Chapter 4

Multi-item Supply Chain Management models

4.1 Introduction

The inventory models are normally developed on the basic assumption that the retailer is paid for the units of the item immediately after the units are received. However, it may not be true for today's competitive business transactions. Nowadays, it is normally found that the supplier allows a certain fixed time period (termed as credit period) to its retailers for settling the amount that the retailer owes to the supplier for the item supplied. The trade credit is a supplier's short-term loan to the retailer, allowing the retailer to delay payment of an invoice. Also the retailer can allow trade credit for the customers to increase his/her demand of the items.

Nowadays, promotional effort strategy is essential in the oligopoly marketing system. This strategy is utilised by both large and small business houses to inform, persuade and remind customers about the products and services they have to offer. Without business promotion, companies would be stagnant and would lack substantial growth in the sale of their brands, because their brands would have low visibility in the market. As a result of promotional efforts, customers are informed about new products and are also reminded about existing products. This effort can help companies to introduce new uses for old products in an effort to gain a new segment of market. Recently, the study of inventory control problems incorporating promotional cost dependent demand is made by some researchers

[136, 153, 193]. Again, all these studies are normally made only from the retailers point of view. But in present competitive market, decision has to be made from the supply chain point of view, where the profit of the retailer as well as the profits of all the partners of the supply chain will improve.

In Tsao's [184] and Huang et al.'s [78] investigation, this problem (Two level supply chain with retailer's credit period and promotional cost sharing) is modelled as a profit maximization problem and is analysed under two distinct scenarios: Non-Coordination Scenario (NCS) and Coordination Scenario (CS). However, the following limitations are found in Tsao's and Huang et al.'s work, which are as follows:

- The supplier does not hold any product, but the holding cost of the supplier is considered.
- The interest earned and the interest paid by the retailer and the supplier are not properly calculated.
- They did not consider the uncertain situation due to volatile market, i.e., changing bank interests at regular intervals, uncertain available resource (i.e., budget).

Correcting and removing the above limitations, the following things are incorporated in the first model (Model 4.1) of this chapter:

- The two-level trade credit is considered, i.e., the retailer and the customer both enjoyed credit period.
- Demand of the items are influenced by both customer credit period and promotional cost.
- Uncertain resource in the form of budget is considered.
- Models are discussed in crisp and different imprecise environments.
- A soft computing technique PSO is implemented and used to solve the model under imprecise parameters.

Nowadays, with the advent of multinationals in developing countries, there is a stiff competition amongst the companies for capturing the market of a product. Market price plays key role in stimulating demand of a product. As a result, to boost the demand, some manufacturers offer price discount in the form of putting additional materials in every unit pack, bringing down the unit price for a certain period of time. Obviously, the demand increases due to low price. After that specified period of time, the manufacturer withdraws the additional amount and thus unit price increases. By this process, the demand increases due to the fact that some customers have already accustomed with the product during the price discount period and do not switch over to other products though price discount is withdrawn. This process of boosting a product is commonly practiced by different manufacturers specially when a product is newly launched in the market. Some research works have already been done in this direction [68, 99, 138]. But till now a little attention has been paid on supply chain research papers explicitly incorporating the above type of boosting and change in demand specially on supply chain models under promotional cost sharing. Price discount policy plays an important role in improving channel performance [79, 83, 137, 154, 189, 197, 198, 207]. Also promotional effort strategy is an essential part in coordination mechanism [78, 151, 184]. Tsao [184] discussed how channel coordination can be achieved using promotional cost sharing and cash discount policy. Huang *et al.* [78] made some correction on the study of Tsao [184]. Though promotional cost sharing is studied by Tsao [184] to improve supplier-retailer channel performance, none has studied two level cash discount policy for the same purpose. In the second model (Model 4.2) of this chapter, an attempt has been made to improve performance of a supplier-retailer channel by simultaneous use of promotional cost sharing, single level trade credit and two level cash discount policy. With these implementations, the model is analyzed in fuzzy environment using GMIV approach and credibility measure approach.

In any supply chain, profit of each party mostly depends on the market demand of the items involved in the chain. Though every item has some base demand in the market, goal of every supply chain is to improve this base demand to survive in the market. Displayed inventory level always influences the customers and accordingly the retailers normally hire a showroom in the market place to attract the customers. This investment is mainly done at the retailer level. Two other factors which highly influence the demand are – advertisement [113, 120] and selling price [113, 143, 190]. Any item is supposed to be sold in maximum retail price printed on the packet, but in reality, it is observed that different retailers give different discounts to attract their customers. Sometime packaging is made with some extra amount which basically decreases the unit price. Free gift/extra amount with a purchase above a predefined minimum amount is another approach of reducing the selling price. Again different multinationals as well as small companies use frequent advertisement to boost the demand of their products to the customers. Though this type of investment reduces the profit from per unit selling, as total demand improves significantly high, the resultant profit of each party increases. But if only one party invests this promotional cost, then, he/she will be the sole decision maker (DM) of the system, which may not satisfy the other party's interest. So a coordination is highly required among all the parties in such a manner that all the parties will share the promotional cost and take part in the marketing decision. Some research articles have already been published incorporating promotional cost sharing in supply chain [23, 150]. In all these studies, it is assumed that a promotional effort influences the demand of an item and promotional cost is a function of this promotional effort. From these studies, it is neither clear how promotional effort actually improves the demand nor how the promotional cost function is estimated. Moreover, none of these studies considered the influence of displayed inventory on the demand, specially for a supply chain management (SCM) system under retailer's two-warehouse facility.

It has already been mentioned that the displayed inventory has significant role in drawing attention to the customers. Due to this reason, it is normally observed that the retailers decoratively displayed their items in the market outlet. In the market place, it is very difficult to acquire sufficient space to store and display the units. Normally the retailer distributes the display area among different items depending upon the amount of dependency of the demands on displayed units of different items [16, 114, 117]. Due to insufficient size of the market outlet they hire a warehouse with sufficiently large capacity, little away from the market. Items are initially stored in this warehouse and transferred to the market outlet for sell in a regular time interval. So the retailer uses two warehouses to run his/her business smoothly. There are some inventory control models incorporating this phenomenon [12, 139, 140, 200, 212]. In most of these models, it is noticed that the items are ordered separately and are transferred from the warehouse to the market outlet individually, i.e., order of an item is placed when it's inventory level reaches the reorder level and its units are shifted from warehouse as soon as units at market outlet vanishes or reaches a fixed lower level [88, 113, 117]. But ordering an item as well as its shipment involves some costs, known as ordering cost and transportation cost which are considerable amounts. Simultaneous ordering and transportation of different items may reduce this cost significantly [117]. But a little attention have been paid to develop SCM in this direction, specially under promotional cost sharing. Moreover, though impreciseness of inventory parameters is a well established phenomenon [68, 113, 117], it is not reflected widely in the literature of SCM specially in the models under promotional cost sharing [150]. Incorporating the above mentioned shortcomings, in the third model (Model 4.3) of this chapter, a two-level SCM model is proposed where a retailer collects different items from a wholesaler and sells to its customers using two rented warehouses.

After production of an item it reaches to the customers through different agents, like, supplier, wholesaler, retailer etc., and totality of such a system is known as supply chain. In every supply chain, goal of each party is to improve his/her profit. Due to this reason, in present day competitive market, each party offers some sort of credit period to its purchaser to improve the sale. But sale of each party mainly depends on the demand of the item to the customers. Due to this reason, getting credit opportunity from its wholesaler, the retailer offers some credit opportunity to its customers. But customers are basically floating in nature and their is no guaranty that all the customers will obey the business ethics. A portion of the customers may not pay the credit amount at the end of the credit period. Due to this credit risk, the retailer normally offers a partial credit period to its customers, i.e., credit opportunity is offered on a portion of the amount purchased by any customer. On the other hand, to improve the demand, the retailer uses some promotional activities, like, local advertisement, offering price discount, free gift etc. and the cost due to these activities is known as promotional cost. During the last decade, several research papers have been published reflecting some portions of this real life phenomenon [78, 150, 184, 185].

Again, in present day volatile market, inflation plays a major role in marketing decision [24, 77, 102, 156, 167, 201]. Demands of most of the items in the market are price sensitive in nature, except medicine and life support items. So price sensitive demand is influenced by the inflation also. Also price discount policy is the most effective promotional activity to boost the base demand of an item. But none of the existing studies of the supply chain models reflects this real life phenomenon, specially under trade credit policy and promotional cost sharing. The retailer usually orders different items at a regular time interval, may be termed as basic period (BP) [117] and cycle length of any item is an integer multiple of this BP. Though joint replenishment policy is reflected in few two level supply chain models [6, 117], it has not been reflected in multi-level supply chains under trade credit and promotional cost sharing. Moreover, most of the existing supply chain models are developed in crisp environment, though impreciseness of different parameters of any supply chain is a well established phenomenon [69, 118, 126, 138, 151]. To overcome these shortcomings, in the fourth model (Model 4.4) of this chapter, a multi-item supplier-wholesaler-retailer-customers supply chain is proposed incorporating inflationary effects where each parties offers a partial trade credit period to his/her purchasers. Items are replenished by the retailer using BP policy.

4.2 Model 4.1: Uncertain Multi-item Supply Chain with Two Level Trade Credit Under Promotional Cost Sharing [1]

4.2.1 Assumptions and Notations

The following notations are used in this model:

Notation	Meaning
c_i	supplier's purchase cost of the item i.
w_i	retailer's purchase cost of the item i, which is a mark-up m_s of c_i; i.e., $w_i = m_s c_i$.
r_i	retailer's selling price of the item i, which is a mark-up m_r of w_i; i.e., $r_i = m_r w_i$.
A_R	major setup cost of the retailer per order.
A_S	major setup cost of the supplier per order.
$a_{R,i}$	minor setup cost of the retailer for adding the item i into the order.
$a_{S,i}$	minor setup cost of the supplier for adding the item i into the order.
$h_{R,i}$	retailer's holding cost per unit for the item i, which is a mark-up m_h of w_i; i.e., $h_{R,i} = m_h w_i$.
T	replenishment cycle length.
T^l	optimal value of T for the coordination scenario.
T^t	optimal value of T for the non-coordination scenario.
t_R	customers' credit period offered by the retailer.
t_R^l	optimal value of t_R for the coordination scenario.
t_R^t	optimal value of t_R for the non-coordination scenario.
t_S	retailer's credit period offered by the supplier.
Q_i	order quantity for the item i.
Q_i^l	optimal value of Q_i for the coordination scenario.
Q_i^t	optimal value of Q_i for the non-coordination scenario.

[1]This model has been published in **Computers & Industrial Engineering, 2018, 118, 451-463**, *Elsevier*, with title "*Uncertain multi-item supply chain with two level trade credit under promotional cost sharing*"

Notation	Meaning
ρ_i	retailer promotional effort for the item i, $\rho_i \geq 1$.
ρ_i^l	optimal value of ρ_i for the coordination scenario.
ρ_i^t	optimal value of ρ_i for the non-coordination scenario.
ξ_i	basic demand for the item i.
I_p	rate of interest paid to the bank.
I_e	rate of interest earned from the bank.
F	fraction of the retailer's promotional cost shared by the supplier.
B_R	retailer's total purchase cost.
B_R^m	retailer's maximum budget.
$C_i(\rho_i, \xi_i)$	annual promotional effort cost for the item i, where $C_i(\rho_i, \xi_i) = K_i(\rho_i - 1)^2 \xi_i^{\alpha_i}$, K_i is a positive constant and α_i is a constant.
Π_j	annual profit, $j = R$ for the retailer, $j = S$ for the supplier and $j = C$ for the channel.
$E[\tilde{Z}]$	expected value of a fuzzy number $\tilde{Z}$.
$E[\check{Z}]$	expected value of a rough variable $\check{Z}$.

Symbols ˜ and ˇ are used on the top of above symbols to indicate fuzzy and rough parameters respectively.

This model is developed under the following assumptions:

1. The retailer adopts joint multi-item replenishment policy.

2. No shortages are allowed.

3. The supplier provides a credit period t_S for the retailer.

4. The retailer also provides a credit period t_R for the customer, which magnify the base demand ξ_i with λt_R where λ is a parameter, so chosen to best fit the demand function.

5. The promotional effort ρ_i for the item i also magnify the basic demand ξ_i, with $(\rho_i - 1)$.

6. So introduction of promotional cost and customers' credit period changes the base demand ξ_i, of i-th item to $(\rho_i + \lambda t_R)\xi_i = \rho_i' \xi_i$, where $\rho_i' = (\rho_i + \lambda t_R)$.

7. The promotional effort cost is an increasing function of promotional effort and basic demand, $C_i(\rho_i, \xi_i) = K_i(\rho_i - 1)^2 \xi_i^{\alpha_i}$, where K_i is a positive constant and α_i is a constant [97].

4.2.2 Mathematical Formulation of the Model

Here, a supplier-retailer supply chain is considered where the supplier supplies n items to the retailer under joint replenishment policy and the supplier does not hold any product. The retailer adopts a promotional cost $K_i(\rho_i-1)^2\xi_i^{\alpha_i}$ to increase the base demand (ξ_i) of the i-th item and annual increase of the demand is $(\rho_i-1)\xi_i$. The supplier offers a credit period t_S to the retailer. Due to this facility, the retailer also offers a credit period t_R to the customers to increase the demand of the items. Increase of base demand ξ_i of the i-th item due to the credit period t_R is assumed as $\lambda t_R\xi_i$, where, λ is a parameter used to best fit the demand function. So the resultant demand of i-th item, due to introduction of promotional cost and credit period given to the customers, is increased as $\rho_i'\xi_i=(\rho_i+\lambda t_R)\xi_i$. So effective demand of i-th item $D_i=\rho_i'\xi_i$. Here the cycle length T, the promotional effort ρ_i, $i=1,2,...,n$ and the credit period t_R are decision variables.

4.2.2.1 Retailer's Profit

The order quantity, $Q_i=\rho_i'\xi_i T$

The inventory level of i-th item at any time t, $q_i(t)=Q_i-D_i t$

The major set-up cost per unit time $=\dfrac{A_R}{T}$

The minor set-up cost for the i^{th} item per unit time $=\dfrac{a_{R,i}}{T}$

The selling price for the i^{th} item per unit time $=\dfrac{r_iQ_i}{T}=r_i\rho_i'\xi_i$

The purchase price for the i^{th} item per unit time $=\dfrac{w_iQ_i}{T}=w_i\rho_i'\xi_i$

The promotional cost for the i^{th} item per unit time $=K_i(\rho_i-1)^2\xi_i^{\alpha_i}$

The total holding cost for the i^{th} item

$$= h_{R,i}\int_0^T q_i(t)dt = h_{R,i}\int_0^T (Q_i-D_it)dt = \frac{\rho_i'\xi_iT^2}{2}h_{R,i}$$

Therefore, the holding cost for i^{th} item per unit time $=\dfrac{\rho_i'\xi_iT}{2}h_{R,i}$.

Calculation of interest to be paid and interest earned for i^{th} item:

Assuming $t_R<t_S$, the total interest paid by the retailer for the i^{th} item per unit time $(TIP_{R,i})=\frac{1}{T}(IP_1+IP_2+IP_3)$,

where, IP_1 = Interest to be paid due to the cumulative units stocked during $[t_S, T]$

$$= \int_{t_S}^{T} q_i(t) w_i I_p dt = w_i I_p \frac{D_i}{2}(T - t_S)^2$$

IP_2 = Interest to be paid due to the units sold during $[t_S, T]$

$$= \int_{t_S}^{T} D_i t_R w_i I_p dt = w_i I_p D_i t_R (T - t_S)$$

IP_3 = Interest to be paid due to the units sold during $[t_S - t_R, t_S]$

$$= \int_{t_S - t_R}^{t_S} D_i(t + t_R - t_S) w_i I_p dt = w_i I_p D_i \frac{t_R^2}{2}$$

Hence,

$$\begin{aligned} TIP_{R,i} &= w_i I_p \frac{D_i}{2T}\Big[(T - t_S)^2 + 2t_R(T - t_S) + t_R^2\Big] \\ &= w_i I_p \frac{\rho_i' \xi_i}{2T}(T + t_R - t_S)^2 \end{aligned}$$

Now, total interest earned by the retailer for the i^{th} item per unit time $(TIE_{R,i})$

$$\begin{aligned} &= \frac{1}{T} \times \text{Interest to be earned due to normal selling during } [0, t_S - t_R] \\ &= \frac{1}{T} \int_{0}^{t_S - t_R} D_i\big\{t_S - (t + t_R)\big\} w_i I_e dt = w_i I_e \frac{\rho_i' \xi_i}{2T}(t_S - t_R)^2 \end{aligned}$$

Hence, the retailer's average profit is as follows:

$$\begin{aligned} \Pi_R(\rho_i, T, t_R) &= -\frac{A_R}{T} + \sum_{i=1}^{n} \Big[(r_i - w_i)\rho_i'\xi_i - \frac{a_{R,i}}{T} - \frac{\rho_i'\xi_i T}{2} h_{R,i} - K_i(\rho_i - 1)^2 \xi_i^{\alpha_i} \\ &\quad - w_i I_p \frac{\rho_i'\xi_i}{2T}(T + t_R - t_S)^2 + w_i I_e \frac{\rho_i'\xi_i}{2T}(t_S - t_R)^2\Big] \qquad (4.1) \end{aligned}$$

where, $\rho_i' = \rho_i + \lambda t_R$.

4.2.2.2 Supplier's Profit

The major set-up cost per unit time $= \dfrac{A_S}{T}$
The minor set-up cost for the i^{th} item per unit time $= \dfrac{a_{S,i}}{T}$
The selling price for the i^{th} item per unit time $= \dfrac{w_i Q_i}{T} = w_i \rho_i' \xi_i$
The purchase price for the i^{th} item per unit time $= \dfrac{c_i Q_i}{T} = c_i \rho_i' \xi_i$
There is no holding cost for the supplier, i.e., holding cost $= 0$
The total interest paid by the supplier for i^{th} item per unit time $(TIP_{S,i})$
$= \dfrac{1}{T} Q_i c_i t_s I_p = \rho_i' \xi_i c_i t_s I_p$
Hence, the supplier's average profit is as follows:

$$\Pi_S(\rho_i, T, t_R) = -\frac{A_S}{T} + \sum_{i=1}^{n} \Big[(w_i - c_i)\rho_i' \xi_i - \frac{a_{S,i}}{T} - \rho_i' \xi_i c_i t_S I_p \Big] \tag{4.2}$$

4.2.2.3 Channel Profit

The channel profit, i.e., the sum of the retailer's and the supplier's average profit is as follows:

$$\Pi_C(\rho_i, T, t_R) = \Pi_R(\rho_i, T, t_R) + \Pi_S(\rho_i, T, t_R) \tag{4.3}$$

4.2.2.4 Non-Coordination Scenario

In this scenario, the retailer determines the optimal promotional effort, the replenishment cycle and the retailer's credit period to maximize the retailer's profit per unit time $\Pi_R(\rho_i, T, t_R)$. The following lemma is considered to derive the condition of the optimal solution.

Lemma 4.1. The solution of $\partial \Pi_R(\rho_i, T, t_R)/\partial \rho_i = 0$, for $i = 1, 2, ..., n$; $\partial \Pi_R(\rho_i, T, t_R)/\partial T = 0$ and $\partial \Pi_R(\rho_i, T, t_R)/\partial t_R = 0$ is maximal for $\Pi_R(\rho_i, T, t_R)$, iff $V_R \le 0$ and $W_R \le 0$;

$$\text{where,}\quad V_R = -\frac{2\Big[A_R + \sum_{i=1}^{n} a_{R,i}\Big]}{T^3} - \sum_{i=1}^{n} w_i (I_p - I_e) \frac{\rho_i' \xi_i}{T^3} (t_S - t_R)^2 + \sum_{i=1}^{n} \Big[\Big\{ -\frac{\xi_i}{2}(h_{R,i} + w_i I_p) + w_i (I_p - I_e) \frac{\xi_i}{2T^2} (t_S - t_R)^2 \Big\}^2 \Big/ 2K_i \xi_i^{\alpha_i} \Big]$$

$$\text{and } W_R = -\sum_{i=1}^{n} w_i I_p \frac{\xi_i}{T}\Big\{\rho_i' + 2\lambda(T + t_R - t_S)\Big\} + \sum_{i=1}^{n} w_i I_e \frac{\xi_i}{T}\Big\{\rho_i' - 2\lambda(t_S - t_R)\Big\}$$
$$+ \sum_{i=1}^{n} \Big\{ - w_i I_p \xi_i + w_i (I_p - I_e)\frac{\xi_i}{T}(t_S - t_R)\Big\}^2 \Big/ 2K_i \xi_i^{\alpha_i} - \frac{V_R' \times W_R'}{V_R}$$

$$\text{where, } V_R' = -\sum_{i=1}^{n} \frac{\lambda\xi_i}{2}(h_{R,i} + w_i I_p) - \sum_{i=1}^{n} w_i (I_p - I_e)\frac{\xi_i(t_S - t_R)}{2T^2}\Big\{2\rho_i' - \lambda(t_S - t_R)\Big\}$$
$$+ \sum_{i=1}^{n}\Big[\Big\{ - w_i I_p \xi_i + w_i (I_p - I_e)\frac{\xi_i}{T}(t_S - t_R)\Big\}$$
$$\times \Big\{ - \frac{\xi_i}{2}(h_{R,i} + w_i I_p) + w_i (I_p - I_e)\frac{\xi_i}{2T^2}(t_S - t_R)^2\Big\}\Big/ 2K_i \xi_i^{\alpha_i}\Big]$$
$$\text{and } W_R' = V_R'.$$

Proof. The Hessian matrix for Π_R is

$$D_{n+2} = \begin{bmatrix} \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_1^2} & 0 & \cdots & 0 & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_1 \partial T} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_1 \partial t_R} \\ 0 & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_2^2} & \cdots & 0 & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_2 \partial T} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_2 \partial t_R} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_n^2} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_n \partial T} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_n \partial t_R} \\ \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial T \partial \rho_1} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial T \partial \rho_2} & \cdots & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial T \partial \rho_n} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial T^2} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial T \partial t_R} \\ \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial t_R \partial \rho_1} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial t_R \partial \rho_2} & \cdots & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial t_R \partial \rho_n} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial t_R \partial T} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial t_R^2} \end{bmatrix}$$

Since $\frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_i^2} < 0$ for $i = 1, 2, ..., n$, so $(-1)^i.|D_i| > 0$ for $i = 1, 2, ..., n$. If $(-1)^i.|D_i| > 0$ for $i = n+1$ and $i = n+2$, then there must be a solution of the given set of equations to maximize $\Pi_R(\rho_i, T, t_R)$.

Multiplying each element in each row i, $(i = 1, 2, ..., n)$ of D_{n+2} by $-\Big(\frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial T \partial \rho_i} \Big/ \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_i^2}\Big)$ and adding it to the corresponding element in $(n+1)th$ row, the above matrix becomes

$$D_{n+2} = \begin{bmatrix} \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_1^2} & 0 & \cdots & 0 & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_1 \partial T} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_1 \partial t_R} \\ 0 & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_2^2} & \cdots & 0 & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_2 \partial T} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_2 \partial t_R} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_n^2} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_n \partial T} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial \rho_n \partial t_R} \\ 0 & 0 & \cdots & 0 & V_R & V_R' \\ \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial t_R \partial \rho_1} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial t_R \partial \rho_2} & \cdots & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial t_R \partial \rho_n} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial t_R \partial T} & \frac{\partial^2 \Pi_R(\rho_i,T,t_R)}{\partial t_R^2} \end{bmatrix}$$

$$\begin{aligned} \text{where, } V_R &= \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial T^2} - \sum_{i=1}^{n}\Big[\Big(\frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_i\partial T}\Big)^2\Big/\frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_i^2}\Big] \\ &= -\frac{2\Big[A_R+\sum_{i=1}^{n}a_{R,i}\Big]}{T^3} - \sum_{i=1}^{n} w_i(I_p-I_e)\frac{\rho_i'\xi_i}{T^3}(t_S-t_R)^2 \\ &+\sum_{i=1}^{n}\Big[\Big\{-\frac{\xi_i}{2}(h_{R,i}+w_iI_p)+w_i(I_p-I_e)\frac{\xi_i}{2T^2}(t_S-t_R)^2\Big\}^2\Big/2K_i\xi_i^{\alpha_i}\Big] \end{aligned}$$

$$\begin{aligned} \text{and } V_R' &= \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial T\partial t_R} - \sum_{i=1}^{n}\Big[\frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_i\partial t_R}.\frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_i\partial T}\Big/\frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_i^2}\Big] \\ &= -\sum_{i=1}^{n}\frac{\lambda\xi_i}{2}(h_{R,i}+w_iI_p) - \sum_{i=1}^{n} w_i(I_p-I_e)\frac{\xi_i(t_S-t_R)}{2T^2}\Big\{2\rho_i'-\lambda(t_S-t_R)\Big\} \\ &+\sum_{i=1}^{n}\Big[\Big\{-w_iI_p\xi_i+w_i(I_p-I_e)\frac{\xi_i}{T}(t_S-t_R)\Big\} \\ &\times\Big\{-\frac{\xi_i}{2}(h_{R,i}+w_iI_p)+w_i(I_p-I_e)\frac{\xi_i}{2T^2}(t_S-t_R)^2\Big\}\Big/2K_i\xi_i^{\alpha_i}\Big] \end{aligned}$$

Now, multiplying each element in each column i, $(i = 1, 2, ..., n)$ of D_{n+2} by $-\Big(\frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_i\partial T}\Big/\frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_i^2}\Big)$ and adding it to the corresponding element in $(n+1)th$ column, the matrix reduces to

$$D_{n+2} = \begin{bmatrix} \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_1^2} & 0 & \cdots & 0 & 0 & \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_1\partial t_R} \\ 0 & \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_2^2} & \cdots & 0 & 0 & \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_2\partial t_R} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_n^2} & 0 & \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_n\partial t_R} \\ 0 & 0 & \cdots & 0 & V_R & V_R' \\ \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial t_R\partial\rho_1} & \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial t_R\partial\rho_2} & \cdots & \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial t_R\partial\rho_n} & W_R' & \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial t_R^2} \end{bmatrix}$$

where,

$$\begin{aligned} W_R' &= \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial T\partial t_R} - \sum_{i=1}^{n}\Big[\frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_i\partial t_R}.\frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_i\partial T}\Big/\frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_i^2}\Big] \\ &= V_R'. \end{aligned}$$

Now, multiplying each element in each row i, $(i = 1, 2, ..., n)$ of D_{n+2} by $-\Big(\frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial t_R\partial\rho_i}$

$/\frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_i^2}\Big)$ and in $(n+1)th$ row by $-(W_R'/V_R)$ and adding it to the corresponding element in $(n+2)th$ row, the following matrix is obtained

$$D_{n+2}=\begin{bmatrix} \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_1^2} & 0 & \cdots & 0 & 0 & \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_1\partial t_R} \\ 0 & \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_2^2} & \cdots & 0 & 0 & \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_2\partial t_R} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_n^2} & 0 & \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_n\partial t_R} \\ 0 & 0 & \cdots & 0 & V_R & V_R' \\ 0 & 0 & \cdots & 0 & 0 & W_R \end{bmatrix}$$

where,

$$\begin{aligned} W_R &= \frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial t_R^2}-\sum_{i=1}^{n}\Big[\Big(\frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_i\partial t_R}\Big)^2\Big/\frac{\partial^2\Pi_R(\rho_i,T,t_R)}{\partial\rho_i^2}\Big]-\frac{V_R'\times W_R'}{V_R} \\ &= -\sum_{i=1}^{n} w_iI_p\frac{\xi_i}{T}\Big\{\rho_i'+2\lambda(T+t_R-t_S)\Big\}+\sum_{i=1}^{n} w_iI_e\frac{\xi_i}{T}\Big\{\rho_i'-2\lambda(t_S-t_R)\Big\} \\ &\quad +\sum_{i=1}^{n}\Big\{-w_iI_p\xi_i+w_i(I_p-I_e)\frac{\xi_i}{T}(t_S-t_R)\Big\}^2\Big/2K_i\xi_i^{\alpha_i}-\frac{V_R'\times W_R'}{V_R} \end{aligned}$$

Hence the required condition is proved. □

Let ρ_i^t, T^t, t_R^t be the optimal decision of the retailer in this scenario. Then the retailer's profit in this scenario is $\Pi_R(\rho_i^t,T^t,t_R^t)$.

4.2.2.5 Coordination Scenario

In this scenario, the supplier likes to be a decision maker and so offers to pay a percentage of the promotional cost F to the retailer. Then the retailer's and the supplier's profits are as follows:

$$\begin{aligned} \Pi_R^F(\rho_i,T,t_R) &= -\frac{A_R}{T}+\sum_{i=1}^{n}\Big[(r_i-w_i)\rho_i'\xi_i-\frac{a_{R,i}}{T}-\frac{\rho_i'\xi_iT}{2}h_{R,i}-(1-F)K_i(\rho_i-1)^2\xi_i^{\alpha_i} \\ &\quad -w_iI_p\frac{\rho_i'\xi_i}{2T}(T+t_R-t_S)^2+w_iI_e\frac{\rho_i'\xi_i}{2T}(t_S-t_R)^2\Big] \qquad (4.4) \\ \Pi_S^F(\rho_i,T,t_R) &= -\frac{A_S}{T}+\sum_{i=1}^{n}\Big[(w_i-c_i)\rho_i'\xi_i-\frac{a_{S,i}}{T}-FK_i(\rho_i-1)^2\xi_i^{\alpha_i}-\rho_i'\xi_ic_it_SI_p\Big] \qquad (4.5) \end{aligned}$$

Thus the channel profit is

$$\Pi_C(\rho_i, T, t_R) = \Pi_R^F(\rho_i, T, t_R) + \Pi_S^F(\rho_i, T, t_R) \tag{4.6}$$

As the supplier and the retailer both are decision maker, in this case the channel profit is optimized for marketing decision. To derive the condition of the optimal solution the following lemma is considered.

Lemma 4.2. The solution of $\partial\Pi_C(\rho_i, T, t_R)/\partial\rho_i = 0$, for $i = 1, 2, ..., n$; $\partial\Pi_C(\rho_i, T, t_R)/\partial T = 0$ and $\partial\Pi_C(\rho_i, T, t_R)/\partial t_R = 0$ is maximal for $\Pi_C(\rho_i, T, t_R)$, iff $V_C \leq 0$ and $W_C \leq 0$;

$$\begin{aligned}
\text{where, } V_C &= -\frac{2\Big[(A_R + A_S) + \sum_{i=1}^{n}(a_{R,i} + a_{S,i})\Big]}{T^3} - \sum_{i=1}^{n} w_i(I_p - I_e)\frac{\rho_i'\xi_i}{T^3}(t_S - t_R)^2 \\
&+ \sum_{i=1}^{n}\Big[\Big\{-\frac{\xi_i}{2}(h_{R,i} + w_i I_p) + w_i(I_p - I_e)\frac{\xi_i}{2T^2}(t_S - t_R)^2\Big\}^2 \Big/ 2K_i\xi_i^{\alpha_i}\Big] \\
\text{and } W_C &= -\sum_{i=1}^{n} w_i I_p \frac{\xi_i}{T}\Big\{\rho_i' + 2\lambda(T + t_R - t_S)\Big\} + \sum_{i=1}^{n} w_i I_e \frac{\xi_i}{T}\Big\{\rho_i' - 2\lambda(t_S - t_R)\Big\} \\
&+ \sum_{i=1}^{n}\{-w_i I_p \xi_i + w_i(I_p - I_e)\frac{\xi_i}{T}(t_S - t_R)\}^2/2K_i\xi_i^{\alpha_i} - \frac{V_C' \times W_C'}{V_C} \\
\text{where, } V_C' &= -\sum_{i=1}^{n}\frac{\lambda\xi_i}{2}(h_{R,i} + w_i I_p) - \sum_{i=1}^{n} w_i(I_p - I_e)\frac{\xi_i(t_S - t_R)}{2T^2}\{2\rho_i' - \lambda(t_S - t_R)\} \\
&+ \sum_{i=1}^{n}[\{-w_i I_p \xi_i + w_i(I_p - I_e)\frac{\xi_i}{T}(t_S - t_R)\} \\
&\times\{-\frac{\xi_i}{2}(h_{R,i} + w_i I_p) + w_i(I_p - I_e)\frac{\xi_i}{2T^2}(t_S - t_R)^2\}/2K_i\xi_i^{\alpha_i}] \\
\text{and } W_C' &= V_C'.
\end{aligned}$$

Proof. The proof is similar to that in Lemma 4.1. □

Let ρ_i^l, T^l, t_R^l be the optimal decision of the coordination scenario. Then the retailer's profit and the supplier's profit in this scenario are $\Pi_R^F(\rho_i^l, T^l, t_R^l)$ and $\Pi_S^F(\rho_i^l, T^l, t_R^l)$ respectively.

The retailer's and the supplier's profits under the non-coordination scenario are viewed as the lower bounds for the model under the coordination scenario. Let $\Pi_R = \Pi_R(\rho_i^t, T^t, t_R^t)$ and $\Pi_S = \Pi_S(\rho_i^t, T^t, t_R^t)$.

Proposition 4.1. (a) Profits for both parties increase under the coordination scenario, when the fraction of the retailer's promotional cost is determined to be within the appropriate range (F_{min}, F_{max}), where

$$\begin{aligned} F_{min} &= \Bigg\{\Pi_R + \frac{A_R}{T^l} - \sum_{i=1}^{n}\Big[(r_i - w_i)\rho_i''\xi_i - \frac{a_{R,i}}{T^l} - \frac{\rho_i''\xi_i T^l}{2}h_{R,i} - K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i} \\ &\quad -w_i I_p \frac{\rho_i''\xi_i}{2T^l}(T^l + t_R^l - t_S)^2 + w_i I_e \frac{\rho_i''\xi_i}{2T^l}(t_S - t_R^l)^2\Big]\Bigg\} \Big/ \sum_{i=1}^{n} K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i} \\ \& \ F_{max} &= \Bigg\{-\frac{A_S}{T^l} + \sum_{i=1}^{n}\Big[(w_i - c_i)\rho_i''\xi_i - \frac{a_{S,i}}{T^l} - \rho_i''\xi_i c_i t_S I_p\Big] \\ &\quad -\Pi_S\Bigg\} \Big/ \sum_{i=1}^{n} K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i} \quad \text{where, } \rho_i'' = \rho_i^l + \lambda t_R^l. \end{aligned}$$

(b) When the supplier and the retailer have the same bargaining power, the appropriate fraction of the promotional cost sharing is $F = (F_{max} + F_{min})/2$.

Proof. (a) From

$$\begin{aligned} &\Pi_R^F(\rho_i^l, T^l, t_R^l) - \Pi_R \geq 0 \\ \Rightarrow\ & -\frac{A_R}{T^l} + \sum_{i=1}^{n}\Big[(r_i - w_i)\rho_i''\xi_i - \frac{a_{R,i}}{T^l} - \frac{\rho_i''\xi_i T^l}{2}h_{R,i} - (1 - F)K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i} \\ & -w_i I_p \frac{\rho_i''\xi_i}{2T^l}(T^l + t_R^l - t_S)^2 + w_i I_e \frac{\rho_i''\xi_i}{2T^l}(t_S - t_R^l)^2\Big] \geq \Pi_R \\ & \text{where, } \rho_i'' = \rho_i^l + \lambda t_R^l. \end{aligned}$$

$$\begin{aligned} \Rightarrow\ F \geq & \Bigg\{\Pi_R + \frac{A_R}{T^l} - \sum_{i=1}^{n}\Big[(r_i - w_i)\rho_i''\xi_i - \frac{a_{R,i}}{T^l} - \frac{\rho_i''\xi_i T^l}{2}h_{R,i} - K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i} \\ & -w_i I_p \frac{\rho_i''\xi_i}{2T^l}(T^l + t_R^l - t_S)^2 + w_i I_e \frac{\rho_i''\xi_i}{2T^l}(t_S - t_R^l)^2\Big]\Bigg\} \Big/ \sum_{i=1}^{n} K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i} \end{aligned}$$

Therefore, F_{min} is obtained. Also from $\Pi_S^F(\rho_i^l, T^l, t_R^l) - \Pi_S \geq 0$, F_{max} can be obtain.

