DIRECTORATE OF DISTANCE EDUCATION

VIDYASAGAR UNIVERSITY
MIDNAPORE-721 102

M. Sc. in Mathematics

Part - II

Paper - VIII

Module No. 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100

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M.Sc. Course

in

Applied Mathematics with Oceanology

and

Computer Programming

Part - II

Paper - VIII **Group - A**

Module No. 85

Mathematical Methods

STRUCTURE

1.1 Introduction :

Transform Concept :

We define the integral transform $F(\alpha)$ of a function $f(x)$ by integral

$$F(\alpha) = \int_a^b k(\alpha, x) f(x) dx \qquad (1)$$

where $k(\alpha, x)$ being a known function of α and x, called the kernel of the transform and α is the transform parameter. When a and b are both finite we shall speak of $F(\alpha)$ as the *finite integral transform* of $f(x)$. On the other hand, if $a = 0$ or $-\infty$ and $b = \infty$, the transform (1) is called *infinite integral transform*.

Example on an infinite integral transform :

I. Fourier Exponential Transform : $F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$

II. Fourier Sine Transform : $F_s(\alpha) = \sqrt{2/\pi} \int_0^{\infty} f(x) \sin \alpha x \, dx$

III. Fourier Cosine Transform : $F_c(\alpha) = \sqrt{2/\pi} \int_0^{\infty} f(x) \cos \alpha x \, dx$

IV. Laplace Transform : $F(s) = \int_0^{\infty} e^{-st} f(t) \, dt$

V. Hankel Transform : $F_n(\alpha) = \int_0^{\infty} r J_n(\alpha r) f(r) \, dr$

where $J_n(\alpha r)$ being the Bessel function of the first kind of order n.

Example on Finite Integral Transforms :

Hankel Transform : $F_n(\alpha) = \int_0^{a} r J_n(\alpha r) f(r) \, dr$

where $J_n(\alpha r)$ being the Bessel function of the first kind of order n.

1.2 Objectives :

Utility of an Integral Transform :

By the use of Integral Transform ordinary and partial differential equations can be reduced to algebric and ordinary differential equations respectively, which are very easier to solve than solving the original ones. Another importance of Integral transforms is that they provide powerful operational methods for solving initial value problems and initial-boundary value problems for linear differential and integral equations.

Keywords :

Integral Transform, Fourier transform, Fourier sine and cosine transform, Convolution theorem, Parseval's theorem.

1.3 Fourier Transform : (Exponential Fourier Transform)

Definition :

If (i) $f(x)$ and $f'(x)$ are piecewise continuous in the interval $(-\infty, \infty)$

and (ii) $f(x)$ is absolutely integrable in $(-\infty, \infty)$ then

$$F(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{i\alpha x}dx$$

is called the Fourier Transform of $f(x)$.

Explanation : The above integral exists, if $f(x)$ is integrable in any finite interval and the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is absolutely convergent. Since if $f(x)$ is integrable in any finite interval then $f(x)e^{i\alpha x}$ is also integrable in the same interval and if $\int_{-\infty}^{\infty}|f(x)|\,dx$ is convergent, then the integral $\int_{-\infty}^{\infty} f(x)\,e^{i\alpha x}dx$ is also convergent due to the following inequality :

$$\left|\int_{-\infty}^{\infty} f(x)\,e^{i\alpha x}dx\right| \le \int_{-\infty}^{\infty}|f(x)|\left|e^{i\alpha x}\right|dx = \int_{-\infty}^{\infty}|f(x)|\,dx$$

1.4 Some Elementary Properties on Fourier Transform :

(a) . The fourier transform of a function, if exists, is bounded.

Proof : Let $f(\alpha)$ be the fourier transform of a function $f(x)$. Then

$$F(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{i\alpha x}dx$$

Since $F(\alpha)$ exists, the integral $\int_{-\infty}^{\infty}|f(x)|\,dx$ is convergent and hence $\int_{-\infty}^{\infty}|f(x)|\,dx \le \beta$, a positive constant.

Now $|F(\alpha)| \le \left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty}|f(x)|\,\left|e^{i\alpha x}\right|dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|f(x)|\,dx \le \frac{\beta}{\sqrt{2\pi}}$, since $\left|e^{itx}\right|=1$

Hence the fourier transform of $f(x)$ is bounded.

(b) **Fourier Transform is linear** i.e., if $F_1(\alpha)$ and $F_2(\alpha)$ are the fourier transforms of the two functions $f_1(x)$ and $f_2(x)$ respectively, then the fourier transform of $a_1 f_1(x)+a_2 f_2(x)$ is $a_1F_1(\alpha)+a_2F_2(\alpha)$, where a_1 and a_2 are two complex constants. **[Linearity Property]**

Proof : Let us suppose that both $F_1(\alpha)$ and $F_2(\alpha)$ exist, then both $f_1(x)$ and $f_2(x)$ are integrable in any finite interval and the integrals $\int_{-\infty}^{\infty}|f_1(x)|\,dx$ and $\int_{-\infty}^{\infty}|f_2(x)|\,dx$ are convergent. This indicates that the function $a_1f_1(x)+a_2f_2(x)$ is also integrable in any finite interval and

$$\int_{-\infty}^{\infty}|a_1f_1(x)+a_2f_2(x)|\,dx \le |a_1|\int_{-\infty}^{\infty}|f_1(x)|\,dx+|a_2|\int_{-\infty}^{\infty}|f_2(x)|\,dx,$$ a bounded quantity.

Hence fourier transform of $a_1f_1(x)+a_2f_2(x)$ exists and is given by

$$F[a_1f_1(x)+a_2f_2(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[a_1f_1(x)+a_2f_2(x)]\,e^{i\alpha x}dx$$

$$=a_1\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f_1(x)e^{i\alpha x}dx+a_2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f_2(x)\,e^{i\alpha x}dx$$

$$=a_1F_1(\alpha)+a_2F_2(\alpha)$$

Hence fourier transform is linear.

(c) If $F(\alpha)$ is the fourier transform of $f(x)$, then $F(\alpha)e^{i\alpha x}$ is the fourier transform of $f(x-a)$, where 'a' is real. **[Shifting Property]**

Proof : Let us assume that $F(\alpha)$ exists, then $f(x)$ is integrable in any finite interval and the integral $\int_{-\infty}^{\infty}|f(x)|\,dx$ is convergent. This indicates that $f(x-a)$ is also integrable in any finite interval and since

$\int_{-\infty}^{\infty}|f(x-a)|\,dx=\int_{-\infty}^{\infty}|f(t)|\,dt$, where $t=x-a$ and the integral $\int_{-\infty}^{\infty}|f(x-a)|\,dx$ is convergent. Hence fourier transform of $f(x-a)$ exists and is given by

$$F[f(x-a)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-a)\,e^{i\alpha x}dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\alpha(t+a)} dt, \quad t = x - a$$

$$= e^{i\alpha a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\alpha t} dt$$

$$= e^{i\alpha a} F(\alpha)$$

(d) If $F(\alpha)$ is the fourier transform of $f(x)$, then $F(\alpha + a)$ is the fourier transform of $f(x)\, e^{iax}$, where 'a' is real.

Proof : Let us assume $F(\alpha)$ exists. In above property (c) we have seen that the fourier transform of $f(x)\, e^{iax}$ exists and is given by

$$F\{f(x)\, e^{iax}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{iax} e^{i\alpha x} dx = \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} f(x)\, e^{i(\alpha+a)x} dx = F(\alpha + a)$$

Since in the expression, $F(\alpha) = \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x} dx$

Replacing α by $\alpha + a$ in above, we get

$$F(\alpha + a) = \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} f(x)\, e^{i(\alpha+a)x} dx$$

(e) If $F(\alpha)$ is the fourier transform of $f(x)$, then the fourier transform of $f(ax)$ is $\frac{1}{a} F(\alpha/a)$, where 'a' is real and $a \neq 0$. **[change of scale]**.

Proof : Let us assume that $F(\alpha)$ exists, then $f(x)$ is integrable in any finite interval and the integral $\int_{-\infty}^{\infty} |f(x)|\, dx$ is convergent. This indicates that $f(ax)$ is also integrable in any finite interval. Hence fourier transform of $f(ax)$ exists and is given by

$$F[f(ax)] = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} f(ax)\, e^{i\alpha x} dx = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{\alpha}{a}\right)t} \frac{dt}{a}, \quad \text{where } t = ax.$$

$$= \frac{1}{a} \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} f(t)\, e^{i(\alpha/a)t} dt$$

$$= \frac{1}{a} F(\alpha/a)$$

1.5 Continuity and Differentiability of Fourier Transform :

Therem 1 : If the fourier transform $F(\alpha)$ of a function $f(x)$ e: ' ;ts, then $F(\alpha)$ is a continuous function of α.

Proof : Since fourier transform of $f(x)$ exists, $f(x)$ is integrable in any finite interval and the integral $\int_{-\infty}^{\infty}|f(x)|\,dx$ is convergent. We further assume that $f(x)$ is bounded in any finite interval. Since the integral $\int_{-\infty}^{\infty}|f(x)|\,dx$ is convergent, corresponding to any arbitary positive $\in$ there exists a number X (>0) such that

$$\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{-X}|f(x)|\,dx < \in/5 \qquad (1)$$

$$\left(\frac{1}{\sqrt{2\pi}}\right)\int_{X}^{\infty}|f(x)|\,dx < \in/5 \qquad (2)$$

Now $F(\alpha+h)-F(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{i(\alpha+h)x}dx-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\alpha x}dx$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-X}f(x)\,e^{i(\alpha+h)x}dx+\frac{1}{\sqrt{2\pi}}\int_{-X}^{X}f(x)e^{i(\alpha+h)x}dx+\frac{1}{\sqrt{2\pi}}\int_{X}^{\infty}f(x)e^{i(\alpha+h)x}dx$$

$$-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-X}f(x)\,e^{i\alpha x}dx-\frac{1}{\sqrt{2\pi}}\int_{-X}^{X}f(x)e^{i\alpha x}dx-\frac{1}{\sqrt{2\pi}}\int_{-X}^{\infty}f(x)e^{i\alpha x}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-X}f(x)\,e^{i\alpha x}\left(e^{ihx}-1\right)dx+\phi(h+k)-\phi(h)+\frac{1}{\sqrt{2\pi}}\int_{X}^{\infty}f(x)e^{i\alpha x}\left(e^{ihx}-1\right)dx \qquad (3)$$

$$\phi(h)=\frac{1}{\sqrt{2\pi}}\int_{-X}^{X}f(x)\,e^{i\alpha x}dx \quad .$$

Since $f(x)$ is bounded and integrable in the finite interval $(-X, X)$ and $e^{i\alpha x}$ is a continuous function of x and α in the intervals $(-X<x<X)$, $(-\infty<\alpha<\infty)$. $\phi(\alpha)$ is the continuous function of α. Therefore corresponding to the arbitary positive $\in$, which we have already chosen, there exists a positive number δ such that

$$|\phi(\alpha+h)-\phi(\alpha)|<\in/5 \qquad (4) \text{ whenever } |h|<\delta.$$

Therefore from (3) we get,

$$|F(\alpha+h)-F(\alpha)| \le |\phi(\alpha+h)-\phi(\alpha)| + \left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{-X}|f(x)|\,|e^{i\alpha x}|(|e^{ikh}|+1)\,dx$$

$$+ \frac{1}{\sqrt{2\pi}}\int_{X}^{\infty}|f(x)|\,|e^{i\alpha x}|(|e^{ikh}|+1)\,dx$$

$$< \in/5 + 2\in/5 + 2\in/5 = \in \qquad \text{by (1), (2) \& (4) \& } |h|<\delta$$

$$\Rightarrow |F(\alpha+h)-F(\alpha)| < \in \text{ and } |h|<\delta$$

This implies that $F(\alpha)$ is a continuous function of α.

Therem 2 : If the fourier transform of a function $f(x)$ and $xf(x)$ exist, then the derivative of $F(\alpha)$, the fourier transform of $f(x)$, exists and is given by $F'(\alpha) = F[ix f(x)]$.

Generalising the above theorem we get,

$F^m(\alpha) = F[(ix)^m f(x)]$ where $F^m(\alpha)$ means that m times differentiation of $F(\alpha)$, m being any positive integer.

Fourier Transform of Derivatives :

Theorem 1 : If in any finite interval a function $f(x)$ is continuous and its derivative is piecewise continuous, the integrals $\int_{-\infty}^{\infty} f(x)\,dx$ and $\int_{-\infty}^{\infty} f'(x)\,dx$ are absolutely convergent and $f(x)\to 0$ as $|x|\to\infty$, then $F[f'(x)] = -i\alpha\, F(\alpha)$, where $F(\alpha)$ is the fourier transform of $f(x)$.

Theorem 2 : If a function $f(x)$ and its derivatives upto order $(n-1)$ are continuous in any finite interval, its n-th derivative is piecewise continuous in any finite interval, the integrals $\int_{-\infty}^{\infty} f^m(x)\,dx$ are absolutely convergent for $m = 0, 1, 2, \ldots, n$ and $f^m(x)\to 0$ as $|x|\to\infty$ for $m = 0, 1, 2, \ldots, n-1$, then

$F[f^n(x)] = (-i\alpha)^n F(\alpha)$, where $f^n(x)$ means that the n-th derivative of $f(x)$.

Examples :

Ex.-1 : Find fourier transform of $e^{-a|x|}$, $a>0$.

Solu : $F\left[e^{-a|x|}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}e^{i\alpha x}\,dx$ (by the definition of fourier transform) we know that $|x| = \begin{cases} -x, & x < 0 \\ x, & x > 0 \end{cases}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{ax} e^{i\alpha x} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(a+i\alpha)x}}{a+i\alpha} \right]_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-x(a-i\alpha)}}{a-i\alpha} \right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{a+i\alpha} + \frac{1}{a-i\alpha} \right], \text{ since } a > 0.$$

$$= \sqrt{2/\pi} \left[\frac{a}{a^2 + \alpha^2} \right]$$

Ex.-2 : Find the fourier transform of $e^{-a^2x^2}$, $a>0$.

Solu : Now, $F\left[e^{-a^2x^2} \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{i\alpha x} dx$ [by the defination of fourier transform]

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - i\alpha x)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{i\alpha}{2a}\right)^2 - \frac{\alpha^2}{4a^2}} dx$$

$$= e^{-\frac{\alpha^2}{4a^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{i\alpha}{2a}\right)^2} dx$$

$$= e^{-\frac{\alpha^2}{4a^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-p^2} \frac{dp}{a}, \quad p = ax - \frac{i\alpha}{2a}$$

$$= e^{-\frac{\alpha^2}{4a^2}} \frac{2}{a\sqrt{2\pi}} \int_{0}^{\infty} e^{-p^2} dp$$

$$= e^{-\frac{\alpha^2}{4a^2}} \frac{2}{a\sqrt{2\pi}} \int_{0}^{\infty} e^{-r} r^{\frac{1}{2}-1} \frac{dr}{2}, \quad r = p^2$$

$$= e^{-\frac{\alpha^2}{4a^2}} \frac{1}{a\sqrt{2\pi}} \Gamma\left(\tfrac{1}{2}\right)$$

$$= e^{-\frac{\alpha^2}{4a^2}} \frac{1}{a\sqrt{2\pi}} \sqrt{\pi}, \qquad \text{Since } \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{1}{a\sqrt{2}} e^{-\frac{\alpha^2}{4a^2}}$$

Ex.3 : Prove that the fourier transform of $1/x$ is $i\sqrt{\pi/2}\,\text{sgn}(k)$ where $\text{sgn}(k)$ signum function.

Solu :

$$F\{1/x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{x} e^{i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{1}{x} e^{i\alpha x} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{x} e^{i\alpha x} dx$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{x'} e^{i\alpha x'} dx' + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{x} e^{i\alpha x} dx$$

[Put $\alpha = -x'$ in the first Integral]

$$= \left(\frac{1}{\sqrt{2\pi}}\right) \int_0^{\infty} \frac{e^{i\alpha x} - e^{i\alpha x}}{x} dx = \frac{2i}{\sqrt{2\pi}} \int_0^{\infty} \frac{\sin \alpha x}{x} dx$$

$$= \frac{2i}{\sqrt{2\pi}} \text{sgn}(\alpha) \int_0^{\infty} \frac{\sin(|\alpha| x)}{x} dx, \quad (|\alpha| x = y)$$

$$= \frac{2i}{\sqrt{2\pi}} \text{sgn}(\alpha) \int_0^{\infty} \frac{\sin y}{y} dy$$

$$= \frac{2i}{\sqrt{2\pi}} \text{sgn}(\alpha) \pi/2, \quad \text{since} \int_0^{\infty} \frac{\sin y}{y} dy = \pi/2$$

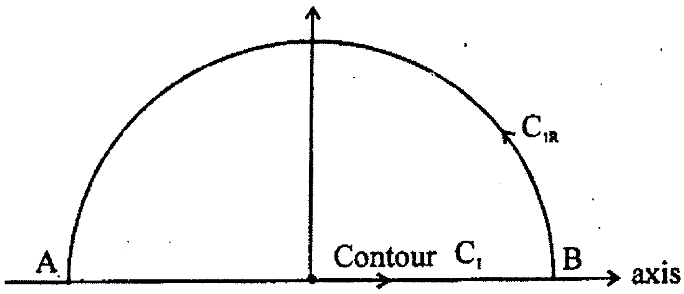
$$= i\sqrt{\pi/2}\ \text{sgn}(\alpha)$$

Ex.- Find Fourier Transform of $\frac{a}{x^2 + a^2}$, $a > 0$.

Solu : Now the fourier transform of $\frac{a}{x^2 + a^2}$ is

$$F\left[\frac{a}{x^2 + a^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a \frac{e^{i\alpha x}}{x^2 + a^2} dx$$

To evaluate this integral we apply the theory of complex variable. We integrate the function $f(z)=\frac{ae^{i\alpha z}}{z^2+a^2}$ of complex variable z around the closed contour C_1 or C_2 according as $\alpha>0$ or <0 in the complex z-plane. C_1 consists of straight line segment AB joining the points $-R$ to R and a semi-circular arc $C_{IR}:|z|=R,\ 0<\arg z<\pi$; C_2 consists of straight line segment CD joining the points R to $-R$ and a semi-circular are $C_{2R}:|z|=R,\ -\pi<\arg z<0$.



(a) For $\alpha>0$. The only singularity of the function $f(z)$ that lies inside C_1 for sufficiently large R is at $z=ia$, which is a simple pole, and the residue of $f(z)$ at this pole is $\frac{1}{2i}e^{-a\alpha'}$.

Therefore by Cauchy's Residue theorem we have

$$\int_{-R}^{R}\frac{ae^{i\alpha x}}{x^2+a^2}\,dx+\int_{0}^{\pi}\frac{ae^{i\alpha R(\cos\theta+i\sin\theta)}}{R^2e^{2i\theta}+a^2}\,i\,\mathrm{Re}^{i\theta}\,d\theta=\pi e^{-a\alpha} \qquad (1)$$

Since on AB, $z=x$, and on C_{IR}, $z=Re^{i\theta}$

$$\text{Now }\left|\int_{C_{IR}}f(z)dz\right|\le\int_{0}^{\pi}\frac{ae^{-\alpha R\sin\theta}}{\left|R^2e^{2i\theta}+a^2\right|}\,d\theta<\int_{0}^{\pi}\frac{aRd\theta}{\left|R^2e^{2i\theta}+a^2\right|}$$

since for $0<\theta<\pi$, $\sin\theta>0$ and therefore $e^{-KR\sin\theta}<1$

$$<\int_{0}^{\pi}\frac{aRd\theta}{R^2}=\frac{\pi a}{R}\to 0\ as\ R\to\infty.$$

$$\therefore\ \int_{C_{IR}}f(z)\,dz\to 0\ as\ R\to\infty.$$

Therefore from (i) proceeding to the limit $R\to\infty$, we get

$$\int_{-\infty}^{\infty}\frac{ae^{i\alpha x}}{x^2+a^2}dx=\pi e^{-a|\alpha|} \qquad (2)$$

(b) For $\alpha<0$,

In this case the only singularity of the function $f(z)$ that lies inside C_2 is at $z=-ia$, which is also a simple pole, and the residue of $f(z)$ at this pole is $\frac{-ae^{-|\alpha|a}}{2ia}$ (since K is negative we can write $K = -|K|$). Therefore by cauchy's residue theorem, we have

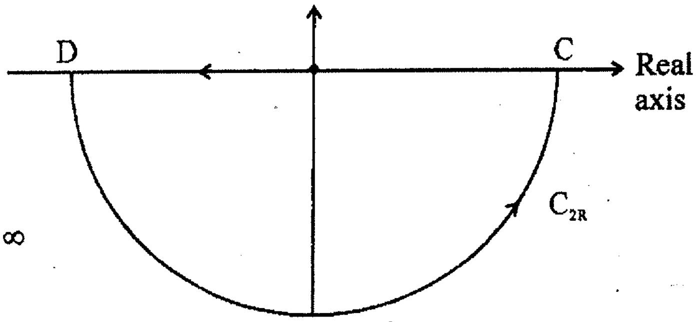
$$-\int_{-R}^{R}\frac{ae^{i\alpha x}}{x^2+a^2}\,dx+\int_{-\pi}^{0}\frac{ae^{i\alpha R(\cos\theta+i\sin\theta)}}{R^2e^{2i\theta}+a^2}\,i\,\mathrm{R}e^{i\theta}\,d\theta=-\pi e^{-a|\alpha|} \qquad (3)$$

The integration of C_{2R} for large R becomes

$$\left|\int_{C_{2R}} f(z)dz\right| = \left|\int_{-\pi}^{0}\frac{ae^{i|\alpha|R(\cos\theta+i\sin\theta)}}{R^2e^{2i\theta}}\,i\,\mathrm{R}e^{i\theta}\,d\theta\right|$$

$$\le \int_{-\pi}^{0}\frac{ae^{|\alpha|R\sin\theta}}{R}\,d\theta,\ Put\ \theta=-\phi.$$

$$=\int_{0}^{\pi}\frac{ae^{-|\alpha|R\sin\phi}}{R}\,d\phi<\frac{a\pi}{R}\to 0\ as\ R\to\infty$$



Contour C_2

Since $e^{-|\alpha|R\sin\phi}<1$, as $\sin\phi>0$ for $0<\phi<\pi$.

Therefore from (3) proceeding to the limit $R\to\infty$, we get

$$\int_{-\infty}^{\infty}\frac{ae^{i\alpha x}}{x^2+a^2}\,dx=\pi e^{-a|\alpha|} \qquad (4)$$

Consequently whether $K>0$ or <0, we have

$$\int_{-\infty}^{\infty}\frac{ae^{i\alpha x}}{x^2+a^2}\,dx=\pi e^{-a|\alpha|} \text{ and therefore } F\left[\frac{a}{x^2+a^2}\right]=\frac{1}{\sqrt{2\pi}}\pi e^{-a|\alpha|}=\sqrt{\frac{\pi}{2}}e^{-a|\alpha|}$$

1.6 Inverse Fourier Transform :

If $F(\alpha)$ is the Fourier Transform of a function $f(x)$, then by inverse Fourier Transform of $F(\alpha)$ we mean a function $G(x)$ of real variable x denoted by $F^{-1}[F(\alpha)]$ and defined by

$$G(x)=F^{-1}[F(\alpha)]=\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty}F(\alpha)\,e^{-i\alpha x}d\alpha$$

If $f(x)$ satisfies certain conditions then $G(x)=f(x)$, at points of continuity of $f(x)$

$$= \frac{1}{2}[f(x+0)+f(x-0)]$$, at points of finite discontinuty of $f(x)$.

Theorem : [Riemann – Lebesgue's Theorem]

If $f(x)$ satisfies Dirichelet's conditions in $-\infty < x < \infty$, the integral $\int_{-\infty}^{\infty}|f(x)|\,dx$ exists and $F(\alpha)$ be the Fourier Transform of $f(x)$, then $F(\alpha) \to 0$ as $|\alpha| \to \infty$, i.e., $\underset{|\alpha| \to \infty}{Lt} F(\alpha) = 0$.

Theorem : [Fourier inversion theorem]

If $f(x)$ satisfies Dirichelet's conditions in $-\infty < x < \infty$ and the integral $\int_{-\infty}^{\infty}|f(x)|\,dx$ exists, then

$\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x} d\alpha = \frac{1}{2}[f(x+0)+f(x-0)]$ where $F(\alpha)$ is the Fourier transform of $f(x)$.

Theorem : [Fourier Integral theorem]

If $f(x)$ satisfis Dirichelet's conditions in $-\infty < x < \infty$ and the integral $\int_{-\infty}^{\infty}|f(x)|\,dx$ exists, then

$$\frac{1}{2}[f(x-0)+f(x+0)] = \left(\frac{1}{\pi}\right)\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} f(t)\cos[\alpha(t-x)]\,dt$$

Proof : From Fourier inversion theorem, we have

$$\frac{1}{2}[f(x+0)+f(x-0)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x} d\alpha = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\alpha\, e^{-i\alpha x}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, f(t)\, e^{i\alpha t}\right)$$

[Using the value of $F(\alpha)$]

$$= \frac{1}{2\pi}\int_{-\infty}^{0} d\alpha \int_{-\infty}^{\infty} dt\, f(t)\, e^{i\alpha(t-x)} + \frac{1}{2\pi}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} dt\, f(t)\, e^{i\alpha(t-x)}$$

$$= \frac{1}{2\pi}\int_0^{\infty} d\alpha' \int_{-\infty}^{\infty} dt\, f(t)\, e^{-i\alpha'(t-x)} + \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} dt\, f(t)\, e^{i\alpha(t-x)}$$ Put $\alpha = -\alpha'$ in first Integral

$$= \frac{1}{2\pi}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} dt\, f(t)\, e^{-i\alpha(t-x)} + \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} dt\, f(t)\, e^{i\alpha(t-x)}$$

$$= \frac{2}{2\pi}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} dt\, f(t)\, \frac{1}{2}\left[e^{i\alpha(t-x)} + e^{-i\alpha(t-x)}\right]$$

$$=\frac{1}{\pi}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} dt\, f(t) \cos\left[\alpha\,(t-x)\right] \quad \text{(Proved)}$$

1.7 Convolution Theorem and Passeval's Theorem for Fourier Transform :

Definition : The function $h(x)$ or $f*g(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy$$

is called the conolution or Faltung of the two functions $f(x)$ and $g(x)$.

The above integral exists if both the functions $f(x)$ and $g(x)$ are integrable in any finite interval and the integrals $\int_{-\infty}^{\infty}|f(x)|\,dx$ and $\int_{-\infty}^{\infty}|\,g(x)\,|\,dx$ exists.

Convolution theorem or Faltung theorem :

If $F(\alpha)$ and $G(\alpha)$ are the fourier transforms of the functions $f(x)$ and $g(x)$, then the product $F(\alpha)$ $G(\alpha)$ is the Fourier Transform of the convolution product $h(x)$ or $f*g$ i.e.,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{i\alpha x}\, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy = F(\alpha)\,G(\alpha).$$

By Fourier inversion theorem the above can be written as

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)\,G(\alpha)\,e^{i\alpha x}d\alpha = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy.$$

Proof : Let $H(\alpha)$ be the Fourier transform of the convolution $h(x)$ of the two functions $f(x)$ and $g(x)$ where

$$h(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$$

therefore, $H(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x)\,e^{i\alpha x}dx$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{i\alpha x}\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy \quad \text{[Using the value of } h(x)\text{]}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} dz\, e^{i\alpha(y+z)}\int_{-\infty}^{\infty} f(z)\,g(y)\,dy \quad \text{[Put } x-y=z,\ dx=dz\text{]}$$

$$= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z)\, e^{i\alpha z} dz \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)\, e^{i\alpha y} dy \right)$$ [Assuming that the Changing the Order of Integration is pcrmissiblc]

$$= F(\alpha)\, G(\alpha)$$

Hence the theorem.

Theorem : PARSEVAL'S IDENTITY :

If $F(\alpha)$ is the Fourier Transform of $f(x)$, then $\int_{-\infty}^{\infty} |F(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} |f(x)|^2 dx$

Proof : The convolution of the two functions $f(x)$ and $g(x)$ is given by

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, g(x-y) dy \qquad (1)$$

If $F(\alpha)$, $G(\alpha)$ and $H(\alpha)$ be the Fourier Transforms of $f(x)$, $g(x)$ and $h(x)$ respectively, then according to the convolution theorem,

$$H(\alpha) = F(\alpha)\, G(\alpha) \qquad (2)$$

By Fourier inversion theorem, we have

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\alpha)\, e^{-i\alpha x} d\alpha \qquad (3)$$

Here assume that both $f(x)$, and $g(x)$ are continuous functions of x and therefore $h(x)$ is also a continuous function of x.

Using (1), (2) the equation (3) becomes,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)\, G(\alpha) e^{-i\alpha x} d\alpha$$

Putting $x = 0$ in above, we get

$$\int_{-\infty}^{\infty} f(y)\, g(-y)\, dy = \int_{-\infty}^{\infty} F(\alpha)\, G(\alpha)\, d\alpha \qquad (4)$$

Again let $g(-y) = \bar{f}(y)$, where the bar indicates complex conjugate of $f(y)$.

Then $G(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(-u)\, e^{-i\alpha u} du$ Put $x = -u$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(u) e^{-i\alpha u} du = \overline{F}(\alpha) \qquad (5)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

which is the complex conjugate of $F(\alpha)$.

Using (5), (4) becomes

$$\int_{-\infty}^{\infty} f(y) \bar{f}(y)\, dy = \int_{-\infty}^{\infty} F(\alpha) \overline{F}(\alpha)\, d\alpha$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(y)|^2 dy = \int_{-\infty}^{\infty} |F(\alpha)|^2 d\alpha$$

Examples on Fourier Inversion Formula and application of convolution theorems :

Ex.- Find the Fourier Transform of the function $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

Hence evaluate $\int_0^{\infty} \frac{\sin x}{x} dx$.

Solu : 1st Part

By the definition of Fourier Transform we have

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1} f(x) e^{i\alpha x} dx + \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} f(x) e^{i\alpha x} dx + \frac{1}{\sqrt{2\pi}} \int_{1}^{\infty} f(x) e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\alpha x}}{i\alpha} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\alpha} - e^{-i\alpha}}{2i} \right] \left(\frac{2}{\alpha} \right)$$

$$= \frac{2}{\alpha} \cdot \frac{1}{\sqrt{2\pi}} \sin \alpha = \sqrt{\frac{2}{\pi}} \frac{\sin \alpha}{\alpha}$$

$$\therefore\ F(\alpha)=\begin{cases}\sqrt{\frac{2}{\pi}}\dfrac{\sin\alpha}{\alpha}, & \alpha>0\\ \sqrt{\frac{2}{\pi}}, & \alpha=0\end{cases}$$

2nd Part :

By Fourier inversion theorem, we get the following at places of continuity of $f(x)$.

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\alpha)\,e^{-i\alpha x}dx=\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{\sin\alpha}{\alpha}e^{-i\alpha x}d\alpha$$

The point $x=0$ being a point of continuity of $f(x)$, putting $x=0$ in the above equation, we get

$$f(0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\alpha}{\alpha}d\alpha=\frac{1}{\pi}\int_{-\infty}^{0}\frac{\sin\alpha}{\alpha}.d\alpha+\frac{1}{\pi}\int_{0}^{\infty}\frac{\sin\alpha}{\alpha}d\alpha$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\frac{\sin\alpha}{\alpha}d\alpha+\frac{1}{\pi}\int_{0}^{\infty}\frac{\sin\alpha}{\alpha}d\alpha$$ [changing α by $-\alpha$ in the Ist integral]

$$=\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\alpha}{\alpha}d\alpha$$

$$\Rightarrow 1=\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\alpha}{\alpha}d\alpha \qquad \text{since } f(0)=1$$

$$\Rightarrow \frac{\pi}{2}=\int_{0}^{\infty}\frac{\sin\alpha}{\alpha}d\alpha \qquad \textbf{(Ans)}$$

Ex.: Find the Fourier Transform of $f(x)=\begin{cases}1-x^2, & |x|\le 1\\ 0, & |x|>1\end{cases}$

Hence evaluate $\displaystyle\int_{0}^{\infty}\frac{x\cos x-\sin x}{x^3}\cos(x/2)dx$.

Solu : (1st Part)

By the definition of Fourier Transform we have

$$F(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{i\alpha x}dx=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\left(1-x^2\right)e^{i\alpha x}dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[(1-x^2)\,\frac{e^{i\alpha x}}{i\alpha}\right]_{-1}^{1}+\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}2x\frac{e^{i\alpha x}}{i\alpha}dx=\frac{2}{\sqrt{2\pi}}\left[\frac{xe^{i\alpha x}}{(i\alpha)^2}-\frac{e^{i\alpha x}}{(i\alpha)^3}\right]_{-1}^{1}$$

$$=\sqrt{\frac{2}{\pi}}\left[\left(\frac{e^{i\alpha}+e^{-i\alpha}}{i^2\alpha^2}\right)+\left(\frac{e^{i\alpha}-e^{-i\alpha}}{i\alpha^3}\right)\right]$$

$$=-\sqrt{\frac{2}{\pi}}\frac{2\cos\alpha}{\alpha^2}+\sqrt{\frac{2}{\pi}}\frac{2}{\alpha^3}\sin\alpha=-\sqrt{\frac{2}{\pi}}\frac{2}{\alpha^3}(\alpha\cos\alpha-\sin\alpha)$$

2nd Part : By Fourier inversion theorem we have the following at places of continuity of $f(x)$.

$$f(x)=\left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}F(\alpha)\,e^{-i\alpha x}d\alpha$$

$$=-\left(1/\sqrt{2\pi}\right)\left(\sqrt{2/\pi}\right)\int_{-\infty}^{\infty}\frac{2}{\alpha^3}(\alpha\cos\alpha-\sin\alpha)e^{-i\alpha x}d\alpha$$

At $x=1/2$, the function $f(x)$ is continuous, so we have

$$f(1/2)=-2/\pi\int_{-\infty}^{\infty}\frac{1}{\alpha^3}(\alpha\cos\alpha-\sin\alpha)e^{-i\alpha/2}d\alpha$$

$$\Rightarrow 1-1/4=-2/\pi\int_{-\infty}^{\infty}\frac{1}{\alpha^3}(\alpha\cos\alpha-\sin\alpha)\{\cos(\alpha/2)-i\sin(\alpha/2)\}\,d\alpha\,.$$

Equating real parts in above we get

$$-\frac{3\pi}{8}=\int_{-\infty}^{\infty}\frac{1}{\alpha^3}(\alpha\cos\alpha-\sin\alpha)\cos(\alpha/2)d\alpha$$

$$\therefore\int_{-\infty}^{\infty}\frac{1}{\alpha^3}(\alpha\cos\alpha-\sin\alpha)\cos(\alpha/2)d\alpha=-\frac{3\pi}{16}.$$ [since the integrand is an even function]

Ex.: Use Parseval's identity to prove that

$$\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)}=\frac{\pi}{ab(a+b)}$$

Solu : We consider two function $\begin{cases} f(x) = e^{-a|x|}, & a > 0 \\ g(x) = e^{-b|x|}, & b > 0 \end{cases}$

Let us denote the Fourier transforms of $f(x)$ & $g(x)$ are $F(\alpha)$ and $G(\alpha)$.

Then $F(\alpha) = \sqrt{2/\pi}\,\dfrac{a}{\alpha^2 + a^2}$, $G(\alpha) = \sqrt{2/\pi}\,\dfrac{b}{\alpha^2 + b^2}$

Here α, a, b are all real so $\bar{f}(x) = f(x)$ & $\bar{g}(x) = g(x)$

Therefore from the Parseval's identity, we get,

$$\int_{-\infty}^{\infty} e^{-a|x|}\, e^{-b|x|} dx = \int_{-\infty}^{\infty} \sqrt{2/\pi}\,\frac{a}{\alpha^2 + a^2} \cdot \sqrt{2/\pi}\,\frac{b}{\alpha^2 + b^2}\, d\alpha$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\left(\alpha^2 + a^2\right)\left(\alpha^2 + b^2\right)}\, d\alpha = \frac{\pi}{2ab} \int_{-\infty}^{\infty} e^{-(a+b)|x|} dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{d\alpha}{\left(\alpha^2 + a^2\right)\left(\alpha^2 + b^2\right)} = \frac{\pi}{2ab}\, 2\int_{0}^{\infty} e^{-(a+b)|x|} dx$$ [As the integrand is an even function]

$$= \frac{\pi}{ab}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^{\infty} = \frac{\pi}{ab(a+b)}$$

$$\therefore \int_{-\infty}^{\infty} \frac{d\alpha}{\left(\alpha^2 + a^2\right)\left(\alpha^2 + b^2\right)} = \frac{\pi}{ab(a+b)}$$

1.8 Fourier Sine Transform and Cosine Transform :

Definition : If a function $f(x)$ is defined in the interval $(0, \infty)$, then its fourier sine transform denoted by $F_S(\alpha)$ defined by the integral $F_S(\alpha) = \sqrt{2/\pi}\int_0^{\infty} f(x)\, \sin(\alpha x)\, dx \rightarrow (1)$, provided the integral exists.

