

2019

B.Sc.

3rd Semester Examination

MATHEMATICS (Honours)

Paper - C 5-T

**Theory of Real Functions and
Introduction to Metric Space**

Full Marks : 60

Time : 3 Hours

*The figures in the margin indicate full marks.
Candidates are required to give their answers
in their own words as far as practicable.
Illustrate the answers wherever necessary.*

Unit - 1 [Marks : 21]

1. Answer any *three* questions : $2 \times 3 = 6$

- (a) Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function in $[0, 1]$ prove that there exist a point c in $[0, 1]$ such that $f(c) = c$.
- (b) State Cauchy's criteria for the existence of $\lim_{x \rightarrow \infty} f(x)$.

[Turn Over]

- (c) Give example of functions f and g which are not continuous at a point $c \in \mathbb{R}$ but the sum $f + g$ is continuous at c .
- (d) What do you mean by removable discontinuity of a function at an interior point of an interval?

(e) Let $f(x) = \begin{cases} 2, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} - \mathbb{Q} \end{cases}$

Prove that f is discontinuous at every point c in \mathbb{R} .

2. Answer any *one* question : 5×1 = 5

- (a) Let $D \subset \mathbb{R}$, and f and g be functions on D to \mathbb{R} and $g(x) \neq 0$ for all $x \in D$. Let $c \in D'$ and $\lim_{x \rightarrow c} f(x) = l$, $\lim_{x \rightarrow c} g(x) = m \neq 0$. Then

prove that $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{l}{m}$. 5

- (b) A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} - \mathbb{Q} \end{cases} \text{ then show that } f \text{ is}$$

continuous at 0 and f has a discontinuity of the 2nd kind at every other point in \mathbb{R} . 5

3. Answer any *one* question : 10×1=10

- (a) (i) Let $A, B \subset \mathbb{R}$ and $f: A \rightarrow \mathbb{R}$, $g: B \rightarrow \mathbb{R}$ be functions s.t $f(A) \subset B$. Let $C \in A$ and f is continuous at c and g is continuous at $f(c) \in B$. Then the composite function $g \circ f: A \rightarrow \mathbb{R}$ is continuous at C . 5
- (ii) Let $D \subset \mathbb{R}$ and $f: D \rightarrow \mathbb{R}$ be functions s.t $f(x) \geq 0 \quad \forall x \in D$ and f is continuous on D . Then \sqrt{f} is continuous on D . Hence prove that $h(x) = \sqrt{x^3 + 3}$, $x \in \mathbb{R}$ is continuous on \mathbb{R} . 3+2
- (b) (i) Let I be a bounded interval and $f: I \rightarrow \mathbb{R}$ be uniformly continuous on I . Then prove that f is bounded on I . Show that $\cos \frac{1}{x}$ is not uniformly continuous. 4+2
- (ii) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be strictly increasing and continuous and let $S = f(\mathbb{R})$. Then $f^{-1}: S \rightarrow \mathbb{R}$ is also strictly increasing and continuous. 4

[Turn Over]

Unit 2 [Marks : 14]

4. Answer any *two* questions : 2×2=4

(a) Prove that $\frac{\tan x}{x} > \frac{x}{\sin x}, 0 < x < \frac{\pi}{2}$

(b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = |x| + |x - 1|$, $x \in \mathbb{R}$. Find the derived function f' and specify the domain of f' .

(c) Verify Rolle's theorem for $f(x) = \sin\left(\frac{1}{x}\right)$ on $\left[\frac{1}{3\pi}, \frac{1}{2\pi}\right]$.

5. Answer any *two* questions : 5×2=10

(a) Use MVT to prove the inequality

$$\frac{x}{\sqrt{1-x^2}} \leq \sin^{-1} x < x \text{ if } 0 \leq x < 1. \text{ When does}$$

the equality hold. 4+1

(b) A function f is twice differentiable on $[a, b]$ and $f(a) = f(b) = 0$ and $f(c) < 0$ for some c in (a, b) . Prove that there is at least one point ξ in (a, b) for which $f''(\xi) > 0$. 5

- (c) When a function is said to be differentiable at a point ? Let I be an interval and $c \in I$. Let the functions $f: I \rightarrow \mathbb{R}$ and $g: I \rightarrow \mathbb{R}$ be differentiable at c . Then prove that $f.g$ is differentiable at c and

$$(fg)'(c) = f'(c)g(c) + g'(c)f(c). \quad 1+4$$

Unit 3 [Marks : 14]

6. Answer any *two* questions : 2×2=4

- (a) State Taylor's theorem with Cauchy's form of remainder.
- (b) Find the local extremum points of the function

$$f(x) = \frac{x^2}{(1-x)^3}.$$

- (c) A function f is differentiable on $[0, 2]$ and $f(0) = 0$, $f(1) = 2$, $f(2) = 1$. Prove that $f'(c) = 0$ for some c in $(0, 2)$.

7. Answer any *one* question : 10×1=10

- (a) (i) State and prove the Maclaurin's theorem with Cauchy's form of remainder.

[Turn Over]

(ii) Use Taylor's theorem to prove that

$$1 + \frac{x}{2} - \frac{x^3}{8} < \sqrt{1+x} < 1 + \frac{x}{2}, \text{ if } x > 0. \quad 6+4$$

(b) (i) If $f'(x) = (x-a)^{2n}(x-b)^{2m+1}$ where m, n are positive integers, show that f has neither a maxima nor a minima at 'a' and f has a minimum at 'b'. 5

(ii) Let $f: I \rightarrow \mathbb{R}$ be such that f has a local extrema at an interior point c of I . If $f'(c)$ exists then prove that $f'(c) = 0$. 5

Unit 4 [Marks : 11]

8. Answer any three questions : 2×3=6

(a) Prove that in any discrete metric space, the sets are closed.

(b) Let (M, d) be a metric space. Then prove that $\forall A, B \in \mathcal{M}, \quad ACB \Rightarrow S(A) \leq S(B)$ where $S(X)$ denote the diameter of a set X .

(c) Let (M, d) be a metric space. Then prove that (M, \sqrt{d}) is also a metric space.

(d) Let $X = \mathbb{N}$, the set of natural number, and d ye defined by $d(m, n) = \left| \frac{1}{m} - \frac{1}{n} \right|$, $m, n \in \mathbb{N}$, then prove that (X, d) is a discrete metric space.

(e) Let M be a non-empty set for $x, y \in M$, and

$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$ then prove that (M, d) is a metric space.

9. Answer any *one* question : 1×5=5

(a) State and prove that Hausdorff property.

(b) Let $\{A_n\}$ be a sequence of open sets in \mathbb{R}_u such that each A_n is dense in \mathbb{R}_u . Prove that $\bigcap_{n=1}^{\infty} A_n$ is dense in \mathbb{R}_u .
