2019

B.Sc. (Honours)

4th Semester Examination

MATHEMATICS

Paper - C9T

(Multivariate Calculus)

Full Marks : 60 Time : 3 Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable. Illustrate the answers wherever necessary.

Unit - I

1. Answer any ***three*** questions : 2×3

(a) Show that the limit exists at the origin but the repeated limit does not, for the function

$$f(x, y)=\begin{cases} x\sin\frac{1}{y}+y\sin\frac{1}{x}, & xy\neq 0 \\ 0 & , xy=0 \end{cases}$$

[Turn Over]

(b) For $F(x, y) = x^4 y^2 \sin^{-1}\frac{y}{x}$ show that

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = 6F$$

(c) Define directional derivative of the function $f(x, y)$ at the point (a, b). Obtain partial derivative as a special case of it.

(d) Is $f(x, y) = |y|(1+x)$ differentiate at (0, 0) ?

(e) Find the maximum or minimum value of

$f(x, y) = x^3 + y^3 - 3axy.$

2. Answer any ***one*** question : 5×1

(a) State and prove sufficient condition for differentiability of a function $f(x, y)$ at a point (a, b).

(b) Let $(a, b) \in D$, the domain of definition of f. If $f_x(a, b)$ exist and $f_y(x, y)$ is continuous at (a, b) then show that $f(x, y)$ is differentiable at (a, b).

3. Answer any *one* question 10×1

(a) (i) Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 225$, $z = 0$.

(ii) If z be a differentiable function of x and y and if $x = c\cosh(u)\cos(v)$, $y = c\ \sin hv$ $\sin v$ then prove that

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\frac{1}{2}c^2\ (\cosh 2u - \cos 2v)$$

$$\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)$$ 5+5

(b) (i) Define total differential of a function $f(x, y, z)$.

Approximate the change in the hypotenuse of a right angled triangle whose sides are 6 and 8 cm, when the shorter side is lengthened by $\left(\frac{1}{4}\text{cm}\right)$ and the longer is shortened by $\left(\frac{1}{8}\text{cm}\right)$.

[Turn Over]

(ii) Prove that the volume of the greatest rectangular parallelopiped, that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, is $\frac{8abc}{3\sqrt{3}}$. (2+3)+5

Unit - II

4. Answer any ***two*** questions : 2×2

(a) Let

$$f(x, y) = \begin{cases} \frac{1}{2}, & y = \text{rational} \\ x, & y = \text{irrational} \end{cases}$$

verify whether $\int_0^1 dy \int_0^1 f(x, y)dx$ exists or not.

(b) Evaluate $\int_0^\infty \frac{\sin rx}{x} dx$ from $\int_0^\infty \int_0^\infty e^{-xy} \sin rx \, dx \, dy$ with the help of change of order of integration.

(c) Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ over the region R bounded by $xy = 1$, $y = 0$, $y = x$, $x = 2$.

5. Answer any ***two*** questions : 5×2

(a) Show in a diagram the field of integration of the integral $\int_0^1 \left(\int_x^{1/x} \frac{y\,dy}{(1+xy)^2(1+y^2)} \right) dx$ and by changing the order of integration, show that the value of the integral is $\frac{\pi - 1}{4}$.

(b) Are the two iterated integrals $\int_1^\infty dx \int_1^\infty \frac{x-y}{(x+y)^3}\,dy$ and $\int_1^\infty dy \int_1^\infty \frac{x-y}{(x+y)^3}\,dx$ equal? Justify your answer.

(c) Evaluate

$$\iiint_E \sqrt{a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2 - a^2b^2z^2}\,dx\,dy\,dz$$

[Turn Over]

where E is the region bounded by the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Unit - III

6. Answer any ***three*** questions : 2×3

(Symbols have their usual meaning)

(a) Find the total work done in moving a particle in a force field given by

$$F=(2x-y+z)\hat{i}+(x+y-z)\hat{j}+(3x-2y-5z)\hat{k},$$

along a circle C in the xy-plane $x^2+y^2=9$, $z=0$.

(b) Evaluate the vector line integral $\int_C \vec{F}\times d\vec{x}$ where

$\vec{F}=Z\hat{i}$ and C is the part of the circular helix $\vec{x}=b\cos t\hat{i}+b\sin t\hat{j}+c\,t\hat{k}$ between the points $(-b, 0, \pi c)$ and $(b, 0, 0)$.

(c) Prove that $\vec{\nabla}.\left[r\vec{\nabla}\left(\frac{1}{r^3}\right)\right]=3r-4$, where $\vec{r}$ is the position vector and $r=|\vec{r}|$

(d) Find the equation of the tangent plane to the surface $xyz = 4$ at the point (1, 2, 2).

(e) If $\Delta\phi = (2xyz^3, x^2z^3, 3x^2yz^2)$ and

$\phi(1,-2,2) = 4$, find the function ϕ.

7. Answer any ***one*** question : 10×1

(a) (i) If $\vec{\nabla}.\vec{E} = 0, \vec{\nabla}.\vec{H} = 0, \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{H}}{\partial t}$ and

$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{E}}{\partial t}$, then show that

$$\nabla^2\vec{H} = \frac{\partial^2\vec{H}}{\partial t^2} \text{ and } \nabla^2\vec{\mathrm{E}} = \frac{\partial^2\vec{\mathrm{E}}}{\partial \mathrm{t}^2}.$$

(ii) Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},\ r = |\vec{r}\,|$ and $f(x)$ is a scalar function possessing first and 2nd order derivatives prove that

$$\nabla^2 f(x) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}.$$

[*Turn Over*]

If $\nabla^2 f(r) = 0$, show that $f(r) = A + \frac{B}{r}$ where A and B are arbitrary constants.

(b) (i) Prove that

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla}^2 \vec{A}$$

(ii) Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$.

If $f(r) = \log r$ and $g(r) = 1/r$, $r \neq 0$. Satisfy $2\vec{\nabla} f + h(r)\vec{\nabla} g = 0$ then find $h(r)$.

Unit - IV

8. Answer any ***two*** questions : 2×2

(a) Evaluate

$$\int_S (x^2 dy\, dz + y^2 z\, dz\, dx + 2z\,(xy - x - y)\, dx\, dy)$$

where S is the surface of the cube

$$0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$$

(b) Show that $\iint_S \vec{r}.d\vec{s} = 3v$ where v is the volume enclosed by the closed surface S and $\vec{r}$ has its usual meaning.

(c) (i) State Green's theorem in the plane.

(ii) If S be any closed surface enclosing a volume V and $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, prove that $\iint_S \vec{F}.\hat{n}\, ds = 6V$.

9. Answer any ***one*** question : 5×1

(a) Evaluate $\iint_S \vec{F}.\hat{n}\, dS$, where

$\vec{F} = x\hat{i} - y\hat{j} + (z^2 - 1)\hat{k}$, S is the surface of the region bounded by $x^2 + y^2 = 4$, $z = 0$, $z = 4$ in the first octant.

(b) Verify Green's theorem in the plane for $\oint (xy + y^2)dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

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