Chapter 4

Deterioration of fixed lifetime products in an inventory model*

4.1 Introduction

The effect of deterioration in any system is very important. It is necessary to maintain product's deterioration as instances fruits and vegetables, which are deteriorates over time, for any system. Ghare and Schrader (1963) investigated a model for exponentially decaying inventory model. In this direction, Philip (1974) deduced an inventory model, where three-parameter weibull distribution rate is without any shortages. Shah (1977) extended Philip's model (1974) by introducing shortages. Aggarwal and Jaggi (1995) surveyed an ordering inventory model with deteriorating items and permissible delay-in-payments. Sarker *et al.* (1997) developed an lot-size inventory model in which demand was assumed as inventory-level dependent demand. Liao *et al.* (2000) investigated an deteriorating inventory model with inflation and permissable delay-in-payments. Chang *et al.*

*A part of this work, presented in this chapter, is published in *Annals of Operations Research*, 229, 677-702, 2015.

(2001) specified deteriorating products based on trade-credit policy. Manna and Chaudhuri (2006) introduced an ordering inventory model that describes ramp type demand, time-dependent deterioration, and shortages. Several research works are done by Sana (2008), Skouri *et al.* (2011), Sett *et al.* (2012), and Sarkar and Sarkar (2013a). Sarkar and Sarkar (2013) presented an improved inventory model with probabilistic deterioration. Sarkar *et al.* (2013) observed an time-dependent inventory model for decay. They considered component cost and selling-price as a continuous rate of time. Sarkar and Sarkar (2013b) developed an inventory model with infinite replenishment rate, stock-dependent demand, time-varying deterioration rates and partial backlogging. Sarkar (2013) discussed a production-inventory model in which deterioration of products is assumed as probabilistic. Shah *et al.* (2013) derived a non-instantaneous deteriorating inventory model where generalized type deterioration is taken.

Most of the inventory models are formulated based on the consideration that the retailer must pay the purchasing amount for product instantly while the retailer receives it from the supplier. Generally, the supplier provides a fixed time period to retailer to adjust the amount. This time-period is called trade-credit-period. Interest is dully charged if the payment is not settled within the duration of trade-credit-period. In this direction, Goyal (1985) found out an EOQ model with permissible delay-in-payments. Khouja and Mehrez (1996) obtained an optimal inventory policy under different credit policies. Sarker *et al.* (2000) discussed supply chain models for perishable products with permissible delay-in-payments. Abad and Jaggi (2003) developed an inventory system for adjusting the length of credit-period for a seller when end customers demand is price sensitive. Thangam and Uthaykumar (2008) obtained an EPQ model to derive a partial trade-credit policy. Teng (2009) studied some optimal ordering techniques for a retailer who provides different trade-credits to their consumers. Sarkar *et al.* (2010) obtained a single-level trade-credit policy where retailers are allowed a period by the supplier with some discount rates. They considered different types of deterministic demand patterns in which delay-periods and different discounts rates on purchasing cost with the impact of inflation. Khanra *et al.* (2013) provided an inventory system that allows time-dependent demand, trade-credit policy, and shortages. Chen *et al.* (2014) proposed an EOQ by considering the strategy that suppliers offer retailer a fully permissible delay if retailers orders greater than or equal to a pre-assumed quantity. On the other hand, if the retailers order quantity is less than that pre-assigned quantity, then the retailer obtain a partial-payment, and enjoy a permissible delay of periods for the remaining balance.

All of the above mentioned inventory models were developed by assuming only single-level trade-credit policy. In many recent researches, it is assumed that the supplier offers the retailer a full trade-credit-period but the retailer offers customers a partial trade-credit-period. In addition, customers must pay for products during purchasing of that product. For this result, the retailer can delay the payment upto the last minute of permissible delay-period provided by the supplier. Under this assumption, the retailer can obtain more profit. Chung (2011) addressed a simplified solution procedure for the optimal replenishment decision under two-level of trade-credit policy. Ho (2011) obtained an integrated-inventory model for price, credit-linked demand, and two-level trade-credit strategy. Sarkar (2012a) extended the existing literatures by adding stock-dependent demand and imperfect production for two progressive periods. Mahata (2012) discussed an EPQ model for constant deteriorating products with retailer's partial trade-credit strategy. He wrote exponential deterioration in his model, but he used a constant deterioration. The purpose of this model is to extend his model with the time-varying deterioration for fixed lifetime products. Chen and Wang (2012) described the effects of trade-credit and limited liability in a two-level supply chain with budget constraint. They obtained that trade-credit contract can create huge importance in a supply chain with budget constraint and partly coordinate the supply chain. Sarkar (2012b) developed two-level trade-credit policy with time-varying deterioration rate and time-dependent demand. Soni (2013) obtained some replenishment techniques with deteriorating products for trade-credit policy, and limited capability. Chung and Cárdenas-Barrón (2013) developed a deteriorating inventory system for stock-dependent demand and trade-credit system. Ouyang *et al.* (2013) surveyed a comprehensive extension of optimal replenishment decisions under two-level of trade-credit policy depending on order quantity. Li *et al.* (2014) formulated different inventory models with two-level of trade-credit linked to order quantity. Sarkar *et al.* (2014) described a business-strategy that suppliers offer credit-period to motivate customers for buying more items. They considered this policy, the production of defective items and the inspection policy where order quantity and lead time are considered as decision variables. Researchers such as (Chung *et al.* (2014), Wu *et al.* (2014)) obtained some inventory models for deteriorating products with two-level trade-credit policy. See Table 4.1 for contribution of various authors.

Table 4.1: Contribution of various authors

Author(s)	**Single-level trade-credit policy**	**Two-level trade-credit policy**	**Delay -in- payments**	**Variable deterio- ration**	**Other deterio- rations**
Ghare and Schrader (1963)					$\surd$
Aggarwal and Jaggi (1995)			$\surd$		$\surd$
Sarker *et al.* (2000)	$\surd$		$\surd$		

Author(s)	**Single-level trade-credit policy**	**Two-level trade-credit policy**	**Delay -in- payments**	**Variable deterio- ration**	**Other deterio- rations**
Liao *et al.* (2000)			$\surd$		$\surd$
Abad and Jaggi (2003)		$\surd$			
Manna and Chaudhuri (2006)				$\surd$	
Sana (2008)			$\surd$	$\surd$	
Thangam and Uthaykumar (2008)		$\surd$			
Teng (2009)		$\surd$	$\surd$		
Sarkar *et al.* (2010)	$\surd$		$\surd$		
Chung (2011)		$\surd$			
Mahata (2012)		$\surd$			$\surd$
Sarkar (2012a)			$\surd$		
Sett *et al.* (2012)				$\surd$	
Sarkar (2012b)		$\surd$	$\surd$		
Sarkar (2013)				$\surd$	
Ouyang *et al.* (2013)		$\surd$			
Soni (2013)		$\surd$			$\surd$

Author(s)	**Single-level trade-credit policy**	**Two-level trade-credit policy**	**Delay -in- payments**	**Variable deterio- ration**	**Other deterio- rations**
Shah *et al.* (2013)				√	
Sarkar and Sarkar (2013a)				√	
Sarkar and Sarkar (2013b)				√	
Sarkar and Sarkar (2013)				√	
Sarkar *et al.* (2013)					√
Khanra *et al.* (2013)	√		√		
Sarkar *et al.* (2014)			√		
This chapter		√	√	√	

This chapter extends Mahata's model (2012) [Mahata, G.C. (2012). An EPQ-based inventory model for exponentially deteriorating items under retailer partial trade-credit policy in supply chain. *Expert Systems with Applications*, 39(3), 3537-3550.] by assuming time-varying deterioration. He wrote exponential deterioration in the title of his model but he considered constant deterioration. The proposed model considers time-varying deterioration for the fixed lifetime products. In addition, the model assumes that the supplier offers the retailer a full trade-credit-period but the retailer provided customers a partial trade-credit-period. Retailer's trade-credit period is not necessarily longer than customers trade-credit-period. Under these assumptions, this model formulates a cost minimization problem. The model has been solved by classical optimization technique. Some numerical examples and graphical representations are given.

4.2 Mathematical model

To develop this model, following notation are used.

T_1	cycle length in years (decision variable)
D	demand rate per year (units/year)
P_1	production rate
A_1	ordering cost per order ($/order)
h	holding cost per unit per year without interest charges ($/unit/year)
c	unit purchasing cost per item ($/item)
s	unit selling-price per item ($/item)
α_1	customers fraction of the total paying amount owed to retailer
M_1	trade-credit-period of retailer given by supplier in years (year)
N_1	customers trade-credit-period given by retailer in years (year)
I_{e1}	interest earned per year from customers to the retailer ($/year)
I_{c1}	interest charged per year by the supplier to the retailer ($/year)
$\theta_1(t)$	deterioration rate, $0 < \theta_1(t) < 1$

t_1 time at which the production stops in a cycle (year)

$TRC(T_1)$ annual total cost ($)

$\acute{T_1}$ optimal cycle time in years (year)

This chapter is formed based on the following assumptions:

1. Suppliers offer full trade-credit policy to retailers.

2. Retailers offer partial trade-credit policy to their customers.

3. Customers can make a partial payment to retailers when items are sold. Then customers must pay the rest amount within the trade-credit-period offered by retailers. For the above reason, retailers can achieve more interest from the consumer's payment with rate I_{e1}.

4. For the case $T_1 \geq M_1$, the account is adjusted at $T_1 = M_1$. Retailers must pay the interest charges on the products with rate I_{c1}.

5. When $T_1 \leq M_1$, the account is settled at $T_1 = M_1$. There is no need for retailers to pay the interest.

6. Demand rate D and production rate P_1 are constant.

7. The deterioration rate is time-dependent as $\theta_1(t) = \frac{1}{1+L-t}$, where $L > t$ and L is the maximum lifetime of products at which the total on-hand inventory deteriorates. When t increases, $\theta_1(t)$ increases and $\mathrm{Lim}_{t \to L}\theta_1(t) \to 1$. [See for instance Sarkar (2012b)]

8. Time horizon is infinite.

9. Shortages are not allowed and the lead time is negligible.

The production level starts at $t = 0$ and increases up to the time $t = t_1$. During time interval $[0, t_1]$, the inventory system is affected by production, demand, and deterioration. After the time $t = t_1$, the inventory level decreases to $t = T_1$ for deterioration and absorption rate. The graphical representation of this inventory system is given in Figure 4.1.