(b) The following relations are found from (a):

$$\begin{aligned} F_{max} - F &= \Big[\Pi_S^F(\rho_i^l, T^l, t_R^l) - \Pi_S\Big] \Big/ \sum_{i=1}^{n} K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i} \\ \Rightarrow \triangle\Pi_S^F &= (F_{max} - F)\sum_{i=1}^{n} K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i} \end{aligned}$$

Similarly,

$$\triangle\Pi_R^F = (F - F_{min})\sum_{i=1}^{n} K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i}$$

Now, $\triangle\Pi_S^F(F) \times \triangle\Pi_R^F(F)$

$$\begin{aligned}
&= \Big[(F_{max} - F)\sum_{i=1}^{n} K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i}\Big] \times \Big[(F - F_{min})\sum_{i=1}^{n} K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i}\Big] \\
&= (F_{max} - F)(F - F_{min})\Big\{\sum_{i=1}^{n} K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i}\Big\}^2 \\
&= (F_{max}.F - F_{max}.F_{min} - F^2 + F.F_{min}) \times \Big\{\sum_{i=1}^{n} K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i}\Big\}^2 \\
&= \Big[-\Big(F - \frac{F_{max} + F_{min}}{2}\Big)^2 + \frac{(F_{max} - F_{min})^2}{4}\Big] \times \Big\{\sum_{i=1}^{n} K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i}\Big\}^2.
\end{aligned}$$

Thus, the appropriate fraction of the promotional cost sharing is obtained as $F = (F_{max} + F_{min})/2$. □

4.2.2.6 Models in Different Environments and under Different Constraints

The proposed model can be formulated in different environments (crisp/imprecise) depending upon the estimation of different inventory parameters. Also different constraints can be incorporated with the proposed model to make it more realistic. In those cases closed form solution is difficult to derive but the model can be solved by any soft computing technique like PSO. Mathematical form of different models are presented below.

Model 4.1.1: Crisp Objective (without any constraint): For this model Π_R and Π_C are to be optimized for Non-Coordination Scenario (NCS) and Coordination Scenario (CS) respectively. Here t_S is known and t_R is unknown; i.e., supplier gives fixed credit period to the retailer and the retailer decides how much credit period will be given to the customers to maximize profit. The problem is as follows:

$$\begin{aligned}
&\text{Determine} \quad \rho_i (i = 1, 2, ..., n), T, t_R \text{ to} \\
\text{For NCS:} \quad &\text{Maximize} \quad \Pi_R \qquad (4.7)\\
\text{For CS:} \quad &\text{Maximize} \quad \Pi_C \qquad (4.8)
\end{aligned}$$

This model is solved following PSO technique (cf. § 2.2.2.1) and GRG technique (cf. § 2.2.1.1) using LINGO 14.0 software. From previous discussion, it is clear that optimal solution of both the scenarios of this model exists under certain conditions as stated in the lemmas.

Model 4.1.1.1: Crisp Objective with Crisp Budget: In real life, it may happen that the budget of the retailer is limited. So, a budget constraint is considered in this model. The retailer's total purchase cost (B_R) is given by

$$B_R = \sum_{i=1}^{n} w_i Q_i \tag{4.9}$$

Clearly this amount should not exceed the upper limit of the retailer's budget B_R^m. So the problem of this model takes the following form:
For NCS and CS, the problems are (4.7) and (4.8) respectively with contraint

$$\sum_{i=1}^{n} w_i Q_i \leq B_R^m \tag{4.10}$$

This model is solved following PSO technique (cf. § 2.2.2.1) and GRG technique (cf. § 2.2.1.1) using LINGO 14.0 software.

Model 4.1.1.2: Crisp Objective with Fuzzy Budget: Taking B_R^m as fuzzy, denoted by $\tilde{B}_R^m$, the corresponding fuzzy model of Model 4.1.1.1 is defined as:
For NCS and CS, the problems are (4.7) and (4.8) respectively with contraint

$$\sum_{i=1}^{n} w_i Q_i \leq \tilde{B}_R^m \tag{4.11}$$

where, $\tilde{B}_R^m = (B_{R1}, B_{R2}, B_{R3})$ is the imprecise available budget (TFN type) of the retailer. For this model, credibility measure (cf. § 2.1.2.5) is used to deal with fuzzy constraint [60, 150]. For any fuzzy event $\tilde{A}$, it is known that $Cr(\tilde{A}) + Cr(\bar{\tilde{A}}) = 1$ [107] and using this phenomenon the above constraint reduces to

$$Cr\left\{\sum_{i=1}^{n} w_i Q_i \leq \tilde{B}_R^m\right\} > 0.5 \tag{4.12}$$

Here, the credibility measure of the constraint are made using (2.7). This model is solved using PSO technique.

Model 4.1.1.3: Crisp Objective with Rough Budget: Here B_R^m is considered as rough variable and is denoted by $\check{B}_R^m$. For this model, trust measure (cf. § 2.1.3) is used to deal with rough constraint [150]. For any rough event $\check{A}$, it is known that $Tr(\check{A}) + Tr(\bar{\check{A}}) = 1$ [107]. Using this phenomenon, the above problem reduces to

For NCS and CS, the problems are (4.7) and (4.8) respectively with constraint

$$Tr\left\{\sum_{i=1}^{n} w_i Q_i \leq \breve{B}_R^m\right\} > 0.5 \tag{4.13}$$

where, $\breve{B}_R^m = ([B_{R1}, B_{R2}][B_{R3}, B_{R4}])$, $0 \leq B_{R3} \leq B_{R1} \leq B_{R2} \leq B_{R4}$ is rough variable. Here, the trust measure of the constraints are made using (2.24). This model is solved using PSO technique.

Model 4.1.2: Fuzzy Objective (without any constraint): In present volatile market situation, it is observed that bank interests are normally changed frequently. So it is reasonable to consider different interest rates as imprecise parameters. As fuzzy optimization is made by expert's opinion and not much past data is required for the purpose so it is better to estimate the bank interest rates as fuzzy numbers [66]. For this reason, the rate of interest paid to the bank (I_p) and the rate of interest earned from the bank (I_e) are considered as fuzzy numbers in this scenario. With these parameters, due to same reason the promotional cost coefficients (K_i, $i = 1, 2, ..., n$) are also considered as fuzzy. According to these assumptions, the individual profits and the channel profit are reduces to fuzzy numbers and represented by

$$\begin{aligned}
\tilde{\Pi}_R(\rho_i, T, t_R) &= -\frac{A_R}{T} + \sum_{i=1}^{n}[(r_i - w_i)\rho_i'\xi_i - \frac{a_{R,i}}{T} - \frac{\rho_i'\xi_i T}{2} h_{R,i} - \tilde{K}_i(\rho_i - 1)^2 \xi_i^{\alpha_i} \\
&\quad - w_i \tilde{I}_p \frac{\rho_i'\xi_i}{2T}(T + t_R - t_S)^2 + w_i \tilde{I}_e \frac{\rho_i'\xi_i}{2T}(t_S - t_R)^2] \qquad (4.14) \\
\tilde{\Pi}_S(\rho_i, T, t_R) &= -\frac{A_S}{T} + \sum_{i=1}^{n}[(w_i - c_i)\rho_i'\xi_i - \frac{a_{S,i}}{T} - \rho_i'\xi_i c_i t_S \tilde{I}_p] \qquad (4.15) \\
\tilde{\Pi}_C(\rho_i, T, t_R) &= -\frac{(A_R + A_S)}{T} + \sum_{i=1}^{n}[(r_i - c_i)\rho_i'\xi_i - \frac{(a_{R,i} + a_{S,i})}{T} - \frac{\rho_i'\xi_i T}{2} h_{R,i} \\
&\quad - \tilde{K}_i(\rho_i - 1)^2 \xi_i^{\alpha_i} - w_i \tilde{I}_p \frac{\rho_i'\xi_i}{2T}(T + t_R - t_S)^2 + w_i \tilde{I}_e \frac{\rho_i'\xi_i}{2T}(t_S - t_R)^2 \\
&\quad - \rho_i'\xi_i c_i t_S \tilde{I}_p] \qquad (4.16)
\end{aligned}$$

Considering the fuzzy numbers $\tilde{I}_p$, $\tilde{I}_e$, $\tilde{K}_i$ ($i = 1, 2, ..., n$) as TFNs (I_{p1}, I_{p2}, I_{p3}), (I_{e1}, I_{e2}, I_{e3}), (K_{i1}, K_{i2}, K_{i3}) respectively, the fuzzy numbers $\tilde{\Pi}_R$, $\tilde{\Pi}_S$, $\tilde{\Pi}_C$ becomes TFNs $(\Pi_{R1}, \Pi_{R2}, \Pi_{R3})$, $(\Pi_{S1}, \Pi_{S2}, \Pi_{S3})$, $(\Pi_{C1}, \Pi_{C2}, \Pi_{C3})$ respectively, where

$$\begin{aligned}
\Pi_{Rj} &= -\frac{A_R}{T} + \sum_{i=1}^{n}[(r_i - w_i)\rho_i'\xi_i - \frac{a_{R,i}}{T} - \frac{\rho_i'\xi_i T}{2}h_{R,i} - K_{i(4-j)}(\rho_i - 1)^2\xi_i^{\alpha_i} \\
&\quad -w_i I_{p(4-j)}\frac{\rho_i'\xi_i}{2T}(T + t_R - t_S)^2 + w_i I_{ej}\frac{\rho_i'\xi_i}{2T}(t_S - t_R)^2] \qquad (4.17) \\
\Pi_{Sj} &= -\frac{A_S}{T} + \sum_{i=1}^{n}[(w_i - c_i)\rho_i'\xi_i - \frac{a_{S,i}}{T} - \rho_i'\xi_i c_i t_S I_{p(4-j)}] \qquad (4.18) \\
\Pi_{Cj} &= -\frac{(A_R + A_S)}{T} + \sum_{i=1}^{n}[(r_i - c_i)\rho_i'\xi_i - \frac{(a_{R,i} + a_{S,i})}{T} - \frac{\rho_i'\xi_i T}{2}h_{R,i} \\
&\quad -K_{i(4-j)}(\rho_i - 1)^2\xi_i^{\alpha_i} - w_i I_{p(4-j)}\frac{\rho_i'\xi_i}{2T}(T + t_R - t_S)^2 + w_i I_{ej}\frac{\rho_i'\xi_i}{2T}(t_S - t_R)^2 \\
&\quad -\rho_i'\xi_i c_i t_S I_{p(4-j)}]; \quad \text{for } j = 1, 2, 3. \qquad (4.19)
\end{aligned}$$

For the coordination scenario, the individual profits and the channel profit as fuzzy numbers are represented by

$$\begin{aligned}
\tilde{\Pi}_R^F(\rho_i, T, t_R) &= -\frac{A_R}{T} + \sum_{i=1}^{n}[(r_i - w_i)\rho_i'\xi_i - \frac{a_{R,i}}{T} - \frac{\rho_i'\xi_i T}{2}h_{R,i} - (1 - F)\tilde{K}_i(\rho_i - 1)^2\xi_i^{\alpha_i} \\
&\quad -w_i\tilde{I}_p\frac{\rho_i'\xi_i}{2T}(T + t_R - t_S)^2 + w_i\tilde{I}_e\frac{\rho_i'\xi_i}{2T}(t_S - t_R)^2] \qquad (4.20) \\
\tilde{\Pi}_S^F(\rho_i, T, t_R) &= -\frac{A_S}{T} + \sum_{i=1}^{n}[(w_i - c_i)\rho_i'\xi_i - \frac{a_{S,i}}{T} - F\tilde{K}_i(\rho_i - 1)^2\xi_i^{\alpha_i} - \rho_i'\xi_i c_i t_S\tilde{I}_p] \qquad (4.21) \\
\tilde{\Pi}_C(\rho_i, T, t_R) &= \tilde{\Pi}_R^F(\rho_i, T, t_R) + \tilde{\Pi}_S^F(\rho_i, T, t_R) \qquad (4.22)
\end{aligned}$$

So in this case the problem reduces to

$$\begin{aligned}
&\text{Determine} \quad \rho_i(i = 1, 2, ..., n), T, t_R \text{ to} \\
\text{For NCS:} \quad &\text{Maximize} \quad \tilde{\Pi}_R \qquad (4.23) \\
\text{For CS:} \quad &\text{Maximize} \quad \tilde{\Pi}_C \qquad (4.24)
\end{aligned}$$

Since the objective function of the problem is fuzzy in nature, it cannot be solved using LINGO. Here, PSO (cf. 2.2.2.1) is used for this purpose. This problem is solved following two approaches. In the first approach, the problem is directly solved using PSO and called direct approach. In another approach, the expected values of the fuzzy objectives are optimized using PSO to find optimal decision. This approach is named as expected value optimization approach.

Model 4.1.2.1: Fuzzy Objective with Fuzzy Budget: Here the fuzzy objectives of Model 4.1.2 is optimized under fuzzy budget constraints of Model 4.1.1.2. So the problem for this model is as follows:

For NCS and CS, the problems are (4.23) and (4.24) respectively with constraint (4.12).

Model 4.1.3: Rough Objective (without any constraint): Here the rate of interest paid to the bank (I_p), the rate of interest earned from the bank (I_e) and the promotional cost coefficients ($K_i,\ i = 1, 2, ..., n$) are considered as rough variables. According to these assumptions, the individual profits and the channel profit are reduces to

$$\begin{aligned}\check{\Pi}_R(\rho_i, T, t_R) &= -\frac{A_R}{T} + \sum_{i=1}^{n}[(r_i - w_i)\rho_i'\xi_i - \frac{a_{R,i}}{T} - \frac{\rho_i'\xi_i T}{2}h_{R,i} - \check{K}_i(\rho_i - 1)^2\xi_i^{\alpha_i} \\ &\quad -w_i\check{I}_p\frac{\rho_i'\xi_i}{2T}(T + t_R - t_S)^2 + w_i\check{I}_e\frac{\rho_i'\xi_i}{2T}(t_S - t_R)^2] \qquad (4.25)\\ \check{\Pi}_S(\rho_i, T, t_R) &= -\frac{A_S}{T} + \sum_{i=1}^{n}[(w_i - c_i)\rho_i'\xi_i - \frac{a_{S,i}}{T} - \rho_i'\xi_i c_i t_S\check{I}_p] \qquad (4.26)\\ \check{\Pi}_C(\rho_i, T, t_R) &= -\frac{(A_R + A_S)}{T} + \sum_{i=1}^{n}[(r_i - c_i)\rho_i'\xi_i - \frac{(a_{R,i} + a_{S,i})}{T} - \frac{\rho_i'\xi_i T}{2}h_{R,i} \\ &\quad -\check{K}_i(\rho_i - 1)^2\xi_i^{\alpha_i} - w_i\check{I}_p\frac{\rho_i'\xi_i}{2T}(T + t_R - t_S)^2 + w_i\check{I}_e\frac{\rho_i'\xi_i}{2T}(t_S - t_R)^2 \\ &\quad -\rho_i'\xi_i c_i t_S\check{I}_p] \qquad (4.27)\end{aligned}$$

Considering the rough numbers $\check{I}_p$, $\check{I}_e$, $\check{K}_i$ $(i = 1, 2, ..., n)$ as $([I_{p1}, I_{p2}][I_{p3}, I_{p4}])$, $([I_{e1}, I_{e2}]\ [I_{e3}, I_{e4}])$, $([K_{i1}, K_{i2}][K_{i3}, K_{i4}])$ respectively, the rough numbers $\check{\Pi}_R$, $\check{\Pi}_S$, $\check{\Pi}_C$ becomes $([\Pi_{R1}, \Pi_{R2}][\Pi_{R3}, \Pi_{R4}])$, $([\Pi_{S1}, \Pi_{S2}][\Pi_{S3}, \Pi_{S4}])$, $([\Pi_{C1}, \Pi_{C2}][\Pi_{C3}, \Pi_{C4}])$ respectively, where

$$\begin{aligned}\Pi_{Rj} &= -\frac{A_R}{T} + \sum_{i=1}^{n}[(r_i - w_i)\rho_i'\xi_i - \frac{a_{R,i}}{T} - \frac{\rho_i'\xi_i T}{2}h_{R,i} - K_{i(m-j)}(\rho_i - 1)^2\xi_i^{\alpha_i} \\ &\quad -w_i I_{p(m-j)}\frac{\rho_i'\xi_i}{2T}(T + t_R - t_S)^2 + w_i I_{ej}\frac{\rho_i'\xi_i}{2T}(t_S - t_R)^2] \qquad (4.28)\\ \Pi_{Sj} &= -\frac{A_S}{T} + \sum_{i=1}^{n}[(w_i - c_i)\rho_i'\xi_i - \frac{a_{S,i}}{T} - \rho_i'\xi_i c_i t_S I_{p(m-j)}] \qquad (4.29)\\ \Pi_{Cj} &= -\frac{(A_R + A_S)}{T} + \sum_{i=1}^{n}[(r_i - c_i)\rho_i'\xi_i - \frac{(a_{R,i} + a_{S,i})}{T} - \frac{\rho_i'\xi_i T}{2}h_{R,i} \\ &\quad -K_{i(m-j)}(\rho_i - 1)^2\xi_i^{\alpha_i} - w_i I_{p(m-j)}\frac{\rho_i'\xi_i}{2T}(T + t_R - t_S)^2 + w_i I_{ej}\frac{\rho_i'\xi_i}{2T}(t_S - t_R)^2 \\ &\quad -\rho_i'\xi_i c_i t_S I_{p(m-j)}];\ \text{ where } m = 3 \text{ for } j = 1, 2 \text{ and } m = 7 \text{ for } j = 3, 4. \qquad (4.30)\end{aligned}$$

For the coordination scenario, the individual profits and the channel profit as rough numbers are represented by

$$\begin{aligned}\check{\Pi}_R^F(\rho_i,T,t_R) &= -\frac{A_R}{T}+\sum_{i=1}^{n}[(r_i-w_i)\rho_i'\xi_i-\frac{a_{R,i}}{T}-\frac{\rho_i'\xi_iT}{2}h_{R,i}-(1-F)\check{K}_i(\rho_i-1)^2\xi_i^{\alpha_i} \\ &\quad -w_i\check{I}_p\frac{\rho_i'\xi_i}{2T}(T+t_R-t_S)^2+w_i\check{I}_e\frac{\rho_i'\xi_i}{2T}(t_S-t_R)^2] \quad (4.31)\\ \check{\Pi}_S^F(\rho_i,T,t_R) &= -\frac{A_S}{T}+\sum_{i=1}^{n}[(w_i-c_i)\rho_i'\xi_i-\frac{a_{S,i}}{T}-F\check{K}_i(\rho_i-1)^2\xi_i^{\alpha_i}-\rho_i'\xi_ic_it_S\check{I}_p] \quad (4.32)\\ \check{\Pi}_C(\rho_i,T,t_R) &= \check{\Pi}_R^F(\rho_i,T,t_R)+\check{\Pi}_S^F(\rho_i,T,t_R) \quad (4.33)\end{aligned}$$

So in this case the problem reduces to

	Determine	$\rho_i(i=1,2,...,n),T,t_R$ to	
For NCS:	Maximize	$\check{\Pi}_R$	(4.34)
For CS:	Maximize	$\check{\Pi}_C$	(4.35)

Since the objective function of the problem is rough in nature, it cannot be solved using LINGO. Here PSO (cf. § 2.2.2.1) is used for this purpose. This problem is solved following two approaches. In the first approach, the problem is directly solved using PSO and called direct approach. In another approach, the expected values of the rough objectives are optimized using PSO to find optimal decision. This approach is named as expected value optimization approach.

Model 4.1.3.1: Rough Objective with Rough Budget: Here the rough objectives of Model 4.1.3 is optimized under rough budget constraints of Model 4.1.1.3. The problem for this model is as follows:
For NCS and CS, the problems are (4.34) and (4.35) respectively with constraint (4.13).

4.2.3 Numerical Illustration

The models are illustrated with following set of hypothetical data which are presented below:

Example 4.1. The following parametric values are in appropriate units:
$I_e = 0.08$, $I_p = 0.10$, $m_r = 1.85$, $m_s = 1.6$, $m_h = 0.05$, $c_1 = 6$, $c_2 = 6.25$, $\xi_1 = 90$, $\xi_2 = 120$, $A_R = 275$, $a_{r,1} = 1$, $a_{r,2} = 1$, $A_S = 150$, $a_{s,1} = 0.8$, $a_{s,2} = 0.7$, $\alpha_1 = 1.25$,

Table 4.1: Optimum Results of Model 4.1.1 and Model 4.1.1.1 using PSO technique and Lingo Software

Output Variable	Model 4.1.1				Model 4.1.1.1			
	using PSO		using Lingo		using PSO		using Lingo	
	NCS	CS	NCS	CS	NCS	CS	NCS	CS
Π_R	1989.72	2042.12	1989.72	2042.06	1989.07	2009.27	1989.07	2013.45
Π_S	931.98	980.90	931.85	980.96	923.25	976.47	923.57	973.12
Π_C	2921.70	3023.02	2921.57	3023.02	2912.32	2985.74	2912.64	2986.56
B_R	3574.49	4701.00	3573.60	4702.88	3399.72	3399.61	3400.00	3400.00
ρ_1	1.48	1.63	1.48	1.63	1.47	1.61	1.48	1.61
ρ_2	1.64	1.84	1.64	1.84	1.63	1.79	1.63	1.81
T	1.08	1.21	1.08	1.22	1.03	0.89	1.03	0.89
t_R	0.16	0.73	0.16	0.73	0.17	0.77	0.17	0.69
Q_1	146.89	191.79	146.87	191.85	139.65	140.17	139.78	138.83
Q_2	216.43	285.98	216.36	286.11	205.90	205.40	205.81	206.72
F	-	0.18	-	0.18	-	0.12	-	0.12
V_R	-448.85	-	-449.10	-	-	-	-	-
W_R	-105.69	-	-105.67	-	-	-	-	-
V_C	-	-686.77	-	-469.60	-	-	-	-
W_C	-	-107.77	-	-118.39	-	-	-	-

$\alpha_2 = 1.2$, $K_1 = 2.7$, $K_2 = 2.5$, $t_s = 0.8$, $\lambda = 0.17$, $w_1 = m_s \times c_1$, $w_2 = m_s \times c_2$, $r_1 = m_r \times w_1$, $r_2 = m_r \times w_2$, $h_{R,1} = m_h \times w_1$, $h_{R,2} = m_h \times w_2$.

$F = 0.18$ is taken for the models without constraint (Model 4.1.1, Model 4.1.2, Model 4.1.3) and $F = 0.12$ is taken for the models with budget constraint (Model 4.1.1.1, Model 4.1.1.2, Model 4.1.1.3, Model 4.1.2.1, Model 4.1.3.1) in CS.

(a) Crisp Objective: For Model 4.1.1, the above data set is considered as input. For Model 4.1.1.1, the budget constraint is used, assuming that the budget of retailer is limited. Here, the maximum budget of retailer (B_R^m) is taken as 3400 units for NCS and CS. Both the models are solved using GRG and PSO techniques and results are presented in Table 4.1. In coordination scenario, the Model 4.1.1 is solved due to different values of sharing of promotional cost (F) and results are presented in Table 4.2.

For Model 4.1.1.2, the fuzzy parameter $\tilde{B}_R^m$ is taken as TFN as follows: $\tilde{B}_R^m = (B_{R1}, B_{R2}, B_{R3}) = (3300, 3400, 3500)$ for NCS and CS. All other parameters are same as in crisp model. With these parametric values, the model is solved using PSO technique and results are presented in Table 4.3.

TABLE 4.2: Values of Π_R, Π_S in Model 4.1.1 due to different F in CS using PSO technique

F	Π_R	Π_S	Π_C
0.11	**1983.01**	1040.01	3023.02
0.12	1991.44	1031.58	3023.02
0.15	2016.75	1006.27	3023.02
0.18	2042.12	980.90	3023.02
0.21	2067.40	955.62	3023.02
0.23	2084.31	938.71	3023.02
0.24	2092.70	**930.32**	3023.02

TABLE 4.3: Optimum Results of Model 4.1.1.2 and Model 4.1.1.3 using PSO technique

Output Variable	Model 4.1.1.2		Model 4.1.1.3	
	NCS	CS	NCS	CS
Π_R	1989.07	2009.27	1988.91	2003.40
Π_S	923.25	976.47	926.19	982.18
Π_C	2912.32	2985.74	2915.10	2985.58
ρ_1	1.47	1.61	1.47	1.60
ρ_2	1.63	1.79	1.63	1.81
T	1.03	0.89	1.03	0.89
t_R	0.17	0.77	0.22	0.80
B_R	3399.72	3399.61	3398.96	3399.01
F	-	0.12	-	0.12

For Model 4.1.1.3, the value of rough variable $\check{B}_R^m$ is taken as: $\check{B}_R^m = ([B_{R1}, B_{R2}][B_{R3}, B_{R4}]) = ([3350, 3450][3300, 3500])$, where $0 \leq B_{R3} \leq B_{R1} \leq B_{R2} \leq B_{R4}$ for NCS and CS. All other parametric values are same as in crisp model. With these parametric values, the model is solved using PSO technique and the results are presented in Table 4.3.

(b) Fuzzy Objective: For Model 4.1.2, the fuzzy parameters $\tilde{I}_e$, $\tilde{I}_p$, $\tilde{K}_i$ $(i = 1, 2)$ for both NCS and CS are taken as TFNs as follows: $\tilde{I}_e = (I_{e1}, I_{e2}, I_{e3}) = (0.07, 0.08, 0.09)$, $\tilde{I}_p = (I_{p1}, I_{p2}, I_{p3}) = (0.10, 0.11, 0.12)$, $\tilde{K}_1 = (K_{11}, K_{12}, K_{13}) = (2.6, 2.7, 2.8)$, $\tilde{K}_2 = (K_{21}, K_{22}, K_{23}) = (2.4, 2.5, 2.6)$ and all other parameters are same as in crisp model. With these parametric values the model is solved using PSO technique and the results are presented in Table 4.4. As stated earlier, here two approaches are followed to find marketing decisions – Direct (objective values) optimization approach and Expected values (of the objectives) optimization approach. In the first approach, the fuzzy objectives of the problems are directly

TABLE 4.4: Optimum Results of Model 4.1.2 using PSO technique

Output Variable	NCS	CS
	Direct Approach	
$\tilde{\Pi}_R$; $E[\tilde{\Pi}_R]$	(1958.07, 1987.39, 2016.70); 1987.39	(2011.42, 2053.15, 2094.89); 2053.15
$\tilde{\Pi}_S$; $E[\tilde{\Pi}_S]$	(891.26, 907.65, 924.03); 907.65	(909.45, 934.60, 959.74); 934.60
$\tilde{\Pi}_C$; $E[\tilde{\Pi}_C]$	(2849.34, 2895.03, 2940.73); 2895.03	(2920.87, 2987.75, 3054.63); 2987.75
ρ_1	1.48	1.64
ρ_2	1.64	1.85
T	1.08	1.19
t_R	0.07	0.49
F	-	0.18
	Expected value optimization approach	
$\tilde{\Pi}_R$; $E[\tilde{\Pi}_R]$	(1958.06, 1987.39, 2016.72); 1987.39	(2011.45, 2053.19, 2094.93); 2053.19
$\tilde{\Pi}_S$; $E[\tilde{\Pi}_S]$	(891.17, 907.55, 923.93); 907.55	(909.40, 934.55, 959.71); 934.55
$\tilde{\Pi}_C$; $E[\tilde{\Pi}_C]$	(2849.23, 2894.94, 2940.65); 2894.94	(2920.85, 2987.75, 3054.64); 2987.75
ρ_1	1.48	1.64
ρ_2	1.64	1.85
T	1.08	1.19
t_R	0.07	0.48
F	-	0.18

optimized using PSO to find marketing decisions. In another approach, the expected values of the fuzzy objectives are optimized using PSO to find marketing decisions. It is found from the Table 4.4 that the results obtained using both the approaches are almost same.

For Model 4.1.2.1, the values of fuzzy parameters of the objective functions $\tilde{I}_e$, $\tilde{I}_p$, $\tilde{K}_i$ $(i = 1, 2)$ are taken same as Model 4.1.2 and the value of fuzzy parameter $\tilde{B}_R^m$ of the constraints is taken same as Model 4.1.1.2. All other parametric values are same as in crisp model. This model is solved using PSO technique and the results are presented in Table 4.5.

(c) Rough Objective: For Model 4.1.3, the rough parameters $\check{I}_p$, $\check{I}_e$, $\check{K}_i$ $(i = 1, 2)$ for both NCS and CS are taken as follows: $\check{I}_e = ([I_{e1}, I_{e2}][I_{e3}, I_{e4}]) = ([0.07, 0.08]$ $[0.06, 0.09])$, $\check{I}_p = ([I_{p1}, I_{p2}][I_{p3}, I_{p4}]) = ([0.10, 0.11][0.09, 0.12])$, $\check{K}_1 = ([K_{11}, K_{12}]$ $[K_{13}, K_{14}]) = ([2.6, 2.7][2.5, 2.8])$, $\check{K}_2 = ([K_{21}, K_{22}][K_{23}, K_{24}]) = ([2.4, 2.5][2.3, 2.6])$ and all other parameters are same as in crisp model. With these parametric values the model is solved using PSO technique and results are presented in Table 4.6. As stated earlier, here also two approaches are followed to find marketing decisions - Direct (objective values) optimization approach and Expected values (of

TABLE 4.5: Optimum Results of Model 4.1.2.1 using PSO technique

Output Variable	Scenario	
	NCS	CS
$\tilde{\Pi}_R$	(1957.42, 1986.95, 2016.48)	(1980.55, 2018.28, 2056.01)
$\tilde{\Pi}_S$	(886.18, 902.57, 918.96)	(916.58, 939.29, 962.00)
$\tilde{\Pi}_C$	(2843.60, 2889.52, 2935.43)	(2897.13, 2957.57, 3018.01)
ρ_1	1.48	1.63
ρ_2	1.65	1.83
T	1.04	0.90
t_R	0.06	0.50
F	-	0.12

TABLE 4.6: Optimum Results of Model 4.1.3 using PSO technique

Output Variable	NCS	CS
	Direct Approach	
$\check{\Pi}_R$; $E[\check{\Pi}_R]$	([1980.36,2009.23][1951.48,2038.11]);1994.79	([2035.72,2080.90][1990.55,2126.07]);2058.31
$\check{\Pi}_S$; $E[\check{\Pi}_S]$	([921.82,938.43][905.21,955.03]);930.12	([949.91,975.52][924.31,1001.12]);962.72
$\check{\Pi}_C$; $E[\check{\Pi}_C]$	([2902.17,2947.66][2856.69,2993.14]);2924.91	([2985.64,3056.42][2914.86,3127.19]);3021.03
ρ_1	1.49	1.65
ρ_2	1.65	1.86
T	1.07	1.20
t_R	0.17	0.59
F	-	0.18
	Expected value optimization approach	
$\check{\Pi}_R$; $E[\check{\Pi}_R]$	([1980.36,2009.23][1951.48,2038.10]);1994.79	([2035.59,2080.80][1990.39,2126.00]);2058.19
$\check{\Pi}_S$; $E[\check{\Pi}_S]$	([921.91,938.52][905.30,955.13]);930.21	([950.03,975.64][924.43,1001.24]);962.83
$\check{\Pi}_C$; $E[\check{\Pi}_C]$	([2902.26,2947.75][2856.78,2993.23]);2925.00	([2985.62,3056.43][2914.81,3127.24]);3021.03
ρ_1	1.49	1.65
ρ_2	1.65	1.86
T	1.07	1.20
t_R	0.17	0.60
F	-	0.18

the objectives) optimization approach. In the first approach, the rough objectives of the problem are directly optimized using PSO to find marketing decisions. In another approach, the expected values of the rough objectives are optimized using PSO to find marketing decisions. It is found from the Table 4.6 that the results obtained using both the approaches are almost same.

For Model 4.1.3.1, the values of rough parameters of the objective functions $\check{I}_e$, $\check{I}_p$, $\check{K}_i$ $(i = 1, 2)$ are taken same as Model 4.1.3 and the value of rough parameter $\check{B}_R^m$ of the constraints is taken same as Model 4.1.1.3. All other parametric values

TABLE 4.7: Optimum Results of Model 4.1.3.1 using PSO technique

Output Variable	Scenario	
	NCS	CS
$\check{\Pi}_R$	([1980.06,2008.31][1951.81,2036.56])	([2014.16,2052.35][1975.98,2090.54])
$\check{\Pi}_S$	([914.47,931.07][897.87,947.67])	([941.31,964.14][918.48,986.98])
$\check{\Pi}_C$	([2894.53,2939.38][2849.68,2984.23])	([2955.47,3016.49][2894.45,3077.51])
ρ_1	1.48	1.63
ρ_2	1.64	1.83
T	1.02	0.90
t_R	0.19	0.50
F	-	0.12

are same as in crisp model. This model is solved using PSO technique and the results are presented in Table 4.7.

4.2.4 Discussion

From the optimum results of Example 4.1 in Table 4.1 and 4.2, the following observations are made:

- It is found that the profits for both the parties (i.e., supplier and retailer) increase in the coordination scenario than the non-coordination scenario for a compromise value of F, i.e, if the supplier bears a compromise portion of promotional cost then it is beneficial for both parties. So theoretical expected result agrees with numerical findings.
- It is also found that promotional efforts of all the items are grater than 1. So promotional effort has a positive effect in a supply chain.
- Again in both the scenarios it is observed that $t_R > 0$, which implies that customers' credit period has a significant effect in a supply chain.

Using Proposition 4.1, $F_{min} = 0.118$ and $F_{max} = 0.238$ are obtained. From Table 4.2, it is found that for $F_{min} < F < F_{max}$ profit of both the parties increase to some extent. If $F = 0.11 < F_{min}$, then the retailer's profit Π_R is less in the CS (1983.01) than the NCS (1989.72). Similarly, if $F = 0.24 > F_{max}$, then the supplier's profit Π_S is decreased in the CS (930.32) than the NCS (931.98). The same sort of observations are obtained for the imprecise models also. All these observations agrees with reality.

For imprecise models inventory parameters are imprecise. As a result, calculated profits from the corresponding models are imprecise in nature. For any event which produces imprecise output, none can predict the actual output, but an estimation can be made about the output. Normally actual output occurs near the expected value of the calculated imprecise output. Due to this reason, the imprecise models, Model 4.1.2 and Model 4.1.3, are solved following two approaches - Direct approach and Expected value optimization approach. From Table 4.4 and Table 4.6, it is observed that both the approaches produces almost same result. But in the direct approach as objectives are optimized directly, it can be concluded that the results of the approach are less error-prone. Again in both the approaches the decision variables are crisp. So the decisions for the decision maker (DM) are precise and there is no ambiguity in the marketing decisions. The DM can not find the actual profit in advance but he can estimate that its value will be near the expected value of the calculated imprecise profit.

4.2.5 Managerial Implementation

In the present global marketing system, a manufacturer/supplier firm follows two practices: one is offering of discount to the retailers in the form of trade credit and another is promotion of own goods in the form of advertisement, hoarding etc. separately or jointly with retailers. The problem to the manager/owner of such a said firm is how much to invest as promotional effort and how long trade credit to be offered for maximum profit. His/her other problems are: (i) Nowadays, very often bank interest are changed at the direction of higher authorities (Reserve Bank of India, India) against loans and deposits, (ii) Again cash-in-hand, i.e., available budget for the firm also fluctuates. The mathematical representations of these realistic phenomena are difficult and can only be represented by imprecise variables – fuzzy or rough variables with the help of fluctuating data, if available or by the experienced experts' opinions. Thus this investigation solves the above mentioned problems for the production/supplier firm assuming that the retailer is co-operative and ready to help the supplier in the form of sharing the promotional cost and offering a part of the supplier's credit to his/her customers to boost the market demand.

