2018

CBCS

3rd Semester

MATHEMATICS

PAPER—GE3T

(Honours)

Full Marks : 60

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Answer *all* questions

Differential Equation and Vector Calculus

1. Answer any *ten* questions : 10×2

(a) Prove that $\left[\vec{\alpha}+\vec{\beta},\ \vec{\beta}+\vec{\gamma},\ \vec{\gamma}+\vec{\alpha}\right]=2\left[\vec{\alpha},\vec{\beta},\vec{\gamma}\right]$.

(b) If $\vec{R}$ be a unit vector in the direction of $\vec{r}$, then Prove that $\vec{R}\times\dfrac{d\vec{R}}{dt}=\dfrac{1}{r^2}\vec{r}\times\dfrac{d\vec{r}}{dt}$, where $r=\left|\vec{r}\right|$.

(Turn Over)

(c) State Picard's Theorem.

(d) Calcualte the Wronskian of the set $\{x, x^2, x^3\}$

(e) State the Principle of Superposition for homogeneous equation.

(f) Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

(g) If θ be the angle between two unit vectors $\vec{u}$ and $\vec{v}$, then prove that $2\sin\frac{\theta}{2} = \left|\vec{u} - \vec{v}\right|$.

(h) Define equilibrium points.

(i) Solve the equation $\frac{dx}{dt} = -wy$ and $\frac{dy}{dt} = wx$ and show that the point (x, y) lies on a circle.

(j) Define basic theory of linear systems in normal form of two equations in two unknown functions.

(k) Find the value of the constant d, such that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+d\hat{j}+5\hat{k}$ are coplaner.

(l) Show that the vector $\vec{r}=(x+3y)\hat{i}+(y+az)\hat{j}+(x+az)\hat{k}$ is solenoidal, if $a=-2$.

(m) If S is any closed surface enclosing a volume V and $\vec{F}=x\hat{i}+2y\hat{j}+3z\hat{k}$, Prove that $\int_S \vec{F}\cdot n\, dS = 6V$

(n) Solve : $\left(xy^2-e^{1/x^3}\right)dx-x^2y\,dy=0$.

(o) Find the differential equation, whose Primitives are $(x^2+1)(y^2+1)=C$, where C is arbitrary constant.

2. Answer any *four* questions : 4×5

(a) (i) Solve the equation $\frac{d^2y}{dx^2}+(x-1)^2\frac{dy}{dx}-4(x-1)y=0$ in series about the ordinary point $x=1$.

(ii) Determine the singular point of the following differential equation.

$$x^2(x-1)^2\frac{d^2y}{dx^2}+2(x-2)\frac{dy}{dx}+(x+3)y=0.$$

Also find the indicial equation.

(iii) Evaluate $\int_S \vec{F}\cdot\vec{n}\,ds$, where $\vec{F} = 4xi - 2y^2 j + z^2 k$ and S is the curved surface of the cylinder $x^2 + y^2 = 4$ bounded by the planes $z = 0$, $z = 3$.

(b) (i) Find the directional derivative of $f = xy^2 + yz^2 + zx^2$ at the point (1, −2, 5) in the directioin of x-axis.

(ii) Find the unit vector in the direction of the tangent at any point on the curve given by $\vec{r} = (a \cos t)i + (a \sin t)j + btk$. 3+2

(c) (i) Prove that $\frac{d\vec{n}}{ds} + \tau\vec{b} - \kappa\vec{t}$, where $\vec{t}$, $\vec{b}$, $\vec{n}$ are the unit targent, binormal and principal normal vectors respectively and τ, κ and s are torsion, curvature and arc length respectively.

(ii) Find the magnitude of the volume of the parallelopiped having the vectors $\vec{a} = -3i + 7j + 5k$, $\vec{b} = 5i + 7j - 3k$, and $\vec{c} = 7i - 5j - 3k$ as the concurrent edges. 2+3

(d) Solve the differential equation $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$ by the method of variation parameter.

(e) Verify Green's theorem in the plane for $\oint_c (xy+y^2)dx + x^2dy$ where c is the closed of the region bounded by $y = x$ and $y = x^2$.

(f) Solve by the method of undetermined coefficients $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 8$.

