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Social Constructivism in Teaching Mathematics: An Absorbing Markov Chain Representation

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ABSTRACT

The application of the principles of social constructivism for learning has become very popular during the last decades for teaching mathematics. In this article the process of teaching mathematics on the basis of those principles is represented by an absorbing Markov chain introduced on its steps. Interesting conclusions are produced through this representation and a measure is obtained for the effectiveness of teaching procedure. An example for teaching the derivative to fresher university students is also presented illustrating our results.

Keywords: Social constructivism, Markov chain (MC), transition probabilities, transition matrix, absorbing Markov chain (AMC), fundamental matrix.

Mathematical Subject Classification (2010): 60J20, 97B50

1. Introduction

Mathematics teaching is intended to promote the learning of mathematics. But, while theory provides us with lenses for analyzing learning, the position of mathematics teaching remains theoretically anomalous and underdeveloped. We might see one of the problems to lie in the relationships between learning, teaching and the practice of teaching. Theories help us to analyze, or explain, but they do not provide recipes for action; rarely do they provide direct guidance for practice.

The main theories about learning fall in the following four philosophical frameworks:

- *Behaviorism*, a theory established by the American psychologist John B. Watson (1878-1958), which considers learning as the acquisition of new behavior based on environmental conditions and discounts any independent activities of the mind [1].
- *Cognitivism*, which replaced behaviorism during the 1960's as the dominant theory for the process of learning and argues that knowledge can be seen as a process of symbolic mental constructions and that learning is defined as change in individual's cognitive structures [24]. Changes in the learner's behavior are in fact observed, but only as an indication of what is occurring in his/her mind. In other words, cognitive theories look beyond behavior to explain the brain-based process of learning.

- *Constructivism*, a philosophical framework based on Piaget's theory for learning [10] and formally introduced by von Clasersfeld [15] during the 1970's, which suggests that knowledge is not passively received from the environment, but is actively constructed by the learner through a process of adaptation based on and constantly modified by the learner's experience of the world.
- The Vygotsky's *social development theory* [2], which argues that knowledge is a product of culture and social interaction. Learning takes place when the individuals engage socially to talk and act about shared problems or interests. The synthesis of constructivism with the socio-cultural ideas is known as *social constructivism*. The basic difference between the classical constructivism - which is often termed as *cognitive constructivism* - and the social constructivism is that the former argues that thinking precedes language, whereas the latter supports the exactly inverse argument [8].

Some decades ago the dominant teaching method in mathematics education used to be the *explicit instruction (EI)*, which is mainly based on principles of cognitivism. The teacher is in the "center" of this method and tries with clear statements and explanations of the mathematical context and by supported practice to transfer the new knowledge to students in the best possible way [3]**.** The main criticism against EI is that it may prevent conceptual understanding and critical analysis [13]. As a result many teachers, adopting ideas of constructivism, enriched the EI with a series of challenging questions so that to keep an active discourse with students, as a means to promote mathematical thinking [7]**.** However**,** following the failure of the introduction of the "new mathematics" to school education, constructivism and the socio-cultural theories for learning have become very popular during the last decades as a basis for teaching and learning mathematics, especially among teachers of the elementary and secondary education. New teaching approaches have been introduced, like the *problem-based learning* [18], the *application-oriented teaching* involving mathematical modeling [17], the *inquiry-based learning* through creative exploration [4], the formation of CoP's among students and teachers [23], etc.

In this work an *absorbing Markov chain (AMC)* is introduced on the steps of the process of teaching mathematics which is based on those ideas. Interesting conclusions are derived through this representation and a measure is obtained for the teaching effectiveness. The rest of the article is organized as follows: In section 2 the mathematical background about AMCs needed for the understanding of the paper is presented. In section 3 the AMC model is developed, while an example is presented in section 4 on teaching the derivative on fresher university students illustrating the applicability and usefulness of the AMC model in practice. The article closes with a brief discussion on the perspectives of future research on the subject and the general conclusions presented in section 5.

2. Absorbing Markov chains

A *Markov Chain (MC)* is a stochastic process that moves in a sequence of steps (phases) through a set of states and has a *one-step memory*. That means that the probability of entering a certain state in a certain step depends on the state occupied in the previous step and not in earlier steps. This is known as the *Markov property*. However, for being able to model as many real life situations as possible by using MCs, one could accept in practice that the probability of entering a certain state in a certain step, although it may not be completely independent of previous steps, it mainly depends on the state occupied in the previous step [5]. When the set of states of a MC is a finite set, then we speak about a *finite MC.* For general facts on finite MCs we refer to Chapter 2 of the book [21] and for more details to the book [6].

The basic concepts of MCs were introduced by A. Markov in 1907 on coding literal texts. Since then the MC theory was developed by a number of leading mathematicians, such as A. Kolmogorov, W. Feller, etc. However, only from the 1960's the importance of this theory to the natural, social and most of the applied sciences has been recognized ([21], Chapters 2, 3).