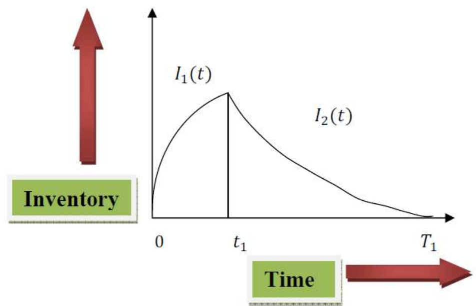

Figure 4.1: Graphical representation of the inventory system

The differential equation of the inventory system within $[0, t_1]$ is

$$\frac{dI_1(t)}{dt} + \theta_1(t)I_1(t) = P_1 - D, \quad 0 \leq t \leq t_1$$

along with the initial condition $I_1(0) = 0$.

In the time interval $[t_1, T_1]$, the inventory level is decreased by both demand and deterioration.

The differential equation of the inventory model is

$$\frac{dI_2(t)}{dt} + \theta_1(t)I_2(t) = -D, \quad t_1 \leq t \leq T_1$$

along with the boundary condition $I_2(T_1) = 0$.

Solutions of above two differential equations are

$$I_1(t) = (P_1 - D)(1 + L - t)\ln\frac{(1 + L)}{(1 + L - t)}$$

and

$$I_2(t) = D(1 + L - t)\ln\frac{(1 + L - t)}{(1 + L - T_1)}$$

By applying continuity condition at t_1, $I_1(t_1) = I_2(t_1)$, one obtain

$$t_1 = (1+L) - (1+L-T_1)^{\frac{D}{P_1}}(1+L)^{\frac{P_1-D}{P_1}}.$$

Based on the trade-credit policy, there are two cases as $M_1 \geq N_1$ and $M_1 < N_1$.

Case 1 $M_1 \geq N_1$

Annual ordering cost is $= \frac{A_1}{T_1}$

Annual stock holding cost without interest charges is

$$\begin{aligned} &= \frac{h}{T_1}\left[\int_0^{t_1} I_1(t)dt + \int_{t_1}^{T_1} I_2(t)dt\right] \\ &= \frac{h(P_1-D)}{T_1}x_1 + \frac{hP_1(1+L-t_1)^2}{2T_1}\ln(1+L-t_1) + \frac{hD}{T_1}y_1 \end{aligned}$$

[See Appendix A2 for x_1 and y_1.]

Deterioration cost is $= \frac{c(P_1t_1 - DT_1)}{T_1}$.

There are four sub-cases for interest charge as follows:

Case 1.1 $M_1 \leq t_1$ i.e., $M_1 \leq t_{M1} \leq T_1$

See Figure 4.2 for the sub-case.

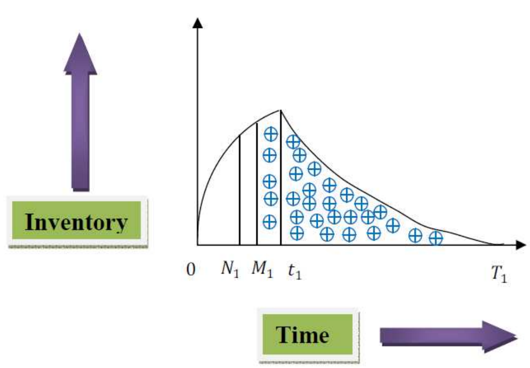

Figure 4.2: Total accumulation of the interest payable in case $T_1 \geq t_{M1}$

Annual interest payable is

$$
\begin{aligned}
&= \frac{cI_{c1}}{T_1}\left[\int_{M_1}^{t_1} I_1(t)dt + \int_{t_1}^{T_1} I_2(t)dt\right] \\
&= \frac{cI_{c1}(P_1 - D)}{T_1}x_2 - \frac{cI_{c1}P(1+L-t_1)^2}{2T_1}\ln(1+L-t_1) - \frac{cI_{c1}D}{T_1}y_1
\end{aligned}
$$

[See Appendix A2 for x_2.]

Case 1.2 $t_1 \leq M_1 \leq T_1$ i.e., $M_1 \leq T_1 \leq t_{M1}$

See Figure 4.3 for this sub-case.

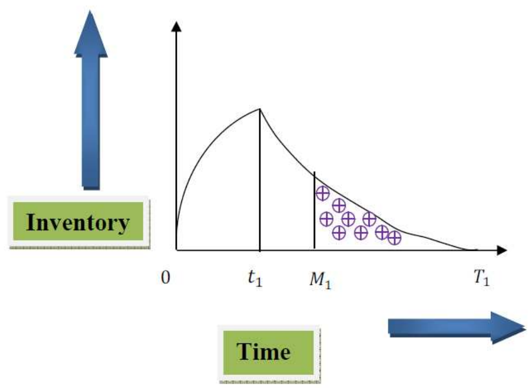

Figure 4.3: Total accumulation of interest payable when $M_1 \leq T_1 \leq t_{M1}$ and $t_1 \leq N_1 \leq M_1$

Annual interest payable is

$$
= \frac{cI_{c1}}{T_1}\left[\int_{M_1}^{T_1} I_2(t)dt\right] = \frac{DcI_{c1}}{2T_1}y_2.
$$

[See Appendix A2 for y_2.]

Case 1.3 $N_1 \leq T_1 \leq M_1$

Annual interest payable is 0.

Case 1.4 $0 < T_1 \leq N_1$

For this case, annual interest payable is 0.

There are four sub-cases for interest earned as follows:

Case 1.(i) $M_1 \leq t_1$ i.e., $M_1 \leq t_{M1} \leq T_1$

See Figure 4.4 for this sub-case.

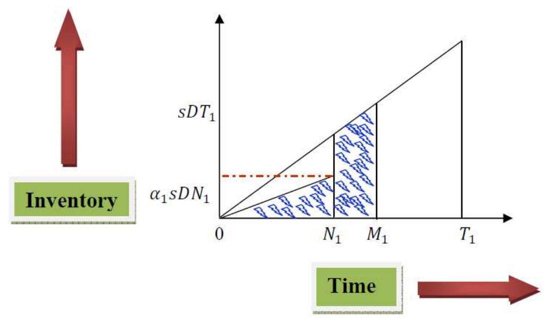

Figure 4.4: Total interest earned in case $M_1 \leq T_1$

Annual interest earned is

$$\begin{aligned} &= \frac{sI_{e1}}{T_1}\left[\frac{DN_1{}^2\alpha_1}{2} + \frac{(DN_1 + DM_1)(M_1 - N_1)}{2}\right] \\ &= \frac{sI_{e1}D}{2T_1}\left[M_1{}^2 - (1-\alpha_1)N_1{}^2\right] \end{aligned}$$

Case 1.(ii) $t_1 \leq M_1 \leq T_1$ i.e., $M_1 \leq T_1 \leq t_{M1}$

As Case 1.(i),

Annual interest received is $= \frac{sI_{e1}D}{2T_1}[M_1{}^2 - (1-\alpha_1)N_1{}^2]$.

Case 1.(iii) $N_1 \leq T_1 \leq M_1$

See Figure 4.5 for this sub-case.

Annual interest earned is

$$\begin{aligned} &= \frac{sI_{e1}}{T_1}\left[\frac{DN_1{}^2\alpha_1}{2} + \frac{(DT_1 + DN_1)(T_1 - N_1)}{2} + (M_1 - T_1)DT_1\right] \\ &= \frac{sI_{e1}D}{2T_1}\left[2M_1T_1 - (1-\alpha_1)N_1{}^2 - T_1{}^2\right] \end{aligned}$$

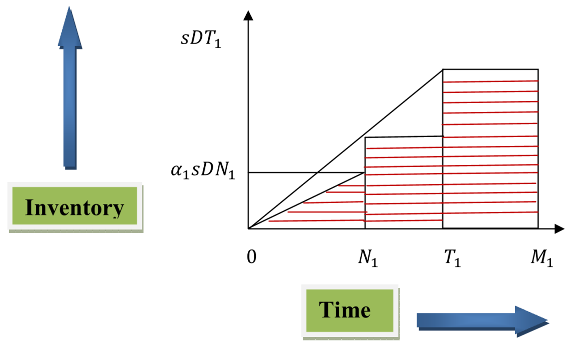

Figure 4.5: Total interest earned in case $N_1 \leq T_1 \leq M_1$

Case 1.(iv) $0 < T_1 \leq N_1$

See Figure 4.6 for this sub-case.

