Chapter 2

Variable demand in a warehouse model *

2.1 Introduction

It is common phenomena that demand is increasing with the increasing time. There are many products for which demand rate depends on time. Demand of items may increase or decrease with time. Many mathematical models have been developed to control inventory by considering constant demand rate while in most of the cases, demand of items increase with time. Harris (1913) first discovered an EOQ for constant demand pattern. Regarding demand as time-dependent, many researchers formulated several inventory models. Hsu and Li (2006) discussed an inventory model for time varying consumer demand. Dye *et al.* (2006) observed an inventory model by including not only cost of lost sales, but also the non-constant purchase cost. They extended their model from a constant demand to any log-concave demand function. Khanra *et al.* (2011) developed an inventory system with time varying demand and delay-in-payments. Sarkar *et al.* (2011a) formed a production-inventory model where demand is assumed as continuous as well as discrete random.

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The defective products are repaired with some fixed cost. Sarkar and Moon (2011) extended Sarkar *et al.*'s (2011a) model with the effect of inflation. They highlighted imperfect items which are reworked at some fixed costs and considered shortages due to the production of imperfect products. The lifetime of defective items followed a Weibull distribution. Sarkar *et al.* (2011b) studied an imperfect production model which produces a single type of items. Their model formulated by time-dependent demand with reliability as a decision variable for inflation and also time value of money.

The loss due to deterioration of items like vegetable, or commodities cannot be ignored. The growth and application of inventory control models regarding deterioration of products is the main concerns of researcher. Many previous studies have been done in this field by assuming constant deterioration. But deterioration of item may vary with time. Using present value concept, many researchers stated about the distribution processing for deterioration. Wee and Law (2001) discussed an inventory model with time-value of money, deterioration, and price-dependent demand. Chu and Chen (2002) developed the inventory holding cost is in proportion to the cost for deteriorated items. By describing time-dependent deterioration, Khanra and Chaudhuri (2003) invented an order-level inventory problem on continuous and quadratic function of time-dependent demand. In their model for infinite and finite time-horizon, the solution of model was discussed analytically. Chern *et al.* (2008) extended previous inventory model by allowing general partial backlogging rate and inflation. Sett *et al.* (2012) depicted a two-warehouse inventory system with quadratic demand which is useful for those items whose demand increases very rapidly. Their study discussed about time varying deterioration rates. Sarkar *et al.* (2012) formulated an optimal inventory replenishment policy with time varying demand and time varying partial backlogging. Sarkar (2012b) constructed an inventory model for time varying demand and deterioration rate. Sarkar and Sarkar

(2013a) developed an inventory model for time-dependent deterioration rate. Their study discussed about inventory-dependent demand function. They considered three possible cases for demand and inventory. Sarkar and Sarkar (2013b) extended earlier literature for infinite replenishment rate by incorporating partial backlogging, stock varying demand, and time dependent deterioration. Sarkar (2013) presented a production model along with SCM. Sarkar and Sarkar (2013c) extended an EMQ model with deterioration and exponential demand. Sarkar *et al.* (2013) discussed a deteriorating inventory system for deteriorating products and time dependent demand. Sarkar *et al.* (2015) depiced an inventory system with both full and partial trade-credit policy.

Pricing is also an important factor in success of business for any item. In general, when selling-price of items decreases, customers are more attracted to that product. Hence, demand rate of products may consider based on price. Wee (1997) analyzed an inventory system for price varying demand of items with variable deterioration and completely backorder. Datta and Paul (2001) derived an inventory model where demand rate was affected by price and stock-level. Goyal and Chang (2009) obtained an ordering-transfer inventory model that provides limited display space and stock-level-dependent demand rate. Sarkar *et al.* (2010a) discovered an inventory model under the assumption that retailers are allowed a period by supplier to obtain trade-credit for goods bought with some discount rates. They developed retailer's optimal replenishment decision under trade-credit policy with inflation. They assumed several types of deterministic demand patterns with the delay-periods and different discounts rates on purchasing cost. Sarkar *et al.* (2010b) developed an imperfect production process for stock-dependent demand. These imperfect items were reworked at some fixed cost for restoring its original quality. In addition, in their model unit production cost is a function of reliability parameter and production rate. Sana (2011) investigated an inventory model to obtain retailer's optimal order quantity with limited display space. In his article, demand of products depends on selling-price, salesmen's initiatives and display stock-level where more stocks of one product forms a negative impact of another products. Sarkar (2012c) assumed an imperfect production process with price and advertising demand pattern under the effect of inflation. To reduce the production of imperfect items, development cost, production cost, and material cost are dependent on reliability in his model. Sarkar (2012a) deduced an inventory framework in which supplier generally offers a delay-period to the retailer to buy more. In this point of view, permissible delay-in-payments are considered along with stock-dependent demand, finite replenishment rate, and the production of defective items. See Table 2.1 for contribution of various authors.

Table 2.1: Contribution of various authors

Author(s)	**Time-dependent demand**	**Price-dependent demand**	**Other demands**	**Random deterio-rations**	**Other deterio-rations**
Harris (1913)			$\surd$		
Wee (1997)		$\surd$			$\surd$
Datta and Paul (2001)		$\surd$			
Wee and Law (2001)		$\surd$			$\surd$
Chu and Chen (2002)	$\surd$			$\surd$	
Khanra and Chaudhuri (2003)	$\surd$				$\surd$
Dye *et al.* (2006)	$\surd$				$\surd$
Hsu and Li (2006)	$\surd$				
Chern *et al.* (2008)			$\surd$		$\surd$

Author(s)	**Time-dependent demand**	**Price-dependent demand**	**Other demands**	**Random deteriorations**	**Other deteriorations**
Goyal and Chang (2009)			√		
Sarkar *et al.* (2010a)		√			
Sarkar *et al.* (2010b)		√			
Khanra *et al.* (2011)	√				√
Sana (2011)		√	√		
Sarkar *et al.* (2011a)			√		√
Sarkar *et al.* (2011b)	√				√
Sarkar and Moon (2011)			√	√	
Sett *et al.* (2012)	√				√
Sarkar *et al.* (2012)	√				√
Sarkar (2012a)			√		
Sarkar (2012b)	√				√
Sarkar (2012c)		√			√
Sarkar (2013)			√		√
Sarkar *et al.* (2013)	√				√
Sarkar and Sarkar (2013a)				√	

Author(s)	**Time-dependent demand**	**Price-dependent demand**	**Other demands**	**Random deterio-rations**	**Other deterio-rations**
Sarkar and Sarkar (2013b)			$\surd$		$\surd$
Sarkar and Sarkar (2013c)			$\surd$	$\surd$	
Sarkar *et al.* (2014)					$\surd$
This chapter	$\surd$	$\surd$		$\surd$	

This chapter presents an inventory model for probabilistic deteriorating rate with several demand function as time and price-dependent, and finite production rate. The display space is taken to be limited. This model includes the number of transfer per order from the warehouse to display area. The main objective of this model is to maximize average profit function over finite planning horizon and obtain the optimal order quantity and the number of transfer per order. In this chapter, there are four cases of demand functions. The average profit function is maximized in each case. Numerical examples and sensitivity analysis are buildup for each demand functions.

2.2 Mathematical model

Following notation are used to formulate this model.

Decision variables

t_1 replenishment cycle time in display area (year)

n integer number of shipments for stocks from warehouse to display area per order

p unit selling-price of stocks per unit (\$/unit)

Parameters

h_1 unit carrying cost per stock in warehouse (\$/unit/unit time)

h unit carrying cost per stock in display area, where $h > h_1$ (\$/unit/unit time)

c unit purchasing cost (\$/unit)

S retailer's ordering cost per order (\$/order)

s fixed cost of stocks per transfer to display area from warehouse (\$/transfer)

T replenishment cycle time in warehouse

Q order quantity placed to the supplier (units)

$I(t)$ inventory level at time t in the display area

R fixed inventory level of stocks in display area for transferring of q items reducing stockout

q stock per transfer to display area from warehouse (units/transfer)

D demand function considered as time-dependent, price-dependent, and time-price-dependent

θ probabilistic deterioration rate, $0 < \theta < 1$

AP_1 average profit for demand function $D(t,p) = x + x_1 + yt - y_1p + zt^2 - z_1p^2$

AP_2 average profit while demand function is $D(t) = x + yt + zt^2$

AP_3 average profit for demand function $D(p) = x_1 - y_1p - z_1p^2$

AP_4 average profit when demand function is $D(t) = x_2e^{y_2t}$

This model is formulated on the basis of following assumptions

1. The model consider a warehouse problem with random deterioration rate θ which follows a uniform distribution.

   The probability density function of deterioration is

$$f(X) = \left\{ \begin{array}{cc} \frac{1}{b-a}, & \text{if } \ X \in [a,b] \\ 0, & \text{otherwise} \end{array} \right\}$$

   where a and $b(> a)$ are two parameters of this distribution and $0 < a < b < 1$.

   Therefore, $\theta = E[f(X)] = \frac{b+a}{2}$.

2. The retailer places an order of quantity Q from a supplier and stores them into the warehouse. These items are transferred to display area from warehouse in equal lots of q until inventory level in warehouse reaches to zero.

3. The transferring time of stocks from warehouse to display area is taken as negligible.

4. The demand function as follows

   $D(t,p) = x + x_1 + yt - y_1p + zt^2 - z_1p^2$, $D(t) = x + yt + zt^2$,

   $D(p) = x_1 - y_1p - z_1p^2$, and $D(t) = x_2e^{y_2t}$. x, y, and z are beginning rate, increasing rate, and rate of change for demand in first and second demand function respectively. x_1, y_1, and z_1 are initial rate, decreasing rate, and rate of change for demand in first and third demand function separately. x_2 is constant parameter and y_2 is increasing rate of demand regarding fourth demand function.

5. Lead time is considered as negligible and shortages are not allowed.

Here, an inventory model related with warehouse and display area are considered. Two types of costs (warehouse cost and display area cost) are given. These costs are used to calculate the profit of the model.

Warehouse cost

When the retailer orders Q items from the supplier, it is instantly supplied to the retailer and the retailer stocks all the items in the warehouse. Now, the items Q can be divided into q equal parts i.e., $Q = nq$ and a part is transferred to the display area when the previous part has just been depleted. The process will continue until the inventory at the warehouse reaches at zero level.

Retailer's ordering cost per order is $= S$.

During the time interval $[0, t_1]$, total item is

$$[q + 2q + 3q + ... + (n-1)q]t_1 \quad = \quad \frac{n(n-1)qt_1}{2}.$$

Hence, the stock holding cost is $= h_1 t_1 \frac{n(n-1)}{2} q$.

Cost at display area

At the time $t = 0$, the level of inventory $I(t)$ starts with a maximum inventory say $\bar{I}$ and then it reaches to R at the end of cycle t_1. Figure 2.1 represents the inventory system.

Case I

In this case, demand rate is considered as a function of price and time. As demand may increase when the selling-price diminishes and vice-versa or it may fluctuate with the change of time. The consideration of time and price-dependent demand is useful for deteriorated items, for example, fashionable goods, fruits, and vegetables. This study discussed an inventory model by assuming demand as a quadratic function of time and price.

i.e.,

$$\begin{aligned} f(t,p) &= D(t,p) = (x + yt + zt^2) + (x_1 - y_1 p - z_1 p^2) \\ &= x + x_1 + yt - y_1 p + zt^2 - z_1 p^2 \end{aligned}$$

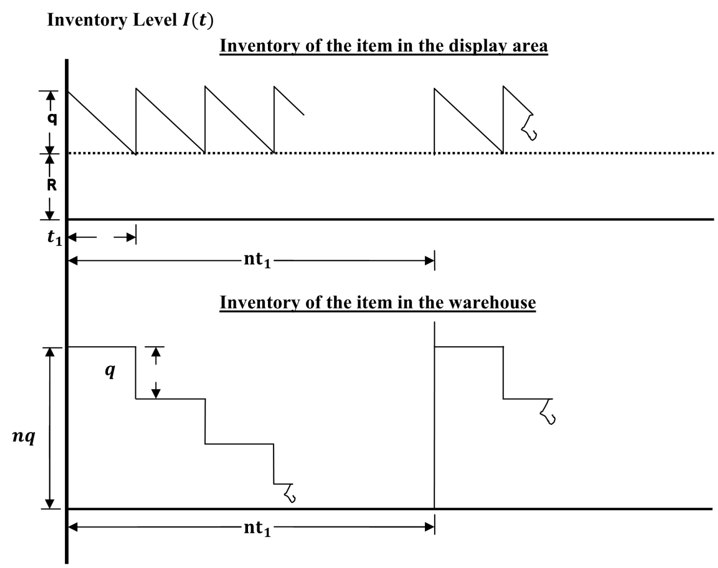

Figure 2.1: Graphical representation of inventory system

The governing differential equation of this inventory model is

$$\begin{aligned} \frac{dI(t)}{dt} + \theta I(t) &= -f(t,p), \quad 0 \le t \le t_1,\ I(t_1) = R \\ &= -(x + x_1 + yt - y_1 p + zt^2 - z_1 p^2) \end{aligned}$$

Using the boundary condition, inventory level *I(t)* as

$$\begin{aligned} I(t) &= \frac{(1 - e^{\theta(t_1 - t)})}{\theta}(y_1 p + z_1 p^2 - x - x_1) + y\Big(\frac{(t_1 e^{\theta(t_1-t)} - t)}{\theta} - \frac{(e^{\theta(t_1-t)} - 1)}{\theta^2}\Big) \\ &+ z\Big(\frac{({t_1}^2 e^{\theta(t_1-t)} - t^2)}{\theta} - \frac{(2t_1 e^{\theta(t_1-t)} - 2t)}{\theta^2} + \frac{(2e^{\theta(t_1-t)} - 2)}{\theta^3}\Big) + Re^{\theta(t_1-t)} \end{aligned}$$

During $[0,t_1]$, the total costs are as follows:

(i) Fixed cost of stocks per transfer to display area from warehouse is $= s$.

