2019

B.Sc.

1st Semester Examination

MATHEMATICS (Honours)

Paper - C2T

(Algebra)

Full Marks : 60 Time : 3 Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable. Illustrate the answers wherever necessary.

Unit - I

(Classical Algebra)

1. Answer any *one* questions : $1\times2=2$

   (a) Find the sum of 99th powers of the roots of the equation $x^7-1=0$.

   (b) z is a variable complex number such that $|z| = 2$. Show that the point $z+\frac{1}{z}$ lies on an ellipse.

[*Turn Over*]

2. Answer any *two* questions : 2×5=10

(a) If $x = \log \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, where θ is real, prove that $\theta = i \log \tan\left(\frac{\pi}{4}+i\frac{x}{2}\right)$.

(b) Show that the equation

$$(x-a)^3+(x-b)^3+(x-c)^3+(x-d)^3=0,$$

where a, b, c, d are not all equal, has only one real root.

(c) If $s_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$ prove that

(i) $s_n > \frac{2n}{n+1}$, if $n>1$

(ii) $\left(\frac{n-s_n}{n-1}\right)^{n-1} > \frac{1}{n}$, if $n>2$.

3. Answer any *one* question : 1×10

(a) (i) If z_1, z_2 and a are complex numbers where $a \neq 0$, show that

but (the p.v. of a^{z_1}). (the p.v. of a^{z_2}) = the p.v of $a^{z_1+z_2}$. 4

(ii) Prove that the minimum value of

$x^2+y^2+z^2$ is $\left(\frac{e}{7}\right)^2$ where x, y, z are positive real numbers subject to the condition $2x+3y+6z=c$, c being a constant. Find the values of x, y, z for which the minimum value is attained. 3

(iii) Solve the equation

$3x^3-26x^2+52x-24=0$ given that the roots are in geometric progression. 3

(b) (i) If α, β, γ be the roots of the equation $x^3+px^2+qx+r=0$, then form the equation whose roots are

$\alpha+\frac{1}{\alpha}, \beta+\frac{1}{\beta}, \gamma+\frac{1}{\gamma}$. 5

(iii) Solve the equation $x^4+12x-5=0$ by Ferrari's method. 5

Unit - II

(Sets and Integers)

4. Answer any *five* questions : $2\times5=10$

(a) If $f: A \to B$ and $g: B \to C$ be two mappings such that $g \circ f: A \to C$ is surjective, then show that g is surjective.

[Turn Over]

(b) Use the 2nd principle of Induction to prove that $\left(3+\sqrt{7}\right)^n+\left(3-\sqrt{7}\right)^n$ is an even integer for all $n \in \mathbb{N}$.

(c) Let $f:\mathbb{R}\to\mathbb{R}$ be a functin defined by $f(x)=x^2, x\in\mathbb{R}$,

and suppose $P=\{x\in\mathbb{R}: O\le x\le 4\}$. Find $f^{-1}[f(P)]$. Is $f^{-1}[f(P)]$ equal to P?

(d) Find the remainder when 777^{777} is divided by 16.

(e) If k be a + ve integer, show that gcd (ka, kb) = k gcd (a, b).

(f) Prove that there is a one-to-one correspondence between the sets (0,1) and [0, 1].

(g) Use division algorithm to prove that the square of any integer is of the form $5k$ or $5k\pm 1, k\in\mathbb{Z}$.

(h) Use the theory of congruences to prove that $17 \mid 2^{3n+1}+3.5^{2n+1}, \forall n\ge 1$.

5. Answer any *one* question : $1\times5=5$

(a) (i) Prove that the product of any m consecutive integers is divisible by m.

(ii) State the fundamental theorem of Arithmetic.

4+1

(b) Let $S = \{x \in \mathbb{R} : -1 < x < 1\}$ and $f : \mathbb{R} \to S$ be defined by $f(x) = \frac{x}{1+|x|}, x \in \mathbb{R}$. Show that f is a bijection and find f^{-1}.

Unit - III

(System of Linear Equations)

6. Answer any *two* questions : 2×2=4

(a) Find the conditions on $a, b \in \mathbb{R}$ so that the set $\{(a,b,1),(b,1,a),(1,a,b)\}$ is linearly dependent in $\mathbb{R}^3$.

(b) Find all real λ for which the rank of matrix A is 2 :

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 3 & \lambda \\ 1 & 1 & 6 & \lambda+1 \end{pmatrix}$$

(c) For what value of k the planes $x - 4y + 5z = k$, $x - y + 2z = 3$ and $2x + y + z = 0$ intersect in a line?

[Turn Over]

7. Answers any *one* question : 1×5

(a) Solve the system of equations

$x_2 + x_3 = a$

$x_1 + x_3 = b$

$x_1 + x_2 = c$ and use the solution to find the inverse of the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. 5

(b) Determine the conditions for which the system of equations

$$x + y + z = b$$
$$2x + y + 3z = b+1$$
$$5x + 2y + az = b^2$$

(a) has only one solution (b) has no solution

(c) has many solutions.

Unit - IV

(Linear Transformation and Eigen Values)

8. Answer any two questions : 2×2=4

(a) A linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_1 - 3x_2 - 2x_3).$$

$(x_1, x_2, x_3) \in \mathbb{R}^3$. Find the matrix of T relative to the ordered bases $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$.

(b) Define an eigen vector of a matrix $A_{n \times n}$ over a field F. Show that there exist many eigen vectors of A corresponding to an eigen value $\lambda \in F$.

(c) Let V be a real vector space with $\{\alpha, \beta, \gamma\}$ as a basis. Prove that the set $\{\alpha + \beta + \gamma,\ \beta + \gamma,\ \gamma\}$ is also *a* basis.

9. Answer any *one* question : $1 \times 10 = 10$

(a) (i) State Cayley-Hamilton theorem and use it to find A^{100}, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$ 1+4

(ii) Show that the intersection of two subspaces of a vector space over a field F is a subspace of V. 3

[Turn Over]

(iii) If, $\alpha = (1, 2, 2)$, $\beta = (0, 2, 1)$, $\gamma = (2, 2, 4)$, determine whether α is a linear combinatin of β and γ. 2

(b) (i) If S be a real skew symmetric matrix of order n, prove that —

(I) the matrix $S - I_n$ is non-singular,

(II) the matrix $(S - I_n)^{-1}\ (S + I_n)$ is orthgonal.

(III) if X be an eigen vector of S with eigen value λ, then X is also an eigen vector of $\left(S - I_n\right)^{-1}\left(S + I_n\right)$ with eigen value $\frac{\lambda + 1}{\lambda - 1}$. 1+2+3

(ii) Let $S = \left\{\left(x, y, z\right) \in \mathbb{R}^3 : x + y + z = 0\right\}$.

Prove that S is a subspace of $\mathbb{R}^3$. Find a basis of S. Determine the dimension of S. 4