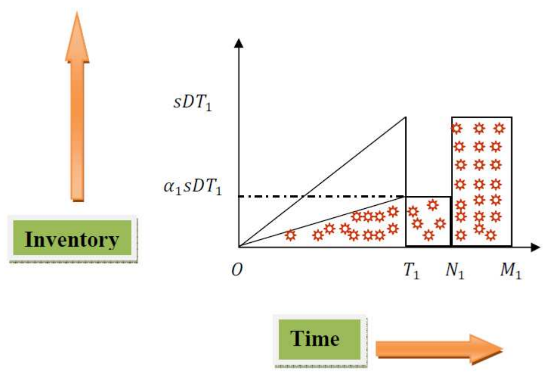

Figure 4.6: Total interest earned in case $0 < T_1 \leq N_1$

Annual interest earned is

$$\begin{aligned}
&= \frac{sI_{e1}}{T_1}\left[\frac{DT_1{}^2\alpha_1}{2} + \alpha_1 DT_1(N_1 - T_1) + (M_1 - N_1)DT_1\right] \\
&= sI_{e1}D\left[M_1 - (1-\alpha_1)N_1 - \frac{\alpha_1 T_1}{2}\right].
\end{aligned}$$

Retailer's annual total cost is

$TRC(T_1) =$ ordering charge + cost for holding+ cost for deterioration + interest payable − interest earned.

$$TRC(T_1) = \left\{ \begin{array}{c} TRC_1(T_1), \text{ if } T_1 \geq t_{M1} \\ TRC_2(T_1), \text{ if } M_1 \leq T_1 \leq t_{M1} \\ TRC_3(T_1), \text{ if } N_1 \leq T_1 \leq M_1 \\ TRC_4(T_1), \text{ if } 0 < T_1 \leq N_1 \end{array} \right\}$$

where cost expressions are

$$\begin{aligned} TRC_1(T_1) &= \frac{A_1}{T_1} + \frac{h(P_1-D)}{T_1}x_1 + (hP_1 + cI_{c1}P_1)\frac{(1+L-t_1)^2}{2T_1}\ln(1+L-t_1) + \frac{hD}{T_1}y_1 \\ &+ \frac{c(P_1t_1 - DT_1)}{T_1} + \frac{cI_{c1}(P_1-D)}{T_1}x_2\ln(1+L-t_1) - \frac{sI_{e1}D}{2T_1}[M_1{}^2 - (1 \\ &- \alpha_1)N_1{}^2] + \frac{cI_{c1}D}{T_1}y_1, \end{aligned}$$

$$\begin{aligned} TRC_2(T_1) &= \frac{A_1}{T_1} + \frac{h(P_1-D)}{T_1}x_1 + \frac{hP_1(1+L-t_1)^2}{2T_1}\ln(1+L-t_1) + \frac{hD}{T_1}y_1 + \frac{DcI_{c1}}{2T_1}y_2 \\ &- \frac{sI_{e1}D}{2T_1}[M_1{}^2 - (1-\alpha_1)N_1{}^2] + \frac{c(P_1t_1 - DT_1)}{T_1}, \end{aligned}$$

$$\begin{aligned} TRC_3(T_1) &= \frac{A_1}{T_1} + \frac{h(P_1-D)}{T_1}x_1 + \frac{hP_1(1+L-t_1)^2}{2T_1}\ln(1+L-t_1) + \frac{hD}{T_1}y_1 \\ &+ \frac{c(P_1t_1 - DT_1)}{T_1} - \frac{sI_{e1}D}{2T_1}[2M_1T_1 - (1-\alpha_1)N_1{}^2 - T_1{}^2], \end{aligned}$$

and

$$\begin{aligned} TRC_4(T_1) &= \frac{A_1}{T_1} + \frac{h(P_1-D)}{T_1}x_1 + \frac{hP_1(1+L-t_1)^2}{2T_1}\ln(1+L-t_1) + \frac{hD}{T_1}y_1 \\ &+ \frac{c(P_1t_1 - DT_1)}{T_1} - sI_{e1}D\left[M_1 - (1-\alpha_1)N_1 - \frac{\alpha_1T_1}{2}\right]. \end{aligned}$$

As, $t_{M1} = (1+L) - (1+L-M_1)^{\frac{P_1}{D}}.(1+L)^{\frac{D-P_1}{D}}$ and from the continuity condition at t_{M1}, $TRC_1(t_{M1}) = TRC_2(t_{M1})$, $TRC_2(t_{M1}) = TRC_3(t_{M1})$, $TRC_3(t_{M1}) = TRC_4(t_{M1})$. $TRC(T_1)$, $TRC_1(T_1)$, $TRC_2(T_1)$, $TRC_3(T_1)$, and $TRC_4(T_1)$ are well defined for $T_1 > 0$.

Case 2. $M_1 < N_1$

Annual ordering cost is $= \frac{A_1}{T_1}$

Annual stock holding cost excluding interest charges is

$$\begin{aligned} &= \frac{h}{T_1}\left[\int_0^{t_1} I_1(t)dt + \int_{t_1}^{T_1} I_2(t)dt\right] \\ &= \frac{h(P_1-D)}{T_1}x_1 + \frac{hP_1(1+L-t_1)^2}{2T_1}\ln(1+L-t_1) + \frac{hD}{T_1}y_1 \end{aligned}$$

Deterioration cost is $= \frac{c(P_1 t_1 - DT_1)}{T_1}$.

There are three sub-cases for interest charges as follows:

Case 2.1 $t_{M1} \le T_1$

See Figure 4.2 for this sub-case.

Annual interest payable is

$$\begin{aligned} &= \frac{cI_{c1}}{T_1}\left[\int_{M_1}^{t_1} I_1(t)dt + \int_{t_1}^{T_1} I_2(t)dt\right] \\ &= \frac{h(P_1-D)}{T_1}x_1 + \frac{hP_1(1+L-t_1)^2}{2T_1}\ln(1+L-t_1) + \frac{hD}{T_1}y_1 \end{aligned}$$

Case 2.2 $M_1 \le T_1 \le t_{M1}$

See Figure 4.3 for this sub-case.

Annual interest payable is $= \frac{cI_{c1}}{T_1}\left[\int_{M_1}^{T_1} I_2(t)dt\right] = \frac{DcI_{c1}}{2T_1}y_2$.

Case 2.3 $0 < T_1 \le M_1$

For this sub-case, the annual interest payable is 0.

There are three sub-cases for interest earned as follows:

Case 2.(i) $t_{M1} \le T_1$

See Figure 4.7 for this sub-case.

Annual interest earned is $= \frac{sI_{e1}DM_1{}^2\alpha_1}{2T_1}$.

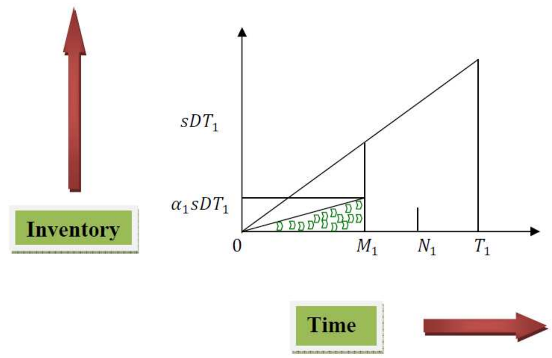

Figure 4.7: Total interest earned in case $t_{M1} \leq T_1$

Case 2.(ii) $M_1 \leq T_1 \leq t_{M1}$

As Case 2.(i), we obtain

Annual interest earned is $= \frac{sI_{e1}DM_1{}^2\alpha_1}{2T_1}$.

Case 2.(iii) $0 < T_1 \leq M_1$

See Figure 4.8 for this case.

Annual interest earned is $= \frac{sI_{e1}}{T_1}\left[\frac{DT_1{}^2\alpha_1}{2} + \alpha_1 DT_1(M_1 - T_1)\right] = sI_{e1}D\alpha_1\left[M_1 - \frac{T_1}{2}\right]$.

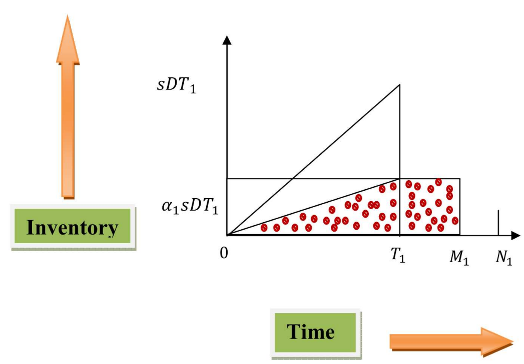

Figure 4.8: Total interest earned in case $0 < T_1 \leq M_1$

Retailer's annual total cost is

$TRC(T_1)$ = cost of ordering + holding charge + decay or deterioration cost + interest payable − interest earned.

$$TRC(T_1) = \left\{ \begin{array}{c} TRC_5(T_1);\ \text{if } T_1 \geq t_{M1} \\ \\ TRC_6(T_1);\ \text{if } M_1 \leq T_1 \leq t_{M1} \\ \\ TRC_7(T_1);\ \text{if } 0 < T_1 \leq M_1 \end{array} \right\}$$

where cost expressions are given by

$$\begin{aligned} TRC_5(T_1) &= \frac{A_1}{T_1} + \frac{(h+cI_{c1})P(1+L-t_1)^2}{2T_1}\ln(1+L-t_1) - \frac{sI_{e1}DM_1{}^2\alpha_1}{2T_1} + \frac{cP_1t_1}{T_1} \\ &- cD + \frac{h(P_1-D)}{T_1}x_1 + \frac{(h+cI_{c1})D}{T_1}y_1 + \frac{cI_{c1}(P_1-D)}{T_1}x_2, \end{aligned}$$

$$\begin{aligned} TRC_6(T_1) &= \frac{A_1}{T_1} + \frac{h(P_1-D)}{T_1}x_1 + \frac{hP_1(1+L-t_1)^2}{2T_1}\ln(1+L-t_1) + \frac{hD}{T_1}y_1 \\ &+ \frac{DcI_{c1}}{2T_1}y_2 - \frac{sI_{e1}DM_1{}^2\alpha_1}{2T_1} + \frac{c(P_1t_1-DT_1)}{T_1}, \end{aligned}$$

and

$$\begin{aligned} TRC_7(T_1) &= \frac{A_1}{T_1} + \frac{h(P_1-D)}{T_1}x_1 + \frac{hP_1(1+L-t_1)^2}{2T_1}\ln(1+L-t_1) + \frac{hD}{T_1}y_1 \\ &- sI_{e1}D\alpha_1\left[M_1 - \frac{T_1}{2}\right] + \frac{c(P_1t_1-DT_1)}{T_1}. \end{aligned}$$

As, $t_{M1} = (1+L)-(1+L-M_1)^{\frac{P_1}{D}}(1+L)^{\frac{D-P}{D}}$ and from the continuity condition at t_{M1}, $TRC_5(t_{M1}) = TRC_6(t_{M1})$ and $TRC_6(t_{M1}) = TRC_7(t_{M1})$. $TRC(T_1)$, $TRC_5(T_1)$, $TRC_6(T_1)$, and $TRC_7(T_1)$ are well defined for $T_1 > 0$.

Lemma

For a continuous function $g(t)$ on (a,b) and $\frac{dg(t)}{dt} = 0$, $g(t)$ will be convex.

Proof

For the proof of this lemma, two cases are described.