(ii) Holding cost is

$$\begin{aligned} = h\int_0^{t_1} I(t)dt &= \frac{hR(e^{\theta t_1} - 1)}{\theta} + \frac{h(y_1 p + z_1 p^2 - x - x_1)}{\theta}\Big(t_1 + \frac{(1 - e^{\theta t_1})}{\theta}\Big) + hy\Big(\frac{t_1 e^{\theta t_1}}{\theta^2} \\ &+ \frac{(1 - e^{\theta t_1})}{\theta^3} - \frac{{t_1}^2}{2\theta}\Big) + hz\Big(\frac{{t_1}^2 e^{\theta t_1}}{\theta^2} - \frac{2t_1 e^{\theta t_1}}{\theta^3} + \frac{(2e^{\theta t_1} - 2)}{\theta^4} - \frac{{t_1}^3}{\theta^3}\Big) \end{aligned}$$

(iii) The revenue per cycle is

$$= (p-c)\int_0^{t_1} D(t,p)dt = (p-c)\int_0^{t_1}(x+x_1+yt-y_1p+zt^2-z_1p^2)dt$$
$$= (p-c)\left((x+x_1-y_1p-z_1p^2)t_1+\frac{y{t_1}^2}{2}+\frac{z{t_1}^3}{3}\right)$$

Equating equation of $I(t)$ and $I(0)=q+R$, we obtain

$$\begin{aligned} q &= \frac{(1-e^{\theta t_1})}{\theta}(y_1p+z_1p^2-x-x_1)+y\Big(\frac{t_1e^{\theta t_1}}{\theta}-\frac{(e^{\theta t_1}-1)}{\theta^2}\Big)+z\Big(\frac{{t_1}^2e^{\theta t_1}}{\theta}-\frac{2t_1e^{\theta t_1}}{\theta^2} \\ &+ \frac{(2e^{\theta t_1}-2)}{\theta^3}\Big)+Re^{\theta t_1}-R \end{aligned}$$

(iv) Stock holding cost in the warehouse is

$$\begin{aligned} = h_1\left[\frac{n(n-1)}{2}q\right]t_1 &= h_1\Big[\frac{n(n-1)}{2}\Big(\frac{(1-e^{\theta t_1})}{\theta}(y_1p+z_1p^2-x-x_1)+y\left(\frac{t_1e^{\theta t_1}}{\theta}-\frac{(e^{\theta t_1}-1)}{\theta^2}\right) \\ &+ z\Big(\frac{{t_1}^2e^{\theta t_1}}{\theta}-\frac{2t_1e^{\theta t_1}}{\theta^2}+\frac{(2e^{\theta t_1}-2)}{\theta^3}\Big)+Re^{\theta t_1}-R\Big)\Big]t_1 \end{aligned}$$

Thus, the average profit per unit time is

$AP_1(n,p,t_1) = \frac{1}{T}$[revenue-(total cost in warehouse)-(total cost in display area)] ($T=nt_1$)

$$\begin{aligned} &= (p-c)\left((x+x_1-y_1p-z_1p^2)+\frac{yt_1}{2}+\frac{z{t_1}^2}{3}\right)-\Big[h_1\Big(\frac{(n-1)}{2}\Big(\frac{(1-e^{\theta t_1})}{\theta}(y_1p+z_1p^2 \\ &- x-x_1)+y\left(\frac{t_1e^{\theta t_1}}{\theta}-\frac{(e^{\theta t_1}-1)}{\theta^2}\right)+z\Big(\frac{{t_1}^2e^{\theta t_1}}{\theta}-\frac{2t_1e^{\theta t_1}}{\theta^2}+\frac{(2e^{\theta t_1}-2)}{\theta^3}\Big)+Re^{\theta t_1}-R\Big)\Big) \\ &+ \frac{S}{nt_1}\Big]-\frac{s}{t_1}-h\Big[\frac{R(e^{\theta t_1}-1)}{\theta t_1}+y\Big(\frac{e^{\theta t_1}}{\theta^2}+\frac{(1-e^{\theta t_1})}{\theta^3t_1}-\frac{t_1}{2\theta}\Big)+\frac{(y_1p+z_1p^2-x-x_1)}{\theta}\Big(1 \\ &+ \frac{(1-e^{\theta t_1})}{\theta t_1}\Big)+z\left(\frac{t_1e^{\theta t_1}}{\theta^2}-\frac{2e^{\theta t_1}}{\theta^3}+\frac{(2e^{\theta t_1}-2)}{\theta^4t_1}-\frac{{t_1}^2}{\theta^3}\right)\Big] \end{aligned}$$

which is to maximize the total profit function. Thus, a lemma is formulated to obtain the global optimum solution.

Lemma 1

$AP_1(n^*, p^*, t_1{}^*)$ will have the global maximum solution where n^*, p^*, and $t_1{}^*$ are optimal values of n, p, and t_1 if following conditions are satisfied

$$(i) 4S\theta < h_1 n^3 t_1 (1 - e^{\theta t_1})(y_1 + 2z_1 p),$$

$$(ii) 4z_1 p + \frac{h_1}{2}\frac{(1-e^{\theta t_1})}{\theta}(y_1 + 2z_1 p) > 2\Big[cz_1 - y_1 - \frac{z_1}{\theta}\Big(\frac{h_1(n-1)}{2}(1-e^{\theta t_1}) + h\left(\frac{1-e^{\theta t_1}}{\theta t_1} + 1\right)\Big)\Big],$$

$$(iii) h_1 \frac{(e^{\theta t_1} - 1)}{\theta}(y_1 + 2z_1 p) M < N\left(\frac{h_1}{2}U + \frac{s}{t_1{}^2}\right),$$

$$(iv) \left(\frac{y}{2} + \frac{2zt_1}{3}\right) + \frac{h}{\theta}\left(\frac{e^{\theta t_1}}{t_1} + \frac{(1-e^{\theta t_1})}{\theta t_1{}^2}\right) < \frac{h_1(1-n)}{2}(y_1 + 2z_1 p)e^{\theta t_1}.$$

[See Appendix A for the values of M, N, and U.]

Proof

The necessary condition for optimal solution of $AP_1(n, p, t_1)$ can be calculated by $\frac{\partial AP_1(n,p,t_1)}{\partial n} = 0$, $\frac{\partial AP_1(n,p,t_1)}{\partial p} = 0$, and $\frac{\partial AP_1(n,p,t_1)}{\partial t_1} = 0$. i.e.,

$$\begin{aligned}\frac{\partial AP_1(n,p,t_1)}{\partial n} &= \frac{S}{n^2 t_1{}^2} - \frac{h_1}{2}\Big[\frac{(1-e^{\theta t_1})}{\theta}(y_1 p + z_1 p^2 - x - x_1) + y\Big(\frac{t_1 e^{\theta t_1}}{\theta} - \frac{(e^{\theta t_1} - 1)}{\theta^2}\Big) \\ &+ z\left(\frac{t_1{}^2 e^{\theta t_1}}{\theta} - \frac{2t_1 e^{\theta t_1}}{\theta^2} + \frac{2(e^{\theta t_1} - 1)}{\theta^3}\right) + Re^{\theta t_1} - R\Big] = 0\end{aligned}$$

The equation $\frac{\partial AP_1(n,p,t_1)}{\partial n}$ gives $n = \sqrt{\frac{2S}{h_1 t_1 f}}$. [See Appendix B for the value of f].

For the decision variable p,

$$\begin{aligned}\frac{\partial AP_1(n,p,t_1)}{\partial p} &= 0 \\ \frac{\partial AP_1(n,p,t_1)}{\partial p} &= 2z_1 p^2 - 2p\left[cz_1 - y_1 - \frac{z_1}{\theta}\left(\frac{h_1(n-1)}{2}(1-e^{\theta t_1}) + h\left(\frac{(1-e^{\theta t_1})}{\theta t_1} + 1\right)\right)\right] - \Big[x \\ &+ x_1 + \frac{yt_1}{2} + \frac{zt_1{}^2}{3} + cy_1 - \frac{y_1}{\theta}\left(\frac{h_1(n-1)}{2}(1-e^{\theta t_1}) + h\left(\frac{(1-e^{\theta t_1})}{\theta t_1} + 1\right)\right)\Big] = 0\end{aligned}$$

Now p^* will be calculated if $\eta(p^*) = 0$ where $\frac{\partial AP_1(n,p,t_1)}{\partial p} = \eta(p)$.

and $\frac{\partial AP_1(n,p,t_1)}{\partial t_1} = 0$ gives

$$\begin{aligned} i.e., \frac{\partial AP_1(n,p,t_1)}{\partial t_1} &= \left(\frac{y}{2} + \frac{2zt_1}{3}\right) + \frac{h_1(n-1)}{2}(y_1p + z_1p^2 - x - x_1)e^{\theta t_1} - (yt_1e^{\theta t_1} + z{t_1}^2e^{\theta t_1} \\ &+ R\theta e^{\theta t_1})\frac{h_1(n-1)}{2} + \frac{(S+sn)}{{t_1}^2} - h\Big[\frac{Re^{\theta t_1}}{t_1} - \frac{(y_1p + z_1p^2 - x - x_1)}{\theta}\Big(\frac{e^{\theta t_1}}{t_1} \\ &+ \frac{(1-e^{\theta t_1})}{\theta {t_1}^2}\Big) - \frac{(Re^{\theta t_1} - 1)}{\theta {t_1}^2} + y\Big(\frac{e^{\theta t_1}}{\theta} - \frac{e^{\theta t_1}}{\theta^2 t_1} - \frac{(1-e^{\theta t_1})}{\theta^3 {t_1}^2} - \frac{1}{2\theta}\Big) - z\Big(\frac{e^{\theta t_1}}{\theta^2} \\ &- \frac{t_1e^{\theta t_1}}{\theta} - \frac{2e^{\theta t_1}}{\theta^3 t_1} + \frac{2t_1}{\theta^3} + \frac{2(e^{\theta t_1} - 1)}{\theta^4 {t_1}^2}\Big)\Big] = 0 \end{aligned}$$

Now ${t_1}^*$ will be calculated if $\xi_1({t_1}^*) = 0$ where $\frac{\partial AP_1(n,p,t_1)}{\partial t_1} = \xi_1(t_1)$.

To verify the sufficient conditions for global optimum solution, the second order partial derivatives of $AP_1(n,p,t_1)$ with respect to n, p, and t_1 are as follows:

$$\frac{\partial^2 AP_1(n,p,t_1)}{\partial n^2} = \frac{-2S}{n^3 t_1},$$

$$\begin{aligned} \frac{\partial^2 AP_1(n,p,t_1)}{\partial {t_1}^2} &= \frac{2z}{3} + \frac{h_1(n-1)}{2}\theta(y_1p + z_1p^2 - x - x_1)e^{\theta t_1} - (y + y\theta t_1 + 2zt_1 + z{t_1}^2\theta \\ &+ R\theta^2)\frac{h_1(n-1)}{2}e^{\theta t_1} - \frac{2(S+sn)}{{t_1}^3} - h\Big[\frac{R\theta e^{\theta t_1}}{t_1} - \frac{2R(e^{\theta t_1} - 1)}{\theta {t_1}^3} + \Big(\frac{2e^{\theta t_1}}{{t_1}^2} \\ &- \frac{\theta e^{\theta t_1}}{t_1} + \frac{2(1-e^{\theta t_1})}{\theta {t_1}^3}\Big)\frac{(y_1p + z_1p^2 - x - x_1)}{\theta} + y\Big(e^{\theta t_1} - \frac{e^{\theta t_1}}{\theta t_1} + \frac{2e^{\theta t_1}}{\theta^2 t_1^2} \\ &+ \frac{2(1-e^{\theta t_1})}{\theta^3 {t_1}^3}\Big) + z\Big(t_1e^{\theta t_1} + \frac{2e^{\theta t_1}}{\theta^2 t_1} - \frac{4e^{\theta t_1}}{\theta^3 {t_1}^2} + \frac{4(e^{\theta t_1} - 1)}{\theta^4 {t_1}^3} - \frac{2}{\theta^3}\Big)\Big], \end{aligned}$$

$$\frac{\partial^2 AP_1(n,p,t_1)}{\partial p^2} = 4z_1p - 2\left[cz_1 - y_1 - \frac{z_1}{\theta}\left(\frac{h_1(n-1)}{2}(1-e^{\theta t_1}) + h\left(\frac{(1-e^{\theta t_1})}{\theta t_1} + 1\right)\right)\right],$$

$$\frac{\partial^2 AP_1(n,p,t_1)}{\partial n \partial t_1} = \frac{h_1}{2}e^{\theta t_1}(y_1p + z_1p^2 - x - x_1 - yt_1 - z{t_1}^2 - R\theta)\frac{s}{{t_1}^2},$$

$$\frac{\partial^2 AP_1(n,p,t_1)}{\partial p \partial t_1} = \left(\frac{y}{2} + \frac{2zt_1}{3}\right) + \frac{h_1(n-1)}{2}(y_1 + 2z_1p)e^{\theta t_1} + \frac{h(y_1 + 2z_1p)}{\theta}\left(\frac{e^{\theta t_1}}{t_1} + \frac{(1-e^{\theta t_1})}{\theta {t_1}^2}\right),$$

and

$$\frac{\partial^2 AP_1(n,p,t_1)}{\partial n \partial p} = \frac{h_1}{2}\left((y_1 + 2z_1p)\frac{(e^{\theta t_1} - 1)}{\theta}\right).$$

The sufficient condition for global optimum solution for this case is all principal minors are alternating in sign.

i.e., the sufficient condition for the optimum solution of $AP_1(n, t_1)$ are $\frac{\partial^2 AP_1(n,p,t_1)}{\partial n^2} < 0$, $\frac{\partial^2 AP_1}{\partial n^2}\frac{\partial^2 AP_1}{\partial p^2} - (\frac{\partial^2 AP_1}{\partial n \partial p})^2 > 0$, and the value of third principal minor i.e., the value of the Hessian matrix $H < 0$.

