycle time (T).

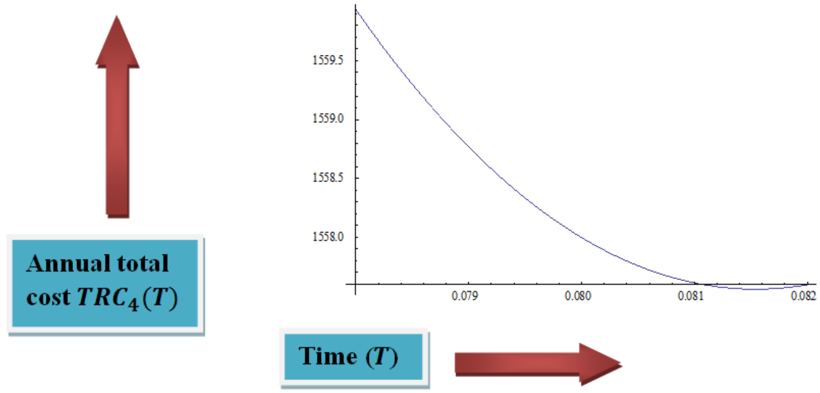

Figure 5.19: Annual total cost $TRC_4(T)$ versus time (T)

Example 5(b)

Let $A_2 = \$100$/order, $I_{c2} = \$0.3/\$/\text{year}$, $P_2 = 3500$ units/year, $M_2 = 0.01$ year, $s = \$100$/unit, $D = 1200$ units/year, $c = \$30$/unit, $h = \$20$/unit/year, $I_{e2} = \$0.16/\$/\text{year}$, $N_2 = 0.3$ year, $\alpha_2 = 0.01$, $\theta_2 = 0.01$, then the retailer's annual total cost is $TRC_5(T) = \$3000.28$ and length of cycle time $T = 0.06$ year. Figure 5.20 shows the minimum of the annual total cost $TRC_5(T)$ at optimal cycle time (T).

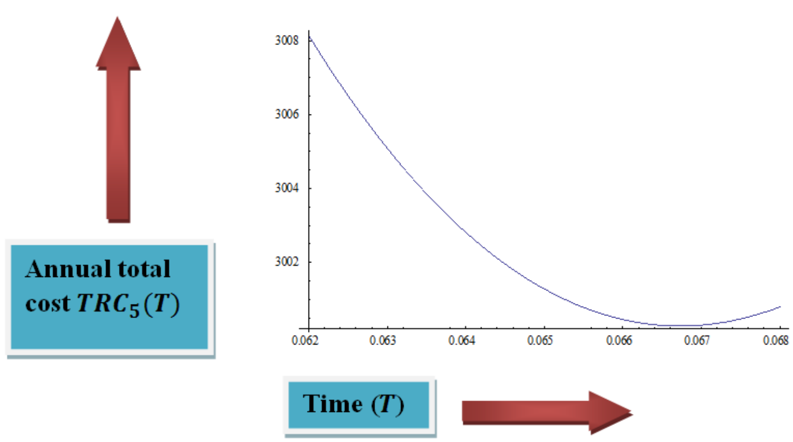

Figure 5.20: Annual total cost $TRC_5(T)$ versus time (T)

Example 6(b)

Let $D = 2000$ units/year, $A_2 = \$160$/order, $P_2 = 4000$ units/year, $s = \$100$/unit, $c = \$30$/unit, $I_{e2} = \$0.13/\$$/year, $h = \$10$/unit/year, $N_2 = 0.6$ year, $M_2 = 0.09$ year, $I_{c2} = \$0.2/\$$/year, $\alpha_2 = 0.01$, $\theta_2 = 0.01$, then the retailer's annual total cost is $TRC_6(T) = \$3487.29$ and length of cycle time $T = 0.09$ year. Figure 5.21 shows the minimum of the annual total cost $TRC_6(T)$ at optimal cycle time (T).