On the basis of present investigation's findings, advice to the manager of a production/supplier firm is to go for co-operative promotional effort with retailer to derive maximum benefits for both sides. Taking some hypothetical data, optimum percentage of sharing has been pointed out. To cover up the uncertain situations due to changes in bank interest and available resource, we have taken some hypothetical imprecise data and derived the optimum share of percentage for maximum benefit to both sides – supplier and retailer. In practical cases, changing these values by real available data or experts' opinions, the firm's manager can have his/her optimum decisions.

4.3 Model 4.2: Fuzzy Optimization for Multi-item Supply Chain with Trade Credit and Two-Level Price Discount Under Promotional Cost Sharing [2]

4.3.1 Assumptions and Notations

The following notations are used in this model:

Notation	Meaning
c_i	supplier's purchase cost of the item i.
w_i	retailer's purchase cost of the item i, which is a mark-up m_s of c_i; i.e., $w_i = m_s c_i$.
r_i	retailer's selling price of the item i, which is a mark-up m_r of w_i; i.e., $r_i = m_r w_i$.
A_R	major setup cost of the retailer per order.
A_S	major setup cost of the supplier per order.
$a_{R,i}$	minor setup cost of the retailer for adding the item i into the order.
$a_{S,i}$	minor setup cost of the supplier for adding the item i into the order.
$h_{R,i}$	retailer's holding cost per unit for the item i, which is a mark-up m_h of w_i; i.e., $h_{R,i} = m_h w_i$.
T	replenishment cycle length.
T^l	optimal value of T for the coordination scenario.
T^t	optimal value of T for the non-coordination scenario.
f_R	fraction of cash discount provided from the retailer to the customer per unit of product, i.e., effective selling price of the retailer is $(1 - f_R)r_i$ per unit for item i.
f_R^l	optimal value of f_R for the coordination scenario.
f_R^t	optimal value of f_R for the non-coordination scenario.

[2] This model has been published in **International Journal of Fuzzy Systems, 2018, 20(5), 1644-1655**, *Springer*, with title "*Fuzzy Optimization for Multi-item Supply Chain with Trade Credit and Two-Level Price Discount Under Promotional Cost Sharing*"

Notation	Meaning
f_S	fraction of cash discount provided from the supplier to the retailer per unit of product, i.e., effective purchase cost of the retailer is $(1 - f_S)w_i$ per unit for item i.
t_S	retailer's credit period offered by the supplier.
Q_i	order quantity for the item i.
Q_i^l	optimal value of Q_i for the coordination scenario.
Q_i^t	optimal value of Q_i for the non-coordination scenario.
ρ_i	retailer promotional effort for the item i, $\rho_i \geq 1$.
ρ_i^l	optimal value of ρ_i for the coordination scenario.
ρ_i^t	optimal value of ρ_i for the non-coordination scenario.
ξ_i	basic demand for the item i.
I_p	rate of interest paid to the bank.
I_e	rate of interest earned from the bank.
F	fraction of the retailer's promotional cost shared by the supplier.
$C_i(\rho_i, \xi_i)$	annual promotional effort cost for the item i, where $C_i(\rho_i, \xi_i) = K_i(\rho_i - 1)^2 \xi_i^{\alpha_i}$, K_i is a positive constant and α_i is a constant.
Π_j	annual profit, $j = R$ for retailer, $j = S$ for supplier and $j = C$ for channel.
$G(\tilde{A})$	GMIV of the fuzzy number $\tilde{A}$.
$Cr(\tilde{A} * \tilde{B})$	credibility measure of a fuzzy event $\tilde{A} * \tilde{B}$, where $*$ is any valid binary operator on the fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

This model is developed under the following assumptions:

1. The retailer adopts joint multi-item replenishment policy.

2. No shortages are allowed.

3. The supplier provides a credit period t_S and a cash discount f_S per unit item for the retailer.

4. The retailer also provides a cash discount f_R per unit item for the customer, which magnify the base demand ξ_i with λf_R, where λ is a parameter.

5. The promotional effort ρ_i for item i also magnify the base demand ξ_i with $(\rho_i - 1)$.

6. So introduction of promotional effort and cash discount given to the customers reduces the base demand ξ_i, of i-th item, to $(\rho_i + \lambda f_R)\xi_i = \rho_i' \xi_i$, where $\rho_i' = (\rho_i + \lambda f_R)$.

7. The promotional effort cost is an increasing function of promotional effort and basic demand and is of the form: $C_i(\rho_i, \xi_i) = K_i(\rho_i - 1)^2 \xi_i^{\alpha_i}$, where K_i is a positive constant and α_i is a constant [97].

4.3.2 Mathematical Formulation of the Model

Here, a supplier-retailer supply chain is considered where supplier supplies n items to the retailer under joint replenishment policy. Retailer adopts a promotional cost $K_i(\rho_i - 1)^2 \xi_i^{\alpha_i}$ to increase the base demand (ξ_i) of the i-th item and annual increase of demand is $(\rho_i - 1)\xi_i$. Supplier offers a credit period t_S and a fraction f_S of cash discount per unit item to the retailer to settle the account. Due to the cash discount, retailer offers a fraction f_R of cash discount per unit item to its customers to increase the demand of the items. Increase of base demand ξ_i of the i-th item due to cash discount f_R is assumed as $\lambda f_R \xi_i$, where, λ is a parameter used to best fit the demand function. So the resultant demand of i-th item, due to introduction of promotional cost and cash discount given to the customers, is increased as $\rho_i' \xi_i = (\rho_i + \lambda f_R)\xi_i$. So effective demand of i-th item $D_i = \rho_i' \xi_i$. The cycle length T, the promotional effort ρ_i, $i = 1, 2, ..., n$ and the fraction of cash discount per unit item f_R are decision variables and this decision is made by the retailer.

4.3.2.1 Retailer's Profit

Order quantity, $Q_i = \rho_i' \xi_i T$

The inventory level of i-th item at any time t, $q_i(t) = Q_i - D_i t$

Major set-up cost per unit time $= \dfrac{A_R}{T}$

Minor set-up cost for i^{th} item per unit time $= \dfrac{a_{R,i}}{T}$

Total selling price for i^{th} item per unit time $= \dfrac{(1 - f_R) r_i Q_i}{T} = (1 - f_R) r_i \rho_i' \xi_i$

Total purchase price for i^{th} item per unit time $= \dfrac{(1 - f_S) w_i Q_i}{T} = (1 - f_S) w_i \rho_i' \xi_i$

Promotional cost for i^{th} item per unit time $= K_i(\rho_i - 1)^2 \xi_i^{\alpha_i}$

Total holding cost for i^{th} item $= h_{R,i}\int_0^T q_i(t)dt = h_{R,i}\int_0^T (Q_i - D_i t)dt = \frac{\rho_i'\xi_i T^2}{2}h_{R,i}$

Therefore, holding cost for i^{th} item per unit time $= \frac{\rho_i'\xi_i T}{2}h_{R,i}$.

Calculation of interest to be paid and interest earned for i^{th} item:

Interest paid by the retailer for i^{th} item per unit time $(TIP_{R,i})$

$$= \frac{1}{T} \times \text{Interest to be paid due to units stocked during } [t_S, T]$$

$$= \frac{1}{T}\int_{t_S}^{T} q_i(t)(1-f_S)w_i I_p dt = (1-f_S)w_i I_p \frac{\rho_i'\xi_i}{2T}(T-t_S)^2$$

Now, interest earned by the retailer for i^{th} item per unit time $(TIE_{R,i})$

$$= \frac{1}{T} \times \text{Interest to be earned due to normal selling during } [0, t_S]$$

$$= \frac{1}{T}\int_0^{t_S} D_i(t_S - t)(1-f_S)w_i I_e dt = (1-f_S)w_i I_e \frac{\rho_i'\xi_i}{2T}t_S^2$$

Hence, the retailer's average profit is as follows:

$$\begin{aligned}\Pi_R(\rho_i, T, f_R) = & -\frac{A_R}{T} + \sum_{i=1}^{n}\Big[\big\{(1-f_R)r_i - (1-f_S)w_i\big\}\rho_i'\xi_i - \frac{a_{R,i}}{T} - K_i(\rho_i - 1)^2\xi_i^{\alpha_i} \\ & - \frac{\rho_i'\xi_i T}{2}h_{R,i} - (1-f_S)w_i I_p \frac{\rho_i'\xi_i}{2T}(T-t_S)^2 \\ & + (1-f_S)w_i I_e \frac{\rho_i'\xi_i}{2T}t_S^2\Big]; \text{ where, } \rho_i' = \rho_i + \lambda t_R \end{aligned} \quad (4.36)$$

4.3.2.2 Supplier's Profit

Major set-up cost per unit time $= \frac{A_S}{T}$

Minor set-up cost for i^{th} item per unit time $= \frac{a_{S,i}}{T}$

Total selling price for i^{th} item per unit time $= \frac{(1-f_S)w_i Q_i}{T} = (1-f_S)w_i\rho_i'\xi_i$

Total purchase price for i^{th} item per unit time $= \frac{c_i Q_i}{T} = c_i\rho_i'\xi_i$

There is no holding cost for supplier, i.e., holding cost = 0

Total interest paid by the supplier for i^{th} item per unit time $(TIP_{S,i})$

$= \frac{1}{T}Q_i t_s c_i I_p = \rho_i'\xi_i t_s c_i I_p$

Hence, supplier's average profit is as follows:

$$\Pi_S(\rho_i, T, f_R) = -\frac{A_S}{T} + \sum_{i=1}^{n}\Big[\big\{(1-f_S)w_i - c_i\big\}\rho_i'\xi_i - \frac{a_{S,i}}{T} - \rho_i'\xi_i t_S c_i I_p\Big] \quad (4.37)$$

4.3.2.3 Channel Profit

The channel profit, i.e., the sum of the retailer's and the supplier's profit is as follows:

$$\Pi_C(\rho_i, T, f_R) = \Pi_R(\rho_i, T, f_R) + \Pi_S(\rho_i, T, f_R) \quad (4.38)$$

4.3.2.4 Non-Coordination Scenario (NCS)

In this scenario, the retailer determines the optimal promotional effort, replenishment cycle and fraction of cash discount given to the customers to maximize the retailer's profit per unit time $\Pi_R(\rho_i, T, f_R)$. The following lemma is considered to derive the condition of the optimal solution.

Lemma 4.3. The solution of $\partial\Pi_R(\rho_i, T, f_R)/\partial\rho_i = 0$, for $i = 1, 2, ..., n$; $\partial\Pi_R(\rho_i, T, f_R)/\partial T = 0$ and $\partial\Pi_R(\rho_i, T, f_R)/\partial f_R = 0$ is maximal for $\Pi_R(\rho_i, T, f_R)$, iff $V_R \leq 0$ and $W_R \leq 0$;

$$\begin{aligned}
\text{where,}\ V_R &= -\frac{2\Big[A_R + \sum_{i=1}^{n} a_{R,i}\Big]}{T^3} - \sum_{i=1}^{n}(1-f_S)w_i(I_p - I_e)\frac{\rho_i'\xi_i}{T^3}t_S^2 \\
&+ \sum_{i=1}^{n}\Big[\Big\{-\frac{\xi_i}{2}\{h_{R,i} + (1-f_S)w_i I_p\} + (1-f_S)w_i(I_p - I_e)\frac{\xi_i t_S^2}{2T^2}\Big\}^2\Big/ 2K_i\xi_i^{\alpha_i}\Big] \\
\text{and}\ W_R &= -\sum_{i=1}^{n} 2\lambda r_i\xi_i + \sum_{i=1}^{n}\Big[r_i^2\xi_i^2\Big/2K_i\xi_i^{\alpha_i}\Big] - \frac{V_R' \times W_R'}{V_R} \\
\text{where,}\ V_R' &= -\sum_{i=1}^{n}\frac{\lambda\xi_i}{2}\Big\{h_{R,i} + (1-f_S)w_i I_p\Big\} + \sum_{i=1}^{n}(1-f_S)w_i(I_p - I_e)\frac{\lambda\xi_i t_S^2}{2T^2} \\
&+ \sum_{i=1}^{n}\Big[\Big\{\frac{r_i\xi_i^2}{2}\{h_{R,i} + (1-f_S)w_i I_p\} - (1-f_S)w_i(I_p - I_e)\frac{r_i\xi_i^2 t_S^2}{2T^2}\Big\}\Big/2K_i\xi_i^{\alpha_i}\Big] \\
\text{and}\ W_R' &= V_R'.
\end{aligned}$$

Proof. The Hessian matrix for Π_R is

$$D_{n+2} = \begin{bmatrix} \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_1^2} & 0 & \cdots & 0 & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_1\partial T} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_1\partial f_R} \\ 0 & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_2^2} & \cdots & 0 & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_2\partial T} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_2\partial f_R} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_n^2} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_n\partial T} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_n\partial f_R} \\ \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial T\partial\rho_1} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial T\partial\rho_2} & \cdots & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial T\partial\rho_n} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial T^2} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial T\partial f_R} \\ \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R\partial\rho_1} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R\partial\rho_2} & \cdots & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R\partial\rho_n} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R\partial T} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R^2} \end{bmatrix}$$

Since $\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i^2} < 0$ for $i = 1, 2, ..., n$, so $(-1)^i.|D_i| > 0$ for $i = 1, 2, ..., n$. If $(-1)^i.|D_i| > 0$ for $i = n+1$ and $i = n+2$, then there must be a solution of the given set of equations to maximize $\Pi_R(\rho_i, T, f_R)$.

Multiplying each element in each row i, $(i = 1, 2, ..., n)$ of D_{n+2} by $-\Big(\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial T\partial\rho_i} \Big/ \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i^2}\Big)$ and adding it to the corresponding element in $(n+1)th$ row, the above matrix becomes

$$D_{n+2} = \begin{bmatrix} \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_1^2} & 0 & \cdots & 0 & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_1\partial T} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_1\partial f_R} \\ 0 & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_2^2} & \cdots & 0 & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_2\partial T} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_2\partial f_R} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_n^2} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_n\partial T} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_n\partial f_R} \\ 0 & 0 & \cdots & 0 & V_R & V_R' \\ \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R\partial\rho_1} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R\partial\rho_2} & \cdots & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R\partial\rho_n} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R\partial T} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R^2} \end{bmatrix}$$

$$\begin{aligned} \text{where, } V_R &= \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial T^2} - \sum_{i=1}^{n}\Big[\Big(\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i\partial T}\Big)^2 \Big/ \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i^2}\Big] \\ &= -\frac{2\Big[A_R + \sum_{i=1}^{n} a_{R,i}\Big]}{T^3} - \sum_{i=1}^{n}(1-f_S)w_i(I_p - I_e)\frac{\rho_i'\xi_i}{T^3}t_S^2 \\ &\quad + \sum_{i=1}^{n}\Big[\Big\{-\frac{\xi_i}{2}\{h_{R,i} + (1-f_S)w_iI_p\} + (1-f_S)w_i(I_p - I_e)\frac{\xi_i t_S^2}{2T^2}\Big\}^2 \Big/ 2K_i\xi_i^{\alpha_i}\Big] \end{aligned}$$

$$\begin{aligned} \text{and } \ V'_R &= \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial T\partial f_R} - \sum_{i=1}^{n}\Big[\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i\partial f_R}.\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i\partial T}\Big/\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i^2}\Big] \\ &= -\sum_{i=1}^{n}\frac{\lambda\xi_i}{2}\Big\{h_{R,i}+(1-f_S)w_iI_p\Big\} + \sum_{i=1}^{n}(1-f_S)w_i(I_p-I_e)\frac{\lambda\xi_i t_S^2}{2T^2} \\ &\quad + \sum_{i=1}^{n}\Big[\Big\{\frac{r_i\xi_i^2}{2}\{h_{R,i}+(1-f_S)w_iI_p\} - (1-f_S)w_i(I_p-I_e)\frac{r_i\xi_i^2t_S^2}{2T^2}\Big\}\Big/2K_i\xi_i^{\alpha_i}\Big] \end{aligned}$$

Now, multiplying each element in each column i, $(i = 1,2,...,n)$ of D_{n+2} by $-\Big(\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i\partial T}\Big/\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i^2}\Big)$ and adding it to the corresponding element in $(n+1)th$ column, the matrix reduces to

$$D_{n+2} = \begin{bmatrix} \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_1^2} & 0 & \cdots & 0 & 0 & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_1\partial f_R} \\ 0 & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_2^2} & \cdots & 0 & 0 & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_2\partial f_R} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_n^2} & 0 & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_n\partial f_R} \\ 0 & 0 & \cdots & 0 & V_R & V'_R \\ \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R\partial\rho_1} & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R\partial\rho_2} & \cdots & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R\partial\rho_n} & W'_R & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R^2} \end{bmatrix}$$

where,

$$\begin{aligned} W'_R &= \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial T\partial f_R} - \sum_{i=1}^{n}\Big[\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i\partial f_R}.\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i\partial T}\Big/\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i^2}\Big] \\ &= V'_R. \end{aligned}$$

Now, multiplying each element in each row i, $(i = 1,2,...,n)$ of D_{n+2} by $-\Big(\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R\partial\rho_i}\Big/\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i^2}\Big)$ and in $(n+1)th$ row by $-(W'_R/V_R)$ and adding it to the corresponding element in $(n+2)th$ row, the following matrix is obtained

$$D_{n+2} = \begin{bmatrix} \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_1^2} & 0 & \cdots & 0 & 0 & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_1\partial f_R} \\ 0 & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_2^2} & \cdots & 0 & 0 & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_2\partial f_R} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_n^2} & 0 & \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_n\partial f_R} \\ 0 & 0 & \cdots & 0 & V_R & V'_R \\ 0 & 0 & \cdots & 0 & 0 & W_R \end{bmatrix}$$

where,

$$\begin{aligned} W_R &= \frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial f_R^2} - \sum_{i=1}^{n}\Big[\Big(\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i\partial f_R}\Big)^2\Big/\frac{\partial^2\Pi_R(\rho_i,T,f_R)}{\partial\rho_i^2}\Big] - \frac{V_R'\times W_R'}{V_R} \\ &= -\sum_{i=1}^{n}2\lambda r_i\xi_i + \sum_{i=1}^{n}\Big[r_i^2\xi_i^2\Big/2K_i\xi_i^{\alpha_i}\Big] - \frac{V_R'\times W_R'}{V_R}. \end{aligned}$$

Hence the required condition is proved. □

Let ρ_i^t, T^t, f_R^t be the optimal decision of the retailer in this scenario. Then the retailer's profit in this scenario is $\Pi_R(\rho_i^t, T^t, f_R^t)$.

4.3.2.5 Coordination Scenario (CS)

In this scenario, the supplier offers to pay a percentage of the promotional cost F. Then the retailer's and the supplier's profits are as follows:

$$\begin{aligned} \Pi_R^F(\rho_i,T,f_R) &= -\frac{A_R}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_R)r_i-(1-f_S)w_i\Big\}\rho_i'\xi_i - (1-F)K_i(\rho_i-1)^2\xi_i^{\alpha_i} \\ &\quad -\frac{a_{R,i}}{T} - \frac{\rho_i'\xi_i T}{2}h_{R,i} - (1-f_S)w_iI_p\frac{\rho_i'\xi_i}{2T}(T-t_S)^2 + (1-f_S)w_iI_e\frac{\rho_i'\xi_i}{2T}t_S^2\Big] \\ \Pi_S^F(\rho_i,T,f_R) &= -\frac{A_S}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_S)w_i-c_i\Big\}\rho_i'\xi_i - \frac{a_{S,i}}{T} - FK_i(\rho_i-1)^2\xi_i^{\alpha_i} - \rho_i'\xi_i t_S c_i I_p\Big] \end{aligned}$$

Thus the channel profit is

$$\Pi_C(\rho_i,T,f_R) = \Pi_R^F(\rho_i,T,f_R) + \Pi_S^F(\rho_i,T,f_R) \tag{4.39}$$

To derive the condition of the optimal solution the following lemma is considered.

Lemma 4.4. The solution of $\partial\Pi_C(\rho_i,T,f_R)/\partial\rho_i = 0$, for $i = 1,2,...,n$; $\partial\Pi_C(\rho_i,T,f_R)/\partial T = 0$ and $\partial\Pi_C(\rho_i,T,f_R)/\partial f_R = 0$ is maximal for $\Pi_C(\rho_i,T,f_R)$, iff $V_C \le 0$ and $W_C \le 0$;

$$\begin{aligned} \text{where, } V_C &= -\frac{2\Big[(A_R+A_S)+\sum_{i=1}^{n}(a_{R,i}+a_{S,i})\Big]}{T^3} - \sum_{i=1}^{n}(1-f_S)w_i(I_p-I_e)\frac{\rho_i'\xi_i}{T^3}t_S^2 \\ &\quad + \sum_{i=1}^{n}\Big[\Big\{-\frac{\xi_i}{2}\{h_{R,i}+(1-f_S)w_iI_p\} + (1-f_S)w_i(I_p-I_e)\frac{\xi_i t_S^2}{2T^2}\Big\}^2\Big/2K_i\xi_i^{\alpha_i}\Big] \\ \text{and } W_C &= -\sum_{i=1}^{n}2\lambda r_i\xi_i + \sum_{i=1}^{n}\Big[r_i^2\xi_i^2\Big/2K_i\xi_i^{\alpha_i}\Big] - \frac{V_C'\times W_C'}{V_C} \end{aligned}$$

$$\text{where, } V'_C = -\sum_{i=1}^{n} \frac{\lambda\xi_i}{2}\Big\{h_{R,i} + (1-f_S)w_i I_p\Big\} + \sum_{i=1}^{n}(1-f_S)w_i(I_p - I_e)\frac{\lambda\xi_i t_S^2}{2T^2}$$
$$+\sum_{i=1}^{n}\Big[\Big\{\frac{r_i\xi_i^2}{2}\{h_{R,i} + (1-f_S)w_i I_p\} - (1-f_S)w_i(I_p - I_e)\frac{r_i\xi_i^2 t_S^2}{2T^2}\Big\}\Big/2K_i\xi_i^{\alpha_i}\Big]$$
$$\text{and } W'_C = V'_C.$$

Proof. The proof is similar to that in Lemma 4.3. □

Let ρ_i^l, T^l, f_R^l be the optimal decision of the coordination scenario. Then the retailer's profit and supplier's profit in this scenario are $\Pi_R^F(\rho_i^l, T^l, f_R^l)$ and $\Pi_S^F(\rho_i^l, T^l, f_R^l)$ respectively.

The retailer's and the supplier's profits under the non-coordination scenario are viewed as the lower bounds for the model under the coordination scenario. Let $\Pi_R = \Pi_R(\rho_i^t, T^t, f_R^t)$ and $\Pi_S = \Pi_S(\rho_i^t, T^t, f_R^t)$.

Proposition 4.2. (a) Profits for both parties increase under the coordination scenario, when the fraction of the retailer's promotional cost is determined to be within the appropriate range (F_{min}, F_{max}), where

$$F_{min} = \Big\{\Pi_R + \frac{A_R}{T^l} - \sum_{i=1}^{n}\Big[\Big\{(1-f_R)r_i - (1-f_S)w_i\Big\}\rho_i''\xi_i - \frac{a_{R,i}}{T^l} - \frac{\rho_i''\xi_i T^l}{2}h_{R,i}$$
$$-K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i} - (1-f_S)w_i I_p\frac{\rho_i''\xi_i}{2T^l}(T^l - t_S)^2 + (1-f_S)w_i I_e\frac{\rho_i''\xi_i}{2T^l}t_S^2\Big]\Big\}$$
$$\Big/\sum_{i=1}^{n}K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i}$$
$$\& \; F_{max} = \Big\{-\frac{A_S}{T^l} + \sum_{i=1}^{n}\Big[\Big\{(1-f_S)w_i - c_i\Big\}\rho_i''\xi_i - \frac{a_{S,i}}{T^l} - \rho_i''\xi_i c_i t_S I_p\Big]$$
$$-\Pi_S\Big\}\Big/\sum_{i=1}^{n}K_i(\rho_i^l - 1)^2\xi_i^{\alpha_i} \quad \text{where, } \rho_i'' = \rho_i^l + \lambda f_R^l.$$

(b) When the supplier and the retailer have the same bargaining power, the appropriate fraction of the promotional cost sharing is $F = (F_{max} + F_{min})/2$.

Proof. (a) From

$$\begin{aligned}&\Pi_R^F(\rho_i^l, T^l, f_R^l) - \Pi_R \geq 0\\ \Rightarrow\ & -\frac{A_R}{T^l} + \sum_{i=1}^{n}\Big[\Big\{(1-f_R)r_i - (1-f_S)w_i\Big\}\rho_i''\xi_i - \frac{a_{R,i}}{T^l} - (1-F)K_i(\rho_i^l-1)^2\xi_i^{\alpha_i}\\ & -\frac{\rho_i''\xi_i T^l}{2}h_{R,i} - (1-f_S)w_i I_p\frac{\rho_i''\xi_i}{2T^l}(T^l - t_S)^2 + (1-f_S)w_i I_e\frac{\rho_i''\xi_i}{2T^l}t_S^2\Big] \geq \Pi_R\\ & \text{where, } \rho_i'' = \rho_i^l + \lambda f_R^l.\end{aligned}$$

$$\begin{aligned}\Rightarrow\ & F \geq \Big\{\Pi_R + \frac{A_R}{T^l} - \sum_{i=1}^{n}\Big[\Big\{(1-f_R)r_i - (1-f_S)w_i\Big\}\rho_i''\xi_i - \frac{a_{R,i}}{T^l} - \frac{\rho_i''\xi_i T^l}{2}h_{R,i}\\ & -K_i(\rho_i^l-1)^2\xi_i^{\alpha_i} - (1-f_S)w_i I_p\frac{\rho_i''\xi_i}{2T^l}(T^l-t_S)^2 + (1-f_S)w_i I_e\frac{\rho_i''\xi_i}{2T^l}t_S^2\Big]\Big\}\\ & \Big/\sum_{i=1}^{n}K_i(\rho_i^l-1)^2\xi_i^{\alpha_i}\end{aligned}$$

Therefore, F_{min} is obtained. Also from $\Pi_S^F(\rho_i^l, T^l, f_R^l) - \Pi_S \geq 0$, F_{max} can be obtain.

(b) The following relations are found from (a):

$$\begin{aligned}F_{max} - F &= \Big[\Pi_S^F(\rho_i^l, T^l, f_R^l) - \Pi_S\Big]\Big/\sum_{i=1}^{n}K_i(\rho_i^l-1)^2\xi_i^{\alpha_i}\\ \Rightarrow \triangle\Pi_S^F &= (F_{max} - F)\sum_{i=1}^{n}K_i(\rho_i^l-1)^2\xi_i^{\alpha_i}\\ \text{Similarly,}\quad \triangle\,\Pi_R^F &= (F - F_{min})\sum_{i=1}^{n}K_i(\rho_i^l-1)^2\xi_i^{\alpha_i}\end{aligned}$$

Now, $\triangle\Pi_S^F(F) \times \triangle\Pi_R^F(F)$

$$\begin{aligned}&= \Big[(F_{max} - F)\sum_{i=1}^{n}K_i(\rho_i^l-1)^2\xi_i^{\alpha_i}\Big] \times \Big[(F - F_{min})\sum_{i=1}^{n}K_i(\rho_i^l-1)^2\xi_i^{\alpha_i}\Big]\\ &= (F_{max} - F)(F - F_{min})\Big\{\sum_{i=1}^{n}K_i(\rho_i^l-1)^2\xi_i^{\alpha_i}\Big\}^2\\ &= (F_{max}.F - F_{max}.F_{min} - F^2 + F.F_{min}) \times \Big\{\sum_{i=1}^{n}K_i(\rho_i^l-1)^2\xi_i^{\alpha_i}\Big\}^2\\ &= \Big[-\Big(F - \frac{F_{max}+F_{min}}{2}\Big)^2 + \frac{(F_{max}-F_{min})^2}{4}\Big] \times \Big\{\sum_{i=1}^{n}K_i(\rho_i^l-1)^2\xi_i^{\alpha_i}\Big\}^2.\end{aligned}$$

Thus, the appropriate fraction of the promotional cost sharing is obtained as $F = (F_{max} + F_{min})/2$. □

4.3.2.6 Fuzzy Model

As discussed in the introduction section that in real life most of the inventory parameters are fuzzy in nature. When some of the inventory parameters are fuzzy in nature model reduces to a fuzzy model. Normally set up cost, holding cost are imprecise in nature. In this model let us consider major set up costs A_R, A_S, minor set up costs $a_{R,i}$, $a_{S,i}$, $i = 1, 2, ..., n$, and holding costs $h_{R,i}$, $i = 1, 2, ..., n$ as fuzzy numbers $\tilde{A}_R$, $\tilde{A}_S$, $\tilde{a}_{R,i}$, $\tilde{a}_{S,i}$, $\tilde{h}_{R,i}$, $i = 1, 2, ..., n$ respectively then profits in both the scenario become imprecise in nature.

Fuzzy model in Non-Coordination Scenario: According to the above assumptions, in this case, the individual profits and the channel profit are reduces to fuzzy numbers and are represented by

$$\begin{aligned}
\tilde{\Pi}_R(\rho_i, T, f_R) &= -\frac{\tilde{A}_R}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_R)r_i - (1-f_S)w_i\Big\}\rho_i'\xi_i - \frac{\tilde{a}_{R,i}}{T} - K_i(\rho_i-1)^2\xi_i^{\alpha_i} \\
&\quad -\frac{\rho_i'\xi_i T}{2}\tilde{h}_{R,i} - (1-f_S)w_i I_p\frac{\rho_i'\xi_i}{2T}(T-t_S)^2 \\
&\quad +(1-f_S)w_i I_e\frac{\rho_i'\xi_i}{2T}t_S^2\Big] \qquad (4.40)\\
\tilde{\Pi}_S(\rho_i, T, f_R) &= -\frac{\tilde{A}_S}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_S)w_i - c_i\Big\}\rho_i'\xi_i - \frac{\tilde{a}_{S,i}}{T} - \rho_i'\xi_i t_S c_i I_p\Big] \qquad (4.41)\\
\tilde{\Pi}_C(\rho_i, T, f_R) &= -\frac{(\tilde{A}_R + \tilde{A}_S)}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_R)r_i - c_i\Big\}\rho_i'\xi_i - \frac{(\tilde{a}_{R,i} + \tilde{a}_{S,i})}{T} - \frac{\rho_i'\xi_i T}{2}\tilde{h}_{R,i} \\
&\quad -K_i(\rho_i-1)^2\xi_i^{\alpha_i} - \rho_i'\xi_i t_S c_i I_p - (1-f_S)w_i I_p\frac{\rho_i'\xi_i}{2T}(T-t_S)^2 \\
&\quad +(1-f_S)w_i I_e\frac{\rho_i'\xi_i}{2T}t_S^2\Big] \qquad (4.42)
\end{aligned}$$

Considering the fuzzy numbers $\tilde{A}_R$, $\tilde{A}_S$, $\tilde{a}_{R,i}$, $\tilde{a}_{S,i}$, $\tilde{h}_{R,i}$, $i = 1, 2, ..., n$ as TFNs (A_{R1}, A_{R2}, A_{R3}), (A_{S1}, A_{S2}, A_{S3}), $(a_{R1,i}, a_{R2,i}, a_{R3,i})$, $(a_{S1,i}, a_{S2,i}, a_{S3,i})$, $(h_{R1,i}, h_{R2,i}, h_{R3,i})$ respectively, the fuzzy numbers $\tilde{\Pi}_R$, $\tilde{\Pi}_S$, $\tilde{\Pi}_C$ becomes TFNs $(\Pi_{R1},\Pi_{R2},\Pi_{R3})$, $(\Pi_{S1},\Pi_{S2},\Pi_{S3})$, $(\Pi_{C1},\Pi_{C2},\Pi_{C3})$ respectively, where

$$\begin{aligned}
\Pi_{R1} &= -\frac{A_{R3}}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_R)r_i - (1-f_S)w_i\Big\}\rho_i'\xi_i - \frac{a_{R3,i}}{T} - \frac{\rho_i'\xi_i T}{2}h_{R3,i} \\
&\quad -K_i(\rho_i-1)^2\xi_i^{\alpha_i} - (1-f_S)w_i I_p\frac{\rho_i'\xi_i}{2T}(T-t_S)^2 + (1-f_S)w_i I_e\frac{\rho_i'\xi_i}{2T}t_S^2\Big] \\
\Pi_{R2} &= -\frac{A_{R2}}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_R)r_i - (1-f_S)w_i\Big\}\rho_i'\xi_i - \frac{a_{R2,i}}{T} - \frac{\rho_i'\xi_i T}{2}h_{R2,i} \\
&\quad -K_i(\rho_i-1)^2\xi_i^{\alpha_i} - (1-f_S)w_i I_p\frac{\rho_i'\xi_i}{2T}(T-t_S)^2 + (1-f_S)w_i I_e\frac{\rho_i'\xi_i}{2T}t_S^2\Big]
\end{aligned}$$

$$\begin{aligned}\Pi_{R3} &= -\frac{A_{R1}}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_R)r_i - (1-f_S)w_i\Big\}\rho_i'\xi_i - \frac{a_{R1,i}}{T} - \frac{\rho_i'\xi_i T}{2}h_{R1,i} \\ &\quad -K_i(\rho_i-1)^2\xi_i^{\alpha_i} - (1-f_S)w_i I_p\frac{\rho_i'\xi_i}{2T}(T-t_S)^2 + (1-f_S)w_i I_e\frac{\rho_i'\xi_i}{2T}t_S^2\Big]\end{aligned}$$

$$\begin{aligned}\Pi_{S1} &= -\frac{A_{S3}}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_S)w_i - c_i\Big\}\rho_i'\xi_i - \frac{a_{S3,i}}{T} - \rho_i'\xi_i t_S c_i I_p\Big] \\ \Pi_{S2} &= -\frac{A_{S2}}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_S)w_i - c_i\Big\}\rho_i'\xi_i - \frac{a_{S2,i}}{T} - \rho_i'\xi_i t_S c_i I_p\Big] \\ \Pi_{S3} &= -\frac{A_{S1}}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_S)w_i - c_i\Big\}\rho_i'\xi_i - \frac{a_{S1,i}}{T} - \rho_i'\xi_i t_S c_i I_p\Big]\end{aligned}$$

$$\begin{aligned}\Pi_{C1} &= -\frac{(A_{R3}+A_{S3})}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_R)r_i - c_i\Big\}\rho_i'\xi_i - \frac{(a_{R3,i}+a_{S3,i})}{T} - \frac{\rho_i'\xi_i T}{2}h_{R3,i} \\ &\quad -K_i(\rho_i-1)^2\xi_i^{\alpha_i} - \rho_i'\xi_i t_S c_i I_p - (1-f_S)w_i I_p\frac{\rho_i'\xi_i}{2T}(T-t_S)^2 + (1-f_S)w_i I_e\frac{\rho_i'\xi_i}{2T}t_S^2\Big] \\ \Pi_{C2} &= -\frac{(A_{R2}+A_{S2})}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_R)r_i - c_i\Big\}\rho_i'\xi_i - \frac{(a_{R2,i}+a_{S2,i})}{T} - \frac{\rho_i'\xi_i T}{2}h_{R2,i} \\ &\quad -K_i(\rho_i-1)^2\xi_i^{\alpha_i} - \rho_i'\xi_i t_S c_i I_p - (1-f_S)w_i I_p\frac{\rho_i'\xi_i}{2T}(T-t_S)^2 + (1-f_S)w_i I_e\frac{\rho_i'\xi_i}{2T}t_S^2\Big] \\ \Pi_{C3} &= -\frac{(A_{R1}+A_{S1})}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_R)r_i - c_i\Big\}\rho_i'\xi_i - \frac{(a_{R1,i}+a_{S1,i})}{T} - \frac{\rho_i'\xi_i T}{2}h_{R1,i} \\ &\quad -K_i(\rho_i-1)^2\xi_i^{\alpha_i} - \rho_i'\xi_i t_S c_i I_p - (1-f_S)w_i I_p\frac{\rho_i'\xi_i}{2T}(T-t_S)^2 + (1-f_S)w_i I_e\frac{\rho_i'\xi_i}{2T}t_S^2\Big]\end{aligned}$$

Now the GMIV [116] of $\tilde{\Pi}_R$, $\tilde{\Pi}_S$, $\tilde{\Pi}_C$ are given by

$$\begin{aligned}G(\tilde{\Pi}_R) &= \frac{1}{6}[\Pi_{R1} + 4\Pi_{R2} + \Pi_{R3}] \\ &= -\frac{G(\tilde{A}_R)}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_R)r_i - (1-f_S)w_i\Big\}\rho_i'\xi_i - \frac{G(\tilde{a}_{R,i})}{T} - \frac{\rho_i'\xi_i T}{2}G(\tilde{h}_{R,i}) \\ &\quad -K_i(\rho_i-1)^2\xi_i^{\alpha_i} - (1-f_S)w_i I_p\frac{\rho_i'\xi_i}{2T}(T-t_S)^2 + (1-f_S)w_i I_e\frac{\rho_i'\xi_i}{2T}t_S^2\Big] \qquad (4.43)\end{aligned}$$

$$\begin{aligned}G(\tilde{\Pi}_S) &= \frac{1}{6}[\Pi_{S1} + 4\Pi_{S2} + \Pi_{S3}] \\ &= -\frac{G(\tilde{A}_S)}{T} + \sum_{i=1}^{n}\Big[\Big\{(1-f_S)w_i - c_i\Big\}\rho_i'\xi_i - \frac{G(\tilde{a}_{S,i})}{T} - \rho_i'\xi_i t_S c_i I_p\Big] \qquad (4.44)\end{aligned}$$

$$
\begin{aligned}
G(\tilde{\Pi}_C) &= \frac{1}{6}[\Pi_{C1} + 4\Pi_{C2} + \Pi_{C3}] \\
&= -\frac{\{G(\tilde{A}_R) + G(\tilde{A}_S)\}}{T} + \sum_{i=1}^{n}\Big[\{(1-f_R)r_i - c_i]\rho_i'\xi_i - \frac{\{G(\tilde{a}_{R,i}) + G(\tilde{a}_{S,i})\}}{T} \\
& -\frac{\rho_i'\xi_i T}{2}G(\tilde{h}_{R,i}) - K_i(\rho_i - 1)^2\xi_i^{\alpha_i} - \rho_i'\xi_i t_S c_i I_p - (1-f_S)w_i I_p \frac{\rho_i'\xi_i}{2T}(T-t_S)^2 \\
& +(1-f_S)w_i I_e \frac{\rho_i'\xi_i}{2T} t_S^2\Big] \qquad (4.45)
\end{aligned}
$$

where, $G(\tilde{A}_R) = \frac{1}{6}(A_{R1} + 4A_{R2} + A_{R3})$, $G(\tilde{A}_S) = \frac{1}{6}(A_{S1} + 4A_{S2} + A_{S3})$, $G(\tilde{a}_{R,i}) = \frac{1}{6}(a_{R1,i} + 4a_{R2,i} + a_{R3,i})$, $G(\tilde{a}_{S,i}) = \frac{1}{6}(a_{S1,i} + 4a_{S2,i} + a_{S3,i})$, $G(\tilde{h}_{R,i}) = \frac{1}{6}(h_{R1,i} + 4h_{R2,i} + h_{R3,i})$, for $i = 1, 2, ..., n$.