Since $\left|\int_0^{\infty} f(x)\, \sin(\alpha x)\, dx\right| \leq \int_0^{\infty} |f(x)||\sin(\alpha x)| dx \leq \int_0^{\infty} |f(x)| dx$

The integrals (1) exists, if $f(x)$ is integrable in any subinterval of $(0, \infty)$ and the integral $\int_0^{\infty} f(x)\, dx$ is absolutely convergent.

Fourier Cosine Transform :

Definition : If a function $f(x)$ is defined in the interval $(0, \infty)$, then its fourier cosine transform denoted by $F_C(\alpha)$ and is defined by the integral $F_C(\alpha) = \sqrt{2/\pi}\int_0^\infty f(x)\cos(\alpha x)\,dx \quad \rightarrow (2)$, provided the integral exists.

Since $\left|\int_0^\infty f(x)\cos(\alpha x)\,dx\right| \le \int_0^\infty |f(x)||\cos(\alpha x)|dx \le \int_0^\infty |f(x)|dx$

The integrals (2) exists, if $f(x)$ is integrable in any subinterval of $(0, \infty)$ and the integral $\int_0^\infty f(x)\,dx$ is absolutely convergent.

Inversion Formulas for Fourier Sine and Cosine Transform :

For Inversion Formula of Fourier sine transform, let us introduced a function $g(x)$ in $(-\infty, \infty)$ and defined as

$$g(x) = \begin{cases} f(x), & x \ge 0 \\ -f(-x), & x < 0 \end{cases}$$

If Fourier sine transform of $f(x)$ exists, then the Fourier Transform $G(\alpha)$ of $g(x)$ exists and is defined by

$$G(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\,e^{i\alpha x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\{-f(-x)\}\,e^{i\alpha x}dx + \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)\,e^{i\alpha x}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty -f(x)\,e^{-i\alpha x}dx + \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)\,e^{i\alpha x}dx$$

$$= \frac{2i}{\sqrt{2\pi}}\int_0^\infty f(x)\,\frac{e^{i\alpha x} - e^{-i\alpha x}}{2i}\,dx \quad \text{[Putting } x \text{ by } -x \text{ in the 1st Integral]}$$

$$= \frac{1}{\sqrt{2\pi}}2i\int_0^\infty f(x)\sin(\alpha x)\,dx = i\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\alpha x)\,dx = iF_S(\alpha)$$

So if $f(x)$ satisfies Dirichelet's Conditions in $(0, \infty)$ and $\int_0^\infty |f(x)|\,dx$ is convergent, then obviously the function $g(x)$ satisfies the same conditions in $(-\infty, \infty)$ and the integral $\int_{-\infty}^{\infty}|g(x)|\,dx$ is convergent.

Therefore by Fourier inversion theorem we have for $x>0$,

$$\frac{1}{2}[f(x+0)+f(x-0)]=\left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}G(\alpha)\,e^{-i\alpha x}d\alpha$$

$$=\left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}i\,F_S(\alpha)\,e^{-i\alpha x}d\alpha, \qquad \text{since } G(\alpha)=iF_S(\alpha)$$

$$=\left(i/\sqrt{2\pi}\right)\left[\int_{-\infty}^{0}F_S(\alpha)\,e^{-i\alpha x}d\alpha+\int_{0}^{\infty}F_S(\alpha)\,e^{-i\alpha x}d\alpha\right]$$

$$=\left(i/\sqrt{2\pi}\right)\left[\int_{0}^{\infty}F_S(-\alpha)\,e^{i\alpha x}d\alpha+\int_{0}^{\infty}F_S(\alpha)\,e^{-i\alpha x}d\alpha\right] \quad \text{[Replacing } \alpha \text{ by } -\alpha \text{ in the 1st integral]}$$

$$=\left(i/\sqrt{2\pi}\right)\left[\int_{0}^{\infty}-F_S(\alpha)\,e^{i\alpha x}d\alpha+\int_{0}^{\infty}F_S(\alpha)\,e^{-i\alpha x}d\alpha\right] \quad \text{[Since } F_S(-\alpha)=-F_S(\alpha)\text{]}$$

$$=-\left(i/\sqrt{2\pi}\right)\int_{0}^{\infty}2\,i\,F_S(\alpha)\,\sin(\alpha x)\,d\alpha,$$

$$=\sqrt{2/\pi}\int_{0}^{\infty}F_S(\alpha)\,\sin(\alpha x)\,d\alpha$$

So, the inversion formula for Fourier Sine Transform becomes

$$\frac{1}{2}[f(x+0)+f(x-0)]=\sqrt{2/\pi}\int_{0}^{\infty}F_S(\alpha)\,\sin(\alpha x)\,d\alpha$$

When $f(x)$ is continuous, then inversion formula becomes

$$f(x)=\sqrt{2/\pi}\int_{0}^{\infty}F_S(\alpha)\,\sin(\alpha x)\,d\alpha$$

Similarly for inversion formula of Fourier Cosine Transform, let us introduced a function $g(x)$ in $(-\infty, \infty)$ and defined as

$$g(x)=\begin{cases} f(x), & x\geq 0 \\ f(-x), & x\leq 0 \end{cases}$$

Now design the same technique as in Fourier Sine Transform we reach the following inversion formula for Fouricr Cosine transform

$$\frac{1}{2}[f(x+0)+f(x-0)]=\sqrt{2/\pi}\int_0^\infty F_C(\alpha)\cos(\alpha x)\,d\alpha$$

When $f(x)$ is continuous, then inversion formula becomes,

$$f(x)=\sqrt{2/\pi}\int_0^\infty F_C(\alpha)\cos\alpha x\,d\alpha$$

1.9 Parseval's Theorem for Fourier Sine and Cosine Transforms :

Let $f(x)$ and $g(x)$ be two functions both defined in the interval $(0, \infty)$ and let their Fourier Sine and cosine transforms be $F_S(\alpha)$, $G_S(\alpha)$ and $F_C(\alpha)$, $G_C(\alpha)$ respectively.

$$\text{Now }\int_0^\infty F_C(\alpha)\,G_C(\alpha)\cos(\alpha x)\,d\alpha=\int_0^\infty d\alpha\,F_C(\alpha)\cos(\alpha x)\sqrt{2/\pi}\int_0^\infty g(t)\cos(\alpha x)\,dt$$

$$=\sqrt{2/\pi}\int_0^\infty dt\,g(t)\int_0^\infty d\alpha\,F_C(\alpha)\cos(\alpha x)\cos(\alpha t)$$

[Assume that changing order of integration is permissable]

$$=\sqrt{2/\pi}\int_0^\infty dt\,g(t)\int_0^\infty F_C(\alpha)\,\frac{1}{2}[\cos\alpha(x+t)+\cos\alpha(x-t)]\,d\alpha$$

$$=\frac{1}{2}\int_0^\infty g(t)dt\left[\sqrt{2/\pi}\int_0^\infty d\alpha\,F_C(\alpha)\cos[\alpha(x+t)]+\sqrt{2/\pi}\int_0^\infty d\alpha\,F_C(\alpha)\cos[\alpha|x-t|]\right]$$

Using the inversion formula for Fourier Cosine Transform we get,

$$\int_0^\infty F_C(\alpha)\,G_C(\alpha)\cos\alpha x\,d\alpha=\frac{1}{2}\int_0^\infty g(t)\,dt[f(x+t)+f(|x-t|)]$$

Now putting $x = 0$ in above we get

$$\int_0^\infty F_C(\alpha)\,G_C(\alpha)d\alpha=\frac{1}{2}\int_0^\infty g(t)\,dt[f(t)+f(t)]=\int_0^\infty f(t)\,g(t)\,dt \qquad (1)$$

Let $g(t)=\bar{g}(t)$, where the bar indicates Complex Conjugate of $g(t)$.

Then $\overline{G}_C(\alpha) = G_C(\alpha)$ (proof is obvious) & where $\overline{G}_C(\alpha)$ is the Complex conjugate of $G_C(\alpha)$.

From (1) using above, we have

$$\int_0^\infty F_C(\alpha)\,\overline{G}_C(\alpha)\,d\alpha = \int_0^\infty f(t)\,\overline{g}(t)dt \qquad (2)$$

Again let $g(t) = f(t)$, then we get the relations,

$$\int_0^\infty F_C(\alpha)\,\overline{F}_C(\alpha)\,d\alpha = \int_0^\infty f(t)\overline{f}(t)\,dt$$

$$\Rightarrow \int_0^\infty |F_C(\alpha)|^2 d\alpha = \int_0^\infty |f(t)|^2 dt \qquad (3)$$

This relation (3) is known as Parseval's relation for Fourier cosine Transform.

The relation (2) is known as Generalized of Parseval's Relation for Fourier cosine Transform.

In the same Fashion, we obtained the Generalised of Parseval's relation and Parseval's relation for Fourier sine Transform as follows.

$$\Rightarrow \int_0^\infty F_S(\alpha)\,\overline{G_S(\alpha)}\,d\alpha = \int_0^\infty f(t)\,\overline{g(t)}\,dt$$

$$\& \int_0^\infty |F_S(\alpha)|^2 d\alpha = \int_0^\infty |f(t)|^2 dt$$

Example on Inversion Formula and Parseval's Relation :

Ex.: Prove the following by using the inversion formula for Fourier Sine & Cosine Transform.

(a) $\displaystyle\int_0^\infty \frac{\cos(\alpha a)}{\alpha^2+b^2}\,d\alpha = \frac{\pi}{2b}e^{-bx},\quad x>0,\ b>0$ (b) $\displaystyle\int_0^\infty \frac{\alpha\sin(\alpha x)}{\alpha^2+b^2}\,d\alpha = \frac{\pi}{2}e^{-bx},\quad x>0,\ b>0$

Solu. : (a) Let us define the function $f(x) = e^{-bx}$, $x>0,\ b>0$

Now taking Fourier Cosine Transform $f_C(\alpha)$ of $f(x)$, we get

$$F_C(\alpha) = \sqrt{2/\pi}\int_0^\infty e^{-bx}\cos(\alpha x)\,d\alpha = \sqrt{2/\pi}\left[e^{-bx}\frac{(-b\cos(\alpha x)+\alpha\sin(\alpha x))}{b^2+\alpha^2}\right]_0^\infty$$

$$= \sqrt{2/\pi}\,\frac{b}{b^2+\alpha^2}$$

Therefore by inversion formula, we get,

$$f(x)=e^{-bx}=\sqrt{2/\pi}\int_0^\infty F_C(\alpha)\cos(\alpha x)\,d\alpha=\sqrt{2/\pi}\sqrt{2/\pi}\int_0^\infty \frac{b\cos(\alpha x)}{b^2+\alpha^2}\,d\alpha$$

$$\Rightarrow \int_0^\infty \frac{b\cos(\alpha x)}{b^2+\alpha^2}\,d\alpha=\frac{\pi}{2b}e^{-bx},\ x>0 \qquad \text{(Proved)}$$

(b) Let us define the function $f(x)=e^{-bx}$, $x>0$. $b>0$ Now taking fourier sine Transform $F_S(\alpha)$ of $f(x)$ we get

$$F_S(\alpha)=\sqrt{2/\pi}\int_0^\infty e^{-bx}\sin(\alpha x)\,d\alpha=\sqrt{2/\pi}\left[\frac{e^{-\alpha x}(-b\sin bx-\alpha\cos(\alpha x))}{b^2+\alpha^2}\right]_0^\infty$$

$$=\sqrt{2/\pi}\frac{\alpha}{b^2+\alpha^2}$$

Therefore by inversion formula we get,

$$f(x)=e^{-bx}=\sqrt{2/\pi}\int_0^\infty F_S(\alpha)\sin(\alpha x)\,d\alpha=\sqrt{2/\pi}\int_0^\infty\sqrt{2/\pi}\frac{\alpha\sin(\alpha x)}{b^2+\alpha^2}\,d\alpha$$

$$\Rightarrow \int_0^\infty \frac{\alpha\sin(\alpha x)}{\alpha^2+b^2}\,d\alpha=\pi/2\,e^{-bx}. \qquad x>0,\ b>0.$$

Ex.: Find the Fourier Cosine Transform of e^{-at^2}

Solu.: By the definition of Fourier Cosine Transform we have,

$$F_C(\alpha)=\sqrt{2/\pi}\int_0^\infty e^{-at^2}\cos(\alpha t)\,dt \qquad (1)$$

Differentiating with respect to α, we obtain

$$\frac{dF_C(\alpha)}{d\alpha}=-\sqrt{2/\pi}\int_0^\infty te^{-at^2}\sin(\alpha t)\,dt=\frac{1}{2a}\sqrt{2/\pi}\int_0^\infty\sin(\alpha t)\,d\left(e^{-at^2}\right)$$

$$=\frac{1}{2a}\sqrt{2/\pi}\left\{\left[e^{-at^2}\sin(\alpha t)\right]_0^\infty-\alpha\int_0^\infty e^{-at^2}\cos(\alpha t)\,dt\right\}$$

$$= \frac{1}{2a}\sqrt{\frac{2}{\pi}}(-\alpha)\int_0^{\infty} e^{-at^2}\cos(\alpha t)\,dt$$

$$= -\frac{\alpha}{2a}F_C(\alpha), \text{ Using (1)}$$

$$\Rightarrow \frac{dF_C(\alpha)}{F_C(\alpha)} = -\frac{\alpha}{2a}d\alpha$$

Integrating, we get $F_C(\alpha) = Ce^{-\alpha^2/4a}$ [C = Integration Constant]

When $\alpha = 0$, from equation (1), we have

$$F_C(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-at^2}dt = \sqrt{\frac{2}{\pi}}\frac{1}{a}\frac{\sqrt{\pi}}{2} = \frac{1}{a\sqrt{2}}$$

Then from (2) we get by putting $\alpha = 0$,

$$\frac{1}{a\sqrt{2}} = C$$

Hence $F_C(\alpha) = \frac{1}{a\sqrt{2}}e^{-\alpha^2/4a}$

Therefore the Fourier Cosine Transform of e^{-at^2} is $\frac{1}{a\sqrt{2}}e^{-\alpha^2/4a}$.

Ex.: If the Fourier Sine Transform of $f(x)$ is $\frac{\alpha}{1+\alpha^2}$, find $f(x)$.

Solu.: From the definition of Inverse Fourier Sine Transform, we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\alpha}{1+\alpha^2}\sin(\alpha x)\,d\alpha \qquad (1)$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{(\alpha^2+1)-1}{\alpha(1+\alpha^2)}\sin(\alpha x)\,d\alpha$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin(\alpha x)}{\alpha}d\alpha - \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin(\alpha x)}{\alpha(1+\alpha^2)}d\alpha$$

$$= \sqrt{2/\pi} \cdot \pi/2 - \sqrt{2/\pi} \int_0^\infty \frac{\sin(\alpha x)}{\alpha(1+\alpha^2)} d\alpha, \qquad \text{since} \int_0^\infty \frac{\sin(\alpha x)}{\alpha} d\alpha = \pi/2$$

$$f(x) = \sqrt{\pi/2} - \sqrt{2/\pi} \int_0^\infty \frac{\sin(\alpha x)}{\alpha(1+\alpha^2)} d\alpha \qquad (2)$$

Differentiating w. to x, we get,

$$\frac{df(x)}{dx} = -\sqrt{2/\pi} \int_0^\infty \frac{\cos(\alpha x)}{1+\alpha^2} d\alpha \qquad (3)$$

Again differentiating $w.r.$ to x, equation (2)

$$\frac{d^2 f(x)}{dx^2} = +\sqrt{2/\pi} \int_0^\infty \frac{\alpha \sin(\alpha x)}{1+\alpha^2} d\alpha \qquad (4)$$

Subtracting (2) from (4) we get,

$$\frac{d^2 f(x)}{dx^2} - f(x) = -\sqrt{\pi/2} + \sqrt{2/\pi}\left[\int_0^\infty \frac{\sin(\alpha x)}{\alpha(1+\alpha^2)} d\alpha + \int_0^\infty \frac{\sin(\alpha x)}{1+\alpha^2} d\alpha\right]$$

$$= -\sqrt{\pi/2} + \sqrt{2/\pi}\left[\int_0^\infty \frac{(1+\alpha^2)\sin(\alpha x)}{\alpha(1+\alpha^2)} d\alpha\right]$$

$$= -\sqrt{\pi/2} + \sqrt{2/\pi} \int_0^\infty \frac{\sin(\alpha x)}{\alpha} d\alpha$$

$$= -\sqrt{\pi/2} + \sqrt{2/\pi}\, \pi/2$$

$$= 0$$

$$\therefore \frac{d^2 f(x)}{dx^2} - f(x) = 0$$

The solution of above differential equation is

$$f(x) = C_1 e^{+x} + C_2 e^{-x} \qquad (5)$$

where C_1 & C_2 are arbitary constant

Differentiating, $\frac{df(x)}{dx} = C_1 e^{+x} - C_2 e^{-x}$ (6)

Put $x = 0$ in (2), $f(0) = \sqrt{\pi/2}$

and put $x = 0$ in (3), $\frac{df(0)}{dx} = -\sqrt{2/\pi} \int_0^\infty \frac{1}{1+\alpha^2} d\alpha = -\sqrt{2/\pi}\left[+a^{-1}\alpha\right]_0^\infty = \sqrt{2/\pi}\,\pi/2 = -\sqrt{\pi/2}$

Using the above results, by putting $x = 0$ in (5) & (6) we have

$$C_1 + C_2 = \sqrt{\pi/2}$$

& $\quad C_1 - C_2 = -\sqrt{\pi/2}$

$\Rightarrow C_1 = 0,\ C_2 = \sqrt{\pi/2}$

Hence $f(x) = \sqrt{\pi/2}\, e^{-x}$.

Ex.: Use Parseval Relation for Fourier Cosine Transform to evaluate the following integrals :

(a) $\int_0^\infty \frac{dt}{\left(a^2+t^2\right)\left(b^2+t^2\right)} = \frac{\pi}{2ab(a+b)}$

(b) $\int_0^\infty \frac{\sin\lambda t\ \sin\mu t}{t^2} dt = \frac{\pi}{2}\min(\lambda,\ \mu)$

Solu.: We consider the two function

$f(x) = e^{-ax},\ x > 0$, and $g(x) = e^{-bx},\ x > 0$

If we denote the Fourier Cosine Transform of $f(x)$ and $g(x)$ are $F_C(\alpha)$ & $G_C(\alpha)$. Then we easily calculated that

$$F_C(\alpha) = \sqrt{2/\pi}\,\frac{a}{a^2+\alpha^2} \text{ and } G_C(\alpha) = \sqrt{2/\pi}\,\frac{a}{b^2+\alpha^2}$$

Therefore from the generalisation of parseval's relation for Fourier Cosine Transform

$$\int_0^\infty F_C(\alpha)\,\overline{G}_C(\alpha)\,d\alpha = \int_0^\infty f(t)\,\overline{g}(t)\,dt$$

Then we have $\frac{2}{\pi}\int_0^{\infty}\frac{ab}{\left(a^2+\alpha^2\right)\left(b^2+\alpha^2\right)}\,d\alpha=\int_0^{\infty}e^{-ax}\,e^{-bx}dx$

$$\Rightarrow \int_0^{\infty}\frac{d\alpha}{\left(a^2+\alpha^2\right)\left(b^2+\alpha^2\right)}=\frac{\pi}{2ab}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^{\infty}$$

$$\Rightarrow \int_0^{\infty}\frac{d\alpha}{\left(a^2+\alpha^2\right)\left(b^2+\alpha^2\right)}=\frac{\pi}{2(a+b)ab} \qquad \text{(Proved)}$$

(b) Let us consider the two function as follows

$$f(x)=\begin{cases}1, & 0<x<\mu\\ 0, & x\geq\mu\end{cases} \qquad g(x)=\begin{cases}1, & 0<x<\lambda\\ 0, & x\geq\lambda\end{cases}$$

Let us also denoted the Fourier Cosine Transform of $f(x)$ and $g(x)$ are $F_C(\alpha)$ & $G_C(\alpha)$ respectively. Then we can calculate

$$F_C(\alpha)=\sqrt{2/\pi}\int_0^{\infty}f(x)\,\cos(\alpha x)\,dx=\sqrt{2/\pi}\int_0^{\mu}1.\,\cos(\alpha x)\,dx=\sqrt{2/\pi}\left[\frac{\sin(\alpha x)}{\alpha}\right]_0^{\mu}=\sqrt{2/\pi}\,\frac{\sin(\alpha\mu)}{\alpha}$$

Similarly $G_C(\alpha)=\sqrt{2/\pi}\int_0^{\infty}g(x)\,\cos(\alpha x)\,dx$

$$=\sqrt{2/\pi}\,\frac{\sin(\alpha\lambda)}{\alpha}$$

Therefore from the generalisation of Parseval's Relation for Fourier Cosine Transform

$$\int_0^{\infty}F_C(\alpha)\,\overline{G}_C(\alpha)d\alpha=\int_0^{\infty}f(t)\,\overline{g}(t)\,dt$$

$$\Rightarrow \frac{2}{\pi}\int_0^{\infty}\frac{\sin(\alpha\lambda)\,\sin(\alpha\mu)}{\alpha^2}d\alpha=\int_0^{\min(\lambda,\mu)}1.\,dx$$

$$\Rightarrow \int_0^{\infty}\frac{\sin(\alpha\lambda)\,\sin(\alpha\mu)}{\alpha^2}\,d\alpha=\frac{\pi}{2}\min(\lambda,\mu) \qquad \text{(Proved)}$$

1.10 Multiple Fourier Transform :

Let us introduced the Fourier Transform of a function of several variables. The methodology of Fourier Transform of a function of single variable can be extended to functions of several variables. Let $f(x, y)$ be a function of two independent variables *x* and *y*, defined in $(-\infty < x < \infty,\ -\infty < y < \infty)$. Let $\bar{f}(\alpha, y)$ be the Fourier Transform of $f(x, y)$ then $\bar{f}(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x, y)\, e^{i\alpha x} dx \rightarrow (1)$ in this case *y* treated as constant.

Again take the Fourier Transform of $\bar{f}(\alpha, y)$ and denote as $\bar{F}(\alpha, \beta)$ and defined as

$$\bar{F}(\alpha, \beta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\alpha, y)\, e^{i\beta y} dy \rightarrow (2)$$

Using (1), (2) becomes

$$\bar{F}(\alpha, \beta) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, e^{i(\alpha x + \beta y)} dx\, dy \rightarrow (3)$$

The above result is called two demensional Fourier Transform of the function $f(x, y)$ of two variables *x* & *y*. In the same fashion, we can find the inversion formula for two dimensional Fourier Transform. Let us assume that $f(x, y)$ is a continuous function of *x* & *y*, then taking the inversion formula of (1) & (2) we get successively,

$$f(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\alpha, y)\, e^{-i\alpha x} d\alpha \rightarrow (4)$$

& $$\bar{f}(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{F}(\alpha, \beta)\, e^{-i\beta y} d\beta \rightarrow (5)$$

Using (5), (4) becomes

$$f(x, y) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\alpha, \beta)\, e^{-i(\alpha x + \beta y)} d\alpha\, d\beta \rightarrow (6)$$

The above results is called two dimensional inversion formula of Fourier Transform of the function $f(x, y)$ of two variables *x* & *y*.

Generalising the above ideas we can obtained the following. Let $f(x_1, x_2, \ldots, x_n)$ be function of *n* independent variables $x_1, x_2, ..., x_n$. Then n dimensional Fourier Transform of the functions $f(x_1, x_2, ..., x_n)$ is defined by the function.

$$F(\alpha_1, \alpha_2, ..., \alpha_n) = \left(1/\sqrt{2\pi}\right)^n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} f(x_1, x_2, ..., x_n)\, e^{i(\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n)} dx_1\, dx_2 ... dx_n$$

If $f(x_1, x_2, ..., x_n)$ be a continuous function of $x_1, x_2, ..., x_n$ in $(-\infty < x_1 < \infty,\ -\infty < x_2 < \infty,\ ...,\ -\infty < x_n < \infty)$ then the inversion formula for n-dimensional Fourier Transform is given by

$$f(x_1, x_2, ..., x_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} F(\alpha_1, \alpha_2, ..., \alpha_n)\, e^{-i(\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n)} d\alpha_1 d\alpha_2 ... d\alpha_n$$

1.11 Solution of Partial Differential Equations by the help of Fourier Transform :

Using suitable Fourier Transform, the partial differential equation of some problems of Physics can be reduces either to an ordinary differential equation or to an algebraie equation, which are very easier to solve than solving the original ones.

Solution of Diffusion Equation (Heat Equation) :

Ex.: Solve the following heat conduction problem given by

PDE : $\dfrac{\partial u}{\partial t} = K\dfrac{\partial^2 u}{\partial x^2},\ -\infty < x < \infty,\ t > 0$

Subject to BCS : $u(x, t)$ and $u_x(x, t)$ both $\to 0$ as $|x| \to \infty$

IC : $u(x, 0) = f(x),\ -\infty < x < \infty$

Solu.: Let us denote the Fourier Transform of $u(x, t)$ with respect to x by $\bar{u}(\alpha, t)$. Then we can write.

$$F\left[\frac{\partial u}{\partial t}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial u}{\partial t} e^{i\alpha x} dx = \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x, t)\, e^{i\alpha x} dx\right] = \frac{\partial}{\partial t}\bar{u}(\alpha, t) \qquad (1)$$

and $$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial x^2} e^{i\alpha x} dx = \frac{1}{\sqrt{2\pi}}\left[\frac{\partial u}{\partial x} e^{i\alpha x}\right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial u}{\partial x} e^{i\alpha x}(i\alpha) dx$$

$$= 0 - \frac{1}{\sqrt{2\pi}}(i\alpha)\int_{-\infty}^{\infty}\frac{\partial u}{\partial x} e^{i\alpha x} dx \qquad \text{[Using BCS]}$$

$$= -\frac{i\alpha}{\sqrt{2\pi}}\left[u(x, t)\, e^{i\alpha x}\right]_{-\infty}^{\infty} + \frac{i\alpha}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} u(x, t)\, e^{i\alpha x}(i\alpha)dx\right]$$

$$= 0 + (i\alpha)^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t)\, e^{i\alpha x} dx$$

$$= -\alpha^2\, \bar{u}(\alpha, t) \qquad (2)$$

Now taking the Fourier Transform of given PDE i.e., $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$

$$\therefore \quad F\left\{\frac{\partial u}{\partial t}\right\} = F\left\{K\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \quad \frac{d}{dt}\bar{u}(\alpha, t) = K\left[F\left\{\frac{\partial^2 u}{\partial x^2}\right\}\right] \qquad \text{Using (1)}$$

$$\Rightarrow \quad \frac{d}{dt}\bar{u}(\alpha, t) = K\left\{-\alpha^2\bar{u}(\alpha, t)\right\} \qquad \text{Using (2)}$$

$$\Rightarrow \quad \frac{d\,\bar{u}(\alpha, t)}{dt} = -K\alpha^2\bar{u}(\alpha, t)$$

$$\Rightarrow \quad \frac{d\,\bar{u}(\alpha, t)}{\bar{u}(\alpha, t)} = -K\alpha^2 dt$$

The solution of above differential equation is

$$\bar{u}(\alpha, t) = A\, e^{-K\alpha^2 t} \qquad (3) \qquad [\text{A = Integration Constant}]$$

Taking Fourier Transform of IC, i.e., $u(x, 0) = f(x)$

Then $F\{u(x, 0)\} = F\{f(x)\}$

$\Rightarrow \bar{u}(\alpha, 0) = \bar{f}(\alpha)$ (4) where $\bar{f}(\alpha)$ is the Fourier transform of $f(x)$.

Put $t = 0$ in (3) and then using (4) we get

$$\bar{u}(\alpha, 0) = A \Rightarrow \bar{f}(\alpha) = A$$

Putting the value of A in (3) we get

$$\bar{u}(\alpha, t) = \bar{f}(\alpha)e^{-K\alpha^2 t} \qquad (5)$$

Taking inversion formula of Fourier Transform ot the equation (5)

$$u(x,\ t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{u}(\alpha,\ t)\,e^{-i\alpha x}d\alpha=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{f}(\alpha)\,e^{-K\alpha^{2}t}\,e^{-i\alpha x}d\alpha \qquad \text{Using (5)}$$

$$u(x,\ t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{f}(\alpha)\,\bar{g}(\alpha)\,e^{-i\alpha x}d\alpha \qquad (6)$$

Where $\bar{g}(\alpha)=e^{-K\alpha^{2}t}$ then $g(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{g}(\alpha)\,e^{-i\alpha x}d\alpha=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-K\alpha^{2}t}\,e^{-i\alpha x}dx$

$$=\frac{e^{-\left(x^{2}/4Kt\right)}}{\sqrt{2kt}} \quad \text{[which shown earlier]}$$

Now using convolution theorem on Fourier Transform to the *equ*ⁿ (6)

$$u(x,\ t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\alpha)\,g(x-\alpha)\,d\alpha$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\alpha)\,e^{-\left\{(x-\alpha)^{2}/4Kt\right\}}\,d\alpha$$

Note : Since the range of spatial variable is infinite, the fourier exponential transform is used rather than the sine or cosine transform.

Ex.: Solve the heat conduction (Flow of heat in a semi-infinite medium) problem described by

PDE : $\frac{\partial u}{\partial t}=K\frac{\partial^{2}u}{\partial x^{2}},\ 0<x<\infty,\ t>0$

BC : $u(0,\ t)=u_{0},\ t\geq 0$

IC : $u(x,\ 0)=0,\ 0<x<\infty$

$u(x,\ t)$ and $\partial u/\partial x$ both tend to zero as $x\to\infty$.

Solu.: Since u is specified at $x=0$, the fourier sine transform is applicable to this problem. Taking Fourier sine transform of the given PDE and using $\bar{u}_{S}(\alpha,\ t)$ is the Fourier sine transform of $u(x,\ t)$ we have

$$F_{S}\left[\frac{\partial u}{\partial t}\right]=F_{S}\left[K\frac{\partial^{2}u}{\partial x^{2}}\right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u(x,\, t)}{\partial t} \sin(\alpha x)\, dx = k\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin \alpha x\, dx$$

$$\Rightarrow \frac{d}{dt}\left[\sqrt{2/\pi} \int_0^\infty u(x,\, t)\, \sin(\alpha x)\, dx\right] = k\sqrt{2/\pi}\left[\frac{\partial u}{\partial x} \sin \alpha x\Big|_0^\infty - \alpha \int_0^\infty \frac{\partial u}{\partial x} \cos \alpha x\, dx\right]$$

$$\Rightarrow \frac{d\,\bar{u}_S(\alpha,\, t)}{dt} = K\sqrt{2/\pi}\left[0 - \alpha \int_0^\infty \frac{\partial u}{\partial x} \cos \alpha x\, dx\right]$$

$$= -K\alpha \sqrt{2/\pi}\left[u(x,\, t)\, \cos(\alpha x)\Big|_0^\infty - \int_0^\infty u(x,t)\, \sin \alpha x(-\alpha)\, dx\right]$$

$$= -K\alpha \sqrt{2/\pi}(-u_0) - k\alpha^2 \sqrt{2/\pi} \int_0^\infty u(x,\, t)\, \sin(\alpha x)\, dx$$

$$= \sqrt{2/\pi}\; k\alpha\, u_0 - k\alpha^2 \bar{u}_S(\alpha,\, t)$$

$$\Rightarrow \frac{d\,\bar{u}_S(\alpha,\, t)}{dt} + k\alpha^2 \bar{u}_S(\alpha,\, t) = \sqrt{2/\pi}\; k\alpha u_0$$

The solution of above differential equation is

$$\bar{u}_S(\alpha,\, t)e^{-K\alpha^2 t} = \sqrt{2/\pi} k\alpha u_0 \int e^{-K\alpha^2 t} dt$$

$$\Rightarrow \bar{u}_S(\alpha,\, t)e^{-K\alpha^2 t} = \sqrt{2/\pi}\; k\alpha u_0 \left[\frac{e^{-K\alpha^2 t}}{-K\alpha^2}\right] + A \qquad \text{[A = Integration constant]}$$

$$\Rightarrow \bar{u}_S(\alpha,\, t)e^{-K\alpha^2 t} = -\sqrt{2/\pi}\, \frac{u_0}{\alpha} e^{-K\alpha^2 t} + A \qquad (1)$$

Now taking the Fourier sine transform of $u(x, 0)=0$.

We get $\quad \bar{u}_S(\alpha,\, 0) = 0 \qquad (2)$

Put $t = 0$ in (1) and then using (2) we get

$$A = \sqrt{2/\pi}\, \frac{u_0}{\alpha}$$

Putting the value of A in (1) we get,

$$\bar{u}_S(\alpha, t)\, e^{-K\alpha^2 t} = \sqrt{2/\pi}\, \frac{u_0}{\alpha}\left[1 - e^{-K\alpha^2 t}\right] \qquad (3)$$

Taking the inversion formula of Fourier sine transform we get

$$u(\alpha, t) = \sqrt{2/\pi} \int_0^\infty \bar{u}_S(\alpha, t) \sin(\alpha x)\, d\alpha = \sqrt{2/\pi} \int_0^\infty \sqrt{2/\pi}\, \frac{u_0}{\alpha}\left(1 - e^{-K\alpha^2 t}\right) \sin(\alpha x)\, d\alpha \quad \text{[using (3)]}$$

$$\Rightarrow u(x, t) = (2/\pi)\, u_0 \int_0^\infty \frac{\sin(\alpha x)}{\alpha} d\alpha - (2/\pi)\, u_0 \int_0^\infty \frac{\sin(\alpha x)}{\alpha} e^{-K\alpha^2 t} d\alpha, \text{ since} \int_0^\infty \frac{\sin(\alpha x)}{\alpha} d\alpha = \pi/2$$

$$= (2/\pi)\, u_0\, (\pi/2) - (2/\pi)\, u_0 \int_0^\infty \frac{\sin(\alpha x)}{\alpha} e^{-K\alpha^2 t} d\alpha$$

Using the Standard Integral $erf(y) = \left(2/\sqrt{\pi}\right) \int_0^y e^{-u^2} du$

and $\quad (\pi/2) erf(y) = \int_0^\infty e^{-\alpha^2} \frac{\sin(2\alpha y)}{\alpha} d\alpha$

The equation (4) becomes,

$$u(x, t) = u_0 - \frac{2}{\pi} u_0 \left[\frac{\pi}{2} erf\left(\frac{x}{\sqrt{2kt}}\right)\right] = u_0\left[1 - erf\left(\frac{x}{\sqrt{2Kt}}\right)\right]$$

Finally, the solution of the heat conduction problem is

$$u(x, t) = u_0 - \frac{2}{\pi} u_0 u(x, t) = u_0\left[1 - erf\left(\frac{x}{\sqrt{2Kt}}\right)\right]$$

Solution of Laplace Equation :

Ex.- Solve the following boundary value problem in the half place $y > 0$, described by

PDE : $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty,\ y > 0$

BCS : $\quad u(x, 0) = f(x), \quad -\infty < x < \infty$

u is bounded as $y \to \infty$; u and $\dfrac{\partial u}{\partial x}$ both vanish as $|x| \to \infty$.

Solu.: Since x has an infinite range of values, we take the Fourier (exponential) transform of PDE with respect to x, we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial x^2}e^{i\alpha x}dx+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial y^2}e^{i\alpha x}dx=0$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\left[\left(\frac{\partial u}{\partial x}e^{i\alpha x}\right)_{-\infty}^{\infty}-(i\alpha)\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}e^{i\alpha x}dx\right]+\frac{d^2}{dy^2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,\,y)\,e^{i\alpha x}dx\right]=0$$

$$\Rightarrow -\frac{i\alpha}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}e^{i\alpha x}dx+\frac{d^2}{dy^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,\,y)\,e^{i\alpha x}dx=0$$

[since u and $\frac{\partial u}{\partial x}$ both vanish as $|x|\to\alpha$]

$$\Rightarrow -\frac{i\alpha}{\sqrt{2\pi}}\left[\left(u\,e^{i\alpha x}\right)_{-\infty}^{\infty}(i\alpha)\int_{-\infty}^{\infty}u\,e^{i\alpha x}dx\right]+\frac{d^2}{dy^2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,\,y)\,e^{i\alpha x}dx\right]=0$$

[since u and $\frac{\partial u}{\partial x}$ both vanish as $|x|\to\alpha$]

$$\Rightarrow i^2\alpha^2\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,\,y)\,e^{i\alpha x}dx\right)+\frac{d^2}{dy^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,\,y)\,e^{i\alpha x}dx\right)=0$$

$$\Rightarrow -\alpha^2\,\bar{u}(\alpha,\,y)+\frac{d^2}{dy^2}\bar{u}(\alpha,\,y)=0 \qquad (1)$$

where $\bar{u}(\alpha,\,y)$ is the Fourier Transform of $u(x,\,y)$.