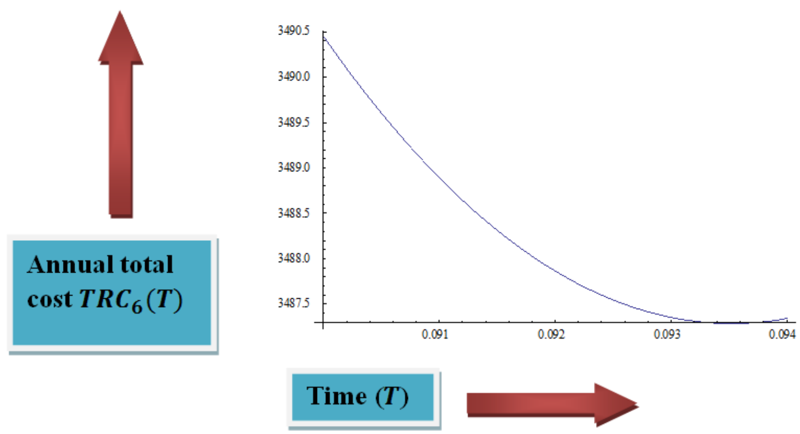

Figure 5.21: Annual total cost $TRC_6(T)$ versus time (T)

Example 7(b)

Let $s = \$90$/unit, $A_2 = \$120$/order, $D = 2000$ units/year , $c = \$30$/unit, $I_{c2} = \$0.2/\$$/year, $P_2 = 4500$ units/year, $I_{e2} = \$0.15/\$$/year, $N_2 = 0.5$ year, $\alpha_2 = 0.01$, $h = \$10$/unit/year, $M_2 = 0.2$ year, $\theta_2 = 0.01$, then the retailer's annual total cost is $TRC_7(T) = \$3168.56$ and length of cycle time $T = 0.07$ year. Figure 5.22 shows the minimum of the annual total cost $TRC_7(T)$ at optimal cycle time (T).

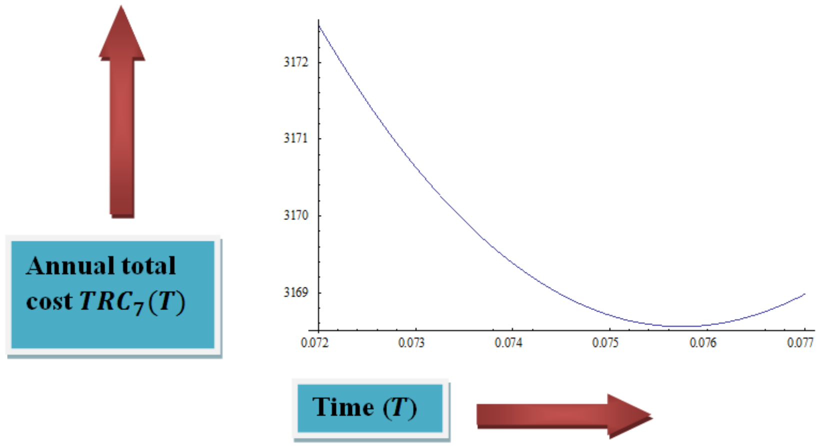

Figure 5.22: Annual total cost $TRC_7(T)$ versus time (T)

Sensitivity Analysis

This section provides sensitivity analysis for the key parameters of this model.

See Table 5.2, Table 5.3, Table 5.4, Table 5.5, Table 5.6, Table 5.7, and Table 5.8 for sensitivity analysis of key parameters for each seven cases.

Table 5.2: Sensitivity analysis for the case $M_2 \le t_{M2} \le T$

Parameters	Changes(in %)	$TRC_1(T)$	Parameters	Changes(in %)	$TRC_1(T)$
	-50%	-80.24		-50%	8.85
	-25%	$-$		-25%	4.53
A_2	$+25\%$	12.18	s	$+25\%$	$-$
	$+50\%$	23.28		$+50\%$	$-$

Parameters	Changes(in %)	$TRC_1(T)$	Parameters	Changes(in %)	$TRC_1(T)$
	-50%	-150.47		-50%	$-$
	-25%	-69.05		-25%	$-$
c	$+25\%$	$-$	h	$+25\%$	-48.81
	$+50\%$	$-$		$+50\%$	-104.04

'$-$' refers to infeasible solution.

- Retailer's annual total cost $TRC_1(T)$ raises while ordering cost A_2 increases.
- As selling-price s increases, Retailer's annual total cost $TRC_1(T)$ decreases gradually. If the selling price increases for 25% and 50%, then this model does not provides any feasible results.
- If the purchasing cost c raises, then retailer's annual total cost $TRC_1(T)$ enhances. If the purchasing cost grows for 25% more, then there is a non-feasible outcome. It is observed that there is a limit to increase the purchasing cost.
- As the holding cost h increases, then retailer's annual total cost $TRC_1(T)$ decreases.