So in this case the problem reduces to

$$
\begin{aligned}
\text{Determine} \quad & \rho_i(i = 1, 2, ..., n), T, f_R \\
\text{to maximize} \quad & \tilde{\Pi}_R = (\Pi_{R1}, \Pi_{R2}, \Pi_{R3}) \qquad (4.46)
\end{aligned}
$$

Till date, fuzzy optimization is not well defined. There exist no method which guarantees optimal solution of any optimization problem having fuzzy objective. Due to this reason, here, the problem is solved using PSO (cf. § 2.2.2.1), where comparisons of the objectives are made following two approaches. Let $\tilde{\Pi}_{Ra}$, $\tilde{\Pi}_{Rb}$ be the two objectives corresponding to two solutions X_a, X_b respectively. Then two approaches of comparison of solutions are listed below:

- In the first approach, GMIV of the objectives are taken to compare the solutions. According to this approach X_a dominates X_b if G($\tilde{\Pi}_{Ra}$)> G($\tilde{\Pi}_{Rb}$). So in this case crisp equivalent of the fuzzy numbers are used for the decision.
- In the second approach, credibility measure of fuzzy event is used to compare the solutions. According to this approach X_a dominates X_b if Cr($\tilde{\Pi}_{Ra} > \tilde{\Pi}_{Rb}$) > 0.5. In this approach no crisp equivalent of fuzzy numbers are used to find marketing decisions. This is a valid fuzzy comparison operation as $Cr(\tilde{A} > \tilde{B}) + Cr(\tilde{A} \leq \tilde{B}) = 1$ [107].

Fuzzy model in Coordination Scenario: For the coordination scenario, the individual profits and the channel profit as fuzzy numbers are represented by

$$\begin{aligned}\tilde{\Pi}_R^F(\rho_i,T,f_R) &= -\frac{\tilde{A}_R}{T}+\sum_{i=1}^{n}\Big[\Big\{(1-f_R)r_i-(1-f_S)w_i\Big\}\rho_i'\xi_i-(1-F)K_i(\rho_i-1)^2\xi_i^{\alpha_i}\\ &\quad -\frac{\tilde{a}_{R,i}}{T}-\frac{\rho_i'\xi_i T}{2}\tilde{h}_{R,i}-(1-f_S)w_iI_p\frac{\rho_i'\xi_i}{2T}(T-t_S)^2\\ &\quad +(1-f_S)w_iI_e\frac{\rho_i'\xi_i}{2T}t_S^2\Big] \qquad (4.47)\end{aligned}$$

$$\begin{aligned}\tilde{\Pi}_S^F(\rho_i,T,f_R) &= -\frac{\tilde{A}_S}{T}+\sum_{i=1}^{n}\Big[\Big\{(1-f_S)w_i-c_i\Big\}\rho_i'\xi_i-\frac{\tilde{a}_{S,i}}{T}-FK_i(\rho_i-1)^2\xi_i^{\alpha_i}\\ &\quad -\rho_i'\xi_i t_S c_i I_p\Big] \qquad (4.48)\\ \tilde{\Pi}_C(\rho_i,T,f_R) &= \tilde{\Pi}_R^F(\rho_i,T,f_R)+\tilde{\Pi}_S^F(\rho_i,T,f_R) \qquad (4.49)\end{aligned}$$

So in this case problem reduces to

$$\begin{aligned}&\text{Determine} && \rho_i(i=1,2,...,n),T,f_R \text{ to}\\ &\text{maximize} && \tilde{\Pi}_C=(\Pi_{C1},\Pi_{C2},\Pi_{C3}) \qquad (4.50)\end{aligned}$$

The problem is solved using proposed PSO where comparisons of the objectives are made by using two approaches – (i) GMIV approach and (ii) Credibility Measure approach, which are discussed above.

4.3.3 Numerical Illustration

The model is illustrated with following set of hypothetical data which are presented below:

Example 4.2. Here two items are considered. The following different parametric values are in appropriate units :
$I_e = 0.08$, $I_p = 0.10$, $m_r = 1.8$, $m_s = 1.6$, $m_h = 0.05$, $c_1 = 7.5$, $c_2 = 6$, $\xi_1 = 60$, $\xi_2 = 80$, $A_R = 275$, $a_{r,1} = 1$, $a_{r,2} = 1$, $A_S = 140$, $a_{s,1} = 0.8$, $a_{s,2} = 0.8$, $\alpha_1 = 1.3$, $\alpha_2 = 1.2$, $K_1 = 2.7$, $K_2 = 2.5$, $t_s = 0.5$, $\lambda = 4.2$, $f_S = 0.25$, $w_1 = m_s \times c_1$, $w_2 = m_s \times c_2$, $r_1 = m_r \times w_1$, $r_2 = m_r \times w_2$, $h_{R,1} = m_h \times w_1$, $h_{R,2} = m_h \times w_2$.

TABLE 4.8: Optimum Results of Crisp Model in NCS and CS using PSO technique and Lingo Software

Output Variable	using PSO		using Lingo	
	NCS	CS	NCS	CS
Π_R	1900.12	1910.33	1900.12	1910.35
Π_S	162.63	170.22	162.61	170.20
Π_C	2062.75	2080.55	2062.73	2080.55
ρ_1	1.55	1.58	1.55	1.58
ρ_2	1.68	1.72	1.68	1.72
T	1.24	1.48	1.24	1.48
f_R	0.09	0.11	0.09	0.11
Q_1	142.56	179.55	142.55	179.46
Q_2	202.63	255.08	202.58	255.04
F_{min}	-	0.0366	-	0.0366
F_{max}	-	0.0774	-	0.0774
F	-	0.06	-	0.06
V_R	-293.48	-	-294.69	-
W_R	-18755.24	-	-18756.22	-
V_C	-	-171.94	-	-172.90
W_C	-	-18583.07	-	-18585.36

TABLE 4.9: Values of Π_R, Π_S due to different F in CS using PSO technique

F	Π_R	Π_S	Π_C
0.03	**1897.23**	183.32	2080.55
0.04	1901.65	178.90	2080.55
0.05	1905.99	174.56	2080.55
0.06	1910.33	170.22	2080.55
0.07	1914.62	165.93	2080.55
0.08	1919.07	**161.48**	2080.55

(a) Crisp Model: The above data set is considered as input for crisp model. This model is solved using PSO technique (cf. § 2.2.2.1) and GRG technique (cf. § 2.2.1.1) using Lingo 14.0 software and the results are presented in Table 4.8. For above Example 4.2, the channel profit is optimized due to sharing of different fraction (F) of promotional cost by the supplier and the results are shown in Table 4.9. PSO is used to find the results of crisp model to established its efficiency (with respect to LINGO) to solve this model. In the ANOVA test (cf. § 4.12) it is established that it's performance is comparable with the established software LINGO (prepared following GRG technique). But LINGO can not be applicable for fuzzy optimization, where PSO can be used to find marketing decision. As a result, here, PSO is also used to find solutions of crisp model.

TABLE 4.10: Results of Fuzzy model following GMIV approach

Output Variable	using PSO		using Lingo	
	NCS	CS	NCS	CS
$\tilde{\Pi}_R$	(1894.20,1900.12,1907.89)	(1904.49,1910.23,1918.27)	(1894.20,1900.12,1907.89)	(1904.52,1910.26,1918.30)
$G(\tilde{\Pi}_R)$	1900.43	1910.61	1900.43	1910.64
$\tilde{\Pi}_S$	(158.61,162.70,165.99)	(166.89,170.32,173.08)	(158.59,162.69,165.98)	(166.86,170.29,173.05)
$G(\tilde{\Pi}_S)$	162.57	170.21	162.56	170.18
$\tilde{\Pi}_C$	(2052.80,2062.83,2073.88)	(2071.38,2080.55,2091.35)	(2052.79,2062.81,2073.87)	(2071.38,2080.55,2091.35)
$G(\tilde{\Pi}_C)$	2063.00	2080.82	2062.99	2080.82
ρ_1	1.55	1.58	1.55	1.58
ρ_2	1.68	1.72	1.68	1.72
T	1.24	1.47	1.24	1.48
f_R	0.09	0.11	0.09	0.11
F	-	0.06	-	0.06

TABLE 4.11: Results of Fuzzy model following Credibility Measure approach

Output Variable	using PSO	
	NCS	CS
$\tilde{\Pi}_R$	(1894.20, 1900.12, 1907.89)	(1904.60, 1910.33, 1918.38)
$\tilde{\Pi}_S$	(158.53, 162.63, 165.92)	(166.78, 170.22, 172.97)
$\tilde{\Pi}_C$	(2052.73, 2062.75, 2073.81)	(2071.38, 2080.55, 2091.35)
ρ_1	1.55	1.58
ρ_2	1.68	1.72
T	1.24	1.48
f_R	0.09	0.11
F	-	0.06

(b) Fuzzy Model: The values of imprecise parameters are as follows: $(A_{R1}, A_{R2}, A_{R3}) = (270, 275, 280)$, $(A_{S1}, A_{S2}, A_{S3}) = (136, 140, 145)$, $(a_{R1,1}, a_{R2,1}, a_{R3,1}) = (0.95, 1, 1.03)$, $(a_{R1,2}, a_{R2,2}, a_{R3,2}) = (0.97, 1, 1.04)$, $(a_{S1,1}, a_{S2,1}, a_{S3,1}) = (0.76, 0.80, 0.83)$, $(a_{S1,2}, a_{S2,2}, a_{S3,2}) = (0.77, 0.80, 0.84)$, $(m_{h1}, m_{h2}, m_{h3}) = (0.048, 0.05, 0.051)$, $h_{Rj,i} = m_{hj} \times w_i$, for $i = 1, 2$ and $j = 1, 2, 3$. All other parametric values are same as crisp model. With these parametric values the model is solved using PSO technique and GRG technique following GMIV approach and the results are presented in Table 4.10. Again this model is solved using PSO technique following credibility measure approach and the results are presented in Table 4.11.

4.3.3.1 ANOVA Test

To compare the efficiency of the algorithm PSO with LINGO, here ANOVA test [85] is done. To perform this test, here, results obtained following these two approaches for four models are considered which are presented in Table 4.12. These two sets of results are considered as two samples ($J = 2$). Clearly size of each sample is $I = 4$. Critical value of the F-ratio is $F(J-1, J(I-1)) = F(1, 6) = 5.99$

TABLE 4.12: Values for ANOVA test

Approach	Obtained optimum value			
	Π_R in NCS	Π_C in CS	$G(\tilde{\Pi}_R)$ in NCS	$G(\tilde{\Pi}_C)$ in CS
PSO	1900.12	2080.55	1900.43	2080.82
Lingo	1900.12	2080.55	1900.43	2080.82

for significance level 0.05. As two samples are same, calculated value of the F-ratio is 0 which is less than the critical value $F(1,6) = 5.99$. So there is no significant difference between the two samples, i.e., performances of the two algorithms are same for solving the proposed model.

4.3.4 Discussion

Using Proposition 4.2, $F_{min} = 0.0366$ and $F_{max} = 0.0774$ are obtained. From Table 4.9, it is found that for $F_{min} < F < F_{max}$ profit of both the parties increase to some extent. If $F = 0.03 < F_{min}$, then the retailer's profit Π_R is less in the CS (1897.23) than the NCS (1900.12). Similarly, if $F = 0.08 > F_{max}$, then the supplier's profit Π_S is less in the CS (161.48) than the NCS (162.63).

From the above results of Example 4.2, the following observations are made:

- It is found that the profits for both the parties (i.e., supplier and retailer) increase in the coordination scenario than the non-coordination scenario for a compromise value of F, i.e, if the supplier bears a compromise portion of promotional cost then it is beneficial for both parties. So theoretical expected result agrees with numerical findings.

- It is also found that promotional efforts of all the items are grater than 1. So promotional effort has a positive effect in a supply chain.

- Again in both the scenarios it is observed that $f_R > 0$, which implies that price discount given to the customers has a significant effect in a supply chain.

- For fuzzy model it is observed that results obtained using both the approaches are all most same. But in credibility approach no crisp equivalent of the objectives are used to find marketing decision. In this respect we can say that the result obtained using credibility approach is more appropriate than GMIV approach.

4.4 Model 4.3: A Two-warehouse Multi-item Supply Chain with Stock Dependent Promotional Demand Under Joint Replenishment Policy: A Mixed-mode ABC Approach

4.4.1 Assumptions and Notations

The following assumptions and notations are used for mathematical formulation of the model. The symbols $\tilde{}$ and $\breve{}$ are used over some parameters/variables to indicate fuzzy and rough quantities respectively.

(i) A wholesaler-retailer supply chain is considered and the inventory system involves N items. The wholesaler supplies the items to the retailer. The retailer has two rented warehouses RW_1 and RW_2. Location of RW_1 is at the heart of the market place and RW_2 is little away from the market place. The holding cost at RW_1 is higher than that at RW_2.

(ii) Storage area of RW_1 and RW_2 are SA_1, SA_2 units, respectively.

(iii) H_p is the planning horizon.

(iv) N_o represents the number of orders, done by the retailer during H_p.

(v) T_o is the basic time interval between two consecutive orders, i.e., $T_o = H_p/N_o$.

(vi) M_t is the number of times the items are transferred by the retailer from RW_2 to RW_1 during T_o.

(vii) B_t is the basic time interval between two consecutive shipments of items by the retailer from RW_2 to RW_1. So $B_t = T_o/M_t$.

(viii) c_{mo} is the major ordering cost of the retailer.

(ix) Z_R is the total profit of the retailer.

(x) Z_W is the total profit of the wholesaler.

(xi) Z_T is the total profit of the retailer and the wholesaler.

For i-th item following notations are used (cf. Figure 4.1):

(xii) n_i is the number of integer multiple of T_o when the replenishment of i-th item is a part of group replenishment.

(xiii) L_i is the retailer's cycle length, i.e., $L_i = n_i T_o$.

(xiv) m_i is the number of integer multiple of B_t when the transfer of i-th item is a part of group transfer from RW_2 to RW_1.

(xv) T_{ti} is the duration between two consecutive shipments of the item by the retailer from RW_2 to RW_1. So $T_{ti} = m_i B_t$.

(xvi) Item is transferred from RW_2 to RW_1 in P_i shipments by the retailer. So

$$P_i = \begin{cases} \left[\dfrac{L_i}{T_{ti}}\right], & \text{if } L_i \text{ is an integer multiple of } T_{ti} \\ \\ \left[\dfrac{L_i}{T_{ti}}\right] + 1, & \text{otherwise} \end{cases}$$

(xvii) Total number of cycles of the retailer is

$$N_i = \begin{cases} \left[\dfrac{H_p}{L_i}\right], & \text{if } H_p \text{ is an integer multiple of } L_i \\ \\ \left[\dfrac{H_p}{L_i}\right] + 1, & \text{otherwise} \end{cases}$$

(xviii) Ll_i is the last cycle length of the retailer, i.e., $Ll_i = H_p - (N_i - 1)L_i$.

(xix) Item is transferred from RW_2 to RW_1 in Pl_i shipments by the retailer in the last cycle. So

$$Pl_i = \begin{cases} \left[\dfrac{Ll_i}{T_{ti}}\right], & \text{if } Ll_i \text{ is an integer multiple of } T_{ti} \\ \\ \left[\dfrac{Ll_i}{T_{ti}}\right] + 1, & \text{otherwise} \end{cases}$$

(xx) In the j-th retailer cycle item is transferred at $t = T_{ij1}, T_{ij2}, ..., T_{ijP_i}$; where $T_{ij1} = (j-1)L_i$, $T_{ijk} = T_{ij1} + (k-1)T_{ti}$, for $k = 1, 2, ..., P_i$ and for the last retailer cycle item is transferred at $t = T_{iN_i1}, T_{iN_i2}, ..., T_{iN_iPl_i}$; where $T_{iN_ik} = (N_i - 1)L_i + (k-1)T_{ti}$, for $k = 1, 2, ..., Pl_i$.

(xxi) Ar_i is the area required to store one unit.

(xxii) A fraction λ_i of SA_1 is allotted for i-th item. So maximum displayed inventory level $Q_{di} = \dfrac{\lambda_i SA_1}{Ar_i}$ and $\sum_{i=1}^{N} \lambda_i = 1$.

(xxiii) Q_{ij} is the order quantity at the beginning of j-th retailer cycle, which is same in all the cycles except the last cycle.

(xxiv) Q_{sijk} is the stock level at RW_1 at the beginning of k-th sub-cycle in j-th retailer cycle, when items are transferred from RW_2 to RW_1. Stock levels are same for all the sub-cycles except in the first sub-cycle where $Q_{sij1} = 0$.

(xxv) c_{pi} is the purchase cost of the retailer per unit item.

(xxvi) s_{pi} is the normal selling price (maximum retail price (MRP)) of the retailer per unit item.

(xxvii) s_{pdi} is the discounted selling price of the retailer per unit item, which is a mark-up m_{kdi} of s_{pi}; i.e., $s_{pdi} = m_{kdi} s_{pi}$.

(xxviii) c_{h1i} and c_{h2i} are the holding costs per unit quantity per unit time at RW_1 and RW_2 respectively. h_{1i} and h_{2i} are the fractions of purchase costs assumed as holding costs at RW_1 and RW_2 respectively. So $c_{h1i} = h_{1i} c_{pi}$ and $c_{h2i} = h_{2i} c_{pi}$.

(xxix) c_{oi} is the minor ordering cost of the item, which is partly constant and partly order quantity dependent and is of the form: $c_{oi} = c_{o1i} + c_{o2i} Q_{ij}$.

(xxx) c_{ti} represents minor transportation cost of the item from RW_2 to RW_1 and is of the form: $c_{ti} = c_{t1i} + c_{t2i} Q_{ij}$.

(xxxi) f_{ri} is the frequency of advertisement per unit time.

(xxxii) c_{ai} represents the advertisement cost per advertisement.

(xxxiii) $q_i(t)$ represents the inventory level at RW_1 at any time t.

(xxxiv) Demand of the item D_i depends on the inventory level $q_i(t)$, frequency of advertisement f_{ri}, discounted selling price s_{pdi} and is of the form: $D_i(t) = \dfrac{(1+f_{ri})^{\alpha} x_i + y_i q_i}{(s_{pdi})^{\gamma}} = A_i + B_i q_i$, where, $A_i = \dfrac{(1+f_{ri})^{\alpha} x_i}{(m_{kdi} s_{pi})^{\gamma}}$, $B_i = \dfrac{y_i}{(m_{kdi} s_{pi})^{\gamma}}$ and x_i, y_i, α, γ are the constants so chosen to best fit the demand function.

(xxxv) M_i retailer cycles are completed during one wholesaler cycle.

(xxxvi) $p_i = \left[\frac{N_i}{M_i}\right]$. If M_i divides N_i (i.e., $M_i|N_i$), then the wholesaler have p_i complete cycles. Otherwise, there are $M_{1i} = N_i - p_i M_i$ retailer cycles in $(p_i + 1)$-th wholesaler cycle.

(xxxvii) c_{pwi} is the purchase cost of the wholesaler per unit item.

(xxxviii) c_{hwi} is the holding cost of the wholesaler per unit item. This holding cost is assumed as the fraction h_{wi} of the purchase cost of the wholesaler c_{pwi}, i.e., $c_{hwi} = h_{wi}c_{pwi}$.

(xxxix) c_{woi} is the wholesaler's ordering cost of the item, which is partly constant and partly order quantity dependent and is of the form: $c_{woi} = c_{wo1i} + c_{wo2i}QW_{ij}$; where QW_{ij} is the order quantity of the wholesaler in the j-th cycle.

4.4.2 Mathematical Formulation of the Model

4.4.2.1 Retailer's Profit

The replenishment of the items at retailer's level is made jointly using a BP policy. Under this policy, the retailer orders different items regularly at a fixed time interval, T_o, called BP. Only those items are included in the order whose level reaches reorder level at the time of the order. So cycle length of each item is an integer multiple of T_o. For i-th item it is assumed that cycle length L_i is a multiple n_i of T_o, i.e., $L_i = n_i T_o$. So order of i-th item is included in every n_i-th order. The i-th item has a replenishment quantity (Q_{ij}) sufficient to meet the demand of the item for an time interval L_i. Among Q_{ij} units initially Q_{di} units are stored in RW_1 and remaining $(Q_{ij} - Q_{di})$ units are stored in RW_2. Items are sold from RW_1 and are filled up from RW_2 using another BP policy having period B_t, i.e., the items are transferred from RW_2 to RW_1 at a regular time interval B_t. The i-th item has a transferred quantity $(Q_{di} - Q_{sijk})$ sufficient to last for exactly an integer multiple (m_i) of B_t, where Q_{sijk} is the quantity in hand at RW_1 before the item is transferred and after shipment the stock at RW_1 reaches Q_{di}, i.e., i-th item is transferred from RW_2 to RW_1 at a regular time interval $m_i B_t$, called sub-cycle. Item stored at RW_1 at the beginning of the last sub-cycle is just sufficient to meet the customer demand during that period, i.e., at the end of each cycle inventory level reaches zero (cf. Figure 4.1).

Formulation for the i-th item in k-th sub-cycle ($k = 1, 2, ..., P_i$) of j-th retailer cycle $[T_{ij1}, T_{i(j+1)1}]$ for $j = 1, 2, ..., N_i - 1$:

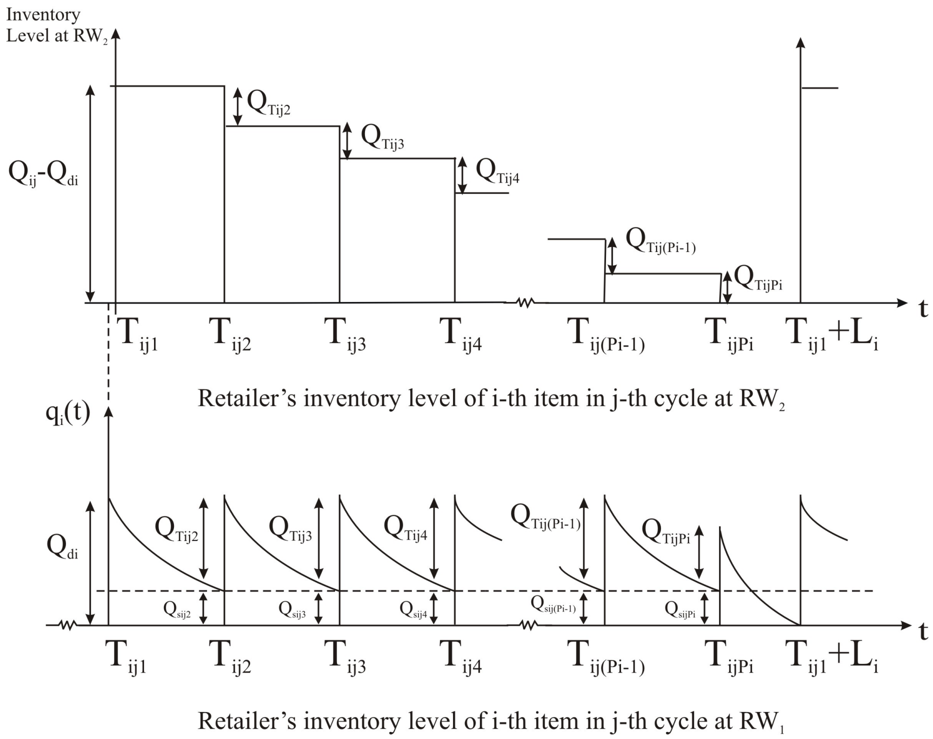

FIGURE 4.1: Retailer's inventory level

Instantaneous state $q_i(t)$ of the i-th item at RW_1 is given by

$$\frac{dq_i(t)}{dt} = -(A_i + B_i q_i), \text{ for } T_{ijk} \le t \le T_{ij(k+1)} \tag{4.51}$$

with boundary conditions $q_i(T_{ijk}) = Q_{di}$, for $k = 1, 2, ..., P_i - 1$.
Solving (4.51), the inventory level $q_i(t)$ can be found as follows:

$$q_i(t) = \frac{1}{B_i}\Big\{ - A_i + (A_i + B_i Q_{di})e^{-B_i(t - T_{ijk})}\Big\} \tag{4.52}$$

and the stock level, when the items are transferred, is as follows:

$$Q_{sij(k+1)} = \frac{1}{B_i}\Big\{ - A_i + (A_i + B_i Q_{di})e^{-B_i T_{ti}}\Big\}, \text{ for } k = 1, 2, ..., P_i - 1. \tag{4.53}$$

where, $Q_{sij1} = 0$.
From RW_2 to RW_1, transferred quantity of i-th item at $t = T_{ijk}$ is given by

$$Q_{Tijk} = Q_{di} - Q_{sijk}, \text{ for } k = 1, 2, ..., P_i - 1. \tag{4.54}$$

Instantaneous state $q_i(t)$ of the i-th item at RW_1 in the last sub-cycle ($k = P_i$) is given by

$$\frac{dq_i(t)}{dt} = -(A_i + B_i q_i), \text{ for } T_{ijP_i} \le t \le T_{ij1} + L_i \tag{4.55}$$

with boundary conditions $q_i(T_{ijP_i}) = Q_{TijP_i} + Q_{sijP_i}$, $q_i(T_{ij1} + L_i) = 0$.
Solving (4.55), the inventory level $q_i(t)$ can be found as follows:

$$\begin{aligned} q_i(t) &= \frac{1}{B_i}\Big[-A_i + \Big\{A_i + B_i q_i(T_{ijP_i})\Big\}e^{-B_i(t-T_{ijP_i})}\Big], \qquad (4.56)\\ &\quad \text{for } T_{ijP_i} \le t \le T_{ij1} + L_i \\ \text{and } q_i(T_{ijP_i}) &= \frac{A_i}{B_i}\Big[e^{B_i\{L_i-(P_i-1)T_{ti}\}} - 1\Big] \qquad (4.57)\end{aligned}$$

Now the transferred quantity of i-th item from RW_2 to RW_1 at $t = T_{ijP_i}$ can be calculated as follows:

$$Q_{TijP_i} = q_i(T_{ijP_i}) - Q_{sijP_i} \tag{4.58}$$

The retailer's order quantity of i-th item in j-th cycle is

$$\begin{aligned} Q_{ij} &= \sum_{k=1}^{P_i-1} Q_{Tijk} + Q_{TijP_i} \\ &= (P_i - 1)\Big[Q_{di} - \frac{1}{B_i}\big\{-A_i + (A_i + B_i Q_{di})e^{-B_i T_{ti}}\big\}\Big] \\ &\quad + \frac{A_i}{B_i}\Big[e^{B_i\{L_i-(P_i-1)T_{ti}\}} - 1\Big] \qquad (4.59)\end{aligned}$$

Holding cost: Holding cost at RW_1 in k-th sub-cycle ($k = 1, 2, ..., P_i - 1$) of j-th retailer cycle ($j = 1, 2, ..., N_i - 1$) is $c_{h1i}H1_{ijk}$, where

$$H1_{ijk} = \int_{T_{ijk}}^{T_{ij(k+1)}} q_i(t)dt = \frac{1}{B_i}\Big[-A_i T_{ti} + \frac{A_i + B_i Q_{di}}{B_i}\{1 - e^{-B_i T_{ti}}\}\Big] \tag{4.60}$$

Holding cost at RW_1 in the last sub-cycle ($k = P_i$) of j-th retailer cycle ($j = 1, 2, ..., N_i - 1$) is $c_{h1i}H1_{ijP_i}$, where

$$\begin{aligned} H1_{ijP_i} &= \int_{T_{ijP_i}}^{T_{ij1}+L_i} q_i(t)dt \\ &= \frac{1}{B_i}\Big[-A_i\{L_i - (P_i-1)T_{ti}\} + \frac{A_i}{B_i}\Big\{e^{B_i\{L_i-(P_i-1)T_{ti}\}} - 1\Big\}\Big] \qquad (4.61)\end{aligned}$$

So, the holding cost at RW_1 in the j-th retailer cycle is $c_{h1i}H1_{ij}$, where

$$H1_{ij} = \sum_{k=1}^{P_i-1} H1_{ijk} + H1_{ijP_i} \tag{4.62}$$

Hence, the holding cost at RW_1 in first (N_i-1) retailer cycles is $c_{h1i}H1F_i$, where

$$\begin{aligned} H1F_i &= \sum_{j=1}^{N_i-1}\Big[\sum_{k=1}^{P_i-1} H1_{ijk} + H1_{ijP_i}\Big] \\ &= (N_i-1)(P_i-1)\frac{1}{B_i}\Big[-A_iT_{ti} + \frac{A_i+B_iQ_{di}}{B_i}\{1-e^{-B_iT_{ti}}\}\Big] \\ &\quad +(N_i-1)H1_{ijP_i} \end{aligned} \tag{4.63}$$

Stock at RW_2 during $T_{ijk} \le t \le T_{ij(k+1)}$ is

$$\begin{aligned} Q2_{ijk} &= Q_{ij} - \sum_{l=1}^{k} Q_{Tijl} \\ &= Q_{ij} - kQ_{di} + \frac{k-1}{B_i}\Big\{-A_i + (A_i+B_iQ_{di})e^{-B_iT_{ti}}\Big\} \end{aligned} \tag{4.64}$$

Therefore, the holding cost at RW_2 in k-th sub-cycle of j-th retailer cycle is $c_{h2i}H2_{ijk}$, where

$$H2_{ijk} = \int_{T_{ijk}}^{T_{ij(k+1)}} Q2_{ijk}dt = Q2_{ijk}T_{ti} \tag{4.65}$$

Hence, the holding cost at RW_2 in first (N_i-1) retailer cycles is $c_{h2i}H2F_i$, where

$$\begin{aligned} H2F_i = \sum_{j=1}^{N_i-1}\sum_{k=1}^{P_i-1} H2_{ijk} &= T_{ti}(N_i-1)\Big[(P_i-1)Q_{ij} - \frac{(P_i-1)P_i}{2}Q_{di} \\ &\quad + \frac{(P_i-2)(P_i-1)}{2B_i}\Big\{-A_i + (A_i+B_iQ_{di})e^{-B_iT_{ti}}\Big\}\Big] \end{aligned} \tag{4.66}$$

Sell revenue: Sell revenue during $T_{ijk} \le t \le T_{ij(K+1)}$ is $s_{pdi}SR_{ijk}$, where

$$SR_{ijk} = \int_{T_{ijk}}^{T_{ij(k+1)}} D_i(t)dt = \frac{A_i+B_iQ_{di}}{B_i}(1-e^{-B_iT_{ti}}) \tag{4.67}$$

$$\text{and } SR_{ijP_i} = \int_{T_{ijP_i}}^{T_{ij1}+L_i} D_i(t)dt = \frac{A_i+B_iq_i(T_{ijP_i})}{B_i}\Big[1-e^{-B_i\{L_i-(P_i-1)T_{ti}\}}\Big] \tag{4.68}$$

So, the sell revenue in first $(N_i - 1)$ retailer cycles is $s_{pdi}SRF_i$, where

$$\begin{aligned} SRF_i &= \sum_{j=1}^{N_i-1}\Big[\sum_{k=1}^{P_i-1} SR_{ijk} + SR_{ijP_i}\Big] \\ &= (N_i-1)(P_i-1)\frac{A_i + B_iQ_{di}}{B_i}(1 - e^{-B_iT_{ti}}) \\ &\quad +(N_i-1)\frac{A_i}{B_i}\Big[e^{B_i\{L_i-(P_i-1)T_{ti}\}} - 1\Big] \end{aligned} \tag{4.69}$$

Purchase cost: Purchase cost in first $(N_i - 1)$ retailer cycles is $c_{pi}PCF_i$, where

$$PCF_i = \sum_{j=1}^{N_i-1} Q_{ij} = (N_i-1)Q_{ij} \tag{4.70}$$

Minor ordering cost: Minor ordering cost in first (N_i-1) retailer cycles is given by

$$OCF_i = \sum_{j=1}^{N_i-1}(c_{o1i} + c_{o2i}Q_{ij}) = (N_i-1)(c_{o1i} + c_{o2i}Q_{ij}) \tag{4.71}$$

Minor transportation cost: Minor transportation cost in first $(N_i - 1)$ retailer cycles is $c_{ti}TCF_i$, where

$$TCF_i = \sum_{j=1}^{N_i-1}\Big[\sum_{k=1}^{P_i-1} Q_{Tijk} + Q_{TijP_i}\Big] = \sum_{j=1}^{N_i-1} Q_{ij} = (N_i-1)Q_{ij} \tag{4.72}$$

Advertisement cost: Advertisement cost in first $(N_i - 1)$ retailer cycles is given by

$$ACF_i = f_{ri}(N_i-1)L_ic_{ai} \tag{4.73}$$

Formulation for the i-th item in k-th sub-cycle $(k = 1, 2, ..., Pl_i)$ of last retailer cycle $[T_{iN_i1}, H_p]$:

Cycle-length of the last retailer cycle is $Ll_i = H_p - (N_i - 1)L_i$. This cycle-length may be different from other retailer cycle. So, the number of sub-cycles is also different. In this retailer cycle, the number of sub-cycle is Pl_i.