The solution of above differential equation (1) is

$$\bar{u}(\alpha,\,y)=A\,e^{-|\alpha|y}+B\,e^{+|\alpha|y} \qquad (1)$$

where A and B are two constants.

Since u must be bounded as $y\to\infty$, $\bar{u}(\alpha,\,y)$ and its Fourier transform should be bounded as $y\to\infty$. That indicates that $B=0$ and consequently the solution $\bar{u}(\alpha,\,y)$ becomes

$$\bar{u}(\alpha,\,y)=Ae^{-|\alpha|y} \qquad (2)$$

Taking Fourier transform of $u\,(x,\,0)=f(x)$ we get

$\bar{u}(\alpha, 0) = \bar{f}(\alpha) \rightarrow (3)$ where $\bar{f}(\alpha)$ is the Fourier transform of $f(x)$

Put $y = 0$ in (2), $\bar{u}(\alpha, 0) = A = \bar{f}(\alpha)$ [using (3)]

Putting the value of A in (2) becomes

$$\bar{u}(\alpha, y) = \bar{f}(\alpha)\, e^{-|\alpha| y} \qquad (4)$$

$$= \bar{f}(\alpha)\, \bar{g}(\alpha) \text{ (Say)} \qquad (5)$$

where $\bar{g}(\alpha) = e^{-|\alpha| y}$

Now taking Fourier inversion formula, we get

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{g}(\alpha)\, e^{-i\alpha x} d\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\alpha| y}\, e^{-i\alpha x} d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{\alpha y - i\alpha x} d\alpha + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\alpha y - i\alpha x} d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} e^{-\alpha y + i\alpha x} d\alpha + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\alpha(y+ix)} d\alpha \right] \quad \text{[Put } \alpha = -\alpha \text{ in 1st Integral]}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-\alpha(y-ix)}}{-(y-ix)} \Bigg|_0^{\infty} + \frac{e^{-\alpha(y+ix)}}{-(y+ix)} \Bigg|_0^{\infty} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{y-ix} + \frac{1}{y+ix} \right] = \frac{2y}{\sqrt{2\pi}(y^2+x^2)} = \sqrt{\frac{2}{\pi}} \frac{y}{y^2+x^2} \qquad (6)$$

Now applying convolution of Fourier transform of equation (5) we get

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \sqrt{\frac{2}{\pi}} \frac{y}{y^2+(x-t)^2} dt \qquad \text{[by (6)]}$$

$$= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{y^2+(x-t)^2} dt \qquad \text{(Answer)}$$

Ex.: Solve the following problem of two dimensional flow of a perfect fluid in a half space, where the fluid is introduced with prescribed velocity through a slit on the boundary.

PDE : $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0, \quad -\infty<x<\infty,\ y\geq 0$

BCS : $\dfrac{\partial\phi}{\partial y}=-f(x),\ |x|<a,\ y=0$

$$=0,\ |x|>a,\ y=0$$

$$\phi(x,\ y)\to 0,\ y\to\infty$$

Also find the solution in the particular case $f(x) = U$.

Solu : Taking the Fourier transform of the given PDE with respect to x, we get

$$F\left[\frac{\partial^2\phi}{\partial x^2}\right]+F\left[\frac{\partial^2\phi}{\partial y^2}\right]=0$$

$$\Rightarrow \frac{d^2\overline{\phi}(\alpha,\ y)}{dy^2}-K^2\overline{\phi}(\alpha,\ y)=0 \qquad (1) \qquad \text{[which is shown earlier example]}$$

where $\overline{\phi}(\alpha,\ y)$ is the Fourier transform of $\phi(x,\ y)$ with respect to x.

Again taking the Fourier transform of the boundary conditions with respect to x we get,

$$\frac{d}{dy}\overline{\phi}(\alpha,\ 0)=-\bar{f}(\alpha) \qquad (2)$$

and $\overline{\phi}(\alpha,\ y)\to 0$ (3) as $y\to\infty$

where $\bar{f}(\alpha)$ is the Fourier Transform of $f(x)$ defined in this problem.

Now the solution of (1) is

$$\overline{\phi}(\alpha,\ y)=Ae^{-|\alpha|y}+Be^{|\alpha|y}$$

Since $\overline{\phi}(\alpha,\ y)\to 0$ as $y\to\infty$ according to the condition (3), then we must have $B = 0$.

$$\therefore\ \overline{\phi}(\alpha,\ y)=Ae^{|\alpha|y} \qquad (4)$$

Differentiating $w\ r$ to y

$$\frac{d\overline{\phi}(\alpha,\ y)}{dy}=-|\alpha|Ae^{-|\alpha|y} \qquad (5)$$

Put $y = 0$ in (5) we get $\frac{d\bar{\phi}(\alpha, 0)}{dy} = -|\alpha| A = -\bar{f}(\alpha)$ [Using (2)]

$$\Rightarrow A = \frac{\bar{f}(\alpha)}{|\alpha|}$$

Hence the equation (4) becoms with the value of A.

$$\bar{\phi}(\alpha, y) = \frac{\bar{f}(\alpha)}{|\alpha|} e^{-|\alpha| y} \qquad (6)$$

Now taking inverse fourier transform of above (6) we get

$$\phi(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\bar{f}(\alpha)}{|\alpha|} e^{-|\alpha| y} e^{-i\alpha x} d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-|\alpha| y} e^{-i\alpha x}}{|\alpha|} d\alpha\, \bar{f}(\alpha)$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty} \frac{e^{-(|\alpha| y + i\alpha x)}}{|\alpha|} d\alpha \int_{-a}^{a} f(x) e^{i\alpha x} dx \qquad (7)$$

where $\bar{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{i\alpha x} dx$

Equation (7) represents the two dimensional flow of a perfect fluid in a half space.

Now when $f(x) = U$ (Const) we have

$$\bar{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} U e^{i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \frac{U}{i\alpha} \left[e^{i\alpha a} - e^{-i\alpha a}\right] = \frac{2U}{\sqrt{2\pi}} \frac{\sin(\alpha a)}{\alpha}$$

Hence the solution (7) in this case becomes.

$$\phi(x, y) = \frac{U}{\pi} \int_{-\infty}^{\infty} \frac{\sin(\alpha a)}{\alpha} \frac{e^{-i\alpha x}}{|\alpha|} e^{-|\alpha| y} d\alpha$$

Solution of wave equation :

Ex.: Find the solution of the following problem of free vibration of a stretched string of infinite length.

PDE : $\frac{\partial^2 u}{\partial x^2} - \frac{1}{C^2} \frac{\partial^2 u}{\partial t^2} = 0, \quad -\infty < x < \infty$

BCS : $u(x, 0) = f(x)$

$$\frac{\partial}{\partial t} u(x, 0) = g(x)$$

u and $\frac{\partial u}{\partial x}$ are both vanish as $|x| \to \infty$.

Solu : Taking the Fourier transform of the given PDE with respect to x, we get

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] - \frac{1}{c^2} F\left[\frac{\partial^2 u}{\partial t^2}\right] = 0$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{i\alpha x} dx - \frac{1}{c^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} e^{i\alpha x} dx = 0$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \left[\frac{\partial u}{\partial x} e^{i\alpha x}\Big|_{-\infty}^{\infty} - (i\alpha) \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{i\alpha x} dx\right] - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{i\alpha x} dx\right] = 0$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} (-i\alpha) \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{i\alpha x} dx - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{i\alpha x} dx\right] = 0$$

$$\Rightarrow (-i\alpha) \frac{1}{\sqrt{2\pi}} \left[u(x, t) e^{i\alpha x}\Big|_{-\infty}^{\infty} - (i\alpha) \int_{-\infty}^{\infty} u(x, t) e^{i\alpha x} dx\right] - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{i\alpha x} dx\right] = 0$$

$$\Rightarrow (-i\alpha)^2 \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{i\alpha x} dx\right) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{i\alpha x} dx\right) = 0$$

$$\Rightarrow c^2\alpha^2 \, \bar{u}(\alpha, t) + \frac{d^2}{dt^2} \bar{u}(\alpha, t) = 0$$

$$\Rightarrow \frac{d^2 \bar{u}(\alpha, t)}{dt^2} + c^2\alpha^2 \, \bar{u}(\alpha, t) = 0$$

where $\bar{u}(\alpha, t)$ is the Fourier transform of $u(x, t)$ $w\, r$ to x,

Taking Fourier transform of the initial condition $u(x, 0) = f(x)$ and $\frac{\partial u(x, 0)}{\partial t} = g(x)$ we get,

$\bar{u}(\alpha, 0) = \bar{f}(\alpha)$ (2) and $\frac{d\,\bar{u}(\alpha, 0)}{dt} = \bar{g}(\alpha)$ (3)

where $\bar{f}(\alpha)$ and $\bar{g}(\alpha)$ are the Fourier transform of $f(x)$ and $g(x)$ respectively.

Now the solution of equation (1) is

$$\bar{u}(\alpha, t) = A\, e^{i\alpha ct} + B\, e^{-i\alpha ct} \tag{4}$$

where A & B are constants

Differentiating $w\ r$ to t to (4) we get

$$\frac{d\,\bar{u}(\alpha, t)}{dt} = (i\alpha c)A\, e^{i\alpha ct} - (i\alpha c)B\, e^{-i\alpha ct} \tag{5}$$

Put $t = 0$ in (4) & (5) we get

$$\bar{u}(\alpha, 0) = A + B = \bar{f}(\alpha) \quad \text{[using (2)]}$$

$$\frac{d\,\bar{u}(\alpha, 0)}{dt} = (i\alpha c)A - (i\alpha c)B = \bar{g}(\alpha) \quad \text{[using (3)]}$$

Solving above, $A = \frac{1}{2}\left[\bar{f}(\alpha) - \frac{i}{\alpha c}\bar{g}(\alpha)\right]$

$$B = \frac{1}{2}\left[\bar{f}(\alpha) - \frac{i}{\alpha c}\bar{g}(\alpha)\right]$$

Putting the values of A & B from above in (4) we get,

$$\bar{u}(\alpha, t) = \frac{1}{2}\left[\bar{f}(\alpha) - \frac{i}{\alpha c}\bar{g}(\alpha)\right] e^{i\alpha ct} + \frac{1}{2}\left[\bar{f}(\alpha) - \frac{i}{2c}\bar{g}(\alpha)\right] e^{-i\alpha ct}$$

$$= \frac{1}{2}\bar{f}(\alpha)\left[e^{i\alpha ct} + e^{-i\alpha ct}\right] - \frac{i}{2\alpha c}\bar{g}(\alpha)\left[e^{i\alpha ct} - e^{-i\alpha ct}\right] \tag{8}$$

Now taking inverse Fourier transform of equation (8), we get

$$u(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{u}(\alpha, t)\, e^{-i\alpha x} d\alpha = \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(\alpha) e^{-i\alpha(x-ct)} d\alpha + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(\alpha) e^{-i\alpha(x+ct)} d\alpha + \right]$$

$$+ \frac{1}{2C}\left[-\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\bar{g}(\alpha)}{\alpha}\left\{e^{-i\alpha(x-ct)} - e^{-i\alpha(x+ct)}\right\} d\alpha\right] \tag{9}$$

Now we have from inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(\alpha)\, e^{-i\alpha x} d\alpha$$

Replacing x by $x-ct$ and $x+ct$ in above respectively, we get

$$f(x-ct)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{f}(\alpha)\,e^{-i\alpha(x-ct)}d\alpha \tag{10}$$

$$\& \quad f(x+ct)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{f}(\alpha)\,e^{-i\alpha(x+ct)}d\alpha \tag{11}$$

Again integrating the following inversion formula we get,

$$g(u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{g}(\alpha)\,e^{-i\alpha u}d\alpha$$

with respect to u between the limits $x-ct$ to $x+ct$, we get

$$\int_{x-ct}^{x+ct} g(u)\,du=\int_{x-ct}^{x+ct} du\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{g}(\alpha)\,e^{-i\alpha u}d\alpha$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\alpha\,\bar{g}(\alpha)\int_{x-ct}^{x+ct}du\,e^{-i\alpha u}$$

[Assuming the changing of order of integration is permissible]

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\alpha\,\bar{g}(\alpha)\left[\frac{e^{-i\alpha u}}{-i\alpha}\right]_{x-ct}^{x+ct}$$

$$=\frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\bar{g}(\alpha)}{\alpha}\left[e^{-i\alpha(x-ct)}-e^{-i\alpha(x+ct)}\right]d\alpha \tag{12}$$

Using (10), (11) & (12), equation (9) becomes

$$u(x,\,t)=\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(u)\,du$$

1.12 Unit Summary :

Many linear boundary value and initial value problems in Applied Mathematics, Physics and Engineering science, can be effectively solved by the help of Fourier Transform, Fourier Cosine and sine transform.

These transforms are very useful for solving differential or integral equations for mainly two reasons, Firstly, these equations are replaced by simple algebric equations, which enables us to find the solution of the transform function. The solution of the given equation is then obtained in the original variables by inverting the transform solution. Secondly, the transform solution combined with the convolution theorem provides an elegant representation of the solution for the boundary value and initial value problems.

1.13. Exercises :

1. Find the Fourier sine transform of $f(x)$, if

$$f(x)=\begin{cases}0, & 0<x<a\\ x, & a\le x\le b\\ 0, & x>b\end{cases}$$

2 Using Parseval's relation for the Fourier cosine transforms of

$$g(x)=e^{-ax},\ f(x)=\begin{cases}1, & 0<x<\lambda\\ 0, & x>\lambda\end{cases}$$

Show that $\int_0^\infty \frac{\sin\lambda\alpha}{\alpha\left(a^2+\alpha^2\right)}d\alpha=\pi/2\left[\frac{1-e^{-a\lambda}}{a^2}\right]$

3. If $a>0$, b is any real or complex, show that

$$\int_{-\infty}^{\infty} e^{-ax^2-2bx}dx=\frac{\sqrt{\pi}}{\sqrt{a}}e^{b^2/a}$$

4. Prove the following relation

$$\int_0^\infty \left|F_S(\alpha)\right|^2 d\alpha=\int_0^\infty \left|f(x)\right|^2 dx$$

5. Using the method of Integral Transform, solve the following potential problem in the semi-infinite strip described by

PDE : $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ 0<x<\infty,\ 0<y<a$

BCS : $u(x,\ 0)=f(x)$

$u(x,\ \infty)=0$

$u(x,\ y)=0,\ 0<y<a,\ 0<x<\infty$

and $\frac{\partial u}{\partial x}$ tends to zero as $x\rightarrow\infty$.

6. Find the temperature u at time t and at a distance x from one end of a semi-infinite rod satisfying the

equation

$$\text{PDE :} \quad \frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty$$

$$\text{BCS :} \quad u(x, 0) = f(x), \ 0 < x < \infty$$

$$u(0, t) = 0$$

$$u, u_x \to 0 \text{ as } x \to 0.$$

7. Compute the displacement $u(x, t)$ of an infinite string using the method of Fourier transform given that the string is initially at rest and that the initial displacement is $f(x)$, $-\infty < x < \infty$

Ans : 1. $F_S(\alpha) = \sqrt{2/\pi}\left[\frac{a\cos(a\alpha) - b\cos(\alpha b)}{\alpha} + \frac{\sin(b\alpha) - \sin(a\alpha)}{\alpha^2}\right]$

5. $u(x, y) = 2/\pi \int_0^\infty f(\zeta)\, d\zeta \int_0^\infty \frac{\sinh(a-y)\alpha}{\sinh a\alpha} \sin(a\zeta)\sin(\alpha x)\, d\alpha$

6. $u(x, t) = \frac{1}{\sqrt{4\pi\lambda t}} \int_0^\infty d\alpha\, f(\alpha)\left[e^{-\frac{(x-\alpha)^2}{4\lambda t}} - e^{-\frac{(x+\alpha)^2}{4\lambda t}}\right]$

7. $u(x, t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$

1.14 References :

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

Part - II

Paper - VIII **Group - A**

Module No. 86
Hankel Transforms

Structure

2.1 Introduction :

The Hankel Transform involving Bessel functions the Kernel arises naturally in the discussion of axisymmetric problems formulated in cylindrical polar co-ordinates. This module deals with the definition and basic operational proporties of the Hankel Transform. A large number of axisymmetric problems in cylindrical polar co-ordinates are solved with the help of the Hankel Transform.

2.2 Objectives :

The Hankel Transform are extremely useful in solving a variety of partial differential equations in cylindrical polar co-ordinates. Also for the PDE when the variables are finite range, we can apply the finite Hankel Transform to solve it.

Keywords :

Hankel Transform, Bessel Function, Finite Hankel Transform, Heaviside Unit Step function.

2.3 Definition of Hankel Transforms :

Hankel Transform of order n of a function $f(r)$, $0 \le r < \infty$, denoted by

$$H_n\{f(r)\} = F_n(\alpha) = \int_0^\infty r f(r) J_n(\alpha r),\ n > -\tfrac{1}{2}$$

where $J_n(\alpha r)$ is the Bessel function of order n and argument αr. Also $J_n(r)$ is defined by

$$J_n(r) = \left(\frac{r}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \left(\frac{r}{2}\right)^{2k}$$

2.4 Inversion Formula of Hankel Transforms :

If the integral $\int_0^\infty f(r)dr$ is absolutely convergent and $f(r)$ is continuous in the neighbourhood of r, then

$$f(r) = \int_0^\infty \alpha\, J_n(\alpha r) F_n(\alpha) d\alpha$$

where $F_n(\alpha)$ is the Hankel Transform of order n of the function $f(r)$.

Theorem : If the integral $\int_0^\infty f(r)\,dr$ is absolutely convergent and $f(r)$ is continuous in the neighbourhood of r, then $$f(r) = \int_0^\infty \alpha\, F_0(\alpha) J_0(\alpha r) d\alpha$$

when $F_0(\alpha)$ is the Hankel Transform of order 0 of the function $f(r)$.

Proof : We can write $f(r) = f\left(\sqrt{x^2+y^2}\right) = g(x, y)$ (say). (1)

where $x = r\cos\theta,\ y = r\sin\theta,\ 0 \le r < \infty,\ 0 \le \theta < 2\pi$.

the two dimensional Fourier transform $\bar{g}(k,l)$ of the function $g(x, y)$ is

$$\bar{g}(k,l)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,y)\,e^{i(kx+ly)}dx\,dy$$

$$=\frac{1}{2\pi}\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi}f(r)\,e^{ir\alpha\cos(\theta-\theta_1)}r\,dr\,d\theta$$

where $\alpha=\sqrt{k^2+l^2}$ and $\theta_1=\tan^{-1}(l/k)$. Here we have taken the line joining (0, 0) and (k, l) in the xy-plane as the initial line.

$$=\frac{1}{2\pi}\int_0^{\infty}r\,f(r)\,2\pi\,J_0(\alpha r)dr,\ Since\int_0^{2\pi}e^{i\lambda\cos\theta}d\theta=2\pi\,J_0(\lambda)$$

$$=\int_0^{\infty}r\,f(r)J_0(\alpha r)dr.$$

Therefore $\bar{g}(k,l)=\int_0^{\infty}rf(r)\,J_0(\alpha r)dr=F_0(\alpha)$ (2)

By fourier inversion theorem we get,

$$g(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{g}(k,l)\,e^{-i(kx+ly)}dk\,dl$$

$$=\frac{1}{2\pi}\int_{\alpha=0}^{\infty}\int_{\phi=0}^{2\pi}F_0(\alpha)\,e^{-i\alpha r\cos(\phi-\phi_1)}\alpha\,d\alpha\,d\phi,\qquad\text{[using (2)]}$$

Put $k=\alpha\cos\phi$, $l=\alpha\sin\phi$, $k^2+l^2=\alpha^2$, $\phi=\tan^{-1}(l/k)$

and $0\le\alpha<\infty$, $0\le\phi<2\pi$

$$=\frac{1}{2\pi}\int_{\alpha=0}^{\infty}\alpha F_0(\alpha)\,d\alpha\int_{\phi=0}^{2\pi}\bar{e}^{i\alpha r\cos(\theta-\theta_1)}d\phi$$

$$=\frac{1}{2\pi}\int_{\alpha=0}^{\infty}\alpha F_0(\alpha)\,d\alpha\,2\pi\,J_0(-\alpha r)\qquad\text{since}\int_0^{2\pi}e^{-i\lambda\cos\theta}d\theta=2\pi\,J_0(-\lambda)$$

$$=\int_0^{\infty}\alpha\,F_0(\alpha)J_0(-\alpha r)d\alpha$$

where the line joining (0, 0) and the point (x, y) in kl-plane has been taken as initial line

$k = \alpha \cos\phi,\ l = \alpha \sin\phi.$

Since by (1), $f(r) = g(x,y)$ and $J_0(-\lambda) = J_0(\lambda)$ the ɔve relation gives,

$$f(r) = \int_0^\infty \alpha F_0(\alpha)\, J_0(\alpha r) d\alpha.$$

which is an inversion formula of Hankel transform.

We can conclude that the Hankel transform has its own inverse

2.5 Hankel Transform of Derivatives :

Basically Hankel transforms of derivatives is needed for solving Physical & Mechanical problems or in broad sense boundary value problems.

Ex. Prove that $H_n\left\{\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{n^2}{r^2} f\right\} = -\alpha^2 F_n(\alpha)$, provided both $(i)\ rf'(r)$ and $(ii)\ rf(r)$ tend to zero as $r \to 0$ and $r \to \infty$.

Solu : Now $H_n\left\{\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{n^2}{r^2} f(r)\right\}$

$$= H_n\left\{\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right\} - H_n\left\{\frac{n^2}{r^2} f(r)\right\}$$

$$= H_n\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) f(r)\right\} - H_n\left\{\frac{n^2}{r^2} f(r)\right\}$$

$$= \int_0^\infty J_n(\alpha r).r.\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) f(r)\right\} dr - \int_0^\infty \frac{n^2}{r^2} f(r) r\, J_n(\alpha r)\, dr$$

$$= \int_0^\infty J_n(\alpha r)\left\{\frac{d}{dr}\left(r\frac{d}{dr}\right)\right\} f(r) dr - \int_0^\infty \frac{n^2}{r} f(r)\, J_n(\alpha r)\, dr$$

Integrating by parts twice and using the conditions (i) & (ii) we find

$$= \left[J_n(\alpha r) r \frac{d}{dr} f(r)\right]_0^\infty - \int_0^\infty \frac{d}{d\alpha}(J_n(\alpha r))\left(r\frac{df(r)}{dr}\right) dr - \int_0^\infty \frac{n^2}{r} f(r) J_n(\alpha r) dr$$

$$= 0 - \int_0^\infty \alpha\, J_0'(\alpha r)\, r \frac{df}{dr}\, dr - \int_0^\infty \frac{n^2}{r} f(r)\, J_n(\alpha r)\, dr \qquad \text{Using (i)}$$

$$= -\left[f(r)\, \alpha r J_n'(\alpha r)\right]_0^\infty + \int_0^\infty f(r) \frac{d}{dr}\left[\alpha r J_n'(\alpha r)\right] dr - \int_0^\infty \frac{n^2}{r} f(r) J_n(\alpha r)\, dr,$$

$$= 0 + \int_0^\infty f(r) \frac{d}{dr}\left[\alpha r J_0'(\alpha r)\right] dr - \int_0^\infty \frac{n^2}{r} f(r)\, J_n(\alpha r)\, dr.$$

$$= \int_0^\infty (-1)\, r \left(\alpha^2 - \frac{n^2}{r^2}\right) J_n(\alpha r) f(r) dr - \int_0^\infty \frac{n^2}{r} f(r) J_n(\alpha r) dr$$

$$= -\alpha^2 \int_0^\infty r f(r) J_n(\alpha r) dr \qquad \text{since } \frac{d}{dr}\{\alpha r\, J_n'(\alpha r)\} = -r\left(\alpha^2 - \frac{n^2}{r^2}\right) J_n(\alpha r)$$

$$= -\alpha^2 F_n(\alpha)$$

$$\therefore\ H_n\left\{\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{n^2}{r^2} f(r)\right\} = -\alpha^2 F_n(\alpha).$$

In particular cases for $n = 0$ and $n = 1$, we have

$$H_0\left\{\frac{1}{r}\frac{d}{dr}\left(r \frac{d}{dr}\right) f(r)\right\} = -\alpha^2 H_0\{f(r)\} = -\alpha^2 F_0(\alpha) \qquad \text{and}$$

$$H_1\left\{\frac{1}{r}\frac{d}{dr}\left(r \frac{d}{dr}\right) f(r) - \frac{1}{r^2} f(r)\right\} = -\alpha^2 H_1\{f(r)\} = -\alpha^2 F_1(\alpha)$$

Result : $\dfrac{d}{dr}\{r J_0'(r)\} = -r\left(1^2 - \dfrac{n^2}{r^2}\right) J_n(r)$

Proof : The Bessel equation is $\dfrac{d^2 x}{dr^2} + \dfrac{1}{r}\dfrac{dx}{dr} + \left(1 - \dfrac{n^2}{r^2}\right) x = 0$

Since $J_n(r)$ is the solution of Bessel equation, so,

$$\frac{d^2}{dr^2} J_n(\alpha) + \frac{1}{r}\frac{d}{dr} J_n(r) + \left(1 - \frac{n^2}{r^2}\right) J_n(r) = 0$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left\{r\frac{dJ_n(r)}{dr}\right\}+\left(1-\frac{n^2}{r^2}\right)J_n(r)=0$$

$$\Rightarrow \frac{d}{dr}\left\{r\frac{dJ_n(r)}{dr}\right\}=-r\left(1-\frac{n^2}{r^2}\right)J_n(r) \qquad \text{(Proved)}$$

Ex.: Find the zero-order Hankel Transform of the following functions.

(a) $\frac{e^{-r}}{r}$ (ii) $H(a-r)$, $H(r)$ = Heaviside unit function $=\begin{cases}1, & r\geq 0.\\ 0, & r<0.\end{cases}$

Solu.: (a) Let us denote the zero-order Hankel Transform of given function is

$H_o\left\{\frac{e^{-r}}{r}\right\}$ or $F_o(\alpha)$ and defined by

$$H_o\left\{\frac{e^{-r}}{r}\right\}=F_o(\alpha)=\int_0^\infty \frac{e^{-r}}{r}r\,J_o(\alpha r)dr=\int_0^\infty e^{-r}J_o(\alpha r)dr$$

$$=\int_0^\infty e^{-r}\left\{1-\frac{r^2\alpha^2}{(1!)^2 2^2}+\frac{r^4\alpha^4}{(2!)^2 2^4}-\ldots\right\}dr$$

$$=\left.\frac{e^{-r}}{-1}\right|_0^\infty-\frac{\alpha^2}{(1!)^2 2^2}\int_0^\infty r^2e^{-r}dr+\frac{\alpha^4}{(2!)^2 2^4}\int_0^\infty r^4e^{-r}dr-\ldots$$

$$=1-\frac{\alpha^2 2!}{(1!)^2 2^2}+\frac{\alpha^4 4!}{(2!)^2 2^4}-\ldots$$

$$=1-\frac{\alpha^2}{2}+\frac{3}{8}\alpha^4-\ldots$$

$$=\left(1+\alpha^2\right)^{-1/2},\ |\alpha|<1.$$

(b) Let us denote the zero-order Hankel Transform of given function is

$$H_0\{H(a-r)\}=\int_0^\infty J_0(\alpha r)\,r\,H(a-r)\,dr \qquad (1)$$

Since $H(a-r)=\begin{cases}1, & a-r\geq 0\\ 0, & a-r<0\end{cases}$

$$=\begin{cases}1, & r\leq a\\ 0, & r>a\end{cases}$$

Using above, equation (1) becomes

$$H_0\{H(a-r)\}=\int_0^a J_0(\alpha r)\,1.r\,dr+\int_a^\infty J_0(\alpha r)\,r.0\,dr=\int_0^a J_0(\alpha r)\,r\,dr$$

$$=\int_0^{a\alpha} J_0(t)\frac{t}{\alpha^2}dt \qquad \text{Put } \alpha r=t$$

$$=\frac{1}{\alpha^2}\int_0^{a\alpha} t\left[1-\frac{t^2}{4}+\frac{t^4}{64}-\ldots\right]dt$$

$$=\frac{1}{\alpha^2}\left[\frac{t^2}{2}-\frac{t^4}{16}+\frac{t^6}{6.64}-\ldots\right]_0^{a\alpha}$$

$$=\frac{a^2}{2}-\frac{a^4\alpha^2}{16}+\frac{a^6\alpha^4}{6\times 64}-\ldots$$

$$=\frac{a}{\alpha}\left[\frac{a\alpha}{2}-\frac{1}{2!}\frac{(a\alpha)^3}{2}+\frac{1}{2!\,3!}\left(\frac{a\alpha}{2}\right)^5-\ldots\right]$$

$$=\frac{a}{\alpha}J_1(a\alpha).$$

Ex.: Find $H_1\left\{\frac{e^{-r}}{r}\right\}$

Solu.: By definition of 1st order Hankel Transform we have

$$H_1\left\{\frac{e^{-r}}{r}\right\}=\int_0^\infty J_1(\alpha r)\frac{e^{-r}}{r}.rdr$$

$$=\int_0^\infty e^{-r}\left[\frac{r\alpha}{2}-\frac{1}{2!}\frac{(r\alpha)^3}{2^3}+\frac{1}{2!\,3!}\frac{(\alpha r)^5}{2^5}-\ldots\right]dr$$

$$=\frac{\alpha}{2}-\frac{3}{8}\alpha^3+\frac{5}{16}\alpha^5-\ldots$$

$$=\frac{1}{\alpha}\left[\frac{\alpha^2}{2}-\frac{3}{8}\alpha^4+\frac{5}{16}\alpha^6-\ldots\right]$$

$$=\frac{1}{\alpha}\left[1-\left(1-\frac{\alpha^2}{2}+\frac{3}{8}\alpha^4-\frac{5}{16}\alpha^6+\ldots\right)\right]$$

$$=\frac{1}{\alpha}\left[1-\left(1+\alpha^2\right)^{-1/2}\right],\ |\alpha|<1.$$

Ex.: Show that $H_0\{e^{-ar}\}=\dfrac{a}{\left(a^2+\alpha^2\right)^{3/2}},\ a>0$

Solu.: By the definition of zero-order Hankel Transform we have

$$H_0\{e^{-ar}\}=\int_0^\infty re^{-ar}J_0(\alpha r)\,dr$$

$$=\int_0^\infty e^{-ar}\left[r-\frac{r^3\alpha^2}{2^2}+\frac{r^5\alpha^4}{(2!)^2 2^4}-\ldots\right]dr$$

$$=\int_0^\infty e^{-ar}r\,dr-\frac{\alpha^2}{2^2}\int_0^\infty e^{-ar}r^3dr+\frac{\alpha^4}{(2!)^2 2^4}\int_0^\infty e^{-ar}r^5dr-\ldots$$

$$=\frac{1!}{a^2}-\frac{3!}{a^4}\frac{\alpha^2}{2^2}+\frac{5!}{a^6}\frac{\alpha^4}{(2!)^2 2^4}-\ldots \qquad \text{[Using Gamma function]}$$

$$=\frac{1}{a^2}\left[1-\frac{3}{2}\left(\frac{\alpha}{a}\right)^2+\frac{15}{8}\left(\frac{\alpha}{a}\right)^4-\ldots\right]$$

$$=\frac{1}{a^2}\cdot\frac{1}{\left[1+\left(\frac{\alpha}{a}\right)^2\right]^{3/2}}=\frac{a^3}{a^2\left(a^2+\alpha^2\right)^{3/2}}=\frac{a}{\left(a^2+\alpha^2\right)^{3/2}}$$

Ex.: Prove that $H_n\left\{\frac{1}{r}f(r)\right\}=\frac{\alpha}{2n}\left[F_{n-1}(\alpha)+F_{n+1}(\alpha)\right],\ (n\neq 0)$

Solu.: By definition of nth order Hankel Transform we have

$$H_n\left\{\frac{1}{r}f(r)\right\}=\int_0^\infty J_n(\alpha r)\,r.\frac{1}{r}f(r)dr$$

$$=\int_0^\infty\frac{J_n(\alpha r)}{r}\,rf(r)dr$$

$$=\int_0^\infty\frac{\alpha r}{2n}\left[J_{n+1}(\alpha r)+J_{n-1}(\alpha r)\right]f(r)dr$$

$\left[\text{Since by recurrence relation of Bessel function } J_{n+1}(r)+J_{n-1}(r)=\frac{2n}{r}J_n(r)\right]$

$$=\frac{\alpha}{2n}\left[\int_0^\infty rJ_{n+1}(\alpha r)f(r)dr+\int_0^\infty rJ_{n-1}(\alpha r)f(r)dr\right]$$

$$=\frac{\alpha}{2n}\left[F_{n+1}(\alpha)+F_{n-1}(\alpha)\right]$$

2.6 Hankel Transform of Derivatives of Function :

Ex.: Prove that $H_n\{f'(r)\}=\frac{\alpha}{2n}\left[(n-1)F_{n+1}(\alpha)-(n+1)F_{n-1}(\alpha)\right]$

provided $f(r)$ is such that $\underset{r\to 0}{Lt}\ rf(r)=0$ and $\underset{r\to\infty}{Lt}\ rf(r)=0$ hold.

Solu.: By the definition of n-order Hankel Transform we have,

$$H_n\{f'(r)\}=\int_0^\infty J_n(\alpha r)\,rf'(r)\,dr$$

$$=\left[rf(r)J_n(\alpha r)\right]_0^\infty-\int_0^\infty f(r)\frac{d}{dr}\{rJ_n(\alpha r)\}dr$$

$$= -\int_0^\infty f(r)\,\{J_n(\alpha r) + \alpha r\, J_n'(\alpha r)\}\, dr, \qquad (1) \quad [Since,\ rf(r) \to 0,\ as\ r \to 0\ \&\ r \to \infty]$$

We know that from the recurrence relation

$$J_{n-1}(r) - J_{n+1}(r) = 2J_n'(r)\ \&\ J_{n+1}(r) + J_{n-1}(r) = \frac{2n}{r} J_n(r)$$

Using the above relations in (1)

$$H_n\{f'(r)\} = -\int_0^\infty f(r)\left[\frac{\alpha r}{2n}\{J_{n-1}(\alpha r) + J_{n+1}(\alpha r)\} + \frac{\alpha r}{2n}\{J_{n-1}(\alpha r) - J_{n+1}(\alpha r)\}\right] dr$$

$$= \frac{\alpha}{2n}\left[\int_0^\infty (n-1) J_{n+1}(\alpha r) r f(r) dr - \int_0^\infty (n+1) J_{n-1}(\alpha r) r f(r) dr\right.$$

$$= \frac{\alpha}{2n}\left[(n-1)\, F_{n+1}(\alpha) - (n+1)\, F_{n-1}(\alpha)\right]$$

When $n = 1$, the above result indicates,

$$H_1\{f'(r)\} = -\alpha H_0\{f(r)\}.$$

Ex.: Prove that $H_n\{r^n H(a-r)\} = \dfrac{a^{n+1}}{\alpha} J_{n+1}(a\alpha)$

Solu.: $H_n\{r^n H(a-r)\} = \int_0^\infty r^n r. J_n(\alpha r)\, H(a-r)\, dr$

$$= \int_0^a r^{n+1} J_n(\alpha r).1.dr + \int_a^\infty r^{n+1} J_n(\alpha r).0 dr$$

$$= \int_0^a r^{n+1} J_n(\alpha r)\, dr,\ a > 0 \qquad \text{sinc } H(a-r) = \begin{cases} 1, & r \le a \\ 0, & r > a \end{cases}$$

$$= \int_0^a r^{n+1}\left(\frac{\alpha r}{2}\right)^n \left[\frac{1}{n!} - \frac{1}{(n+1)!}\left(\frac{\alpha r}{2}\right)^2 + \frac{1}{2!(n+2)!}\left(\frac{\alpha r}{2}\right)^4 - \ldots\right] dr$$

$$=\left(\frac{\alpha}{2}\right)^{n}\frac{a^{2n+2}}{n!(2n+2)}-\left(\frac{\alpha}{2}\right)^{n+2}\frac{a^{2n+4}}{(n+1)!(2n+4)}+\left(\frac{\alpha}{2}\right)^{n+4}\frac{a^{2n+6}}{2!(n+2)!(2n+6)}-\dots$$

$$=\frac{a^{n+1}}{\alpha}\left[\left(\frac{a\alpha}{2}\right)^{n+1}\frac{1}{(n+1)!}-\frac{1}{(n+2)!}\left(\frac{a\alpha}{2}\right)^{n+3}+\frac{1}{2!}\frac{1}{(n+3)!}\left(\frac{a\alpha}{2}\right)^{n+5}-\dots\right]$$

$$=\frac{a^{n+1}}{\alpha}J_{n+1}(a\alpha)$$

2.7 The Parseval's Relation for Hankel Transform :

If $H_n\{f(r)\}=F(\alpha)$ and $H_n\{g(r)\}=G(\alpha)$ then

$$\int_0^\infty r\,f(r)g(r)dr=\int_0^\infty \alpha\,F(\alpha)\,G(\alpha)d\alpha$$

Proof : We have $\int_0^\infty \alpha\,F(\alpha)\,G(\alpha)d\alpha=\int_0^\infty \alpha\,F(\alpha)d\alpha\int_0^\infty J_n(\alpha r)\,rg(r)dr$

[By the definition of inverse Hankel Transform]

$$=\int_0^\infty r\,g(r)\,dr\int_0^\infty \alpha\,F(\alpha)\,J_n(\alpha r)dr$$

$$=\int_0^\infty r\,g(r)\,dr\,f(r)$$ [Again by the definition of inverse Hankel Transform]

$$\Rightarrow\int_0^\infty \alpha F(\alpha)\,G(\alpha)d\alpha=\int_0^\infty r\,f(r)\,g(r)dr.$$ (proved)

2.8 Example of Partial Differential Equation on Hankel Transform :

Ex.: Find the solution of the following problem of free symmetric vibration of a stretched membrane of infinite extent.