Table 5.3: Sensitivity analysis for the case $M_2 \leq T \leq t_{M2}$

Parameters	Changes(in %)	$TRC_2(T)$	Parameters	Changes(in %)	$TRC_2(T)$
	-50%	-15287.6		-50%	38.83
	-25%	-36.51		-25%	20.05
A_2	$+25\%$	32.30	s	$+25\%$	$-$
	$+50\%$	$-$		$+50\%$	$-$

Parameters	Changes(in %)	$TRC_2(T)$	Parameters	Changes(in %)	$TRC_2(T)$
	-50%	-106.242		-50%	46.47
	-25%	$-$		-25%	$-$
c	$+25\%$	$-$	h	$+25\%$	$-$
	$+50\%$	66.82		$+50\%$	$-$

'$-$' refers to infeasible solution.

- While ordering cost A_2 raises, the annual total cost $TRC_2(T)$ inclined.
- An increasing value in selling-price s decreases retailer's annual total cost $TRC_2(T)$.
- An enhancing value in the purchasing cost c increases retailer's annual total cost $TRC_2(T)$.

Table 5.4: Sensitivity analysis for the case $N_2 \leq T \leq M_2$

Parameters	Changes(in %)	$TRC_3(T)$	Parameters	Changes(in %)	$TRC_3(T)$
	-50%	-82.81		-50%	$-$
	-25%	-38.74		-25%	$-$
A_2	$+25\%$	34.95	c	$+25\%$	33.40
	$+50\%$	$-$		$+50\%$	64.34
	-50%	84.53		-50%	7.97
	-25%	42.37		-25%	4.01
s	$+25\%$	-42.53	h	$+25\%$	-4.06
	$+50\%$	-85.20		$+50\%$	-8.19

'$-$' refers to infeasible solution.

- As ordering cost A_2 increases, retailer's annual total cost $TRC_3(T)$ also increases.

- Retailer's annual total cost $TRC_3(T)$ decreases as the selling price s raises. In this case, the selling-price is much sensitive rather than any other key parameters.

- An inclining purchasing cost c increases retailer's annual total cost $TRC_3(T)$.

- If holding cost h diminishes, retailer's annual total cost is also $TRC_3(T)$ raises. For this case, the positive percentage change is maximum than the negative percentage change.

Table 5.5: Sensitivity analysis for the case $0 < T \leq N_2$

Parameters	Changes(in %)	$TRC_4(T)$	Parameters	Changes(in %)	$TRC_4(T)$
	-50%	-81.27		-50%	$-$
	-25%	-37.04		-25%	-42.28
A_2	$+25\%$	32.45	c	$+25\%$	36.70
	$+50\%$	61.64		$+50\%$	69.58
	-50%	88.30		-50%	8.61
	-25%	44.15		-25%	4.35
s	$+25\%$	-44.16	h	$+25\%$	-4.43
	$+50\%$	-88.33		$+50\%$	-8.95

'$-$' refers to infeasible solution.

- The negative percentage change for ordering cost A_2 is much more than the positive percentage change. Retailer's annual total cost $TRC_4(T)$ raises when ordering cost A_2 increases.

- As selling-price s raises, retailer's annual total cost $TRC_4(T)$ depletes. The positive percentage change and negative percentage changes are almost similar for this case.

- Retailer's annual total cost $TRC_4(T)$ enhances while the purchasing cost c raises.

- The positive percentage change in holding cost h is maximum than the negative percentage change in this case. An inclining value in holding cost h depletes retailer's annual total cost $TRC_4(T)$.

Table 5.6: Sensitivity analysis for the case $T \geq t_{M2}$

Parameters	Changes(in %)	$TRC_5(T)$	Parameters	Changes(in %)	$TRC_5(T)$
A_2	-50%	-31.23	c	-50%	-286.83
	-25%	-15.41		-25%	-137.58
	$+25\%$	15.08		$+25\%$	123.66
	$+50\%$	29.88		$+50\%$	225.61
s	-50%	0.038	h	-50%	202.19
	-25%	0.019		-25%	114.70
	$+25\%$	-0.019		$+25\%$	-125.25
	$+50\%$	-0.038		$+50\%$	-255.45

'$-$' refers to infeasible solution.