Instantaneous state $q_i(t)$ of the i-th item at RW_1 is given by

$$\frac{dq_i(t)}{dt} = -(A_i + B_iq_i), \text{ for } T_{iN_ik} \le t \le T_{iN_i(k+1)} \tag{4.74}$$

with boundary conditions $q_i(T_{iN_ik}) = Q_{di}$, for $k = 1, 2, ..., Pl_i - 1$.
Solving (4.74), the inventory level $q_i(t)$ can be found as follows:

$$q_i(t) = \frac{1}{B_i}\{-A_i + (A_i + B_iQ_{di})e^{-B_i(t-T_{iN_ik})}\} \quad (4.75)$$

and the stock level, when the items are transferred, is as follows:

$$Q_{siN_i(k+1)} = \frac{1}{B_i}\{-A_i + (A_i + B_iQ_{di})e^{-B_iT_{ti}}\}, \text{ for } k = 1, 2, ..., Pl_i - 1. \quad (4.76)$$

where, $Q_{siN_i1} = 0$.
With similar explanations, the inventory level $q_i(t)$ for $T_{iN_iPl_i} \leq t \leq H_p$ is as follows:

$$q_i(t) = \frac{1}{B_i}\Big[-A_i + \big\{A_i + B_iq_i(T_{iN_iPl_i})\big\}e^{-B_i(t-T_{iN_iPl_i})}\Big] \quad (4.77)$$

$$\text{and } q_i(T_{iN_iPl_i}) = \frac{A_i}{B_i}\Big[e^{B_i\{Ll_i-(Pl_i-1)T_{ti}\}} - 1\Big] \quad (4.78)$$

The retailer's order quantity of i-th item in last retailer cycle is

$$\begin{aligned} Q_{iN_i} &= \sum_{k=1}^{Pl_i-1} Q_{TiN_ik} + Q_{TiN_iPl_i} \\ &= (Pl_i - 1)\Big[Q_{di} - \frac{1}{B_i}\{-A_i + (A_i + B_iQ_{di})e^{-B_iT_{ti}}\}\Big] \\ &\quad + \frac{A_i}{B_i}\Big[e^{B_i\{Ll_i-(Pl_i-1)T_{ti}\}} - 1\Big] \end{aligned} \quad (4.79)$$

Holding cost: Holding cost at RW_1 in k-th sub-cycle ($k = 1, 2, ..., Pl_i - 1$) of last retailer cycle ($j = N_i$) is $c_{h1i}H1_{iN_ik}$, where

$$H1_{iN_ik} = \int_{T_{iN_ik}}^{T_{iN_i(k+1)}} q_i(t)dt = \frac{1}{B_i}\Big[-A_iT_{ti} + \frac{A_i + B_iQ_{di}}{B_i}\{1 - e^{-B_iT_{ti}}\}\Big] \quad (4.80)$$

Holding cost at RW_1 in the last sub-cycle ($k = Pl_i$) of last retailer cycle ($j = N_i$) is $c_{h1i}H1_{iN_iPl_i}$, where

$$\begin{aligned} H1_{iN_iPl_i} = \int_{T_{iN_iPl_i}}^{H_p} q_i(t)dt &= \frac{1}{B_i}\Big[-A_i\{Ll_i - (Pl_i - 1)T_{ti}\} \\ &\quad + \frac{A_i}{B_i}\Big\{e^{B_i\{Ll_i-(Pl_i-1)T_{ti}\}} - 1\Big\}\Big] \end{aligned} \quad (4.81)$$

Hence, the holding cost at RW_1 in the last retailer cycle is $c_{h1i}H1L_i$, where

$$\begin{aligned} H1L_i &= \sum_{k=1}^{Pl_i-1} H1_{iN_ik} + H1_{iN_iPl_i} \\ &= (Pl_i - 1)\frac{1}{B_i}\Big[-A_iT_{ti} + \frac{A_i + B_iQ_{di}}{B_i}\{1 - e^{-B_iT_{ti}}\}\Big] + H1_{iN_iPl_i} \quad (4.82) \end{aligned}$$

Stock at RW_2 during $T_{iN_ik} \le t \le T_{iN_i(k+1)}$ is

$$\begin{aligned} Q2_{iN_ik} &= Q_{iN_i} - \sum_{l=1}^{k} Q_{TiN_il} \\ &= Q_{iN_i} - kQ_{di} + \frac{k-1}{B_i}\big\{-A_i + (A_i + B_iQ_{di})e^{-B_iT_{ti}}\big\} \quad (4.83) \end{aligned}$$

Therefore, the holding cost at RW_2 in k-th sub-cycle of last retailer cycle is $c_{h2i}H2_{iN_ik}$, where

$$H2_{iN_ik} = \int_{T_{iN_ik}}^{T_{iN_i(k+1)}} Q2_{iN_ik}dt = Q2_{iN_ik}T_{ti} \quad (4.84)$$

Hence, the holding cost at RW_2 in last retailer cycle is $c_{h2i}H2L_i$, where

$$\begin{aligned} H2L_i = \sum_{k=1}^{Pl_i-1} H2_{iN_ik} &= T_{ti}\Big[(Pl_i - 1)Q_{iN_i} - \frac{(Pl_i - 1)Pl_i}{2}Q_{di} \\ &+ \frac{(Pl_i - 2)(Pl_i - 1)}{2B_i}\big\{-A_i + (A_i + B_iQ_{di})e^{-B_iT_{ti}}\big\}\Big] \quad (4.85) \end{aligned}$$

Sell revenue: Sell revenue during $T_{iN_ik} \le t \le T_{iN_i(K+1)}$ is $s_{pdi}SR_{iN_ik}$, where

$$SR_{iN_ik} = \int_{T_{iN_ik}}^{T_{iN_i(k+1)}} D_i(t)dt = \frac{A_i + B_iQ_{di}}{B_i}(1 - e^{-B_iT_{ti}}) \quad (4.86)$$

$$\text{and } SR_{iN_iPl_i} = \int_{T_{iN_iPl_i}}^{H_p} D_i(t)dt = \frac{A_i}{B_i}\Big[e^{B_i\{Ll_i-(Pl_i-1)T_{ti}\}} - 1\Big] \quad (4.87)$$

So, the sell revenue in last retailer cycle is $s_{pdi}SRL_i$, where

$$\begin{aligned} SRL_i &= \sum_{k=1}^{Pl_i-1} SR_{iN_ik} + SR_{iN_iPl_i} \\ &= (Pl_i - 1)\frac{A_i + B_iQ_{di}}{B_i}(1 - e^{-B_iT_{ti}}) + \frac{A_i}{B_i}\Big[e^{B_i\{Ll_i-(Pl_i-1)T_{ti}\}} - 1\Big] \quad (4.88) \end{aligned}$$

Purchase cost: Purchase cost in last retailer cycle is $c_{pi}PCL_i$, where

$$PCL_i = Q_{iN_i} \quad (4.89)$$

Minor ordering cost: Minor ordering cost in last retailer cycle is given by

$$OCL_i = c_{o1i} + c_{o2i}Q_{iN_i} \tag{4.90}$$

Minor transportation cost: Minor transportation cost in last retailer cycle is $c_{ti}TCL_i$, where

$$TCL_i = Q_{iN_i} \tag{4.91}$$

Advertisement cost: Advertisement cost in last retailer cycle is given by

$$ACL_i = f_{ri}Ll_ic_{ai} \tag{4.92}$$

Major ordering cost: The major ordering cost of the retailer through the whole planning horizon is

$$MOC = c_{mo}N_o \tag{4.93}$$

4.4.2.2 Wholesaler's Profit

If $M_i|N_i$ (M_i divides N_i), then there are p_i full cycles in wholesaler's inventory period. Otherwise, there are $M_{1i}(= N_i - p_iM_i)$ retailer cycles in $(p_i + 1)$-th wholesaler cycle with p_i full cycles, where $p_i = \left[\frac{N_i}{M_i}\right]$ and $[x]$ represents integral part of x. Wholesaler's inventory level of i-th item in k-th wholesaler-cycle is shown in Figure 4.2.

The order quantity of the wholesaler for i-th item in j-th cycle is given by

$$QW_{ij} = Q_{i\{(j-1)M_i+1\}} + Q_{i\{(j-1)M_i+2\}} + \ldots + Q_{i\{jM_i\}} \tag{4.94}$$

The order quantity of the wholesaler for i-th item in last cycle is given by

$$\text{If } M_i|N_i,\ QW^l_{ip_i} = Q_{i\{(p_i-1)M_i+1\}} + Q_{i\{(p_i-1)M_i+2\}} + \ldots + Q_{i\{p_iM_i\}} \tag{4.95}$$

$$\text{If } M_i \nmid N_i,\ QW^l_{i(p_i+1)} = Q_{i\{p_iM_i+1\}} + Q_{i\{p_iM_i+2\}} + \ldots + Q_{i\{p_iM_i+M_{1i}\}} \tag{4.96}$$

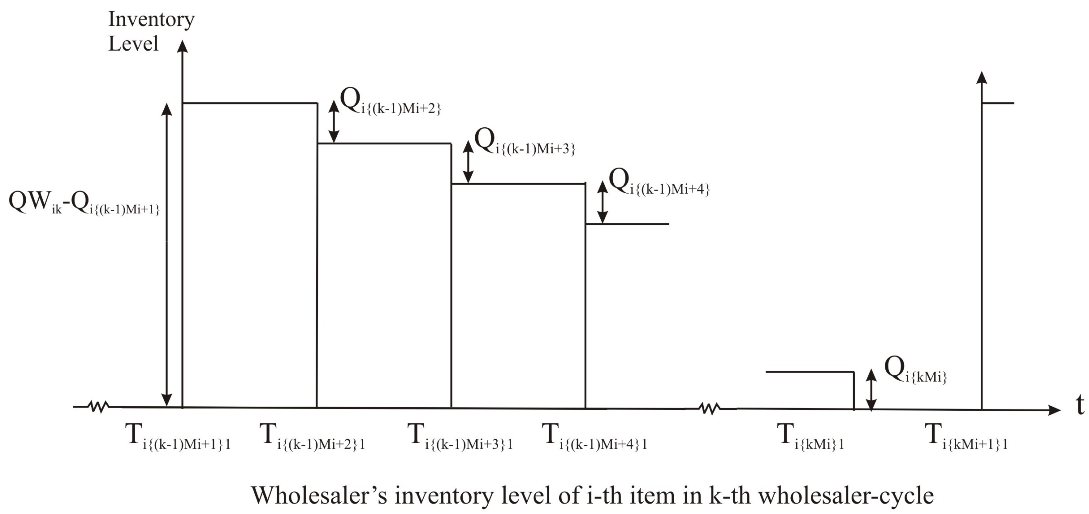

FIGURE 4.2: Wholesaler's inventory level

The total order quantity of the wholesaler can be calculated as follows.

$$TQW_i = \sum_{j=1}^{N_i} Q_{ij} \tag{4.97}$$

Holding cost: The holding amount of the wholesaler for i-th item in j-th cycle is given by

$$HW_{ij} = \sum_{k=(j-1)M_i+1}^{jM_i} Q_{ik}(T_{ik1} - T_{wj}), \text{ where, } T_{wj} = T_{i\{(j-1)M_i+1\}1} \tag{4.98}$$

The holding amount of the wholesaler for i-th item in last cycle is given by

$$\text{If } M_i|N_i,\ HW^l_{ip_i} = \sum_{k=(p_i-1)M_i+1}^{p_iM_i} Q_{ik}(T_{ik1} - T_{wp_i}), \tag{4.99}$$

where, $T_{wp_i} = T_{i\{(p_i-1)M_i+1\}1}$

$$\text{If } M_i \nmid N_i,\ HW^l_{i(p_i+1)} = \sum_{k=p_iM_i+1}^{p_iM_i+M_{1i}} Q_{ik}(T_{ik1} - T_{wp_i+1}), \tag{4.100}$$

where, $T_{wp_i+1} = T_{i\{p_iM_i+1\}1}$

Hence, the total holding cost of the wholesaler is

$$HCW_i = \begin{cases} c_{hwi}\Big(\sum_{j=1}^{p_i-1} HW_{ij} + HW^l_{ip_i}\Big), & \text{if } M_i|N_i \\ c_{hwi}\Big(\sum_{j=1}^{p_i} HW_{ij} + HW^l_{i(p_i+1)}\Big), & \text{if } M_i \nmid N_i \end{cases} \tag{4.101}$$

Sell revenue: The sell revenue of the wholesaler for i-th item is

$$SRW_i = c_{pi} TQW_i \tag{4.102}$$

Purchase cost: The purchase cost of the wholesaler for i-th item is

$$PCW_i = c_{pwi} TQW_i \tag{4.103}$$

Ordering cost: The ordering cost of the wholesaler for i-th item is

$$OCW_i = \begin{cases} \sum_{j=1}^{p_i} (c_{wo1i} + c_{wo2i} QW_{ij}), \text{ if } M_i | N_i \\ \sum_{j=1}^{p_i} (c_{wo1i} + c_{wo2i} QW_{ij}) + (c_{wo1i} + c_{wo2i} QW^l_{i(p_i+1)}), \text{ if } M_i \nmid N_i \end{cases} \tag{4.104}$$

4.4.2.3 Promotional Cost

Promotional cost is an important part in any marketing system. In most of the research papers, promotional cost is considered as the function of promotional effort which increases the base demand of the item [97, 150]. But in these studies no proper guideline is outlined about the actual process of the improvement of the demand of an item by any promotional effort and the exact amount of the cost behind this promotional effort. In this study two promotional efforts are used-one is advertisement and other is price discount. Let us assume that the MRP per unit of the i-th item is s_{pi}. To increase the demand of the items, the retailer sells the product in a discounted price s_{pdi}. Clearly the differences between the sales revenue with discounted price and the sales revenue with normal price is the promotional cost associated with this promotional activity. Again cost of different advertisements is the promotional cost associated with the advertisement related promotional activities. So total promotional cost associated with the promotional activity, PRC, is given by

$$PRC = \sum_{i=1}^{N} \Big[(s_{pi} - s_{pdi})(SRF_i + SRL_i) + (ACF_i + ACL_i) \Big] \tag{4.105}$$

4.4.2.4 Crisp Model

The total profit gained by the retailer through the whole planning horizon is given by

$$\begin{aligned} Z_R &= \sum_{i=1}^{N}\Big[s_{pdi}(SRF_i+SRL_i)-c_{pi}(PCF_i+PCL_i)-c_{h1i}(H1F_i+H1L_i) \\ &\quad -c_{h2i}(H2F_i+H2L_i)-(OCF_i+OCL_i)-(TCF_i+TCL_i) \\ &\quad -(ACF_i+ACL_i)\Big]-MOC \end{aligned} \tag{4.106}$$

The total profit gained by the wholesaler through the whole planning horizon is given by

$$Z_W = \sum_{i=1}^{N}\Big[SRW_i-PCW_i-HCW_i-OCW_i\Big] \tag{4.107}$$

The channel profit of both the retailer and wholesaler is

$$Z_T = Z_R+Z_W \tag{4.108}$$

If the wholesaler does not coordinate with the retailer, then this situation is termed as *Non-Coordination Scenario* (NCS). Here, the retailer is the leader decision maker and the wholesaler is the follower and hence the problem in this scenario is as follows.

$$\left.\begin{array}{ll} \text{To determine} & N_o, M_t, n_i, m_i, f_{ri}, m_{kdi}, \lambda_i;\ \text{for } i=1,2,...,N \\ \text{Maximize} & Z_R \end{array}\right\} \tag{4.109}$$

Depending upon the retailer's decision, the wholesaler tries to improve his/her profit. So the problem of the wholesaler mathematically takes the following form:

$$\left.\begin{array}{ll} \text{To determine} & M_i;\ \text{for } i=1,2,...,N \\ \text{Maximize} & Z_W \end{array}\right\} \tag{4.110}$$

If the wholesaler shares some portion of the promotional cost, then this scenario is termed as *Coordination Scenario* (CS). Let us consider that the wholesaler shares F fraction of the promotional cost. So the retailer have gained the same amount. Therefore, the profits of the retailer, the wholesaler and the channel profit are respectively

$$\begin{aligned} Z_R &= \sum_{i=1}^{N}\Big[s_{pdi}(SRF_i+SRL_i)-c_{pi}(PCF_i+PCL_i)-c_{h1i}(H1F_i+H1L_i) \\ &\quad -c_{h2i}(H2F_i+H2L_i)-(OCF_i+OCL_i)-(TCF_i+TCL_i) \\ &\quad -(ACF_i+ACL_i)\Big]-MOC+F.PRC \qquad (4.111) \\ Z_W &= \sum_{i=1}^{N}\Big[SRW_i-PCW_i-HCW_i-OCW_i\Big]-F.PRC \qquad (4.112) \\ Z_T &= Z_R+Z_W \qquad (4.113) \end{aligned}$$

In this scenario, the retailer and the wholesaler jointly determine the marketing decision and hence the problem mathematically takes the following form:

$$\left.\begin{array}{ll} \text{To determine} & N_o, M_t, n_i, m_i, f_{ri}, m_{kdi}, \lambda_i, M_i;\ \text{for } i=1,2,...,N \\ \text{Maximize} & Z_T \end{array}\right\} \qquad (4.114)$$

4.4.2.5 Fuzzy Model

It has already been mentioned about the impreciseness of different parameters of any inventory system [113, 116, 118]. Due to fluctuating world economy different costs changes frequently. In the proposed model, the fractions h_{1i}, h_{2i} in the holding cost functions of the retailer, the constants c_{o1i}, c_{o2i} in the minor ordering cost function of the retailer, the advertisement cost c_{ai}, the fraction h_{wi} in the holding cost function of the wholesaler, the constants c_{wo1i}, c_{wo2i} in the ordering cost function of the wholesaler are assumed as the triangular fuzzy numbers (TFNs) [115, 127, 214] $\widetilde{h}_{1i}$, $\widetilde{h}_{2i}$, $\widetilde{c}_{o1i}$, $\widetilde{c}_{o2i}$, $\widetilde{c}_{ai}$, $\widetilde{h}_{wi}$, $\widetilde{c}_{wo1i}$, $\widetilde{c}_{wo2i}$ respectively, for $i=1,2,...,N$, where $\widetilde{h}_{1i}=(h_{11i},h_{12i},h_{13i})$, $\widetilde{h}_{2i}=(h_{21i},h_{22i},h_{23i})$, $\widetilde{c}_{o1i}=(c_{o11i},c_{o12i},c_{o13i})$, $\widetilde{c}_{o2i}=(c_{o21i},c_{o22i},c_{o23i})$, $\widetilde{c}_{ai}=(c_{a1i},c_{a2i},c_{a3i})$, $\widetilde{h}_{wi}=(h_{w1i},h_{w2i},h_{w3i})$, $\widetilde{c}_{wo1i}=(c_{wo11i},c_{wo12i},c_{wo13i})$, $\widetilde{c}_{wo2i}=(c_{wo21i},c_{wo22i},c_{wo23i})$. Hence, the profits in both the scenarios become fuzzy in nature.

In NCS, the individual profits and the channel profit are represented by

$$\begin{aligned} \widetilde{Z}_R &= \sum_{i=1}^{N}\Big[s_{pdi}(SRF_i+SRL_i)-c_{pi}(PCF_i+PCL_i)-\widetilde{c}_{h1i}(H1F_i+H1L_i) \\ &\quad -\widetilde{c}_{h2i}(H2F_i+H2L_i)-(\widetilde{OCF}_i+\widetilde{OCL}_i)-(TCF_i+TCL_i) \\ &\quad -(\widetilde{ACF}_i+\widetilde{ACL}_i)\Big]-MOC \qquad (4.115) \end{aligned}$$

$$\widetilde{Z}_W = \sum_{i=1}^{N}\Big[SRW_i - PCW_i - \widetilde{HCW}_i - \widetilde{OCW}_i\Big] \quad (4.116)$$
$$\widetilde{Z}_T = \widetilde{Z}_R + \widetilde{Z}_W \quad (4.117)$$

In NCS, since the retailer is the leader and the wholesaler is the follower so the retailer determines the marketing decision at first. So the mathematical problem in this scenario is as follows.

$$\left.\begin{array}{ll}\text{To determine} & N_o, M_t, n_i, m_i, f_{ri}, m_{kdi}, \lambda_i;\ \text{for } i = 1,2,...,N \\ \text{Maximize} & \widetilde{Z}_R\end{array}\right\} \quad (4.118)$$

Depending upon the retailer's decision, the wholesaler tries to improve his/her profit. So the problem of the wholesaler mathematically takes the following form:

$$\left.\begin{array}{ll}\text{To determine} & M_i;\ \text{for } i = 1,2,...,N \\ \text{Maximize} & \widetilde{Z}_W\end{array}\right\} \quad (4.119)$$

In CS, the individual profits and the channel profit are represented by

$$\begin{aligned}\widetilde{Z}_R = {} & \sum_{i=1}^{N}\Big[s_{pdi}(SRF_i + SRL_i) - c_{pi}(PCF_i + PCL_i) - \widetilde{c}_{h1i}(H1F_i + H1L_i) \\ & -\widetilde{c}_{h2i}(H2F_i + H2L_i) - (\widetilde{OCF}_i + \widetilde{OCL}_i) - (TCF_i + TCL_i) \\ & -(\widetilde{ACF}_i + \widetilde{ACL}_i)\Big] - MOC + F.\widetilde{PRC} \end{aligned} \quad (4.120)$$
$$\widetilde{Z}_W = \sum_{i=1}^{N}\Big[SRW_i - PCW_i - \widetilde{HCW}_i - \widetilde{OCW}_i\Big] - F.\widetilde{PRC} \quad (4.121)$$
$$\widetilde{Z}_T = \widetilde{Z}_R + \widetilde{Z}_W \quad (4.122)$$

In CS, the retailer and the wholesaler jointly takes the marketing decision to improve the channel profit as well as individual profit. So in this scenario, the problem mathematically takes the following form:

$$\left.\begin{array}{ll}\text{To determine} & N_o, M_t, n_i, m_i, f_{ri}, m_{kdi}, \lambda_i, M_i;\ \text{for } i = 1,2,...,N \\ \text{Maximize} & \widetilde{Z}_T\end{array}\right\} \quad (4.123)$$

As the fuzzy variables are taken as TFNs, the individual profits and the total profit becomes also TFNs as $\widetilde{Z}_R = (Z_{R1}, Z_{R2}, Z_{R3})$, $\widetilde{Z}_W = (Z_{W1}, Z_{W2}, Z_{W3})$ and $\widetilde{Z}_T = (Z_{T1}, Z_{T2}, Z_{T3})$; where

$$\begin{aligned} Z_{Rj} &= \sum_{i=1}^{N}\Big[s_{pdi}(SRF_i+SRL_i)-c_{pi}(PCF_i+PCL_i)-c_{h1(4-j)i}(H1F_i+H1L_i) \\ &-c_{h2(4-j)i}(H2F_i+H2L_i)-\{OCF_{(4-j)i}+OCL_{(4-j)i}\}-(TCF_i+TCL_i) \\ &-\{ACF_{(4-j)i}+ACL_{(4-j)i}\}\Big]-MOC+F.PRC_j \qquad (4.124) \end{aligned}$$

$$Z_{Wj} = \sum_{i=1}^{N}\Big[SRW_i-PCW_i-HCW_{(4-j)i}-OCW_{(4-j)i}\Big]-F.PRC_{4-j} \qquad (4.125)$$

$$Z_{Tj} = Z_{Rj}+Z_{Wj} \qquad (4.126)$$

These expressions can be used to find TFNs of the profit functions in CS for $j = 1, 2, 3$. The same expressions can be used in NCS by taking $F = 0$.

4.4.2.6 Rough Model

Another approach of estimation of vague parameters is the use of rough set theory [105]. Some inventory models have already been published following rough estimation of imprecise parameters, like holding cost, ordering cost, etc [69, 126, 150]. In the proposed model, the fractions h_{1i}, h_{2i} in the holding cost functions of the retailer, the constants c_{o1i}, c_{o2i} in the minor ordering cost function of the retailer, the advertisement cost c_{ai}, the fraction h_{wi} in the holding cost function of the wholesaler, the constants c_{wo1i}, c_{wo2i} in the ordering cost function of the wholesaler are assumed as the rough numbers $\check{h}_{1i}$, $\check{h}_{2i}$, $\check{c}_{o1i}$, $\check{c}_{o2i}$, $\check{c}_{ai}$, $\check{h}_{wi}$, $\check{c}_{wo1i}$, $\check{c}_{wo2i}$ respectively, for $i = 1, 2, ..., N$, where $\check{h}_{1i} = ([h_{11i}, h_{12i}][h_{13i}, h_{14i}])$, $\check{h}_{2i} = ([h_{21i}, h_{22i}][h_{23i}, h_{24i}])$, $\check{c}_{o1i} = ([c_{o11i}, c_{o12i}][c_{o13i}, c_{o14i}])$, $\check{c}_{o2i} = ([c_{o21i}, c_{o22i}][c_{o23i}, c_{o24i}])$, $\check{c}_{ai} = ([c_{a1i}, c_{a2i}][c_{a3i}, c_{a4i}])$, $\check{h}_{wi} = ([h_{w1i}, h_{w2i}][h_{w3i}, h_{w4i}])$, $\check{c}_{wo1i} = ([c_{wo11i}, c_{wo12i}][c_{wo13i}, c_{wo14i}])$, $\check{c}_{wo2i} = ([c_{wo21i}, c_{wo22i}][c_{wo23i}, c_{wo24i}])$. Hence, the profits in both the scenarios become rough in nature.

In NCS, the individual profits and the channel profit are represented by

$$\begin{aligned} \check{Z}_R &= \sum_{i=1}^{N}\Big[s_{pdi}(SRF_i+SRL_i)-c_{pi}(PCF_i+PCL_i)-\check{c}_{h1i}(H1F_i+H1L_i) \\ &-\check{c}_{h2i}(H2F_i+H2L_i)-(O\check{C}F_i+O\check{C}L_i)-(TCF_i+TCL_i) \\ &-(A\check{C}F_i+A\check{C}L_i)\Big]-MOC \qquad (4.127) \end{aligned}$$

$$\check{Z}_W = \sum_{i=1}^{N}\Big[SRW_i-PCW_i-H\check{C}W_i-O\check{C}W_i\Big] \qquad (4.128)$$

$$\check{Z}_T = \check{Z}_R+\check{Z}_W \qquad (4.129)$$

In NCS, since the retailer is the leader and the wholesaler is the follower so the retailer determines the marketing decision at first. So the mathematical problem in this scenario is as follows.

$$\left.\begin{array}{ll}\text{To determine} & N_o, M_t, n_i, m_i, f_{ri}, m_{kdi}, \lambda_i;\ \text{for } i = 1, 2, ..., N \\ \text{Maximize} & \check{Z}_R\end{array}\right\} \tag{4.130}$$

Depending upon the retailer's decision, the wholesaler tries to improve his/her profit. So the problem of the wholesaler mathematically takes the following form:

$$\left.\begin{array}{ll}\text{To determine} & M_i;\ \text{for } i = 1, 2, ..., N \\ \text{Maximize} & \check{Z}_W\end{array}\right\} \tag{4.131}$$

In CS, the individual profits and the channel profit are represented by

$$\begin{aligned}\check{Z}_R &= \sum_{i=1}^{N}\Big[s_{pdi}(SRF_i + SRL_i) - c_{pi}(PCF_i + PCL_i) - \check{c}_{h1i}(H1F_i + H1L_i) \\ &\quad -\check{c}_{h2i}(H2F_i + H2L_i) - (O\check{C}F_i + O\check{C}L_i) - (TCF_i + TCL_i) \\ &\quad -(A\check{C}F_i + A\check{C}L_i)\Big] - MOC + F.P\check{R}C \qquad (4.132)\\ \check{Z}_W &= \sum_{i=1}^{N}\Big[SRW_i - PCW_i - H\check{C}W_i - O\check{C}W_i\Big] - F.P\check{R}C \qquad (4.133)\\ \check{Z}_T &= \check{Z}_R + \check{Z}_W \qquad (4.134)\end{aligned}$$

In CS, the retailer and the wholesaler jointly takes the marketing decision to improve the channel profit as well as individual profit. So in this scenario problem mathematically takes the following form:

$$\left.\begin{array}{ll}\text{To determine} & N_o, M_t, n_i, m_i, f_{ri}, m_{kdi}, \lambda_i, M_i;\ \text{for } i = 1, 2, ..., N \\ \text{Maximize} & \check{Z}_T\end{array}\right\} \tag{4.135}$$

For the rough variables, the individual profits and the total profit also becomes rough numbers as $\check{Z}_R = ([Z_{R1}, Z_{R2}][Z_{R3}, Z_{R4}])$, $\check{Z}_W = ([Z_{W1}, Z_{W2}][Z_{W3}, Z_{W4}])$ and $\check{Z}_T = ([Z_{T1}, Z_{T2}][Z_{T3}, Z_{T4}])$; where

TABLE 4.13: Input data of Crisp model for Example 4.3 and Example 4.6

i	x_i	y_i	c_{pi}	s_{pi}	h_{1i}	h_{2i}	c_{o1i}	c_{o2i}	c_{t1i}	c_{t2i}	c_{ai}	A_{ri}	c_{pwi}	h_{wi}	c_{wo1i}	c_{wo2i}
1	150	0.15	1.52	3.95	0.05	0.04	7.5	0.024	7	0.05	7	0.32	1.02	0.02	26	0.01
2	200	0.30	1.45	3.77	0.10	0.02	5.5	0.014	20	0.03	12	0.35	0.85	0.02	24	0.01
3	175	0.12	1.58	4.11	0.08	0.06	6.1	0.015	7	0.05	10	0.31	0.92	0.02	28	0.01
4	240	0.28	1.49	3.87	0.05	0.02	6.6	0.020	22	0.03	11	0.29	0.87	0.02	25	0.01

$$Z_{Rj} = \sum_{i=1}^{N}\Big[s_{pdi}(SRF_i + SRL_i) - c_{pi}(PCF_i + PCL_i) - c_{h1(m-j)i}(H1F_i + H1L_i) - c_{h2(m-j)i}(H2F_i + H2L_i) - \{OCF_{(m-j)i} + OCL_{(m-j)i}\} - (TCF_i + TCL_i) - \{ACF_{(m-j)i} + ACL_{(m-j)i}\}\Big] - MOC + F.PRC_j \quad (4.136)$$

$$Z_{Wj} = \sum_{i=1}^{N}\Big[SRW_i - PCW_i - HCW_{(m-j)i} - OCW_{(m-j)i}\Big] - F.PRC_{m-j} \quad (4.137)$$

$$Z_{Tj} = Z_{Rj} + Z_{Wj} \quad (4.138)$$

These expressions can be used to find rough numbers of the profit functions in CS for $j = 1, 2, 3, 4$; where $m = 3$ for $j = 1, 2$ and $m = 7$ for $j = 3, 4$. The same expressions can be used in NCS by taking $F = 0$.

4.4.3 Numerical Illustration and Discussion

The model is illustrated with a set of hypothetical test data for different environments (crisp/fuzzy/rough). Two examples are considered to illustrate the crisp model. The fuzzy and the rough models are discussed using two separate examples. In this section, the numerical results in different scenarios for different examples are obtained using MMCABC approach (cf. § 2.2.2.4).

Example 4.3. **(For the crisp model)** In this example, 3 items are considered, i.e., $N = 3$. The input data for different items ($i = 1, 2, 3$) are presented in the first three rows of Table 4.13. Other parametric values are: $c_{mo} = 5$, $\alpha = 0.38$, $\gamma = 1.8$, $H_p = 20$, $SA_1 = 90$.

In NCS, optimizing retailer's profit with these data, the best found retailer's profit Z_R and the corresponding values of the decision variables N_o, M_t, n_i, m_i, f_{ri}, m_{kdi}, λ_i (for $i = 1, 2, ..., N$) are tabulated in Table 4.14. The obtained values of the decision variables N_o, M_t, n_i, m_i, f_{ri}, m_{kdi}, λ_i (for $i = 1, 2, ..., N$) of the retailer

TABLE 4.14: Results of Crisp model in NCS for Example 4.3

Item (i)	n_i	m_i	f_{ri}	N_o	M_t	m_{kdi}	λ_i	M_i	Z_R	Z_W	Z_T
1	1	2	1			0.974	0.656	2			
2	2	1	0	4	2	0.960	0.199	1	1777.50	413.35	2190.86
3	1	1	0			1.000	0.145	4			

TABLE 4.15: Values of Z_R and Z_W for different F of Crisp model in CS for Example 4.3

F	Z_R	Z_W	Z_T
0.12	**1743.62**	821.95	2565.57
0.13	1788.10	777.47	2565.57
0.14	1842.91	722.66	2565.57
0.15	1883.50	682.07	2565.57
0.16	1932.19	633.38	2565.57
0.17	1983.67	581.90	2565.57
0.18	2034.59	530.98	2565.57
0.19	2079.28	486.29	2565.57
0.20	2125.19	440.38	2565.57
0.21	2174.97	**390.60**	2565.57

Bold face indicates the values of profit less than the NCS

are taken to optimize wholesaler's profit. The profit amount of the wholesaler Z_W and the corresponding total profit Z_T are presented in Table 4.14. According to the wholesaler's decision, the values of M_i are also presented in Table 4.14.

In CS, a parametric study on F is done and the results are presented in Table 4.15. From this table, the appropriate range of F can be obtained. The appropriate range of F is $(0.13, 0.20)$, because out of this range, the profits of either the retailer or the wholesaler decreases in the CS than the NCS. So any value of F outside of this range is not applicable simultaneously to both the parties (the retailer and the wholesaler). From Table 4.15, it is found that if $F = 0.12$, then the retailer's profit in CS (1743.62) decreases than that in the NCS (1777.50). Again, if $F = 0.21$, then the wholesaler's profit in CS (390.60) is less than that in the NCS (413.35). Taking $F = 0.17$, the total profit of the retailer and the wholesaler is optimized and the corresponding results are presented in Table 4.16. From this table, it is clear that for $F = 0.17$ the profits of both the parties is far better than the NCS.

Example 4.4. **(For the fuzzy model)** The input values of fuzzy parameters $\widetilde{h}_{1i}$, $\widetilde{h}_{2i}$, $\widetilde{c}_{o1i}$, $\widetilde{c}_{o2i}$, $\widetilde{c}_{ai}$, $\widetilde{h}_{wi}$, $\widetilde{c}_{wo1i}$, $\widetilde{c}_{wo2i}$ (for the item $i = 1, 2, 3$) are presented in Table 4.17. All other parametric values are same as in the Example 4.3 for the crisp model.