(a) $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}-\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}=0,\quad 0\le r<\infty$

(b) $u(r,0)=f(r)$ (c) $u_t(r,0)=g(r)$

where $u(r,t)$ is the transverse displacement of the membrane.

Solu.: The given differential equation is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0, \quad 0\le r<\infty \tag{1}$$

Taking Hankel transform of order zero of equation (1), with respect to r, we get,

$$H_0\left\{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\right\}=0$$

$$\Rightarrow \int_0^\infty\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right) r\, J_0(\alpha r)dr-\frac{1}{c^2}\int_0^\infty\frac{\partial^2 u}{\partial t^2} r\, J_0(\alpha r)\, dr=0$$

$$\Rightarrow -\alpha^2\bar{u}_0(\alpha)-\frac{1}{c^2}\frac{d^2}{dt^2}\int_0^\infty u(r,t)\, r\, J_0(\alpha r)dr=0$$

$$\text{Since} \int_0^\infty\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right) r\, J_0(\alpha r)dr=-\alpha^2\bar{u}_0(\alpha)$$

$$\Rightarrow \alpha^2\bar{u}_0(\alpha)+\frac{1}{c^2}\frac{d^2}{dt^2}\bar{u}_0(\alpha,t)=0 \tag{2}$$

where $\bar{u}_0(\alpha,t)$ is the Hankel transform of $u(r,t)$.

Here we assume that the function $u(r,t)$ is such that

$$r\frac{\partial u}{\partial r}\to 0 \text{ as } r\to 0 \text{ and } r\to\infty.$$

Taking Hankel transform initial conditions (b) & (c) we get,

$$\bar{u}_0(\alpha,0)=\bar{f}_0(\alpha) \tag{3}$$

$$\& \quad \frac{\partial}{\partial t}\bar{u}_0(\alpha,0)=\bar{g}_0(\alpha) \tag{4}$$

where $\bar{f}_0(\alpha)$ & $\bar{g}_0(\alpha)$ are Hankel transforms of $f(r)$ & $g(r)$ respectively.

Now the solution of equation of (1) is

$$\bar{u}_0(\alpha,t)=A\cos(\alpha ct)+B\sin(\alpha ct) \tag{5}$$

where A & B are two constants, i.e., they are independent of t.

Put $t=0$, in (5) and using (3) & (4) we get

$A = \bar{f}_0(\alpha)$ & $B = \dfrac{\bar{g}_0(\alpha)}{\alpha c}$.

Therefore, the solution for $\bar{u}_0(\alpha, t)$ given by (5) becomes.

$$\bar{u}_0(\alpha,t) = \bar{f}_0(\alpha)\cos(\alpha ct) + \frac{\bar{g}_0(\alpha)}{\alpha c}\sin(\alpha ct) \qquad (6)$$

Taking inversion formula of Hankel of (6) we get the desired solution of the problem.

$$u(r,t) = \int_0^\infty \bar{u}_0(\alpha,t)\,\alpha\, J_0(\alpha r)d\alpha = \int_0^\infty\left[\bar{f}_0(\alpha)\cos(\alpha ct) + \frac{\bar{g}_0(\alpha)}{\alpha c}\sin(\alpha ct)\right]\alpha\, J_0(\alpha r)\, d\alpha$$

$$= \int_0^\infty \bar{f}_0(\alpha)\cos(\alpha ct)\,\alpha\, J_0(\alpha r)d\alpha + \int_0^\infty \frac{\bar{g}_0(\alpha)}{\alpha c}\sin(\alpha ct)\,\alpha\, J_0(\alpha r)d\alpha$$

$$= \int_0^\infty \alpha\, J_0(\alpha r)\cos(\alpha ct)d\alpha\left\{\int_0^\infty f(r)\, r\, J_0(\alpha r)dr\right\} + \frac{1}{c}\int_0^\infty J_0(\alpha r)\sin(\alpha ct)\left\{\int_0^\infty r g(r) J_0(\alpha r)\, dr\right\}$$

2.9 Finite Hankel Transform :

Definition and Inversion Formula :

The finite Hankel Transform of order *n* of a function *f(r)*, $0 \le r \le a$, denoted by $H_{n,i}[f(r)]$ or $\bar{f}_n(\alpha_i)$, is defined by

$$H_{n,i}[f(r)] = \bar{f}_n(\alpha_i) = \int_0^a r f(r) J_n(r\alpha_i)dr,$$

where α_i is the root of the transcendental equation $J_n(a\alpha_i) = 0$. Obviously $J_n(x)$ being the Bessel function of order *n* and argument *x*.

The inversion formula for finite Hankel Transform is stated in the following theorem without any proof.

Theovem : If *f(r)* satisfies Dirichelet's condition in (0,a) and if its finite Hankel transform of order *n* is given by

$$\bar{f}_n(\alpha_i) = \int_0^a r f(r) J_n(r\alpha_i)\, dr$$

where α_i is a root of the transcendental equation $J_n(a\alpha_i) = 0$, then at any point of the interval $(0, a)$ at which the function *f(x)* is continuous

$$f(r)=\frac{2}{a^2}\sum_i \bar{f}_n(\alpha_i)\frac{J_n(r\alpha_i)}{\left[J_n'(a\alpha_i)\right]^2}$$

where the sum is taken over all positive roots of the equation $J_n(a\alpha_i)=0$.

2.10 Finite Hankel Transform of Derivatives and Examples :

Ex.: Find out the Finite Hankel Transform of $\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{n^2}{r^2}f$, where $f(r)$ is a function of r defined in the interval $(0, a)$, restricting n to the case $n\ge 0$.

Solu :- Now $H_{n,i}\left[\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{n^2}{r^2}f\right]=\int_0^a r\left[\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{n^2}{r^2}f\right]J_n(r\alpha_i)dr$

$$=\int_0^a r\frac{d^2f}{dr^2}J_n(r\alpha_i)\,dr+\int_0^a\frac{df}{dr}J_n(r\alpha_i)\,dr-\int_0^a\frac{n^2}{r^2}rf(r)J_n(r\alpha_i)\,dr \qquad (1)$$

Now $\int_0^a r\frac{d^2f}{dr^2}J_n(r\alpha_i)dr=\left[rJ_n(r\alpha_i)\frac{df}{dr}\right]_0^a-\int_0^a\frac{d}{dr}\{rJ_n(r\alpha_i)\}\frac{df}{dr}dr$

$$=-\int_0^a\frac{d}{dr}\{rJ_n(r\alpha_i)\}\,df(r),\ Since\ J_n(a\alpha_i)=0$$

$$=-\left[\frac{d}{dr}\{rJ_n(r\alpha_i)\}f(r)\right]_0^a+\int_0^a\frac{d^2}{dr^2}\{rJ_n(r\alpha_i)\}f(dr)$$

$$=-\left[\{J_n(r\alpha_i)+r\alpha_i J_n'(r\alpha_i)\}f(r)\right]_0^a+\int_0^a\frac{d^2}{dr^2}\{rJ_n(r\alpha_i)\}f(dr)$$

$$=-a\alpha_i J_n'(a\alpha_i)f(a)+\int_0^a\frac{d^2}{dr^2}\{rJ_n(r\alpha_i)\}f(r)dr \qquad (2)$$

since $J_n(a\alpha_i)=0,\ n\ge 0$

Again $\int_0^a\frac{df}{dr}J_n(r\alpha_i)\,dr=\left[f(r)J_n(r\alpha_i)\right]_0^a-\int_0^a f\frac{d}{dr}\{J_n(r\alpha_i)\}\,dr$

$$= -\int_0^a f \frac{d}{dr}\{J_n(r\alpha_i)\}\, dr \tag{3}$$

Since $J_n(a\alpha_i) = 0,\ n \geq 0,$

Using (2) & (3), (1) becomes,

$$H_{n,i}\left[\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{n^2}{r^2} f\right] = -a\alpha_i\, J_n'(a\alpha_i) f(a) + \int_0^a \frac{d^2}{dr^2}\{rJ_n(r\alpha_i)\} f(r) dr$$

$$-\int_0^a f \frac{d}{dr}\{J_n(r\alpha_i)\}\, dr - \int_0^a \frac{n^2}{r^2} r f(r) J_n(r\alpha_i)\, dr$$

$$= \int_0^a f(r)\left[\frac{d^2}{dr^2}\{r\, J_n(r\alpha_i)\} - \frac{d}{dr}\{J_n(r\alpha_i)\} - \frac{n^2}{r^2} r\, J_n(r\alpha_i)\right] dr - a\alpha_i\, J_n'(a\alpha_i) f(a)$$

$$= \int_0^a r f(r)\left[\frac{d^2}{dr^2} J_n(r\alpha_i) + \frac{1}{r}\frac{d}{dr} J_n(r\alpha_i) - \frac{n^2}{r^2}\, J_n(r\alpha_i)\right] dr - a\alpha_i\, J_n'(a\alpha_i) f(a)$$

$$= \int_0^{a\alpha_i} \frac{p}{u} f\left(\frac{p}{\alpha_i}\right)\left[\frac{d^2}{dp^2} J_n(p) + \frac{1}{p}\frac{d}{dp} J_n(p) - \frac{n^2}{p^2} J_n(p)\right] \alpha_i^2 dp - a\alpha_i\, J_n'(a\alpha_i) f(a)$$

where $p = r\alpha_i$

$$= -a\alpha_i\, J_n'(a\alpha_i) f(a) - \alpha_i^2 \int_0^a r f(r) J_n(r\alpha_i)\, dr,$$

{since Bessel function of order n satisfies the equation

$$= -a\alpha_i\, J_n'(a\alpha_i) f(a) - \alpha_i^2 \bar{f}_n(\alpha_i) \qquad \frac{d^2}{dp^2} J_n(p) + {}^1\!/_p \frac{d}{dp} J_n(p) + \left(1 - \frac{v^2}{p^2}\right) J_n(p) = 0\}$$

Hence we have derived the relation

$$H_{n,i}\left[\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{n^2}{r^2} f\right] = -a\alpha_i\, J_n'(a\alpha_i) f(a) - \alpha_i^2 \bar{f}_n(\alpha_i)$$

which is valid for $n \geq 0$.

For the particular case $n=0$, the above relation becomes

$$H_{0,i}\left[\frac{d^2 f}{dr^2}+\frac{1}{r}\frac{df}{dr}\right]=-a\alpha_i\, J_0'(a\alpha_i)f(a)-\alpha_i^2\bar{f}_0(\alpha_i)$$

$$=a\alpha_i\, J_1(a\alpha_i)f(a)-\alpha_i^2\bar{f}_0(\alpha_i) \qquad \text{since } J_0'(a\alpha_i)=-J_1(a\alpha_i)$$

Ex.: Solve the following problem of conduction of heat in an infinite circular cylinder

$$(i)\ \frac{\partial u}{\partial t}=\lambda\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right),\ 0\le r\le a,$$

$$(ii)\ u(a,t)=0 \quad (iii)\ u(r,0)=f(r),\ 0\le r\le a.$$

Solution : The given differential equation is

$$\frac{\partial u}{\partial t}=\lambda\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial t}\right), \qquad (1) \qquad 0\le r\le a$$

Taking finite Hankel Transform of order zero of the given equation (1) with respect to r,

$$\int_0^a\left(\frac{\partial u}{\partial t}\right)r\,J_0(r\alpha_i)dr=\lambda\int_0^a\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial t}\right)r\,J_0(r\alpha_i)dr$$

$$\Rightarrow\frac{d}{dt}\{\bar{u}_0(\alpha_i,t)\}=\lambda\left[-a\alpha_i\, J_0'(a\alpha_i)\ u(a,t)-\alpha_i^2\bar{u}_0(\alpha_i,t)\right]$$

[Apply Hankel Transform of zeroth order]

$$\Rightarrow\frac{d}{dt}\{\bar{u}_0(\alpha_i,t)\}=-\lambda\alpha_i^2\,\bar{u}_0(\alpha_i,t) \qquad \text{since } u(a,t)=0$$

$$\Rightarrow\frac{d}{dt}\bar{u}_0(\alpha_i,t)+\lambda\alpha_i^2\,\bar{u}_0(\alpha_i,t)=0 \qquad (2)$$

where $\bar{u}_0(\alpha_i,t)$ denotes the finite Hankel Transform of order zero or the function $u(r,t)$ and α_i being the root of the equation $J_0(ax)=0$ in x.

Also taking finite Hankel Transform of order zero of the initial condition (iii) we get,

$$\bar{u}_0(\alpha_i,0)=\bar{f}_0(\alpha_i) \qquad (3)$$

where $\bar{f}_0(\alpha_i)$ is the finite Hankel Transform of order zero of the functions $f(r)$.

Now the solution of equation (2) is

$$\bar{u}_0(\alpha_i,t) = A e^{-\lambda\alpha_i^2 t} \qquad (4)$$

where A is a constant, i.e., independent of t. Put $t = 0$ in (4)

$$\bar{u}_0(\alpha_i,0) = A = \bar{f}_0(\alpha_i), \text{ Using (3)}$$

Therefore the solution for $\bar{u}_0(\alpha_i,t)$ given by (4) becomes

$$\bar{u}_0(\alpha_i,t) = \bar{f}_0(\alpha_i) e^{-\lambda\alpha_i^2 t} \qquad (5)$$

Taking inversion of this according to the finite inversion formula, we get,

$$u(r,t) = \frac{2}{a^2}\sum_i \bar{f}_0(\alpha_i) e^{-\lambda\alpha_i^2 t} \frac{J_0(r\alpha_i)}{[J_0'(a\alpha_i)]^2}$$

$$= \frac{2}{a^2}\sum_i \frac{J_0(r\alpha_i)}{[J_1(a\alpha_i)]^2} e^{-\lambda\alpha_i^2 t} \int_0^a r f(r) J_0(r\alpha_i) dr \qquad (6)$$

Substituting for $\bar{f}_0(\alpha_i)$ according to the definition and using the relation $J_0'(x) = -J_1(x)$.

Ex.: Solve the equation $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0,\ 0 \le r < \infty,\ z \ge 0.$

Boundary conditions are $u = u_0$ (constant) where $z = 0,\ 0 \le r < 1.$

$$\frac{\partial u}{\partial z} = 0, \text{ where } z = 0, r > 1.$$

Given that $\int_0^\infty \frac{\sin \alpha}{\alpha} J_0(\alpha r)\, d\alpha = \frac{\pi}{2},\ 0 \le r \le 1$

and $\int_0^\infty J_0(\alpha r) \sin \alpha\, d\alpha = 0,\ r > 1.$

Solu.: The given equation is $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0.$ (1)

Taking Hankel Transform of zeroth order w.r to r to the given equation, we get

$$H_0\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right\} + H_0\left\{\frac{\partial^2 u}{\partial z^2}\right\} = 0$$

$$\Rightarrow -\alpha^2 \bar{u}_0(\alpha, z) + \frac{d^2}{dz^2} \bar{u}_0(\alpha, z) = 0 \qquad (2)$$

where $\bar{u}_0(\alpha, z) = \int_0^\infty u(r, z)\, r\, J_0(\alpha r) dr$

Now the solution of differential equation (2) is

$$\bar{u}_0(\alpha, z) = Ae^{\alpha z} + Be^{-\alpha z} \qquad (3)$$

To exist the above solution $\bar{u}_0(\alpha, z)$ must vanish as z tends to infinity. Therefore $\bar{u}_0(\alpha, z)$ must also vanish as z tends to infinity, $\therefore A = 0$.

Then equation (3) reduces to $\bar{u}_0(\alpha, z) = Be^{-\alpha z}$ (4)

where B is independent of z, i.e., B is a function of α only.

Now by inversion formula, we have

$$u(r, z) = \int_0^\infty \bar{u}_0(\alpha, z)\, \alpha\, J_0(\alpha r) d\alpha = \int_0^\infty Be^{-\alpha z} \alpha\, J_0(\alpha r) d\alpha \qquad (5)$$

Now applying the boundary conditions we have from (5), when $z = 0$,

$$u(r, 0) = \int_0^\infty \alpha B\, J_0(\alpha r) d\alpha = u_0, \quad 0 \le r \le 1, \qquad (6)$$

and $\left(\frac{\partial u(r,z)}{\partial z}\right)_{z=0} = \int_0^\infty -\alpha^2 B\, J_0(\alpha r)\, d\alpha = 0, \; r > 1,$ (7)

Comparing (6) & (7) with the given integrals i.e.,

$$\int_0^\infty J_0(\alpha r) \frac{\sin \alpha}{\alpha} d\alpha = \pi/2, \; 0 \le r < 1$$

$$\& \int_0^\infty J_0(\alpha r) \sin \alpha \, d\alpha = 0, \; r > 1$$

We have $B = \frac{2u_0}{\pi} \frac{\sin \alpha}{\alpha^2}$. (8)

Now putting the value of B in (5), we get the required solution

$$u(r,z)=\int_0^\infty \frac{2u_0}{\pi}\frac{\sin\alpha}{\alpha^2}\,\alpha\, J_0(\alpha r)\, e^{-\alpha z} d\alpha$$

$$=\frac{2u_0}{\pi}\int_0^\infty \frac{\sin\alpha}{\alpha}\, e^{-\alpha z}\, J_0(\alpha r)\, d\alpha.$$

2.11 Unit Summary :

At the end of the discussion, the gist of this module are depicted as follows :

(i) We have seen that a wide variety of physical problems solved by the Hankel Transform.

(ii) Some realistic problems in PDE with finite range of integration can be solved by Finite Hankel Transform.

2.12 Exercises

1. Show that $H_0\left\{\frac{1}{r}\right\}=\frac{1}{\alpha}$ by using the fact that the Hankel Transform is its own inverse.

2. Prove that $H_n\left\{r^n e^{-r^2}\right\}=\frac{\alpha^n}{2^{n+1}}e^{-\alpha^2/4}$

3. Prove that $H_0\left\{\frac{e^{-ar}}{r}\right\}=\frac{1}{\left(\alpha^2+a^2\right)^{1/2}},\ a>0.$

4. Prove that $H_n\{f(ar)\}=\frac{1}{a^2}F_n(\alpha/a),\ a>0$

5. Show that $H_n\{1/r\, J_{n+1}(ar)\}=\frac{\alpha^n}{a^{n+1}}H(a-\alpha),\ a>0$, $H(r)$ is heaviside step function.

6. Solve the following problem of free symmetric vibration of a stretched circular membrane

PDE: $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0,\ \ 0\le r<a$

Bcs: $u(a,t)=0$

$u(r,0)=f(r)$

$$\frac{\partial}{\partial t}u(r,0) = g(r)$$

where $u(r, t)$ is the transverse displacement of the membrane.

$$\left[\text{Solu}: u(r,t) = \frac{2}{a^2}\sum_i \frac{J_0(r\alpha_i)}{[J_1(a\alpha_i)]^2}\cos(c\alpha_i t)\int_0^a u f(u) J_0(u\alpha_i)du\right.$$

$$\left. + \frac{2}{ca^2}\sum_i \frac{J_0(r\alpha_i)}{[J_1(a\alpha_i)]^2}\frac{\sin(c\alpha_i t)}{\alpha_i}\int_0^a u g(u) J_0(u\alpha_i)du\right]$$

2.13 References :

1. Lokenath Debnath, Integral Transforms and their Applications, CRC Press, New York, Londan, Tokyo, 1995.

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M.Sc. Course

in

Applied Mathematics with Oceanology

and

Computer Programming

Part - II

Paper - VIII **Group - A**

Module No. 87

Laplace Transforms

Structure

3.1 Introduction :

Most of the physical problems are generated by ordinary or partial differential equations with appropriate initial or boundary conditions. Generally these problems are formulated as initial value problems, boundary value problems for applied and engineering sciences in realistic sense. The laplace transform method is particularly useful for finding solutions of these problems. The method is very effective for the solution of the response of a linear system governed by an ordinary differential equation.

3.2 Objectives :

A large number of realistic problems in sciences and Engineering involve the solution of linear ordinary and partial differential equations. The Laplace Transform technique is one of the powerful tools for solving realistic problems involving PDE or ODE, particularly initial or boundary value problems. It reduces the solution of PDE to the solution of ODE and then to the solution of an algebric equation. This method has a particular advantage a in finding the solution of an initial value problem or boundary value problem, without finding the general solution and then using the given IC or BC for evaluating arbitany constants.

Keywords :

Laplace Transform, Perodic Function, Inverse Laplace Transform, Convolution Theorem, Complex Inversion Formula.

3.3 Definition of the Laplace Transform :

Let $f(t)$ be a function of t specified for $t \geq 0$. Then the laplace transform of $f(t)$ is denoted by $L\{f(t)\}$ or $F(p)$ and is defined by the integral as follows :

$$L\{f(t)\} = F(p) = \int_0^\infty e^{-pt} f(t)dt, \qquad (1)$$

provided the integral exists. Let us assume that the parameter p is complex and consequently the laplace transform of $f(t)$ i.e., $F(p)$ is a function of complex variable p.

3.4 Sufficient Conditions For Existence of Laplace Transforms :

A class of functions $f(t)$, for which their Laplace Transform $L\{f(t)\}$ or $F(p)$ exists, satisfying the following properties

1. $f(t)$ is sectionally continuous function on every finite interval of t for $t > 0$.

2. $f(t)$ is a function of exponential order γ as $t \to \infty$, i.e., $\left|e^{-\gamma t} f(t)\right| < M$ or $|f(t)| < Me^{\gamma t}$, where M is a real constants and γ also positive constant.

Example on exponential order :

Ex-1 $f(t) = t^2$ is of exponential order γ as $t \to \infty$, since $\left|t^2\right| = t^2 < e^{3t}$ for all $t > 0$.

Ex-2 $f(t) = e^{t^3}$ is not of exponential order since $\left|e^{-\gamma t} e^{t^3}\right| = e^{t^3 - \gamma t}$ can be made larger than any given constant by increasing t.

Intuitively, functions of exponential order cannot "grow" in absolute value more rapidly than $Me^{\gamma t}$ as t increases. In practice, however, this is no restriction since M and γ can be as large as desired. Bounded functions, such as Sin at or Cos at, are of exponential order.

Theorem 1 : If a real-valued function $f(t)$ of real variable is sectionally continuons in any finite interval of t and is of exponential order $O\left(e^{\gamma t}\right)$ at $t \to 0$ when $t \geq 0$, then the integral $\int_0^\infty e^{-pt} f(t)dt$ converges in the domain Real $(p) > \gamma$.

Proof : Since $f(t)$ is of exponential order $O\left(e^{\gamma t}\right)$ at $t \to \infty$, there exist two real positive constants M and γ such that

$$|f(t)| \leq Me^{\gamma t}$$

Therefore,

$$\left|\int_0^\infty e^{-pt} f(t)dt\right| \leq \int_0^\infty \left|e^{-(x+iy)t}\right| |f(t)| dt,\ p = x+iy$$

$$\leq \int_0^\infty e^{-xt} \left|e^{-iyt}\right| Me^{\gamma t} dt$$

$$\leq M\int_0^\infty e^{-(x-\gamma)t} dt,\ since \left|e^{-iyt}\right| = 1$$

which exists if $x > \gamma$ i.e., Real $(p) > \gamma$.

Hence the integral $\int_0^\infty e^{-pt} f(t)dt$ exists in the domain Real $(p) > \gamma$.

3.5 Laplace Transform of derivatives :

Theorem 2 : If $f(t)$ is continuous and is of exponential order $O\left(e^{\gamma t}\right)$ at $t \to \infty$ and $f'(t)$ is piecewise continuous in any finite interval of t, then the laplace transform of $f'(t)$ exists for Real $(p) > \gamma$ and is given by

$$L\{f'(t)\} = pF(p) - f(o).$$

Proof : According to the definition of laplace transform we can write,

$$L\{f'(t)\} = \int_0^\infty f'(t)e^{-pt} dt = \underset{T\to\infty}{Lt} \int_0^T f'(t)\, e^{-pt} dt \qquad (1)$$

provided the limit exists.

Since $f'(t)$ is piecewise continuous in any finite interval of t, the finite interval (0, T) can be broken up into a finite number of sub-intervals in each of which $f'(t)$ is continuous. Let there be n such subintervals and (b_{r-1}, b_r) be the r-th subinterval where $b_0 = 0$ and $b_n = T$.

Therefore, we can write

$$\int_0^T e^{-pt} f'(t)\,dt = \sum_{r=1}^{n} \int_{b_{r-1}}^{b_r} f'(t) e^{-pt}\,dt$$

$$= \sum_{r=1}^{n} \left\{ \left[e^{-pt} f(t)\right]_{b_{r-1}}^{b_r} + p \int_{b_{r-1}}^{b_r} f(t) e^{-pt}\,dt \right\}$$

$$= \sum_{r=1}^{n} \left[e^{-pb_r} f(b_r - 0) - e^{-pb_{r-1}} f(b_{r-1} + 0)\right] + p \sum_{r=1}^{n} \int_{b_{r-1}}^{b_r} f(t) e^{-pt}\,dt$$

$$= e^{-pT} f(T) - f(0) + p \int_0^T f(t) e^{-pt}\,dt \qquad (2)$$

Since $f(t)$ is continuous, $f(b_r - 0) = f(b_r + 0) = f(b_r)$

Now

$$\left|e^{-pT} f(T)\right| = \left|e^{-(x+iy)T}\right| \left|f(T)\right|,\ p = x + iy$$

$$\le e^{-xT} M e^{\gamma t} = M e^{-(x-\gamma)T} \to 0 \text{ as } T \to \infty,$$

if $x > \gamma$ i.e., Real $(p) > \gamma$.

Therefore, $\underset{T\to\infty}{Lt}\, e^{-pT} f(T) = 0$ for Real $(p) > \gamma$. (3)

Also since $f(t)$ is continuous and is of exponential order $O(e^{\gamma t})$ at $t \to \infty$, its Laplace transform exists for Real $(p) > \gamma$, and so the following limit exists and is equal to $F(p)$.

$$\underset{T\to\infty}{Lt} \int_0^T f(t) e^{-pt}\,dt = F(p) \text{ for Real } (p) > \gamma \qquad (4)$$

By the use of (2), the equation (1) can be written as

$$L\{f'(t)\} = \underset{T\to\infty}{Lt}\, e^{-pT} f(t) - f(0) + p \underset{T\to\infty}{Lt} \int_0^T f(t) e^{-pt}\,dt.$$

$$= pF(p) - f(0), \text{ using (3) \& (4) for Real } (p) > \gamma,$$

Hence the theorem.

Generalising the above idea of the theorem we can obtained the Laplace transform of the n-th derivative of the function is given in the following theorem.

Theorem 3 : If the $(n-1)$th derivative of a function $f(t)$ is continuous, its n-th derivative is piecewise continuous in any finite interval of t and $f(t)$, $f'(t)$, ... $f^{(n-1)}(t)$ are each of exponential order $O(e^{\gamma t})$ at $t \to \infty$, then the Laplace Transform of $f^n(t)$ exists for Real $(p) > \gamma$ and is given by

$$L\{f^n(t)\} = p^n F(p) - p^{n-1} f(0) - p^{n-2} f'(0) - \ldots - f^{(n-1)}(0),$$

where $F(p)$ is the Laplace Transform of $f(t)$.

3.6 Some Important Properties of Laplace Transform :

1. Linearity Property : If $L\{f_r(t)\} = F_r(p)$, which exists for Real $(p) > \gamma_r$, $r = 1, 2, \ldots, n$ then $L\{C_1 f_1(t) + C_2 f_2(t) + \ldots + C_n f_n(t)\} = C_1 F_1(p) + C_2 F_2(p) + \ldots + C_n F_n(p)$ which exists for Real $(p) > \max(\gamma_1, \gamma_2, \ldots, \gamma_n)$ and $C_1, C_2, \ldots, C_n$ are n constants.

Proof : Now we have

$$L[C_1 f_1(t) + C_2 f_2(t) + \ldots + C_n f_n(t)]$$

$$= \int_0^\infty e^{-pt}[C_1 f_1(t) + C_2 f_2(t) + \ldots + C_n f_n(t)]\, dt$$

$$C_1 \int_0^\infty e^{-pt} f_1(t)\, dt + C_2 \int_0^\infty e^{-pt} f_2(t)\, dt + \ldots + C_n \int_0^\infty e^{-pt} f_n(t)\, dt$$

$$= C_1 F_1(p) + C_2 F_2(p) + \ldots + C_n F_n(p)$$

which exists in the common region of existence of the integrals $\int_0^\infty e^{-pt} f_r(t) dt$. This common region is the domain at complex p-plane given by Real $(p) > \max(\gamma_1, \gamma_2, \ldots, \gamma_n)$.

2. Change of Scale Property :

If $L\{f(t)\} = F(p)$, which exists for Real $(p) > \gamma$, then for any real positive α, $L\{f(\alpha t)\} = \frac{1}{\alpha} F(p/\alpha)$

Proof : Since the Laplace Transform of $f(t)$ exists for Real $(p) > \gamma$, the function $f(t)$ is of exponential order

$O(e^{\gamma t})$ at $t \to \infty$. This implies that there exists a positive constant M such that

$$|f(t)| \le Me^{\gamma t}$$

Now replacing t by αt we get $|f(\alpha t)| \le Me^{\alpha\gamma t}$.

$$\left|\int_0^\infty f(\alpha t)e^{-pt}dt\right| \le \int_0^\infty |f(\alpha t)|\left|e^{-(x+iy)t}\right|dt \le M\int_0^\infty e^{\gamma\alpha t}e^{-xt}dt = M\int_0^\infty e^{-(x-\alpha\gamma)t}dt$$

which exists for $x > \gamma\alpha$ or Real $(p) > \gamma\alpha$.

Hence the laplace transform of $f(\alpha t)$ exists for Real $(p) > \gamma\alpha$.

Now $L\{f(\alpha t)\} = \int_0^\infty e^{-pt} f(\alpha t)dt$, put $\alpha t = \beta$

$$= \int_0^\infty e^{-\frac{p}{\alpha}\beta} f(\beta)\frac{1}{\alpha}d\beta = \frac{1}{\alpha}\int_0^\infty e^{-\frac{p}{\alpha}\beta} f(\beta)d\beta = \frac{1}{\alpha}F(p/\alpha).$$

since we have $F(p) = \int_0^\infty f(t)e^{-pt}dt$ then $F(p/\alpha) = \int_0^\infty e^{-\frac{p}{\alpha}t} f(t)dt.$

3. Shifting Property :

If $L\{f(t)\} = F(p)$, which exists for Real $(p)>\gamma$, then for any complex constant a, we have $L\{e^{at}f(t)\} = F(p-a)$, which exists for Real $(p)>\gamma +$ Real (a).

Proof : Since the laplace transform of $f(t)$ exists for Real $(p) > \gamma$, the function $f(t)$ is of exponential order $O(e^{\gamma t})$ at $t \to \infty$. This implies that there exists a positive constant M such that $|f(t)| \le Me^{\gamma t}$.

Now let $g(t) = e^{at}f(t)$ then $|g(t)| = |e^{at}||f(t)| = |e^{(x+iy)t}||f(t)|$

$$\le |e^{xt}| Me^{\gamma t} \qquad \text{put } a = x + iy$$

$$\therefore |g(t)| \le Me^{(x+\gamma)t}$$

This implies that $g(t)$ is of exponential order $O(e^{(x+\gamma)t})$. So its laplace transform exists for Real $(p)>\gamma +$ Real (a).

Now $$L\{e^{at}f(t)\} = \int_0^\infty e^{-pt}e^{at}f(t)dt = \int_0^\infty e^{-(p-a)t}f(t)dt = F(p-a)$$

since we have $F(p)=\int_0^\infty f(t)e^{-pt}dt$ then $F(p-a)=\int_0^\infty e^{-(p-a)t}f(t)dt$.

4. Translation Property :

If $L\{f(t)\}=F(p)$, which exists for Real $(p)>\gamma$, and $H(t)$ is the unit step function, then for $\alpha>0$, $L\{f(t-\alpha)H(t-\alpha)\}=e^{-p\alpha}F(p)$ which exist for Real $(p)>\gamma$.

Proof : Since the laplace transform of $f(t)$ exists for Real $(p)>\gamma$, the function $f(t)$ is of exponential order $O(e^{\gamma t})$ at $t\to\infty$. This implies that there exists a positive constant M such that $|f(t)|\le Me^{\gamma t}$. Replacing t by $t-\alpha$ we have

$$|f(t-\alpha)|\le Me^{\gamma(t-\alpha)} = Me^{\gamma t}e^{-\gamma\alpha}$$

which implies that $f(t-\alpha)H(t-\alpha)$ is also of exponential order $O(e^{\gamma t})$ at $t\to\infty$.

Again $$H(t-\alpha)=\begin{cases}1, & t-\alpha>0\\ 0, & t-\alpha<0\end{cases} \Rightarrow H(t-\alpha)=\begin{cases}1, & t>\alpha\\ 0, & t<\alpha\end{cases}$$

Now $$L\{f(t-\alpha)H(t-\alpha)\}=\int_0^\infty f(t-\alpha)H(t-\alpha)\,e^{-pt}dt$$

$$=\int_0^\infty f(t-\alpha)H(t-\alpha)\,e^{-pt}dt+\int_\alpha^\infty f(t-\alpha)H(t-\alpha)\,e^{-pt}dt$$

$$=\int_0^\infty f(t-\alpha).\ 0\ .\ e^{-pt}dt+\int_\alpha^\infty f(t-\alpha).\ 1\ .\ e^{-pt}dt$$

$$=\int_\alpha^\infty f(t-\alpha)\,e^{-pt}dt=\int_0^\infty f(u)\,e^{-p(u+\alpha)}du, \quad \text{put } u=t-\alpha$$

$$=e^{-p\alpha}\int_0^\infty f(u)e^{-pu}du=e^{-p\alpha}F(p).$$

5. The Laplace Transform of an Integral :

If $L\{f(t)\} = F(p)$, which exists for Real $(p) > \gamma$, and $g(t) = \int_0^t f(u)\,du$,

then $L\{g(t)\} = L\left\{\int_0^t f(x)du\right\} = \frac{1}{p}F(p)$, which exists for Real $(p) > \gamma$.

Proof : Since the laplace transform of $f(t)$ exists for Real $(p) > \gamma$, the function $f(t)$ is of exponential order $O\left(e^{\gamma t}\right)$ at $t \to \infty$. This implies that there exists a positive constant M such that

$$|f(t)| \le Me^{\gamma t}, \text{ and so } |g(t)| = \left|\int_0^t f(u)du\right| \le \int_0^t |f(u)|\,du \le M\int_0^t e^{\gamma u}du$$

$$\le \frac{M}{\gamma}\left(e^{\gamma t} - 1\right) \le \frac{M}{\gamma}e^{\gamma t}$$

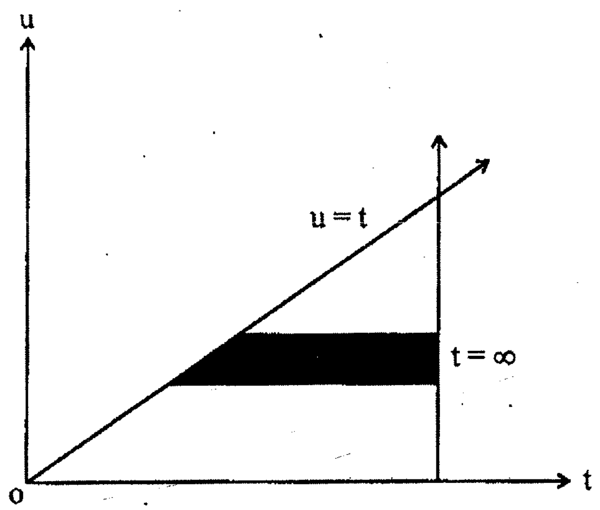
Hence $g(t)$ is of same exponential order as $f(t)$ and so its laplace transform exists and is given by

$$L\{g(t)\} = L\left\{\int_0^t f(u)du\right\} = \int_0^\infty e^{-pt}\left(\int_0^t f(u)du\right)dt$$

$$= \int_{u=0}^{\infty} f(u)\,du \int_{t=u}^{\infty} e^{-pt}dt$$

$$= \int_{u=0}^{\infty} f(u)du \left[\frac{e^{-pt}}{-p}\right]_u^\infty$$

$$= \frac{1}{p}\int_0^\infty f(u)\,e^{-pu}du = \frac{1}{p}F(p), \text{ which exists for Real } (p) > \gamma.$$



[Changing the Order of Integration]

$$\begin{Bmatrix} t : u \text{ to } \infty \\ u : 0 \text{ to } \infty \end{Bmatrix}$$

6. Multiplication by t^n :

If $L\{f(t)\} = F(p)$, which exists for Real $(p) > \gamma$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{dp^n} F(p) = (-1)^n F^n(p).$$

Proof : Since the laplace transform of $f(t)$ exists for Real $(p) > \gamma$, and also assume that the laplace transform of $t^n f(t)$ exists. To establish the result, we use the Mathematical Induction.