Now, $\quad \frac{\partial^2 AP_1(n,p,t_1)}{\partial n^2} = \frac{-2S}{n^3 t_1} < 0.$

To show the condition of second principal minor, if $\frac{\partial^2 AP_1}{\partial n^2} > \frac{\partial^2 AP_1}{\partial n \partial p}$ and $\frac{\partial^2 AP_1}{\partial p^2} > \frac{\partial^2 AP_1}{\partial n \partial p}$, then the condition holds.

Now, $\quad \frac{\partial^2 AP_1(n,p,t_1)}{\partial n \partial p} = \frac{h_1}{2}\left((y_1 + 2z_1 p)\frac{(e^{\theta t_1} - 1)}{\theta}\right).$

which can be written as

$$\frac{\partial^2 AP_1(n,p,t_1)}{\partial n \partial p} = \frac{\partial^2 AP_1(n,p,t_1)}{\partial n^2} - \xi_1.$$

where

$$\xi_1 = \frac{h_1}{2}\frac{(1 - e^{\theta t_1})}{\theta}(y_1 + 2z_1 p) - \frac{2S}{n^3 t_1}$$

In this case, $\frac{\partial^2 AP_1}{\partial n^2} > \frac{\partial^2 AP_1}{\partial n \partial p}$ will hold if $\xi_1 > 0$.

Now $\xi_1 > 0$ exists, when

$$4S\theta < h_1 n^3 t_1 (1 - e^{\theta t_1})(y_1 + 2z_1 p)$$

Similarly,

$$\frac{\partial^2 AP_1(n,p,t_1)}{\partial n \partial p} = \frac{\partial^2 AP_1(n,p,t_1)}{\partial p^2} - \xi_2$$

where

$$\xi_2 = N + \frac{h_1}{2}\frac{(1 - e^{\theta t_1})}{\theta}(y_1 + 2z_1 p)$$

Now $\frac{\partial^2 AP_1}{\partial p^2} > \frac{\partial^2 AP_1}{\partial n \partial p}$ will exist if $\xi_2 > 0$.

i.e., if

$$2\left[cz_1 - y_1 - \frac{z_1}{\theta}\left(\frac{h_1(n-1)}{2}(1-e^{\theta t_1}) + h\left(\frac{1-e^{\theta t_1}}{\theta t_1}+1\right)\right)\right] < 4z_1 p + \frac{h_1(1-e^{\theta t_1})(y_1+2z_1 p)}{2\theta}$$

Similar as above, value of third principal minor i.e., $H < 0$ will hold if

$$h_1\frac{(e^{\theta t_1}-1)}{\theta}(y_1+2z_1 p)M < N\left(\frac{h_1}{2}U + \frac{s}{{t_1}^2}\right)$$

and

$$\left(\frac{y}{2}+\frac{2zt_1}{3}\right) + \frac{h}{\theta}\left(\frac{e^{\theta t_1}}{t_1} + \frac{(1-e^{\theta t_1})}{\theta {t_1}^2}\right) < \frac{h_1(1-n)}{2}(y_1+2z_1 p)e^{\theta t_1}$$

Therefore, $AP_1(n^*, p^*, {t_1}^*)$ will have the global maximum (where n^*, p^*, and ${t_1}^*$ are optimal values of n, p, and t_1) if the conditions hold.

Case II

This section provides demand function is time-dependent. As time increases, the demand of each product increases. To show this matter, the demand is considered as quadratic function of time.

For the demand function $f(t) = D(t) = x + yt + zt^2$

The governing differential equation of this inventory model is

$$\begin{aligned}\frac{dI(t)}{dt} + \theta I(t) &= -f(t), \quad 0 \le t \le t_1, \; I(t_1) = R \\ &= -(x + yt + zt^2)\end{aligned}$$

Using the boundary condition, the inventory level *I(t)* as

$$I(t) = Re^{\theta(t_1-t)} + (1-e^{\theta(t_1-t)})\left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right) + (t_1 e^{\theta(t_1-t)} - t)\left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) - \frac{z}{\theta}\left(t^2 - {t_1}^2 e^{\theta(t_1-t)}\right)$$

During $[0,t_1]$, the total costs are as follows:

(i) Fixed cost of stocks per transfer to display area from warehouse $= s$.

(ii) Holding cost is

$$
\begin{aligned}
= h\int_0^{t_1} I(t)dt &= \frac{-hR(1-e^{\theta t_1})}{\theta} + h\left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right)\left(t_1 + \frac{(1-e^{\theta t_1})}{\theta}\right) - h\Big(\frac{t_1(1-e^{\theta t_1})}{\theta} \\
&+ \frac{t_1^2}{2}\Big)\left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) - \frac{hz}{\theta}\Big(\frac{{t_1}^2(1-e^{\theta t_1})}{\theta} + \frac{{t_1}^3}{3}\Big)
\end{aligned}
$$

(iii) The revenue per cycle is

$$
= (p-c)\int_0^{t_1} D(t)dt = (p-c)\int_0^{t_1}(x+yt+zt^2)dt = (p-c)\left(xt_1 + y\frac{t_1^2}{2} + z\frac{t_1^3}{3}\right)
$$

From $I(0)=q+R$, one has

$$
\begin{aligned}
q+R &= Re^{\theta t_1} + (1-e^{\theta t_1})\left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right) + t_1e^{\theta t_1}\left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) + \frac{z}{\theta}{t_1}^2e^{\theta t_1} \\
i.e., \quad q &= Re^{\theta t_1} + (1-e^{\theta t_1})\left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right) + t_1e^{\theta t_1}\left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) + \frac{z}{\theta}{t_1}^2e^{\theta t_1} - R
\end{aligned}
$$

(iv) Stock holding cost in the warehouse is

$$
\begin{aligned}
= h_1\left[\frac{n(n-1)}{2}q\right]t_1 &= h_1\Big[\frac{n(n-1)}{2}\Big(Re^{\theta t_1} + (1-e^{\theta t_1})\left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right) \\
&+ t_1e^{\theta t_1}\left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) + \frac{z}{\theta}{t_1}^2e^{\theta t_1} - R\Big)\Big]t_1
\end{aligned}
$$

The average profit function per unit time is

$AP_2(n,t_1) = \frac{1}{T}$[revenue-(total cost in warehouse)-(total cost in display area)] (where T=nt_1)

$$
\begin{aligned}
&= (p-c)\left(x + y\frac{t_1}{2} + z\frac{t_1^2}{3}\right) - \Big[\frac{S}{nt_1} + h_1\Big(\frac{(n-1)}{2}\Big(Re^{\theta t_1} + (1-e^{\theta(t_1-t)})\left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right) \\
&+ t_1e^{\theta t_1}\left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) + \frac{z}{\theta}{t_1}^2e^{\theta t_1} - R\Big)\Big)\Big] - \frac{s}{t_1} - h\Big[-\frac{R(1-e^{\theta t_1})}{\theta t_1} + \left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right)\Big(1 \\
&+ \frac{(1-e^{\theta t_1})}{\theta t_1}\Big) - \left(\frac{(1-e^{\theta t_1})}{\theta} + \frac{t_1}{2}\right)\left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) - \frac{z}{\theta}\Big(\frac{t_1^2}{3} + \frac{t_1(1-e^{\theta t_1})}{\theta}\Big)\Big]
\end{aligned}
$$

which is to maximize with respect to the decision variables n and t_1. We have made the following lemma to make the global optimum solution for it.

Lemma 2

$AP_2(n^*, {t_1}^*)$ will have the global maximum (where n^* and ${t_1}^*$ are optimal values of n and t_1) if following conditions are satisfied

$$(i)\frac{S}{n^2t_1^2}+\frac{h}{2}e^{\theta t_1}\left(z{t_1}^2+\left(y+\frac{2z}{\theta}\right)t_1+\left(R\theta+\frac{y}{\theta}\right)\right)>\frac{2S}{n^3t_1}+he^{\theta t_1}\Big(\frac{z(\theta t_1+1)}{\theta^2}+\frac{e^{-\theta t}\left(\frac{y}{\theta}-\frac{2z}{\theta^2}-x\right)}{2}\Big)$$

and

$$(ii)\frac{2(p-c)z}{3}+\frac{2hz}{3\theta}+\frac{S}{n^2{t_1}^2}+\alpha e^{\theta(t_1-t)}+(\beta+\gamma)e^{\theta t_1}+\delta>\frac{2}{{t_1}^3}\left(\frac{S}{n}+s\right)+\frac{h_1Re^{\theta t_1}\theta}{2}[(n-1)\theta-1]$$

Proof

From the necessary condition of the optimal solution, $\frac{\partial AP_2(n,t_1)}{\partial n}=0$.

i.e.,

$$\begin{aligned}\frac{\partial AP_2(n,t_1)}{\partial n} &= \frac{S}{n^2t_1}-\frac{h_1}{2}\Big[Re^{\theta t_1}+(1-e^{\theta t_1})\left(\frac{y}{\theta^2}+\frac{2z}{\theta^3}-\frac{x}{\theta}\right)+t_1e^{\theta t_1}\Big(\frac{y}{\theta}+\frac{2z}{\theta^2}\Big)+\frac{zt_1^2e^{\theta t_1}}{\theta}\\ &- R\Big]=0\end{aligned}$$

which gives

$$n=\sqrt{\frac{2S}{h_1t_1(Re^{\theta t_1}+(1-e^{\theta t_1})\left(\frac{y}{\theta^2}+\frac{2z}{\theta^3}-\frac{x}{\theta}\right)+t_1e^{\theta t_1}\Big(\frac{y}{\theta}+\frac{2z}{\theta^2}\Big)+\frac{zt_1^2e^{\theta t_1}}{\theta}-R)}}$$

For the second decision variable t_1, $\frac{\partial AP_2(n,t_1)}{\partial t_1}=0$ gives

$$i.e., a_1{t_1}^6+a_2{t_1}^5+a_3{t_1}^4+a_4{t_1}^3+a_5{t_1}^2+a_6=0$$

[See Appendix C for the values of $\alpha, \beta, \gamma, \delta, a_1, a_2, a_3, a_4, a_5$, and a_6.]

Now ${t_1}^*$ will be obtained if $\xi_2({t_1}^*)=0$ where $\frac{\partial AP_2(n,t_1)}{\partial t_1}=\xi_2(t_1)$.

To obtain the global maximum, one has to check the sufficient conditions. Thus, the second order partial derivatives of $AP_2(n,t_1)$ with respect to n and t_1 are calculated which are as follows:

$$\frac{\partial^2 AP_2(n,t_1)}{\partial n^2} = \frac{-2S}{n^3t_1},$$

$$\frac{\partial^2 AP_2(n,t_1)}{\partial {t_1}^2} = \frac{\partial^2 \lambda_1}{\partial {t_1}^2} + \frac{\partial^2 \lambda_2}{\partial {t_1}^2}$$

where

$$\begin{aligned}\frac{\partial^2 \lambda_1}{\partial {t_1}^2} &= \frac{2(p-c)z}{3} - \frac{2S}{n{t_1}^3} - \frac{h_1(n-1)}{2}\Big[R\theta^2 e^{\theta t_1} - \Big(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\Big)e^{\theta(t_1-t)}\theta + \Big(\frac{y}{\theta} - \frac{2z}{\theta^2}\Big)e^{\theta t_1}\theta(t_1\theta \\ &+ 2) + \frac{z}{\theta}e^{\theta t_1}(4t_1\theta + {t_1}^2\theta^2 + 2)\Big]\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 \lambda_2}{\partial {t_1}^2} &= \frac{hR}{\theta}\left[\frac{2(1-e^{\theta t_1})}{{t_1}^3} + \frac{2\theta e^{\theta t_1}}{{t_1}^2} - \frac{\theta^2 e^{\theta t_1}}{t_1}\right] - \frac{2s}{t_1^3} - h\left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right)\Big[\frac{2e^{\theta t_1}}{{t_1}^2} + \frac{2(1-e^{\theta t_1})}{\theta {t_1}^3} \\ &- \frac{\theta e^{\theta t_1}}{t_1}\Big] - h\left(y - \frac{2z}{\theta}\right)e^{\theta t_1} + \frac{hz}{\theta}\left(-t_1\theta e^{\theta t_1} - 2e^{\theta t_1} + \frac{2}{3}\right).\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 AP_2(n,t_1)}{\partial n \partial t_1} &= -\frac{S}{n^2{t_1}^2} - \frac{h_1}{2}\Big[R\theta e^{\theta t_1} - \left(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\right)e^{\theta(t_1-t)} + \Big(\frac{y}{\theta} - \frac{2z}{\theta^2}\Big)e^{\theta t_1}(1+\theta t_1) \\ &+ \frac{z}{\theta}t_1 e^{\theta t_1}(\theta t_1 + 2)\Big]\end{aligned}$$

The sufficient conditions for the optimum solution of $AP_2(n,t_1)$ are $\frac{\partial^2 AP_2(n,t_1)}{\partial n^2} < 0$ and $\frac{\partial^2 AP_2}{\partial n^2}\frac{\partial^2 AP_2}{\partial {t_1}^2} - (\frac{\partial^2 AP_2}{\partial n \partial t_1})^2 > 0$.