- The negative percentage change for ordering cost A_2 is maximum than the positive percentage change. Retailer's annual total cost $TRC_5(T)$ enhances when ordering cost A_2 raises.

- The positive percentage change and negative percentage change for s are similar. As the selling-price raises then retailer's annual total cost $TRC_5(T)$ diminishes.

- For the parameter c, this model is very much sensitive for negative percentage change instead of positive percentage change. While the purchasing cost c raises, retailer's annual total cost

$TRC_5(T)$ is also increases.

- In this case, the positive percentage change is greater than the negative percentage change for the parameter h. An inclining value in holding cost h depletes retailer's annual total cost $TRC_5(T)$.

Table 5.7: Sensitivity analysis for the case $M_2 \leq T \leq t_{M2}$

Parameters	Changes(in %)	$TRC_6(T)$	Parameters	Changes(in %)	$TRC_6(T)$
	-50%	-23.47		-50%	–
	-25%	–		-25%	–
A_2	$+25\%$	10.59	c	$+5\%$	–
	$+50\%$	–		$+50\%$	-37.71
	-50%	1.23		-50%	10.03
	-25%	0.62		-25%	5.09
s	$+25\%$	-0.62	h	$+25\%$	-5.27
	$+50\%$	-1.25		$+50\%$	-10.75

'–' refers to infeasible solution.

- When the ordering cost A_2 raises, retailer's annual total cost $TRC_6(T)$ is also increases.

- The negative percentage change is lower than positive percentage change for the parameter selling price s. As the selling-price s raises, retailer's annual total cost $TRC_6(T)$ diminishes.

- The positive percentage change is maximum than the negative percentage change for the parameter holding cost h. An raising value in this parameter diminishes retailer's annual total cost $TRC_6(T)$.

Table 5.8: Sensitivity analysis for the case $0 < T \leq M_2$

Parameters	Changes(in %)	$TRC_7(T)$	Parameters	Changes(in %)	$TRC_7(T)$
A_2	-50%	-30.72	c	-50%	$-$
	-25%	-14.00		-25%	-16.10
	$+25\%$	12.26		$+25\%$	13.95
	$+50\%$	23.30		$+50\%$	26.45
s	-50%	-2.44	h	-50%	3.28
	-25%	1.22		-25%	1.65
	$+25\%$	-1.22		$+25\%$	-1.69
	$+50\%$	-2.45		$+50\%$	-3.41

'$-$' refers to infeasible solution.

- The negative percentage change is much more rather than the positive change for ordering cost A_2. Retailer's annual total cost $TRC_7(T)$ increases whenever ordering cost A_2 increases.
- If selling-price s raises, retailer's annual total cost $TRC_7(T)$ diminishes.
- While the purchasing cost c inclined, then retailer's annual total cost $TRC_7(T)$ is also raises.
- This model is very much sensitive for positive change instead of the negative change for holding cost h. An raising value in holding cost h depletes retailer's annual total cost $TRC_7(T)$.

5.4 Concluding remarks and future works

In this chapter, retailer's annual total cost is minimized by using retailer's full trade-credit policy along with partial trade-credit policy. In this chapter, several trade-credit cases is analyzed analytically. The chapter ended with various numerical findings. The model saved costs from the exiting literature. In future, some extension can be done to this model by incorporating backlogging and probabilistic demand. It will be a good research if preservation cost is added to reduce the deterioration of products.

5.5 Appendix

Appendix A3

Case A $M_2 \geq N_2$

Case A.(1) $M_2 \leq t_1$ or $M_2 \leq t_{M2} \leq T$

Now

$$\frac{dTRC_1(T)}{dT} = \frac{j_1(T)}{T^2}$$

where

$$\begin{aligned} j_1(T) &= -A_2 - \frac{h(P_2-D){t_1}^2}{2} + (h+cI_{c2})DT^2\left(\frac{1}{2} - \frac{3T^2}{4} - \frac{{t_1}^2}{2T^2} - \frac{t_1}{2} + \frac{{t_1}^2}{4}\right) + \frac{sI_{e2}D}{2}[{M_2}^2 - (1 \\ &- \alpha_2){N_2}^2] - \frac{cI_{c2}(P_2-D)({t_1}^2 - {M_2}^2)}{2} - cP_2t_1 \end{aligned}$$

To observe optimal value of T say T_1^*, one can solve the equation $j_1(T) = 0$.