We have $F(p) = \int_0^\infty e^{-pt} f(t) dt$

Then by Leibnitz's Rule for differentiating under the integral sign, we have

$$\frac{dF(p)}{ds} = F'(p) = \frac{d}{dp}\int_0^\infty e^{-pt} f(t)\, dt = \int_0^\infty \frac{\partial}{\partial p}\left(e^{-pt}\right) f(t)\, dt$$

$$= -\int_0^\infty e^{-pt}\left(t f(t)\right) dt$$

$$= -L\{t f(t)\}$$

Thus, $$L\{t f(t)\} = -\frac{dF(p)}{dp} = -F'(p) \qquad (1)$$

Which proves the theorem for n = 1.

Now assume the results true for n = k, i.e., assume

$$\int_0^\infty e^{-pt}\{t^k f(t)\}dt = (-1)^k F^k(p) \qquad (2)$$

then $$\frac{d}{dp}\int_0^\infty e^{-pt}\{t^k f(t)\}\,dt = (-1)^k \frac{d}{dp}F^k(p)$$

$$\Rightarrow -\int_0^\infty e^{-pt}\{t^{k+1} f(t)\}\,dt = (-1)^k F^{k+1}(p) \quad \text{by Leibnitz's Rule}$$

$$\Rightarrow \int_0^\infty e^{-pt}\{t^{k+1} f(t)\}\,dt = (-1)^{k+1} F^{k+1}(p) \qquad (3)$$

It follows that if (2) is true, i.e., if the theorem holds for $n = k$, then (3) is true, i.e., the theorem holds for $n = k+1$. But by (1) the theorem is true for $n = 1$. Hence it is true for $n = 1 + 1 = 2$ and $n = 2+1=3$, etc., and thus for all positive integer values of n.

Example : Laplace Transforms of Some Elementary Functions :

Ex – 1 : Find out the laplace transforms of following :

(a) 1 (b) t (c) e^{at} (d) $\sin at$ (e) $\cos at$ (f) $\sinh at$ (g) $\cosh at$ (h) t^n, $n > -1$.

Ans: (a) $$L\{1\} = \int_0^\infty e^{-pt}\, 1\, dt = \underset{T\to\infty}{Lt}\int_0^T e^{-pt}dt = \underset{T\to\infty}{Lt}\left[\frac{e^{-pt}}{-p}\right]_0^T = \underset{T\to\infty}{Lt}\left[\frac{1-e^{-pt}}{p}\right] = \frac{1}{p},\ p > 0.$$

(b) $$L\{t\} = \int_0^\infty e^{-pt}\, t\, dt = \underset{T\to\infty}{Lt}\int_0^T t\, e^{-pt}dt = \underset{T\to\infty}{Lt}\left[t.\frac{e^{-pt}}{-p} - 1.\frac{e^{-pt}}{-p^2}\right]_0^T$$

$$= \underset{T\to\infty}{Lt}\left[T\frac{e^{-pT}}{-p} - \frac{e^{-pT}}{p^2} + \frac{1}{p^2}\right] = \frac{1}{p^2},\ p > 0.$$

(c) $$L\{e^{at}\} = \int_0^\infty e^{-pt}\, e^{at}\, dt = \underset{T\to\infty}{Lt}\int_0^T e^{-(p-a)t}dt = \underset{T\to\infty}{Lt}\left[\frac{e^{-(p-a)t}}{-(p-a)}\right]_0^T = \frac{1}{p-a},\ \text{Real}(p) > \text{Real}(a).$$

(d) $L\{\sin at\} = \int_0^\infty \sin at\, e^{-pt} dt = \underset{T\to\infty}{Lt} \int_0^T \sin at\, e^{-pt} dt$

$$= \underset{T\to\infty}{Lt} \left[\frac{e^{-pt}}{p^2+a^2}(-p \sin at - a \cos at)\right]_0^T = \frac{a}{p^2+a^2}, \text{ Real } (p) > 0$$

(e) $L\{\cos at\} = \int_0^\infty \cos at\, e^{-pt} dt = \underset{T\to\infty}{Lt} \int_0^T \cos at\, e^{-pt} dt$

$$= \underset{T\to\infty}{Lt} \left[\frac{e^{-pt}}{p^2+a^2}(-p \cos at + a \sin at)\right]_0^T = \frac{p}{p^2+a^2}, \text{ Real } (p) > 0$$

(f) $L\{\sinh at\} = \int_0^\infty \sinh at\, e^{-pt} dt = \underset{T\to\infty}{Lt} \int_0^T \frac{e^{at}-e^{-at}}{2} e^{-pt} dt$

$$= \frac{1}{2} \underset{T\to\infty}{Lt} \int_0^T \left\{e^{-(p-a)t} - e^{-(p+a)t}\right\} dt$$

$$= \frac{1}{2} \underset{T\to\infty}{Lt} \left[\frac{e^{-(p-a)t}}{-(p-a)} - \frac{e^{-(p+a)t}}{-(p+a)}\right]_0^T = \frac{1}{2}\left[\frac{-1}{p+a} + \frac{1}{p-a}\right] = \frac{a}{p^2-a^2}, \text{ Real } (p) > |a|.$$

(e) $L\{\cosh at\} = \int_0^\infty \cosh at\, e^{-pt} dt = \underset{T\to\infty}{Lt} \int_0^T \frac{e^{at}+e^{-at}}{2} e^{-pt} dt$

$$= \frac{1}{2} \underset{T\to\infty}{Lt} \int_0^T \left\{e^{-(p-a)t} + e^{-(p+a)t}\right\} dt = \frac{p}{p^2-a^2}, \text{ Real } (p) > |a|.$$

(h) Let $f(t) = t^n,\ n > -1$.

If $-1 < n < 0$, then $f(t) \to \infty$ as $t \to 0^+$. Hence $f(t)$ is not a piecewise continuous function in any finite integral of t and so the condition of existence of its laplace transform is not satisfied. Now we can show that its laplace transform still exists.

Using Gamma function we have

$$\int_0^\infty t^n e^{-pt} dt = \frac{\Gamma(n+1)}{p^{n+1}} \qquad \text{for Real } (p) > 0,$$

$$= \frac{n!}{p^{n+1}}$$

Ex-2 : Find out the laplace transform of the following :

(a) $t^n e^{at}$, (b) $e^{at} \sin bt$ (c) $\cos bt\, e^{at}$ (d) $t \sin at$ (e) $t \cos at$.

Ans : (a) Let $f(t) = t^n$ then $L\{f(t)\} = L\{t^n\} = \frac{n!}{p^{n+1}}$ for Real (p) > 0.

Then by shifting property, $L\{t^n e^{at}\} = \frac{n!}{(p-a)^{n+1}}$. since, $L\{f(t)\, e^{at}\} = F(p-a)$.

(b) Let $f(t) = \sin bt$ then $L\{\sin bt\} = \frac{b}{p^2 + b^2} = F(p)$

Then by shifting property, we have $L\{f(t)e^{at}\} = F(p-a)$

$$\Rightarrow L\{(\sin\ bt)e^{at}\} = \frac{b}{(p-a)^2 + b^2}.$$

(c) Let $f(t) = \cos bt$, then $L\{f(t)\} = L\{\cos\ bt\} = \frac{p}{p^2 + b^2}$

Then by shifting property, we have $L\{f(t)e^{at}\} = F(p-a)$

$$\Rightarrow L\ \{e^{at} \cos\ bt\} = \frac{(p-a)}{(p-a)^2 + b^2}$$

(d) Let us define the laplace transform of $t \sin at$,

$$L\{t\ \sin\ at\} = L\left[\frac{1}{2i}\{t(e^{iat} - e^{-iat})\}\right]$$

$$= \frac{1}{2i}\left[L\{te^{iat}\} - L\{te^{-iat}\}\right]$$

$$= \frac{1}{2i}\left[\frac{1!}{(p-ia)^2} - \frac{1!}{(p+ia)^2}\right]$$ [by shifting property], since $L\{t\} = \frac{1!}{p^2}$

$$= \frac{2ap}{(p^2+a^2)^2}$$

(e) By the same way as described in (d), we have $L\{t \cos at\} = \frac{p^2-a^2}{(p^2+a^2)^2}$.

Ex-3 : Find $L\{e^{-2t}(3 \cos 6t - 5 \sin 6t)\}$

Solu : $L\{e^{-2t}(3 \cos 6t - 5 \sin 6t)\} = L\{e^{-2t} 3 \cos 6t\} - L\{e^{-2t} 5 \sin 6t\}$ [by shifting property]

$$= 3L\{e^{-2t} \cos 6t\} - 5L\{e^{-2t} \sin 6t\} \quad (1)$$

Now $L\{\cos 6t\} = \frac{p}{p^2+36}$ & $L\{\sin 6t\} = \frac{6}{p^2+36}$

By the help of above and using shifting property (1) becomes

$$L\{e^{-2t}(3 \cos 6t - 5 \sin 6t)\} = 3.\frac{p+2}{(p+2)^2+36} - 5.\frac{6}{(p+2)^2+36}$$

$$= \frac{3p+6-30}{p^2+4p+4+36} = \frac{3p-24}{p^2+4p+40}$$

Ex-4 : Find $L\{g(t)\}$ where $g(t) = \begin{cases} \cos(t - 2p/3), & (t > 2p/3) \\ 0, & (t < 2p/3) \end{cases}$

Solu : Let $f(t) = \cos t$

So $F(p) = \frac{p}{p^2+1}$

Now $L\{g(t)\} = e^{-\frac{2\pi}{3}p} F(p)$ [shifting property]

$$= \frac{pe^{-(2\pi/3)p}}{p^2+1}$$

Ex-5 : Given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}(1/p)$, find $L\left\{\frac{\sin at}{t}\right\}$

$$L\left\{\frac{\sin at}{at}\right\}=\frac{1}{a}L\left\{\frac{\sin at}{t}\right\}=\frac{1}{a}\tan^{-1}\left\{\frac{1}{\left(p/a\right)}\right\}=\frac{1}{a}\tan^{-1}\left(a/p\right)$$

$$\therefore\ L\left\{\frac{\sin at}{t}\right\}=\tan^{-1}\left(a/p\right).$$

3.7 Theorems :

Theorem 1 : If $F(p)$ is the laplace transform of a function $f(t)$, which is piecewise continuons in any finite interval of t and is of exponential order $O\left(e^{\gamma t}\right)$ at $t\to\infty$, then (i) $\underset{p\to\infty}{Lt}\ F(p)=0$ and

(ii) $\underset{p\to\infty}{Lt}\ pF(p)=f(0)$ **[Initial value theorem]**

Proof : (i) According to the definition of laplace transform we have

$$F(p)=\int_0^\infty e^{-pt}f(t)dt$$

Let $|f(t)|\le Me^{\gamma t}$, Therefore $F(p)\le\left|\int_0^\infty e^{-pt}f(t)dt\right|$

$$\le\int_0^\infty\left|e^{-pt}\right|\left|f(t)\right|dt$$

$$\le\int_0^\infty\left|e^{-(x+iy)t}\right|\left|f(t)\right|dt,\ p=x+iy$$

$$\le\int_0^\infty e^{-xt}\left|e^{-iyt}\right|\left|f(t)\right|dt$$

$$\le\int_0^\infty e^{-xt}e^{\gamma t}dt$$

$$\le M\left[\frac{e^{-(x-\gamma)}}{-(x-\gamma)}\right]_0^\infty=\frac{M}{x-\gamma}\ \text{for } x>\gamma.$$

$$\to 0 \text{ as } x\to\infty.$$

Hence $F(p)\to 0$ as $x\to\infty$.

when $x \to \infty$ then $p \to \infty$ and so $\underset{p\to\infty}{Lt}\ F(p) = 0.$

(ii) Let us suppose that $f'(t)$ is piecewise continuous in any finite interval of t, and given that $f(t)$ is continuous and of exponential $O(e^{\gamma t})$ at $t \to \infty$. Then the laplace transform of $f'(t)$ can be defined as follows:

$$L\{f'(t)\} = \int_0^\infty f'(t)\, e^{-pt} dt$$

$$= \left[f(t)\, e^{-pt}\right]_0^\infty - \int_0^\infty (-p) f(t)\, e^{-pt} dt$$

$$= -f(0) + p\int_0^\infty f(t)\, e^{-pt} dt$$

$$= -f(0) + pF(p)$$

Since $f'(t)$ is piecewise continuous and of exponential order then

$$L\{f'(t)\} = \int_0^\infty e^{-pt} f'(t)\, dt \to 0 \text{ as } p \to \infty.$$

$$\therefore\ pF(p) = f(0).$$

Note : If $f(t)$ is not continuous at $t = o$ the required result still holds, and we must have write

$L(f'(t)) = pF(p) - f(0^+)$ where $\underset{t\to 0}{Lt}\ f(t) = f(0^+)$ exists but is not equal to $f(0)$.

Theorem 2 : (Final Value Theorem)

If $f(t)$ is continuous and is of exponential order $O(e^{\gamma t})$ at $t \to \infty$ and $f'(t)$ is piecewise continuous in any finite interval of t, then $\underset{p\to\infty}{Lt}\ pF(p) = \underset{t\to\infty}{Lt}\ f(t) = f(\infty)$

Proof : We have $L\{f'(t)\} = \int_0^\infty e^{-pt} f'(t) dt$

$$= \left[e^{-pt} f(t)\right]_0^\infty - \int_0^\infty (-p) e^{-pt} f(t) dt$$

$$= -f(0) + p\int_0^\infty f(t)\, e^{-pt} dt$$

$$L\{f'(t)\} = -f(0) + pF(p) \qquad (1)$$

Again
$$\underset{p\to 0}{Lt}\, L\{f'(t)\} = \underset{p\to 0}{Lt} \int_0^\infty e^{-pt} f'(t)dt$$

$$= \int_0^\infty f'(t)dt = [f(t)]_0^\infty = f(\infty) - f(0) \qquad (2)$$

Now taking the limit $p \to 0$ in (1) and using (2)

$$\underset{p\to 0}{Lt}\, L\{f'(t)\} = \underset{p\to 0}{Lt}[pF(p) - f(0)]$$

$$\Rightarrow f(\infty) - f(0) = \underset{p\to 0}{Lt}\, pF(p) - f(0)$$

$$\Rightarrow \underset{p\to 0}{Lt}\, pF(p) = f(\infty) = \underset{t\to\infty}{Lt}\, f(t) \quad \text{(Proved)}$$

Another Important Property of laplace transform :

Theorem : If a function $\frac{f(t)}{t}$ satisfies the conditions of its laplace transform and $L\{f(t)\} = F(p)$, which exists for Real $(p) > \gamma$, then $L\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(u)du.$

Proof : Let $g(t) = \frac{f(t)}{t} \Rightarrow f(t) = t\,g(t)$, taking the laplace transform of both sides and $L\{f(t)\} = F(p)$ and $L\{g(t)\} = G(p)$.

$$\therefore\ L\{f(t)\} = L\{t\,g(t)\}$$

$$\Rightarrow F(p) = (-1)\frac{d}{dp}\{G(p)\} = -\frac{dG(p)}{dp}$$

Then integrating to the limit p to ∞. we get

$$\int_p^\infty F(u)du = -\int_p^\infty \frac{dG(p)}{dp}dp = [G(p)]_\infty^p = G(p) - G(\infty).$$

$$\therefore\ \int_p^\infty F(u)du = G(p) - G(\infty).$$

Again $\underset{p\to\infty}{Lt}\ G(p) = G(\infty) = 0$. (Assuming it)

$$\therefore \int_p^{\infty} F(u)du = G(p) = L\left\{\frac{f(t)}{t}\right\}.$$

Ex-6 : Find $L\left\{\frac{e^{-t} - e^{-3t}}{t}\right\}$

Solu : Let $f(t) = e^{-t} - e^{-3t}$

$$\text{Then } L\{f(t)\} = F(p) = \int_0^{\infty}\left(e^{-t} - e^{-3t}\right) e^{-pt} dt = \frac{1}{p+1} - \frac{1}{p+3}.$$

Now we have

$$L\left\{\frac{e^{-t} - e^{-3t}}{t}\right\} = L\left\{\frac{f(t)}{t}\right\} = \int_p^{\infty} F(u)du = \int_p^{\infty}\left(\frac{1}{u+1} - \frac{1}{u+3}\right) du$$

$$= \left[\log(u+1) - \log(u+3)\right]_p^{\infty}$$

$$= \left[\log\left(\frac{u+1}{u+3}\right)\right]_p^{\infty} = \left[\log\left(\frac{1+1/u}{1+3/u}\right)\right]_p^{\infty}$$

$$= \log 1 - \log\left(\frac{1+1/p}{1+3/p}\right) = -\log\left(\frac{p+1}{p+3}\right) = \log\left(\frac{p+3}{p+1}\right).$$

Ex-7 : Evaluate $\int_0^{\infty} t\, e^{-3t} \sin t\, dt$

Solu : Now $L\{\sin t\} = \frac{1}{p^2+1}$

then $F(p) = L\{t \sin t\} = (-1)\frac{d}{dp}\left(\frac{1}{p^2+1}\right) = \frac{2p}{\left(p^2+1\right)^2}$

$$\int_0^{\infty}(t \sin t)\, e^{-3t} dt = \int_0^{\infty} f(t)\, e^{-3t} dt = F(3) = \frac{2\times 3}{\left(3^2+1\right)^2} = \frac{6}{100} = \frac{3}{50} \qquad \text{[Using (1)]}$$

3.8 Periodic Function :

A function $f(t)$ is said to be periodic with period T if it satisfies $f(t+T)=f(t)$. In general, $f(t+nT)=f(t)$, for n = 1, 2, 3, ...

Property of laplace transform on Periodic Function :

Theorem : If the function $f(t)$ has period $T>0$ show that

$$F(p)=\frac{1}{1-e^{-pT}}\int_0^T f(t)\,e^{-pt}dt.$$

Solu : $$L\{f(t)\}=\int_0^\infty f(t)\,e^{-pt}dt=\int_0^T f(t)\,e^{-pt}dt+\int_T^{2T} f(t)\,e^{-pt}dt+\int_{2T}^{3T} f(t)\,e^{-pt}dt+\ ...$$

$$L\{f(t)\}=\int_0^T f(t)e^{-pt}dt+\int_T^{2T} f(t)e^{-pt}dt+\int_{2T}^{3T} f(t)e^{-pt}dt+\ ...$$

$$=I_1+I_2+I_3+\ ... \qquad (1)$$

Now $$I_2=\int_T^{2T} f(t)\,e^{-pt}dt=\int_0^T f(u+T)\,e^{-p(u+T)}du=e^{-pT}\int_0^T f(u)\,e^{-pu}du=e^{-pT}I_1.$$

Similarly $I_3=\int_{2T}^{3T} f(t)\,e^{-pt}dt=e^{-2pT}I_1$ and so on

Using above, equation (1) becomes

$$L\{f(t)\}=I_1+e^{-pT}I_1+e^{-2pT}I_1+\ ...$$

$$=\left(1+e^{-pT}+e^{-2pT}+\ ...\right)I_1$$

$$=\frac{1}{1-e^{-pT}}\int_0^T f(t)e^{-pt}dt$$

So $F(p)=\frac{1}{1-e^{-pT}}\int_0^T f(t)e^{-pt}dt.$ (Proved)

Ex-8 : The function $f(t)$ is defined as follows :

$$f(t)=\begin{cases}\sin t, & 0<t<\pi\\ 0, & \pi<t<2\pi\end{cases}$$

Find $L\{f(t)\}$.

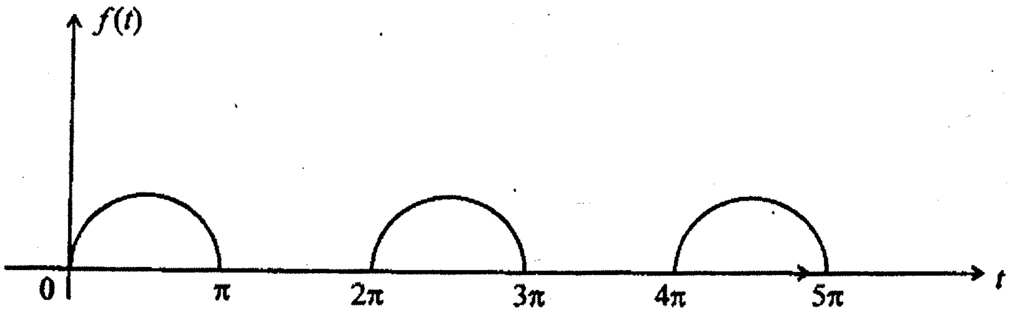
Solu : Here the function $f(t)$ is periodiocity $2\pi = T$.

Then $F(p) = \frac{1}{1-e^{-pT}} \int_0^T f(t)\, e^{-pt} dt$

$$= \frac{1}{1-e^{-2\pi p}} \int_0^{\pi} \sin t\, e^{-pt} dt + \frac{1}{1-e^{-2\pi p}} \int_{\pi}^{2\pi} 0.\, e^{-pt} dt$$

$$= \frac{1}{1-e^{-2\pi p}} \int_0^{\pi} e^{-pt} \sin t\, dt = \frac{1}{1-e^{-2\pi p}} \left[\frac{e^{-pt}(-p\sin t - \cos t)}{p^2+1} \right]_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi p}} \left[\frac{e^{-p\pi}+1}{p^2+1} \right] = \frac{1}{(p^2+1)(1-e^{-\pi p})}$$



The graph of the function $f(t)$ is of ten called a half wave rectified sine curve.

Ex-9 : Let $f(t) = \frac{g'(t)}{t}$ and g(0)=0, show that $L\{f(t)\} = \int_p^{\infty} u\bar{g}(u) du$.

Solu : $L\{f(t)\} = \int_0^{\infty} e^{-pt} f(t) dt = \int_0^{\infty} g'(t) \frac{e^{-pt}}{t} dt$

$$= \int_0^{\infty} g'(t) \left[\int_p^{\infty} e^{-ut} du \right] dt, \text{ Since} \int_p^{\infty} e^{-ut} du = \left. \frac{e^{-ut}}{-t} \right]_p^{\infty} = \frac{e^{-pt}}{t}.$$

$$= \int_p^{\infty} du \left[\int_0^{\infty} g'(t)\, e^{-ut} dt \right]$$

$$= \int_p^\infty du \left[g(t)e^{-ut}\Big|_{t=o}^{\infty} - (-u)\int_0^\infty g(t)\, e^{-ut}\, dt \right]$$

$$= \int_p^\infty du \left[0 + u\int_0^\infty g(t)\, e^{-ut}\, dt \right], \text{ since } g(0) = 0$$

$$= \int_p^\infty u\, \bar{g}(u) du, \qquad \text{where } g(u) = \int_0^\infty g(t)\, e^{-ut}\, dt.$$

Ex-10 : If $f(t) = \int_0^t \frac{g(u)}{u} du$, show that $L\{f(t)\} = (1/p)\int_p^\infty G(u)\, du$ where $G(u) = L[g(u)]$.

Solu : By definition of Laplace Transform we have

$$F(p) = L\{f(t)\} = \int_0^\infty f(t)\, e^{-pt} dt = \int_0^\infty e^{-pt} dt \left(\int_0^t \frac{g(u)}{u} du \right)$$

$$= \left[\frac{e^{-pt}}{-p} \int_0^t \frac{g(u)}{u} du \right]_0^\infty - \int_0^\infty \frac{e^{-pt}}{-p} \left(\frac{g(t)}{t} \right) dt$$

$$= 0 + \frac{1}{p}\int_0^\infty g(t) \frac{e^{-pt}}{t} dt$$

$$= \frac{1}{p}\int_0^\infty g(t) dt \left(\int_p^\infty e^{-ut} du \right), \qquad \text{since } \int_p^\infty e^{-ut} du = \frac{e^{-pt}}{t}$$

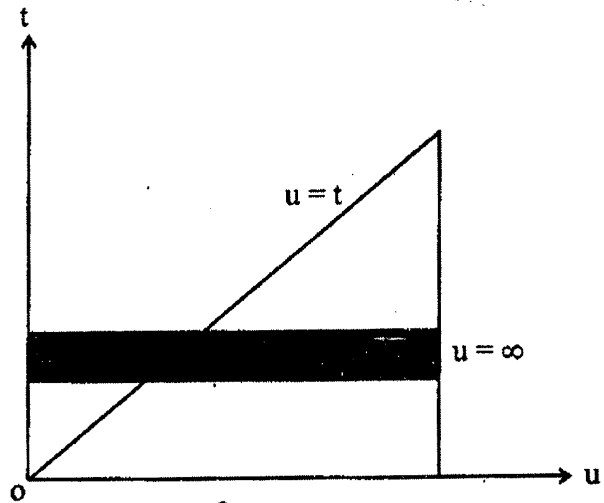
$$= \frac{1}{p}\int_p^\infty du \left(\int_0^\infty g(t)\, e^{-ut}\, dt \right) = \frac{1}{p}\int_p^\infty G(u) du.$$

Ex-11 : If $f(t) = \int_t^\infty \frac{g(u)}{u} du$, show that $L\{f(t)\} = \frac{1}{p}\int_0^p G(u)\, du$, where $G(u) = \int_0^\infty g(u)\, e^{-ut} dt$.

Solu : By definition $L\{f(t)\} = \int_0^\infty f(t)\, e^{-pt} dt = \int_{t=0}^\infty e^{-pt} \left(\int_{u=t}^\infty \frac{g(u)}{u} du \right) dt$

$$= \int_{u=0}^\infty \frac{g(u)}{u} du \int_{t=0}^u e^{-pt} dt = \int_{u=o}^\infty \frac{g(u)}{u} du \left[\frac{e^{-pt}}{-p} \right]_0^u$$

$$=\int_0^{\infty}\frac{g(u)}{u}\left(\frac{1-e^{-pu}}{p}\right)=\frac{1}{p}\int_0^{\infty}\frac{g(u)}{u}\left(1-e^{-pu}\right)du$$



u : 0 to ∞
t : 0 to u

$$=\left(\tfrac{1}{p}\right)\int_0^{\infty}g(u)\left[\int_0^{p}e^{-uq}dq\right]du,\qquad since\int_0^{p}e^{-uq}dq=\frac{1-e^{-up}}{u}$$

$$=\left(\tfrac{1}{p}\right)\int_0^{p}\left[\int_0^{\infty}g(u)\,e^{-uq}du\right]dq$$

$$=\left(\tfrac{1}{p}\right)\int_0^{p}G(q)\,dq\qquad \text{where}\quad G(q)=\int_0^{\infty}g(u)\,e^{-uq}du.$$

$$\therefore\; L\left\{\int_t^{\infty}\frac{g(u)}{u}du\right\}=\frac{1}{p}\int_0^{p}G(u)du.$$

Ex-12 : Evaluate $L\left\{\int_0^{t}\frac{\sin u}{u}du\right\}$ by the help of Initial Value Theorem.

Solu : Let $f(t)=\frac{\sin t}{t},\; L\left\{\frac{\sin t}{t}\right\}=F(p).$

Now we have $L\{\sin t\} = \frac{1}{p^2+1} \Rightarrow L\left\{t.\frac{\sin t}{t}\right\} = \frac{1}{p^2+1}$

$$\therefore\ L\{t.f(t)\} = \frac{1}{p^2+1}$$

$$\Rightarrow\ (-1)\frac{d}{dp}\{F(p)\} = \frac{1}{p^2+1}$$

Integrating w.r. to p we get,

$$F(p) = -\tan^{-1} p + c,\ (c = \text{Integrating Constant})$$

$$\Rightarrow pF(p) = cp - p\tan^{-1} p \qquad (1)$$

Now by Initial Value Theorem $\underset{p\to\infty}{Lt}\ pF(p) = f(0)$

$$\therefore\ \underset{p\to\infty}{Lt}\ pF(p) = \underset{p\to\infty}{Lt}\ p\left(c - \tan^{-1} p\right) = f(0) = \underset{t\to 0}{Lt}\frac{\sin t}{t} = 1$$

$$\Rightarrow\ \underset{p\to\infty}{Lt}\ p\left(c - \tan^{-1} p\right) = 1$$

$$\Rightarrow\ \underset{p\to\infty}{Lt}\left(c - \tan^{-1} p\right) = \underset{p\to\infty}{Lt}\ 1/p$$

$$\Rightarrow\ c - \tan^{-1}(\infty) = 0$$

$$\Rightarrow\ c - \pi/2 = 0 \Rightarrow c = \pi/2$$

Using the value of $c = \pi/2$ in (1) we get,

$$pF(p) = p\left[\pi/2 - \tan^{-1} p\right]$$

$$\Rightarrow F(p) = \pi/2 - \tan^{-1} p = \tan^{-1}(1/p).$$

$$\therefore\ L\{f(t)\} = L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}(1/p)$$

$$L\left\{\int_0^t \frac{\sin u}{u} du\right\} = \frac{F(p)}{p} = \frac{\tan^{-1}(1/p)}{p} = \frac{1}{p}\tan^{-1}(1/p). \quad \text{(Ans)}$$

Ex-13 : Find the laplace transform of $J_0(t)$ by using IVP.

Solu : The Bessel's equation of order *n* is

$$t^2\frac{d^2x}{dt^2}+t\frac{dx}{dt}+\left(t^2-n^2\right)x=0$$

Put $n=0$, $t^2\frac{d^2x}{dt^2}+t\frac{dx}{dt}+t^2x=0$ (order zero)

As $J_0(t)$ is the solution of Bessel function of order zero.

$$\therefore\ t^2\frac{d^2}{dt^2}J_0(t)+\frac{d}{dt}J_0(t)+t\,J_0(t)=0.$$

Now taking the laplace transform of above we get,

$$L\left\{t\frac{d^2J_0(t)}{dt^2}\right\}+L\left\{\frac{d}{dt}J_0(t)\right\}+L\{t\,J_0(t)\}=0. \qquad (1)\ \text{[by Linearity Prop.]}$$

Now $\quad L\{J_0(t)\}=J_0(p)=\int_0^\infty J_0(t)\,e^{-pt}dt.$

$$L\{t\,J_0(t)\}=(-1)\frac{d}{dp}\{J_0(p)\}=-J_0'(p) \qquad (2)$$

$$L\left\{\frac{d\,J_0(t)}{dt}\right\}=\int_0^\infty\frac{d\,J_0(t)}{dt}e^{-pt}dt=\left[J_0(t)\,e^{-pt}\right]_0^\infty-\int_0^\infty(-p)e^{-pt}J_0(t)dt$$

$$=0-J_0(0)+p\int_0^\infty J_0(t)\,e^{-pt}dt$$

$$=p\,J_0(p)-1 \qquad (3)\ \text{Since } J_0(0)=1$$

$$L\left\{\frac{d^2J_0(t)}{dt^2}\right\}=\int_0^\infty\frac{d^2J_0(t)}{dt^2}\,e^{-pt}dt$$

$$=\left[\frac{d\,J_0(t)}{dt}e^{-pt}\right]_0^\infty+p\int_0^\infty\frac{d\,J_0(t)}{dt}e^{-pt}dt$$

$$=-\frac{d\,J_0(0)}{dt}+p\left[p\,J_0(p)-1\right] \qquad \text{using (3)}$$

$$= p^2 J_0(p) - p \qquad (4) \text{ Since } J_0(0) = 1.$$

$$L\left\{t\frac{d^2 J_0(t)}{dt^2}\right\} = -\frac{d}{dp}\left[p^2 J_0(p) - p\right] = -2p\, J_0(p) - p^2 J_0'(p) + 1$$

$$= 1 - 2p\, J_0(p) - p^2 J_0'(p) \qquad (5)$$

Using (2), (3) & (5), in (1), we get,

$$\{1 - 2p\, J_0(p) - p^2\, J_0'(p)\} + \{p\, J_0(p) - 1\} - J_0'(p) = 0$$

$$\Rightarrow (p^2 + 1)\, J_0'(p) = -p\, J_0(p)$$

$$\Rightarrow \frac{d\, J_0(p)}{J_0(p)} = -\frac{2p}{2(p^2+1)}dp$$

Integrating, $\log[J_0(p)] = \log\left[c(p^2+1)^{-1/2}\right]$, ($c$ = Integrating Constant)

$$\Rightarrow J_0(p) = \frac{c}{\sqrt{p^2+1}}$$

$$\Rightarrow p\, J_0(p) = \frac{cp}{\sqrt{p^2+1}} \qquad (6)$$

$$\underset{p\to\infty}{Lt}\; p\, J_0(p) = \underset{p\to\infty}{Lt}\; \frac{cp}{\sqrt{p^2+1}} = c. \qquad (7)$$

Also by IVP, $\underset{p\to\infty}{Lt}\; pF(p) = f(0)$

$$\Rightarrow \underset{p\to\infty}{Lt}\; p\, J_0(p) = J_0(0) = 1 \qquad (8)$$

Using (8), (7) becomes $c = 1$.

Hence from (6), $p\, J_0(p) = \frac{p}{\sqrt{p^2+1}}$

$$\Rightarrow J_0(p) = \frac{1}{\sqrt{p^2+1}}.$$

$$\therefore\ L\{J_0(t)\} = \frac{1}{\sqrt{p^2+1}}$$

Ex-14 : Show that $\int_0^\infty J_0(t)\,dt = 1.$

Solu : Let us define the laplace transform of $J_0(t)$

i.e., $\therefore\ L\{J_0(t)\} = \int_0^\infty J_0(t)\,e^{-pt}dt.$

$$\Rightarrow J_0(p) = \int_0^\infty J_0(t)\,e^{-pt}dt$$

$$\Rightarrow \frac{1}{\sqrt{p^2+1}} = \int_0^\infty J_0(t)\,e^{-pt}dt \qquad \text{since } J_0(p) = \frac{1}{\sqrt{p^2+1}}$$

Put $p = 0$ in both sides

$$1 = \int_0^\infty J_0(t)dt$$

$$\therefore\ \int_0^\infty J_0(t)dt = 1.$$

Ex-15 : Find the laplace transform of $e^{-at}J_0(at)$

Solu : We have $L\{J_0(t)\} = \frac{1}{\sqrt{p^2+1}} = J_0(p).$

$$L\{J_0(at)\} = \frac{1}{a}J_0\left(\frac{p}{a}\right) = \frac{1}{a}\frac{1}{\sqrt{\left(\frac{p}{a}\right)^2+1}} = \frac{1}{\sqrt{p^2+a^2}} = \bar{J}_0(p)$$

Now $$L\{e^{-at}J_0(at)\} = \bar{J}_0(p+a) = \frac{1}{\sqrt{(p+a)^2+a^2}} = \frac{1}{\sqrt{p^2+2pa+2a^2}}$$

Ex-16 : Find $L\{t\,J_1(t)\}$

Solu : We know that $J_0'(t) = -J_1(t).$

$$L\{J_0'(t)\} = -L\{J_1(t)\}$$

$$\Rightarrow L\{J_1(t)\} = -L\{J_0'(t)\} = -[p\,J_0(p)-1], \quad \text{where } J_0(p) = L\{J_0(t)\}$$

$$= 1 - \frac{p}{\sqrt{p^2+1}} \qquad \text{since } J_0(p) = \frac{1}{\sqrt{p^2+1}}$$

$$L\{t\,J_1(t)\} = (-1)\frac{d}{dp}\left(1 - \frac{p}{\sqrt{p^2+1}}\right) = +\left[\frac{d}{dp}\left(\frac{p}{\sqrt{p^2+1}}\right)\right]$$

$$= \frac{\sqrt{p^2+1} - \frac{p}{2}.\ 2p\left(p^2+1\right)^{-\frac{1}{2}}}{p^2+1} = \frac{1}{\left(p^2+1\right)^{\frac{3}{2}}}$$

$$\therefore\ L\{t\,J_1(t)\} = \frac{1}{\left(p^2+1\right)^{\frac{3}{2}}}.$$

3.9. Inverse Laplace Transform :

Definition : If the laplace transform of a function $f(t)$ is $F(p)$, i.e., if $L\{f(t)\} = F(p)$, then $f(t)$ is called an inverse laplace transform of $F(p)$ and we write symbolically $f(t) = L^{-1}\{F(p)\}$ where L^{-1} is called the inverse laplace transformation operator.