Now

$$\frac{\partial^2 AP_2(n,t_1)}{\partial n^2} = \frac{-2S}{n^3 t_1} < 0$$

We have to show $\frac{\partial^2 AP_2}{\partial n^2}\frac{\partial^2 AP_2}{\partial {t_1}^2} - (\frac{\partial^2 AP_2}{\partial n \partial t_1})^2 > 0$.

For the proof of this above condition, if $\frac{\partial^2 AP_2}{\partial n^2} > \frac{\partial^2 AP_2}{\partial n \partial t_1}$ and $\frac{\partial^2 AP_2}{\partial {t_1}^2} > \frac{\partial^2 AP_2}{\partial n \partial t_1}$, then the conditions hold.

Now

$$\begin{aligned}\frac{\partial^2 AP_2(n,t_1)}{\partial n \partial t_1} &= -\frac{S}{n^2{t_1}^2} - \frac{h_1}{2}\Big[R\theta e^{\theta t_1} - \left(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\right)e^{\theta(t_1-t)} + \Big(\frac{y}{\theta} - \frac{2z}{\theta^2}\Big)e^{\theta t_1}(\theta t_1 + 1) \\ &+ \frac{z}{\theta}t_1 e^{\theta t_1}(\theta t_1 + 2)\Big]\end{aligned}$$

which can be written as

$$\frac{\partial^2 AP_2(n,t_1)}{\partial n \partial t_1} = \frac{\partial^2 AP_2(n,t_1)}{\partial n^2} - \xi_3.$$

where

$$\begin{aligned}\xi_3 &= \frac{S}{n^2{t_1}^2} + \frac{h_1}{2}\left[R\theta e^{\theta t_1} - \left(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\right)e^{\theta(t_1-t)} + \left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right)e^{\theta t_1}(\theta t_1 + 1) + \frac{z}{\theta}t_1 e^{\theta t_1}(\theta t_1 + 2)\right] \\ &- \frac{2S}{n^3 t_1}\end{aligned}$$

$\frac{\partial^2 AP_2}{\partial n^2} > \frac{\partial^2 AP_2}{\partial n \partial t_1}$ will hold if $\xi_3 > 0$.

Now $\xi_3 > 0$ will exist if

$$\frac{S}{n^2 t_1^2} + \frac{h}{2}e^{\theta t_1}\left(z{t_1}^2 + \left(y + \frac{2z}{\theta}\right)t_1 + \left(R\theta + \frac{y}{\theta}\right)\right) > \frac{2S}{n^3 t_1} + he^{\theta t_1}\left(\frac{z(\theta t_1 + 1)}{\theta^2} + \frac{e^{-\theta t}\left(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\right)}{2}\right)$$

Similarly,

$$\frac{\partial^2 AP_2(n,t_1)}{\partial n \partial t_1} = \frac{\partial^2 AP_2(n,t_1)}{\partial {t_1}^2} - \xi_4$$

where

$$\begin{aligned}\xi_4 &= \frac{2(p-c)z}{3} - \frac{2S}{n{t_1}^3} - \frac{h_1(n-1)}{2}\Big[R\theta^2 e^{\theta t_1} - \left(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\right)e^{\theta(t_1-t)}\theta + \Big(\frac{y}{\theta} - \frac{2z}{\theta^2}\Big)e^{\theta t_1}\theta(t_1\theta \\ &+ 2) + \frac{z}{\theta}e^{\theta t_1}(4t_1\theta + {t_1}^2\theta^2 + 2)\Big] - \frac{2s}{t_1^3} + \frac{hR}{\theta}\Big[\frac{2(1-e^{\theta t_1})}{{t_1}^3} + \frac{2\theta e^{\theta t_1}}{{t_1}^2} - \frac{\theta^2 e^{\theta t_1}}{t_1}\Big] - h\Big(\frac{y}{\theta^2} - \frac{2z}{\theta^3} \\ &- \frac{x}{\theta}\Big)\Big[\frac{2e^{\theta t_1}}{{t_1}^2} + \frac{2(1-e^{\theta t_1})}{\theta {t_1}^3} - \frac{\theta e^{\theta t_1}}{t_1}\Big] - h\left(y - \frac{2z}{\theta}\right)e^{\theta t_1} + \frac{hz}{\theta}\left(-t_1\theta e^{\theta t_1} - 2e^{\theta t_1} + \frac{2}{3}\right) + \frac{S}{n^2{t_1}^2} \\ &+ \frac{h_1}{2}\Big[R\theta e^{\theta t_1} - \left(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\right)e^{\theta(t_1-t)} + \left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right)e^{\theta t_1}(\theta t_1 + 1) + \frac{z}{\theta}t_1 e^{\theta t_1}(\theta t_1 + 2)\Big]\end{aligned}$$

Now $\frac{\partial^2 AP_2}{\partial {t_1}^2} > \frac{\partial^2 AP_2}{\partial n \partial t_1}$ will exist if $\xi_4 > 0$.

i.e., if

$$\frac{2(p-c)z}{3} + \frac{2hz}{3\theta} + \frac{S}{n^2{t_1}^2} + \alpha e^{\theta(t_1-t)} + (\beta+\gamma)e^{\theta t_1} + \delta > \frac{2}{{t_1}^3}\left(\frac{S}{n} + s\right) + \frac{h_1 R e^{\theta t_1}\theta}{2}[(n-1)\theta - 1]$$

Therefore, $AP_2(n^*, {t_1}^*)$ will have the global maximum (where n^* and ${t_1}^*$ are optimal values of n and t_1) if the conditions hold.

Case III

In this section, demand of products is a function of selling-price. In general, selling-price decreases means demand of products increases and vice-versa. Customers are more affective to that product whose selling-price is low. Therefore, demand can be a function of selling-price. Here, demand is taken to be as quadratic function of selling-price.

The demand function is $f(p) = D(p) = x_1 - y_1 p - z_1 p^2$

The governing differential equation of this inventory model is

$$\begin{aligned} \frac{dI(t)}{dt} + \theta I(t) &= -f(p), \quad 0 \le t \le t_1, \quad I(t_1) = R \\ &= -(x_1 - y_1 p - z_1 p^2) \end{aligned}$$

Utilizing the boundary condition, the inventory level *I(t)* as

$$I(t) = Re^{\theta(t_1 - t)} + \frac{(x_1 - y_1 p - z_1 p^2)(e^{\theta(t_1 - t)} - 1)}{\theta}$$

During $[0,t_1]$, the total costs are as follows:

(i) Fixed cost of stocks per transfer to display area from warehouse $= s$.

(ii) Holding cost is

$$= h \int_0^{t_1} I(t)dt = hR\frac{(e^{\theta t_1} - 1)}{\theta} + \frac{h(x_1 - y_1 p - z_1 p^2)}{\theta}\left(\frac{(e^{\theta} t_1 - 1)}{\theta} - t_1\right)$$

(iii) The revenue per cycle is

$$= (p - c)\int_0^{t_1} D(p)dt = (p - c)\int_0^{t_1} (x_1 - y_1 p - z_1 p^2)dt = (p - c)(x_1 - y_1 p - z_1 p^2)t_1$$

Using *I(0)=q+R*, one has

$$\begin{aligned} q+R &= Re^{\theta t_1} + (x_1 - y_1 p - z_1 p^2)\frac{(e^{\theta t_1}-1)}{\theta} \\ i.e., \quad q &= Re^{\theta t_1} + (x_1 - y_1 p - z_1 p^2)\frac{(e^{\theta t_1}-1)}{\theta} - R \end{aligned}$$

(iv) Stock holding cost in the warehouse is

$$= h_1\left[\frac{n(n-1)}{2}q\right]t_1 = h_1 t_1 \frac{n(n-1)}{2}[Re^{\theta t_1} + (x_1 - y_1 p - z_1 p^2)\frac{(e^{\theta t_1}-1)}{\theta} - R]$$

The average profit function per unit time is

$AP_3(n,p,t_1)$=$\frac{1}{T}$[revenue-(total cost in warehouse)-(total cost in display area)] (where T=nt_1)

$$\begin{aligned} &= (p-c)(x_1 - y_1 p - z_1 p^2) - \Big(\frac{S}{nt_1} + h_1\frac{(n-1)}{2}[Re^{\theta t_1} + \frac{(e^{\theta t_1}-1)}{\theta}(x_1 - y_1 p - z_1 p^2) \\ &- R]\Big) - \frac{s}{t_1} - h\Big[R\frac{(e^{\theta t_1}-1)}{\theta t_1} - \frac{(x_1 - y_1 p - z_1 p^2)}{\theta}\Big(1 - \frac{(e^{\theta t_1}-1)}{\theta t_1}\Big)\Big] \end{aligned}$$

$AP_3(n,p,t_1)$ is to maximize with respect to the decision variables n, p, and t_1. To obtain the global optimum solution, Lemma 3 is formulated.

Lemma 3

$AP_3(n^*,p^*,{t_1}^*)$ will have the global maximum (where n^*, p^*, and ${t_1}^*$ are optimal values of n, p, and t_1) if following conditions are satisfied.

$$(i) 4S\theta > n^3 t_1(e^{\theta t_1}-1)h_1(2z_1 p - y),$$

$$(ii)\frac{2z_1\theta\left(\frac{h}{\theta}+3p\right)}{(e^{\theta t_1}-1)} > \left[h_1\left(z_1(n-1+p) - \frac{y_1}{2}\right) + \frac{2z_1\theta}{\theta t_1}\right],$$

and

$$\begin{aligned} (iii)\left[1 + \frac{sn}{S} + \frac{n{t_1}^3}{2S}\left(\frac{(n-1)\theta^2 e^{\theta t_1}}{2} + h\right)\left(R + \frac{D(p)}{\theta}\right)\right] l_1 l_3 + \frac{n{t_1}^3}{2S}\Big(l_3{l_4}^2 \\ + h{h_1}^2\left(\frac{y_1}{2} + z_1 p^2\right)^2\Big) < {l_5}^2 + \frac{n{t_1}^3 h_1}{S}\left(\frac{y_1}{2} + z_1 p\right) l_4 l_5. \end{aligned}$$

[See Appendix D for the values of $D(p), l_1, l_2, l_3, l_4$, and l_5.]

Proof

From the necessary conditions of the optimal solution, $\frac{\partial AP_3(n,p,t_1)}{\partial n} = 0$.

i.e.,

$$\frac{\partial AP_3(n,p,t_1)}{\partial n} = \frac{S}{n^2{t_1}^2} - \frac{h_1}{2}\left[Re^{\theta t_1} + D(p)\frac{(e^{\theta t_1}-1)}{\theta} - R\right] = 0$$

which gives

$$n = \sqrt{\frac{2S}{h_1{t_1}^2\left[Re^{\theta t_1} + D(p)\frac{(e^{\theta t_1}-1)}{\theta} - R\right]}}$$

For the decision variable p, $\frac{\partial AP_3(n,p,t_1)}{\partial p} = 0$ gives

$$i.e., p = \frac{\theta(2y_1 - cy_1 - x_1 - 2cz_1)}{2z_1\left[\frac{h_1(n-1)}{2}\frac{(e^{\theta t_1}-1)}{\theta} - h\left(\frac{(e^{\theta t_1}-1)}{\theta t_1} - 1\right)\right]}$$

For another decision variable t_1, $\frac{\partial AP_3(n,p,t_1)}{\partial t_1} = 0$ gives

$$i.e., \frac{S + n^2s^2}{{t_1}^2} = \left(R + \frac{D(p)}{\theta}\right)\left[\frac{h_1(n-1)}{2}\theta e^{\theta t_1} - h\left(\frac{e^{\theta t_1}}{t_1} - \frac{(e^{\theta t_1}-1)}{\theta {t_1}^2}\right)\right]$$

Now ${t_1}^*$ will be obtained if $\xi_3({t_1}^*) = 0$, where $\frac{\partial AP_3(n,p,t_1)}{\partial t_1} = \xi_3(t_1)$.