Let us consider a finite MC with n states, say S_1, S_2, …, S_n, where n is a non negative integer, $n \geq 2$. Denote by p_{ij} the *transition probability* from state S_i to state S_j , i, j = 1, 2,…, n ; then the matrix $A= [p_{ij}]$ is called the *transition matrix* of the MC. Since the transition from a state to some other state (including itself) is the certain event, we have that

$$p_{i1} + p_{i2} + \ldots + p_{in} = 1, \text{ for } i=1, 2, \ldots, n \qquad (1).$$

A state of a MC is called *absorbing* if, once entered, it cannot be left. Further a MC is said to be an *absorbing MC (AMC)*, if it has at least one absorbing state and if from every state it is possible to reach an absorbing state, not necessarily in one step. Working with an AMC with k absorbing states, $1 \leq k < n$, one brings its transition matrix A to its *canonical form* A^* by listing the absorbing states first and then makes a partition of A^* as follows

$$A^* = \begin{bmatrix} I_k & | & O \\ - & | & - \\ R & | & Q \end{bmatrix} \qquad (2).$$

In the above partition of A^*, I_k denotes the unitary k X k matrix, O is a zero matrix, R is the (n – k) X k transition matrix from the non-absorbing to the absorbing states and Q is the (n – k) X (n – k) transition matrix between the non absorbing states.

It can be shown ([16], Section 2) that the square matrix $I_{n-k} - Q$, where I_{n-k} denotes the unitary n-k X n-k matrix, is always an invertible matrix. Then, the *fundamental matrix* N of the AMC is defined to be the inverse matrix of $I_{n-k} - Q$. Therefore ([9], Section 2.4)

$$N = [n_{ij}] = (I_{n-k} - Q)^{-1} = \frac{1}{D\ (I_{n-k} - Q)}\, adj\ (I_{n-k} - Q) \qquad (3).$$

In equation (3) $D\ (I_{n-k} - Q)$ and $adj\ (I_{n-k} - Q)$ denote the determinant and the *adjoin* of the matrix $I_{n-k} - Q$ respectively. It is recalled that the adjoin matrix of a matrix M is the matrix of the algebraic complements of the transpose matrix M^t of M, which is obtained by turning the rows of M to columns and vice versa. It is also recalled that the algebraic complement m_{ij}' of an element m_{ij} of M is calculated by the formula

$$m_{ij}' = (-1)^{i+j} D_{ij} \tag{4}$$

In formula (4) D_{ij} denotes the determinant of the matrix obtained by deleting the *i*-th row and the *j*-th column of M.

It is well known ([6], Chapter 3) that the element n_{ij} of the fundamental matrix N gives the mean number of times in state s_i before the absorption, when the starting state of the AMC is s_j, where s_i and s_j are non absorbing states.

3. The Markov chain model for teaching mathematics

3.1. An Instructional Treatment for Teaching Mathematics based on the Principles of Social Constructivism

The steps of typical framework for teaching mathematics based on the principles of social constructivism are the following:

- *Orientation* (S_1): This is the starting step which connects the past with the present learning experiences and focuses student thinking on the learning outcomes of the current activities.
- *Exploration* (S_2): In this step students explore their environment to create a common base of experiences by identifying and developing concepts, processes and skills.
- *Formalization* (S_3): Here students explain and verbalize the concepts that they have been explored and develop new skills and behaviors. The instructor has the opportunity to introduce formal terms, definitions and explanations for the new concepts and processes and to demonstrate new skills or behaviors.
- *Assimilation* (S_4): In that step students develop a deeper and broader conceptual understanding and obtain more information about areas of interest by practicing on their new skills and behaviors.
- *Assessment* (S_5): This is the final step of the teaching process, where learners are encouraged to assess their understanding and abilities and teachers evaluate student skills on the new knowledge.

Depending on the student reactions in the classroom, there are forward or backward transitions between the three intermediate steps (S_2, S_3, S_4) of the above framework during the teaching process, the flow-diagram of which is shown in Figure 1.

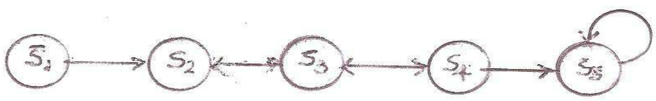

Figure 1: The flow-diagram of the teaching process

It is of worth noticing that the steps S_2, S_3, S_4 are actually the Polya's *consecutive phases* for teaching mathematics with the method of *discovery* [12], which is highly based on *active learning*, an idea adopted from Piaget [10]. The consecutive phases and the active learning are the first two of the Polya's three famous *axioms of learning* [11], for which he used to say that they are not of his own conception, but they have been derived through the experience of centuries. The third axiom is the *best motivation* (a problem, a real life situation, a story, etc.), which is connected to the step of orientation (S_1) and has to do with the suitable "learning situation" that has to be created by the instructor connecting the previous to the new knowledge and increasing the student interest about it.