Ex.: *Since* $L\{e^{-5t}\} = \frac{1}{p+5}$ we can write $L^{-1}\left\{\frac{1}{p+5}\right\} = e^{-5t}$.

The property of inverse laplace transform is same as laplace transform which described earlier, just change L^{-1} instead of L and the corresponding change.

Ex-17 : Given that $L^{-1}\left\{\frac{p}{\left(p^2+1\right)^2}\right\} = \frac{t \sin t}{2}$ then find $L^{-1}\left\{\frac{1}{\left(p^2+1\right)^2}\right\}$

Solu : Let $F(p) = \frac{p}{\left(p^2+1\right)^2}$, $f(t) = \frac{t \sin t}{2}$

Now $L^{-1}\left\{\frac{F(p)}{p}\right\} = L^{-1}\left\{\frac{1}{p}\cdot\frac{p}{(p^2+1)^2}\right\} = \int_0^t f(u)du,$

$\Bigg[$ Using the formula $L\int_0^t f(u)du = \frac{F(p)}{p}$

$$\Rightarrow \int_0^t f(u)du = L^{-1}\left\{\frac{F(p)}{p}\right\}\Bigg]$$

$$= \int_0^t \frac{u \sin u \, du}{2}$$

$$= \tfrac{1}{2}\left[-u \cos u + \sin u\right]_0^t$$

$$= \tfrac{1}{2}\left[\sin t - t \cos t\right].$$

Property : Show that $\int_0^t \left(\int_0^v f(u)du\right)dv = L^{-1}\left\{\frac{F(p)}{p^2}\right\}.$

Let $g(t) = \int_0^t \left(\int_0^v f(u)du\right)dv, \; g(0) = 0.$

$$\Rightarrow g(t) = \int_0^t \phi(v)\, dv \text{ where } \phi(v) = \int_0^v f(u)du \qquad (*)$$

Differentiating w.r. to t, $g'(t) = \phi(t) = \int_0^t f(u)du,$ (Using *)

Again differentiating, w.r. to t, $g''(t) = f(t),$ Also $g'(0) = 0.$

Now taking laplace transform, $L[g''(t)] = L\{f(t)\}$

$$\Rightarrow p^2 L\{g(t)\} - pg(o) - g'(o) = F(p).$$

$$\Rightarrow p^2 L\{g(t)\} = F(p)$$

$$\Rightarrow L\{g(t)\} = \frac{F(p)}{p^2}$$

$$\Rightarrow g(t) = L^{-1}\left\{\frac{F(p)}{p^2}\right\}$$

$$\Rightarrow \int_0^t\left(\int_0^v f(u)du\right)dv = L^{-1}\left\{\frac{F(p)}{p^2}\right\}. \qquad \text{(Proved)}$$

The result can be written $L^{-1}\left\{\frac{F(p)}{p^2}\right\} = \int_0^t\int_0^t f(t)dt^2$

Generalising the above result we have

$$L^{-1}\left\{\frac{F(p)}{p^n}\right\} = \int_0^t\int_0^t\dots\int_0^t f(t)dt^n$$

and for $L^{-1}\left\{\frac{F(p)}{p^3}\right\} = \int_0^t\int_0^v\int_0^w f(u)du\,dv\,dw.$

Ex-18 : Evaluate $L^{-1}\left\{\frac{1}{(p^2+1)p^2}\right\}$

Solu : Let $F(p) = \frac{1}{p^2+1} \Rightarrow f(t) = sin\ t$

Now $L^{-1}\left\{\frac{F(p)}{p^2}\right\} = \int_0^t\int_0^v f(u)du\,dv$

$$\Rightarrow L^{-1}\left\{\frac{1}{(p^2+1)p^2}\right\} = \int_0^t\left(\int_0^v sin\ u\ du\right)dv$$

$$= \int_0^t[-cos\ u]_0^v\ dv = \int_0^t(1-cos\ v)\ dv = [v - sin\ v]_0^t = t - sin\ t. \quad \text{(Ans)}$$

Ex-19 : Evaluate $L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\}$, Given that $L\left\{\frac{t}{2}sin\ t\right\} = \frac{p}{(p^2+1)^2}$

Solu : Let $F(p)=\frac{p}{(p^2+1)^2}$, $\quad f(t)=\frac{t\sin t}{2}$

Given that $L^{-1}\{F(p)\}=f(t)$.

$$\Rightarrow L^{-1}\{F(ap)\}=\tfrac{1}{a}f(t/a).$$

$$L^{-1}\left\{\frac{ap}{(a^2p^2+1)^2}\right\}=(1/a)(\tfrac{1}{2})(t/a)\sin(t/a)=\frac{t\sin(t/a)}{2a^2}$$

Put a = 4, $L^{-1}\left\{\frac{4p}{(16p^2+1)^2}\right\}=\frac{t\sin(t/4)}{2.\ 4^2}=\frac{t\sin(t/4)}{2.\ 16}$

$$\Rightarrow L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\}=(t/4)\sin(t/4).$$ Ans.

Ex-20 : Find $L^{-1}\left\{\frac{6p-4}{p^2-4p+20}\right\}$.

Solu : $L^{-1}\left\{\frac{6p-4}{p^2-4p+20}\right\}=L^{-1}\left\{\frac{6(p-2)+8}{(p-2)^2+16}\right\}=6L^{-1}\left\{\frac{p-2}{(p-2)^2+4^2}\right\}+8L^{-1}\left\{\frac{1}{(p-2)^2+4^2}\right\}$ (1)

Now $L^{-1}\left\{\frac{(p-2)}{(p-2)^2+4^2}\right\}=e^{2t}\cos 4t,$ (2)

since $L\{\cos 4t\}=\frac{p}{p^2+4^2}$, $L\{e^{2t}\cos 4t\}=\frac{p-2}{(p-2)^2+4^2}$

Again $L\{\sin 4t\}=\frac{4}{p^2+4^2}$, $L\{e^{2t}\sin 4t\}=\frac{4}{(p-2)^2+4^2}$

$$\Rightarrow e^{2t}\sin 4t=4L^{-1}\left\{\frac{1}{(p-2)^2+4^2}\right\}$$

$$\therefore \quad \frac{e^{2t} \sin 4t}{4} = L^{-1}\left\{\frac{1}{(p-2)^2+4^2}\right\} \qquad (3)$$

Using (2) & (3), (1) becomes, $L^{-1}\left\{\frac{6p-4}{p^2-4p+20}\right\} = 6.e^{2t} \cos 4t + \frac{8e^{2t}}{4} \sin 4t$

$= 2e^{2t}(3\cos 4t + \sin 4t)$ Ans.

3.10 Convolution Theorem :

If $L\{f(t)\} = F(p)$ and $L\{g(t)\} = G(p)$, which exist respectively in the domains Real $(p) > \gamma_1$ and Real $(p) > \gamma_2$, then $L\{\phi(t)\} = F(p)G(p)$ which exist in the domain Real $(p) > \max(\gamma_1, \gamma_2)$, where $\phi(t)$ is the convolution of the two functions $f(t)$ and $g(t)$ defixed by $\phi(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t (t-\tau)g(\tau)d\tau$.

Proof : Since the laplace transform of $f(t)$ and $g(t)$ are both exists for Real $(p) > \gamma_1$ and Real $(p) > \gamma_2$. Therefore $f(t)$ and $g(t)$ are of exponential orders $O(e^{\gamma_1 t})$ and $O(e^{\gamma_2 t})$ respectively at $t \to \infty$. and there exists two positive constants M_1 and M_2 such that $|f(t)| \le M_1 e^{\gamma_1 t}$, $|g(t)| \le M_2 e^{\gamma_2 t}$

Now $|\phi(t)| \le \left|\int_0^t f(\tau)g(t-\tau)d\tau\right| \le \int_0^t |f(\tau)||g(t-\tau)|d\tau \le \int_0^t M_1 e^{\gamma_1 \tau} M_2 e^{\gamma_2 (t-\tau)} d\tau$

$$\le M_1 M_2 e^{\gamma_2 t} \int_0^t e^{(\gamma_1-\gamma_2)\tau} d\tau = M_1 M_2 e^{\gamma_2 t} \frac{e^{(\gamma_1-\gamma_2)t} - 1}{(\gamma_1-\gamma_2)}$$

$$\le M_1 M_2 \frac{e^{\gamma_1 t} - e^{\gamma_2 t}}{\gamma_1 - \gamma_2}$$

If $\gamma_1 > \gamma_2$ or $\gamma_2 > \gamma_1$, then $|\phi(t)| \le \frac{M_1 M_2}{|\gamma_1 - \gamma_2|} e^{\gamma t} = M e^{\gamma t}$

where $M = \frac{M_1 M_2}{|\gamma_1 - \gamma_2|}$ and $\gamma = max(\gamma_1, \gamma_2)$.

Therefore laplace transform of $\phi(t)$ exists in the domain Real $(p) > \gamma = max(\gamma_1, \gamma_2)$ and is given by

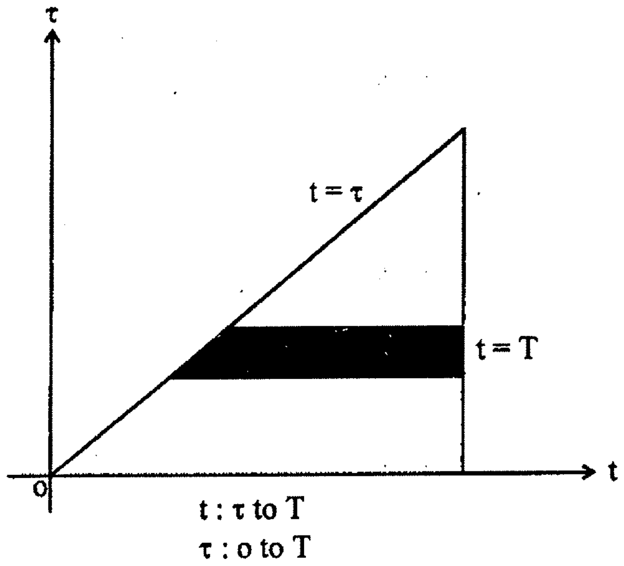
$$L\{\phi(t)\} = \int_0^\infty \phi(t)\, e^{-pt} dt = \underset{T\to\infty}{Lt} \int_0^T \phi(t)\, e^{-pt} dt$$

$$= \underset{T\to\infty}{Lt} \int_{t=0}^{T} e^{-pt}dt \int_{\tau=0}^{t} f(\tau)g(t-\tau)d\tau$$

$$= \underset{T\to\infty}{Lt} \int_{\tau=0}^{T} d\tau \int_{t=\tau}^{T} dt\, e^{-pt} g(t-\tau)f(\tau) \text{ [changing the Order of Integration]}$$

$$= \underset{T\to\infty}{Lt} \int_{\tau=0}^{T} f(\tau)d\tau \int_{t=\tau}^{T} e^{-pt} g(t-\tau)dt$$

$$= \underset{T\to\infty}{Lt} \int_{\tau=0}^{T} f(\tau)d\tau \int_{u=0}^{T-\tau} e^{-p(u+\tau)} g(u)du$$



$$= \underset{T\to\infty}{Lt} \int_{\tau=0}^{T} e^{-p\tau} f(\tau)d\tau \int_{u=0}^{T-\tau} e^{-pu} g(u)du$$

$$= \int_{0}^{\infty} e^{-p\tau} f(\tau)d\tau \int_{0}^{\infty} e^{-pu} g(u)du = F(p)G(p). \text{ [As } T\to\infty, \text{ then } T-\tau\to\infty]$$

$\therefore\ L\{\phi(t)\} = F(p)G(p).$ (Proved)

The above result can be written in the form

$$\phi(t) = L^{-1}\{F(p)\, G(p)\}.$$

Ex-21 : Evaluate $\int_0^t J_0(\tau)J_0(t-\tau)d\tau$.

Solu : Let $f(t)=\int_0^t J_0(\tau)J_0(t-\tau)\,d\tau$.

$$L\{f(t)\}=L\int_0^t J_0(\tau)J_0(t-\tau)\,d\tau$$

$$=L\{J_0(t)\}L\{J_0(t)\} \quad \text{[by convolution theorem]}$$

$$=\frac{1}{\sqrt{p^2+1}}\cdot\frac{1}{\sqrt{p^2+1}} \quad \text{since } L\{J_0(t)\}=\frac{1}{\sqrt{p^2+1}}$$

$$=\frac{1}{p^2+1}$$

$$\Rightarrow f(t)=L^{-1}\left\{\frac{1}{p^2+1}\right\}=\sin t$$

$$\Rightarrow \int_0^t J_0(\tau)J_0(t-\tau)d\tau=\sin t. \quad \text{(Ans)}$$

Ex-22 : Find $L^{-1}\left\{\frac{p}{\left(p^2+a^2\right)^2}\right\}$

Solu : $\left\{\frac{p}{\left(p^2+a^2\right)^2}\right\}=\left\{\frac{p}{p^2+a^2}\cdot\frac{1}{p^2+a^2}\right\}=F(p)G(p).$

Now $F(p)=\frac{p}{p^2+a^2}$, $f(t)=\cos at$

$G(p)=\frac{1}{p^2+a^2}$, $g(t)=\frac{1}{a}\sin at$

Now $L^{-1}\left\{\frac{p}{\left(p^2+a^2\right)^2}\right\}=L^{-1}\{F(p)G(p)\}=\int_0^t f(\tau)g(t-\tau)d\tau$

$$= \int_0^t \cos(a\tau)\frac{1}{a}\sin\{a(t-\tau)\}d\tau$$

$$= \frac{1}{a}\int_0^t \cos\ (a\tau)\ \sin\{a(t-\tau)\}d\tau.$$

$$= \frac{1}{a}\int_0^t cos\ (a\tau)(sin\ at\ cos\ a\tau - cos\ at\ sin\ a\tau)d\tau$$

$$= \frac{sin\ at}{2a}\int_0^t 2cos^2 a\tau d\tau - \frac{cos\ at}{2a}\int_0^t 2sin\ a\tau\ cos\ a\tau d\tau$$

$$= \frac{sin\ at}{2a}\left[\tau + \frac{sin\ 2a\tau}{2a}\right]_0^t - \frac{cos\ at}{2a}\left[\frac{-cos\ 2a\tau}{2a}\right]_0^t$$

$$= \frac{sin\ at}{2a}\left(t + \frac{sin\ 2at}{2a}\right) - \frac{cos\ at}{2a}\left(\frac{1-cos\ 2at}{2a}\right)$$

$$= \frac{t\ sin\ at}{2a} + \frac{sin^2\ at\ cos\ at}{2a^2} - \frac{cos\ at}{2a}\frac{sin^2\ at}{a} = \frac{t\ sin\ at}{2a} \quad \text{(Ans.)}$$

Ex-23 : Evaluate $\int_0^t J_0(\tau)J_1(t-\tau)d\tau$

Solu : We know by convolution theorem $L\{\phi(t)\} = F(p)G(p)$ (1)

Here $\phi(t) = \int_0^t J_0(\tau)J_1(t-\tau)d\tau,\ F(p) = L\{J_0(t)\} = \frac{1}{\sqrt{p^2+1}}$

$$G(p) = L\{J_1(t)\} = -L\{J_0'(t)\}$$

$$= -[p\,J_0(p) - 1]$$

$$= 1 - \frac{p}{\sqrt{p^2+1}}$$

Using the above values,

$$L\int_0^t J_0(\tau)J_1(t-\tau)d\tau = \frac{1}{\sqrt{p^2+1}}\left(1 - \frac{p}{\sqrt{p^2+1}}\right) = \frac{1}{\sqrt{p^2+1}} - \frac{p}{p^2+1}$$

$$\Rightarrow \int_0^t J_0(\tau)J_1(t-\tau)d\tau = L^{-1}\left\{\frac{1}{\sqrt{p^2+1}} - \frac{p}{p^2+1}\right\} = L^{-1}\left\{\frac{1}{\sqrt{p^2+1}}\right\} - L^{-1}\left\{\frac{p}{p^2+1}\right\}$$

$$= J_0(t) - \cos t.$$

$$\therefore \int_0^t J_0(\tau)J_1(t-\tau)d\tau = J_0(t) - \cos t. \text{ (Ans.)}$$

Ex-24 : Find $L^{-1}\left\{\frac{2p^2-4}{(p+1)(p-2)(p-3)}\right\}$. (Problem on Partial Fractions of Laplace Transform)

Solu : We have $\frac{2p^2-4}{(p+1)(p-2)(p-3)} = \frac{A}{p+1} + \frac{B}{p-2} + \frac{C}{p-3}$ (1)

Multiply both sides of (1) by $p+1$ and let $p \to -1$, then

$$A = \underset{p\to -1}{Lt} \frac{2p^2-4}{(p-2)(p-3)} = -1/6$$

Multiply both sides of (1) by $p-2$ and let $p \to 2$, then

$$B = \underset{p\to 2}{Lt} \frac{2p^2-4}{(p+1)(p-3)} = -\frac{4}{3}$$

Multiply both sides of (1) by $p-3$ and let $p \to 3$, then

$$C = \underset{p\to 3}{Lt} \frac{2p^2-4}{(p+1)(p-2)} = \frac{7}{2}$$

Thus $L^{-1}\left\{\frac{2p^2-4}{(p+1)(p-2)(p-3)}\right\} = L^{-1}\left\{\frac{-1/6}{p+1}\right\} + L^{-1}\left\{\frac{-4/3}{p-2}\right\} + L^{-1}\left\{\frac{7/2}{p-3}\right\}$

$$= (-1/6)e^{-t} - (4/3)e^{2t} + (7/2)e^{3t}.$$

Ex-25 : Find $L^{-1}\left\{\frac{3p+1}{(p-1)(p^2+1)}\right\}$.

Solu : $\frac{3p+1}{(p-1)(p^2+1)} = \frac{A}{p-1} + \frac{Bp+C}{p^2+1}$ (1) Multiply both sides by $p-1$ and let $p \to 1$

then $A = \underset{p\to 1}{Lt}\ \dfrac{3p+1}{p^2+1} = 2$ and same way $B = -2, C = 1$

$$L^{-1}\left\{\frac{3p+1}{(p-1)(p^2+1)}\right\} = L^{-1}\left\{\frac{2}{p-1}\right\} + L^{-1}\left\{\frac{-2p+1}{p^2+1}\right\}$$

$$= L^{-1}\left\{\frac{2}{p-1}\right\} + L^{-1}\left\{\frac{-2p}{p^2+1}\right\} + L^{-1}\left\{\frac{1}{p^2+1}\right\} = 2e^{-t} - 2\cos\ t + \sin\ t.$$

3.11 Complex Inversion Formula : (Bromwich's Integral Formula) :

If $F(p) = L\{f(t)\}$ then $L^{-1}\{F(p)\}$ is given by

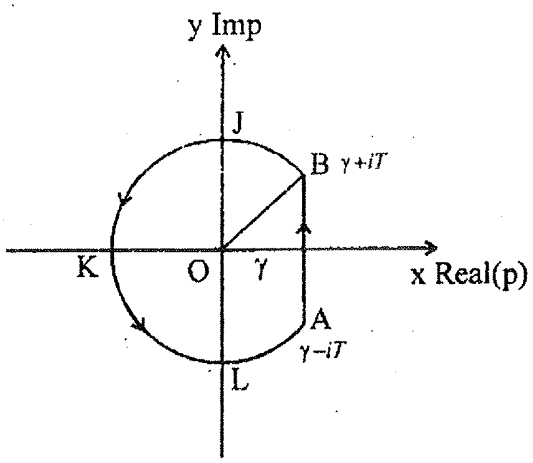
$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(p)\, e^{pt} dp \text{ for } t > 0$$

$$= 0 \text{ for } t < 0$$

where γ is a real constant is choosen in such a way that $p = \gamma$ lies to the right of all the sungularities of $F(p)e^{pt}$. This is known as complex inversion formula or Bromwich's Integral Formula.

Proof : To evaluate (1) let us consider the contour $\dfrac{1}{2\pi i}\oint_c F(p)e^{pt} dp$ (2)

where c is the contour is given by the figure. Then contour c is composed of the line AB and they are $BJKLA$ by Γ of a circle of large radius R having its centre at the origin O.



Figure

$$\oint_c F(p)\,e^{pt}dp = \underset{R\to\infty}{Lt}\left(\int_{AB} + \int_{\Gamma}\right) F(p)e^{pt}dp$$

$$= \underset{R\to\infty}{Lt}\left[\int_{\gamma-iT}^{\gamma+iT} F(p)e^{pt}dp + \int_{\Gamma} F(p)e^{pt}dp\right]$$

$$\therefore \oint_c F(p)\,e^{pt}dp = \underset{R\to\infty}{Lt}\int_{\gamma-iT}^{\gamma+iT} F(p)\,e^{pt}dp = \int_{\gamma-i\infty}^{\gamma+i\infty} F(p)\,e^{pt}dp$$

Now $2\pi i\sum$ Residues of $F(p)e^{pt}$ at the poles of $F(p) = \int_{\gamma-i\infty}^{\gamma+i\infty} F(p)e^{pt}dp$

$$\Rightarrow 2\pi i\sum R = \int_{\gamma-i\infty}^{\gamma+i\infty} F(p)e^{pt}dp$$

$$\Rightarrow \sum R = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(p)e^{pt}dp = f(t)\,.$$

$$\Rightarrow f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(p)e^{pt}dp = \sum R.\ \text{(Proved)}$$

Ex-26 : Evaluate $L^{-1}\left\{\frac{1}{(p+1)(p-2)^2}\right\}$ by using the method of residues.

Solu : According to Complex Inversion formula, we have

$$L^{-1}\left\{\frac{1}{(p+1)(p-2)^2}\right\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{pt}}{(p+1)(p-2)^2}dp$$

$$= \frac{1}{2\pi i}\oint_c \frac{e^{pt}dp}{(p+1)(p-2)^2} = \sum \text{ residues of } \frac{e^{pt}}{(p+1)(p-2)^2} \text{ at poles } p=-1 \text{ and } p=2.$$

Now, residue at simple pole $p=-1$ is

$$\underset{p\to-1}{Lt}(p+1)\left\{\frac{e^{pt}}{(p+1)(p-2)^2}\right\} = \frac{1}{9}e^{-t}$$

and residue at double pole $p=2$ is

$$\underset{p\to 2}{Lt}\frac{1}{1!}\frac{d}{dp}\left[(p-2)^2\left\{\frac{e^{pt}}{(p+1)(p-2)^2}\right\}\right]$$

$$=\underset{p\to 2}{Lt}\frac{d}{dp}\left(\frac{e^{pt}}{p+1}\right)=\underset{p\to 2}{Lt}\frac{t(p+1)e^{pt}-e^{pt}}{(p+1)^2}=\frac{1}{3}te^{pt}-\frac{1}{9}e^{pt}.$$

Then $$L^{-1}\left\{\frac{1}{(p+1)(p-2)^2}\right\}=\sum\text{Residues}=\frac{1}{9}e^{-t}+\frac{1}{3}te^{2t}-\frac{1}{9}e^{2t}\quad\text{(Ans.)}$$

Ex-27 : Evaluate $L^{-1}\left\{\frac{p}{(p+1)^3(p-1)^2}\right\}$ by using the method of residues.

Solu : According to Complex Inversion formula, we have

$$L^{-1}\left\{\frac{p}{(p+1)^3(p-1)^2}\right\}=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{pt}}{(p+1)^3(p-1)^2}dp=\frac{1}{2\pi i}\oint_c\frac{e^{pt}dp}{(p+1)^3(p-1)^2}$$

$$=\sum\text{residues of }\frac{e^{pt}}{(p+1)^3(p-1)^2}\text{ at poles }p=-1\ \&\ p=2.$$

Now, residue at $p=-1$ is

$$\underset{p\to -1}{Lt}\frac{1}{2!}\frac{d^2}{dp^2}\left[(p+1)^3\frac{pe^{pt}}{(p+1)^3(p-1)^2}\right]$$

$$\underset{p\to -1}{Lt}\frac{1}{2}\frac{d^2}{dp^2}\left[\frac{pe^{pt}}{(p-1)^2}\right]=\frac{1}{16}e^{-t}\left(1-2t^2\right)$$

and residue at $p=1$ is

$$\underset{p\to 1}{Lt}\frac{1}{1!}\frac{d}{dp}\left[(p-1)^2\frac{pe^{pt}}{(p+1)^3(p-1)^2}\right]=\underset{p\to 1}{Lt}\frac{d}{dp}\left[\frac{pe^{pt}}{(p-1)^2}\right]=\frac{1}{16}e^{t}(2t-1)$$

Then $$L^{-1}\left\{\frac{p}{(p+1)^3(p-1)^2}\right\}=\sum\text{Residues}=\frac{1}{16}e^{-t}\left(1-2t^2\right)+\frac{1}{16}e^{t}(2t-1).\quad\text{(Ans.)}$$

Some Special Functions :

1. The Error function is defined as $erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du.$

2. The Complementary Error function is defined as $erfc(t) = 1 - erf(t)$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2} du.$$

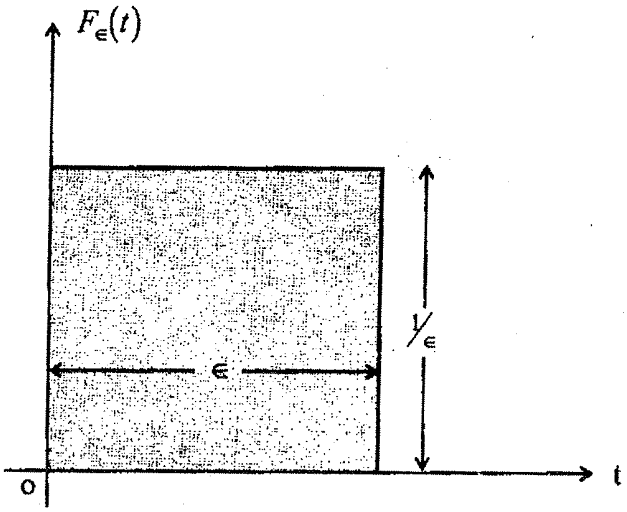
3. <u>The unit Impulse function</u> or <u>Dirac Delta function</u> :

   Consider the function $F_\epsilon(t) = \begin{cases} 1/\epsilon, & 0 \le t \le \epsilon \\ 0, & t > \epsilon \end{cases}$

   where $\epsilon > 0$.

It is geometrically evident that as $\epsilon > 0$, the height of the rectangular shaded region increases indefinitely and the width decreases in such a way that the area is always equal to 1 i.e., $\int_0^\infty F_\epsilon(t)dt = 1.$

In the limifing sense i.e., $\underset{\epsilon \to 0}{Lt} F_\epsilon(t)$ is equal to $\delta(t)$ and is called the unit impulse function or Dirac Delta function and its properties are



1. $\int_0^\infty \delta(t)dt = 1$

2. $\int_0^\infty \delta(t)G(t)dt = G(o),$

3. $\int_0^\infty \delta(t-a)G(t)dt = G(a),$ for any continuous function $G(t)$.

3.12 Applications of Laplace Transform to Ordinary and Partial Differential Equations :

Using the laplace transform, the problems of solving ordinary and partial differential equations, can be reduced to algebric and ordinary differential equations respectively, which are easier to solve than solving the original ones.

A. ODE with Constant Coefficients :

Ex-28 : Find the solution of the equation

$y'' - 3y' + 2y = 4e^{2t}$ where $y = y(t)$. subject to the conditions $y(0) = -3,\ y'(0) = 5.$

Solu : Let $L\{y(t)\} = \bar{y}(p)$

Given that $y'' - 3y' + 2y = 4e^{2t}$

Taking Laplace Transform of above we get,

$$L\{y''(t)\} - 3L\{y'(t)\} + 2L\{y(t)\} = 4L\{e^{2t}\}$$

$$\Rightarrow p^2\bar{y}(p) - py(0) - y'(0) - 3[p\bar{y}(p) - y(0)] + 2\bar{y}(p) = 4.\frac{1}{p-2}$$

$$\Rightarrow p^2\bar{y}(p) + 3p - 5 - 3p\bar{y}(p) + 3(-3) + 2\bar{y}(p) = \frac{4}{p-2}$$

$$\Rightarrow (p^2 - 3p + 2)\bar{y}(p) = \frac{4}{p-2} - 3p + 14$$

$$\Rightarrow \bar{y}(p) = \left(\frac{4}{p-2} - 3p + 14\right)\frac{1}{(p^2 - 3p + 2)} = \left(\frac{4}{(p-2)} - 3p + 14\right)\frac{1}{(p-1)(p-2)}$$

$$= \frac{-3p^2 + 20p - 24}{(p-1)(p-2)^2} = \frac{-7}{p-1} + \frac{4}{p-2} + \frac{4}{(p-2)^2}$$

Now taking inverse laplace transform of above we get,

$$L^{-1}\{\bar{y}(p)\} = L^{-1}\left\{\frac{-7}{p-1}\right\} + L^{-1}\left\{\frac{4}{p-2}\right\} + L^{-1}\left\{\frac{4}{(p-2)^2}\right\}$$

$$\Rightarrow y(t) = -7e^{t} + 4e^{2t} + 4te^{2t}.$$

which is the required solution.

Ex-29 : Solve $y''(t) + a^2 y(t) = f(t),\ y(0) = 1,\ y'(0) = -2.$

Solu : Let $L\{y(t)\} = \bar{y}(p)$

Given that $y''(t) + a^2 y(t) = f(t).$

Taking Laplace Transform of above we get,

$$L\{y''(t)\} + L\{a^2 y(t)\} = L\{f(t)\}$$

$$\Rightarrow p^2\bar{y}(p) - py(0) - y'(0) + a^2\bar{y}(p) = F(p),\ \text{where } F(p) = L\{f(t)\},\ \text{(say)}$$

$$\Rightarrow p^2\bar{y}(p) - p + 2 + a^2\bar{y}(p) = F(p).$$

$$\Rightarrow \bar{y}(p)\left[p^2 + a^2\right] = (p-2) + F(p).$$

$$\Rightarrow \bar{y}(p) = \frac{p-2}{p^2+a^2} + F(p).\frac{1}{p^2+a^2}$$

Now taking inverse laplace transform of above we get

$$L^{-1}\{\bar{y}(p)\} = L^{-1}\left\{\frac{p-2}{p^2+a^2}\right\} + L^{-1}\left\{F(p)\frac{1}{p^2+a^2}\right\}$$

$$\Rightarrow y(t) = L^{-1}\left\{\frac{p}{p^2+a^2}\right\} - \frac{2}{a}L^{-1}\left\{\frac{a}{p^2+a^2}\right\} + L^{-1}\left\{F(p)\frac{a}{p^2+a^2}\right\}\frac{1}{a}$$

$$\Rightarrow y(t) = \cos at - \frac{2\sin at}{a} + \frac{1}{a}\int_0^t f(\tau)\sin\{a(t-\tau)\}d\tau.\ \text{[by convolution theorem]}.$$

which is the required solution.

B. ODE with Variable Coefficients :

Ex-30 : Solve $ty''(t) + y'(t) + 4ty(t) = 0,$ subject to the conditions $y(0) = 3,\ y'(0) = 0$

Solu : Let $L\{y(t)\} = \bar{y}(p)$.

Given that $ty''(t) + y'(t) + 4ty(t) = 0$.

Now taking the laplace transform of above, we get,

$$L\{ty''(t)\} + L\{y'(t)\} + L\{4ty(t)\} = 0.$$

$$\Rightarrow -\frac{d}{dp}\left[p^2\bar{y}(p) - py(0) - y'(0)\right] + \left[p\bar{y}(p) - y(0)\right] - 4\frac{d}{dp}\bar{y}(p) = 0$$

$$\Rightarrow -2p\bar{y}(p) - p^2\frac{d}{dp}\bar{y}(p) + y(0) + p\bar{y}(p) - y(0) - 4\frac{d}{dp}\bar{y}(p) = 0$$

$$\Rightarrow (p^2+4)\frac{d}{dp}\bar{y}(p) + p\bar{y}(p) = 0$$

$$\Rightarrow \frac{d\bar{y}(p)}{\bar{y}(p)} + \frac{p}{p^2+4}dp = 0$$

Integrating we get, $\log \bar{y}(p) + \frac{1}{2}\log(p^2+4) = \log c$, [$c$ = Integration Constant]

$$\Rightarrow \bar{y}(p) = \frac{c}{\sqrt{p^2+4}}$$

Taking the inverse laplace transform of above we get,

$$L^{-1}\{\bar{y}(p)\} = L^{-1}\left\{\frac{c}{\sqrt{p^2+4}}\right\}$$

$\Rightarrow y(t) = c\,J_o(2t)$, since $J_o(t) = L^{-1}\left\{\frac{1}{\sqrt{p^2+1}}\right\}$ and is the Bessel function of order zero.

To find the value of c, we use $y(0) = 3$,

$\therefore\ y(0) = 3 = c\,J_c(2.0) = c\,J_o(0) = c.1$ since $J_0(0) = 1, y(0) = 3$

$\Rightarrow c = 3.$

$\therefore\ y(t) = 3J_o(2t).$

which is the required solution.

Ex-31 : Solve $ty''(t)+2y'(t)+ty(t)=0,\ y(0+)=1,\ y(\pi)=0$.

Solu : Let $L\{y(t)\}=\bar{y}(p)$.

Given that $ty''(t)+2y'(t)+ty(t)=0$.

Now taking the laplace transform of above we get,

$$L\{ty''(t)\}+L\{2y'(t)\}+L\{ty(t)\}=0.$$

$$\Rightarrow -\frac{d}{dp}\left[p^2\bar{y}(p)-py(0)-y'(0)\right]+2\left[p\bar{y}(p)-y(0)\right]-\frac{d}{dp}\bar{y}(p)=0.$$

$$\Rightarrow -2p\bar{y}(p)-p^2\frac{d}{dp}\bar{y}(p)+y(0)+2p\bar{y}(p)-2y(0)-\frac{d}{dp}\bar{y}(p)=0.$$

$$\Rightarrow \left(p^2+1\right)\frac{d}{dp}\bar{y}(p)+y(0)=0.$$

$$\Rightarrow \frac{d\bar{y}(p)}{dp}+\frac{1}{p^2+1}=0$$

Integrating, we get,

$\bar{y}(p)+tan^{-1}p=A,$ A = Integration constant.

$$\Rightarrow \bar{y}(p)=A-tan^{-1}p \qquad (1)$$

Since $\bar{y}(p)=\int_0^\infty y(t)e^{-pt}dt$

$$\bar{y}(\infty)=0.$$

Put $p=\infty$ in (1) we have, $\bar{y}(\infty)=A-tan^{-1}(\infty)$

$$\Rightarrow 0=A-\pi/2$$

$$\Rightarrow A=\pi/2$$

So from (1), $\bar{y}(p)=\pi/2-tan^{-1}p=tan^{-1}(1/p)$

$$L^{-1}\{\bar{y}(p)\}=L^{-1}\left\{tan^{-1}(1/p)\right\}$$

$\Rightarrow y(t)=\frac{sin\ t}{t}$. which is the required solution.

C. Simultaneous ODE

Ex-32. Solve

$$\left.\begin{aligned}\frac{dx}{dt} &= 2x-3y\\ \frac{dy}{dt} &= y-2x\end{aligned}\right\} \text{Subject to } x(0)=8,\ y(0)=3.$$

Solu : Let $L\{x(t)\}=\bar{x}(p),\ L\{y(t)\}=\bar{y}(p)$.

Given that $\frac{dx}{dt}=2x-3y$ (1)

$\frac{dy}{dt}=y-2x$ (2)

Now taking laplace transform of (1) & (2) we get,

$$L\left\{\frac{dx}{dt}\right\}=2L(x)-3L(y)\Rightarrow p\bar{x}(p)-x(0)=2\bar{x}(p)-3\bar{y}(p)$$

$$\Rightarrow (p-2)\bar{x}(p)+3\bar{y}(p)=8 \quad (3)$$

$$L\left\{\frac{dy}{dt}\right\}=L\{y\}-2L\{x\}\Rightarrow p\bar{y}(p)-y(0)=\bar{y}(p)-2\bar{x}(p)$$

$$\Rightarrow 2\bar{x}(p)+(p-1)\bar{y}(p)=3 \quad (4)$$

Solving (3) & (4) we get

$$\bar{x}(p)=\frac{\begin{vmatrix}8 & 3\\ 3 & p-1\end{vmatrix}}{\begin{vmatrix}p-2 & 3\\ 2 & p-1\end{vmatrix}}=\frac{8p-17}{p^2-3p-4}=\frac{8p-17}{(p+1)(p-4)}=\frac{5}{p+1}+\frac{3}{p-4} \quad (5)$$

$$\bar{y}(p)=\frac{\begin{vmatrix}p-2 & 8\\ 2 & 3\end{vmatrix}}{\begin{vmatrix}p-2 & 3\\ 2 & p-1\end{vmatrix}}=\frac{3p-22}{p^2-3p-4}=\frac{3p-22}{(p+1)(p-4)}=\frac{5}{p+1}-\frac{2}{p-4} \quad (6)$$

Now taking laplace transform of (5) & (6) we get,

$$L^{-1}\{\bar{x}(p)\} = L^{-1}\left\{\frac{5}{p+1}\right\} + L^{-1}\left\{\frac{3}{p-4}\right\} \Rightarrow x(t) = 5e^{-t} + 3e^{+4t}$$

$$L^{-1}\{\bar{y}(p)\} = L^{-1}\left\{\frac{5}{p+1}\right\} - L^{-1}\left\{\frac{2}{p-4}\right\} \Rightarrow y(t) = 5e^{-t} - 2e^{4t}$$

$$\therefore \quad \left.\begin{aligned} x(t) &= 5e^{-t} + 3e^{4t} \\ y(t) &= 5e^{-t} - 2e^{4t} \end{aligned}\right\} \text{(Ans)}$$

D. Partial Differential Equations :

Ex-33. A Semi-infinite solid $x > 0$ is initially at temperature zero. At time $t = 0$, a constant temperature $u_o > 0$ is applied and maintained at the face $x = 0$. Find the temperature at any point of the solid at any later time $t > 0$.