To obtain the sufficient conditions, the second order partial derivatives of $AP_3(n,p,t_1)$ with respect to n and t_1 is calculated, which are as follows:

$$\begin{aligned}
\frac{\partial^2 AP_3(n,p,t_1)}{\partial n^2} &= \frac{-2S}{n^3 t_1}, \\
\frac{\partial^2 AP_3(n,p,t_1)}{\partial {t_1}^2} &= -\frac{2(S+n^2s^2)}{{t_1}^3} - \left(R + \frac{D(p)}{\theta}\right)\Big[\frac{h_1(n-1)}{2}\theta^2 e^{\theta t_1} - h\Big(\frac{\theta e^{\theta t_1}}{t_1} - \frac{2e^{\theta t_1}}{{t_1}^2} \\
&+ \frac{2(e^{\theta t_1}-1)}{\theta {t_1}^3}\Big)\Big], \\
\frac{\partial^2 AP_3(n,p,t_1)}{\partial p^2} &= \left(h_1(n-1) + \frac{2h}{\theta t_1}\right)\frac{(e^{\theta t_1}-1)}{\theta} z_1 - 2z_1\left(\frac{h}{\theta} + 3p\right), \\
\frac{\partial^2 AP_3(n,p,t_1)}{\partial n \partial t_1} &= \frac{2ns^2}{{t_1}^2} - \left(R + \frac{D(p)}{\theta}\right)\frac{h_1}{2}\theta e^{\theta t_1}, \\
\frac{\partial^2 AP_3(n,p,t_1)}{\partial p \partial t_1} &= \frac{(y_1 + 2z_1 p)h_1(n-1)\theta e^{\theta t_1}}{2},
\end{aligned}$$

and

$$\frac{\partial^2 AP_3(n,p,t_1)}{\partial n \partial p} = \frac{h_1}{2}\left((y_1 - 2z_1 p)\frac{(e^{\theta t_1}-1)}{\theta}\right).$$

The sufficient conditions for the optimum solution of $AP_3(n,t_1)$ are

$\frac{\partial^2 AP_3(n,p,t_1)}{\partial n^2} < 0$, $\frac{\partial^2 AP_3}{\partial n^2}\frac{\partial^2 AP_3}{\partial p^2} - (\frac{\partial^2 AP_3}{\partial n \partial p})^2 > 0$, and the value of third principal minor is

$$\begin{aligned} H &= \frac{-2S}{n{t_1}^3}\Big[\Big(1 + \frac{h_1(n-1)n{t_1}^3}{4S}(R\theta + D(p))\theta e^{\theta t_1} + \frac{sn}{S} + \frac{n{t_1}^3 h}{2S}\Big(R + \frac{D(p)}{\theta}\Big)l_1 l_3 \\ &+ \frac{n{t_1}^3 h}{2S}{h_1}^2\left(\frac{y_1}{2} + z_1 p\right)^2 - {l_5}^2 - \frac{n{t_1}^3 h_1}{S}\left(\frac{y_1}{2} + z_1 p\right) l_4 l_5 + \frac{n{t_1}^3}{2S} l_3 {l_4}^2\Big)\Big] < 0 \end{aligned}$$

Now $\frac{\partial^2 AP_3(n,p,t_1)}{\partial n^2} = \frac{-2S}{n^3 t_1} < 0$.

We have to show that $\frac{\partial^2 AP_3}{\partial n^2}\frac{\partial^2 AP_3}{\partial p^2} - (\frac{\partial^2 AP_3}{\partial n \partial p})^2 > 0$.

To prove the condition of second principal minor, if $\frac{\partial^2 AP_3}{\partial n^2} > \frac{\partial^2 AP_3}{\partial n \partial p}$ and

$\frac{\partial^2 AP_3}{\partial p^2} > \frac{\partial^2 AP_3}{\partial n \partial p}$, then the conditions hold.

Now

$$\frac{\partial^2 AP_3(n,p,t_1)}{\partial n \partial p} = \frac{h_1}{2}\left((y_1 - 2z_1 p)\frac{(e^{\theta t_1}-1)}{\theta}\right)$$

which can be written as

$$\frac{\partial^2 AP_3(n,p,t_1)}{\partial n \partial p} = \frac{\partial^2 AP_3(n,p,t_1)}{\partial n^2} - \xi_5.$$

where

$$\xi_5 = -\frac{2S}{n^3 t_1} - \frac{h_1}{2}\left((y_1 - 2z_1 p)\frac{(e^{\theta t_1}-1)}{\theta}\right)$$

$\frac{\partial^2 AP_3}{\partial n^2} > \frac{\partial^2 AP_3}{\partial n \partial p}$ will hold if $\xi_5 > 0$.

Now $\xi_5 > 0$ exists, when

$$4S\theta > n^3 t_1 (e^{\theta t_1} - 1) h_1 (2z_1 p - y)$$

Similarly,

$$\frac{\partial^2 AP_3(n,p,t_1)}{\partial n \partial p} = \frac{\partial^2 AP_3(n,p,t_1)}{\partial p^2} - \xi_6$$

where

$$\xi_6 = \frac{(e^{\theta t_1}-1)}{\theta}\left[\left(h_1(n-1)+\frac{2h}{\theta t_1}\right)z_1 - \frac{h_1}{2}(y_1 - 2z_1 p)\right] - 2z_1\left(\frac{h}{\theta}+3p\right)$$

$\frac{\partial^2 AP_3}{\partial p^2} > \frac{\partial^2 AP_3}{\partial n \partial p}$ will exist if $\xi_6 > 0$.

i.e., if

$$\frac{2z_1\theta\left(\frac{h}{\theta}+3p\right)}{(e^{\theta t_1}-1)} > \left[h_1\left(z_1(n-1+p)-\frac{y_1}{2}\right)+\frac{2z_1\theta}{\theta t_1}\right]$$

Similar as above, value of third principal minor i.e., $H < 0$ will be satisfied if

$$\left[1+\frac{sn}{S}+\frac{nt_1{}^3}{2S}\left(\frac{(n-1)\theta^2 e^{\theta t_1}}{2}+h\right)\left(R+\frac{D(p)}{\theta}\right)\right]l_1 l_3 + \frac{nt_1{}^3}{2S}\Bigg(l_3 l_4{}^2 \\ + hh_1{}^2\left(\frac{y_1}{2}+z_1 p^2\right)^2\Bigg) < l_5{}^2 + \frac{nt_1{}^3 h_1}{S}\left(\frac{y_1}{2}+z_1 p\right)l_4 l_5$$

Therefore, $AP_3(n^*, p^*, t_1{}^*)$ will have the global maximum (where n^*, p^*, and $t_1{}^*$ are optimal values of n, p, and t_1) if conditions hold.

Case IV

This section describes that demand of products is exponentially time-dependent. For example, electronic goods, fashionable clothes are those products whose demand rate may fluctuate with time. For new products, initially the demand is very high and then it decreases. That situation of demand can be represented by exponential demand pattern. Therefore, it can be observed that demand of products varies exponentially with time.

In this case, the demand function $f(t) = D(t) = x_2 e^{y_2 t}$

The governing differential equation of this inventory model is

$$\begin{aligned}\frac{dI(t)}{dt} + \theta I(t) &= -f(t), \quad 0 \le t \le t_1, \qquad I(t_1) = R \\ &= -x_2 e^{y_2 t}\end{aligned}$$

Using the boundary condition, the inventory level *I(t)* as

$$I(t) = Re^{\theta(t_1 - t)} + \frac{x_2}{(y_2+\theta)}(e^{(y_2+\theta)t_1 - \theta t} - e^{y_2 t})$$

During $[0,t_1]$, the total costs are as follows:

(i) Fixed cost of stocks per transfer to display area from warehouse = s.

(ii) Holding cost is

$$= h\int_0^{t_1} I(t)dt = \frac{hR}{\theta}(e^{\theta t_1} - 1) + \frac{hx_2}{(y_2+\theta)} e^{(y_2+\theta)t_1}\frac{(1 - e^{-\theta t_1})}{\theta} - \frac{hx_2}{(y_2+\theta)}\frac{(e^{y_2 t_1} - 1)}{y_2}$$

(iii) The revenue per cycle is

$$= (p-c)\int_0^{t_1} D(t)dt = (p-c)\int_0^{t_1} x_2 e^{y_2 t}dt = (p-c)x_2\left[\frac{e^{y_2 t_1}}{y_2} - \frac{1}{y_2}\right]$$

Applying *I(0)=q+R*,

$$\begin{aligned}q + R &= \frac{x_2}{(y_2+\theta)}(e^{(y_2+\theta)t_1} - 1) + Re^{\theta t_1} \\ i.e., \quad q &= \frac{x_2}{(y_2+\theta)}(e^{(y_2+\theta)t_1} - 1) + Re^{\theta t_1} - R\end{aligned}$$

(iv) Stock holding cost in the warehouse is

$$= h_1[\frac{n(n-1)}{2}q]t_1 = h_1 t_1 \frac{n(n-1)}{2}\Big[\frac{x_2}{(y_2+\theta)}(e^{(y_2+\theta)t_1} - 1) + Re^{\theta t_1} - R\Big]$$

The average profit function per unit time is

$AP_4(n,t_1)$=$\frac{1}{T}$[revenue-(total cost in warehouse)-(total cost in display area)] (where T=nt_1)

$$\begin{aligned}&= \frac{(p-c)x_2}{y_2 t_1}(e^{y_2 t_1} - 1) - \Big[\frac{S}{nt_1} + h_1\frac{(n-1)}{2}\Big(\frac{x_2}{(y_2+\theta)}(e^{(y_2+\theta)t_1} - 1) + Re^{\theta t_1} - R\Big)\Big] \\ &- \frac{s}{t_1} - \frac{h}{t_1}\Big[\frac{x_2}{(y_2+\theta)}e^{(y_2+\theta)t_1}\frac{(1 - e^{-\theta t_1})}{\theta} - \frac{x_2}{(y_2+\theta)}\frac{(e^{y_2 t_1} - 1)}{y_2} + \frac{R(e^{\theta t_1} - 1)}{\theta}\Big]\end{aligned}$$

$AP_4(n, t_1)$ is to maximize with respect to the decision variables n and t_1.

To obtain the global maximum solution, Lemma 4 is constructed.

Lemma 4

$AP_4(n^*, t_1{}^*)$ will have the global maximum solution (where n^* and $t_1{}^*$ are optimal values of n and t_1) if following conditions are satisfied

$$(i)e^{\theta t_1}(x_2 e^{y_2 t_1} + R\theta) > \frac{2S(2 - Sn)}{h_1 t_1 n^3}$$

and

$$(ii)\frac{(c-p)x_2 b_1}{y_2} > \frac{2S}{n t_1{}^3} + \frac{h_1(n-1)b_2}{2} + hb_3$$

[See Appendix E for the values of b_1, b_2, and b_3.]

Proof

From the necessary conditions of optimal solution, one has $\frac{\partial AP_4(n,t_1)}{\partial n} = 0$.

i.e.,

$$\frac{\partial AP_4(n, t_1)}{\partial n} = \frac{S}{n^2 t_1} - \frac{h_1}{2}\left(\frac{x_2}{(y_2+\theta)} e^{(y_2+\theta)t_1} + Re^{\theta t_1} - R\right) = 0$$

which implies

$$n = \sqrt{\frac{2S}{h_1 t_1\left(\frac{x_2}{(y_2+\theta)} e^{(y_2+\theta)t_1} + Re^{\theta t_1} - R\right)}}$$

For the other decision variable t_1, $\frac{\partial AP_4(n,t_1)}{\partial t_1} = 0$.

$$i.e., \quad \frac{(p-c)x_2}{y_2}\left(\frac{e^{y_2 t_1} y_2}{t_1} - \frac{(e^{y_2 t_1} - 1)}{t_1^2}\right) + \frac{\frac{S}{t_1{}^2}}{\sqrt{\frac{2S}{h_1 t_1\left(\frac{x_2}{(y_2+\theta)} e^{(y_2+\theta)t_1} + Re^{\theta t_1} - R\right)}}} + \frac{s}{t_1{}^2} - h_1(x_2 e^{(y_2+\theta)t_1}$$

$$+ \quad R\theta e^{\theta t_1})\frac{1}{2}\left(\sqrt{\frac{2S}{h_1 t_1\left(\frac{x_2}{(y_2+\theta)} e^{(y_2+\theta)t_1} + Re^{\theta t_1} - R\right)}} - 1\right) - h\Big[\frac{x_2}{(y_2+\theta)}\Big(-e^{(y_2+\theta)t_1}\frac{(1-e^{\theta t_1})}{\theta t_1{}^2}$$

$$+\frac{e^{(y_2+2\theta)t_1}}{t_1} + \frac{(1-e^{\theta t_1})}{\theta}\frac{e^{(y_2+\theta)t_1}}{t_1}(y_2+\theta)\Big) - e^{y_2 t_1} + Re^{\theta t_1}\Big] = 0$$

Now ${t_1}^*$ can be obtained if $\xi_4({t_1}^*) = 0$ where $\frac{\partial AP_4(n,t_1)}{\partial t_1} = \xi_4(t_1)$.

From the sufficient conditions, the second order partial derivatives of $AP_4(n, t_1)$ with respect to n and t_1 are as follows:

$$\frac{\partial^2 AP_4(n,t_1)}{\partial n^2} = \frac{-2S}{n^3 t_1},$$

$$\begin{aligned}
\frac{\partial^2 AP_4(n,t_1)}{\partial {t_1}^2} &= \frac{(p-c)x_2}{y_2}\left[\frac{2(e^{y_2 t_1}-1)}{{t_1}^3} - \frac{2e^{y_2 t_1} y_2}{{t_1}^2} + \frac{e^{y_2 t_1} {y_2}^2}{t_1}\right] - \frac{2S}{n{t_1}^3} - h_1\frac{(n-1)}{2}[x_2(y_2 \\
&+ \theta)e^{(y_2+\theta)t_1} + Re^{\theta t_1}\theta^2] - h\Big[e^{(y_2+\theta)t_1}(-\theta e^{-\theta t_1} - y_2 e^{y_2 t_1})\frac{x_2}{y_2+\theta} + x_2(y_2 \\
&+ \theta)\Big(\frac{(1-e^{-\theta t_1})}{\theta} + \frac{(1-e^{y_2 t_1})}{y_2}\Big) + 2(e^{-\theta t_1} - e^{y_2 t_1})(y_2+\theta)\frac{x_2}{(y_2+\theta)} \\
&+ R\theta e^{\theta t_1}\Big],
\end{aligned}$$

and

$$\frac{\partial^2 AP_4(n,t_1)}{\partial n \partial t_1} = -\frac{S}{n^2 {t_1}^2} - \frac{h_1}{2}[x_2 e^{(y_2+\theta)t_1} + Re^{\theta t_1}\theta].$$

The sufficient conditions for the optimum solution of $AP_4(n, t_1)$ are $\frac{\partial^2 AP_4(n,t_1)}{\partial n^2} < 0$ and $\frac{\partial^2 AP_4}{\partial n^2}\frac{\partial^2 AP_4}{\partial {t_1}^2} - (\frac{\partial^2 AP_4}{\partial n \partial t_1})^2 > 0$.

$$\frac{\partial^2 AP_4(n,t_1)}{\partial n^2} = \frac{-2S}{n^3 t_1} < 0$$

One has to prove $\frac{\partial^2 AP_4}{\partial n^2}\frac{\partial^2 AP_4}{\partial {t_1}^2} - (\frac{\partial^2 AP_4}{\partial n \partial t_1})^2 > 0$.