3. Answer any *two* questions : 2×10

(a) Consider the linear system

$$\frac{dx}{dt} = 5x + 3y$$
$$\frac{dy}{dt} = 4x + y$$

(i) Show that $x = 3e^{7t}$, $y = 2e^{7t}$ and $x = e^{-t}$, $y = -2e^{-t}$ are solutions of this system. 4

(ii) Show that the above two solutions are linearly independent on every interval $a \le t \le b$ and write the general solution of the system. 3

(iii) Find the solution $x = f(t)$, $y = g(t)$ of the system such that $f(0) = 0$ and $g(0) = 0$. 3

(b) If $\vec{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the following paths C :

(i) $x = t,\ y = t^2,\ z = t^3$

(ii) the straight lines from (0, 0, 0) to (1, 0, 0) then to (1, 1, 0), and then to (1, 1, 1).

(iii) the straight line joining (0, 0, 0) and (1, 1, 1)

3+4+3

(c) (i) Solve the following differential equation

$$\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + 2\gamma w = 0$$

in a series about the ordinary point $z = 0$.

(ii) Solve : $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$ 5+5

(b) Show that $f(x) = \frac{1}{x}$, $x > 0$ is not uniformly continuous in $[0, 1]$.

3. Answer any *one* question : 1×10

(a) (i) Prove that if a real valued function f is continuous on a closed interval $I = [a, b]$, then it is bounded there. 5

(ii) Prove that if a function f is continuous on a closed interval $I = [a, b]$ then f is uniformly continuous on I. 5

(b) (i) Let f be a real-valued continuous function in a closed interval $[a, b]$. Suppose $f(a) \neq f(b)$. Then prove that f assumes every value between $f(a)$ and $f(b)$ at least once. 6

(ii) Apply $\epsilon-\delta$ definition to show that the function

$$f(x) = x\sin\left(\frac{1}{x}\right), \quad x \neq 0$$

$$= 0, \quad x = 0$$

is continuous at $x = 0$. 4

Unit—II

4. Answer any *two* questions : 2×2

(a) State Rolle's theorem.

(b) Is Rolle's theorem applicable to

$f(x) = 1 - x^{2/3}$, $-1 \le x \le 1$? Justify your answer.

(c) For what range of value of x, $f(x) = 2x^3 - 9x^2 + 12x - 3$ decreses as x increses ?

5. Answer any *two* questions : 2×5

(a) Show that $\frac{x}{1+x} < \log(1+x) < x$, if $x > 0$.

(b) In the Mean value theorem

$f(x + h) = f(x) + hf'(x + \theta h)$, $0 < q < 1$

Show that the limiting value of θ as $h \to 0+$ is $\frac{1}{2}$ when $f(x) = \sin x$.

(c) State and prove Lagrange's Mean value theorem.

Unit—III

6. Answer any *two* questions : 2×2

(a) Show that the function $f(x) = x^3 - 3x^2 + 6x + 3$ does not possess any maximum or minimum value.

(b) State Taylor's theorem with Lagrange's form of remainder.

(c) State Maclaurin's theorem with remainder.

7. Answer any *one* question : 1×10

(a) (i) Expand the function $f(x) = \cos x$ in power of x in infinite series. 4

(ii) State and prove Cauchy's mean value thorem. 6

(b) (i) If in the Cauchy's MVT we take $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{2}}$ then prove that c is the geometric mean between a and b. 4

(ii) Find Canchy's Remainder after n terms in the expansions of $(1 + x)^m$ and $\log(1 + x)$ in power of x. 3+3

Unit—IV

8. Answer any *two* questions : 2×3

(a) If (X, d) be a metric space, then show that $\left(X, \frac{d}{1+d}\right)$ is also a metric space.

(b) Define closer of a set in a metric space. Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(c) Let (X, d) be a metric space and $A, B \subseteq X$. Show that diam $(A \cup B) \leq \text{diam}(A) + \text{diam}(B) + d(A, B)$.

9. Answer any *one* question : 1×5

(a) Prove that any finite set has no limit point. 2

(b) In a metric space prove that any open sphere is an open set. 3

Group Theory–1

Unit—I

1. Answer any *two* questions : 2×2

(a) Prove that a group (G, o) is Abelian if $(a \text{ o } b)^{-1} = a^{-1} \text{ o } b^{-1}$ for all $a, b \in G$.