3.2 The AMC Model

We introduce a finite MC having as states S_i, i = 1, 2,…, 5, the corresponding steps of the teaching framework described in section 3.1. From the flow-diagram of Figure 1 it becomes evident that the above chain is an AMC with S_1 being its starting state and S_5 being its unique absorbing state. The minimum number of steps before the absorption is 4 and this happens when we have no backward transitions between the three middle states S_2, S_3 and S_4 of the chain.
The transition matrix of the chain is the matrix

$$A = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & p_{32} & 0 & p_{34} & 0 \\ 0 & 0 & p_{43} & 0 & p_{45} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}, \text{ with } p_{32} + p_{34} = p_{43} + p_{45} = 1.$$

The canonical form of A is the matrix

$$A^* = \begin{array}{c} \\ S_5 \\ - \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} \begin{array}{c} \begin{array}{cccccc} S_5 & & S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 1 & | & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - \\ 0 & | & 0 & 1 & 0 & 0 \\ 0 & | & 0 & 0 & 1 & 0 \\ 0 & | & 0 & p_{32} & 0 & p_{34} \\ p_{45} & | & 0 & 0 & p_{43} & 0 \end{bmatrix} \end{array} = \begin{bmatrix} I_1 & | & O \\ - & | & - \\ R & | & Q \end{bmatrix}.$$

Then $I_4 - Q = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -p_{32} & 1 & -p_{34} \\ 0 & 0 & -p_{43} & 1 \end{bmatrix}$ and

$$D(I_4 - Q)=\begin{vmatrix} 1 & -1 & 0 \\ -p_{32} & 1 & -p_{34} \\ 0 & -p_{43} & 1 \end{vmatrix} = 1 - p_{34}p_{43} - p_{32}.$$

Further, by equation (4), the algebraic complement of the element $m_{11} = 1$ of the transpose matrix

$$(I_4 - Q)^t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & -p_{32} & 0 \\ 0 & -1 & 1 & -p_{43} \\ 0 & 0 & -p_{34} & 1 \end{bmatrix} \text{ is equal to } \begin{vmatrix} 1 & -p_{32} & 0 \\ -1 & 1 & -p_{43} \\ 0 & -p_{34} & 1 \end{vmatrix} = 1 - p_{43}p_{34} - p_{32}.$$

In the same way we calculate the algebraic complements of all the other elements of $(I_4 - Q)^t$ and replacing their values and the value of $D(I_4 - Q)$ to equation (3) we find that

$$N = [n_{ij}] = \frac{1}{1 - p_{34}p_{43} - p_{32}} \begin{bmatrix} 1 - p_{34}p_{43} - p_{32} & 1 - p_{34}p_{32} & 1 & p_{34} \\ 0 & -1 + p_{34}p_{34} & -1 & -p_{34} \\ 0 & p_{32} & 1 & p_{34} \\ 0 & -p_{32}p_{43} & -p_{43} & -1 + p_{32} \end{bmatrix}.$$

Since S_1 is the starting state of the above AMC, it becomes evident that the mean number of steps before the absorption is given by the sum

$$T = n_{11} + n_{12} + n_{13} + n_{14} = \frac{3 - 2p_{43}p_{34} - p_{32} + p_{34}}{1 - p_{34}p_{43} - p_{32}} \qquad (5).$$

It becomes evident that the bigger is T, the more are the student difficulties during the teaching process. Another factor of the student difficulties is the total time spent for the completion of the teaching process. However, the time is usually fixed in a formal teaching procedure in the classroom, which means that in this case T is the unique measure of the student difficulties.

4. A classroom application

The following application took place recently at the Graduate Technological Educational Institute of Western Greece for teaching the concept of the derivative to a group of fresher students of engineering. The instructor used the instructional treatment that has been described in section 3.1 as follows:

Orientation: The student attention was turned to the fact that the definition of the tangent of a circle as a straight line having a unique common point with its circumference does not hold for other curves (e.g. for the parabola). Therefore, there is a need to search for a definition of the tangent covering all cases and in particular of the tangent at a point of the graph of a given function.

Exploration: The discussion in the class led to the conclusion that the tangent at a point A of the graph of a given function $y=f(x)$ can be considered as the limit position of the secant line of the graph through the points A(a, $f(a)$) and B(b, $f(b)$), when the point B is moving approaching to A either from the left, or from the right (Figure 2). But the slope of the secant line AB is equal to $\frac{f(b)-f(a)}{b-a}$, therefore the slope of the tangent of the graph at A is equal to the limit of the above ratio when b tends to a.