To justify above condition, if $\frac{\partial^2 AP_4}{\partial n^2} > \frac{\partial^2 AP_4}{\partial n \partial t_1}$ and $\frac{\partial^2 AP_4}{\partial {t_1}^2} > \frac{\partial^2 AP_4}{\partial n \partial t_1}$, then optimality conditions for second principal minor are satisfied.

$$\frac{\partial^2 AP_4(n,t_1)}{\partial n \partial t_1} = -\frac{S}{n^2 {t_1}^2} - \frac{h_1}{2}[x_2 e^{(y_2+\theta)t_1} + Re^{\theta t_1}\theta]$$

which can be written as

$$\frac{\partial^2 AP_4(n,t_1)}{\partial n \partial t_1} = \frac{\partial^2 AP_4(n,t_1)}{\partial n^2} - \xi_7.$$

where

$$\xi_7 = \frac{2S}{n^3 t_1} - \frac{S}{n^2 {t_1}^2} - \frac{h_1}{2}[x_2 e^{(y_2+\theta)t_1} + Re^{\theta t_1}\theta]$$

$\frac{\partial^2 AP_4}{\partial n^2} > \frac{\partial^2 AP_4}{\partial n \partial t_1}$ will hold if $\xi_7 > 0$.

Now $\xi_7 > 0$ will exist if

$$e^{\theta t_1}(x_2 e^{y_2 t_1} + R\theta) > \frac{2S(2 - Sn)}{h_1 t_1 n^3}.$$

Similarly,

$$\frac{\partial^2 AP_4(n, t_1)}{\partial n \partial t_1} = \frac{\partial^2 AP_4(n, t_1)}{\partial {t_1}^2} - \xi_8$$

where

$$\begin{aligned}\xi_8 &= \frac{(c-p)x_2}{y_2}\left[\frac{2(e^{y_2 t_1} - 1)}{{t_1}^3} - \frac{2e^{y_2 t_1} y_2}{{t_1}^2} + \frac{e^{y_2 t_1}{y_2}^2}{t_1}\right] + \frac{2S}{n{t_1}^3} + h_1\frac{(n-1)}{2}[Re^{\theta t_1}\theta^2 + x_2(y_2 \\ &+ \theta)e^{(y_2+\theta)t_1}] + h\Big[e^{(y_2+\theta)t_1}(-\theta e^{-\theta t_1} - y_2 e^{y_2 t_1})\frac{x_2}{y_2+\theta} + \left(\frac{(1-e^{-\theta t_1})}{\theta} + \frac{(1-e^{y_2 t_1})}{y_2}\right)x_2(y_2 \\ &+ \theta) + 2(e^{-\theta t_1} - e^{y_2 t_1})(y_2+\theta)\frac{x_2}{(y_2+\theta)} + R\theta e^{\theta t_1}\Big] - \frac{S}{n^2{t_1}^2} - \frac{h_1}{2}[x_2 e^{(y_2+\theta)t_1} + Re^{\theta t_1}\theta]\end{aligned}$$

$\frac{\partial^2 AP_4}{\partial {t_1}^2} > \frac{\partial^2 AP_4}{\partial n \partial t_1}$ will exist if $\xi_8 > 0$, i.e., if

$$\frac{(c-p)x_2 b_1}{y_2} > \frac{2S}{n{t_1}^3} + \frac{h_1(n-1)b_2}{2} + hb_3$$

Therefore, $AP_4(n^*, {t_1}^*)$ will have the global maximum (where n^* and ${t_1}^*$ are optimal values of n and t_1) solution if the conditions hold.

2.3 Numerical examples

By using numerical data from Goyal and Chang (2009) model, this chapter formulates the optimal value of average profit and optimal order quantity.

Example 1(a) The values for the parameters are taken as follows:

h = \$0.6/unit/unit time, h_1 = \$0.3/unit/unit time, c = \$1.0/unit, $x = 100$, $y = 10$, $z = 0.1$, $x_1 = 800$, $y_1 = 50$, $z_1 = 0.1$, $R = 1$ unit, $a = 0.1$, $b = 0.2$, s = \$10/transfer, S = \$100/order. Optimal solution is AP_1 = \$3432.74, $n^* = 10$, ${t_1}^* = 0.33$ year, p^* = \$9.3/unit, and optimal order quantity $Q^* = 114.325$ units. Figure 2.2, Figure 2.3, and Figure 2.4 shows the optimality of average Profit (AP_1).

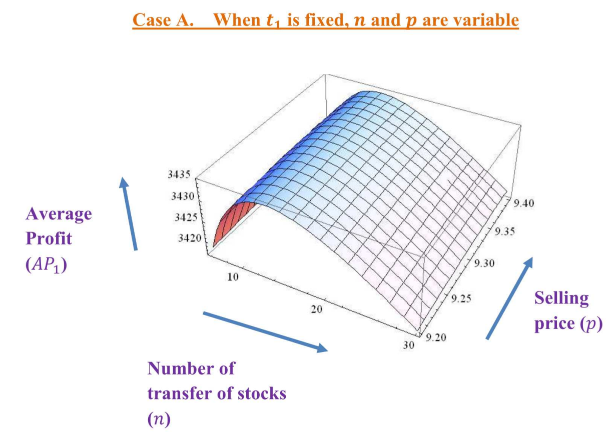

Figure 2.2: Average Profit (AP_1) versus number of transfer of stocks (n) and selling-price (p)

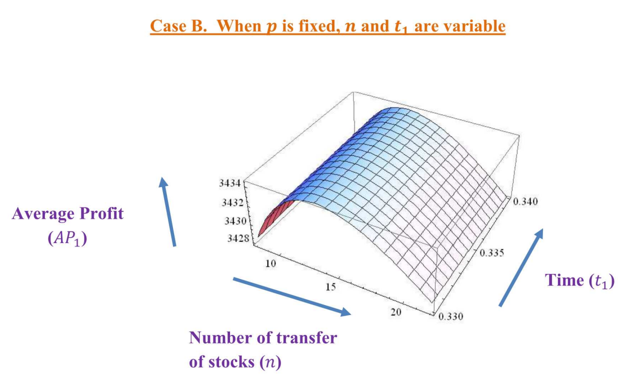

Figure 2.3: Average Profit (AP_1) versus number of transfer of stocks (n) and time (t_1)

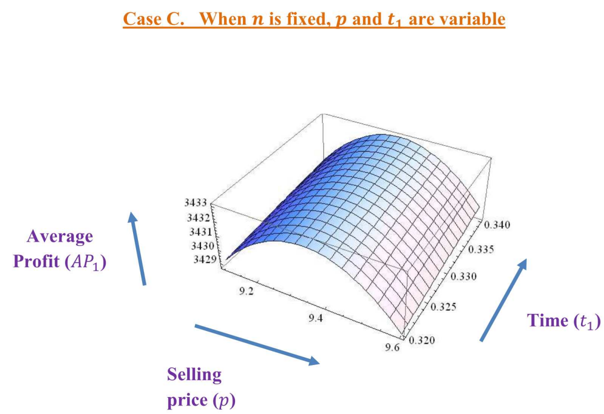

Figure 2.4: Average Profit (AP_1) versus selling-price (p) and time (t_1)

Example 2(a) The values for the parameters are taken as follows:

h = \$0.6/unit/unit time, h_1 = \$0.3/unit/unit time, c = \$1.0/unit/unit time, p = \$3.0/unit, $x = 3000$, $y = 40$, $z = 0.1$, $R = 1$ unit, $a = 0.1$, $b = 0.2$, s = \$10/transfer, S = \$100/order. Then the optimal solution is $AP_2 = \$2983.81$, $n^* = 3$, ${t_1}^* = 0.206$ year, and optimal order quantity $Q^* = 1069.69$ units. Figure 2.5 shows the optimality of average Profit (AP_2).

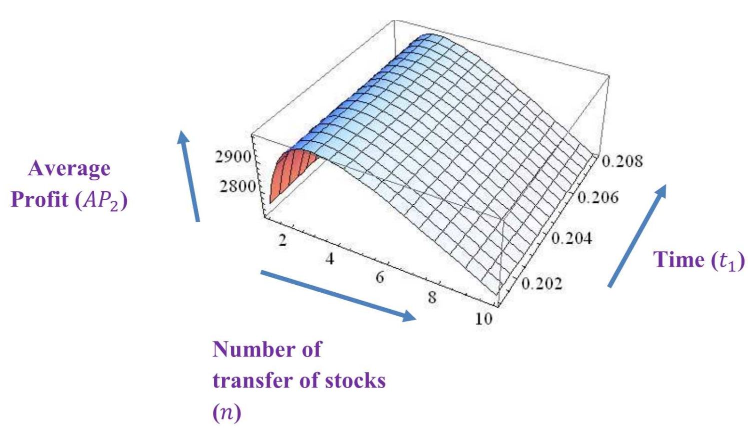

Figure 2.5: Average Profit (AP_2) versus number of transfer of stocks (n) and time (t_1)

Example 3(a) The values for the parameters are taken as follows:

h = \$0.6/unit/unit time, h_1 = \$0.3/unit/unit time, c = \$1.0/unit, $x_1 = 700$, $y_1 = 40$, $z_1 = 0.1$, $R = 1$ unit, $a = 0.1$, $b = 0.2$, s = \$10/transfer, S = \$100/order. Then the optimal solution is AP_3 = \$2473.03, $n^* = 3$, and p^* = \$9.1/unit, $t_1{}^* = 0.42$ year, and optimal order quantity $Q^* = 426.406$ units. Figure 2.6, Figure 2.7, and Figure 2.8 shows the optimality of average Profit (AP_3).

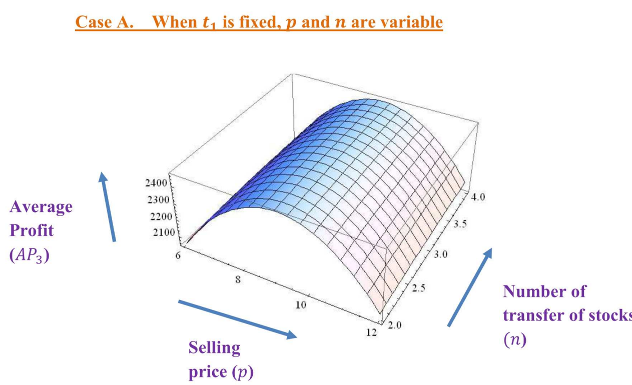

Figure 2.6: Average Profit (AP_3) versus selling-price (p) and number of transfer of stocks (n)

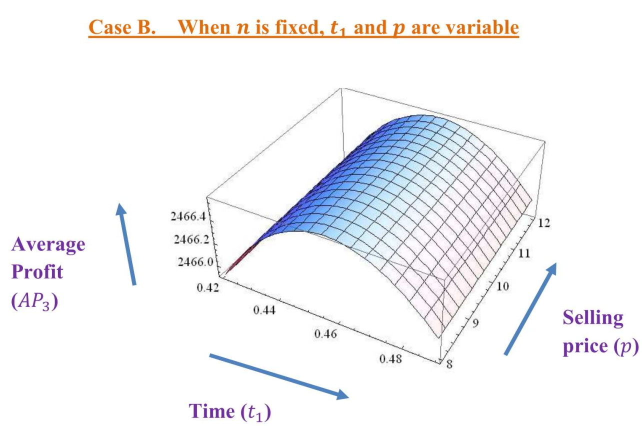

Figure 2.7: Average Profit (AP_3) versus time (t_1) and selling-price (p)

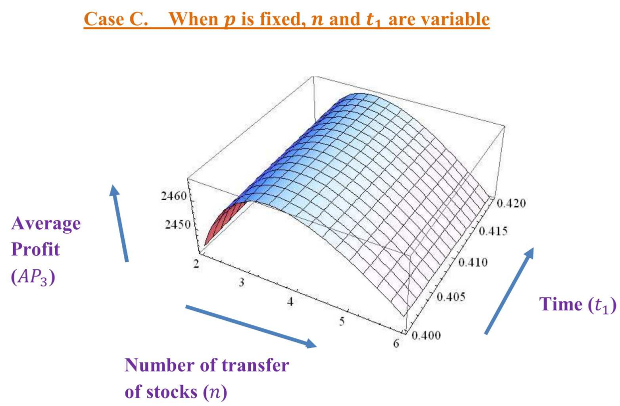

Figure 2.8: Average Profit (AP_3) versus number of transfer of stocks (n) and time (t_1)

Example 4(a) The values for the parameters are taken as follows:

h = \$0.6/unit/unit time, h_1 = \$0.3/unit/unit time, p = \$3/unit, c = \$1.0/unit, $x_2 = 2000$, $y_2 = 0.1$, $R = 1$ unit, $a = 0.1$, $b = 0.2$, s = \$10/transfer, S = \$100/order. Then the optimal solution is AP_4 = \$2634.91, $n^* = 2$, and ${t_1}^* = 0.3$ year, and optimal order quantity $Q^* = 934.702$ units. Figure 2.9 shows the optimality of average Profit (AP_4).