TABLE 4.16: Results of Crisp model in CS for Example 4.3

Item (i)	n_i	m_i	f_{ri}	N_o	M_t	m_{kdi}	λ_i	M_i	Z_R	Z_W	Z_T
1	2	2	2			0.744	0.491	2			
2	3	1	2	8	2	0.617	0.337	1	1983.67	581.90	2565.57
3	1	1	1			0.648	0.171	4			

TABLE 4.17: Input data of Fuzzy model for Example 4.4

Input Variable	Item $i = 1$	Item $i = 2$	Item $i = 3$
$\widetilde{h}_{1i}$	(0.048, 0.050, 0.052)	(0.098, 0.100, 0.102)	(0.078, 0.080, 0.082)
$\widetilde{h}_{2i}$	(0.038, 0.040, 0.042)	(0.018, 0.020, 0.022)	(0.058, 0.060, 0.062)
$\widetilde{c}_{o1i}$	(7.48, 7.50, 7.51)	(5.49, 5.50, 5.52)	(6.08, 6.10, 6.11)
$\widetilde{c}_{o2i}$	(0.023, 0.024, 0.025)	(0.013, 0.014, 0.015)	(0.014, 0.015, 0.016)
$\widetilde{c}_{ai}$	(6.5, 7, 7.5)	(11.5, 12, 12.5)	(9.5, 10, 10.5)
$\widetilde{h}_{wi}$	(0.018, 0.020, 0.022)	(0.018, 0.020, 0.022)	(0.018, 0.020, 0.022)
$\widetilde{c}_{wo1i}$	(25.5, 26, 26.5)	(23.5, 24, 24.5)	(27.5, 28, 28.5)
$\widetilde{c}_{wo2i}$	(0.009, 0.010, 0.011)	(0.009, 0.010, 0.011)	(0.009, 0.010, 0.011)

TABLE 4.18: Results of Fuzzy model in NCS

Item (i)	n_i	m_i	f_{ri}	N_o	M_t	m_{kdi}	λ_i	M_i	Profit Values
1	1	2	1			0.974	0.404	2	$\widetilde{Z}_R$ =(1755.01, 1776.62, 1798.28)
2	2	1	0	4	2	0.970	0.196	1	$\widetilde{Z}_W$ =(390.97, 400.20, 409.43)
3	1	2	0			1.000	0.400	4	$\widetilde{Z}_T$ =(2145.98, 2176.82, 2207.71)

TABLE 4.19: Results of Fuzzy model in CS

Item (i)	n_i	m_i	f_{ri}	N_o	M_t	m_{kdi}	λ_i	M_i	Profit Values
1	2	2	2			0.744	0.495	2	$\widetilde{Z}_R$ =(1901.96, 1985.41, 2068.95)
2	3	1	2	8	2	0.618	0.336	1	$\widetilde{Z}_W$ =(556.50, 580.20, 603.90)
3	1	1	1			0.654	0.169	4	$\widetilde{Z}_T$ =(2458.46, 2565.61, 2672.86)

With similar explanations as in the crisp model, the results in the NCS and CS of the fuzzy model are obtained for the above set of parametric values and are presented in Table 4.18 and Table 4.19 respectively. In this model also same trend of results is obtained as in the crisp model.

Example 4.5. **(For the rough model)** The input values of rough parameters $\check{h}_{1i}$, $\check{h}_{2i}$, $\check{c}_{o1i}$, $\check{c}_{o2i}$, $\check{c}_{ai}$, $\check{h}_{wi}$, $\check{c}_{wo1i}$, $\check{c}_{wo2i}$ (for the item $i = 1, 2, 3$) are presented in Table 4.20. All other parametric values are same as in the Example 4.3 for the crisp model.

With similar explanations as in the crisp model, the results in the NCS and CS

TABLE 4.20: Input data of Rough model for Example 4.5

Input Variable	Item $i = 1$	Item $i = 2$	Item $i = 3$
$\check{h}_{1i}$	([0.049,0.051][0.048,0.052])	([0.099,0.101][0.098,0.102])	([0.079,0.081][0.078,0.082])
$\check{h}_{2i}$	([0.039,0.041][0.038,0.042])	([0.019,0.021][0.018,0.022])	([0.059,0.061][0.058,0.062])
$\check{c}_{o1i}$	([7.49,7.51][7.48,7.52])	([5.49,5.51][5.48,5.52])	([6.09,6.11][6.08,6.12])
$\check{c}_{o2i}$	([0.023,0.025][0.022,0.026])	([0.013,0.015][0.012,0.016])	([0.014,0.016][0.013,0.017])
$\check{c}_{ai}$	([6.5,7.5][6,8])	([11.5,12.5][11,13])	([9.5,10.5][9,11])
$\check{h}_{wi}$	([0.019,0.021][0.018,0.022])	([0.019,0.021][0.018,0.022])	([0.019,0.021][0.018,0.022])
$\check{c}_{wo1i}$	([25.5,26.5][25,27])	([23.5,24.5][23,25])	([27.5,28.5][27,29])
$\check{c}_{wo2i}$	([0.009,0.011][0.008,0.012])	([0.009,0.011][0.008,0.012])	([0.009,0.011][0.008,0.012])

TABLE 4.21: Results of Rough model in NCS

Item (i)	n_i	m_i	f_{ri}	N_o	M_t	m_{kdi}	λ_i	M_i	Profit Values
1	1	2	1			0.974	0.333	2	$\check{Z}_R$ =([1758.14, 1791.28][1741.57, 1807.84])
2	2	1	0	4	2	0.958	0.200	1	$\check{Z}_W$ =([409.20, 422.00][402.79, 428.41])
3	1	2	0			1.000	0.467	4	$\check{Z}_T$ =([2167.34, 2213.28][2144.36, 2236.25])

TABLE 4.22: Results of Rough model in CS

Item (i)	n_i	m_i	f_{ri}	N_o	M_t	m_{kdi}	λ_i	M_i	Profit Values
1	2	2	2			0.744	0.494	2	$\check{Z}_R$ =([1912.29, 2057.26][1839.81, 2129.74])
2	3	1	2	8	2	0.619	0.335	1	$\check{Z}_W$ =([561.54, 600.20][542.21, 619.53])
3	1	1	1			0.648	0.171	4	$\check{Z}_T$ =([2473.83, 2657.45][2382.02, 2749.27])

of the rough model are obtained for the above set of parametric values and are presented in Table 4.21 and Table 4.22 respectively. In this model also same trend of results is obtained as in the crisp model.

Example 4.6. **(For the crisp model)** In this example, 4 items are considered. The input data for first 3 items are same as in Example 4.3 and the input data for fourth item (i.e., $i = 4$) are presented in Table 4.13. All other parametric values are also same as in Example 4.3.

The results for this Example 4.6 in NCS and CS of the crisp model are presented in Table 4.35 and Table 4.36 respectively. In this example also, same trend of results is obtained as found in the Example 4.3 for the crisp model.

In the results of different examples in NCS it is observed that for some items $m_{kdi} = 1$ and for some items $f_{ri} = 0$. So, when the retailer is the decision maker and the wholesaler is the follower, then some promotional effort for some items may not be beneficial for the retailer. From all the above illustration, it is clear

TABLE 4.23: Results of Crisp model in NCS for Example 4.6

Item (i)	n_i	m_i	f_{ri}	N_o	M_t	m_{kdi}	λ_i	M_i	Z_R	Z_W	Z_T
1	1	1	1			0.994	0.229	2			
2	2	1	0	4	2	0.958	0.150	1	2714.00	735.70	3449.71
3	1	1	0			1.000	0.090	4			
4	2	2	1			0.981	0.531	1			

TABLE 4.24: Results of Crisp model in CS for Example 4.6

Item (i)	n_i	m_i	f_{ri}	N_o	M_t	m_{kdi}	λ_i	M_i	Z_R	Z_W	Z_T
1	2	1	2			0.729	0.084	3			
2	4	1	1	11	2	0.630	0.150	1	3092.10	1020.41	4112.51
3	2	1	1			0.668	0.088	3			
4	4	2	3			0.607	0.677	1			

that the frequencies of advertisement of different items are positive in CS, i.e., the promotional effort using advertisement is beneficial for both the parties when joint decision is made. Also mark-up of selling price is less than 1, for all the items in CS. So, the price discount policy is also beneficial for the supply chain, when the joint decision is made using the promotional cost sharing. Moreover, it is also established that the promotional cost sharing is beneficial for all the parties involved in the chain as all of them can take part in the marketing decision. For some items, it is observed that $n_i > 1$, $m_i > 1$. So, BP policy is beneficial for the retailer in CS as well as in NCS.

4.5 Model 4.4: A Multi-item Supply Chain with Multi-level Trade Credit Policy Under Inflation: A Mixed-mode ABC Approach

4.5.1 Assumptions and Notations

The following assumptions and notations are used for mathematical formulation of the model.

(i) Inventory system involves N items.

(ii) H is the finite planning horizon.

(iii) K orders are done by the retailer during the planning horizon H.

(iv) T_o is the basic time interval between two consecutive orders of the retailer, i.e., $T_o = H/K$.

(v) L_i is the cycle length of the retailer for i-th item, i.e., $L_i = n_i T_o$; where, n_i is the number of integer multiple of T_o.

(vi) Total number of retailer-cycle for the i-th item

$$N_i = \begin{cases} \left[\dfrac{H}{L_i}\right], & \text{if } H \text{ is an integer multiple of } L_i \\ \left[\dfrac{H}{L_i}\right] + 1, & \text{otherwise} \end{cases}$$

where, $[x]$ represents integral part of x.

(vii) Ll_i is the last cycle length of the retailer for the i-th item, i.e., $Ll_i = H - (N_i - 1)L_i$.

(viii) $q_{ij}(t)$ is the retailer's inventory level for the i-th item in j-th retailer-cycle at any time t.

(ix) Q_{ij} is the retailer's order quantity for the i-th item in j-th cycle.

(x) Q_i is the total order quantity of the retailer for i-th item for the planning horizon H.

(xi) Q_{di} is the discounted order quantity level, i.e., the order level below which no credit opportunity is allowed to the retailer.

(xii) f_{ri} is the frequency of advertisement of the i-th item per unit time.

(xiii) $R = d - I$, where d is the discount rate and I is the inflation rate.

(xiv) $c_{pi}e^{-(j-1)RL_i}$ is the present value of unit purchase cost of the retailer for i-th item in j-th cycle.

(xv) $s_{pi}e^{-(j-1)RL_i}$ is the present value of the normal selling price (maximum retail price (MRP)) of the retailer for i-th item in j-th cycle per unit item.

(xvi) $s_{pdi}e^{-(j-1)RL_i}$ is the present value of the discounted selling price of the retailer for i-th item in j-th cycle per unit item; where, s_{pdi} is a mark-up m_{kdi} of s_{pi}, i.e., $s_{pdi} = m_{kdi}s_{pi}$.

(xvii) $c_{oi}e^{-(j-1)RL_i}$ is the present value of ordering cost of the retailer for i-th item in j-th cycle.

(xviii) $c_{ai}e^{-(j-1)RL_i}$ is the present value of advertisement cost of the retailer per advertisement for i-th item in j-th cycle.

(xix) $c_{hi}e^{-(j-1)RL_i}$ is the present value of unit holding cost of the retailer for i-th item in j-th cycle; where, c_{hi} is a mark-up m_{hi} of c_{pi}, i.e., $c_{hi} = m_{hi}c_{pi}$.

(xx) t_S is the credit period offered by the supplier to the wholesaler for each wholesaler-cycle.

(xxi) t_W is the credit period offered by the wholesaler to the retailer for each retailer-cycle.

(xxii) t_R is the credit period offered by the retailer to the customers for each retailer-cycle.

(xxiii) The supplier offers a partial trade credit period t_S to the wholesaler on α_S-fraction of the total purchase amount, i.e., the wholesaler has to pay the $(1 - \alpha_S)$-fraction of the total purchase amount. α_S is a positive real number between 0 and 1.

(xxiv) The wholesaler offers a partial trade credit period t_W to the retailer on α_W-fraction of the total purchase amount, i.e., the retailer has to pay the $(1-\alpha_W)$-fraction of the total purchase amount. α_W is a positive real number between 0 and 1.

(xxv) The retailer also offers a partial trade credit period t_R to the customers on α_R-fraction of the total purchase amount, i.e., the customer has to pay the $(1-\alpha_R)$-fraction of the total purchase amount. α_R is a positive real number between 0 and 1.

(xxvi) I_p is the rate of interest paid to the bank.

(xxvii) I_e is the rate of interest earned from the bank.

(xxviii) Demand of the i-th item in j-th retailer cycle D_{ij} is considered of the form:

$$D_{ij}(t) = \frac{(1+f_{ri})^{\gamma_1}}{\{s_{pdi}e^{-(j-1)RL_i}\}^{\delta}}\left[a_i(1+t_R)^{\gamma_2} - b_i t\right] = A_i - B_i t$$

where, $A_i = \frac{(1+f_{ri})^{\gamma_1}}{\{s_{pdi}e^{-(j-1)RL_i}\}^{\delta}}a_i(1+t_R)^{\gamma_2}$, $B_i = \frac{(1+f_{ri})^{\gamma_1}}{\{s_{pdi}e^{-(j-1)RL_i}\}^{\delta}}b_i$ and $s_{pdi} = m_{kdi}s_{pi}$. The demand depends on frequency of advertisement, trade credit period offered by the retailer, selling price of the retailer. Demand of the item decreases with time due to the obsolescence/new-arrivals etc. The arbitrary constants a_i, b_i, γ_1, γ_2, δ are so chosen to best fit the demand function.

(xxix) For the i-th item, M_i retailer cycles are completed during one wholesaler cycle.

(xxx) $p_i = \left[\frac{N_i}{M_i}\right]$. If M_i divides N_i (i.e., $M_i|N_i$), then the wholesaler have p_i complete cycles for the i-th item. Otherwise, there are $M_{1i} = N_i - p_i M_i$ retailer cycles in (p_i+1)-th wholesaler cycle.

(xxxi) $c_{pwi}e^{-(k-1)M_iL_iR}$ is the present value of the unit purchase cost of the wholesaler for the i-th item in the k-th wholesaler cycle.

(xxxii) $c_{owi}e^{-(k-1)M_iL_iR}$ is the present value of the ordering cost of the wholesaler for the i-th item in the k-th wholesaler cycle.

(xxxiii) $c_{hwi}e^{-(k-1)M_iL_iR}$ is the present value of the unit holding cost of the wholesaler for the i-th item in the k-th wholesaler cycle; where, c_{hwi} is a mark-up m_{hwi} of c_{pwi}, i.e., $c_{hwi} = m_{hwi}c_{pwi}$.

4.5.2 Mathematical Formulation of the Model

4.5.2.1 Retailer's Profit

The joint replenishment of the items is made by the retailer using a BP policy. Under this policy, the retailer orders different items regularly at a fixed time interval, T_o, called BP. At the time of order, only those items are included in the order whose inventory level reaches reorder level. So, the cycle length of each item is an integer multiple of T_o. For i-th item, it is assumed that cycle length L_i is an integer multiple n_i of T_o, i.e., $L_i = n_i T_o$. Here, it is assumed that the supplier offers a partial trade credit period t_S in payment to the wholesaler on α_S-fraction of the total purchase amount, i.e., the wholesaler has to pays off $(1-\alpha_S)$-fraction of the total purchase amount at the time of purchase of the item. Similarly, the wholesaler also offers partial trade credit period t_W $(< t_S)$ to the retailer on α_W-fraction of the total purchase amount. Due to this facility, the retailer also offers partial trade credit period t_R $(< t_W)$ to the customers on α_R-fraction of the total purchase amount.

Inventory level of the retailer of i-th item in j-th cycle ($j = 1, 2, ..., N_i - 1$): Instantaneous state $q_{ij}(t)$ of the i-th item in j-th retailer-cycle is given by

$$\frac{dq_{ij}(t)}{dt} = -D_{ij}(t), \text{ for } (j-1)L_i \le t \le jL_i \tag{4.139}$$

with boundary conditions $q_{ij}((j-1)L_i) = Q_{ij}$ and $q_{ij}(jL_i) = 0$.
Solving (4.139), the inventory level $q_{ij}(t)$ can be found as follows:

$$q_{ij}(t) = A_i(jL_i - t) - \frac{B_i}{2}(j^2L_i^2 - t^2) \tag{4.140}$$

and the retailer's order quantity for i-th item in j-th retailer cycle is given by

$$Q_{ij} = A_iL_i - \frac{B_iL_i^2}{2}(2j-1) \tag{4.141}$$

Purchase cost: In the first $(N_i - 1)$ retailer cycles, the purchase cost of the retailer for i-th item is given by

$$PC_i = \sum_{j=1}^{N_i-1} c_{pi}e^{-(j-1)RL_i}Q_{ij} = c_{pi}\left[\left(A_iL_i + \frac{B_iL_i^2}{2}\right)S_1 - B_iL_i^2S_2\right] \tag{4.142}$$

$$\text{where, } S_1 = \sum_{j=1}^{N_i-1} e^{-(j-1)RL_i} = \frac{1-e^{-(N_i-1)RL_i}}{1-e^{-RL_i}}$$
$$S_2 = \sum_{j=1}^{N_i-1} je^{-(j-1)RL_i} = \frac{1-e^{-(N_i-1)RL_i}}{(1-e^{-RL_i})^2} - \frac{(N_i-1)}{(1-e^{-RL_i})}e^{-(N_i-1)RL_i}$$

Sell revenue: In the first $(N_i - 1)$ retailer cycles, the sell revenue of the retailer for i-th item is given by

$$SR_i = \sum_{j=1}^{N_i-1} s_{pdi}e^{-(j-1)RL_i}Q_{ij} = s_{pdi}\left[\left(A_iL_i + \frac{B_iL_i^2}{2}\right)S_1 - B_iL_i^2S_2\right] \quad (4.143)$$

Ordering cost: In the first $(N_i - 1)$ retailer cycles, the ordering cost is given by

$$OC_i = \sum_{j=1}^{N_i-1} c_{oi}e^{-(j-1)RL_i} = c_{oi}S_1 \quad (4.144)$$

Advertisement cost: In the first $(N_i - 1)$ retailer cycles, the advertisement cost is given by

$$AC_i = \sum_{j=1}^{N_i-1} c_{ai}e^{-(j-1)RL_i}f_{ri}L_i = c_{ai}f_{ri}L_iS_1 \quad (4.145)$$

Holding cost: Holding cost of the retailer for i-th item in j-th retailer cycle is given by

$$HC_{ij} = \int_{(j-1)L_i}^{jL_i} c_{hi}e^{-(j-1)RL_i}q_{ij}(t)dt$$
$$= c_{hi}e^{-(j-1)RL_i}\left[\frac{A_iL_i^2}{2} - \frac{B_iL_i^3}{6}(3j-1)\right] \quad (4.146)$$

Therefore, the total holding cost of the retailer in first $(N_i - 1)$ retailer cycles is given by

$$HC_i = \sum_{j=1}^{N_i-1} HC_{ij} = c_{hi}\left[\left(\frac{A_iL_i^2}{2} + \frac{B_iL_i^3}{6}\right)S_1 - \frac{B_iL_i^3}{2}S_2\right] \quad (4.147)$$

Interest earned and paid: If the retailer orders minimum quantity of products (discounted order quantity) for i-th item Q_{di}, then he/she is eligible for credit opportunity from the wholesaler. So, depending upon the retailer's order quantity, the following two cases arise:

- Case-1: $Q_{ij} < Q_{di}$

- Case-2: $Q_{ij} \geq Q_{di}$

Case-1: $Q_{ij} < Q_{di}$
Since, the retailer's order quantity for i-th item in j-th cycle (Q_{ij}) is less than Q_{di}, he/she is not eligible for any credit opportunity, whereas the retailer offers a credit period t_R to the customers on α_R-fraction on the total purchase amount. Total interest earned for i-th item in j-th cycle is

$$TIE_{ij} = 0$$

Total interest paid for i-th item in j-th cycle is

$$TIP_{ij} = I_p(IP_1 + IP_2) \tag{4.148}$$

$$\begin{aligned} \text{where, } IP_1 &= \text{Interest to be paid due to the stock units during } [(j-1)L_i, jL_i] \\ &= \int_{(j-1)L_i}^{jL_i} c_{pi} e^{-(j-1)RL_i} q_{ij}(t)dt \\ &= c_{pi} e^{-(j-1)RL_i} \frac{L_i^2}{6} \Big[3A_i - B_i L_i (3j-1) \Big] \end{aligned} \tag{4.149}$$

$$\begin{aligned} IP_2 &= \text{Interest to be paid due to the customers' credit opportunity for the sold} \\ &\quad \text{units during } [(j-1)L_i, jL_i] \\ &= \int_{(j-1)L_i}^{jL_i} c_{pi} e^{-(j-1)RL_i} \alpha_R D_{ij}(t) t_R dt \\ &= c_{pi} e^{-(j-1)RL_i} \alpha_R t_R \Big[A_i L_i - \frac{B_i L_i^2}{2}(2j-1) \Big] \end{aligned} \tag{4.150}$$

Case-2: $Q_{ij} \geq Q_{di}$
The retailer obtain a grace period t_W from the wholesaler on α_W-fraction of the total purchase amount and offers a credit period t_R to the customers on α_R-fraction of the total purchase amount. According to the assumption, the retailer has to pay $(1 - \alpha_W)$-fraction of the total purchase amount at the receiving time of the units of the item using a bank loan (for i-th item and j-th retailer cycle) called as Initial Bank Loan (IBL), which is given by

$$IBL = (1 - \alpha_W) c_{pi} e^{-(j-1)RL_i} Q_{ij} \tag{4.151}$$

To repay the IBL, here it is assumed that the retailer pays off the purchase cost of the sold units immediately to the bank. The rest portion of the selling price is used to meet the other regular expenditures, like, holding cost, ordering cost etc., to run the business. Let R_1 and R_2 are the collected revenues for the sold units during the time period $[(j-1)L_i, (j-1)L_i + t_R]$ and $[(j-1)L_i, (j-1)L_i + t_W]$ respectively; where,

$$\begin{aligned} R_1 &= \int_{(j-1)L_i}^{(j-1)L_i+t_R} c_{pi} e^{-(j-1)RL_i}(1-\alpha_R) D_{ij}(t) dt \\ &= c_{pi} e^{-(j-1)RL_i}(1-\alpha_R)\Big[A_i - B_i(j-1)L_i - \frac{B_i}{2} t_R\Big] \end{aligned} \tag{4.152}$$

$$\begin{aligned} R_2 &= \int_{(j-1)L_i}^{(j-1)L_i+t_W} c_{pi} e^{-(j-1)RL_i}(1-\alpha_R) D_{ij}(t) dt \\ &\quad + \int_{(j-1)L_i}^{(j-1)L_i+t_W-t_R} c_{pi} e^{-(j-1)RL_i}\alpha_R D_{ij}(t) dt \\ &= c_{pi} e^{-(j-1)RL_i}\Big[\Big\{A_i - B_i(j-1)L_i\Big\}(t_W - \alpha_R t_R) - \frac{B_i}{2}\Big\{t_W^2 + \alpha_R(-2t_W t_R + t_R^2)\Big\}\Big] \end{aligned} \tag{4.153}$$

According to the values of R_1, R_2 and IBL, the following three cases may arise:

- Case-2.1: $IBL \le R_1$
- Case-2.2: $R_1 < IBL \le R_2$
- Case-2.3: $IBL \ge R_2$

Case-2.1: $IBL \le R_1$

In this situation, the IBL of the retailer should be made before the time t_R. Let T_1 be the time at which the IBL should be made and is given by

$$\int_{(j-1)L_i}^{(j-1)L_i+T_1} c_{pi} e^{-(j-1)RL_i}(1-\alpha_R) D_{ij}(t) dt = IBL$$

$$\text{i.e.,} \quad c_{pi} e^{-(j-1)RL_i}(1-\alpha_R)\Big[A_i T_1 - B_i(j-1)L_i T_1 - \frac{B_i}{2} T_1^2\Big] = IBL$$

$$\text{i.e.,} \quad B_i T_1^2 - 2g_1 T_1 + g_2 = 0 \tag{4.154}$$

where, $g_1 = A_i - B_i(j-1)L_i$ and $g_2 = \dfrac{2 \times IBL}{c_{pi} e^{-(j-1)RL_i}(1-\alpha_R)}$

$$\text{i.e.,} \quad T_1 = \frac{g_1 + \sqrt{g_1^2 - B_i g_2}}{B_i} \tag{4.155}$$

Now, the total interest earned for the i-th item in the j-th retailer cycle is

$$TIE_{ij} = I_e(IE_1 + IE_2) \tag{4.156}$$

where, IE_1 = Interest earned due to customers' instant payment for the sold units during $[(j-1)L_i + T_1, (j-1)L_i + t_W]$

$$\begin{aligned}
&= \int_{(j-1)L_i+T_1}^{(j-1)L_i+t_W} c_{pi}e^{-(j-1)RL_i}(1-\alpha_R)D_{ij}(t)\Big\{(j-1)L_i + t_W - t\Big\}dt \\
&= c_{pi}e^{-(j-1)RL_i}(1-\alpha_R)\Bigg[A_i\Big\{(j-1)L_i + t_W\Big\}(t_W - T_1) \\
&\quad -\frac{1}{2}\Big\{A_i + B_i((j-1)L_i + t_W)\Big\}\Big\{((j-1)L_i + t_W)^2 - ((j-1)L_i + T_1)^2\Big\} \\
&\quad +\frac{B_i}{3}\Big\{((j-1)L_i + t_W)^3 - ((j-1)L_i + T_1)^3\Big\}\Bigg]
\end{aligned} \tag{4.157}$$

IE_2 = Interest earned due to the customers' repayment for the sold units during $[(j-1)L_i, (j-1)L_i + t_W - t_R]$

$$\begin{aligned}
&= \int_{(j-1)L_i}^{(j-1)L_i+t_W-t_R} c_{pi}e^{-(j-1)RL_i}\alpha_R D_{ij}(t)\Big\{(j-1)L_i + t_W - t_R - t\Big\}dt \\
&= c_{pi}e^{-(j-1)RL_i}\alpha_R\Bigg[A_i\Big\{(j-1)L_i + t_W - t_R\Big\}(t_W - t_R) \\
&\quad -\frac{1}{2}\Big\{A_i + B_i((j-1)L_i + t_W - t_R)\Big\}\Big\{((j-1)L_i + t_W - t_R)^2 - ((j-1)L_i)^2\Big\} \\
&\quad +\frac{B_i}{3}\Big\{((j-1)L_i + t_W - t_R)^3 - ((j-1)L_i)^3\Big\}\Bigg]
\end{aligned} \tag{4.158}$$

Now, the total interest to be paid for the i-th item in the j-th retailer cycle is

$$TIP_{ij} = I_p(IP_1 + IP_2 + IP_3 + IP_4) \tag{4.159}$$

where, IP_1 = Interest to be paid due to the IBL

$$\begin{aligned}
&= \int_{(j-1)L_i}^{(j-1)L_i+T_1}\Bigg[IBL - \int_{\xi=(j-1)L_i}^{t} c_{pi}e^{-(j-1)RL_i}(1-\alpha_R)D_{ij}(\xi)d\xi\Bigg]dt \\
&= IBL \times T_1 - c_{pi}e^{-(j-1)RL_i}(1-\alpha_R)\Bigg[(j-1)L_i\Big\{\frac{B_i}{2}(j-1)L_i - A_i\Big\}T_1 \\
&\quad +\frac{A_i}{2}\Big\{((j-1)L_i + T_1)^2 - ((j-1)L_i)^2\Big\} \\
&\quad -\frac{B_i}{6}\Big\{((j-1)L_i + T_1)^3 - ((j-1)L_i)^3\Big\}\Bigg]
\end{aligned} \tag{4.160}$$

$$
\begin{aligned}
IP_2 &= \text{Interest to be paid due to the customers' credit opportunity for the sold} \\
&\quad \text{units during } [(j-1)L_i + t_W - t_R, (j-1)L_i + t_W] \\
&= \int_{(j-1)L_i+t_W-t_R}^{(j-1)L_i+t_W} c_{pi} e^{-(j-1)RL_i} \alpha_R D_{ij}(t)(t + t_R - t_W) dt \\
&= c_{pi} e^{-(j-1)RL_i} \alpha_R \Bigg[A_i (t_R - t_W) t_R \\
&\quad + \frac{1}{2} \Big\{ A_i - B_i (t_R - t_W) \Big\} \Big\{ ((j-1)L_i + t_W)^2 - ((j-1)L_i + t_W - t_R)^2 \Big\} \\
&\quad - \frac{B_i}{3} \Big\{ ((j-1)L_i + t_W)^3 - ((j-1)L_i + t_W - t_R)^3 \Big\} \Bigg] \qquad (4.161)
\end{aligned}
$$

$$
\begin{aligned}
IP_3 &= \text{Interest to be paid due to the customers' credit period for the sold units} \\
&\quad \text{during } [(j-1)L_i + t_W, jL_i] \\
&= \int_{(j-1)L_i+t_W}^{jL_i} c_{pi} e^{-(j-1)RL_i} \alpha_R D_{ij}(t) t_R dt \\
&= c_{pi} e^{-(j-1)RL_i} \alpha_R t_R \Bigg[A_i (L_i - t_W) - \frac{B_i}{2} \Big\{ (jL_i)^2 - ((j-1)L_i + t_W)^2 \Big\} \Bigg] \qquad (4.162)
\end{aligned}
$$

$$
\begin{aligned}
IP_4 &= \text{Interest to be paid due to the stock units during } [(j-1)L_i + t_W, jL_i] \\
&= \int_{(j-1)L_i+t_W}^{jL_i} c_{pi} e^{-(j-1)RL_i} q_{ij}(t) dt \\
&= c_{pi} e^{-(j-1)RL_i} \Bigg[jL_i (A_i - \frac{B_i}{2} jL_i)(L_i - t_W) - \frac{A_i}{2} \Big\{ (jL_i)^2 - ((j-1)L_i + t_W)^2 \Big\} \\
&\quad + \frac{B_i}{6} \Big\{ (jL_i)^3 - ((j-1)L_i + t_W)^3 \Big\} \Bigg] \qquad (4.163)
\end{aligned}
$$

Case-2.2: $R_1 < IBL \leq R_2$

In this situation, the IBL of the retailer should be made after the time t_R and before the time t_W. Let T_1 be the time at which the IBL should be made and is given by

$$\int_{(j-1)L_i}^{(j-1)L_i+T_1} c_{pi}e^{-(j-1)RL_i}(1-\alpha_R)D_{ij}(t)dt + \int_{(j-1)L_i}^{(j-1)L_i+T_1-t_R} c_{pi}e^{-(j-1)RL_i}\alpha_R D_{ij}(t)dt = IBL$$

i.e.,
$$c_{pi}e^{-(j-1)RL_i}(1-\alpha_R)\left[A_iT_1 - B_i(j-1)L_iT_1 - \frac{B_i}{2}T_1^2\right] + c_{pi}e^{-(j-1)RL_i}\alpha_R\left[A_i(T_1-t_R) - B_i(j-1)L_i(T_1-t_R) - \frac{B_i}{2}(T_1-t_R)^2\right] = IBL$$

i.e.,
$$B_iT_1^2 - 2g_1T_1 + g_2 = 0 \tag{4.164}$$

where, $g_1 = A_i - B_i(j-1)L_i + B_i\alpha_R t_R$

and $g_2 = 2(1-\alpha_W)Q_{ij} + \alpha_R t_R\left\{2A_i - 2B_i(j-1)L_i + B_it_R\right\}$

i.e.,
$$T_1 = \frac{g_1 + \sqrt{g_1^2 - B_ig_2}}{B_i} \tag{4.165}$$

Now, the total interest earned for the i-th item in the j-th retailer cycle is

$$TIE_{ij} = I_e(IE_1 + IE_2) \tag{4.166}$$

where, IE_1 is given by (4.157).

and IE_2 = Interest earned due to the customers' repayment for the sold units during $[(j-1)L_i + T_1 - t_R, (j-1)L_i + t_W - t_R]$

$$= \int_{(j-1)L_i+T_1-t_R}^{(j-1)L_i+t_W-t_R} c_{pi}e^{-(j-1)RL_i}\alpha_R D_{ij}(t)\left\{(j-1)L_i + t_W - t_R - t\right\}dt$$

$$= c_{pi}e^{-(j-1)RL_i}\alpha_R\Big[A_i\left\{(j-1)L_i + t_W - t_R\right\}(t_W - T_1) - \frac{1}{2}\left\{A_i + B_i((j-1)L_i + t_W - t_R)\right\}\left\{((j-1)L_i + t_W - t_R)^2 - ((j-1)L_i + T_1 - t_R)^2\right\} + \frac{B_i}{3}\left\{((j-1)L_i + t_W - t_R)^3 - ((j-1)L_i + T_1 - t_R)^3\right\}\Big] \tag{4.167}$$

Now, the total interest to be paid for the i-th item in the j-th retailer cycle is

$$TIP_{ij} = I_p(IP_1 + IP_2 + IP_3 + IP_4) \tag{4.168}$$

where, IP_1 = Interest to be paid due to the IBL

$$= \int_{(j-1)L_i}^{(j-1)L_i+t_R} \left[IBL - \int_{\xi=(j-1)L_i}^{t} c_{pi} e^{-(j-1)RL_i} (1-\alpha_R) D_{ij}(\xi) d\xi \right] dt$$
$$+ \int_{(j-1)L_i+t_R}^{(j-1)L_i+T_1} \left[IBL - R_1 - \int_{\xi=(j-1)L_i+t_R}^{t} c_{pi} e^{-(j-1)RL_i} (1-\alpha_R) D_{ij}(\xi) d\xi \right.$$
$$\left. - \int_{\xi=(j-1)L_i}^{t-t_R} c_{pi} e^{-(j-1)RL_i} \alpha_R D_{ij}(\xi) d\xi \right] dt \quad (4.169)$$

$$= IBL \times T_1 - R_1(T_1 - t_R) - c_{pi} e^{-(j-1)RL_i} \alpha_R \frac{B_i}{2} t_R (T_1 - t_R)^2$$
$$- c_{pi} e^{-(j-1)RL_i} (1-\alpha_R) \left[(j-1)L_i t_R \left\{ \frac{B_i}{2}(j-1)L_i - A_i \right\} \right.$$
$$\left. + \frac{A_i}{2} \left\{ ((j-1)L_i + t_R)^2 - ((j-1)L_i)^2 \right\} - \frac{B_i}{6} \left\{ ((j-1)L_i + t_R)^3 - ((j-1)L_i)^3 \right\} \right]$$
$$- c_{pi} e^{-(j-1)RL_i} \left[\left\{ (j-1)L_i + t_R \right\} (T_1 - t_R) \left\{ \frac{B_i}{2}((j-1)L_i + t_R) - A_i \right\} \right.$$
$$+ \frac{A_i}{2} \left\{ ((j-1)L_i + T_1)^2 - ((j-1)L_i + t_R)^2 \right\}$$
$$\left. - \frac{B_i}{6} \left\{ ((j-1)L_i + T_1)^3 - ((j-1)L_i + t_R)^3 \right\} \right] \quad (4.170)$$

IP_2, IP_3, IP_4 are same as in Case-2.1.

