 UG/3rd Sem/MATH(H)/19

**2019**

**B.Sc.**

**3rd Semister Examination**

# MATHEMATICS (Honours)

**Paper - GE 3-T**

Full Marks : 60 Time : 3 Hours

*The figures in the margin indicate full marks.*
*Candidates are required to give their answers in their own words as far as practicable.*
*Illustrate the answers wherever necessary.*

## Differenrial Equation and Vector Calculus

1. Answer any ten questions : 10×2

(a) Determine whether the set {1 – x, 1 + x, 1 – 3x} is linearly dependent on $(-\infty, \infty)$.

(b) Find $\frac{1}{D^2+4}(\sin 2x)$, where $D^2 = \frac{d^2}{dx^2}$.

(c) Find the vector area of the triangle, the position vectors of whose vertices are $\hat{i}+\hat{j}+2\hat{k}$, $2\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-\hat{j}-\hat{k}$.

[ Turn Over ]

(d) If $\vec{a}+\vec{b}+\vec{c}=0$, then prove that $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$.

(e) Find the equilibrium point of the system of differential equations $\dot{x}=e^{x-1}-1$ and $\dot{y}=ye^{x}$.

(f) Stae the principle of superposition for homogeneous equation.

(g) If $u$ and $v$ be two independent solutions of the linear equation $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$, then the wornskian w(u, v) is given by w(u, v) = $Ae^{-\int Pdx}$, where A is a constant.

(h) Show that the three vectors $4\hat{i}+2\hat{j}+\hat{k}$, $2\hat{i}-\hat{j}+3\hat{k}$ and $8\hat{i}+7\hat{k}$ are coplanar.

(i) Find the directional derivative of $\phi=x^2yz+4xz^2$ at (1, –2, –1) in the direction $2\hat{i}-\hat{j}-2\hat{k}$.

(j) Verify that x = 0 is a singular point of the differential equation

$$2x^2\frac{d^2w}{dx^2}+x\frac{dw}{dx}-(x+1)w=0.$$

(k) Find constants a, b and c so that

$$\vec{F} = (x + 2y + az)\hat{i} - (bx - 3y - z)$$

$$\hat{j} + (4x + cy + 2z)\hat{k}$$ is irrotational.

(l) A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of its velocity and acceleration at time t = 1 in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$.

(m) State Lipschitz condition.

(n) Define Euler-cauchy type of equation.

(o) If $\frac{d\vec{a}}{dt} = \vec{r} \times \vec{a}$ and $\frac{d\vec{b}}{dt} = \vec{r} \times \vec{b}$, then show that

$$\frac{d}{dt}[\vec{a} \times \vec{b}] = \vec{r} \times \vec{a} \times \vec{b}$$

where $\vec{r}$ is a constant vector and $\vec{a}$ and $\vec{b}$ are the vector function of a scalar variable t...

2. Answer any four questions : 4×5

(a)

Solve the differential equation $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$ by the method of variation parameter.

*[ Turn Over ]*

(b) Solve : $\frac{dx}{y-zx}=\frac{dy}{x+yz}=\frac{dz}{x^2+y^2}$

(c) Solve : $\frac{dx}{dt}-7x+y=0$; $\frac{dy}{dt}-2x-5y=0$

(d) Show that the volume of the parallelopiped, whose edges are represented by $3\hat{i}+2\hat{j}-4\hat{k}$, $3\hat{i}+\hat{j}+3\hat{k}$ and $\hat{i}-2\hat{j}+\hat{k}$ is 49 cubic units.

(e) (i) Evaluate $\int_{(0,0)}^{(2,1)}(10x^4-2xy^3)dx-3x^2y^2dy$ along the path $x^4-6xy^3=4y^2$.

(ii) Show that $\vec{\nabla}r^n=nr^{n-2}\bar{r}$.

(f) (i) If W be the wronskian of the functions 1, x, $x^2$, ...., $x^{n-1}$ for $n>1$, then show that W = 0! 1! 2! ..... (n − 1)!

(ii) Obtain the differential equation of all circles, each of which touches the axis of x at the origin.

3. Answer any two questions : 2×10

(a) Find the series solution of the equation $(x^2+1)y_2+xy_1-xy=0$ about $x=0$.

(b) (i) If $\vec{r} = (a\cos t)\hat{i} + (a\sin t)\hat{j} + (at\tan\alpha)\hat{k}$, then

prove that $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = a^3\tan\alpha$. 5

(ii) Prove that $\int_1^2 \vec{r} \times \frac{d^2\vec{r}}{dt^2}\,dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$,

where $\vec{r}(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$. 5

(c) (i) If $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$, evaluate $\iint_S \vec{F}.\hat{n}ds$

where S is the surface of the cube bounded by x = 0, x = 1, y = 0, y = 1, z = 0, z = 1. 5

(ii) Given the space curve x = t, y = $t^2$, $z = \frac{2}{3}t^3$, find the curvature k and the torsion $\Upsilon$. 5

(d) Show that the model represented by

$\frac{dx}{dt} = x(4 - x - y)$ ; $\frac{dy}{dt} - y(15 - 5x - 3y)$,

$x \geq 0$, $y \geq 0$ has a position of equilibrium and this position is stable.