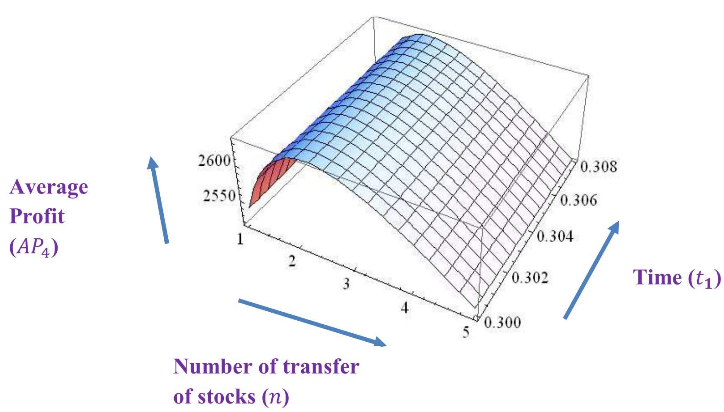

Figure 2.9: Average Profit (AP_4) versus number of transfer of stocks (n) and time (t_1)

Case Study

This model determined retailer's optimal ordering quantity through multi-delivery policy by assuming demand of products is time, price, and time-price dependent. In addition, it is also considered that deterioration rate follows uniform distribution. This model highlights the concept of uniformly distributed deterioration. This fact can be easily observed by considering a real example. Frozen fish is one of the examples of this concept. Generally, small fishes are preserved in ice. In that case, deterioration rate of fishes is constant throughout the time. For this reason, deterioration of products is measured as uniform in this model.

Example 1(b) The values for the parameters are taken as follows:

h = \$0.8/unit/unit time, h_1 = \$0.1/unit/unit time, c = \$3.0/unit, $x = 200$, $y = 30$, $z = 0.2$, $x_1 = 1000$, $y_1 = 80$, $z_1 = 0.3$, $R = 2$ unit, $a = 0.2$, $b = 0.3$, s = \$20/transfer, S = \$90/order. Then the optimal solution is AP_1 = \$2611.79, $n^* = 13$, ${t_1}^* = 0.4$ year, p^* = \$9/unit, and optimal order quantity $Q^* = 347$ units. Figure 2.10, Figure 2.11, and Figure 2.12 shows the optimality of average Profit (AP_1).

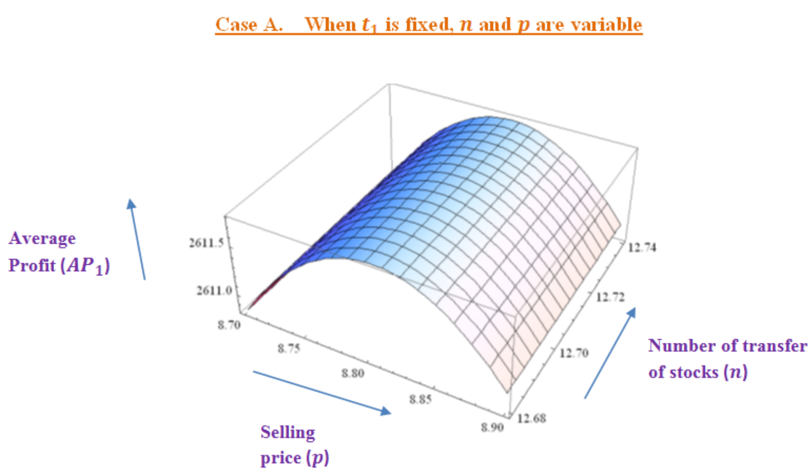

Figure 2.10: Average Profit (AP_1) versus selling-price (p) and number of transfer of stocks (n)

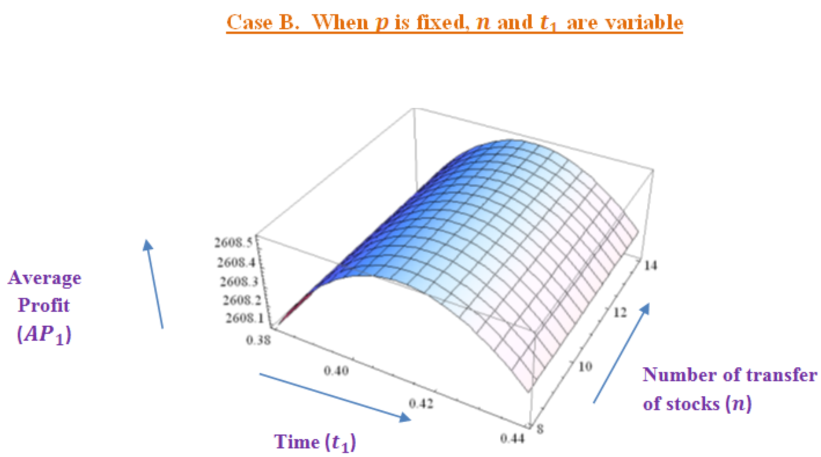

Figure 2.11: Average Profit (AP_1) versus time (t_1) and number of transfer of stocks (n)

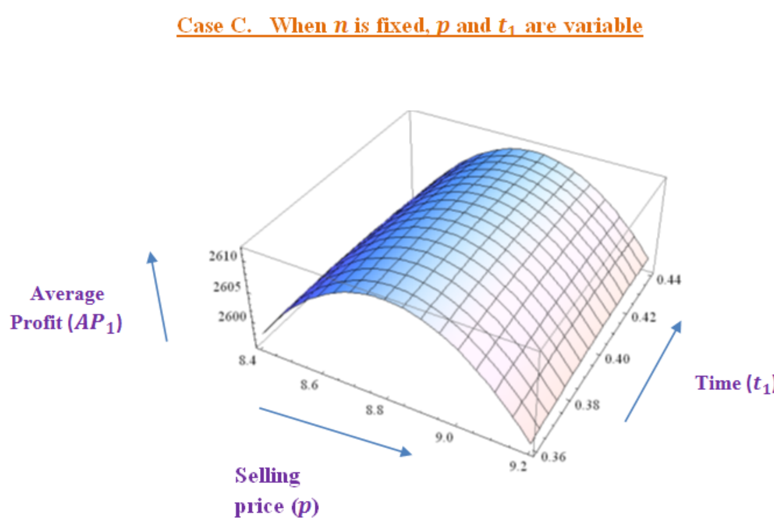

Figure 2.12: Average Profit (AP_1) versus selling-price (p) and time (t_1)

Example 2(b) The values for the parameters are taken as follows:

$h = \$0.8$/unit/unit time, $h_1 = \$0.1$/unit/unit time, $p = \$5$/unit, $c = \$3.0$/unit, $x = 1000$, $y = 50$, $z = 0.2$, $R = 2$ unit, $a = 0.2$, $b = 0.3$, $s = \$20$/transfer, $S = \$90$/order. Then the optimal solution is $AP_2 = \$1703.83$, $n^* = 5$, ${t_1}^* = 0.25$ year, and optimal order quantity $Q^* = 1298$ units. Figure 2.13 shows the optimality of average Profit (AP_2).

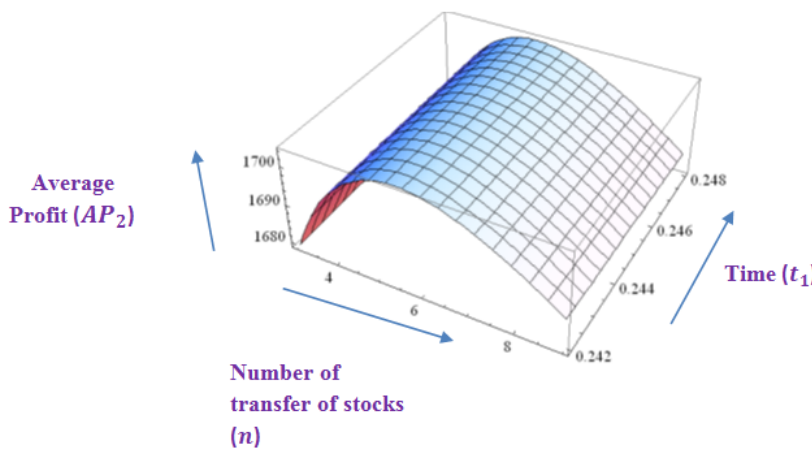

Figure 2.13: Average Profit (AP_2) versus number of transfer of stocks (n) and time (t_1)

Example 3(b) The values for the parameters are taken as follows:

$h = \$0.8$/unit/unit time, $h_1 = \$0.1$/unit/unit time, $c = \$3.0$/unit, $x_1 = 420$, $y_1 = 50$, $z_1 = 0.2$, $R = 2$ unit, $a = 0.2$, $b = 0.3$, $s = \$20$/transfer, $S = \$90$/order. Then the optimal solution is $AP_3 = \$234.01$, $n^* = 6$, $p^* = \$6$/unit, ${t_1}^* = 0.62$ year, and optimal order quantity $Q^* = 455$ units. Figure 2.14, Figure 2.15, and Figure 2.16 shows the optimality of average Profit (AP_3).

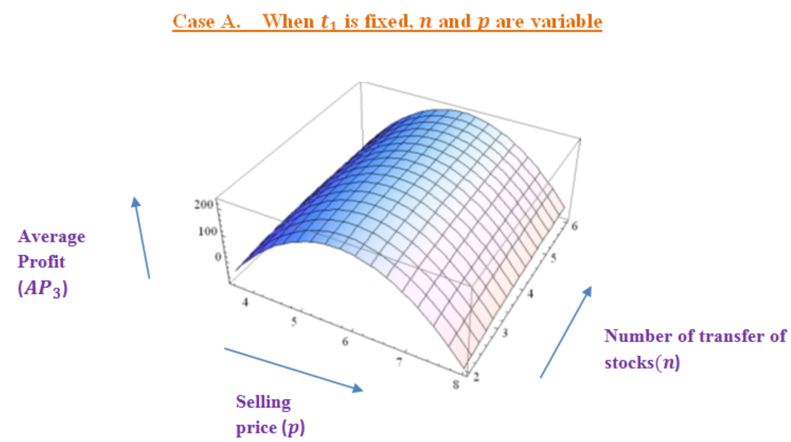

Figure 2.14: Average Profit (AP_3) versus selling-price (p) and number of transfer of stocks (n)

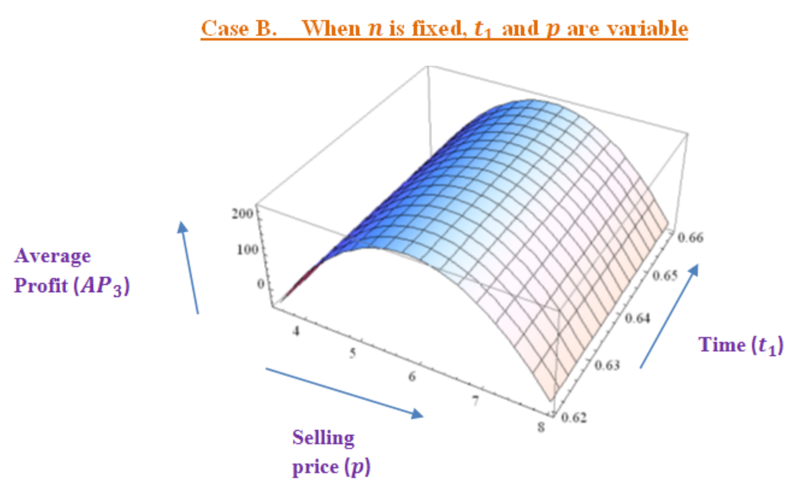

Figure 2.15: Average Profit (AP_3) versus selling-price (p) and time (t_1)

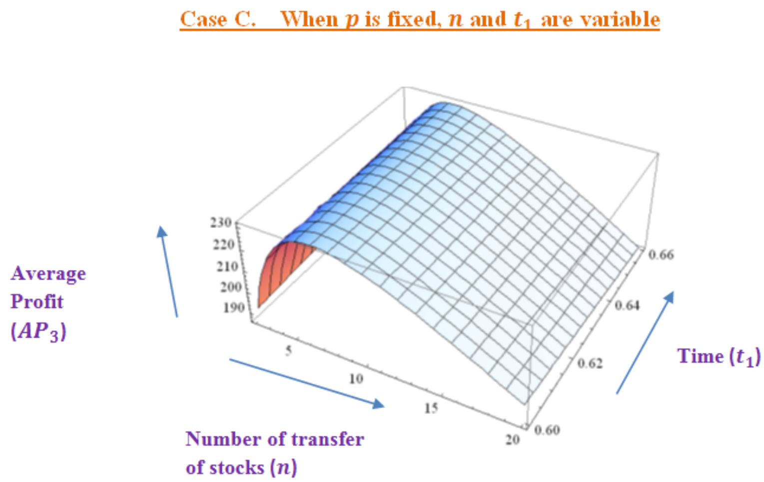

Figure 2.16: Average Profit (AP_3) versus number of transfer of stocks (n) and time (t_1)

Example 4(b) The values for the parameters are taken as follows:

$h = \$0.8$/unit/unit time, $h_1 = \$0.1$/unit/unit time, $p = \$5$/unit, $c = \$3.0$/unit, $x_2 = 500$, $y_2 = 0.2$, $R = 2$ unit, $a = 0.2$, $b = 0.3$, $s = \$20$/transfer, $S = \$90$/order. Then the optimal solution is $AP_4 = \$813.24$, $n^* = 5$, and ${t_1}^* = 0.4$ year, and optimal order quantity $Q^* = 1096$ units. Figure 2.17 shows the optimality of average Profit (AP_4).