Case-2.3: $IBL > R_2$

In this situation, the IBL of the retailer should be made after the time t_W. Let T_1 be the time at which the IBL should be made and is given by

$$\int_{(j-1)L_i}^{(j-1)L_i+T_1} c_{pi} e^{-(j-1)RL_i} (1-\alpha_R) D_{ij}(t) dt$$
$$+ \int_{(j-1)L_i}^{(j-1)L_i+t_W-t_R} c_{pi} e^{-(j-1)RL_i} \alpha_R D_{ij}(t) dt = IBL \quad (4.171)$$

But T_1 is need not be calculated in this case, since T_1 is greater than t_W, so all the dues must be paid by the retailer at t_W. No interest will be earn in this situation, i.e.,

$$TIE_{ij} = 0 \quad (4.172)$$

The total interest to be paid for i-th item in j-th retailer cycle is

$$TIP_{ij} = I_p(IP_1 + IP_2 + IP_3 + IP_4) \quad (4.173)$$

where, IP_1 = Interest to be paid due to IBL upto t_W

$$= \int_{(j-1)L_i}^{(j-1)L_i+t_R}\left[IBL - \int_{\xi=(j-1)L_i}^{t} c_{pi}e^{-(j-1)RL_i}(1-\alpha_R)D_{ij}(\xi)d\xi\right]dt$$
$$+ \int_{(j-1)L_i+t_R}^{(j-1)L_i+t_W}\left[IBL - R_1 - \int_{\xi=(j-1)L_i+t_R}^{t} c_{pi}e^{-(j-1)RL_i}(1-\alpha_R)D_{ij}(\xi)d\xi\right.$$
$$\left. - \int_{\xi=(j-1)L_i}^{t-t_R} c_{pi}e^{-(j-1)RL_i}\alpha_R D_{ij}(\xi)d\xi\right]dt$$

$$= IBL \times t_W - R_1(t_W - t_R) - c_{pi}e^{-(j-1)RL_i}\alpha_R\frac{B_i}{2}t_R(t_W - t_R)^2$$
$$-c_{pi}e^{-(j-1)RL_i}(1-\alpha_R)\left[(j-1)L_it_R\left\{\frac{B_i}{2}(j-1)L_i - A_i\right\}\right.$$
$$\left.+\frac{A_i}{2}\left\{((j-1)L_i + t_R)^2 - ((j-1)L_i)^2\right\} - \frac{B_i}{6}\left\{((j-1)L_i + t_R)^3 - ((j-1)L_i)^3\right\}\right]$$
$$-c_{pi}e^{-(j-1)RL_i}\left[\left\{(j-1)L_i + t_R\right\}(t_W - t_R)\left\{\frac{B_i}{2}((j-1)L_i + t_R) - A_i\right\}\right.$$
$$+\frac{A_i}{2}\left\{((j-1)L_i + t_W)^2 - ((j-1)L_i + t_R)^2\right\}$$
$$\left.-\frac{B_i}{6}\left\{((j-1)L_i + t_W)^3 - ((j-1)L_i + t_R)^3\right\}\right] \quad (4.174)$$

IP_2, IP_3, IP_4 are same as in Case-2.1.

Inventory level of the retailer of i-th item in last cycle ($j = N_i$):
The last cycle length of the retailer is calculated as follows.

$$Ll_i = H - (N_i - 1)L_i \quad (4.175)$$

Instantaneous state $q_{iN_i}(t)$ of the i-th item in the last retailer-cycle is given by

$$\frac{dq_{iN_i}(t)}{dt} = -D_{iN_i}(t), \text{ for } (N_i - 1)L_i \le t \le H \quad (4.176)$$

with boundary conditions $q_{iN_i}((N_i-1)L_i) = Q_{iN_i}$ and $q_{iN_i}(H) = 0$.
Solving (4.176), the inventory level $q_{iN_i}(t)$ can be found as follows:

$$q_{iN_i}(t) = A_i(H - t) - \frac{B_i}{2}(H^2 - t^2) \quad (4.177)$$

and the retailer's order quantity for the i-th item in the last retailer cycle is given by

$$Q_{iN_i} = A_i L l_i - \frac{B_i}{2}\{H^2 - (N_i - 1)^2 L_i^2\} \tag{4.178}$$

Purchase cost: In the last retailer cycle, the purchase cost of the retailer for the i-th item is given by

$$PCL_i = c_{pi} e^{-(N_i-1)RL_i} Q_{iN_i} \tag{4.179}$$

Sell revenue: In the last retailer cycle, the sell revenue of the retailer for the i-th item is given by

$$SRL_i = s_{pdi} e^{-(N_i-1)RL_i} Q_{iN_i} \tag{4.180}$$

Ordering cost: In the last retailer cycle, the ordering cost is given by

$$OCL_i = c_{oi} e^{-(N_i-1)RL_i} \tag{4.181}$$

Advertisement cost: In the last retailer cycle, the advertisement cost is given by

$$ACL_i = c_{ai} e^{-(N_i-1)RL_i} f_{ri} L l_i \tag{4.182}$$

Holding cost: Holding cost of the retailer for the i-th item in the last retailer cycle is given by

$$\begin{aligned} HCL_i = HC_{iN_i} &= \int_{(N_i-1)L_i}^{H} c_{hi} e^{-(N_i-1)RL_i} q_{iN_i}(t)dt \\ &= c_{hi} e^{-(N_i-1)RL_i}\Big[H\Big(A_i - \frac{B_i}{2}H\Big)Ll_i - \frac{A_i}{2}\Big\{H^2 - (N_i-1)^2 L_i^2\Big\} \\ &\quad + \frac{B_i}{6}\Big\{H^3 - (N_i-1)^3 L_i^3\Big\}\Big] \end{aligned} \tag{4.183}$$

Interest earned and paid: The expressions of the interest earned and the interest paid for the last retailer cycle are almost same as the previous section. Only changed expressions (where, $j = N_i$) are given as follows:

In Case-1, IP_1 and IP_2 are given by the following expressions.

$$\begin{aligned} IP_1 &= \text{Interest to be paid due to the stock units during } [(j-1)L_i, H] \\ &= \int_{(j-1)L_i}^{H} c_{pi} e^{-(j-1)RL_i} q_{ij}(t)dt \\ &= c_{pi} e^{-(j-1)RL_i} \Big[H\Big(A_i - \frac{B_i}{2}H\Big)Ll_i - \frac{A_i}{2}\Big\{H^2 - ((j-1)L_i)^2\Big\} \\ &\quad + \frac{B_i}{6}\Big\{H^3 - ((j-1)L_i)^3\Big\}\Big] \end{aligned} \tag{4.184}$$

$$\begin{aligned} IP_2 &= \text{Interest to be paid due to the customers' credit period for the sold units} \\ &\quad \text{during } [(j-1)L_i, H] \\ &= \int_{(j-1)L_i}^{H} c_{pi} e^{-(j-1)RL_i} \alpha_R D_{ij}(t) t_R dt \\ &= c_{pi} e^{-(j-1)RL_i} \alpha_R t_R \Big[A_i Ll_i - \frac{B_i}{2}\Big\{H^2 - ((j-1)L_i)^2\Big\}\Big] \end{aligned} \tag{4.185}$$

In Case-2.1, IP_3 and IP_4 are given by the following expressions.

$$\begin{aligned} IP_3 &= \text{Interest to be paid due to the customers' credit period for the sold units} \\ &\quad \text{during } [(j-1)L_i + t_W, H] \\ &= \int_{(j-1)L_i + t_W}^{H} c_{pi} e^{-(j-1)RL_i} \alpha_R D_{ij}(t) t_R dt \\ &= c_{pi} e^{-(j-1)RL_i} \alpha_R t_R \Big[A_i(Ll_i - t_W) - \frac{B_i}{2}\Big\{H^2 - ((j-1)L_i + t_W)^2\Big\}\Big] \end{aligned} \tag{4.186}$$

$$\begin{aligned} IP_4 &= \text{Interest to be paid due to the stock units during } [(j-1)L_i + t_W, H] \\ &= \int_{(j-1)L_i + t_W}^{H} c_{pi} e^{-(j-1)RL_i} q_{ij}(t)dt \\ &= c_{pi} e^{-(j-1)RL_i} \Big[H(A_i - \frac{B_i}{2}H)(Ll_i - t_W) - \frac{A_i}{2}\Big\{H^2 - ((j-1)L_i + t_W)^2\Big\} \\ &\quad + \frac{B_i}{6}\Big\{H^3 - ((j-1)L_i + t_W)^3\Big\}\Big] \end{aligned} \tag{4.187}$$

In Case-2.2 and Case-2.3, the expressions for IP_3 and IP_4 are given by the equations (4.186) and (4.187) respectively. Remaining all the expressions can be found from the previous section by putting $j = N_i$.

4.5.2.2 Wholesaler's Profit for the i-th item

If $M_i|N_i$ (M_i divides N_i), then there are p_i full cycles in wholesaler's inventory period. Otherwise, there are $M_{1i}(= N_i - p_iM_i)$ retailer cycles in $(p_i + 1)$-th wholesaler cycle with p_i full cycles, where $p_i = \left[\frac{N_i}{M_i}\right]$ and $[x]$ represents integral part of x.

The total order quantity of the i-th item for the wholesaler is

$$TQW_i = \sum_{j=1}^{N_i} Q_{ij} \tag{4.188}$$

The order quantity of the wholesaler for the i-th item in the k-th cycle is given by

$$QW_{ik} = Q_{i\{(k-1)M_i+1\}} + Q_{i\{(k-1)M_i+2\}} + \dots + Q_{i\{kM_i\}} \tag{4.189}$$

The order quantity of the wholesaler for the i-th item in the last cycle is given by

$$\text{If } M_i|N_i,\ QW^l_{ip_i} = Q_{i\{(p_i-1)M_i+1\}} + Q_{i\{(p_i-1)M_i+2\}} + \dots + Q_{i\{p_iM_i\}} \tag{4.190}$$

$$\text{If } M_i \nmid N_i,\ QW^l_{i(p_i+1)} = Q_{i\{p_iM_i+1\}} + Q_{i\{p_iM_i+2\}} + \dots + Q_{i\{p_iM_i+M_{1i}\}} \tag{4.191}$$

Purchase cost: The purchase cost of the wholesaler for the i-th item is

$$PCW_i = \begin{cases} \sum_{k=1}^{p_i} c_{pwi}e^{-(k-1)M_iL_iR}QW_{ik}, \text{ if } M_i|N_i \\ \sum_{k=1}^{p_i} c_{pwi}e^{-(k-1)M_iL_iR}QW_{ik} + c_{pwi}e^{-p_iM_iL_iR}QW^l_{i(p_i+1)}, \text{ if } M_i \nmid N_i \end{cases} \tag{4.192}$$

Sell revenue: The sell revenue of the wholesaler for the i-th item is

$$SRW_i = \sum_{j=1}^{N_i} c_{pi}e^{-(j-1)RL_i}Q_{ij} \tag{4.193}$$

Ordering cost: The ordering cost of the wholesaler for the i-th item is

$$OCW_i = \begin{cases} \sum_{k=1}^{p_i} c_{owi}e^{-(k-1)M_iL_iR}, \text{ if } M_i|N_i \\ \sum_{k=1}^{p_i+1} c_{owi}e^{-(k-1)M_iL_iR}, \text{ if } M_i \nmid N_i \end{cases} \tag{4.194}$$

Holding cost: The holding amount of the wholesaler for the i-th item in the k-th cycle is given by

$$HW_{ik} = \sum_{j=(k-1)M_i+1}^{kM_i} Q_{ij}\Big[jL_i - \{(k-1)M_i + 1\}L_i\Big] \tag{4.195}$$

The holding amount of the wholesaler for the i-th item in the last cycle is given by

$$\text{If } M_i|N_i,\ HW^l_{ip_i} = \sum_{j=(p_i-1)M_i+1}^{p_iM_i} Q_{ij}\Big[jL_i - \{(p_i-1)M_i + 1\}L_i\Big], \tag{4.196}$$

$$\text{If } M_i \nmid N_i,\ HW^l_{i(p_i+1)} = \sum_{j=p_iM_i+1}^{p_iM_i+M_{1i}} Q_{ij}\Big[jL_i - \{p_iM_i + 1\}L_i\Big], \tag{4.197}$$

Hence, the total holding cost of the wholesaler is

$$HCW_i = \begin{cases} \sum_{k=1}^{p_i} c_{hwi}e^{-(k-1)M_iL_iR}HW_{ik}, & \text{if } M_i|N_i \\ \sum_{k=1}^{p_i} c_{hwi}e^{-(k-1)M_iL_iR}HW_{ik} + c_{hwi}e^{-p_iM_iL_iR}HW^l_{i(p_i+1)}, & \text{if } M_i \nmid N_i \end{cases} \tag{4.198}$$

Interest earned and paid: Supplier offers a delay period t_S to the wholesaler for α_S-fraction of the total purchase amount, i.e., the wholesaler has to pay the $(1-\alpha_S)$-fraction of the total purchase amount at the time of receiving of the items immediately. To pay this amount, the wholesaler takes an initial bank loan (for i-th item and k-th wholesaler cycle) which is given by

$$IBL_W = (1-\alpha_S)QW_{ik} \tag{4.199}$$

Let $[\frac{t_S}{L_i}] = n$.

$$\text{Again, let } Q_{i\{(k-1)M_i+1\}} + Q_{i\{(k-1)M_i+2\}} + \ldots + Q_{i\{(k-1)M_i+m\}} \le IBL_W$$
$$\text{and } Q_{i\{(k-1)M_i+1\}} + Q_{i\{(k-1)M_i+2\}} + \ldots + Q_{i\{(k-1)M_i+m+1\}} > IBL_W$$

Then IBL will be made at m-th or $(m+1)$-th retailer cycle of k-th wholesaler cycle. Remaining amount (Q_e) of IBL after the payment during the m-th retailer cycle of k-th wholesaler cycle is given by

$$Q_e = IBL_W - \Big[Q_{i\{(k-1)M_i+1\}} + Q_{i\{(k-1)M_i+2\}} + \ldots + Q_{i\{(k-1)M_i+m\}}\Big] \tag{4.200}$$

According to the values of Q_e, the following four cases may arise:

	Condition	IBL_W will be made at time
Case-1:	$Q_e = 0$	$\{(k-1)M_i + m - 1\}L_i + t_W$
Case-2:	$0 < Q_e < (1-\alpha_W)Q_{i\{(k-1)M_i+m+1\}}$	$\{(k-1)M_i + m\}L_i$
Case-3:	$Q_e = (1-\alpha_W)Q_{i\{(k-1)M_i+m+1\}}$	$\{(k-1)M_i + m\}L_i$
Case-4:	$Q_e > (1-\alpha_W)Q_{i\{(k-1)M_i+m+1\}}$	$\{(k-1)M_i + m\}L_i + t_W$

Case-1: $Q_e = 0$

In this case, IBL of the wholesaler due to instant payment to the supplier will be made at time $\{(k-1)M_i + m - 1\}L_i + t_W$, i.e., after time $(m-1)L_i + t_W$ from the starting point of k-th wholesaler cycle.

Interest to be paid by the wholesaler and the interest earned by the wholesaler are as follows.

$$IPW = I_p c_{pwi} e^{-(k-1)M_i L_i R}(IPW_1 + IPW_2) \tag{4.201}$$
$$IEW = I_e c_{pwi} e^{-(k-1)M_i L_i R}(IEW_1 + IEW_2) \tag{4.202}$$

where, IPW_1, IPW_2, IEW_1 and IEW_2 are given by the expressions in the following subcases.

Case-1.1: $nL_i + t_W \le t_S$

Case-1.1.1: $m - 1 \le n$

$$Q_1 = QW_{ik} - \left(Q_{i\{(k-1)M_i+1\}} + Q_{i\{(k-1)M_i+2\}} + \ldots + Q_{i\{(k-1)M_i+n+1\}}\right) \tag{4.203}$$

is the rest amount after the payment of the wholesaler to the supplier during $(n+1)$-th retailer cycle of the k-th wholesaler cycle.

$$\begin{aligned} IPW_1 &= \text{Interest to be paid due to the IBL} \\ &= \left[IBL_W - (1-\alpha_W)Q_{i\{(k-1)M_i+1\}}\right]t_W + \left[IBL_W - Q_{i\{(k-1)M_i+1\}}\right](L_i - t_W) \\ &\quad + \left[IBL_W - Q_{i\{(k-1)M_i+1\}} - (1-\alpha_W)Q_{i\{(k-1)M_i+2\}}\right]t_W \\ &\quad + \left[IBL_W - Q_{i\{(k-1)M_i+1\}} - Q_{i\{(k-1)M_i+2\}}\right](L_i - t_W) + \ldots \\ &\quad + \Big[IBL_W - Q_{i\{(k-1)M_i+1\}} - Q_{i\{(k-1)M_i+2\}} - \ldots - Q_{i\{(k-1)M_i+m-1\}} \\ &\quad - (1-\alpha_W)Q_{i\{(k-1)M_i+m\}}\Big]t_W \end{aligned}$$

$$
\begin{aligned}
= \; & IBL\{(m-1)L_i + t_W\} \\
& -(1-\alpha_W)\Big[Q_{i\{(k-1)M_i+1\}} + Q_{i\{(k-1)M_i+2\}} + \dots + Q_{i\{(k-1)M_i+m\}}\Big]t_W \\
& -\Big[(m-1)Q_{i\{(k-1)M_i+1\}} + (m-2)Q_{i\{(k-1)M_i+2\}} + \dots + 1.Q_{i\{(k-1)M_i+m-1\}}\Big]L_i
\end{aligned}
$$

$$
\begin{aligned}
IPW_2 \; = \; & \text{Interest to be paid due to the stock units during} \\
& [(k-1)M_iL_i + t_S, kM_iL_i] \\
= \; & Q_1\{(n+1)L_i - t_S\} + \Big[Q_1 - (1-\alpha_W)Q_{i\{(k-1)M_i+n+2\}}\Big]t_W \\
& +\Big[Q_1 - Q_{i\{(k-1)M_i+n+2\}}\Big](L_i - t_W) \\
& +\Big[Q_1 - Q_{i\{(k-1)M_i+n+2\}} - (1-\alpha_W)Q_{i\{(k-1)M_i+n+3\}}\Big]t_W \\
& +\Big[Q_1 - Q_{i\{(k-1)M_i+n+2\}} - Q_{i\{(k-1)M_i+n+3\}}\Big](L_i - t_W) + \dots \\
& +\Big[Q_1 - Q_{i\{(k-1)M_i+n+2\}} - \dots - Q_{i\{(k-1)M_i+M_i-1\}} - (1-\alpha_W)Q_{i\{(k-1)M_i+M_i\}}\Big]t_W
\end{aligned}
$$

$$
\begin{aligned}
= \; & Q_1\Big[(M_i-1)L_i + t_W - t_S\Big] \\
& -\Big[(M_i-n-2)Q_{i\{(k-1)M_i+n+2\}} + (M_i-n-3)Q_{i\{(k-1)M_i+n+3\}} + \dots + 1.Q_{i\{kM_i-1\}}\Big]L_i \\
& -(1-\alpha_W)\Big[Q_{i\{(k-1)M_i+n+2\}} + Q_{i\{(k-1)M_i+n+3\}} + \dots + Q_{i\{(k-1)M_i+M_i\}}\Big]t_W \qquad (4.204)
\end{aligned}
$$

Wholesaler will earn interest, if $m-1 < n$ and it should be zero, if $m-1 = n$. The wholesaler's IBL will be made at time $\{(k-1)M_i + m - 1\}L_i + t_W$, i.e., $(k-1)M_iL_i+(m-1)L_i+t_W$. Thus, the wholesaler will earn interest on the payments of the retailer during the time $[\{(k-1)M_i + m\}L_i, (k-1)M_iL_i + t_S]$.

$$
\begin{aligned}
IEW_1 \; = \; & \text{Interest earned due to the instant payment of the retailer for the sold} \\
& \text{units during } [\{(k-1)M_i + m\}L_i, (k-1)M_iL_i + t_S] \\
= \; & (1-\alpha_W)\Big[Q_{i\{(k-1)M_i+m+1\}}(t_S - mL_i) + Q_{i\{(k-1)M_i+m+2\}}\{t_S - (m+1)L_i\} \\
& + \dots + Q_{i\{(k-1)M_i+n+1\}}(t_S - nL_i)\Big] \qquad (4.205)
\end{aligned}
$$

$$\begin{aligned} IEW_2 &= \text{Interest earned due to the repayment of the retailer for the sold units} \\ &\quad \text{during } [\{(k-1)M_i + m\}L_i, (k-1)M_iL_i + t_S] \\ &= \alpha_W \Big[Q_{i\{(k-1)M_i+m+1\}}(t_S - mL_i - t_W) \\ &\quad + Q_{i\{(k-1)M_i+m+2\}}\{t_S - (m+1)L_i - t_W\} \\ &\quad + \ldots + Q_{i\{(k-1)M_i+n+1\}}(t_S - nL_i - t_W) \Big] \end{aligned} \tag{4.206}$$

Case-1.1.2: $m - 1 > n$

The wholesaler will repay the IBL after the time $(k-1)M_iL_i + t_S$. At this time, the wholesaler has to pay the credit amount offered by the supplier. So, the wholesaler must take another loan to repay the credit amount. Also, there is no opportunity for earning interest in this situation.

$$\begin{aligned} & IPW_1 = \text{Interest to be paid due to the IBL up to } t_S \\ &= \Big[IBL_W - (1-\alpha_W)Q_{i\{(k-1)M_i+1\}}\Big]t_W + \Big[IBL_W - Q_{i\{(k-1)M_i+1\}}\Big](L_i - t_W) \\ &+ \Big[IBL_W - Q_{i\{(k-1)M_i+1\}} - (1-\alpha_W)Q_{i\{(k-1)M_i+2\}}\Big]t_W \\ &+ \Big[IBL_W - Q_{i\{(k-1)M_i+1\}} - Q_{i\{(k-1)M_i+2\}}\Big](L_i - t_W) + \ldots \\ &+ \Big[IBL_W - Q_{i\{(k-1)M_i+1\}} - Q_{i\{(k-1)M_i+2\}} - \ldots - Q_{i\{(k-1)M_i+n\}} \\ &- (1-\alpha_W)Q_{i\{(k-1)M_i+n+1\}}\Big]t_W \\ &+ \Big[IBL_W - Q_{i\{(k-1)M_i+1\}} - Q_{i\{(k-1)M_i+2\}} - \ldots - Q_{i\{(k-1)M_i+n+1\}}\Big](t_S - nL_i - t_W) \end{aligned}$$

$$\begin{aligned} &= IBL \times t_S - \Big[Q_{i\{(k-1)M_i+1\}} + Q_{i\{(k-1)M_i+2\}} + \ldots + Q_{i\{(k-1)M_i+n+1\}}\Big](t_S - \alpha_W t_W) \\ &- \Big[1.Q_{i\{(k-1)M_i+2\}} + 2.Q_{i\{(k-1)M_i+3\}} + \ldots + n.Q_{i\{(k-1)M_i+n+1\}}\Big]L_i \end{aligned} \tag{4.207}$$

At the time $(k-1)M_iL_i + t_S$, the amount of the wholesaler's existing initial bank loan and the second loan is equal to the value of the wholesaler's stocked units Q_1. Thus, during $[(k-1)M_iL_i + t_S, kM_iL_i]$, the wholesaler has to pay interest for the stocked units. IPW_2 is given by (4.204).

No interest will be earned in this case. So, $IEW_1 = IEW_2 = 0$.

Case-1.2: $nL_i + t_W > t_S$

Case-1.2.1: $m - 1 < n$

IPW_1 is given by (4.204). At the time $(k-1)M_iL_i + t_S$, the wholesaler has to pay the unpaid amount to the supplier by a loan from a bank. The unpaid amount is

$$\begin{aligned} Q_2 &= QW_{ik} - \Big(Q_{i\{(k-1)M_i+1\}} + Q_{i\{(k-1)M_i+2\}} + \ldots + Q_{i\{(k-1)M_i+n\}}\Big) \\ &\quad -(1-\alpha_W)Q_{i\{(k-1)M_i+n+1\}} \end{aligned} \tag{4.208}$$

$$\begin{aligned} IPW_2 &= \text{Interest to be paid due to the stock units/ credit payments during} \\ &\quad [(k-1)M_iL_i + t_S, kM_iL_i] \\ &= Q_2(nL_i + t_W - t_S) + \Big[Q_2 - \alpha_W Q_{i\{(k-1)M_i+n+1\}}\Big](L_i - t_W) \\ &\quad + \Big[Q_2 - \alpha_W Q_{i\{(k-1)M_i+n+1\}} - (1-\alpha_W)Q_{i\{(k-1)M_i+n+2\}}\Big]t_W \\ &\quad + \Big[Q_2 - \alpha_W Q_{i\{(k-1)M_i+n+1\}} - Q_{i\{(k-1)M_i+n+2\}}\Big](L_i - t_W) + \ldots \\ &\quad + \Big[Q_2 - \alpha_W Q_{i\{(k-1)M_i+n+1\}} - Q_{i\{(k-1)M_i+n+2\}} - \ldots - Q_{i\{(k-1)M_i+M_i-1\}} \\ &\quad -(1-\alpha_W)Q_{i\{(k-1)M_i+M_i\}}\Big]t_W \end{aligned}$$

$$\begin{aligned} &= Q_2(nL_i + t_W - t_S) + \Big[Q_2 - \alpha_W Q_{i\{(k-1)M_i+n+1\}}\Big](M_i - n - 1)L_i \\ &\quad -(1-\alpha_W)\Big[Q_{i\{(k-1)M_i+n+2\}} + Q_{i\{(k-1)M_i+n+3\}} + \ldots + Q_{i\{(k-1)M_i+M_i\}}\Big]t_W \\ &\quad -\Big[(M_i-n-2)Q_{i\{(k-1)M_i+n+2\}} + (M_i-n-3)Q_{i\{(k-1)M_i+n+3\}} + \ldots + 1.Q_{i\{kM_i-1\}}\Big]L_i \end{aligned} \tag{4.209}$$

IEW_1 is given by (4.205).

$$\begin{aligned} IEW_2 &= \text{Interest earned due to the repayment of the retailer for the sold units} \\ &\quad \text{during } [\{(k-1)M_i + m\}L_i, (k-1)M_iL_i + t_S] \\ &= \alpha_W\Big[Q_{i\{(k-1)M_i+m+1\}}(t_S - mL_i - t_W) + Q_{i\{(k-1)M_i+m+2\}}\{t_S - (m+1)L_i - t_W\} \\ &\quad + \ldots + Q_{i\{(k-1)M_i+n\}}\{t_S - (n-1)L_i - t_W\}\Big] \end{aligned} \tag{4.210}$$

Case-1.2.2: $m-1 \geq n$

$$\begin{aligned}
IPW_1 &= \text{Interest to be paid due to the IBL up to } t_S \\
&= \Big[IBL_W - (1-\alpha_W)Q_{i\{(k-1)M_i+1\}}\Big]t_W + \Big[IBL_W - Q_{i\{(k-1)M_i+1\}}\Big](L_i - t_W) \\
&\quad + \Big[IBL_W - Q_{i\{(k-1)M_i+1\}} - (1-\alpha_W)Q_{i\{(k-1)M_i+2\}}\Big]t_W \\
&\quad + \Big[IBL_W - Q_{i\{(k-1)M_i+1\}} - Q_{i\{(k-1)M_i+2\}}\Big](L_i - t_W) + \ldots \\
&\quad + \Big[IBL_W - Q_{i\{(k-1)M_i+1\}} - Q_{i\{(k-1)M_i+2\}} - \ldots - Q_{i\{(k-1)M_i+n\}} \\
&\quad - (1-\alpha_W)Q_{i\{(k-1)M_i+n+1\}}\Big](t_S - nL_i)
\end{aligned}$$

$$\begin{aligned}
&= IBL \times t_S - (1-\alpha_W)Q_{i\{(k-1)M_i+n+1\}}(t_S - nL_i) \\
&\quad - \Big[Q_{i\{(k-1)M_i+1\}} + Q_{i\{(k-1)M_i+2\}} + \ldots + Q_{i\{(k-1)M_i+n\}}\Big](t_S - \alpha_W t_W) \\
&\quad - \Big[1.Q_{i\{(k-1)M_i+2\}} + 2.Q_{i\{(k-1)M_i+3\}} + \ldots + (n-1).Q_{i\{(k-1)M_i+n\}}\Big]L_i \quad (4.211)
\end{aligned}$$

IPW_2 is given by (4.209). No interest will be earned in this case, i.e., $IEW_1 = IEW_2 = 0$.

Case-2: $0 < Q_e < (1-\alpha_W)Q_{i\{(k-1)M_i+m+1\}}$

In this case, initial bank loan due to the instant payment to the supplier will be made at time $\{(k-1)M_i + m\}L_i$, i.e., after time mL_i from the starting point of the k-th wholesaler cycle.

Interest to be paid and the interest earned by the wholesaler are as follows.

$$IPW = I_p c_{pwi} e^{-(k-1)M_i L_i R}(IPW_1 + IPW_2) \quad (4.212)$$

$$IEW = I_e c_{pwi} e^{-(k-1)M_i L_i R}(IEW_1 + IEW_2 + IEW_3) \quad (4.213)$$

where, IPW_1, IPW_2, IEW_1, IEW_2 and IEW_3 are given by the expressions in the following subcases.

Case-2.1: $nL_i + t_W \leq t_S$

Case-2.1.1: $m \le n$

$$\begin{aligned} IPW_1 &= \text{Interest to be paid due to the IBL} \\ &= \Big[IBL_W - (1-\alpha_W)Q_{i\{(k-1)M_i+1\}}\Big]t_W + \Big[IBL_W - Q_{i\{(k-1)M_i+1\}}\Big](L_i - t_W) \\ &\quad + \Big[IBL_W - Q_{i\{(k-1)M_i+1\}} - (1-\alpha_W)Q_{i\{(k-1)M_i+2\}}\Big]t_W \\ &\quad + \Big[IBL_W - Q_{i\{(k-1)M_i+1\}} - Q_{i\{(k-1)M_i+2\}}\Big](L_i - t_W) + \ldots \\ &\quad + \Big[IBL_W - Q_{i\{(k-1)M_i+1\}} - Q_{i\{(k-1)M_i+2\}} - \ldots - Q_{i\{(k-1)M_i+m\}}\Big](L_i - t_W) \end{aligned}$$

$$\begin{aligned} &= IBL \times mL_i + \alpha_W\Big[Q_{i\{(k-1)M_i+1\}} + Q_{i\{(k-1)M_i+2\}} + \ldots + Q_{i\{(k-1)M_i+m\}}\Big]t_W \\ &\quad - \Big[mQ_{i\{(k-1)M_i+1\}} + (m-1)Q_{i\{(k-1)M_i+2\}} + \ldots + 1.Q_{i\{(k-1)M_i+m\}}\Big]L_i \end{aligned} \quad (4.214)$$

IPW_2 is given by (4.204).

$$\begin{aligned} IEW_1 &= \text{Interest earned due to the instant payment of the retailer for the sold} \\ &\quad \text{units during } [\{(k-1)M_i + m + 1\}L_i, (k-1)M_iL_i + t_S] \\ &= (1-\alpha_W)\Big[Q_{i\{(k-1)M_i+m+2\}}\{t_S - (m+1)L_i\} \\ &\quad + Q_{i\{(k-1)M_i+m+3\}}\{t_S - (m+2)L_i\} \\ &\quad + \ldots + Q_{i\{(k-1)M_i+n+1\}}(t_S - nL_i)\Big] \end{aligned} \quad (4.215)$$

IEW_2 is given by (4.206).

$$\begin{aligned} IEW_3 &= \text{Interest earned due to the excess amount after the repayment of the} \\ &\quad \text{IBL at } \{(k-1)M_i + m\}L_i \\ &= \Big[(1-\alpha_W)Q_{i\{(k-1)M_i+m+1\}} - Q_e\Big](t_S - mL_i) \end{aligned} \quad (4.216)$$

Case-2.1.2: $m > n$

IPW_1 and IPW_2 are given by (4.207) and (4.204) respectively. No interest will be earn in this case, i.e., $IEW_1 = IEW_2 = IEW_3 = 0$.

Case-2.2: $nL_i + t_W > t_S$

Case-2.2.1: $m \le n$

In this case, IPW_1, IPW_2, IEW_1, IEW_2 and IEW_3 are given by (4.214), (4.209), (4.215), (4.210) and (4.216) respectively.

Case-2.2.2: $m > n$

In this case, IPW_1 and IPW_2 are given by (4.211) and (4.209) respectively. No interest will be earn in this case, i.e., $IEW_1 = IEW_2 = IEW_3 = 0$.

Case-3: $Q_e = (1-\alpha_W)Q_{i\{(k-1)M_i+m+1\}}$
In this case also initial bank loan will be made at time $\{(k-1)M_i + m\}L_i$, as Case-2. But here no amount will be excess after the repayment of the initial bank loan. All expressions are same as Case-2 except IEW_3. Here, $IEW_3 = 0$, for all the sub-cases.

Case-4: $Q_e > (1-\alpha_W)Q_{i\{(k-1)M_i+m+1\}}$
In this case, initial bank loan due to the instant payment to the supplier will be made at time $\{(k-1)M_i + m\}L_i + t_W$.
Interest to be paid and the interest to be earned by the wholesaler are as follows.

$$IPW = I_p c_{pwi} e^{-(k-1)M_i L_i R}(IPW_1 + IPW_2) \quad (4.217)$$
$$IEW = I_e c_{pwi} e^{-(k-1)M_i L_i R}(IEW_1 + IEW_2 + IEW_3) \quad (4.218)$$

where, IPW_1, IPW_2, IEW_1, IEW_2 and IEW_3 are given by the expressions in the following subcases.
Case-4.1: $nL_i + t_W \le t_S$
Case-4.1.1: $m \le n$

$$\begin{aligned}
IPW_1 &= \text{Interest to be paid due to the IBL}\\
&= \Big[IBL_W - (1-\alpha_W)Q_{i\{(k-1)M_i+1\}}\Big]t_W + \Big[IBL_W - Q_{i\{(k-1)M_i+1\}}\Big](L_i - t_W)\\
&\quad + \Big[IBL_W - Q_{i\{(k-1)M_i+1\}} - (1-\alpha_W)Q_{i\{(k-1)M_i+2\}}\Big]t_W\\
&\quad + \Big[IBL_W - Q_{i\{(k-1)M_i+1\}} - Q_{i\{(k-1)M_i+2\}}\Big](L_i - t_W) + \ldots\\
&\quad + \Big[IBL_W - Q_{i\{(k-1)M_i+1\}} - Q_{i\{(k-1)M_i+2\}} - \ldots - Q_{i\{(k-1)M_i+m\}}\\
&\quad - (1-\alpha_W)Q_{i\{(k-1)M_i+m+1\}}\Big]t_W
\end{aligned}$$

$$\begin{aligned}
&= IBL(mL_i + t_W)\\
&\quad - (1-\alpha_W)\Big[Q_{i\{(k-1)M_i+1\}} + Q_{i\{(k-1)M_i+2\}} + \ldots + Q_{i\{(k-1)M_i+m+1\}}\Big]t_W\\
&\quad - \Big[mQ_{i\{(k-1)M_i+1\}} + (m-1)Q_{i\{(k-1)M_i+2\}} + \ldots + 1.Q_{i\{(k-1)M_i+m\}}\Big]L_i \quad (4.219)
\end{aligned}$$

IPW_2 is given by (4.204) and IEW_1 is given by (4.215).

$$\begin{aligned} IEW_2 &= \text{Interest earned due to the repayment of the retailer for the sold units} \\ &\quad \text{during } [\{(k-1)M_i+m+1\}L_i, (k-1)M_iL_i+t_S] \\ &= \alpha_W\Big[Q_{i\{(k-1)M_i+m+2\}}\{t_S-(m+1)L_i-t_W\} \\ &\quad +Q_{i\{(k-1)M_i+m+3\}}\{t_S-(m+2)L_i-t_W\} \\ &\quad +\ldots+Q_{i\{(k-1)M_i+n+1\}}(t_S-nL_i-t_W)\Big] \end{aligned} \tag{4.220}$$

$$\begin{aligned} IEW_3 &= \text{Interest earned due to the excess amount after the repayment of the} \\ &\quad \text{IBL at } \{(k-1)M_i+m\}L_i+t_W \\ &= \Big[Q_{i\{(k-1)M_i+m+1\}}-Q_e\Big](t_S-mL_i-t_W) \end{aligned} \tag{4.221}$$

Case-4.1.2: $m > n$

IPW_1 and IPW_2 are given by (4.207) and (4.204) respectively. No interest will be earned in this case, i.e., $IEW_1 = IEW_2 = IEW_3 = 0$.