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*[ Turn Over ]*

## Group Theory - I

### Unit - I

1. Answer any *two* questions : $2\times2=4$

   (a) Prove that (Z, +) is the semigroup.

   (b) If each element, except the identity, of a group be of order 2, prove that the group is abelian.

   (c) Define symmetric group $S_3$. What is the identity element of this group.

2. Answer any *one* question : $5\times1=5$

   (a) In a group $(G, \circ)$ in which $(a \circ b)^3 = a^3 \circ b^3$ and $(a \circ b)^5 = a^5 \circ b^5$ for all $a, b \in G$, prove that the group is abelian.

   (b) Prove that the set D of all odd integers forms a commutative group with respect to * defined by a * b = a + b − 1 for a, b $\in D$.

## Unit - II

3. Answer any *two* questions : 2×2=4

(a) Find the Cyclic subgroups of the group $(s, \circ)$, where $S = \{1, i, -1, -i\}$.

(b) Give an example to show that the union of two subgroups of a group may not be a sub group.

(c) Let (G, o) be a group and KCHCG. If (H, o) be a subgroup of (G, o) and (K, o) be a subgroup of (H, o), prove that (K, o) is a subgroup of (G, o).

4. Answer any *two* questions : 2×5=10

(a) If $H$ and $K$ happens to be a subgroup of G, then prove that $\circ(HK) = \frac{\circ(H)\cdot\circ(K)}{\circ(H \cap K)}$.

(b) Let G be a group in which $(ab)^3 = a^3b^3$ for all a, $b \in G$. Show that $H = \{x^6 : x \in G\}$ is a subgroup of G.

[ Turn Over ]

(c) Let (G, o) be a group and H be a non-empty subset of G. A relation ρ defined on G by "$a\rho b$ if and only if $a \circ b^{-1} \in H$ for a, $b \in G$, is an equivalence relation on G. Prove that (H, o) is a subgroup of (G, o).

## Unit - III

5. Answer any *two* questions : 2×2=4

(a) Let $G$ be a group and $H$ be a subgroup of G. Let $a \in G - H$. The prove that $aH \cap H = \phi$.

(b) G is a cyclic group of order 30 generated by a. Find the order of the cyclic subgroup generated by $a^{18}$.

(c) Find the inverse of the permutation :

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 5 & 6 & 1 \end{pmatrix}$$

6. Answer any *one* question : 1×10=10

(a) (i) Prove that every group of prime order is cyclic.

(ii) A cyclic group of finite order n has one and only one subgroup of order d for every positive divisor and of n. 5

(b) (i) Let G = <a> be a cyclic group of order n. Prove that every subgroup H of G is of the form <$a^m$> where m is a divisor of n. 5

(ii) State and prove Fermat's Little theorem. 5

**Unit - IV**

7. Answer any *two* questions : 2×2=4

(a) Let $G_1, G_2$ be two groups and $Z(G_1)$, $Z(G_2)$ be their respective centres. Prove that $Z(G_1) \times Z(G_2)$ is a centre of the group $G_1 \times G_2$.

(b) Find the order of the element $\frac{2}{3}+Z$ in the quotient group Q/Z.

(c) Let H be a subgroup of a group G and [G : H] = 2. Show that H is a normal subgroup of G

8. Answer any *one* question : 1×10=10

(a) (i) Show that subgroup H of a group G is normal if $aHa^{-1} = H$ for every a in G 5

[ Turn Over ]

(ii) Let $H$ be a subgroup of a group G and [G : H] = 2. Then show that H is normal in G 5

(b) Let a, $b \in \mathbb{R}$ and a mapping $T_{a,b} : \mathbb{R} \to \mathbb{R}$ is defined by $T_{ab}(x) = ax + b$, $x \in R$. Let $G = \{T_{ab} : a \neq 0\}$. Prove that (G, o) is a group where o is the composition of mappings. Let $H = \{T_{ab} : a = 1\}$. Prove that H is a normal subgroup of G

## Unit - V

9. Answer any *two* questions : 2×2=4

(a) Let G = (Z, +) and a mapping $\phi : G \to G$ be defined by $\phi(x) = x + 1$, $x \in G$. Examine if $\phi$ is a homomorphism.

(b) Show that the groups (Q, +) and (R, +) are not isomorphic.

(c) Let (G, o) be a group and a mapping $\phi : G \to G$ is defined by $\phi(x) = x^2, x \in G$. Prove that $\phi$ is homomorphism if G is commutative.