They are given as

Case 1 $M_1 \geq N_1$

Case 2 $M_1 < N_1$

In case $M_1 \geq N_1$, there are four subcases which are as follows:

Case 1.(a) $T_1 \geq t_{M1}$

Case 1.(b) $M_1 \leq T_1 \leq t_{M1}$

Case 1.(c) $N_1 \leq T_1 \leq M_1$

Case 1.(d) $0 < T_1 \leq N_1$

On the other hand, for the case $M_1 < N_1$, there are three subcases, which are as follows:

Case 2.(a) $T_1 \geq t_{M1}$

Case 2.(b) $M_1 \leq T_1 \leq t_{M1}$

Case 2.(c) $0 < T_1 \leq M_1$

The proof of this lemma, i.e., the convexity of all cost functions $TRC_1(T_1)$, $TRC_2(T_1)$, $TRC_3(T_1)$, $TRC_4(T_1)$, $TRC_5(T_1)$, $TRC_6(T_1)$, and $TRC_7(T_1)$ are illustrated in Appendix B2.

4.3 Numerical examples

Some numerical examples are given to illustrate this model. To obtain the retailer's annual total cost, this chapter considers numerical data from Mahata (2012).

Example 1(a)

Let A_1 = \$200/order, P_1 = 3000 units/year, D = 2500 units/year, h = \$15/unit/year, I_{c1} =0.15/year, I_{e1} = \$0.1/year, M_1 = 0.1 year, N_1 = 0.05 year, α_1 = 0.05, s = \$75/unit, c = \$50/unit, L = 0.6 year, then the optimal solution is $TRC_1(T_1)$ = \$879.571 and cycle time T_1 = 0.2 year. Figure 4.9 indicates the minimum of the annual total cost $TRC_1(T_1)$ at the optimal cycle time (T_1).

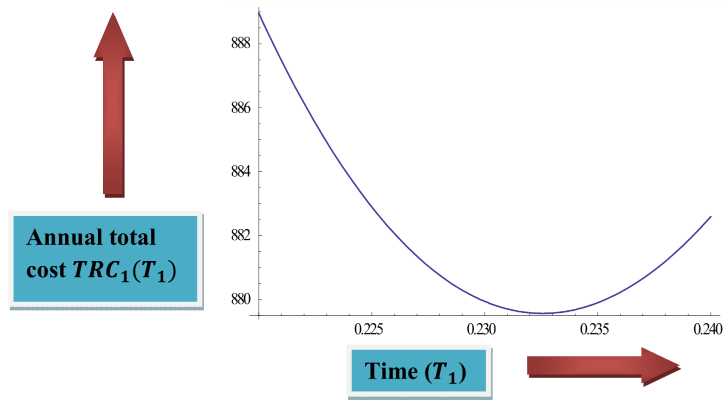

Figure 4.9: Annual total cost $TRC_1(T_1)$ versus time (T_1)

Example 2(a)

Let $A_1 = \$150$/order, $P_1 = 3000$ units/year, $D = 2500$ units/year, $h = \$15$/unit/year, $I_{c1} = \$0.15$/year, $I_{e1} = \$0.1$/year, $c = \$50$/unit, $M_1 = 0.13$ year, $N_1 = 0.05$ year, $s = \$75$/unit, $\alpha_1 = 0.05$, $L = 1$ year, then the optimal solution is $TRC_2(T) = \$1201.45$ and cycle time $T_1 = 0.13$ year. Figure 4.10 indicates the minimum of the annual total cost $TRC_2(T_1)$ at the optimal cycle time (T_1).

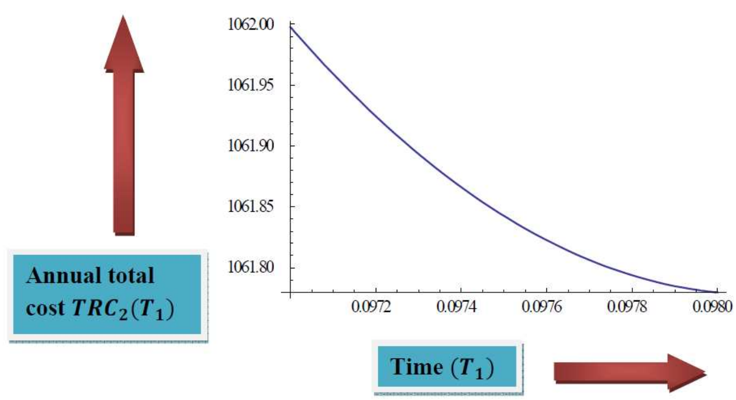

Figure 4.10: Annual total cost $TRC_2(T_1)$ versus time (T_1)

Example 3(a)

Let A_1 = \$100/order, P_1 = 4000 units/year, D = 2500 units/year, c = \$50/unit, I_{c1} = \$0.15/year, I_{e1} = \$0.1/year, M_1 = 0.15 year, N_1 = 0.05 year, s = \$75/unit, $\alpha_1 = 0.05$, $L = 1$ year, h = \$15/unit/year, then the optimal solution is $TRC_3(T_1)$ = \$906.81 and cycle time $T_1 = 0.06$ year. Figure 4.11 indicates the minimum of the annual total cost $TRC_3(T_1)$ at the optimal cycle time (T_1).

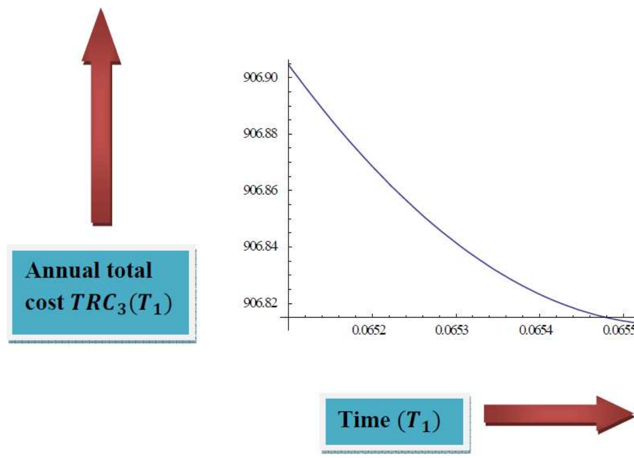

Figure 4.11: Annual total cost $TRC_3(T_1)$ versus time (T_1)

Example 4(a)

Let A_1 = \$50/order, P_1 = 4000 units/year, D = 2500 units/year, s = \$100/unit, I_{c1} = \$0.15/year, I_{e1} = \$0.1/year, h = \$15/unit/year, M_1 = 0.14 year, N_1 = 0.08 year, $\alpha_1 = 0.05$, c = \$50/unit, $L = 1$ year, then the optimal solution is $TRC_4(T_1)$ = \$374.73 and cycle time $T_1 = 0.05$ year. Figure 4.12 indicates the minimum of the annual total cost $TRC_4(T_1)$ at the optimal cycle time (T_1).

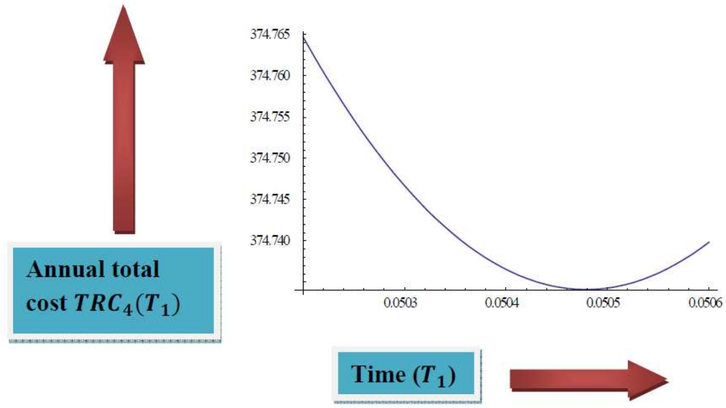

Figure 4.12: Annual total cost $TRC_4(T_1)$ versus time (T_1)

Example 5(a)

Let $A_1 = \$150$/order, $P_1 = 3000$ units/year, $D = 2500$ units/year, $c = \$50$/unit, $I_{c1} = \$0.15$/year, $I_{e1} = \$0.1$/year, $M_1 = 0.1$ year, $h = \$15$/unit/year, $N_1 = 0.5$ year, $\alpha_1 = 0.05$, $s = \$75$/unit, $L = 0.6$ year, then the optimal solution is $TRC_5(T_1) = \$951.37$ and cycle time $T_1 = 0.2$ year. Figure 4.13 indicates the minimum of the annual total cost $TRC_5(T_1)$ at the optimal cycle time (T_1).

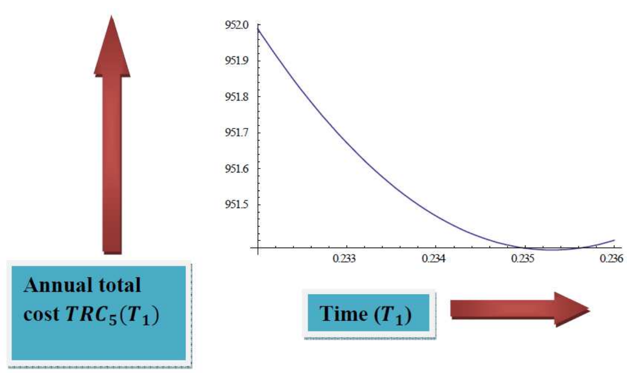

Figure 4.13: Annual total cost $TRC_5(T_1)$ versus time (T_1)

Example 6(a)

Let A_1 = \$100/order, P_1 = 3500 units/year, h = \$15/unit/year, D = 2500 units/year, I_{c1} = \$0.16/year, I_{e1} = \$0.15/year, s = \$75/unit, $M_1 = 0.13$ year, $N_1 = 0.2$ year, $\alpha_1 = 0.05$, c = \$50/unit, $L = 1$ year, then the optimal solution is $TRC_6(T_1)$ = \$2413.96 and cycle time $T_1 = 0.1$ year. Figure 4.14 indicates the minimum of the annual total cost $TRC_6(T_1)$ at the optimal cycle time (T_1).

