বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

M.Sc. Examinations 2020
Semester IV
Subject: APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING
Paper: MTM 401
(Functional Analysis)

(Theory)

Full Marks: 40 **Time: 2hrs.**

Candidates are required to give their answers in their own words as far as practicable.

Answer any one of the following:

1. (a) Let X be an infinite dimensional normed space and Y be a non-zero normed space. Construct an operator $T: X \to Y$ which is linear but not bounded.

   (b) Let $T_1 \in B(V_1, V_2), T_2 \in B(V_2, V_3)$ where V_1, V_2, V_3 are normed spaces. Prove that $T_2T_1 \in B(V_1, V_3)$ and $\| T_2T_1 \| \leq \| T_2 \| \| T_1 \|$.

2. (a) Let V, W be normed spaces and $F: V \to W$ be linear. Show that F is continuous if and only if for every Cauchy sequence $\{x_n\}$ in V the sequence $\{F(x_n)\}$ is Cauchy in W.

   (b) Let $X = C[a, b]$ with the norm given by $\| x \|_1 = \int_a^b |x(t)| \, dt, \ x \in X$. If $k(.,.)$ is a continuous function on $[a, b] \times [a, b]$ and $F(x)(s) = \int_a^b k(s,t)x(t)dt, \ x \in X, s \in [a, b]$, then show that $F \in B(X)$ and $\| F \| = \sup\{\int_a^b |k(s,t)| \ ds : t \in [a, b]\}$.

3. (a) Let X be a linear space over $\mathbb{C}$ and f be a complex linear functional over X. Then $Re(f)$ is a real linear functional on X, regarded as a linear space over $\mathbb{R}$. Show that

   (i) Ref determines as follows: $f(x) = Re\, f(x) - i\, Ref(ix) \ x \in X$.

(ii) If $\|\|$ is a norm on X, then $\| Re(f) \|=\| f \|$.

(b) Let $X = C[0,1]$ with the supremum norm. Consider the sequence $x_n(t) = \frac{t^n}{n^4}, t \in [0,1]$. Check whether the series $\sum_{n=1}^{\infty} x_n$ is summable in X.

4. (a) Let X and Y be Banach spaces and $A \in BL(X,Y)$. Show that there is a constant $c > 0$ such that $\| Ax \| \geq c \| x \|$ for all $x \in X$ if and only if $Ker(A) = \{0\}$ and $Ran(A)$ is closed in X.

(b) Give an example to show that the completeness of the domain is an essential requirement in the Uniform Boundedness Principle.

5. (a) If A^* is the adjoint of the operator $A: H \to H$ then show that $\| A \|=\| A^* \|$, where H is a Hilbert space.

(b) Let H be a Hilbert space and $E \subset$ H. Prove that $\overline{span(E)} = E^{\perp\perp}$.

6. (a) Suppose $\mathrm{X} = \mathrm{C}^1[0,1]$, i.e. the set of all functions $\mathrm{f}: [0,1] \to \mathbb{C}$ such that f' exists and is continuous. Let $\mathrm{Y} = \mathrm{C}[0,1]$ and let X and Y be equipped with supremum norm. Define $\mathrm{A}: \mathrm{X} \to \mathrm{Y}$ by $\mathrm{Af} = \mathrm{f}'$. Show that the graph of A is closed.

(b) Let $X = C^1[0,1]$ equipped with the norm $\| x \|=\| x \|_\infty + \| x' \|_\infty$ and $Y = C[0,1]$ equipped with the supremum norm. Check whether the linear operator $F: X \to Y$ defined by $F(x) = x$ is continuous. Is F an open map?

7. (a) Let Y be a normed space and Y_0be a dense subspace of Y. Suppose Z is a Banach space and $T \in BL(Y_0, Z)$. Prove that there exist a unique $\tilde{T} \in BL(Y, Z)$ such that $\tilde{T}|Y_0 = T$.

(b) If X is a normed space, M is a closed subspace of X, $x_0 \in X \backslash M$ and $d = dist(x_0, M)$, show that there is an $f \in X^*$ such that $f(x_0) = 1, f(x) = 0$ for all $x \in M$ and $\|f\| = d^{-1}$.

8. (a) Let X and Y be Banach spaces and $F: X \to Y$ be linear. Let $\{g_s\} \subset Y^*$ such that for every nonzero y in Y, there is some s with $g_s(y) \neq 0$. Prove that F is continuous if and only if $g_s \circ F$ is continuous for every s.

(b) Suppose $\{T_n: n = 1,2, \dots\} \subset BL(X,Y)$ is a sequence of bounded operators, where X is a Banach space and Y is anormed space, and suppose that the sequence $\{T_n x\}_{n=1}^{\infty}$ is a convergent sequence in Y, for each $x \in X$. Show that the equation $Tx = \lim_{n\to\infty} T_n x$ defines a bounded operator $T \in BL(X,Y)$.

9. (a) Let X and Y be inner product spaces. Then a linear map $F: X \to Y$ satisfies $\langle F(x), F(y) \rangle = \langle x, y \rangle$ for all $x, y \in X$ if and only if it satisfies $\|F(x)\| = \|x\|$ for all $x \in X$, where the norms on X and Y are induced by the respective inner products.

(b) Let $P \in BL(H)$ be a nonzero projection on a Hilbert space H and $\|P\| = 1$. Then show that P is an orthogonal projection.

10. (a) Let $T_n \in BL(X, Y)$, $(n \geq 1)$ where X and Y are normed spaces. If $\{T_n x\}$ converges weakly in Y for every $x \in X$, then there exist unique $T \in BL(X, Y)$ such that $T_n \to T$ weakly.

(b) Let H, K be Hilbert spaces and $T_n (n \geq 1), T \in BL(H, K)$. Show that $T_n \to T$ strongly does not necessarily imply that $T_n^* \to T^*$ strongly. Here $*$ denotes the adjoint of the corresponding operator.

11. (a) Let H be a (complex) Hilbert space, $T \in BL(H)$ and $< Tx, x >= 0$ for all $x \in H$. Then show that $T = 0$. Also, give an example to show that this result is not true for real Hilbert space.

(b) Show that $Ran(T) = Ran(T^*)$ if $T \in BL(H)$ is normal and H is a Hilbert space.

12. (a) Let $\{u_\alpha\}$ be an orthonormal set in an inner product space X and $x \in X$. Show that the set $G_x = \{u_\alpha : < x, u_\alpha > \neq 0\}$ is countable.

(b) Let X be an inner product space and $E \subseteq X$ be closed under scalar multiplication. Show that $x \perp E$ if and only if $dist(x, E) = \| x \|$, $x \in X$.