C/18/B.Sc./2nd Sem/STSH/C3T

2018

2nd Semester

STATISTICS

PAPER—C3T

(Honours)

Full Marks : 60

Time : 3 Hours

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

(Mathematical Analysis)

1. Answer any *ten* questions : 10×2

(a) State the completeness property of real numbers.

(b) Define neighbourhood of a point. How does it differ from a deleted neighbourhood ?

(c) State squeeze theorem for convergence of a sequence of real numbers.

(Turn Over)

(d) Show that the real sequence $\{x_n\}_{n \geq 1}$ where $x_n = n^{1/n}$ converges to 1 as $n \to \infty$.

(e) Check the convergence/divergence of the series

$$\sum_n x_n = \sum_n \frac{1}{n^p} \text{ where } p > 0.$$

(f) Show that every monotone decreasing sequence that unbounded below diverges to $-\infty$.

(g) Give an example of divergent sequences $\{u_n\}$ and $\{v_n\}$ such that the sequence $\{u_n v_n\}$ is convergent.

(h) If $\sum_{n=1}^{\infty} u_n$ is a convergent series of positive real numbers, then prove that $\sum_{n=1}^{\infty} u_n^2$ is convergent.

(i) Define continuity of a real valued function with an example.

(j) A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x, & x \in \mathbb{R} \\ 0, & x \in \mathbb{R} - Q \end{cases}$$

Show that f is continuous at 0.

(k) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that f is differentiable on $\mathbb{R}$

(l) Write the geometrical interpretation of mean value theorem.

(m) Give an example (with justification) where $\sum_n a_n$ and $\sum_n b_n$ are both convergent but $\sum a_n b_n$ is divergent.

(n) State the necessary and sufficient conditions for a function of a single variable to have a local extremum.

(o) Prove that a function f is differentiable at a point 'c' $\Rightarrow$ function f is continuous at 'c'.

2. Answer any *four* questions : 4×5

(a) (i) Prove that the set of all rational numbers $\mathbb{Q}$ is not bounded above.

(ii) Show that every convergent sequence is bounded. 3+2

(b) State Ratio test to test the convergence of a series.

Use it to prove that $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$ is divergent.

(c) (i) If $f(x) = [x]$, show that $\lim_{x \to 2} f(x)$ does not exist.

(ii) Given : $f(x) = 1$ if $x \in Q$

$= 0$ if $x \in R - Q$

where Q = set of ratonal numbers and

R–Q = set of irrational numbers.

Does $\lim_{x \to c} f(x)$ exist $\forall c \in R$? 2+3

(d) (i) A function f is twice differentiable on [a, b] and $f(a) = f(b) = 0$. If $f(c) > 0$ for some $c \in (a, b)$, then prove that there exists of point $\eta \in (a, b)$ such that $f''(\eta) < 0$.

(ii) Verify Rolle's theorem for the following function :

$f(x) = x^2 - 5x + 10, x \in [2, 3].$ 4+1

(e) (i) State and prove fundamental theorem of integral calculus.

(ii) Define Beta and Gamma functions. 4+1

(f) (i) Prove that the integral $\int_0^{\infty}\left(\frac{1}{1+x}-\frac{1}{e^x}\right)\frac{1}{x}dx$ is convergent.

(ii) Evaluate $\int_0^{\infty}\sqrt{x}e^{-x^3}dx$. 3+2

3. Answer any *two* questions : 2×10

(a) (i) Let $f(x, y) = \begin{cases} xy\dfrac{x^2+k^2y^2}{x^2+y^2}, & \text{where } x^2+y^2 \neq 0 \\ 0, & \text{where } x=0, y=0 \end{cases}$

Prove that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

(ii) If $u = \frac{1}{\sqrt{1-2xy+y^2}}$, prove that

$$\frac{\partial}{\partial x}\left\{\left(1-x^2\right)\frac{\partial u}{\partial x}\right\}+\frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right)=0.$$

(b) (i) Show that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots$ converges to $\log_2$, but $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{4}+\cdots\cdots$ converges to $\frac{1}{2}\log_{12}$.

(ii) Obtain the Taylor's infinite series expansion of $\ln(1 + x)$; $x > -1$. 4+6

(c) (i) Check the convergence of the Beta integral

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

(ii) Find the value of the integral $\iint_E e^{\frac{y-x}{y+x}}dxdy$ where E is the triangle with vertices at (0, 0), (0, 1) and (1, 0). 5+5

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