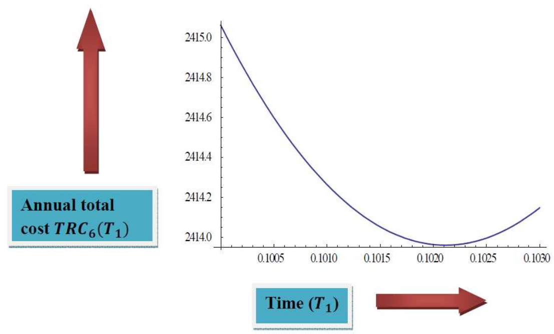

Figure 4.14: Annual total cost $TRC_6(T_1)$ versus time (T_1)

Example 7(a)

Let A_1 = \$50/order, P_1 = 4000 units/year, s = \$100/unit, I_{c1} = \$0.24/year, c = \$50/unit, I_{e1} = \$0.15/year, D = 2500 units/year, $M_1 = 0.8$ year, $N_1 = 0.9$ year, $\alpha_1 = 0.02$, h = \$15/unit/year, $L = 1$ year, then the optimal solution is $TRC_7(T_1)$ = \$1240.07 and cycle time $T_1 = 0.05$ year. Figure 4.15 indicates the minimum of the annual total cost $TRC_7(T_1)$ at the optimal cycle time (T_1).

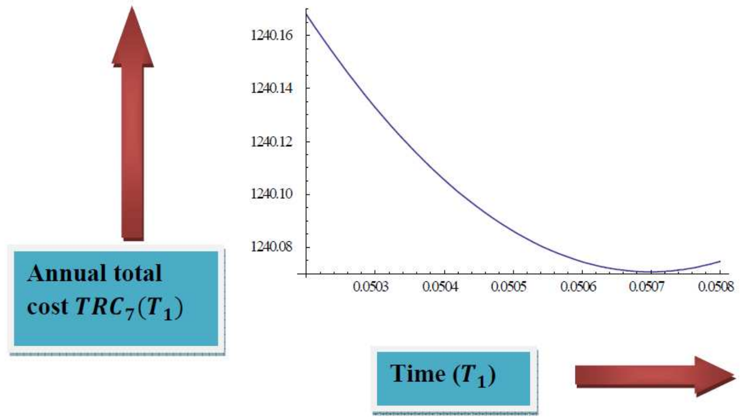

Figure 4.15: Annual total cost $TRC_7(T_1)$ versus time (T_1)

Case Study

This model described time dependent deterioration for fixed lifetime products. Two types of trade credit policy are included in this model. Suppliers offer full trade credit policy to retailers. In spite of that, retailers offer partial trade credit policy to their customers. In this model, main factor is time dependent deterioration for fixed lifetime products. Fruits and vegetables are examples of such products. These products are deteriorates over time. Each Fruits and vegetables has their own shelf life. In this model, those shelf lives are namely taken to be as fixed lifetime of products. For example, fruit like orange can be stored maximum 5 days. After that, it is no longer to eat. On the other hand, if we consider some vegetables such as corn and mushrooms, those will last for 1-2 days.

Example 1(b)

Let $A_1 = \$300$/order, $P_1 = 3100$ units/year, $D = 1000$ units/year, $h = \$10$/unit/year, I_{c1} =0.09/year, $I_{e1} = \$0.04$/year, $M_1 = 0.3$ year, $N_1 = 0.3$ year, $\alpha_1 = 0.01$, $s = \$400$/unit, $c = \$20$/unit, $L = 0.3$ year, then the optimal solution is $TRC_1(T_1) = \$10623.7$ and cycle time $T_1 = 0.7$ year. Figure 4.16 indicates the minimum of the annual total cost $TRC_1(T_1)$ at the optimal cycle time (T_1).

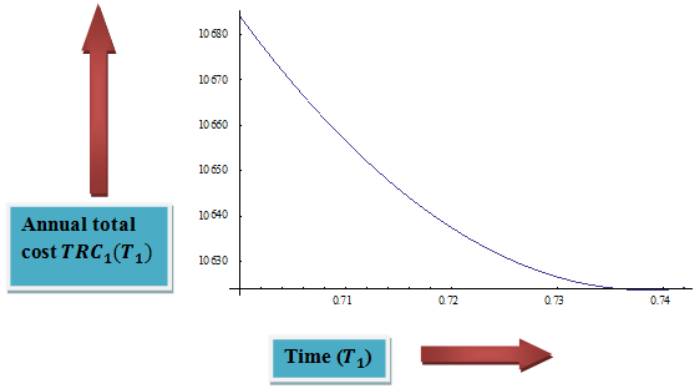

Figure 4.16: Annual total cost $TRC_1(T_1)$ versus time (T_1)

Example 2(b)

Let A_1 = \$200/order, P_1 = 4200 units/year, D = 1000 units/year, h = \$10/unit/year, I_{c1} = \$0.15/year, I_{e1} = \$0.3/year, c = \$30/unit, M_1 = 0.1 year, N_1 = 0.01 year, s = \$80/unit, $\alpha_1 = 0.01$, L = 0.3 year, then the optimal solution is $TRC_2(T)$ = \$2903.72 and cycle time T_1 = 0.1 year. Figure 4.17 indicates the minimum of the annual total cost $TRC_2(T_1)$ at the optimal cycle time (T_1).

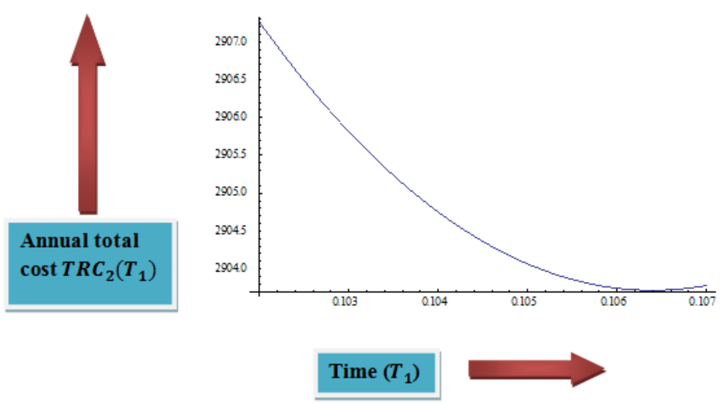

Figure 4.17: Annual total cost $TRC_2(T_1)$ versus time (T_1)

Example 3(b)

Let $A_1 = \$120$/order, $P_1 = 4500$ units/year, $D = 2000$ units/year, $c = \$40$/unit, $I_{c1} = \$0.2$/year, $I_{e1} = \$0.09$/year, $M_1 = 0.13$ year, $N_1 = 0.01$ year, $s = \$80$/unit, $\alpha_1 = 0.01$, $L = 0.3$ year, $h = \$14$/unit/year, then the optimal solution is $TRC_3(T_1) = \$2090.15$ and cycle time $T_1 = 0.06$ year. Figure 4.18 indicates the minimum of the annual total cost $TRC_3(T_1)$ at the optimal cycle time (T_1).

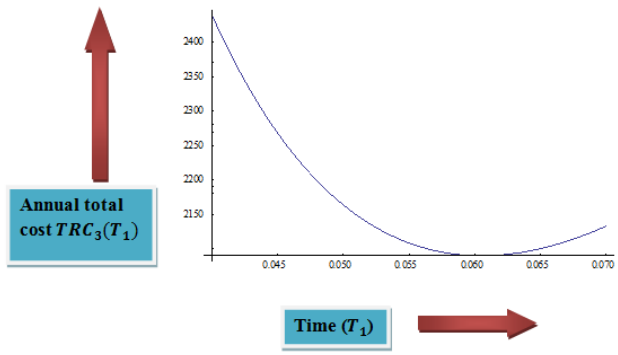

Figure 4.18: Annual total cost $TRC_3(T_1)$ versus time (T_1)

Example 4(b)

Let $A_1 = \$40$/order, $P_1 = 4500$ units/year, $D = 900$ units/year, $s = \$90$/unit, $I_{c1} = \$0.3$/year, $I_{e1} = \$0.2$/year, $h = \$10$/unit/year, $M_1 = 0.15$ year, $N_1 = 0.1$ year, $\alpha_1 = 0.01$, $c = \$40$/unit, $L = 0.3$ year, then the optimal solution is $TRC_4(T_1) = \$725.92$ and cycle time $T_1 = 0.05$ year. Figure 4.19 indicates the minimum of the annual total cost $TRC_4(T_1)$ at the optimal cycle time (T_1).

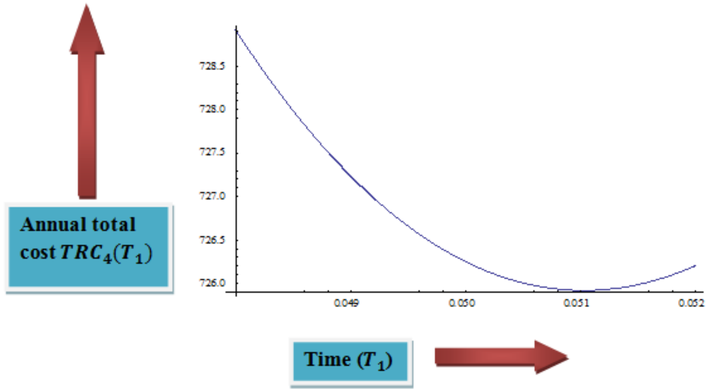

Figure 4.19: Annual total cost $TRC_4(T_1)$ versus time (T_1)

Example 5(b)

Let $A_1 = \$90$/order, $P_1 = 4500$ units/year, $D = 1200$ units/year, $c = \$40$/unit, $I_{c1} = \$0.2$/year, $I_{e1} = \$0.16$/year, $M_1 = 0.16$ year, $h = \$10$/unit/year, $N_1 = 0.2$ year, $\alpha_1 = 0.01$, $s = \$80$/unit, $L = 0.4$ year, then the optimal solution is $TRC_5(T_1) = \$11924.2$ and cycle time $T_1 = 0.4$ year. Figure 4.20 indicates the minimum of the annual total cost $TRC_5(T_1)$ at the optimal cycle time (T_1).