From that equation, it can be concluded that

$$\frac{dj_1(T)}{dT} > 0, \quad \text{if} \quad T > 0.$$

$j_1(T)$ is an inclined function on $[0, \infty)$, so $\frac{dTRC_1(T)}{dT}$ is an increasing function on $[0, \infty)$.

Using Lemma, $TRC_1(T)$ is considered as a convex function on $[0, \infty)$.

In addition as $\lim T \to \infty$, then $j_1(T) \to \infty$.

From equation of $j_1(T)$,

$$j_1(0) = -A_2 - \frac{h(P_2-D){t_1}^2}{2} + \frac{sI_{e2}D}{2}[{M_2}^2 - (1-\alpha_2){N_2}^2] - cP_2t_1 - \frac{cI_{c2}(P_2-D)({t_1}^2-{M_2}^2)}{2}$$

Then

$$\begin{aligned} \frac{dTRC_1(T)}{dT} &< 0; \quad \text{if} \quad T \in [0, T_1^*), \\ &= 0; \quad \text{if} \quad T = T_1^*, \\ &> 0; \quad \text{if} \quad T \in (T_1^*, \infty). \end{aligned}$$

By using the intermediate value theorem, there exists a unique optimal solution which is T_1^*.

Case A.(2) $M_2 \le T \le t_{M2}$

$$\frac{dTRC_2(T)}{dT} = \frac{j_2(T)}{T^2}$$

where

$$\begin{aligned} j_2(T) &= -A_2 - \frac{h(P_2-D){t_1}^2}{2} + (h+cI_{c2})DT^2\left(\frac{1}{2} - \frac{3T^2}{4}\right) + (h+cI_{c2})DT^2\Big(\frac{({t_1}^2+{M_2}^2)}{4} \\ &- \frac{({t_1}^2+{M_2}^2)}{2T^2} - \frac{(t_1+M_2)}{2}\Big) - cP_2t_1 + \frac{sI_{e2}D}{2}[{M_2}^2 - (1-\alpha_2){N_2}^2] \end{aligned}$$

To figure out optimal value of T say T_2^*, one can calculate the equation $j_2(T) = 0$.

From that equation

$$\frac{dj_2(T)}{dT} > 0, \quad \text{if} \quad T > 0.$$

$j_2(T)$ is an increasing function over the interval $[0, \infty)$, so $\frac{dTRC_2(T)}{dT}$ is an increasing function on $[0, \infty)$.

Using Lemma, $TRC_2(T)$ is taken to be as a convex function on $[0, \infty)$.

In addition as $\lim T \to \infty$, then $j_2(T) \to \infty$.

$$j_2(0) = -A_2 - \frac{h(P_2-D){t_1}^2}{2} - cP_2t_1 + \frac{sI_{e2}D}{2}[{M_2}^2 - (1-\alpha_2){N_2}^2].$$

Then

$$
\begin{aligned}
\frac{dTRC_2(T)}{dT} &< 0; \quad \text{if } \; T \in [0, T_2^*), \\
&= 0; \quad \text{if } \; T = T_2^*, \\
&> 0; \quad \text{if } \; T \in (T_2^*, \infty).
\end{aligned}
$$

Using intermediate value theorem, an optimal solution T_2^* exists and is unique.

Case A.(3) $N_2 \leq T \leq M_2$

$$\frac{dTRC_3(T)}{dT} = \frac{j_3(T)}{T^2}$$

where

$$
\begin{aligned}
j_3(T) &= -A_2 - \frac{h(P_2 - D){t_1}^2}{2} + hDT^2\left(\frac{1}{2} - \frac{3T^2}{4} - \frac{t_1}{2} - \frac{{t_1}^2}{2T^2} + \frac{{t_1}^2}{4}\right) - cP_2t_1 + \frac{sI_{e2}D}{2}({M_2}^2 - (1 \\
&- \alpha_2){N_2}^2)
\end{aligned}
$$

To find out the optimal value of T say T_3^*, one can solve the equation $j_3(T) = 0$.