Solu : The boundary - value problem for the determination of the temperature $u(x, t)$ at any point x and any time t is $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$, $\qquad x > 0, t > 0. \qquad (1)$

$u(x, 0) = 0, u(0, t) = u_0$, $|u(x,t)| < M$, This last condition expresses the requirement that the temperature is bounded $\forall\ x, t$.

Taking laplace transform of (1) we get, w.r. to, t.

$$L\left\{\frac{\partial u}{\partial t}\right\} = L\left\{k\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow p\bar{u}(x,p) - u(x,0) = k\frac{d^2}{dx^2}\bar{u}(x,p) \qquad \text{where} \quad \bar{u}(x,p) = \int_0^\infty u(x,t)e^{-pt}dt$$

$$\Rightarrow \frac{d^2}{dx^2}\bar{u}(x,p) - \frac{p}{k}\bar{u}(x,p) = 0, \text{ since } u(x,o) = 0, \qquad (2)$$

Also $\qquad \bar{u}(0,p) = L\{u(0,t)\} = \int_0^\infty u(0,t)e^{-pt}dt = \int_0^\infty u_0 e^{-pt}dt = \dfrac{u_0}{p} \qquad (3)$

and $\bar{u}(x,p)$ is required to be bounded.

Solving (2) we find,

$$\bar{u}(x,p) = c_1 e^{\sqrt{p/k}\,x} + c_2 e^{-\sqrt{p/k}\,x} \qquad (4)$$

Since $\bar{u}(x,p)$ is bounded as $x \to \infty$, then equation is also bounded and for this $c_1 = 0$.

$$\therefore \quad \bar{u}(x,p) = c_2 e^{-\sqrt{p/k}\,x}$$

Put $x = 0$ in above, $\bar{u}(o,p) = c_2$

Using (3), we have from above, $c_2 = \dfrac{u_0}{p}$

$\therefore$ Equation (5) becomes, $\bar{u}(x,p) = \dfrac{u_0}{p} e^{-\sqrt{(p/k)}x}$

Now taking inverse laplace transform of above we get,

$$L^{-1}\{\bar{u}(x,p)\} = L^{-1}\left\{\frac{u_0}{p} e^{-\sqrt{(p/k)}x}\right\} = u_0 L^{-1}\left\{\frac{1}{p} e^{-\sqrt{(p/k)}x}\right\}$$

$$\Rightarrow u(x,t) = u_0 \, erfc\left(x/_{2\sqrt{kt}}\right) = u_0\left[1 - \frac{2}{\sqrt{\pi}} \int_0^{(x/_{2\sqrt{kt}})} e^{-u^2 du}\right].$$

3.13 Unit Summary :

At the end of the discussion, whole unit is summarised as follows :

1. The laplace transform have been used to solve the PDE with initial or boundary value problems.
2. The laplace transform have been used to solve the ODE with initial or boundary value problems.
3. The laplace transform have been utilized to solve certain definite integrals.

3.14 Exercises :

Ex- 1 Find $L\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$

2. Find $L^{-1}\left\{\dfrac{1}{p^3(p^2+1)}\right\}$

3. Find $L^{-1}\left\{\dfrac{1}{(p+1)(p^2+1)}\right\}$

4. Find $\int_0^{\tau} \sin \tau \cos (t-\tau) d\tau$

5. If $L^{-1}\left\{\frac{e^{-\frac{1}{p}}}{\sqrt{p}}\right\}=\frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$ then find $L^{-1}\left\{\frac{e^{-\frac{a}{p}}}{\sqrt{p}}\right\}$

6. Find $L^{-1}\left\{\frac{1}{\left(p^2+1\right)^2}\right\}$ by using the method of residues

7. Find $L^{-1}\left\{log\left(\frac{p+b}{p+a}\right)\right\}$, $a>0,\ b>0$

8. Solve $y'''(t)-3y''(t)+3y'(t)-y=t^2e^t$, $y(0)=1,\ y'(0)=0,\ y''(0)=-2.$

9. Find the solution of the equation $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$, $x>0,\ t>0$, which remains bounded for $x\geq 0$, and obeys the following initial and boundary conditions, $u(x,0)=0,\ u(0,t)=f(t).$

Ans : 1. $\frac{72}{p^5}-\frac{12}{p^4}+\frac{4}{p+3}-\frac{10}{p^5+25}+\frac{3p}{p^2+4}.$

2. $\frac{t^2}{2}+cos\ t-1.$

3. $\frac{1}{2}sin\ t-cos\ t+e^{-t}.$

4. $\frac{t}{2}sin\ t.$

5. $\frac{cos\ 2\sqrt{at}}{\sqrt{\pi t}}.$

6. $\frac{1}{2}(sin\ t-t\ cos\ t).$

7. $\frac{e^{-at}-e^{-bt}}{2}.$

8. $y(t)=e^t-te^t-\frac{t^2e^t}{2}+\frac{t^5e^t}{60}.$

9. $u(x,t)=\frac{x}{2a\sqrt{\pi}}\int_0^t\frac{f(\tau)}{(t-\tau)^{3/2}}e^{-\frac{x^2}{4a^2(t-\tau)}}d\tau.$

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M.Sc. Course in
Applied Mathematics with Oceanology
and
Computer Programming

Part - II

Paper - VIII Group - A

Module No. 88
Integral Equations

Structure :

4.1 Introduction :

This module deals with as Integral equation where the unknown function is presence under the Integral signs. Integral equation occur naturally in many fields of Mechanics and Mathematical Physics. They also arise as representation formulas for the solutions of differential equations. Indeed, a differential equation can be replaced by an Integral equation which incorporates its boundary conditions. As such, each solution of the Integral equation automatically satisfies these boundary conditions. Integral equations also from one of the most useful tools in many branches of pure analysis, such as the theories of functional analysis and stochastic process.

4.2 Objectives :

The main objective of this unit is defined as : (1) Many physical problems which are usually solved by differential equation methods can be solved more effectively by Integral equation method.

(2) Some problems around in many applied fields, particularly in Applied Mathemetics, Theoretical Mechanics and Mathematical Physics can be solved only by an Integral equation methods and that kind of problem can not be solved by standard methods of differential equations.

Keywords :

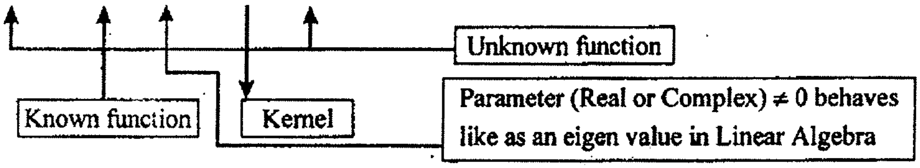
Fredholm Integral Equation, Volterra Integral Equation, Abel Integral Equation, Fredholm Alternative, Eigen Value & Eigen Vector.

4.3 Definition of Integral Equation :

An integral equation is an equation in which an unknown function appears under one or more integral sign.

Example :

$$y(x) = f(x) + \lambda \int k(x,t)\, y(t)\, dt$$



The most general form of an integral equation is

$$h(x)\, y(x) = f(x) + \lambda \int_{a}^{b(x)} k(x,\ t)\, y(t)\, dt \qquad (1)$$

where $h(x), f(x), k(x,\ t)$ are given function and λ, a are constant, b either constant or variable function of x, and $y(x)$ is the unknown function. The function $k(x,\ t)$ is called the Kernel, $a \le x \le b,\ a \le t \le b$.

Linear and non linear Integral equations :

Linear Integral Equation :

An integral equation is called linear if only linear operations are performed in it upon the unknown function.

Example : 1. $f(x)=\int_{a}^{x} k(x, t)y(t)\, dt$

2. $y(x)=f(x)+\lambda\int_{a}^{b} k(x, t)y(t)\, dt$

Non Lincar Integral Equation :

An integral equation which is non-linear is known as a non-linear integral equation.

Example : $y(x)=\int_{a}^{b} k(x, t)\left[y(t)\right]^2\, dt$

4.4 Classification of Integral Equations :

1. Fredholm Integral Equation
2. Volterra Integral Equation.

1. Fredholm Integral Equation :

When the upper limit of an integral equation (1) is constant i.e., $b(x)=b$, then the integral equation is said to be Fredholm Integral Equation

Example : $h(x)\, y(x)=f(x)+\lambda\int_{a}^{b} k(x, t)y(t)dt$

Volterra Integral Equation :

When the upper limit of an integral equation (1) is x, i.e., $b(x)=x$, then the integral equation is said to be Volterra Integral Equation

Example : $h(x)\, y(x)=f(x)+\lambda\int_{a}^{x} K(x, t)y(t)\, dt$

Integral Equation of 1st Kind :

If an integral equation, the function to be determined appears only under the integral sign, then the integral equation is said to be of 1st kind.

Hence, when $h(x)=0$, the integral equation (1) reduces to

$$0=f(x)+\lambda\int_{a}^{b(x)} k(x, t)y(t)\, dt$$

This integral equation is called integral equation of 1st kind.

Fredholm Integral Equation of 1st Kind :

A linear integral equation of the form $f(x)+\lambda\int_{a}^{b}k(x,\ t)y(t)\ dt=0$, is known as Fredholm Integral equation of 1st kind.

Volterra Integral Equation of 1st Kind :

A linear integral equation of the form $f(x)+\lambda\int_{a}^{x}k(x,\ t)y(t)\ dt=0$, is known as Volterra Integral equation of 1st kind.

Integral Equation of 2nd Kind :

If an integral equation, the function to be determined appears under the integral sign as well as outside also, then it is said to be of an Integral Equation of 2nd kind.

Hence when $h(x)=1$, the integral equation (1) reduces to

$$y(x)=f(x)+\lambda\int_{a}^{b(x)}k(x,\ t)y(t)\ dt$$

This integral equation is called integral equation of 2nd kind.

Fredholm Integral Equation of 2nd Kind :

A linear integral equation of the form $y(x)=f(x)+\lambda\int_{a}^{b}k(x,\ t)y(t)\ dt$ is known as Fredholm Integral equation of 2nd kind.

Volterra Integral Equation of 2nd Kind :

A linear integral equation of the form $y(x)=f(x)+\lambda\int_{a}^{x}k(x,\ t)y(t)\ dt$ is known as Volterra Integral Equation of 2nd kind.

Homogeneous Integral Equation :

An integral equation of the form [i.e., put $f(x)=0,\ h(x)=1$ in (1) *IE*] $y(x)=\lambda\int_{a}^{b(x)}k(x,\ t)y(t)\ dt$ is known homogeneous integral equation.

Homogeneous Fredholm Integral Equation of 2nd Kind :

An integral equation of the form $y(x)=\lambda\int_{a}^{b}k(x,\ t)y(t)\ dt$ is known as homogeneous Fredholm Integral equation of 2nd kind.

Homogeneous Volterra Integral equation of 2nd Kind :

An integral equation of the form $y(x)=\lambda\int_a^x k(x,t)y(t)\,dt$ is known as Homogeneous Volterra Integral equation of 2nd kind.

Non Homogeneous Integral equation :

An integral equation of the form [i.e., put $h(x)=1$ in *IE* (1)] $y(x)=f(x)+\lambda\int_a^{b(x)} k(x,t)y(t)\,dt$ is known as Non Homogeneous Integral equation.

Non Homogeneous Fredholm Integral equation of 2nd Kind :

An integral equation of the form $y(x)=f(x)+\lambda\int_a^b k(x,t)y(t)\,dt$ is known as Non Homogeneous Fredholm Integral equation of 2nd kind.

Non homogeneous Volterra Integral equation of 2nd Kind :

An integral equation of the form $y(x)=f(x)+\lambda\int_a^x K(x,t)y(t)\,dt$ is known as Non homogeneous Volterra Integral equation of 2nd kind.

Integral equation of 3rd Kind :

When $h(x)\neq 0$ in *IE* (1), then integral equation is known as Integral equation of 3rd kind :

Singular Integral Equation :

An integral equation is said to be Singular Integral equation when one or both the limits of integration become infinite or when the kernel has a singularity within the range of integration at one or more points.

Example 1 : $y(x)=f(x)+\lambda\int_{-\infty}^{\infty} e^{-|x-t|}y(t)\,dt$

2 : $f(x)=\int_0^x \frac{1}{(x-t)^{\alpha}}y(t)dt,\ 0<\alpha<1$

4.5 Different Kinds of Kernels :

1. **Difference Kernel :** If the kernel depends on the difference $(x-t)$, then the kernel is said to be difference kernel.

Example 1 : Kernel $k(x, t) = \frac{1}{(x-t)^{\alpha}}$

2 : $y(x) = f(x) + \lambda \int_0^x \cos(x-t)\, y(t)\, dt$

2. Symmetric Kernels : A complex valued function $k(x, t)$ is called symmetric if $k(x, t) = k^*(t, x)$ where $k^*(t, x)$ denotes the conjugate of $k(x, t)$. And for a real symmtric kernel $k(x, t) = k(t, x)$.

Example 1 : Kernel $k(x, t) = \sin(x+t)$

2 : Kernel $k(x, t) = xt + x^2 t^2$.

3. Separable or Degenerate Kernel : A kernel $k(x, t)$ is said to be separable or degenerate if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of x only and a function of t only, i.e., $k(x, t) = \sum_{i=1}^{n} a_i(x)\, b_i(t)$

Example 1. Kernel $k(x, t) = 3xt$, but $k(x, t) = e^{xt}$ is not separable.

4.6 Solution of an Integral Equation :

Def.: Consider the linear integral equations

$$h(x)\, y(x) = f(x) + \lambda \int_a^b k(x, t)\, y(t)\, dt \qquad (*)$$

and $$h(x)\, y(x) = f(x) + \lambda \int_a^x k(x, t)\, y(t)\, dt \qquad (**)$$

A solution of the integral equation (*) or (**) is a function $y(x)$, which, when substituted into the equation, reduces it to an identity (with respect to x).

Ex.: 1. Show that the function $y(x) = (1+x^2)^{\frac{3}{2}}$ is a solution of the Volterra integral equation

$$y(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2}\, y(t)\, dt.$$

Solu.: Given integral equation is $y(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2}\, y(t)\, dt$ (1)

Also given, $y(x)=\left(1+x^2\right)^{-3/2}$ (2)

From (2), $y(t)=\left(1+t^2\right)^{-3/2}$ (3)

Now, R.H.S. of (1) $= \dfrac{1}{\left(1+x^2\right)}-\displaystyle\int_0^x \frac{t}{1+x^2}\left(1+t^2\right)^{-3/2}\,dt$, using (3)

$$= \frac{1}{1+x^2}-\frac{1}{1+x^2}\int_0^x \frac{t}{\left(1+t^2\right)^{3/2}}\,dt$$

$$= \frac{1}{1+x^2}-\frac{1}{1+x^2}\int_0^{x^2}(1+k)^{-3/2}\frac{1}{2}\,dk, \quad \text{Putting } t^2=k$$

$$= \frac{1}{1+x^2}-\frac{1}{1+x^2}\frac{1}{2}\left[\frac{(1+k)^{-1/2}}{-1/2}\right]_0^{x^2}$$

$$= \frac{1}{1+x^2}-\frac{1}{1+x^2}\left[\frac{\left(1+x^2\right)^{-1/2}}{-1/2}+\frac{1}{1/2}\right]\frac{1}{2}$$

$$= \frac{1}{1+x^2}-\frac{1}{1+x^2}\left[1-\frac{1}{\left(1+x^2\right)^{1/2}}\right]=\frac{1}{\left(1+x^2\right)^{3/2}}=y(x)=L.H.S \quad \text{by (2)}$$

Hence $y(x)=\left(1+x^2\right)^{-3/2}$ is a solution of given integral equation.

4.7 Applications to Ordinary Differential Equations :

Ex. 1: Reduce the ODE $\dfrac{d^2y}{dx^2}=\lambda y(x)$ to an Integral Equation.

Solu.: Let us write the given ODE $\dfrac{d^2y}{dx^2}=\lambda y(x)=F(x)$ (say). Now integrating to above within the limits a to x *w. r.* to x.

$$\int_a^x \frac{d}{dx}\left(\frac{dy}{dx}\right) dx = \int_a^x F(\zeta)\, d\zeta$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_a^x = \int_a^x F(\zeta)\, d\zeta$$

$$\Rightarrow \frac{dy}{dx} - y'(a) = \int_a^x F(\zeta)\, d\zeta$$

$$\Rightarrow \frac{dy}{dx} = \int_{\zeta=a}^x F(\zeta)\, d\zeta + C_1 \qquad [\text{Let } y'(a) = C_1 = \text{Const.}]$$

Again integrating to above *w. r.* to x within the limits a to x. we get

$$\int_a^x \frac{dy(x)}{dx} dx = \int_{\zeta=a}^x G(\zeta)\, d\zeta + C_1 x + C_1 a, \qquad \text{Let} \int_a^x F(\zeta)\, d\zeta = G(x)$$

$$y(x) = \int_{\zeta=a}^x \int_{t=a}^{\zeta} F(t) dt + C_1 x + C_2$$

$$= \lambda \int_{\zeta=a}^x \int_{t=a}^{\zeta} y(t) dt\, d\zeta + C_1 x + C_2 \quad ....(*)$$

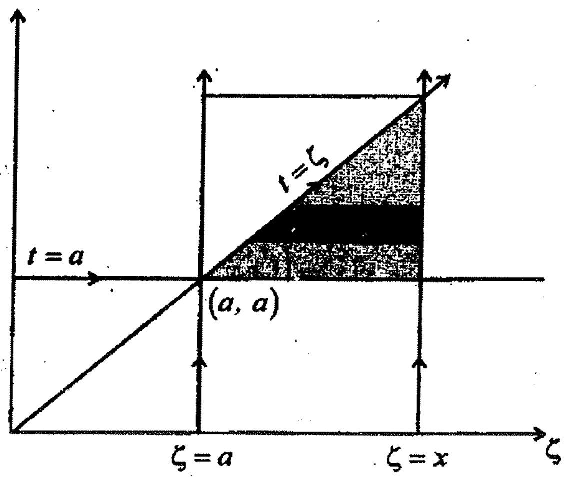
Now $\displaystyle \int_{\zeta=a}^x \int_{t=a}^{\zeta} y(t) dt\, d\zeta = \int_{t=a}^x y(t) \int_{\zeta=t}^x d\zeta\, dt$

[Changing the Order of Integration]

$$= \int_{t=a}^x (x-t) y(t) dt$$

Using above in (*) we get

$$y(x) = \lambda \int_{t=a}^x (x-t) y(t) dt + C_1 x + C_2$$



$\zeta = t$ to x
$t = a$ to x

Ex.-3 Reduce the following BVP to an IE $y''(x) = -\lambda y(x)$ with boundary conditions $y(0) = 0,\ y(1) = 0$.

Solu.: The ODE is $y''(x) = -\lambda\, y(x)$.

Now let us write $y''(x) = -\lambda y(x) = f(x)$ (say)

Now integrating the above equation within the limits 0 to x, we get,

$$\int_0^x y''(x)\,dx = \int_0^x f(\zeta)\,d\zeta$$

$$\Rightarrow y'(x) - y'(0) = \int_0^x f(\zeta)\,d\zeta$$

$$\Rightarrow y'(x) = C_1 + \int_0^x f(\zeta)\,d\zeta \qquad [Let\ y'(0) = C_1]$$

$$\Rightarrow y'(x) = C_1 + F(x), \qquad Let\ F(x) = \int_0^x f(\zeta)\,d\zeta \ \ldots\ldots\ldots\ (*)$$

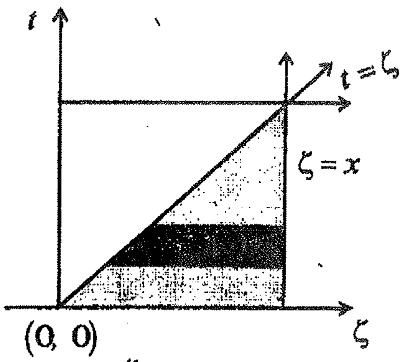
Again integrating the above equation within the same limits

$$\int_0^x y'(x)dx = \int_0^x C_1 dx + \int_0^x F(\zeta)\,d\zeta$$

$$\Rightarrow y(x) - y(0) = C_1 x + \int_{\zeta=0}^{x}\int_{t=0}^{\zeta} f(t)\,dt\,d\zeta \qquad \text{[Using (*)]}$$

$$\Rightarrow y(x) = C_1 x + \int_{t=0}^{x}\int_{\zeta=t}^{x} f(t)\,d\zeta\,dt$$

[Changing the Order of Integration]



ζ : t to x
t : 0 to x

$$\Rightarrow y(x) = C_1 x + \int_{t=0}^{x} f(t)\,(x-t)\,dt$$

$$\Rightarrow y(x) = C_1 x - \lambda\int_0^x y(t)\,(x-t)\,dt \qquad (1) \qquad \text{since } f(t) = -\lambda y(t)$$

Again $y(1) = 0$, so put $x = 1$ in above and using y (1) = 0 we get

$$y(1) = 0 = C_1 - \lambda\int_0^1 y(t)\,(1-t)\,dt$$

$$\Rightarrow C_1 = \lambda\int_0^1 y(t)\,(1-t)\,dt \qquad (2)$$

Using (2), (1) becomes, $y(x) = x\lambda\int_0^1 y(t)\,(1-t)dt - \lambda\int_0^x y(t)\,(x-t)dt$

$$\Rightarrow y(x) = \lambda x \int_0^x (1-t)\, y(t)\, dt + \lambda x \int_x^1 (1-t)\, y(t)\, dt - \lambda \int_0^x y(t)\,(x-t) dt$$

$$= \lambda \int_0^x (\cancel{x} - xt - \cancel{x} + t)\, y(t)\, dt + \lambda x \int_x^1 (1-t)\, y(t)\, dt$$

$$\Rightarrow y(x) = \lambda \int_0^x t(1-x)\, y(t)\, dt + \lambda \int_x^1 x\,(1-t)\, y(t)\, dt$$

$$= \lambda \int_0^1 k(x,\, t)\, y(t)\, dt$$

where the kernel $k(x, t)$ is given by

$$k(x,\, t) = \begin{cases} t\,(1-x), & 0 < t \le x \\ x\,(1-t), & x \le t < 1 \end{cases}$$

Ex.-4 : Reduce the following problem to an IE, $\frac{d^2y}{dx^2} + y(x) = \cos x$ with $y(0) = 0,\ y'(0) = 0$

Solu.: The ODE is $\frac{d^2y}{dx^2} + y(x) = \cos x$

Integrating the above *w. r.* to x within the limits 0 to x we get,

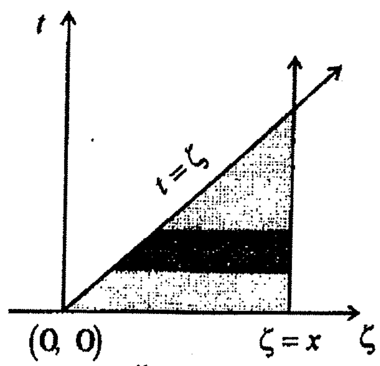
$$\int_0^x \frac{d^2y}{dx^2}\, dx + \int_0^x y(t)\, dt = \int_0^x \cos\zeta\, d\zeta$$

$$\Rightarrow y'(x) - y'(0) = -\int_0^x y(\zeta)\, d\zeta + \int_0^x \cos\zeta\, d\zeta$$

$$\Rightarrow y'(x) = -\int_0^x y(\zeta)\, d\zeta + \sin x$$

Again integrating *w. r.* to t, 0 to x.

$$\int_0^x y'(x) dx = -\int_{\zeta=0}^{x} \int_{t=0}^{\zeta} y(t) dt\, d\zeta + \int_0^x \sin t\, dt$$



$\zeta : t$ to x
$t : 0$ to x

$$\Rightarrow y(x) - y(0) = -\int_{t=0}^{x} y(t) \int_{\zeta=t}^{x} d\zeta\, dt + [-\cos t]_0^x$$

[Changing the Order of Integration]

$$\Rightarrow y(x) = -\int_{t=0}^{x}(x-t)\,y(t)\,dt + (1-\cos x)$$

$$\Rightarrow y(x) = \int_{t=0}^{x}(t-x)\,y(t)\,dt + (1-\cos x)$$

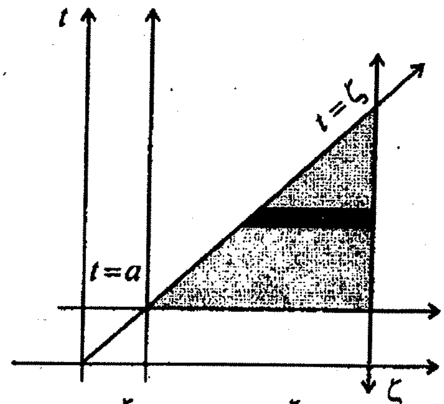
$$\Rightarrow y(x) = \int_{t=0}^{x}k(x,\,t)\,dt + f(x),$$

where $f(x) = 1-\cos x,$

$k(x,\,t) = t-x,$

Ex.-5 : Reduce the BVP $\dfrac{d^2y}{dx^2} = \lambda\,y(x);\ a \le x \le b$ with boundary conditions $y(a)=0$ and $y(b)=0$ to the IE in the form $y(x) = \lambda\int_a^b k(x,\,t)\,y(t)\,dt$

Solu.: Integrating the equation $\dfrac{d^2y}{dx^2} = \lambda\,y(x)$ with respect to x between the limit a to x we get,



$$\int_a^x \frac{d^2y}{dx^2}\,dx = \lambda\int_a^x y(\zeta)\,d\zeta$$

$$\Rightarrow y'(x) - y'(a) = \lambda\int_a^x y(\zeta)\,d\zeta$$

$$\Rightarrow y'(x) = C + \lambda\int_a^x y(\zeta)\,d\zeta, \qquad Let\ \ y'(a) = C_1$$

Again integrating w. r. to x between the limits a to x we get

$$y(x) - y(a) = C(x-a) + \lambda\int_a^x\int_a^\zeta y(t)\,dt\,d\zeta$$

$$\Rightarrow y(x) = C(x-a) + \lambda\int_{t=a}^{x} y(t)\int_{\zeta=t}^{x} d\zeta\,dt \qquad \text{since } y(a) = 0$$

$$\Rightarrow y(x) = C(x-a) + \lambda\int_a^x y(t)\,(x-t)\,dt \qquad (1) \qquad \text{[Changing the Order of Integration]}$$

Given $y(b) = 0$,

$$\therefore\ y(b)=0=C(b-a)+\lambda\int_a^b y(t)\,(b-t)\,dt$$

$$\Rightarrow\ C=\frac{\lambda}{-b+a}\int_a^b (b-t)\,y(t)\,dt$$

Using the value of C in (1) we get

$$y(x)=\lambda\frac{(x-a)}{(-b+a)}\int_a^b (b-t)y(t)dt+\lambda\int_a^x (x-t)\,y(t)\,dt$$

$$=\frac{\lambda}{(a-b)}\left[\int_a^x (b-t)(x-a)y(t)dt+\int_x^b (x-a)(b-t)y(t)dt+\int_a^x (a-b)(x-t)y(t)dt\right]$$

$$=\frac{\lambda}{a-b}\left[\int_a^x (bx-xt-ab+at+ax-bx-at+bt)y(t)dt+\int_x^b (x-a)(b-t)y(t)dt\right]$$

$$=\frac{\lambda}{a-b}\left[\int_a^x (b-x)\,(t-a)\,y(t)dt+\int_x^b (x-a)\,(b-t)\,y(t)dt\right]$$

$$y(x)=\lambda\int_a^b k(x,\,t)\,y(t)dt$$

where $k(x,\,t)=\begin{cases}\dfrac{(b-x)\,(t-a)}{(a-b)}, & a\le t\le x\\[2ex] \dfrac{(x-a)\,(b-t)}{(a-b)}, & x\le t\le b\end{cases}$

Ex.- 6: Reduce the BVP $\dfrac{d^2y}{dx^2}+\lambda xy=1$, in $0\le x\le 1$ with boundary conditions $y(0)=0,\ y(l)=1$, to an IE and find Kernel.

Solution : The ODE $\dfrac{d^2y}{dx^2}+\lambda xy=1$ (1)

Now integrating *w. r.* to x to (1) to the limits 0 to x, we get

$$\int_0^x \frac{d^2y}{dx^2}dx+\lambda\int_0^x \zeta\,y(\zeta)\,d\zeta=x$$

$$\Rightarrow\ y'(x)-y'(0)+\lambda\int_0^x \zeta\,y(\zeta)\,d\zeta=x$$

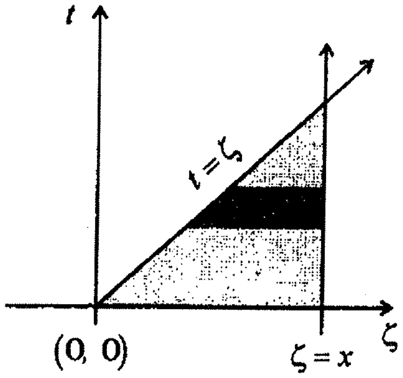
$$\Rightarrow y'(x)+\lambda\int_0^x \zeta\, y(\zeta)\, d\zeta = x + C, \qquad \text{let } y'(0)=C$$

Again integrating *w. r.* to x between the limits 0 to x.

$$y(x)-y(0)+\lambda\int_0^x d\zeta \int_0^\zeta t y(t)\, dt = \frac{x^2}{2}+Cx$$

$$\Rightarrow y(x)+\lambda \int_{t=0}^{x}\int_{\zeta=t}^{x} d\zeta\, t y(t)\, dt = \frac{x^2}{2}+Cx$$

[Changing the Order of Integration]



$\zeta : t$ to x
$t : 0$ to x

$$\Rightarrow y(x)+\lambda\int_0^x (x-t)\, t\, y(t) dt = \frac{x^2}{2}+Cx \qquad (2)$$

Put $x = l$, and $y(l) = 1$

$$1+\lambda\int_0^l (l-t)\, t\, y(t) dt = \frac{l^2}{2}+Cl$$

$$\Rightarrow C = \left(\frac{1}{l}-\frac{l^2}{2}\right)+\lambda\int_0^l (l-t)\, t\, y(t)\, dt$$

Using above, equation (2) becomes

$$y(x)+\lambda\int_0^x (x-t)\, t\, y(t) dt = \frac{x^2}{2}+x\left(\frac{1}{l}-\frac{l^2}{2}\right)+\lambda x\int_0^l (l-t)\, t\, y(t) dt$$

$$\Rightarrow y(x) = -\lambda\int_0^x (x-t)\, t\, y(t) dt+\lambda\int_0^x x(l-t)\, t\, y(t) dt+\lambda\int_x^l x(l-t)\, t\, y(t) dt+\frac{x^2}{2}+x\left(\frac{1}{l}-\frac{l^2}{2}\right)$$

$$\Rightarrow y(x) = \frac{x^2}{2}+x\left(\frac{1}{l}-\frac{l^2}{2}\right)+\lambda\int_0^x \left(xt-t^2+xl-xt\right) t\, y(t)\, dt+\lambda\int_x^l x(l-t)\, t\, y(t)\, dt$$

$$= \frac{x^2}{2}+x\left(\frac{1}{l}-\frac{l^2}{2}\right)+\lambda\int_0^x \left(xl-t^2\right) t\, y(t)\, dt+\lambda\int_x^l x(l-t)\, t\, y(t)\, dt$$

$$\Rightarrow y(x) = f(x)+\lambda\int_0^l k(x, t)\, y(t)\, dt$$

where $\quad f(x) = \dfrac{x^2}{2}+x\left(\dfrac{1}{l}-\dfrac{l^2}{2}\right),\ k(x, t) = \begin{cases} (xl-t^2)t, & 0\le t\le x \\ x(l-t)t, & x\le t\le l \end{cases}$

Ex.- 7: Reduce General BVP $\frac{d^2y}{dx^2}+A(x)\frac{dy}{dx}+B(x)y=g(x)$ with boundary conditions $y(a)=C_1,\ y'(a)=C_2$ to an IE and find Kernel

Solu.: The general 2nd order ODE is $\frac{d^2y}{dx^2}+A(x)\frac{dy}{dx}+B(x)y=g(x)$ (1)

Now integrating (1) *w. r.* to x to the limit a to x, we get

$$\int_a^x \frac{d^2y}{dx^2}dx+\int_a^x A(\zeta)\,y'(\zeta)\,d\zeta+\int_a^x B(\zeta)\,y(\zeta)\,d\zeta=\int_a^x g(\zeta)\,d\zeta$$

$$\Rightarrow y'(x)-y'(a)+\left[A(\zeta)\,y(\zeta)\right]_a^x-\int_a^x A'(\zeta)\,y(\zeta)\,d\zeta+\int_a^x B(\zeta)\,y(\zeta)\,d\zeta=\int_a^x g(\zeta)\,d\zeta$$

$$\Rightarrow y'(x)-C_2+A(x)\,y(x)-A(a)\,y(a)+\int_a^x\left[B(\zeta)-A'(\zeta)\right]y(\zeta)d\zeta=\int_a^x g(\zeta)\,d\zeta$$

Again integrating *w. r.* to x within the limits a to x to the above, we get

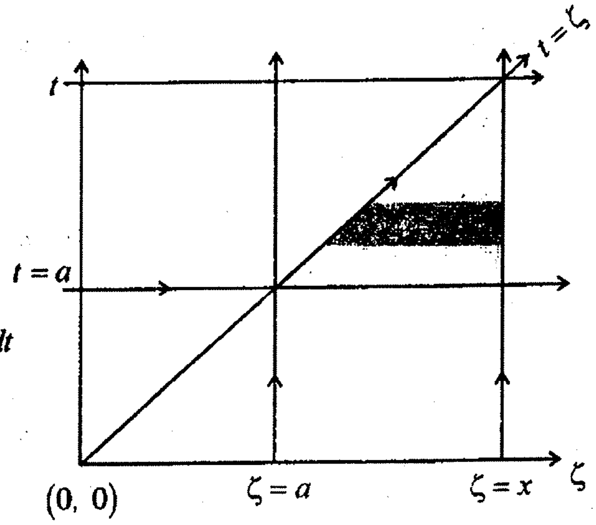
$$y(x)-y(a)-C_2(x-a)+\int_a^x A(\zeta)\,y(\zeta)d\zeta-C_1A(a)(x-a)+\int_{\zeta=a}^{x}\int_{t=a}^{\zeta}\left[B(t)-A'(t)\right]y(t)dt\,d\zeta$$

$$=\int_{\zeta=a}^{x}\int_{t=a}^{\zeta}g(t)dt\,d\zeta$$

[Changing the Order of Integration]

$$=y(x)-C_1-C_2(x-a)-C_1A(a)(x-a)+\int_a^x A(\zeta)\,y(\zeta)d\zeta$$

$$+\int_{t=a}^{x}\int_{\zeta=t}^{x}d\zeta\left[B(t)-A'(t)\right]y(t)\,dt=\int_{t=a}^{x}g(t)\,dt\int_{\zeta=t}^{x}d\zeta$$



ζ : t to x
t : a to x

$$\Rightarrow y(x)=C_1+C_1A(a)(x-a)+C_2(x-a)+\int_{t=a}^{x}(x-t)\,g(t)dt$$

$$+\int_a^x A(t)\,y(t)dt+\int_a^x\left[B(t)-A'(t)\right](x-t)\,y(t)\,dt$$

$$\Rightarrow y(x)=f(x)+\int_a^x k(x,\,t)\,y(t)\,dt$$

where $f(x)=C_1+C_1\,A(a)\,(x-a)+C_2(x-a)+\int_a^x(x-t)\,g(t)\,dt$

and Kernel, $k(x,\ t) = A(t) + [B(t) - A'(t)]\ (x - t)$

Therefore the reduced integral equation (2) is of Volterra Integral equation of 2nd kind.