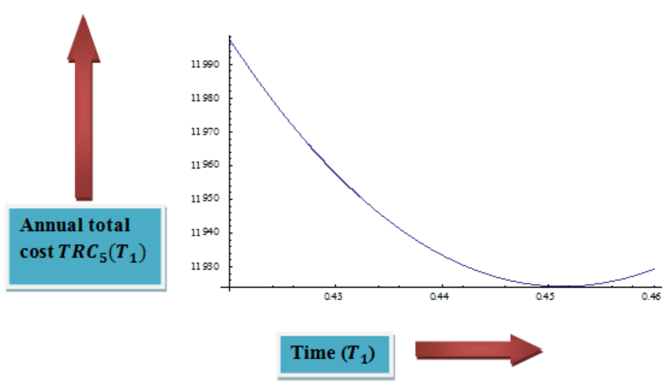

Figure 4.20: Annual total cost $TRC_5(T_1)$ versus time (T_1)

Example 6(b)

Let $A_1 = \$200$/order, $P_1 = 3000$ units/year, $h = \$10$/unit/year, $D = 1500$ units/year, $I_{c1} = \$0.2$/year, $I_{e1} = \$0.12$/year, $s = \$100$/unit, $M_1 = 0.08$ year, $N_1 = 0.3$ year, $\alpha_1 = 0.02$, $c = \$40$/unit, $L = 0.4$ year, then the optimal solution is $TRC_6(T_1) = \$3492.73$ and cycle time $T_1 = 0.1$ year. Figure 4.21 indicates the minimum of the annual total cost $TRC_6(T_1)$ at the optimal cycle time (T_1).

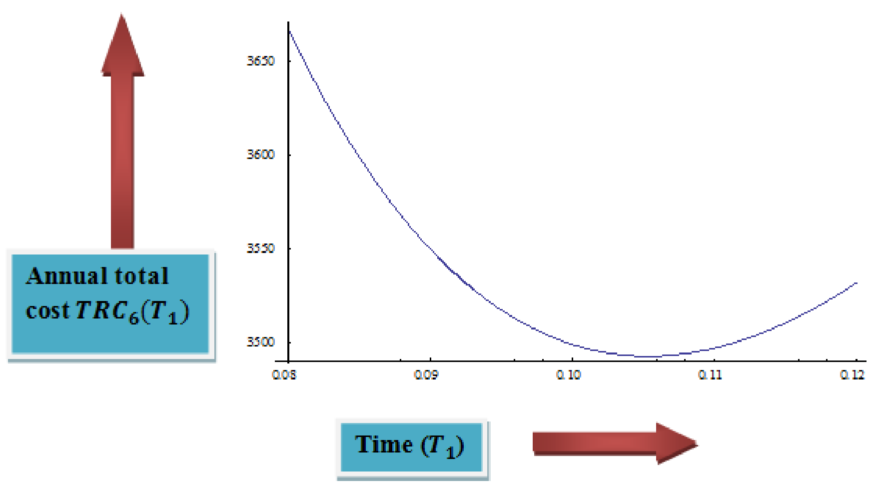

Figure 4.21: Annual total cost $TRC_6(T_1)$ versus time (T_1)

Example 7(b)

Let $A_1 = \$100$/order, $P_1 = 4500$ units/year, $s = \$90$/unit, $I_{c1} = \$0.3$/year, $c = \$30$/unit, $I_{e1} = \$0.2$/year, $D = 2000$ units/year, $M_1 = 0.2$ year, $N_1 = 0.4$ year, $\alpha_1 = 0.01$, $h = \$10$/unit/year, $L = 0.4$ year, then the optimal solution is $TRC_7(T_1) = \$2610.55$ and cycle time $T_1 = 0.07$ year. Figure 4.22 indicates the minimum of the annual total cost $TRC_7(T_1)$ at the optimal cycle time (T_1).

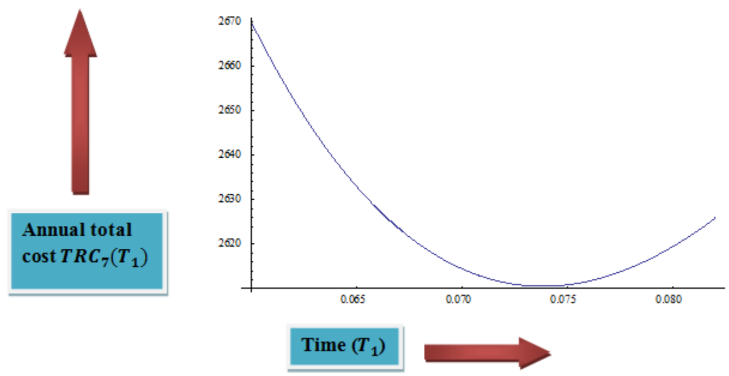

Figure 4.22: Annual total cost $TRC_7(T_1)$ versus time (T_1)

4.4 Concluding remarks and future works

This chapter mainly extended the research works of Mahata (2012). In Mahata's (2012) model, exponential deterioration was written, but he used constant deterioration. In this chapter, time varying deterioration is added for fixed lifetime items. For future research, by incorporating of some more realistic assumptions, such as shortages, and controllable lead time would be more perfect to extend this model.

4.5 Appendices

Appendix A2

$$\begin{aligned} x_1 &= \left(\frac{\ln(1+L)}{2}+\frac{1}{4}\right)(2+2L-t_1)t_1-\ln(1+L)\frac{(1+L)^2}{2}, \\ y_1 &= \left(\frac{\ln(1+L-T_1)}{2}+\frac{1}{4}\right)(2+2L-t_1-T_1)(t_1-T_1)-\ln(1+L-T_1)\frac{(1+L-T_1)^2}{2}, \end{aligned}$$

$$
\begin{aligned}
x_2 &= \left(\frac{\ln(1+L)}{2} - \frac{1}{4}\right)(2+2L-t_1-M_1)(t_1-M_1) - \ln(1+L-M_1)\frac{(1+L-M_1)^2}{2}, \\
y_2 &= [-(T_1-M_1)(2+2L-M_1-T_1)\left(\frac{1}{2} + \ln(1+L-T_1)\right) + \ln(1+L-M_1)(1+L-M_1)^2 \\
&- \ln(1+L-T_1)(1+L-T_1)^2].
\end{aligned}
$$

Appendix B2

Case 1 $M_1 \geq N_1$

Case 1.1 $M_1 \leq t_1$ or $M_1 \leq t_{M1} \leq T_1$

$$
\frac{dTRC_1(T_1)}{dT_1} = \frac{g_1(T_1)}{{T_1}^2}
$$

where

$$
\begin{aligned}
g_1(T_1) &= (h+cI_{c1})DT_1\Big[(2+2L-t_1-T_1)\left(\frac{(t_1-T_1)}{2(1+L-T_1)} - \frac{\ln(1+L-T_1)}{2}\right) - (1+L-T_1) \\
&- \ln(1+L-T_1)(t_1+1+L-2T_1)\Big] - A_1 - h(P_1-D)\Big[\Big(\frac{1}{4} + \frac{\ln(1+L)}{2}\Big)(2+2L-t_1)t_1 \\
&- \ln(1+L)\frac{(1+L)^2}{2}\Big] - \left(\frac{hP_1}{2} + cI_{c1}P_1\right)(1+L-t_1)^2\frac{\ln(1+L-t_1)}{2} - (h+cI_{c1})D\Big[\\
&- (1+L-T_1)^2\frac{\ln(1+L-T_1)}{2} + \left(\frac{\ln(1+L-T_1)}{2} + \frac{1}{4}\right)(2+2L-t_1-T_1)(t_1-T_1)\Big] \\
&- cP_1t_1 + \frac{sI_{e1}D}{2}({M_1}^2 - (1-\alpha_1){N_1}^2) - \Big[\Big(\frac{\ln(1+L)}{2} - \frac{1}{4}\Big)(2+2L-t_1-M_1)(t_1-M_1) \\
&- \ln(1+L-M_1)\frac{(1+L-M_1)^2}{2}\Big]cI_{c1}(P_1-D) - cI_{c1}D\Big[\Big(\frac{1}{4} + \frac{\ln(1+L-T_1)}{2}\Big)(2+2L \\
&- t_1-T_1)(T_1-t_1) + \ln(1+L-T_1)\frac{(1+L-T_1)^2}{2}\Big]
\end{aligned}
$$

For obtaining optimal value of T_1 say $\acute{T_1}$, one can solve the equation $g_1(T_1) = 0$.

Now $\frac{dg_1(T_1)}{dT_1} > 0$ if $T_1 > 0$. As $g_1(T_1)$ is an increasing function on $[0, \infty)$, then $\frac{dTRC_1(T_1)}{dT_1}$ is an increasing function throughout the interval $[0, \infty)$. With the help of this Lemma, $TRC_1(T_1)$ is said to be a convex function on $[0, \infty)$.

Additionally, as $\lim T_1 \to \infty$, then $g_1(T_1) \to \infty$.

$$\begin{aligned} g_1(0) &= -\Big[A_1 + h(P_1 - D)\left[\left(\frac{\ln(1+L)}{2} + \frac{1}{4}\right)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\right] + hD\Big[\\ &- \ln(1+L)\frac{(1+L)^2}{2} + \left(\frac{\ln(1+L)}{2} + \frac{1}{4}\right)(2+2L-t_1)t_1\Big] + \ln(1+L-t_1)cP_1t_1\Big(\frac{hP_1}{2} \\ &+ cI_{c1}P_1\Big)\frac{(1+L-t_1)^2}{2} + cI_{c1}(P_1 - D) - \left[\left(\frac{\ln(1+L)}{2} - \frac{1}{4}\right)(2+2L-t_1-M_1)(t_1\right. \\ &- \left. M_1) - \frac{sI_{e1}D({M_1}^2 - (1-\alpha_1){N_1}^2)}{2} - (1+L-M_1)^2\frac{\ln(1+L-M_1)}{2}\right] \\ &+ \frac{cI_{c1}P_1(1+L-t_1)^2}{2}\ln(1+L-t_1)\Big] \end{aligned}$$

Then

$$\begin{aligned} \frac{dTRC_1(T_1)}{dT_1} &< 0; \quad \text{if} \quad T_1 \in [0, \acute{T_1}), \\ &= 0; \quad \text{if} \quad T_1 = \acute{T_1}, \\ &> 0; \quad \text{if} \quad T_1 \in (\acute{T_1}, \infty). \end{aligned}$$

By applying the intermediate value theorem, there exists a a unique optimal solution which is $\acute{T_1}$.