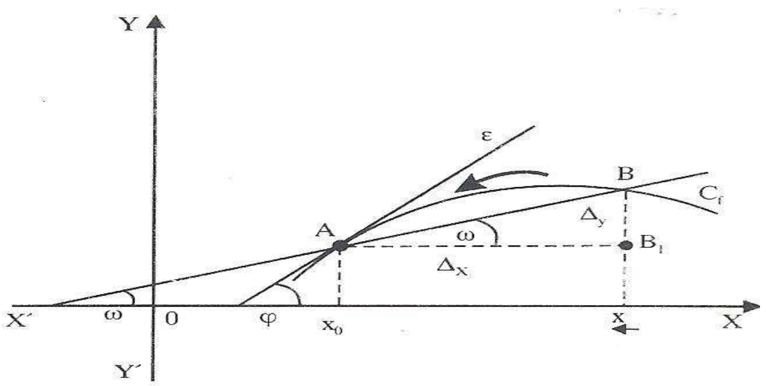

Figure 2: Tangent at a point of the graph of a given function

Formalization: Based on what it has been discussed at the step of exploration, the instructor presented the formal definition of the derivative number $f'(a)$ at a point A(a, $f(a)$) of a given function $y=f(x)$ as the limit (if there exists) of $\frac{f(b)-f(a)}{b-a}$ when $a \to b$, and of the tangent of the graph of $y=f(x)$ at A as the straight line through A with slope $f'(a)$. Some examples followed of calculating the derivative at a given point of a function and the tangent of its graph at this point. Then the definition of the derivative function $y' = f'(x)$ of the function $y=f(x)$ was given and suitable examples were presented to show that its domain is a subset of the domain of $y=f(x)$.

Assimilation: Here the fact that the derivative $y' = f'(x)$ expresses the rate of change of the function $y=f(x)$ with respect to x was emphasized and its physical meaning was also presented connected to the speed and the acceleration at a moment of time of a moving object under the action of a steady force. The fundamental properties of the derivatives followed (sum, product, composite function, etc.) as well as a list of formulas calculating the derivatives of the basic functions and applications of them.

Assessment: At the end of the teaching process a number of exercises and problems analogous to those solved in the classroom were given to students on the purpose of checking at home their understanding of the subject. A week later a written test was performed in the classroom enabling the instructor to assess the student progress.

Another important thing remaining for the instructor was to evaluate the student difficulties during the teaching process, which could help him in reorganizing properly his plans for teaching the same subject in future. This was succeeded with the help of the developed in section 3.2 MC representation of the teaching process in the following way: The instructor observed that the student reactions during the teaching process led to 2 transitions of the discussion from state S_3 (formalization) back to state S_2 (exploration). Therefore, since from state S_2 the chain moves always to S_3 (Figure 1), we had 3 in total transitions from S_2 to S_3. The instructor also observed 3 transitions from S_4 (assimilation) back to S_3. Therefore, since from state S_3 the chain moves always to state S_4 (Figure 1), we had 4 in total transitions from S_3 to S_4. In other words we had 3+3 = 6 in total "arrivals" to S_3, 2 "departures" from S_3 to S_2 and 4 "departures" from S_3 to S_4. Therefore $p_{32}=\frac{2}{6}$ and $p_{34}=\frac{4}{6}$. In the same way one finds that $p_{43}=\frac{3}{4}$ and $p_{45}=\frac{1}{4}$. Replacing the above values of the transition probabilities to equation (5) one finds that the mean number T of steps before the absorption of the MC is equal to 14. Consequently, since the minimum number of steps before the absorption is 4, the students faced significant difficulties during the teaching process. This means that the instructor should find ways to improve his teaching procedure for the same subject in future.

5. Discussion and conclusions

The theory of MCs, being a smart combination of Linear Algebra and Probability, offers ideal conditions for the study and mathematical modelling of a certain kind of situations depending on random variables. In the paper at hands a mathematical representation of the teaching procedure of mathematics based on the principles of the social constructivism for learning was developed with the help of the theory of AMCs. This representation enables the instructor to evaluate the student difficulties during the teaching process. This is very useful for reorganizing the plans for teaching the same subject in future. An application of the AMC was also presented on teaching the concept of the derivative to fresher engineering students. Although the development of the AMC model was proved to be quite laborious requiring the calculation of 17 in total determinants of third order (the determinant of the matrix $I_4 - Q$ and the algebraic complements of its transpose matrix), its final application is very simple. The only thing needed for this purpose is the calculation by the instructor of the transitions of the AMC from S_3 back to S_2 and from S_4 back to S_3. Several other applications of MCs to education have been attempted by the present author in earlier works (e.g. see Chapters 2 and 3 of the book [21]) and it is hoped that this research could be continued in future combined with corresponding fuzzy logic models [19, 20, 22].

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