Case-4.2: $nL_i + t_W > t_S$

Case-4.2.1: $m \leq n$

In this case, IPW_1 and IPW_2 are given by (4.219) and (4.209) respectively. IEW_1 is given by (4.215).

$$\begin{aligned} IEW_2 &= \text{Interest earned due to the repayment of the retailer for the sold units} \\ &\quad \text{during } [\{(k-1)M_i+m+1\}L_i, (k-1)M_iL_i+t_S] \\ &= \alpha_W\Big[Q_{i\{(k-1)M_i+m+2\}}\{t_S-(m+1)L_i-t_W\} \\ &\quad +Q_{i\{(k-1)M_i+m+3\}}\{t_S-(m+2)L_i-t_W\} \\ &\quad +\ldots+Q_{i\{(k-1)M_i+n\}}\{t_S-(n-1)L_i-t_W\}\Big] \end{aligned} \tag{4.222}$$

IEW_3 is given by (4.221).

Case-4.2.2: $m > n$

In this case, IPW_1 and IPW_2 are given by (4.211) and (4.209) respectively. No interest will be earned in this case, i.e., $IEW_1 = IEW_2 = IEW_3 = 0$.

4.5.2.3 Promotional Cost

Promotional cost is an important part in any marketing system. In most of the research papers, promotional cost is considered as the function of promotional effort which increases the base demand of the item [97, 150]. But in these studies no proper guideline is outlined about the actual process of the improvement of the demand of an item by any promotional effort and the exact amount of the cost behind this promotional effort. In this study, two promotional efforts are used- one is advertisement and other is price discount. Let us assume that the MRP per unit of the i-th item is s_{pi}. To increase the demand of the item, the retailer sells the product at a discounted price s_{pdi}. Clearly the difference between the sales revenue with discounted price and the sales revenue with the normal price (MRP) is the promotional cost associated with this promotional activity. Again cost of different advertisements is the promotional cost associated with the advertisement related promotional activities. So total promotional cost associated with the promotional activity, PRC, is given by

$$PRC = \sum_{i=1}^{N} \Big[(s_{pi} - s_{pdi})(SR_i + SRL_i) + (AC_i + ACL_i) \Big] \tag{4.223}$$

4.5.2.4 Crisp Model

The total interest earned and the total interest paid by the retailer for the i-th item in the total planning horizon are as follows.

$$TIE_i = \sum_{j=1}^{N_i} TIE_{ij} \text{ and } TIP_i = \sum_{j=1}^{N_i} TIP_{ij}$$

The total profit gained by the retailer through the whole planning horizon is given by

$$\begin{aligned} Z_R = & \sum_{i=1}^{N} \Big[(SR_i + SRL_i) - (PC_i + PCL_i) - (OC_i + OCL_i) - (AC_i + ACL_i) \\ & -(HC_i + HCL_i) + TIE_i - TIP_i \Big] \end{aligned} \tag{4.224}$$

The total profit gained by the wholesaler through the whole planning horizon is given by

$$Z_W = \sum_{i=1}^{N}\Big[SRW_i - PCW_i - OCW_i - HCW_i + TIEW_i - TIPW_i\Big] \quad (4.225)$$

The channel profit of both the retailer and the wholesaler is

$$Z_T = Z_R + Z_W \quad (4.226)$$

If the wholesaler does not share the cost behind the promotional activities of the retailer, then the retailer is the leader and the wholesaler is the follower, i.e., depending upon the decision of the retailer, the wholesaler will determine the marketing decision. It is already mentioned that this scenario is called NCS and so in this scenario the problem of the retailer and the wholesaler can be mathematically represented as:

$$\left.\begin{array}{ll}\text{Retailer determine} & K, t_R, n_i, f_{ri}, m_{kdi};\ \text{for } i = 1, 2, ..., N \\ \text{to maximize} & Z_R\end{array}\right\} \quad (4.227)$$

$$\left.\begin{array}{ll}\text{Wholesaler determine} & M_i;\ \text{for } i = 1, 2, ..., N \\ \text{to maximize} & Z_W\end{array}\right\} \quad (4.228)$$

If the wholesaler shares a compromise portion of the promotional cost for the joint marketing decision, then the scenario is termed as CS. Let us consider that the wholesaler shares F fraction of the promotional cost. So the retailer have gained the same amount. Therefore, the profits of the retailer, the wholesaler and the channel profit are respectively

$$\begin{aligned} Z_R &= \sum_{i=1}^{N}\Big[(SR_i + SRL_i) - (PC_i + PCL_i) - (OC_i + OCL_i) - (AC_i + ACL_i) \\ &\quad -(HC_i + HCL_i) + TIE_i - TIP_i\Big] + F.PRC \quad (4.229) \\ Z_W &= \sum_{i=1}^{N}\Big[SRW_i - PCW_i - OCW_i - HCW_i + TIEW_i - TIPW_i\Big] \\ &\quad -F.PRC \quad (4.230) \\ Z_T &= Z_R + Z_W \quad (4.231) \end{aligned}$$

In this scenario, the retailer and the wholesaler jointly determine the marketing decision and hence the problem mathematically takes the following form:

$$\left.\begin{array}{ll}\text{Determine} & K, t_R, n_i, f_{ri}, m_{kdi}, M_i;\ \text{for } i = 1, 2, ..., N \\ \text{to maximize} & Z_T\end{array}\right\} \tag{4.232}$$

4.5.2.5 Fuzzy Model

It has already been mentioned about the impreciseness of different parameters of any inventory system [113, 115, 118]. Due to the fluctuating world economy different costs involves in any supply chain changes frequently. In this model, the ordering cost of the retailer c_{oi}, the advertisement cost of the retailer c_{ai}, the mark-up of holding cost of the retailer m_{hi}, the ordering cost of the wholesaler c_{owi}, the mark-up of holding cost of the wholesaler m_{hwi} are assumed as the triangular fuzzy numbers (TFNs) [115, 127, 214] $\widetilde{c}_{oi}$, $\widetilde{c}_{ai}$, $\widetilde{m}_{hi}$, $\widetilde{c}_{owi}$, $\widetilde{m}_{hwi}$ respectively, for $i = 1, 2, ..., N$, where $\widetilde{c}_{oi} = (c_{o1i}, c_{o2i}, c_{o3i})$, $\widetilde{c}_{ai} = (c_{a1i}, c_{a2i}, c_{a3i})$, $\widetilde{m}_{hi} = (m_{h1i}, m_{h2i}, m_{h3i})$, $\widetilde{c}_{owi} = (c_{ow1i}, c_{ow2i}, c_{ow3i})$, $\widetilde{m}_{hwi} = (m_{hw1i}, m_{hw2i}, m_{hw3i})$. Therefore, the holding cost of the retailer and the wholesaler are also fuzzy in nature, i.e., $\widetilde{c}_{hi} = \widetilde{m}_{hi}c_{pi}$ and $\widetilde{c}_{hwi} = \widetilde{m}_{hwi}c_{pwi}$. Hence, the profits in both the scenarios become fuzzy in nature. In NCS, the individual profits and the channel profit are represented by

$$\begin{aligned}\widetilde{Z}_R &= \sum_{i=1}^{N}\Big[(SR_i + SRL_i) - (PC_i + PCL_i) - (\widetilde{OC}_i + \widetilde{OCL}_i) - (\widetilde{AC}_i + \widetilde{ACL}_i) \\ &\quad -(\widetilde{HC}_i + \widetilde{HCL}_i) + TIE_i - TIP_i\Big] &(4.233)\\ \widetilde{Z}_W &= \sum_{i=1}^{N}\Big[SRW_i - PCW_i - \widetilde{OCW}_i - \widetilde{HCW}_i + TIEW_i - TIPW_i\Big] &(4.234)\\ \widetilde{Z}_T &= \widetilde{Z}_R + \widetilde{Z}_W &(4.235)\end{aligned}$$

In NCS, since the retailer is the leader and the wholesaler is the follower, so the retailer determines the marketing decision at first. So the mathematical problem in this scenario is as follows:

$$\left.\begin{array}{ll}\text{Retailer determine} & K, t_R, n_i, f_{ri}, m_{kdi};\ \text{for } i = 1, 2, ..., N \\ \text{to maximize} & \widetilde{Z}_R\end{array}\right\} \tag{4.236}$$

$$\left.\begin{array}{ll}\text{Wholesaler determine} & M_i;\ \text{for } i = 1, 2, ..., N \\ \text{to maximize} & \widetilde{Z}_W\end{array}\right\} \tag{4.237}$$

In CS, the individual profits and the channel profit are represented by

$$\widetilde{Z}_R = \sum_{i=1}^{N}\Big[(SR_i + SRL_i) - (PC_i + PCL_i) - (\widetilde{OC}_i + \widetilde{OCL}_i) - (\widetilde{AC}_i + \widetilde{ACL}_i) -(\widetilde{HC}_i + \widetilde{HCL}_i) + TIE_i - TIP_i\Big] + F.\widetilde{PRC} \quad (4.238)$$

$$\widetilde{Z}_W = \sum_{i=1}^{N}\Big[SRW_i - PCW_i - \widetilde{OCW}_i - \widetilde{HCW}_i + TIEW_i - TIPW_i\Big] -F.\widetilde{PRC} \quad (4.239)$$

$$\widetilde{Z}_T = \widetilde{Z}_R + \widetilde{Z}_W \quad (4.240)$$

where,

$$\widetilde{PRC} = \sum_{i=1}^{N}\Big[(s_{pi} - s_{pdi})(SR_i + SRL_i) + (\widetilde{AC}_i + \widetilde{ACL}_i)\Big] \quad (4.241)$$

In CS, the retailer and the wholesaler jointly takes the marketing decision to improve the channel profit as well as individual profit. So in this scenario, the problem mathematically takes the following form:

$$\left.\begin{array}{ll}\text{Determine} & K, t_R, n_i, f_{ri}, m_{kdi}, M_i;\ \text{for } i = 1,2,...,N \\ \text{to maximize} & \widetilde{Z}_T\end{array}\right\} \quad (4.242)$$

As the fuzzy variables are taken as TFNs, the individual profits and the total profit becomes also TFNs as $\widetilde{Z}_R = (Z_{R1}, Z_{R2}, Z_{R3})$, $\widetilde{Z}_W = (Z_{W1}, Z_{W2}, Z_{W3})$ and $\widetilde{Z}_T = (Z_{T1}, Z_{T2}, Z_{T3})$; where

$$Z_{Rj} = \sum_{i=1}^{N}\Big[(SR_i + SRL_i) - (PC_i + PCL_i) - (OC_{(4-j)i} + OCL_{(4-j)i}) -(AC_{(4-j)i} + ACL_{(4-j)i}) - (HC_{(4-j)i} + HCL_{(4-j)i}) + TIE_i - TIP_i\Big] +F.PRC_j \quad (4.243)$$

$$Z_{Wj} = \sum_{i=1}^{N}\Big[SRW_i - PCW_i - OCW_{(4-j)i} - HCW_{(4-j)i} + TIEW_i - TIPW_i\Big] -F.PRC_{4-j} \quad (4.244)$$

$$Z_{Tj} = Z_{Rj} + Z_{Wj} \quad (4.245)$$

These expressions can be used to find TFNs of the profit functions in CS for $j = 1,2,3$. The same expressions can be used in NCS by taking $F = 0$.

4.5.2.6 Rough Model

Another approach of estimation of vague parameters is the use of rough set theory [105]. Some inventory models have already been published following rough estimation of imprecise parameters, like ordering cost, holding cost etc [69, 126, 150]. In this model, the ordering cost of the retailer c_{oi}, the advertisement cost of the retailer c_{ai}, the mark-up of holding cost of the retailer m_{hi}, the ordering cost of the wholesaler c_{owi}, the mark-up of holding cost of the wholesaler m_{hwi} are assumed as the rough numbers [115, 127, 214] $\check{c}_{oi}$, $\check{c}_{ai}$, $\check{m}_{hi}$, $\check{c}_{owi}$, $\check{m}_{hwi}$ respectively, for $i = 1, 2, ..., N$, where $\check{c}_{oi} = ([c_{o1i}, c_{o2i}][c_{o3i}, c_{o4i}])$, $\check{c}_{ai} = ([c_{a1i}, c_{a2i}][c_{a3i}, c_{a4i}])$, $\check{m}_{hi} = ([m_{h1i}, m_{h2i}][m_{h3i}, m_{h4i}])$, $\check{c}_{owi} = ([c_{ow1i}, c_{ow2i}][c_{ow3i}, c_{ow4i}])$, $\check{m}_{hwi} = ([m_{hw1i}, m_{hw2i}][m_{hw3i}, m_{hw4i}])$. Therefore, the holding cost of the retailer and the wholesaler are also rough in nature, i.e., $\check{c}_{hi} = \check{m}_{hi}c_{pi}$ and $\check{c}_{hwi} = \check{m}_{hwi}c_{pwi}$. Hence, the profits in both the scenarios become rough in nature.

In NCS, the individual profits and the channel profit are represented by

$$\begin{aligned} \check{Z}_R &= \sum_{i=1}^{N}\Big[(SR_i + SRL_i) - (PC_i + PCL_i) - (\check{OC}_i + \check{OCL}_i) - (\check{AC}_i + \check{ACL}_i) \\ &\quad -(\check{HC}_i + \check{HCL}_i) + TIE_i - TIP_i\Big] \qquad (4.246) \\ \check{Z}_W &= \sum_{i=1}^{N}\Big[SRW_i - PCW_i - \check{OCW}_i - \check{HCW}_i + TIEW_i - TIPW_i\Big] \qquad (4.247) \\ \check{Z}_T &= \check{Z}_R + \check{Z}_W \qquad (4.248) \end{aligned}$$

In NCS, since the retailer is the leader and the wholesaler is the follower, so the retailer determines the marketing decision at first. So the mathematical problem in this scenario is as follows:

$$\left.\begin{array}{ll} \text{Retailer determine} & K, t_R, n_i, f_{ri}, m_{kdi};\ \text{for } i = 1, 2, ..., N \\ \text{To maximize} & \check{Z}_R \end{array}\right\} \qquad (4.249)$$

$$\left.\begin{array}{ll} \text{Wholesaler determine} & M_i;\ \text{for } i = 1, 2, ..., N \\ \text{To maximize} & \check{Z}_W \end{array}\right\} \qquad (4.250)$$

In CS, the individual profits and the channel profit are represented by

$$\begin{aligned}
\check{Z}_R &= \sum_{i=1}^{N}\Big[(SR_i + SRL_i) - (PC_i + PCL_i) - (\check{OC}_i + \check{OCL}_i) - (\check{AC}_i + \check{ACL}_i) \\
&\quad -(\check{HC}_i + \check{HCL}_i) + TIE_i - TIP_i\Big] + F.\check{PRC} \quad (4.251)\\
\check{Z}_W &= \sum_{i=1}^{N}\Big[SRW_i - PCW_i - \check{OCW}_i - \check{HCW}_i + TIEW_i - TIPW_i\Big] \\
&\quad -F.\check{PRC} \quad (4.252)\\
\check{Z}_T &= \check{Z}_R + \check{Z}_W \quad (4.253)
\end{aligned}$$

where,

$$\check{PRC} = \sum_{i=1}^{N}\Big[(s_{pi} - s_{pdi})(SR_i + SRL_i) + (\check{AC}_i + \check{ACL}_i)\Big] \quad (4.254)$$

In CS, the retailer and the wholesaler jointly takes the marketing decision to improve the channel profit as well as individual profit. So in this scenario, the problem mathematically takes the following form:

$$\left.\begin{array}{ll}\text{Determine} & K, t_R, n_i, f_{ri}, m_{kdi}, M_i;\ \text{for } i = 1, 2, ..., N \\ \text{to maximize} & \check{Z}_T\end{array}\right\} \quad (4.255)$$

For the rough variables, the individual profits and the total profit also becomes rough numbers as $\check{Z}_R = ([Z_{R1}, Z_{R2}][Z_{R3}, Z_{R4}])$, $\check{Z}_W = ([Z_{W1}, Z_{W2}][Z_{W3}, Z_{W4}])$ and $\check{Z}_T = ([Z_{T1}, Z_{T2}][Z_{T3}, Z_{T4}])$; where

$$\begin{aligned}
Z_{Rj} &= \sum_{i=1}^{N}\Big[(SR_i + SRL_i) - (PC_i + PCL_i) - (OC_{(m-j)i} + OCL_{(m-j)i}) \\
&\quad -(AC_{(m-j)i} + ACL_{(m-j)i}) - (HC_{(m-j)i} + HCL_{(m-j)i}) + TIE_i - TIP_i\Big] \\
&\quad +F.PRC_j \quad (4.256)\\
Z_{Wj} &= \sum_{i=1}^{N}\Big[SRW_i - PCW_i - OCW_{(m-j)i} - HCW_{(m-j)i} + TIEW_i - TIPW_i\Big] \\
&\quad -F.PRC_{m-j} \quad (4.257)\\
Z_{Tj} &= Z_{Rj} + Z_{Wj} \quad (4.258)
\end{aligned}$$

These expressions can be used to find rough numbers of the profit functions in CS for $j = 1, 2, 3, 4$; where $m = 3$ for $j = 1, 2$ and $m = 7$ for $j = 3, 4$. The same expressions can be used in NCS by taking $F = 0$.

TABLE 4.25: Input data of Crisp model for Example 4.7 (for $i = 1, 2, 3$) and Example 4.10 (for $i = 1, 2, 3, 4$)

Item (i)	a_i	b_i	c_{pi}	s_{pi}	m_{hi}	c_{oi}	c_{ai}	Q_{di}	c_{pwi}	c_{owi}	m_{hwi}
1	270	0.25	1.65	4.29	0.05	95	14	40	0.90	50	0.03
2	90	0.20	1.35	3.51	0.05	130	10	60	0.72	80	0.03
3	150	0.19	1.45	3.77	0.05	120	11	45	0.78	70	0.03
4	220	0.22	1.55	4.03	0.05	100	12	55	0.82	75	0.03

TABLE 4.26: Results of Crisp model in NCS for Example 4.7

Item (i)	n_i	f_{ri}	K	m_{kdi}	t_R	M_i	Z_R	Z_W	Z_T
1	1	4		0.954		2			
2	2	1	13	0.993	1.17	2	5696.50	756.90	6453.40
3	2	2		0.986		1			

4.5.3 Numerical Illustration and Discussion

The proposed model is discussed with two sets of hypothetical data. The first set of data is considered as Example 4.7 for 3 items (i.e., $i = 1, 2, 3$) and the second set of data is considered as Example 4.10 with adding 4-th item in Example 4.7 (i.e., $i = 1, 2, 3, 4$). In this section, the numerical results in different scenarios for different examples are obtained using MMCABC approach (cf. § 2.2.2.4).

Example 4.7. **(For the crisp model)** In this example, 3 items are considered, i.e., $N = 3$. The input data for different items ($i = 1, 2, 3$) are presented in the first three rows of Table 4.25. Other parametric values are: $R = 0.03$, $I_p = 0.10$, $I_e = 0.08$, $\alpha_S = 0.60$, $\alpha_W = 0.55$, $\alpha_R = 0.42$, $\delta = 2.03$, $\gamma_1 = 0.25$, $\gamma_2 = 0.25$, $H = 30$, $t_S = 3$, $t_W = 2$.

If the wholesaler does not share any part of the promotional cost, then the retailer is the sole decision maker and the wholesaler is the follower. This situation is known as NCS. Optimizing retailer's profit with the above hypothetical data, the best found retailer's profit Z_R and the corresponding values of the decision variables n_i, f_{ri}, m_{kdi} (for $i = 1, 2, ..., N$), K, t_R are tabulated in Table 4.26. The obtained values of the decision variables n_i, f_{ri}, m_{kdi} (for $i = 1, 2, ..., N$), K, t_R of the retailer are taken to optimize wholesaler's profit. The profit amount of the wholesaler Z_W and the corresponding total profit Z_T are presented in Table 4.26. According to the wholesaler's decision, the values of M_i are also presented in Table 4.26.

TABLE 4.27: Values of Z_R and Z_W for different F of Crisp model in CS for Example 4.7

F	Z_R	Z_W	Z_T
0.06	**5619.44**	1161.65	6781.10
0.07	5750.28	1030.82	6781.10
0.08	5880.26	900.84	6781.10
0.09	6010.48	770.62	6781.10
0.10	6141.36	**639.73**	6781.10

Bold face indicates the values of profit less than the NCS

TABLE 4.28: Results of Crisp model in CS for Example 4.7

Item (i)	n_i	f_{ri}	K	m_{kdi}	t_R	M_i	Z_R	Z_W	Z_T
1	1	5		0.705		2			
2	3	1	14	0.760	0.36	1	5880.54	900.55	6781.10
3	1	4		0.703		2			

In CS, a parametric study on F is done and the results are presented in Table 4.27. The suitable range of F for sharing the percentage of promotional cost can be obtained from this table and the suitable range is $(0.07, 0.09)$. Out of this range, the profits of either the retailer or the wholesaler decreases in the CS than the NCS. So, the value of F out of this range is not beneficial for the chain. From Table 4.27, it is found that if $F = 0.06$, then the retailer's profit in CS (5619.44) decreases than that in the NCS (5696.50). Again, if $F = 0.10$, then the wholesaler's profit in CS (639.73) is less than that in the NCS (756.90). Taking $F = 0.08$, the total profit of the retailer and the wholesaler is optimized and the corresponding results are presented in Table 4.28. From this table, it is clear that for $F = 0.08$ the profits of both the parties is far better than the NCS.

If the selling price elasticity (δ) changes, then its effect on the result are shown in Table 4.29. It is observed from the table that the reduced selling price mark-up m_{kdi} and the customers's credit period t_R are decreased with the increase of δ. In fact the increase of δ decreases the market demand of the items. So to keep the demand high m_{kdi} is decreased. Again decrease of m_{kdi} decreases revenue and so t_R is slightly decreased to make a balance between the demand and the revenue. But as expected, in resultant effect, the profit decreases with the increase of δ.

Again it is observed that the frequency of advertisement increases with the increase of γ_1 and the profits of both the retailer and the wholesaler are also increases. The effects on the results due to the increase of γ_1 are shown in Table 4.30.

TABLE 4.29: Parametric study of δ in CS for Example 4.7

δ	Item (i)	n_i	f_{ri}	K	m_{kdi}	t_R	M_i	Z_R	Z_W	Z_T
	1	1	5		0.705		2			
2.03	2	3	1	14	0.760	0.36	1	5880.54	900.55	6781.10
	3	1	4		0.703		2			
	1	1	5		0.701		2			
2.04	2	3	1	14	0.755	0.35	1	5843.85	894.32	6738.17
	3	1	4		0.699		2			
	1	1	5		0.696		2			
2.05	2	3	1	14	0.751	0.33	1	5807.45	888.71	6696.17
	3	1	4		0.694		2			
	1	1	5		0.692		2			
2.06	2	3	1	14	0.747	0.32	1	5772.78	882.27	6655.05
	3	1	4		0.689		2			
	1	1	5		0.687		2			
2.07	2	3	1	14	0.743	0.31	1	5737.54	877.27	6614.81
	3	1	4		0.685		2			

TABLE 4.30: Parametric study of γ_1 in CS for Example 4.7

γ_1	Item (i)	n_i	f_{ri}	K	m_{kdi}	t_R	M_i	Z_R	Z_W	Z_T
	1	1	5		0.705		2			
0.25	2	3	1	14	0.760	0.36	1	5880.54	900.55	6781.10
	3	1	4		0.703		2			
	1	1	6		0.705		2			
0.26	2	3	1	14	0.759	0.36	1	6042.99	948.71	6991.70
	3	1	4		0.703		2			
	1	1	6		0.705		2			
0.27	2	3	1	14	0.759	0.36	1	6235.10	982.41	7217.51
	3	1	4		0.703		2			
	1	1	7		0.703		2			
0.28	2	2	2	14	0.699	0.34	1	6435.40	1027.48	7462.88
	3	1	5		0.701		2			
	1	1	7		0.703		2			
0.29	2	2	2	14	0.699	0.34	1	6669.31	1068.38	7737.68
	3	1	5		0.701		2			

TABLE 4.31: Parametric study of γ_2 in CS for Example 4.7

γ_2	Item (i)	n_i	f_{ri}	K	m_{kdi}	t_R	M_i	Z_R	Z_W	Z_T
	1	1	5		0.705		2			
0.25	2	3	1	14	0.760	0.36	1	5880.54	900.55	6781.10
	3	1	4		0.703		2			
	1	1	5		0.716		2			
0.26	2	3	1	14	0.765	0.44	1	5913.19	911.37	6824.56
	3	1	4		0.714		2			
	1	1	5		0.726		2			
0.27	2	3	1	14	0.771	0.52	1	5952.34	922.82	6875.16
	3	1	4		0.724		2			
	1	1	5		0.736		2			
0.28	2	3	1	14	0.777	0.59	1	5996.85	935.95	6932.80
	3	1	4		0.734		2			
	1	1	5		0.748		2			
0.29	2	3	1	14	0.783	0.68	1	6048.52	948.92	6997.44
	3	1	4		0.746		2			

Results are obtained due to the variation of γ_2 and the obtained results are presented in Table 4.31. It is observed from the table that the retailer gives more credit period (t_R) to the customers with the increase of γ_2 and also the profits of both the parties (i.e., the retailer and the wholesaler) increases with γ_2. Also it is observed that m_{kdi} increases with the increase of γ_2. In fact if γ_2 increases then to take its advantage t_R increases. But increase of t_R decreases the profit to some extent. To make a balance between the profit and the promotional cost, m_{kdi} increases slightly. All these observations agree with reality.

Example 4.8. **(For the fuzzy model)** The input values of fuzzy parameters $\widetilde{c}_{oi}$, $\widetilde{c}_{ai}$, $\widetilde{m}_{hi}$, $\widetilde{c}_{owi}$, $\widetilde{m}_{hwi}$ (for the item $i = 1, 2, 3$) are presented in Table 4.32. All other parametric values are same as in the Example 4.7 for the crisp model.

With similar explanations as in the crisp model, the results in the NCS and CS of the fuzzy model are obtained for the above set of parametric values and are presented in Table 4.33. In this model also same trend of results is obtained as in the crisp model.

Example 4.9. **(For the rough model)** The input values of rough parameters $\check{c}_{oi}$, $\check{c}_{ai}$, $\check{m}_{hi}$, $\check{c}_{owi}$, $\check{m}_{hwi}$ (for the item $i = 1, 2, 3$) are presented in Table 4.32. All other parametric values are same as in the Example 4.7 of crisp model.

TABLE 4.32: Input data of Fuzzy and Rough model for Example 4.8 and Example 4.9 respectively

	Input Variable	Item $i = 1$	Item $i = 2$	Item $i = 3$
Fuzzy	$\widetilde{c}_{oi}$	(93, 95, 97)	(128, 130, 132)	(118, 120, 122)
	$\widetilde{c}_{ai}$	(13.5, 14, 14.5)	(9.5, 10, 10.5)	(10.5, 11, 11.5)
	$\widetilde{m}_{hi}$	(0.048, 0.050, 0.052)	(0.048, 0.050, 0.052)	(0.048, 0.050, 0.052)
	$\widetilde{c}_{owi}$	(49, 50, 51)	(79, 80, 81)	(69, 70, 71)
	$\widetilde{m}_{hwi}$	(0.028, 0.030, 0.032)	(0.028, 0.030, 0.032)	(0.028, 0.030, 0.032)
Rough	$\check{c}_{oi}$	([94,96][93,97])	([129,131][128,132])	([119,121][118,122])
	$\check{c}_{ai}$	([13.5,14.5][13,15])	([9.5,10.5][9,11])	([10.5,11.5][10,12])
	$\check{m}_{hi}$	([0.049,0.051][0.048,0.052])	([0.049,0.051][0.048,0.052])	([0.049,0.051][0.048,0.052])
	$\check{c}_{owi}$	[49.5,50.5][49,51])	([79.5,80.5][79,81])	([69.5,70.5][69,71])
	$\check{m}_{hwi}$	([0.029,0.031][0.028,0.032])	([0.029,0.031][0.028,0.032])	([0.029,0.031][0.028,0.032])

TABLE 4.33: Results of Fuzzy model for Example 4.8

	Item (i)	n_i	f_{ri}	K	m_{kdi}	t_R	M_i	Profit Values
NCS	1	1	4	13	0.954	1.17	2	$\widetilde{Z}_R$ =(5562.02, 5696.50, 5830.98)
	2	2	1		0.993		2	$\widetilde{Z}_W$ =(740.18, 756.90, 773.63)
	3	2	2		0.986		1	$\widetilde{Z}_T$ =(6302.20, 6453.40, 6604.60)
CS	1	1	5	14	0.705	0.36	2	$\widetilde{Z}_R$ =(5688.35, 5880.47, 6072.60)
	2	3	1		0.760		1	$\widetilde{Z}_W$ =(871.34, 900.62, 929.91)
	3	1	4		0.703		2	$\widetilde{Z}_T$ =(6559.68, 6781.10, 7002.51)

TABLE 4.34: Results of Rough model for Example 4.9

	Item (i)	n_i	f_{ri}	K	m_{kdi}	t_R	M_i	Profit Values
NCS	1	1	4	13	0.954	1.17	2	$\check{Z}_R$ =([5592.93,5800.07][5489.36,5903.64])
	2	2	1		0.993		2	$\check{Z}_W$ =([748.10,764.82][739.74,773.18])
	3	2	2		0.986		1	$\check{Z}_T$ =([6341.03,6564.89][6229.10,6676.82])
CS	1	1	5	14	0.700	0.34	2	$\check{Z}_R$ =([5774.35,6078.85][5622.10,6231.10])
	2	2	1		0.700		1	$\check{Z}_W$ =([824.47,863.39][805.00,882.85])
	3	1	4		0.700		2	$\check{Z}_T$ =([6598.81,6942.24][6427.10,7113.95])

With similar explanations as in the crisp model, the results in the NCS and CS of the rough model are obtained for the above set of parametric values and are presented in Table 4.34. In this model also same trend of results is obtained as in the crisp model.

Example 4.10. **(For the crisp model)** In this example, 4 items are considered. The input data for first 3 items are same as in Example 4.7 and the input data for fourth item (i.e., $i = 4$) are presented in Table 4.25. All other parametric values are also same as in Example 4.7.

TABLE 4.35: Results of Crisp model in NCS for Example 4.10

Item (i)	n_i	f_{ri}	K	m_{kdi}	t_R	M_i	Z_R	Z_W	Z_T
1	1	4		0.933		2			
2	2	1	13	0.978	1.00	2	8536.66	1200.25	9736.91
3	2	2		0.972		1			
4	1	4		0.933		2			

TABLE 4.36: Results of Crisp model in CS for Example 4.10

Item (i)	n_i	f_{ri}	K	m_{kdi}	t_R	M_i	Z_R	Z_W	Z_T
1	1	5		0.700		2			
2	3	1	14	0.757	0.32	1	8813.56	1551.54	10365.10
3	1	4		0.697		2			
4	1	5		0.693		2			

The results for this Example 4.10 in NCS and CS of the crisp model are presented in Table 4.35 and Table 4.36 respectively. In this example also, same trend of results is obtained as found in the Example 4.7 for the crisp model.

From all the above illustration, it is clear that the reduced mark-up (m_{kdi}) is less in the CS than the NCS and also the mark-up is less than 1, for all the items, in both the scenarios. So the price discount policy is beneficial for the supply chain when the joint decision is made using the promotional cost sharing. Again for all the items, frequency of advertisement (f_{ri}) are positive in both the scenarios and the frequency in CS is higher than the NCS. So, the advertisement for different items at a regular time interval is also beneficial when the joint decision is made using the promotional cost sharing. Again customers' credit period offered by the retailer (t_R) is positive in both the scenarios, i.e., trade credit is also beneficial for the supply chain. Moreover, it is also established that promotional cost sharing is beneficial for all the parties involved in the chain as all of them can take part in the marketing decision. For some items, it is observed that $n_i > 1$. So, BP policy is beneficial for the retailer in NCS as well as in CS.

4.6 Conclusion

In Tsao's work [184], a supplier-retailer inventory system with multiple items was studied where the supplier provides an interest-free credit period for the retailer to compensate him for making promotional efforts to stimulate the demand for each item. This problem is modeled as a profit maximization problem and is analyzed under two distinct scenarios: the non-coordination scenario and the coordination scenario. Two channel coordination mechanisms are discussed and some theoretical conclusions are drawn. Huang et al. [78] shown in their note that there exist some nontrivial flaws in Tsao's work. They identified and corrected those flaws and derived theoretical conclusions to replace the invalid conclusions in Tsao [184]. But still there were some flaws in Huang et al.'s study. Those are corrected in Model 4.1. Moreover, here, the multi-item supply chain is introduced under two level trade credit policy and promotional cost sharing with budget constraints in crisp as well as in imprecise environments (fuzzy, rough). It is established that if the supplier shares a part of the promotional cost, then the channel profit as well as the individual profits increase. It is also established that the customers' credit period has sufficient significance in a supply chain.

In the Model 4.2, a two level multi-item supply chain is introduced under two level price discount policy and promotional cost sharing. It is established that if the supplier shares a part of the promotional cost, then the channel profit as well as the individual profit increase. It is also established that the price discount given to the customers has sufficient significance in a supply chain. The model can be extended to multi-level price discount policy under multi-level promotional cost sharing in crisp and imprecise environments. At length an approach is proposed for fuzzy optimization problems where no crisp equivalent of the objectives are required to find optimal decision. This approach can be used to solve optimization problems in other discipline also.

In the Model 4.3, a multi-item wholesaler-retailer supply chain is proposed under retailer's two warehouse facility and joint replenishment policy. The wholesaler and the retailer ordered the items under joint replenishment policy. Retailer uses a separate BP policy to transfer the units from the storehouse to the market showroom. Demand is influenced by the inventory level, frequency of advertisement as well as the selling price. The total cost due to the advertisements and the reduced selling price is considered as the promotional cost. The model is analysed in both the NCS and the CS in crisp as well as in imprecise environments. The following concluding remarks can be drawn from the study:

- For price sensitive demand, the price discount is the most attractive promotional approach.

- Advertisement of items in a regular manner is beneficial for any supply chain.
- Promotional cost sharing is beneficial for both the parties in a wholesaler-retailer supply chain.
- Display area of the show room should be properly distributed among the items for the better return.
- Joint replenishment policy is beneficial for both the retailer as well as the wholesaler.
- A heuristic search approach, MMCABC, appropriate for mixed integer optimization problem is proposed and tested. The algorithm is capable of solving constrained/unconstrained optimization problems in crisp as well as in imprecise environments.

In the Model 4.4, a multi-item supplier-wholesaler-retailer-customers supply chain with partial trade credit policy at each level is considered. Here, the planning horizon is fixed and due to the increase of product's price with time in volatile market situation, the inflationary effect is considered. Retailer orders the different items at a regular time interval using BP policy. Demand is influenced by the time, the frequency of advertisement, customers' credit period offered by the retailer as well as the selling price. The total cost due to the advertisements and the reduced selling price is considered as the promotional cost. The model is analysed in both the NCS and the CS in crisp as well as in imprecise environments. The following concluding remarks can be drawn from the study.

- For price sensitive demand, the price discount is the most attractive promotional approach.
- Inflation is also an important part due to the increase of price of the product with time.
- Advertisement of items in a regular manner is beneficial for any supply chain.
- Trade credit period given to the customers by the retailer is also beneficial for any supply chain.
- Promotional cost sharing is beneficial for both the parties as all of them can take part in joint marketing decision.

- Joint replenishment policy is beneficial for the retailer as it significantly reduces the ordering cost and holding cost.

- A heuristic search approach, MMCABC, appropriate for mixed integer optimization problem is proposed and tested. The algorithm is capable of solving constrained/unconstrained optimization problems in crisp as well as in imprecise environments.