Case 1.(b) $M_1 \le T_1 \le t_{M1}$

$$\frac{dTRC_2(T_1)}{dT_1} = \frac{g_2(T_1)}{{T_1}^2}$$

where

$$\begin{aligned} g_2(T_1) &= -A_1 - h(P_1 - D)\left[\left(\frac{\ln(1+L)}{2} + \frac{1}{4}\right)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\right] - \ln(1+L \\ &- t_1)\frac{hP_1(1+L-t_1)^2}{2} + hD\Big[\ln(1+L-T_1)\frac{(1+L-T_1)^2}{2} - \Big(\frac{1}{4} + \frac{\ln(1+L-T_1)}{2} + \Big)(2 \\ &+ 2L - t_1 - T_1)(t_1 - T_1)\Big] - \frac{DcI_{c1}}{2}\Big[-(T_1 - M_1)(2+2L-M_1-T_1)\left(\frac{1}{2} + \ln(1+L-T_1)\right) \\ &+ \ln(1+L-M_1)(1+L-M_1)^2 - (1+L-T_1)^2\ln(1+L-T_1)\Big] + \frac{sI_{e1}D}{2}[{M_1}^2 - (1-\alpha_1){N_1}^2] \\ &- cP_1t_1 + (2L+2-T_1-M_1)\frac{DcI_{c1}T_1}{2}\Big[(2T_1 + M_1 - 2L - 2) - (3T_1 - M_1 - 2L - 2)\ln(1+L \\ &- T_1) - \Big(\frac{(T_1 - M_1)}{1+L-T_1} + \ln(1+L-T_1)\Big)\Big] \end{aligned}$$

To calculate optimal value of T_1 say $\acute{T_2}$, one can solve the equation $g_2(T_1) = 0$.

Now $\frac{dg_2(T_1)}{dT_1} > 0, \quad \text{if} \ \ T_1 > 0.$

As $g_2(T_1)$ is an increasing function over the interval $[0, \infty)$, so $\frac{dTRC_2(T_1)}{dT_1}$ is an increasing function on $[0, \infty)$. Using Lemma, $TRC_2(T_1)$ is justified as a convex function on $[0, \infty)$.

In addition, as $\lim T_1 \to \infty$, then $g_2(T_1) \to \infty$.

$$
\begin{aligned}
g_2(0) &= -\Big[A_1 + h(P_1 - D)\Big[\Big(\frac{\ln(1+L)}{2} + \frac{1}{4}\Big)(2 + 2L - t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\Big] + \ln(1+L \\
&- t_1) - hD\Big[\ln(1+L)\frac{(1+L)^2}{2} - \Big(\frac{hP_1(1+L-t_1)^2}{2}\frac{\ln(1+L)}{2} - \frac{1}{4}\Big)(2+2L-t_1)t_1\Big] \\
&+ \frac{DcI_{c1}}{2}\Big[M_1(2+2L-M_1)\left(\frac{1}{2} + \ln(1+L)\right) + (1+L-M_1)^2\ln(1+L-M_1) \\
&- \ln(1+L)(1+L)^2\Big] - \frac{sI_{e1}D}{2}[{M_1}^2 - (1-\alpha_1){N_1}^2] + cP_1t_1\Big]
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{dTRC_2(T_1)}{dT_1} &< 0; \quad \text{if} \ \ T_1 \in [0, \acute{T_2}), \\
&= 0; \quad \text{if} \ \ T_1 = \acute{T_2}, \\
&> 0; \quad \text{if} \ \ T_1 \in (\acute{T_2}, \infty).
\end{aligned}
$$

Using intermediate value theorem, a unique optimal solution $\acute{T_2}$ exists.

Case 1.(c) $N_1 \leq T_1 \leq M_1$

$$
\frac{dTRC_3(T_1)}{dT_1} = \frac{g_3(T_1)}{{T_1}^2}
$$

where

$$
\begin{aligned}
g_3(T_1) &= hDT_1\Big[(2+2L-t_1-T_1)\left(\frac{(t_1-T_1)}{2(1+L-T_1)}-\frac{\ln(1+L-T_1)}{2}\right)-(1+L-T_1)-\ln(1\\
&+ L-T_1)(t_1+1+L-2T_1)\Big]-A_1-h(P_1-D)\Big[\Big(\frac{\ln(1+L)}{2}+\frac{1}{4}\Big)(2+2L-t_1)t_1\\
&- \frac{(1+L)^2}{2}\ln(1+L)\Big]-\frac{hP_1(1+L-t_1)^2}{2}\ln(1+L-t_1)-\frac{sI_{e1}D}{2}(1-\alpha_1){N_1}^2\\
&- hD\Big[-\frac{(1+L-T_1)^2}{2}\ln(1+L-T_1)+\Big(\frac{1}{4}+\frac{\ln(1+L-T_1)}{2}\Big)(2+2L-t_1\\
&- T_1)(t_1-T_1)\Big]+\frac{sI_{e1}D{T_1}^2}{2}-cP_1t_1
\end{aligned}
$$

For determining optimal value of T_1 say $\acute{T_3}$, one can solve the equation $g_3(T_1)=0$.

Now $\frac{dg_3(T_1)}{dT_1}>0$ if $T_1>0$.

As $g_3(T_1)$ is an increasing function over the interval $[0,\infty)$, hence $\frac{dTRC_3(T_1)}{dT_1}$ is an increasing function on $[0,\infty)$. Using the statement of Lemma, $TRC_3(T_1)$ is considered as a convex function over $[0,\infty)$. In addition, as $\lim T_1\to\infty$, then $g_3(T_1)\to\infty$.

Now

$$
\begin{aligned}
g_3(0) &= -\Big[A_1+h(P_1-D)\left[\left(\frac{\ln(1+L)}{2}+\frac{1}{4}\right)(2+2L-t_1)t_1-\ln(1+L)\frac{(1+L)^2}{2}\right]+\frac{sI_{e1}D}{2}(1\\
&- \alpha_1){N_1}^2+cP_1t_1+hD\Big[\Big(\frac{\ln(1+L)}{2}+\frac{1}{4}-\ln(1+L)\frac{(1+L)^2}{2}\Big)(2+2L-t_1)t_1\Big]\\
&+ \frac{hP_1(1+L-t_1)^2}{2}\ln(1+L-t_1)\Big]
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{dTRC_3(T_1)}{dT_1} &< 0; \quad \text{if} \quad T_1\in[0,\acute{T_3}),\\
&= 0; \quad \text{if} \quad T_1=\acute{T_3},\\
&> 0; \quad \text{if} \quad T_1\in(\acute{T_3},\infty)
\end{aligned}
$$

Again using the intermediate value theorem, it concludes that a unique optimal solution $\acute{T_3}$ exists.

Case 1.(d) $0 < T_1 \leq N_1$

$$\frac{dTRC_4(T_1)}{dT_1} = \frac{g_4(T_1)}{{T_1}^2}$$

where

$$\begin{aligned}
g_4(T_1) &= hDT_1\Big[(2+2L-t_1-T_1)\Big(\frac{(t_1-T_1)}{2(1+L-T_1)} - \frac{\ln(1+L-T_1)}{2}\Big) - (1+L-T_1) - \ln(1+L \\
&- T_1)(t_1+1+L-2T_1)\Big] - A_1 - h(P_1-D)\Big[\Big(\frac{\ln(1+L)}{2} + \frac{1}{4}\Big)(2+2L-t_1)t_1 - \frac{(1+L)^2}{2}\ln(1 \\
&+ L)\Big] + \frac{sI_{e1}D\alpha_1{T_1}^2}{2} - cP_1t_1 - hD\Big[\Big(\frac{1}{4} + \frac{\ln(1+L-T_1)}{2}\Big)(2+2L-t_1-T_1)(t_1-T_1) - \ln(1 \\
&+ L-T_1)\frac{(1+L-T_1)^2}{2}\Big] - \frac{hP_1(1+L-t_1)^2}{2}\ln(1+L-t_1)
\end{aligned}$$

To find out optimal value of T_1 say $\acute{T_4}$, one can calculate the equation $g_4(T_1) = 0$.

Now $\frac{dg_4(T_1)}{dT_1} > 0$ if $T_1 > 0$.

As $g_4(T_1)$ is an increasing function on $[0, \infty)$, so $\frac{dTRC_4(T_1)}{dT_1}$ is an increasing function throughout the interval $[0, \infty)$. Then by using the Lemma, $TRC_4(T_1)$ is taken to be as a convex function on $[0, \infty)$. In addition, as $\lim T_1 \to \infty$, then $g_4(T_1) \to \infty$.

$$\begin{aligned}
g_4(0) &= -\Big[A_1 + h(P_1-D)\Big[\Big(\frac{\ln(1+L)}{2} + \frac{1}{4}\Big)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\Big] + hD\Big[-\ln(1 \\
&+ L)\frac{(1+L)^2}{2} + \Big(\frac{\ln(1+L)}{2} + \frac{1}{4}\Big)(2+2L-t_1)t_1\Big] + cP_1t_1 + \frac{hP_1(1+L-t_1)^2}{2}\ln(1+L-t_1)\Big]
\end{aligned}$$

Then

$$\begin{aligned}
\frac{dTRC_4(T_1)}{dT} &< 0; \quad \text{if } T_1 \in [0, \acute{T_4}), \\
&= 0; \quad \text{if } T_1 = \acute{T_4}, \\
&> 0; \quad \text{if } T_1 \in (\acute{T_4}, \infty)
\end{aligned}$$

Using the intermediate value theorem, a unique optimal solution $\acute{T_4}$ exists.