10. Answer any *one* question : 1×5=5

(a) Let G = $S_3$, $G' = (\{1, -1\}, \bullet)$ and let $\varphi : G \to G'$ be defined by

$\varphi(\alpha) = 1$ if $\alpha$ be an even permutation in $S_3$

$= -1$ if $\alpha$ be an odd permutation in $S_3$.

Examine if $\varphi$ is a homomorphism.

(b) Let $H \subset K \subset G$ and H is normal in K, K is normal in G and also H is normal in G. Show that K/H is normal in G/H and $\frac{G/H}{K/H} \simeq G/K$.

## Theory of Real Function and Introduction

### Unit - I

1. Answer any *three* questions : 3×2=6

(a) Given an example of jump discontinuity of a function.

(b) Find the points of discontinuity of $f(x) = \frac{1}{1+\frac{1}{x}}$.

[ Turn Over ]

(c) Show that the absolute value function $f(x) = |x|$ is continuous at every point $c \in R$.

(d) If $\lim_{x \to a} |f(x)| = 0$ then show that $\lim_{x \to a} f(x) = 0$.

(e) State intermediate value theorem.

2. Answer any *one* question : $1 \times 5 = 5$

(a) If f(x) and g(x) are two real valued functions of x defined on an interval I such that $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then prove that $\lim_{x \to a} \{f(x).g(x)\} = lm$, $a \in I$.

(b) Discuss the continuety of the function

$$f(x) = \begin{cases} 2, x^2 > 4 \\ 3, x^2 = 4 \\ 0, x^2 < 4 \end{cases}$$

State the type of discontinuity if f(x) is discontinuous any-where.

3. Answer any one question : [1×10 = 10]

(a) (i) If f(x) = x and g(x) = sin x, show that both f and g are uniformly continuous on R, but that their product fg is not uniformly continuous on R. 5

(ii) Show that

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+n}}\right)=1 \quad 5$$

(b) (i) Prove that if f be continuous on a closed interval, then it assumes its least upper bound and its greatest lower bound in that interval. 6

(ii) Prove that if a function f is uniformly continuous in a certain interval I, then it is necessarily continuous on I. 4

## Unit - II [Marks : 14]

4. Answer any *two* questions : 2×2=4

(a) Show that $x > \sin x$ for $0 < x < \frac{\pi}{2}$.

(b) What is the geometrical interpretation of the Rolle's theorem.

(c) Find the point of relative extrema of the following function on the specified domain :

$f(x) = 1 - (x - 1)^{2/3}$ for $0 \le x \le 2$.

5. Answer any two questions : [2×5 = 10]

(a) State and prove Rolle's theorem.

(b) Verify Lagrange's Mean value theorrem for the function $f(x) = x(x - 1)(x - 3)$ in [0, 4].

(c) In the mean value theorem $f(a + h) = f(a) + h \cdot f'(a+\theta h)$ if $a = 1$, $h = 3$ and $f(x) = \sqrt{x}$, find $\theta$.

**Unit - III [Marks : 14]**

6. Answer any *two* questions : 2×2=4

(a) Show that the maximum value of $(x)^{\frac{1}{x}}$ is $(e)^{\frac{1}{e}}$.

(b) Examine the function $f(x) = 4 - 3(x - 2)^{2/3}$ for maxima and minima at $x = 2$.

(c) State Maclaetrin's theorem with cauchy's form of remainder.

7. Answer any one question : [1×10 = 10]

(a) (i) Find the minimum and maximum value of

$$f(x) = 3x + \frac{2}{3x} \text{ for all } x \in R - \{0\}.$$

(ii) Assuming the validity of expansion, show that

$$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} - \frac{2^2 x^5}{5!} + .....$$

(b) (i) Show that $\frac{\tan x}{x} > \frac{x}{\sin x}$ for $0 < x < \frac{\pi}{2}$.

(ii) Show that $R_n \to 0$ as $n \to \infty$ for the expansions of $(1 + x)^m$ in a given range of validity, where $R_n$ is the remainder after *n* terms.

**Unit - IV [Marks : 11]**

8. Answer any *three* questions : 3×2=6

(a) Define dense set.

(b) Let (X, d) be a metric space. Then show that diameter $\delta(A) = 0$ iff $A \subset X$ is a singleton set.

(c) Define open ball. Give an example.

*[ Turn Over ]*

(d) Give two examples of separable metric spaces.

(e) Let (X, d) be a metric space. Then prove that

(i) the null set $\phi$ is closed,

(ii) X is closed.

9. Answer any one question : [1×5 = 5]

(a) Prove that every non-empty open set in the real line is the union of a countable collection of mutually disjoint open intervals.

(b) If d is a metric on a set X, then proved that $d_1$ and $d_2$ are metric where

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

$$d_2(x, y) = \min\{1, d(x, y)\} \text{ for } x, y \in x.$$

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