From that equation

$$\frac{dj_3(T)}{dT} > 0, \quad \text{if} \quad T > 0.$$

$j_3(T)$ is an increasing function during the interval $[0, \infty)$, hence $\frac{dTRC_3(T)}{dT}$ is an increasing function on $[0, \infty)$.

Using Lemma, $TRC_3(T)$ is said to be a convex function on $[0, \infty)$.

As $\lim T \to \infty$, so $j_3(T) \to \infty$.

$$j_3(0) = -A_2 - \frac{h(P_2 - D){t_1}^2}{2} - cP_2t_1 + \frac{sI_{e2}D}{2}({M_2}^2 - (1 - \alpha_2){N_2}^2).$$

Therefore

$$\begin{aligned}\frac{dTRC_3(T)}{dT} &< 0; \quad \text{if} \quad T \in [0, T_3^*),\\ &= 0; \quad \text{if} \quad T = T_3^*,\\ &> 0; \quad \text{if} \quad T \in (T_3^*, \infty).\end{aligned}$$

By considering the intermediate value theorem, it can be stated that there is a unique optimal solution namely T_3^*.

Case A.(4) $0 < T \leq N_2$

$$\frac{dTRC_4(T)}{dT} = \frac{j_4(T)}{T^2}$$

Now

$$j_4(T) = -A_2 - \frac{h(P_2 - D){t_1}^2}{2} + hDT^2\left(\frac{1}{2} - \frac{3T^2}{4} - \frac{t_1}{2} - \frac{{t_1}^2}{2T^2} + \frac{{t_1}^2}{4}\right) - cP_2t_1 + \frac{sI_{e2}D\alpha_2T^2}{2}$$

To calculate optimal value of T say T_4^*, one can solve the equation $j_4(T) = 0$.

From that equation

$$\frac{dj_4(T)}{dT} > 0, \quad \text{if} \quad T > 0$$

Hence, $j_4(T)$ is an inclined function over $[0, \infty)$, therefore $\frac{dTRC_4(T)}{dT}$ is an increasing function on $[0, \infty)$.

Using Lemma, $TRC_4(T)$ is taken to be a convex function on $[0, \infty)$.

In addition as $\lim T \to \infty$, then $j_4(T) \to \infty$. From equation of $j_4(T)$

$$j_4(0) = -A_2 - \frac{h(P_2 - D){t_1}^2}{2} - cP_2t_1.$$

Then

$$\begin{aligned}\frac{dTRC_4(T)}{dT} &< 0; \quad \text{if } \ T \in [0, T_4^*), \\ &= 0; \quad \text{if } \ T = T_4^*, \\ &> 0; \quad \text{if } \ T \in (T_4^*, \infty).\end{aligned}$$

With the help of the intermediate value theorem, it can be observed that there exist a unique optimal solution i.e., T_4^*.

Case B $M_2 < N_2$

Case B.(1)$T \geq t_{M2}$

$$\frac{dTRC_5(T)}{dT} = \frac{j_5(T)}{T^2}$$

where

$$\begin{aligned} j_5(T) &= -A_2 - \frac{h(P_2 - D){t_1}^2}{2} + (h + cI_{c2})DT^2\left(\frac{1}{2} - \frac{3T^2}{4} - \frac{{t_1}^2}{2T^2} - \frac{t_1}{2} + \frac{{t_1}^2}{4}\right) \\ &+ \frac{sI_{e2}D\alpha_2 - \frac{cI_{c2}(P_2-D)({t_1}^2-{M_2}^2)}{2} - cP_2t_1{M_2}^2}{2}\end{aligned}$$

To find out optimal value of T say T_5^*, one can solve the equation $j_5(T) = 0$.

From that equation, $\frac{dj_5(T)}{dT} > 0$ if $T > 0$. $j_5(T)$ is an increasing function throughout the interval $[0, \infty)$, hence $\frac{dTRC_5(T)}{dT}$ is an inclined function on $[0, \infty)$.

Using Lemma, $TRC_5(T)$ is considered as a convex function on $[0, \infty)$.