(b) Consider the group $(D, *)$ where D is the set of all odd integers and $a * b = a + b - 1$ for $a, b \in D$. Find 3^{-1}.

(c) Define dihedral group D_3. 2

2. Answer any *one* question : 1×5

(a) Let $G = \{(a, b) \in Q \times Q : b \neq 0\}$. Prove that (G, o) is an abelian group, where o is defoned by (a, b) o (c, d) $= (ad + bc, bd)$ for $(a, b), (c, d)$, on G.

(b) Let X be a non empty set and $P(X)$ be the power set of X. Examine if $P(X)$ is a group under the composition '*' defined by $A * B = A \cap B$, $A, B \in P(X)$. 5

Unit—II

3. Answer any *two* questions : 2×2

(a) Let G be an abelian group. Prove that the subset $H = \{g \in G : g = g^{-1}\}$ forms a subgroup of G.

(b) Let G be a group and $a \in G$. Prove that $Z(G)$, the centre of the group G, is a subgroup of $C(a)$, the centralizer of a.

(c) Let G be a group and H_1, H_2 be two subgroups of G. Then show that $H_1 \cap H_2$ is a subgroup of G.

4. Answer any *two* questions : 2×5

(a) State and prove the necessary and sufficient condition that a non empty subset H of G to be a subgroup of G.

(b) Let G be a group on which $(ab)^3 = a^3b^3$ for all $a, b \in G$. Show that $H = \{x^2 : x \in G\}$ is a subgroup of G.

(c) If H and K be two subgroups of a group G, then show that HK is a subgroup of G iff $HK = KH$.

Unit—III

5. Answer any *two* questions : 2×2

(a) Prove that every cyclic group is Abelian.

(b) A cyclic group G has only one generator. Prove that either $0(G) = 1$ or $0(G) = 2$.

(c) Examine whether the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 1 & 3 & 2 & 8 & 6 & 7 \end{pmatrix}$$ is odd or even.

6. Answer any *one* question : 1×10

(a) (i) Define cyclic group. Prove that every subgroup of a cyclic group is cyclic. 1+4

(ii) Define alternating group. Show that every permutation on a finite set is either a cycle or it can be expressed as a product of disjoint cycles. 1+4

(b) (i) State and Prove Lagrange's theorem for a finite group. 5

(ii) Let $S = \{1, w, w^2, -1, -w, -w^2\}$, where $w = \cos^2\frac{\pi}{3} + \sin^2\frac{\pi}{3}$. Prove that S is a cyclic group under multiplication.

Unit—IV

7. Answer any *two* questions : 2×2

(a) Define external direct product of two groups.

(b) Show that the alternating group A_3 is a normal subgroup of the symmetric group S_3.

(c) State cauchy's theorem for finite abelian group.

8. Answer any *one* question : 1×10

(a) (i) Let G be the group of all $n \times n$ real non-singular matrices and H be the group of all $n \times n$ real orthogonal matrices. Prove that H is a subgroup of G but H is not a normal subgroup of G. 6

(ii) Let M and N be normal subgroups of a group G such that $M \cap N = \{e\}$. Prove that $mn = nm$ for all $m \in M$ and for all $n \in N$. 4

(b) (i) Find the number of elements of order 5 in the group $Z_{15} \times Z_{10}$. 4

(ii) Prove that the group $Z \times Z$ is not cyelic. 3

(iii) Prove that the group $Z_3 \times Z_4$ is cyclic. 3

Unit—V

9. Answer any *two* questions : 2×2

(a) If $f: G \to G'$ be a homomorphism then show that $f(a^{-1}) = [f(a)]^{-1}, \ \forall a \in G$.

(b) Define image and Kernel of a homomorphism. 1+1

(c) Let $\phi : (G, \circ) \to (G', *)$ be an isomorphism.

Then show that $\phi^{-1} : (G', *) \to (G, \circ)$ is also an isomorphism.

10. Answer any *one* question : 1×5

(a) State and prove first isomorphism theorem.

(b) State and prove Cayley theorem for finite group.

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