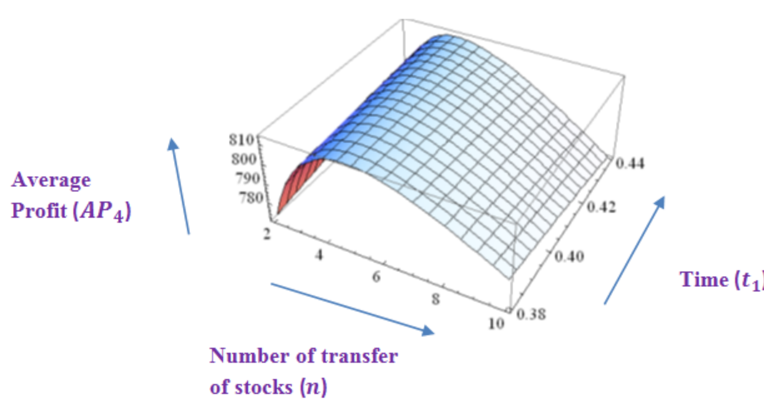

Figure 2.17: Average Profit (AP_4) versus number of transfer of stocks (n) and time (t_1)

Sensitivity Analysis

This section provides the sensitivity analysis of each key parameter. The sensitivity analysis of key parameters for several demand functions are given in the following tables namely Table 2.2, Table 2.3, Table 2.4, and Table 2.5.

Table 2.2: Sensitivity analysis for Case I

Parameters	Changes(in %)	AP_1	Parameters	Changes(in %)	AP_1
h	-50%	0.83	h_1	-50%	0.36
	-25%	0.35		-25%	0.16
	$+25\%$	–		$+25\%$	-0.14
	$+50\%$	–		$+50\%$	-0.27
c	-50%	6.28	s	-50%	0.51
	-25%	3.11		-25%	0.23
	$+25\%$	-3.07		$+25\%$	-0.2
	$+50\%$	-6.10		$+50\%$	-0.39

'–' refers to infeasible solution.

- As unit carrying cost per stock in display area h increases, average profit AP_1 decreases. But, for $+25\%$ and $+50\%$ increase of this parameter, the model does not allow feasible results. This means that, holding cost can be decreased, but one cannot increase it anymore.

- It can be observed if the parameter h_1 i.e., unit carrying cost per stock in warehouse, increases then the average profit AP_1 gradually decreases. The negative percentage change is greater than positive percentage change for h_1. This is the least sensitive parameter among others.

- If purchasing cost c increases, then the average profit AP_1 decreases. In this case, negative percentage change is greater than the positive percentage change for that parameter. It is the most sensitive parameter among others.

- An increasing value in ordering cost s decreases the average profit AP_1. For that parameter s, positive percentage change is less than the negative percentage change. This is also less sensitive parameter among others.

Table 2.3: Sensitivity analysis for Case II

Parameters	Changes(in %)	AP_2	Parameters	Changes(in %)	AP_2
h	-50%	$-$	c	-50%	28.56
	-25%	$-$		-25%	14.28
	$+25\%$	-0.8		$+25\%$	-14.28
	$+50\%$	-1.47		$+50\%$	-28.56
h_1	-50%	2.36	s	-50%	0.95
	-25%	1.02		-25%	0.43
	$+25\%$	-0.79		$+25\%$	-0.38
	$+50\%$	-1.35		$+50\%$	-0.73

'$-$' refers to infeasible solution.

- While the parameter unit carrying cost per stock in display area (i.e., h) decreases for -25% and -50%, this model does not give any feasible solution. But for $+25\%$ and $+50\%$, this model allows feasible results and in that case average profit AP_2 decreases when unit carrying cost per stock in display area h increases.

- As h_1 i.e., unit carrying cost per stock in warehouse, increases, then the average profit AP_2 decreases gradually. The positive percentage change is less than negative percentage change for h_1.

- For the unit purchasing cost c, negative and positive percentage changes are exactly same. An increasing value in purchasing cost c decreases the average profit AP_2. This is the most sensitive parameter comparing with other parameters.

- When ordering cost s increases, the average profit AP_2 decreases. The negative percentage change is bigger in comparing to positive percentage change for s. This is least sensitive among other parameters.

Table 2.4: Sensitivity analysis for Case III

Parameters	Changes(in %)	AP_3	Parameters	Changes(in %)	AP_3
	-50%	-2.96		-50%	0.06
	-25%	-3.4		-25%	-1.89
h	$+25\%$	-4.26	h_1	$+25\%$	-5.78
	$+50\%$	-4.70		$+50\%$	-7.72
	-50%	-3.35		-50%	2.81
	-25%	-3.6		-25%	0.51
s	$+25\%$	-4.07	c	$+25\%$	-7.15
	$+50\%$	-4.31		$+50\%$	-10.47

- If unit carrying cost per stock in display area (i.e., h) increases, the average profit AP_3 decreases. It is found that positive and negative percentage changes are almost double for h.

- For h_1, the positive percentage change is greater than the negative percentage change. The result indicates that average profit AP_3 decreases if h_1 increases.

- While purchasing cost c increases from -50% to $+50\%$, average profit AP_3 decreases. The negative percentage change is smaller than positive percentage change for that parameter. This is the most sensitive parameter comparing to others.

- The increasing value of ordering cost s decreases the average profit AP_3. The negative percentage change is not similar with positive percentage change of s.

Table 2.5: Sensitivity analysis for Case IV

Parameters	Changes(in %)	AP_4	Parameters	Changes(in %)	AP_4
	-50%	-13.60		-50%	12.62
	-25%	-14.93		-25%	-1.82
h	$+25\%$	-17.60	c	$+25\%$	-30.7
	$+50\%$	-18.92		$+50\%$	-45.14
	-50%	-4.29		-50%	-15.63
	-25%	-10.28		-25%	-15.94
h_1	$+25\%$	-22.24	s	$+25\%$	-16.58
	$+50\%$	-28.23		$+50\%$	-16.89

- The negative percentage change and positive percentage change for h is not similar. As unit carrying cost per stock in display area h increases, the average profit AP_4 decreases.

- It can be found that if the parameter h_1 i.e., unit carrying cost per stock in warehouse, increases, then the average profit AP_4 decreases. The negative and positive percentage changes are similar for h_1.

- While purchasing cost c increases, then the average profit AP_4 decreases. From Table 5, it can be concluded that the negative percentage change is higher than positive percentage change.

- An increasing value in ordering cost s decreases the average profit AP_4. In that case, negative and positive percentage changes are close to each other.

2.4 Concluding remarks and future works

This chapter presented four different types of demand functions which are time, selling-price, time-price, and exponentially time-dependent. The main objective of this chapter is to determine retailer's optimal ordering quantity and to maximize average profit function. Additionally, the number of transfers from warehouse to display area is also obtained. In future, this research can be expanded in different ways by considering shortages, discounts, and inflation rates.

2.5 Appendices

Appendix A

$$\begin{aligned}
M &= \left(\frac{y}{2}+\frac{2zt_1}{3}\right)+\frac{h_1(n-1)}{2}(y_1+2z_1p)e^{\theta t_1}+\frac{h(y_1+2z_1p)}{\theta}\left(\frac{e^{\theta t_1}}{t_1}+\frac{(1-e^{\theta t_1})}{\theta {t_1}^2}\right)\\
N &= 4z_1p-2\Big[cz_1-y_1-\frac{z_1}{\theta}\Big(\frac{h_1(n-1)}{2}(1-e^{\theta t_1})+h\Big(\frac{(1-e^{\theta t_1})}{\theta t_1}+1\Big)\Big)\Big]\\
U &= (y_1p+z_1p^2-x-x_1-yt_1-z{t_1}^2-R\theta)e^{\theta t_1}
\end{aligned}$$

Appendix B

$$\begin{aligned}
f &= \Big[\frac{(1-e^{\theta t_1})}{\theta}(y_1p+z_1p^2-x-x_1)+y\left(\frac{t_1e^{\theta t_1}}{\theta}-\frac{(e^{\theta t_1}-1)}{\theta^2}\right)+z\Big(\frac{{t_1}^2e^{\theta t_1}}{\theta}-\frac{2t_1e^{\theta t_1}}{\theta^2}\\
&+ \frac{2(e^{\theta t_1}-1)}{\theta^3}\Big)+Re^{\theta t_1}-R\Big]
\end{aligned}$$

Appendix C

$$
\begin{aligned}
\alpha &= \frac{h_1}{2}(\theta(n-1)-1)\left(\frac{y}{\theta}-\frac{2}{\theta^2}-x\right)\\
\beta &= \left(y-\frac{2z}{\theta}\right)\left(\frac{h_1}{2\theta}(\theta t_1+1-(n-1)\theta(t_1\theta+2)-h)\right)\\
\gamma &= \frac{z}{\theta}\left[h_1({t_1}^2\theta+2t_1-\frac{(n-1)}{2}(4t_1\theta+\theta^2{t_1}^2+2))-h(t_1\theta+2)\right]\\
\delta &= \frac{h}{2}\left[\frac{2(1-e^{\theta t_1})}{{t_1}^3}+\frac{2\theta e^{\theta t_1}}{{t_1}^2}-\frac{\theta^2 e^{\theta t_1}}{t_1}\right]\left[R-\left(\frac{y}{\theta}-\frac{2z}{\theta^2}-x\right)\right]\\
a_1 &= \frac{h_1(n-1)}{4}z\theta^2\\
a_2 &= \frac{h_1(n-1)}{2}\Big((y+2z)\frac{\theta^2}{2}+2z\theta\Big)\\
a_3 &= \frac{h_1(n-1)}{2}\left[(y+2z)\frac{3\theta}{2}+3z\right]-\Big[\frac{h_1(n-1)}{2}\left(\frac{R\theta^3}{2}+\Big(\frac{y}{\theta}-\frac{2z}{\theta^2}-x\Big)e^{-\theta t}\frac{\theta^2}{2}\right)+h\Big(y\\
&- \frac{2z}{\theta}\Big)\frac{\theta}{2}\Big]\\
a_4 &= \Big[\frac{2(p-c)z}{3}-\frac{h_1(n-1)}{2}\Big(R\theta^2+\left(\frac{y}{\theta}-\frac{2z}{\theta^2}-x\right)e^{-\theta t}\theta-2\theta\Big(\frac{y}{\theta}+\frac{2z}{\theta}\Big)-\frac{z}{\theta}\Big)-\frac{hR\theta^2}{2}\\
&+ h\left(\frac{y}{\theta}-\frac{2z}{\theta^2}-x\right)\frac{\theta}{2}-h\left(y-\frac{2z}{\theta}\right)\Big],\\
a_5 &= \Big[(p-c)\frac{y}{2}-\frac{h_1(n-1)}{2}\left[R\theta+\left(\frac{y}{\theta}-\frac{2z}{\theta^2}-x\right)e^{-\theta t}-\left(\frac{y}{\theta}+\frac{2z}{\theta}\right)\right]-\frac{hR\theta}{2}-\frac{h}{2}\Big(\frac{y}{\theta}\\
&- \frac{2z}{\theta^2}\Big)\Big],\\
a_6 &= \frac{S}{n}+s.
\end{aligned}
$$

Appendix D

$$
\begin{aligned}
D(p) &= (x_1-y_1p-z_1p^2).\\
l_1 &= \frac{2(e^{\theta t_1}-1)}{\theta {t_1}^3}-\frac{2e^{\theta t_1}}{\theta^2}+\frac{\theta e^{\theta t_1}}{t_1}.\\
l_2 &= \left(\frac{e^{\theta t_1}}{t_1}-\frac{e^{\theta t_1}-1}{\theta {t_1}^2}\right).\\
l_3 &= \left(2cz_1+h_1(n-1)z_1-6z_1p+\frac{2z_1h}{\theta}l_2\right).
\end{aligned}
$$

$$
\begin{aligned}
l_4 &= \left(-\frac{S}{n^2{t_1}^2} - \frac{h_1}{2}(R\theta + D(p))e^{\theta t_1}\right). \\
l_5 &= (y_1 + 2z_1 p)\left(\frac{h_1(1-n)}{2}e^{\theta t_1} + \frac{h}{\theta}\right) l_2.
\end{aligned}
$$

Appendix E

$$
\begin{aligned}
b_1 &= \left[\frac{2(e^{y_2 t_1} - 1)}{{t_1}^3} - \frac{2e^{y_2 t_1} y_2}{{t_1}^2} + \frac{e^{y_2 t_1} {y_2}^2}{t_1}\right], \\
b_2 &= [x_2(y_2 + \theta)e^{(y_2+\theta)t_1} + Re^{\theta t_1}\theta^2], \\
b_3 &= \Big[e^{(y_2+\theta)t_1}(-\theta e^{-\theta t_1} - y_2 e^{y_2 t_1})\frac{x_2}{y_2+\theta} + x_2(y_2+\theta)\left(\frac{(1-e^{-\theta t_1})}{\theta} + \frac{(1-e^{y_2 t_1})}{y_2}\right) \\
&+ 2(e^{-\theta t_1} - e^{y_2 t_1})(y_2+\theta)\frac{x_2}{(y_2+\theta)} + R\theta e^{\theta t_1}\Big].
\end{aligned}
$$