Thus, $\lim T \to \infty$, then $j_5(T) \to \infty$. From equation of $j_5(T)$,

$$j_5(0) = -A_2 - \frac{h(P_2 - D){t_1}^2}{2} - \frac{cI_{c2}(P_2 - D)({t_1}^2 - {M_2}^2)}{2} - cP_2t_1 + \frac{sI_{e2}D\alpha_2{M_2}^2}{2}.$$

Then

$$\begin{aligned} \frac{dTRC_5(T)}{dT} &< 0; \quad \text{if} \quad T \in [0, T_5^*), \\ &= 0; \quad \text{if} \quad T = T_5^*, \\ &> 0; \quad \text{if} \quad T \in (T_5^*, \infty). \end{aligned}$$

Utilizing the intermediate value theorem, there is a unique optimal solution which is T_5^*.

Case B.(2) $M_2 \leq T \leq t_{M2}$

$$\frac{dTRC_6(T)}{dT} = \frac{j_6(T)}{T^2}$$

where

$$\begin{aligned} j_6(T) &= -A_2 - \frac{h(P_2 - D){t_1}^2}{2} + (h + cI_{c2})DT^2\left(\frac{1}{2} - \frac{3T^2}{4}\right) + (h + cI_{c2})DT^2\Big(\frac{({t_1}^2 + {M_2}^2)}{4} \\ &- \frac{({t_1}^2 + {M_2}^2)}{2T^2} - \frac{(t_1 + M_2)}{2}\Big) - cP_2t_1 + \frac{sI_{e2}D\alpha_2{M_2}^2}{2} \end{aligned}$$

To observe the optimal value of T say T_6^*, one can solve the equation $j_6(T) = 0$.

From that equation, $\frac{dj_6(T)}{dT} > 0$ if $T > 0$.

$j_6(T)$ is an increasing function during the interval $[0, \infty)$, therefore $\frac{dTRC_6(T)}{dT}$ is an increasing function on $[0, \infty)$.

Using Lemma, $TRC_6(T)$ is said to be a convex function on $[0, \infty)$.

Thus, $j_6(T) \to \infty$ as $\lim T \to \infty$.

$$j_6(0) = -A_2 - \frac{h(P_2 - D){t_1}^2}{2} - cP_2t_1 + \frac{sI_{e2}D\alpha_2{M_2}^2}{2}.$$

Then

$$\begin{aligned} \frac{dTRC_6(T)}{dT} &< 0; \quad \text{if} \;\; T \in [0, T_6^*), \\ &= 0; \quad \text{if} \;\; T = T_6^*, \\ &> 0; \quad \text{if} \;\; T \in (T_6^*, \infty). \end{aligned}$$

With the help of the intermediate value theorem, there arise a unique optimal solution i.e., T_6^*.

Case B.(3) $0 < T \leq M_2$

$$\frac{dTRC_7(T)}{dT} = \frac{j_7(T)}{T^2}$$

where

$$j_7(T) = -A_2 - \frac{h(P_2 - D){t_1}^2}{2} + hDT^2\left(\frac{1}{2} - \frac{3T^2}{4} - \frac{{t_1}^2}{2T^2} - \frac{t_1}{2} + \frac{{t_1}^2}{4}\right) - cP_2t_1 + \frac{sI_{e2}D\alpha_2}{2}$$

For obtaining the optimal value of T say T_7^*, one can calculate the equation $j_7(T) = 0$.

From that equation, $\frac{dj_7(T)}{dT} > 0, \quad \text{if} \quad T > 0$.

As $j_7(T)$ is an inclined function over $[0, \infty)$, so $\frac{dTRC_7(T)}{dT}$ is an increasing function on $[0, \infty)$.

By considering the Lemma, $TRC_7(T)$ is taken to be a convex function on $[0, \infty)$.

Hence $j_7(T) \to \infty$ as $\lim T \to \infty$.

$$j_7(0) = -A_2 - \frac{h(P_2 - D){t_1}^2}{2} - cP_2t_1 + \frac{sI_{e2}D\alpha_2}{2}.$$

$$\begin{aligned} \frac{dTRC_7(T)}{dT} &< 0; \quad \text{if} \quad T \in [0, T_7^*), \\ &= 0; \quad \text{if} \quad T = T_7^*, \\ &> 0; \quad \text{if} \quad T \in (T_7^*, \infty). \end{aligned}$$

By utilizing the intermediate value theorem, it can be stated there arise a unique optimal solution which is T_7^*.