Case 2 $M_1 < N_1$

Case 2.(a) $T_1 \geq t_{M1}$

$$\frac{dTRC_5(T_1)}{dT_1} = \frac{g_5(T_1)}{{T_1}^2}$$

where

$$\begin{aligned}
g_5(T_1) &= (h + cI_{c1})DT_1\Big[(2 + 2L - t_1 - T_1)\left(\frac{(t_1 - T_1)}{2(1 + L - T_1)} - \frac{\ln(1 + L - T_1)}{2}\right) - (1 + L - T_1) \\
&- \ln(1 + L - T_1)(t_1 + 1 + L - 2T_1)\Big] - A_1 - \Big(\frac{hP_1}{2} + cI_{c1}P_1\Big)(1 + L - t_1)^2\frac{\ln(1 + L - t_1)}{2} \\
&- h\Big[\Big(\frac{1}{4} + \frac{\ln(1 + L)}{2}\Big)(2 + 2L - t_1)t_1 - \frac{(1 + L)^2}{2}\ln(1 + L)\Big](P_1 - D) - (h + cI_{c1})D\Big[\\
&- \ln(1 + L - T_1)\frac{(1 + L - T_1)^2}{2} + \left(\frac{\ln(1 + L - T_1)}{2} + \frac{1}{4}\right)(2 + 2L - t_1 - T_1)(t_1 - T_1)\Big] \\
&+ \frac{sI_{e1}D\alpha_1{M_1}^2}{2} - cP_1t_1 - cI_{c1}(P_1 - D)\Big[\Big(-\frac{1}{4} + \frac{\ln(1 + L)}{2}\Big)(2 + 2L - t_1 - M_1)(t_1 \\
&- M_1) - \ln(1 + L - M_1)\frac{(1 + L - M_1)^2}{2}\Big] - cI_{c1}D\Big[\Big(\frac{1}{4} + \frac{\ln(1 + L - T_1)}{2}\Big)(2 + 2L \\
&- t_1 - T_1)(T_1 - t_1) + \ln(1 + L - T_1)\frac{(1 + L - T_1)^2}{2}\Big]
\end{aligned}$$

To observe the optimal value of T_1 say $\acute{T_5}$, one can solve the equation $g_5(T_1) = 0$.

Now $\frac{dg_5(T_1)}{dT_1} > 0$ if $T_1 > 0$. As $g_5(T_1)$ is an increasing function during the interval $[0, \infty)$, therefore $\frac{dTRC_5(T_1)}{dT_1}$ is an increasing function $[0, \infty)$. Utilizing Lemma, $TRC_5(T_1)$ is said to be a convex function on $[0, \infty)$.

In addition, $\lim T_1 \to \infty$, then $g_5(T_1) \to \infty$.

$$\begin{aligned}
g_5(0) &= -\Big[A_1 + h(P_1 - D)\Big[\left(\frac{\ln(1 + L)}{2} + \frac{1}{4}\right)(2 + 2L - t_1)t_1 - \ln(1 + L)\frac{(1 + L)^2}{2}\Big] - \frac{sI_{e1}D\alpha_1{M_1}^2}{2} \\
&+ cP_1t_1 + \Big(\frac{hP_1}{2} + cI_{c1}P_1\Big)\frac{(1 + L - t_1)^2}{2}\ln(1 + L - t_1) + hD\Big[\Big(\frac{1}{4} + \frac{\ln(1 + L)}{2}\Big)(2 + 2L - t_1)t_1 \\
&- \ln(1 + L)\frac{(1 + L)^2}{2}\Big] + cI_{c1}(P_1 - D)\Big[\Big(-\frac{1}{4} + \frac{\ln(1 + L)}{2}\Big)(2 + 2L - t_1 - M_1)(t_1 - M_1) - \ln(1 \\
&+ L - M_1)\frac{(1 + L - M_1)^2}{2}\Big]\Big]
\end{aligned}$$

Then

$$\begin{aligned}\frac{dTRC_5(T_1)}{dT_1} &< 0; \quad \text{if} \quad T_1 \in [0, \acute{T_5}),\\ &= 0; \quad \text{if} \quad T_1 = \acute{T_5},\\ &> 0; \quad \text{if} \quad T_1 \in (\acute{T_5}, \infty)\end{aligned}$$

Using the intermediate value theorem, there is a unique optimal solution $\acute{T_5}$.

Case 2.(b) $M_1 \leq T_1 \leq t_{M1}$

$$\frac{dTRC_6(T_1)}{dT_1} = \frac{g_6(T_1)}{{T_1}^2}$$

where

$$\begin{aligned}g_6(T_1) &= -\Big(A_1 + h(P_1 - D)\Big[\Big(\frac{\ln(1+L)}{2} + \frac{1}{4}\Big)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\Big] + \ln(1\\ &+ L - t_1)\frac{hP_1(1+L-t_1)^2}{2} - hD\Big[\ln(1+L-T_1)\frac{(1+L-T_1)^2}{2} - \Big(\frac{\ln(1+L-T_1)}{2}\\ &+ \frac{1}{4}\Big)(2+2L-t_1-T_1)(t_1-T_1)\Big] + \frac{DcI_{c1}}{2}\Big[-(T_1-M_1)(2+2L-M_1-T_1)\Big(\frac{1}{2}\\ &+ \ln(1+L-T_1)\Big) + \ln(1+L-M_1)(1+L-M_1)^2 - (1+L-T_1)^2\ln(1+L-T_1)\Big]\\ &- \frac{sI_{e1}D{M_1}^2\alpha_1}{2} + cP_1t_1 - \Big[(2T_1+M_1-2L-2) - (3T_1-M_1-2L-2)\ln(1+L\\ &- T_1) - (2L+2-T_1-M_1)\Big(\ln(1+L-T_1) + (T_1-M_1)\frac{1}{1+L-T_1}\Big)\Big]\frac{DcI_{c1}T_1}{2}\Big)\end{aligned}$$

To obtain optimal value of T_1 say $\acute{T_6}$, one can solve the equation $g_6(T_1) = 0$.

Now $\frac{dg_6(T_1)}{dT_1} > 0$ if $T_1 > 0$.

As $g_6(T_1)$ is an increasing function over the interval $[0, \infty)$, hence $\frac{dTRC_6(T_1)}{dT_1}$ is an increasing function on $[0, \infty)$. Using Lemma, $TRC_6(T_1)$ is considered to be a convex function on $[0, \infty)$. In addition,

$\lim T_1 \to \infty$, then $g_6(T_1) \to \infty$.

$$
\begin{aligned}
g_6(0) &= -\Big[A_1 - h(P_1-D)\Big[\Big(\frac{\ln(1+L)}{2}+\frac{1}{4}\Big)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\Big] - \ln(1+L \\
&- t_1)\frac{hP_1(1+L-t_1)^2}{2} + hD\Big[\ln(1+L)\frac{(1+L)^2}{2} - \Big(\frac{\ln(1+L)}{2}-\frac{1}{4}\Big)(2+2L-t_1)t_1\Big] \\
&- \frac{DcI_{c1}}{2}\Big[M_1(2+2L-M_1)\left(\frac{1}{2}+\ln(1+L)\right) + (1+L-M_1)^2\ln(1+L-M_1) \\
&- \ln(1+L)(1+L)^2\Big] + \frac{sI_{e1}D{M_1}^2\alpha_1}{2} - cP_1t_1\Big]
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{dTRC_6(T_1)}{dT_1} &< 0; \quad \text{if} \;\; T_1 \in [0, \acute{T_6}), \\
&= 0; \quad \text{if} \;\; T_1 = \acute{T_6}, \\
&> 0; \quad \text{if} \;\; T_1 \in (\acute{T_6}, \infty)
\end{aligned}
$$

Using the intermediate value theorem, a unique optimal solution $\acute{T_6}$ exists.

Case 2.(c) $0 < T_1 \leq M_1$

$$
\frac{dTRC_7(T_1)}{dT_1} = \frac{g_7(T_1)}{{T_1}^2}
$$

where

$$
\begin{aligned}
g_7(T_1) &= hDT_1\Big[(2+2L-t_1-T_1)\left(\frac{(t_1-T_1)}{2(1+L-T_1)} - \frac{\ln(1+L-T_1)}{2}\right) - (1+L-T_1) - \ln(1+L \\
&- T_1)(t_1+1+L-2T_1)\Big] - A - h(P_1-D)\Big[\left(\frac{\ln(1+L)}{2}+\frac{1}{4}\right)(2+2L-t_1)t_1 - \ln(1 \\
&+ L)\frac{(1+L)^2}{2}\Big] + \frac{sI_{e1}D{T_1}^2\alpha_1}{2} - cP_1t_1 - \ln(1+L-t_1)\frac{hP_1(1+L-t_1)^2}{2} - hD\Big[\\
&- \ln(1+L-T_1)\frac{(1+L-T_1)^2}{2} + \Big(\frac{1}{4}+\frac{\ln(1+L-T_1)}{2}\Big)(2+2L-t_1-T)(t_1 \\
&- T_1)\Big]
\end{aligned}
$$

To calculate optimal value of T_1 say $\acute{T_7}$, one can solve the equation $g_7(T_1) = 0$.

Now $\frac{dg_7(T_1)}{dT_1} > 0, \;\; \text{if} \;\; T_1 > 0.$

As $g_7(T_1)$ is an inclined function on $[0,\infty)$, so $\frac{dTRC_7(T_1)}{dT_1}$ is an increasing function on $[0,\infty)$. Using the Lemma, $TRC_7(T_1)$ is taken to be as a convex function on $[0,\infty)$.

In addition, as $\lim T_1 \to \infty$, then $g_7(T_1) \to \infty$.

$$\begin{aligned} g_7(0) &= -\Big[A_1 + h(P_1 - D)\Big[\left(\frac{\ln(1+L)}{2} + \frac{1}{4}\right)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\Big] + \ln(1+L \\ &- t_1)\frac{hP_1(1+L-t_1)^2}{2} + hD\Big[-\ln(1+L)\frac{(1+L)^2}{2} + \Big(\frac{\ln(1+L)}{2} + \frac{1}{4}\Big)(2+2L-t_1)t_1\Big] \\ &+ cP_1t_1\Big] \end{aligned}$$

Then

$$\begin{aligned} \frac{dTRC_7(T_1)}{dT_1} &< 0; \quad \text{if} \quad T_1 \in [0, \acute{T}_7), \\ &= 0; \quad \text{if} \quad T_1 = \acute{T}_7, \\ &> 0; \quad \text{if} \quad T_1 \in (\acute{T}_7, \infty) \end{aligned}$$

Using the intermediate value theorem, there is a unique optimal solution $\acute{T}_7$.