4.8 Solution of Integral Equations :

Resolvent Kernel Method or Neumann Series Solution :

Let us consider the VIE of 2nd kind given by

$$u(x) = f(x) + \lambda \int_a^x k(x,\ t)\ u(t)dt \qquad (1)$$

Our main interest to find the solution of (1).

Let us assume the following series form of $u(x)$ and is defined as

$$u(x) = u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \lambda^3 u_3(x) + \ldots + \lambda^n u_n(x) + \ldots \qquad (2)$$

Using (2), (1) becomes,

$$u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \lambda^3 u_3(x) + \ldots = f(x) + \lambda \int_a^x k(x,\ t)\ [u_0(t) + \lambda u_1(t) + \lambda^2 u_2(t) + \lambda^3 u_3(t) + \ldots]\ dt$$

Equating like powers of λ, we get

$$u_0(x) = f(x) \qquad (3)$$

$$u_1(x) = \int_a^x k(x,\ t)\ u_0(t)dt \qquad (4)$$

$$u_2(x) = \int_a^x k(x,\ t)\ u_1(t)dt \qquad (5)$$

... ...

... ...

$$u_n(x) = \int_a^x k(x,\ t)\ u_{n-1}(t)dt \qquad (6)$$

Now from (3) & (4) we get

$$u_1(x) = \int_a^x k(x,\ t) f(t)dt = \int_a^x k_1(x,\ t) f(t)dt \qquad (7)$$

assume that $k(x,\ t) = k_1(x,\ t)$

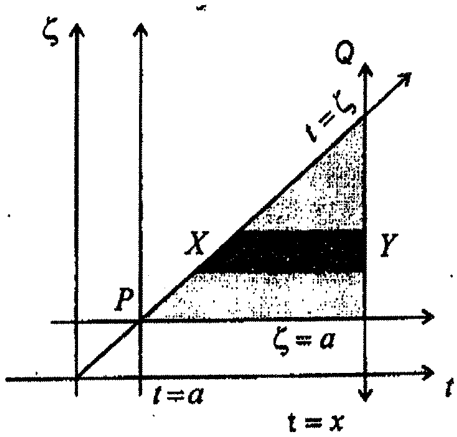
Using (7), (5) becomes,

$$u_2(x) = \int_{t=a}^{x} k(x, t)\left[\int_{\zeta=a}^{t} k_1(t, \zeta) f(\zeta) d\zeta\right] dt$$

$$u_2(x) = \int_{\zeta=a}^{x} f(\zeta)\left[\int_{t=\zeta}^{x} k(x, t)\, k_1(t, \zeta) dt\right] d\zeta \qquad (8)$$

[Assuming the Changing of Order of Integration]

Let $k_2(x, \zeta) = \int_{\zeta}^{x} k(x, t)\, k_1(t, \zeta) dt$



$\therefore\ u_2(x) = \int_{a}^{x} f(\zeta)\, k_2(x, \zeta)\, d\zeta$

In the similar way we can derived

$k_{n+1}(x, t) = \int_{t}^{x} k(x, y)\, k_n(y, t)\, dy$

That is, we get the series of Kernel as follows

$k_1(x, t) = k(x, t)$

$k_2(x, t) = \int_{t}^{x} k(x, y)\, k_1(y, t)\, dy$

... ...

$k_{n+1}(x, t) = \int_{t}^{x} k(x, y)\, k_n(y, t)\, dy$

$\therefore$ The solution is given by

$$u(x) = u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \lambda^3 u_3(x) + \ldots + \lambda^n u_n(x) + \ldots$$

$$= f(x) + \lambda\int_{a}^{x} k_1(x, t) f(t) dt + \lambda^2\int_{a}^{x} k_2(x, t) f(t) dt + \lambda^3\int_{a}^{x} k_3(x, t) f(t) dt$$

$$+ \ldots + \lambda^n \int_{a}^{x} k_n(x, t) f(t) dt + \ldots$$

$$= f(x) + \lambda\int_{a}^{x}\left[k_1 + \lambda k_2 + \lambda^2 k_3 + \ldots + \lambda^{n-1} k_n + \ldots\right] f(t) dt$$

$$= f(x) + \lambda\int_{a}^{x}\sum_{n=0}^{\infty} \lambda^n k_{n+1}(x, t) f(t)\, dt$$

$$\Rightarrow u(x) = f(x) + \lambda \int_a^x \Gamma(x, t; \lambda) f(t)\, dt$$

or, $u(x) = f(x) + \lambda \int_a^x R(x, t; \lambda) f(t)\, dt$

where $\Gamma(x, t; \lambda)$ or $R(x, t; \lambda)$ is called the Resolvent Kernel and the infinite series is also converges.

Solution of Fredholm Integral equation with the help of Resolvent Kernel :

Let $u(x) = f(x) + \lambda \int_a^b k(x, t)\, u(t)\, dt \rightarrow (1)$ be given Fredholm integral equation. Let $k_n(x, t)$ be the iterated Kernel and let $R(x, t; \lambda)$ be the Resolvent Kernel of (1). Then we have

$$R(x, t; \lambda) = \sum_{n=0}^{\infty} \lambda^n k_{n+1}(x, t) \qquad (2)$$

Suppose the sum of infinite series (2) exists and so $R(x, t; \lambda)$ can be obtained in the closed form. Then, the required solution of (1) is given by

$$u(x) = f(x) + \lambda \int_a^b R(x, t; \lambda) f(t)\, dt$$

Ex.- 8 Solve the Integral Equation

$$u(x) = f(x) + \lambda \int_o^x e^{x-t}\, u(t)\, dt$$

Solu.: Here $k(x, t) = e^{x-t} = k_1(x, t)$

we know that $k_{n+1}(x, t) = \int_t k(x, y) k_n(y, t)\, dy$

$$k_2(x, t) = \int_t^x k(x, y) k_1(y, t)\, dy = \int_t^x e^{x-y} e^{y-t} dy = \int_t^x e^{x-t} dy = (x-t)\, e^{x-t}$$

$$k_3(x, t) = \int_t^x k(x, y) k_2(y, t)\, dy = \int_t^x e^{x-y}(y-t)\, e^{y-t} dy = \int_t^x (y-t)\, e^{x-t} dy = \frac{1}{2!}(x-t)^2\, e^{x-t}$$

$$k_4(x, t) = \int_t^x k(x, y) k_3(y, t)\, dy = \int_t^x e^{x-y} \frac{(y-t)^2}{2!}\, e^{y-t} dy = \frac{1}{2!}\int_t^x e^{x-t}(y-t)^2 dy = \frac{1}{3!}(x-t)^3\, e^{x-t}$$

The solution of the given equation is

$$u(x) = f(x) + \lambda \int_a^x \Gamma(x, t; \lambda)\, f(t)\, dt$$

where $\Gamma(x, t; \lambda) = \sum_{n=0}^{\infty} \lambda^n k_{n+1}(x, t)$

$$= k_1(x, t) + \lambda k_2(x, t) + \lambda^2 k_3(x, t) + \lambda^3 k_4(x, t) + \ldots$$

$$= e^{x-t} + \lambda e^{x-t}(x-t) + \lambda^2 \frac{(x-t)^2}{2!} e^{x-t} + \lambda^3 \frac{(x-t)^3}{3!} e^{x-t} + \ldots$$

$$= e^{x-t}\, e^{\lambda(x-t)} = e^{(1+\lambda)(x-t)}$$

$$\therefore\; u(x) = f(x) + \lambda \int_a^x e^{(1+\lambda)(x-t)} f(t)\, dt$$

Ex.: 9. Solve the integral equation $u(x) = (1+x) + \lambda \int_o^x (x-t)\, u(t)\, dt$

Solu. : Here $K(x, t) = (x-t) = K_1(x, t)$

Also we know that $\quad k_{n+1}(x, t) = \int_t^x k(x, y) k_n(y, t)\, dy$

$$k_2(x, t) = \int_t^x (x-y)\, k_1(y, t)\, dy$$

$$= \int_t^x (x-y)(y-t)\, dy = \left[-xty + (x+t)\frac{y^2}{2} - \frac{y^3}{3}\right]_t^x$$

$$= \frac{(x-t)^3}{3\,!}$$

$$k_3(x, t) = \int_t^x k(x, y)\, k_2(y, t)\, dy$$

$$= \int_t^x (x-y)\, \frac{(y-t)^3}{3!}\, dy = \frac{(x-t)^5}{5!}$$

... ...

$$k_{n+1}(x, t) = \frac{(x-t)^{2n+1}}{(2n+1)!}$$

The Resoluent Kernel $\Gamma(x, t; \lambda) = \sum_{n=0}^{\infty} \lambda^n\, k_{n+1}(y, t)$

$$= k_1(x, t) + \lambda k_2(x, t) + \lambda^2 k_3(x, t) + \ldots$$

$$= (x-t) + \lambda \frac{(x-t)^3}{3!} + \lambda^2 \frac{(x-t)^5}{5!} + \ldots$$

The solution is given by

$$u(x) = (1+x) + \lambda \int_0^x \Gamma(x, t; \lambda)(1+t)\, dt$$

$$= (1+x) + \lambda \int_0^x \left[(x-t) + \lambda \frac{(x-t)^3}{3!} + \lambda^2 \frac{(x-t)^5}{5!} + \ldots\right](1+t)\, dt$$

Now $\int_0^x (x-t)(1+t)\, dt = \int_0^x x(1+t)\, dt - \int_0^x t(1+t)\, dt$

$$= \left[xt + \frac{xt^2}{2} - \frac{t^2}{2} - \frac{t^3}{3}\right]_0^x = \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$\int_0^x \frac{(x-t)^3}{3!}(1+t)\, dt = \frac{1}{3!}\int_0^x (x-t)^3 dt + \frac{1}{3!}\int_0^x t(x-t)^3 dt = \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$\int_0^x \frac{(x-t)^5}{5!}(1+t)\, dt = \frac{1}{5!}\int_0^x (x-t)^5 dt + \frac{1}{5!}\int_0^x t(x-t)^5 dt = \frac{x^6}{6!} + \frac{x^7}{7!}$$

Hence $u(x) = (1+x) + \lambda\left(\frac{x^2}{2!} + \frac{x^3}{3!}\right) + \lambda^2\left(\frac{x^4}{4!} + \frac{x^5}{5!}\right) + \lambda^3\left(\frac{x^6}{6!} + \frac{x^7}{7!}\right) + \ldots$

If we assume $\lambda = 1$, then the solution becomes

$$u(x) = (1+x) + \left(\frac{x^2}{2!} + \frac{x^3}{3!}\right) + \left(\frac{x^4}{4!} + \frac{x^5}{5!}\right) + \left(\frac{x^6}{6!} + \frac{x^7}{7!}\right) + \ldots$$

$$= e^x$$

Ex.: 10. Solve the integral equation by Resolvent kernel Method

$$u(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 xt\, y(t)\, dt$$

Soln. : Given that $u(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 xt\, y(t)\, dt$

Comparing above with $u(x) = f(x) + \lambda \int_0^1 k(x, t)\, u(t)\, dt$

We get, $f(x) = \frac{5x}{6}$, $k(x, t) = xt$, $\lambda = \frac{1}{2}$

So, $k(x, t) = xt = k_1(x, t)$

Also we know that $k_{n+1}(x, t) = \int_0^1 k(x, y)\, k_n(y, t)\, dy$

$$k_2(x, t) = \int_0^1 k(x, y)\, k_1(y, t)\, dy = \int_0^1 (xy)\,(yt)\, dy = xt\left[\frac{y^3}{3}\right]_0^1 = \frac{xt}{3}$$

$$k_3(x, t) = \int_0^1 k(x, y)\, k_2(y, t)\, dy = \int_0^1 (xy)\left(\frac{yt}{3}\right) dy = xt\left[\frac{y^3}{3}\right]_0^1 = \left(\frac{1}{3}\right)^2 xt$$

Therefore, in general, $k_{n+1}(x, t) = \left(\frac{1}{3}\right)^n xt$

Now, the Resolvent Kernel $R(x, t; \lambda)$ is given by

$$R(x, t; \lambda) = \sum_{n=0}^{\infty} \lambda^n k_{n+1}(x, t) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{1}{3}\right)^n xt$$

$$= \left(\frac{1}{2}\right)^0 \left(\frac{1}{3}\right)^0 xt + \left(\frac{1}{2}\right)^1 \left(\frac{1}{3}\right)^1 xt + \left(\frac{1}{2}\right)^2 \left(\frac{1}{3}\right)^2 xt + \ldots$$

$$= xt\left[1 + \left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)^2 + \ldots\right] = xt\frac{1}{1 - \frac{1}{6}} = \left(\frac{6}{5}\right)xt$$

Finally, the required solution of given integral equation is given by

$$y(x) = f(x) + \lambda \int_0^1 R(x, t; \lambda) f(t)\, dt$$

$$y(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 \frac{6}{5} xt\, \frac{5t}{6}\, dt = \frac{5x}{6} + \frac{1}{2} x \int_0^1 t^2 dt$$

$$= \frac{5x}{6} + \frac{1}{2} x . \frac{1}{3} = x$$

$\therefore y(x) = x$ is the required solution.

4.9 Solution of Integral Equation by Laplace Transform :

Solution Technique of Difference Type Kernel involving in Integral Equation :

When an integral equations involves the difference type kernel then we use the laplace transform technique or more accurately define the convolution theorem on laplace transform. So we define the convolution and convolution theorem.

Convolution : If $f(t)$ and $g(t)$ be two functions of t, then the convolution $f*g$ is defined as

$$f*g=\int_0^{\tau} f(\tau)\,g(t-\tau)\,d\tau$$

$$=\int_0^{\tau} f(t-\tau)\,g(\tau)\,d\tau$$

Convolution Theorem : If $F(p)$ is the Laplace transform of $f(t)$ and $G(p)$ is the Laplace transform of $g(t)$ then $F(p)\,G(p)=L\{f*g\}=L\int_0^{t} f(\tau)\,g(t-\tau)\,d\tau$

$$=L\int_0^{t} f(t-\tau)\,g(\tau)\,d\tau$$

Ex.: 11. Solve $y(x)=f(x)+\lambda\int_0^{x} k(x,\,t)\,u(t)dt$ in which the kernel $k(x,\,t)$ is of difference type say $k(x,\,t)=k(x-t)$

Solu.: Here $$u(x)=f(x)+\lambda\int_0^{x} k(x,\,t)\,u(t)\,dt \qquad (1)$$

$$\Rightarrow u(x)=f(x)+\lambda\int_0^{x} k(x-t)\,u(t)\,dt \qquad (2)$$

Let,

$$\left.\begin{aligned} L\{u(x)\}&=U(p),\\ L\{f(x)\}&=F(p)\\ L\{k(x)\}&=K(p)\end{aligned}\right\} \qquad (3)$$

$$\therefore\ k*u=\int_0^{x} k(x-t)\,u(t)\,dt$$

$$L\{k*u\}=L\int_0^{x} k(x-t)\,u(t)\,dt$$

$$\Rightarrow K(p)\,U(p) = L\int_0^x k(x-t)\,u(t)\,dt \qquad (4)$$

Taking Laplace Transform of (2), $L\,\{u(x)\} = L\,\{f(x)\} + \lambda L\int_0^x k(x-t)\,u(t)\,dt$

Using (3) & (4), we get, $U(p) = F(p) + \lambda\,K(p)\,U(p)$

$$\Rightarrow [1-\lambda K(p)]\,U(p) = F(p)$$

$$\Rightarrow U(p) = F(p)/[1-\lambda K(p)],\ [1-\lambda K(p)\neq 0]$$

Taking inverse Lappace transform of above, we get,

$$L^{-1}\,\{U(p)\} = L^{-1}\left\{\frac{F(p)}{1-\lambda K(p)}\right\}$$

$$\Rightarrow u(x) = L^{-1}\left\{\frac{F(p)}{1-\lambda K(p)}\right\}$$ is the required solution.

Ex.: 12. Use LT to solve the integral equation

$$u(x) = f(x) + \lambda\int_0^x e^{x-t}\,u(t)\,dt$$

Solu.: Given that $u(x) = f(x) + \lambda\int_0^x e^{x-t}\,u(t)\,dt \qquad (1)$

Comparing with $u(x) = f(x) + \lambda\int_0^x k(x,\,t)\,u(t)\,dt$, we get

$$k(x,\,t) = e^{x-t} = k(x-t);\ \text{ so } \ k(x) = e^x$$

Taking LT of (1) we get,

$$L\,\{u(x)\} = L\,\{f(x)\} + \lambda\,L\int_0^x e^{x-t}\,u(t)\,dt$$

$$\Rightarrow U(p) = F(p) + \lambda\,L\int_0^x k(x-t)\,u(t)\,dt,$$

let $L\,\{u(x)\} = U(p)$ & $L\,\{f(x)\} = F(p)$

$\Rightarrow U(p) = F(p) + \lambda K(p)\,U(p)$, using convolution theorem of LT & $L\,\{k(x)\} = K(p)$

Since $k(x)=e^x$ then $L\{k(x)\} = L\{e^x\} = \int_0^\infty e^x e^{-px}dx = \frac{1}{p-1}$

$$\therefore\ U(p)[1-\lambda K(p)] = F(p)$$

$$\Rightarrow\ U(p) = \frac{F(p)}{1-\lambda.\frac{1}{p-1}} = \frac{(p-1)}{(p-\lambda-1)}F(p)$$

Taking inverse Laplace transform of above, we get

$$\Rightarrow\ L^{-1}\{U(p)\} = L^{-1}\left\{\frac{p-1}{p-\lambda-1}F(p)\right\}$$

$$\Rightarrow\ u(x) = L^{-1}\left\{\left(1+\frac{\lambda}{p-\lambda-1}\right)F(p)\right\} = L^{-1}\{F(p)\} + L^{-1}\left\{\frac{\lambda F(p)}{p-\lambda-1}\right\}$$

$$\Rightarrow\ u(x) = f(x) + \lambda L^{-1}\left\{\frac{F(p)}{p-\lambda-1}\right\}$$

By convolution theorem $L^{-1}\left\{\frac{F(p)}{p-\lambda-1}\right\} = L^{-1}\{G(p)\,F(p)\}$ where $G(p) = \frac{1}{p-\lambda-1}$

Now $g(x) = L^{-1}\{G(p)\} = L^{-1}\left\{\frac{1}{p-(1+\lambda)}\right\} = e^{(1+\lambda)\,x}$

$$\therefore\ L^{-1}\{G(p)\,F(p)\} = \int_0^x f(t)\,g(x-t)dt = \int_0^x f(t)\,e^{(1+\lambda)\,(x-t)}dt$$

$$\therefore\ u(x) = f(x) + \lambda\int_0^x f(t)\,e^{(1+\lambda)\,(x-t)}dt$$

which is the required solution

Ex.: 13 Solve the Integral equation by Laplace transform technique.

$$u(x) = x - \int_0^x (x-t)\,u(t)dt$$

Solu.: Here the kernel $k(x,\,t) = x-t = k(x-t)$ and $k(x) = x$.

Given that $u(x) = x - \int_0^x (x-t)\,u(t)dt$ (1)

Taking Laplace transform of (1), we get

$$L\{u(x)\} = L\{x\} - L\int_0^x (x-t)\,u(t)dt = L\{x\} - L\int_0^x k(x-t)\,u(t)dt$$

$$\Rightarrow U(p) = \frac{1}{p^2} - K(p)\,U(p)$$

where $L\{u(x)\} = U(p)$

$L\{k(x)\} = K(p)$

& $L\int_0^x k(x-t)\,u(t)dt = K(p)\,U(p)$

$$\Rightarrow U(p) = \frac{\frac{1}{p^2}}{[1+K(p)]}$$

$$\Rightarrow U(p) = \frac{1}{p^2\left(1+\frac{1}{p^2}\right)}$$

$$\Rightarrow U(p) = \frac{1}{1+p^2}$$

Taking inverse Laplace transform of above we get

$$L^{-1}\{U(p)\} = L^{-1}\left\{\frac{1}{1+p^2}\right\}$$

$\Rightarrow u(x) = \sin x$ is the required solution.

4.10 Method of Successive Approximations :

The method of Successive Approximations for Solving Volterra Integral equation of 2nd kind :

Let the integral equation is $u(x) = f(x) + \lambda\int_0^x k(x, t)\,u(t)dt$ (1). Let $f(x)$ be continuous in $[0, a]$ and $K(x, t)$ be continuous for $0 \le x \le a,\ 0 \le t \le x$.

We start with some functions $u_0(x)$ continuous in $[0, a]$. Replacing $u(t)$ on R.H.S. of (1) by $u_0(x)$ we obtain

$$u_1(x) = f(x) + \lambda\int_0^x k(x, t)\,u_0(t)dt \qquad (2)$$

$u_1(x)$ given by (2) is itself continous in $[0, a]$. Proceeding likewise we arrive at a sequence of functions $u_0(x), u_1(x), \ldots, u_n(x), \ldots$

where $u_n(x) = f(x) + \lambda \int_0^x k(x, t)\, u_{n-1}(t)\, dt$ (3)

In view of continuity of $f(x)$ snd $k(x, t)$, the sequence $\{u_n(x)\}$ converges, as $n \to \infty$ to obtain the solution $u(x)$ of given integral equation (1), i.e., $u(x) = \underset{n\to\infty}{Lt}\, u_n(x)$.

Ex.: 14 Using the method of successive approximation ot solve the VIE of 2nd kind $u(x) = x - \int_0^x (x-t)\, u(t)\, dt$, assuming $u_0(x) = 0$.

Solu.: Given that $u(x) = x - \int_0^x (x-t)\, u(t)\, dt$ and $u_0(x) = 0$.

We know that $u_n(x) = x - \int_0^x (x-t)\, u_{n-1}(t)\, dt$

Put n = 1, 2, 3, ..., successively, we get $u_1(x) = x - \int_0^x (x-t)\, u_0(t)\, dt = x - \int_0^x (x-t)\, 0\, dt = x$

$$u_2(x) = x - \int_0^x (x-t)\, u_1(t)\, dt = x - \int_0^x (x-t)\, t\, dt = x - \left[\frac{xt^2}{2} - \frac{t^3}{3}\right]_0^x$$

$$= x - \frac{x^3}{2} + \frac{x^3}{3} = x - \frac{x^3}{3!}$$

$$u_3(x) = x - \int_0^x (x-t)\left(t - \frac{t^3}{3!}\right) dt = x - \int_0^x \left(xt - \frac{xt^3}{6} - t^2 + \frac{t^4}{6}\right) dt$$

$$= x - \left[\frac{xt^2}{2} - \frac{xt^4}{24} - \frac{t^3}{3} + \frac{t^5}{3}\right]_0^x$$

$$= x - \frac{x^3}{2} + \frac{x^5}{24} + \frac{x^3}{3} - \frac{x^5}{30}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

and so on,

In general, we have

$$u_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

The required solution $u(x)$ of (1) is given by

$$u(x) = \underset{n\to\infty}{Lt}\, u_n(x) \text{ i.e., } u(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

$$\Rightarrow u(x) = \sin x$$

Ex.: 15 Using the method of successive approximation to solve the VIE of 2nd kind $u(x) = 1 + \int_0^x u(t)dt$, assuming $u_0(t) = 0$.

Solu.: Given that $u(x) = 1 + \int_0^x u(t)dt$ (1) and $u_0(t) = 0$

we know that $u_n(x) = 1 + \int_0^x u_{n-1}(t)dt$

Put n = 1, 2, 3, ..., respectively, we get

$$u_1(x) = 1 + \int_0^x 0\, dt = 1$$

$$u_2(x) = 1 + \int_0^x 1\, dt = 1 + x$$

$$u_3(x) = 1 + \int_0^x (1+t)\, dt = 1 + \left[t + \frac{t^2}{2}\right]_0^x = 1 + x + \frac{x^2}{2}$$

$$u_4(x) = 1 + \int_0^x u_3(t)\, dt = 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right) dt = 1 + \left[t + \frac{t^2}{2} + \frac{t^3}{3}\right]_0^x$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

and so on. In general, we have $u_n(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ \dots\ +\frac{x^n}{n!}$.

The required solution $u(x)$ of (1) is given by

$$u(x)=\underset{n\to\infty}{Lt}\,u_n(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ \dots\ +\frac{x^n}{n!}+\ \dots$$

$$=e^x.$$

$\Rightarrow u(x)=e^x$ is the required solution.

Ex.: 16 Using the method of successive approximations to solve the VIE of 2nd kind $u(x)=1+\int_0^x (x-t)\,u(t)dt$, assuming $u_0(x)=1$.

Solu.: Given that $u(x)=1+\int_0^x (x-t)\,u(t)dt$, and $u_0(x)=1$.

we know that $u_n(x)=1+\int_0^x (x-t)\,u_{n-1}(t)dt$

Put n = 1, 2, 3, ... respectively, we get

$$u_1(x)=1+\int_0^x (x-t).1\,dt=1+\left[xt-\frac{t^2}{2}\right]_0^x=1+x^2-\frac{x^2}{2}=1+\frac{x^2}{2}$$

$$u_2(x)=1+\int_0^x (x-t).\left(1+\frac{t^2}{2}\right)dt=1+\int_0^x\left[x+\frac{xt^2}{2}-t-\frac{t^3}{2}\right]dt$$

$$=1+\left[xt+\frac{xt^3}{3.2}-\frac{t^2}{2}-\frac{t^4}{2.4}\right]_0^x$$

$$=1+\left[x^2+\frac{x^4}{3!}-\frac{x^2}{2}-\frac{x^4}{2.4}\right]$$

$$=1+\frac{x^2}{2!}+\frac{x^4}{4!}$$

$$u_3(x)=1+\int_0^x (x-t)\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}\right)dt=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}$$

and so on. In general, we have

$$u_n(x)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ \dots\ +\frac{x^{2n}}{(2n)!}$$

The required solution $u(x)$ of (1) is given by

$$u(x)=\underset{n\to\infty}{Lt}\,u_n(x)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ \dots\ +\frac{x^{2n}}{(2n)!}+\ \dots$$

$$=\cosh(x)$$

$\therefore\ u(x)=\cosh(x)$ is the required solution.

Solution Technique of VIE of 1st Kind by Laplace Transform :

Let us assume that VIE of 1st Kind.

$$f(x)=\lambda\int_0^x k(x,\,t)\,u(t)\,dt \qquad (1)$$

where $k(x,\,t)=k(x-t)$

Let us define $L\{f(x)\}=F(p),\ L\{k(x)\}=K(p),\ L\{u(x)\}=U(p)$

Now taking laplace transform of (1) we have,

$$L\{f(x)\}=\lambda\,L\int_0^x k(x,\,t)\,u(t)dt$$

$F(p)=\lambda\,K(p)\,U(p)$, [by convolution theorem]

$$\Rightarrow\ U(p)=\frac{F(p)}{\lambda\,K(p)}$$

Now taking inverse Laplace transform of above, we get.

$$L^{-1}\{U(p)\}=L^{-1}\left\{\frac{F(p)}{\lambda\,K(p)}\right\}$$

$$\Rightarrow\ u(x)=\frac{1}{\lambda}L^{-1}\left\{\frac{F(p)}{K(p)}\right\}$$

This is the solution

Ex.: 17 Solve the IE $\sin x=\lambda\int_0^x e^{x-t}\,u(t)dt$

Solu.: Given that $\sin x = \lambda \int_0^x e^{x-t}\, u(t)dt$ (1)

Taking Laplace transform of (1) we get,

$$L\{\sin x\} = \lambda\, L \int_0^x e^{x-t}\, u(t)dt$$

$$\Rightarrow \frac{1}{p^2+1} = \lambda \frac{1}{p-1} \cdot U(p)$$ by convolution theorem & $L\{e^x\} = \frac{1}{p-1}$.

$$\Rightarrow U(p) = \frac{1}{\lambda} \frac{p-1}{p^2+1}$$

$$\Rightarrow U(p) = \frac{1}{\lambda}\left(\frac{p}{p^2+1} - \frac{1}{p^2+1}\right)$$

Taking inverse Laplace transform of above, we get,

$$L^{-1}\{U(p)\} = \frac{1}{\lambda} L^{-1}\left\{\frac{p}{p^2+1}\right\} - \frac{1}{\lambda} L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$\Rightarrow u(x) = \frac{1}{\lambda}[\cos x - \sin x]$$

which is the required solution.

Conversation of VIE of 1st Kind to VIE of 2nd Kind :

Let us assume the VIE of 1st kind,

$$f(x) = \lambda \int_0^x k(x, t)\, u(t)\, dt \qquad (1)$$

Now equation (1) can be reduced to a VIE of Second kind when $k(x, x) \neq 0$. Differentiating *w. r.* to x to (1), we get

$$\frac{df(x)}{dx} = \lambda \int_0^x \frac{\partial k(x, t)}{\partial x} u(t)dt + \lambda . \frac{dx}{dx} . k(x, x)\, u(x),$$ [Using Leibnitz formula]

$$\Rightarrow \frac{df(x)}{dx} = \lambda \int_0^x \frac{\partial k(x, t)}{\partial x} u(t)dt + \lambda . k(x, x)\, u(x)$$

Dividing both sides by $\lambda\, k(x, x)$ and we get,

$$\Rightarrow u(x) = \frac{1}{\lambda k(x, x)}\frac{df(x)}{dx} - \frac{\lambda}{\lambda k(x, x)}\int_0^x \frac{\partial k(x, t)}{\partial x} u(t)dt$$

$$\Rightarrow u(x) = F(x) + \int_0^x G(x, t)\, u(t)\, dt \qquad (2)$$

where $F(x) = \frac{1}{\lambda k(x, x)}\frac{df(x)}{dx}$, $G(x, t) = -\frac{1}{k(x, x)}\frac{\partial k(x, t)}{\partial x}$

Equation (2) represents the VIE of 2nd kind.

Ex. 18 : Reduce the following IE $\sin x = \int_0^x e^{x-t}\, u(t)\, dt$ to an IE of 2nd kind and then solve it.

Solu.: Given that $\sin x = \int_0^x e^{x-t}\, u(t)\, dt \qquad (1)$

Comparing (1) with $f(x) = \lambda \int_0^x k(x, t)\, u(t)\, dt$ we get

$f(x) = \sin x,\ k(x, t) = e^{x-t},\ \lambda = 1$.

Also $k(x, x) = e^{x-x} = e^0 = 1 \neq 0$.

Hence the VIE of 1st kind (1) can be reduced to VIE of 2nd kind.

Differentiating *w. r.* to *x* to the (1) we get,

$$\frac{d}{dx}(\sin x) = \int_0^x \frac{\partial (e^{x-t})}{\partial x}\, u(t)\, dt + 1.\frac{dx}{dx}.e^{x-x}\, u(x)$$

$$\Rightarrow \cos x = \int_0^x e^{x-t}\, u(t)dt + u(x)$$

$$\Rightarrow u(x) = \cos x - \int_0^x e^{x-t}\, u(t)dt \qquad (2)$$

This is the VIE of 2nd kind.

For solving the equation (2), we take the Laplace transform of both sides,

$$L\{u(x)\} = L\{\cos x\} - L\int_0^x e^{x-t}\, u(t)\, dt$$

$$\Rightarrow U(p)=\frac{p}{p^2+1}-L\int_0^x k_1(x-t)\,u(t)\,dt \qquad \text{where } k_1(x-t)=e^{x-t}=k_1(x,t)$$

$$\Rightarrow U(p)=\frac{p}{p^2+1}-K_1(p)\,U(p) \qquad \text{[by convolution theorem]}$$

$$\Rightarrow U(p)\left[1+K_1(p)\right]=\frac{p}{p^2+1}$$

$$\Rightarrow U(p)=\frac{p}{p^2+1}\cdot\frac{1}{1+K_1(p)} \qquad L\{k_1(x)\}=L\{e^x\}\Rightarrow K_1(p)=\frac{1}{p-1}$$

$$\Rightarrow U(p)=\frac{p}{(p^2+1)}\frac{1}{\left(1+\frac{1}{p-1}\right)}$$

$$\Rightarrow U(p)=\frac{p-1}{p^2+1}=\frac{p}{p^2+1}-\frac{1}{p^2+1}$$

$$u(x)=L^{-1}\{U(p)\}=L^{-1}\left\{\frac{p}{p^2+1}\right\}-L^{-1}\left\{\frac{1}{p^2+1}\right\}=\cos x-\sin x.$$

Ex. 19: Reduce the VIE of 1st kind $x=\int_0^x \cos(x-t)\,u(t)\,dt$ to an VIE of the 2nd kind and then solve it.

Solu.: Given that $x=\int_0^x \cos(x-t)\,u(t)\,dt$ (1) which is VIE of 1st kind.

Companing with $f(x)=\lambda\int_0^x k(x,t)\,u(t)\,dt$, we get $f(x)=x$, $k(x,t)=\cos(x-t)$, $\lambda=1$

Also $k(x,x)=\cos(x-x)$, $\cos 0=1\neq 0$

Hence the VIE of 1st kind (1) can be reduced to VIE of 2nd kind.

Differentiating w. r to x to the equation (1) we get

$$\frac{d(x)}{dx}=\int_0^x \frac{\partial}{\partial x}\{\cos(x-t)\}\,u(t)\,dt+1.\frac{dx}{dx}.\cos(x-x)\,u(x)$$

$$\Rightarrow 1=\int_0^x\{-\sin(x-t)\}\,u(t)\,dt+u(x)$$

$$\Rightarrow u(x) = 1 + \int_0^x \sin(x-t)\, u(t)\, dt \qquad (2)$$

This is VIE of 2nd kind.

For solving the equation (2), we take the Laplace transform of both sides,

$$L\{u(x)\} = L\{1\} + L\int_0^x \sin(x-t)\, u(t)\, dt$$

$$\Rightarrow U(p) = \frac{1}{p} + L\int_0^x k_1(x, t)\, u(t)\, dt, \qquad k_1(x, t) = \sin(x-t)$$

$$\Rightarrow U(p) = \frac{1}{p} + K_1(p)\, U(p) \quad \text{by convolution theorem and } L\{u(x)\} = U(p)$$

$$\Rightarrow U(p) = \frac{1}{p}\,\frac{1}{1-K_1(p)} \qquad \text{since, } L\{k_1(x)\} = K_1(p)$$

$$\Rightarrow U(p) = \frac{1}{p}\,\frac{1}{1-\frac{1}{p^2+1}} \qquad \text{since, } L\{\sin x\} = \frac{1}{p^2+1} \Rightarrow K_1(p) = \frac{1}{p^2+1}$$

$$\Rightarrow U(p) = \frac{1}{p}\,\frac{p^2+1}{p^2}$$

$$\Rightarrow U(p) = \frac{p^2}{p^3} + \frac{1}{p^3} = \frac{1}{p} + \frac{1}{p^3}$$

$$u(x) = L^{-1}\{U(p)\} = L^{-1}\left\{\frac{1}{p}\right\} + L^{-1}\left\{\frac{1}{p^3}\right\}$$

$$= 1 + \frac{x^2}{2} \qquad \text{since } L\{x^n\} = \frac{n!}{p^{n+1}}$$

$$\therefore u(x) = 1 + \frac{x^2}{2}$$

which is the required solution.

4.11 Generalised ABEl's Integral Famula :

Ex.: 20 The Abel's integral formula is $f(x) = \int_0^x \frac{1}{(x-t)\,\alpha}\, y(t)\, dt,\ 0 < \alpha < 1$. The integral equation is VIE, 1st kind, singularity as well as difference type kernel. Our main interest to find the solution of given integral equation.

Solu.: Given that $f(x)=\int\limits_{0}^{x}\frac{1}{(x-t)\,\alpha}\,y(t)\,dt$ (1)

Multiplying by $\frac{1}{(u-x)^{1-\alpha}}$ to (1) and then integrating *w. r.* to x between the limit 0 to u.

$$\int\limits_{x=0}^{u}\frac{f(x)}{(u-x)^{1-\alpha}}\,dx=\int\limits_{x=0}^{u}\frac{1}{(u-x)^{1-\alpha}}\int\limits_{t=0}^{x}\frac{y(t)}{(x-t)^{\alpha}}\,dt\,dx$$

$$=\int\limits_{t=0}^{u}y(t)\left(\int\limits_{x=t}^{u}\frac{1}{(u-t)^{1-\alpha}\,(x-t)^{\alpha}}dx\right)dt \qquad (2)$$

[by Changing the Order of Integration]

Now inner integral of R. H. S. of (2)

$$\int\limits_{x=t}^{u}\frac{1}{(u-t)^{1-\alpha}\,(x-t)^{\alpha}}dx\,;$$

$$=-\int\limits_{0}^{1}\frac{(t-u)\,dv}{(1-v)^{\alpha}(u-t)^{\alpha}\,\{v\,(u-t)\}^{1-\alpha}}$$

$$=\int\limits_{0}^{1}\frac{(u-t)\,dv}{(1-v)^{\alpha}\,v^{1-\alpha}\,(u-t)}$$

$$=\int\limits_{0}^{1}v^{\alpha-1}\,(1-v)^{-\alpha}dv$$

put $v=\frac{u-x}{u-t}$

$v(u-t)=u-x$

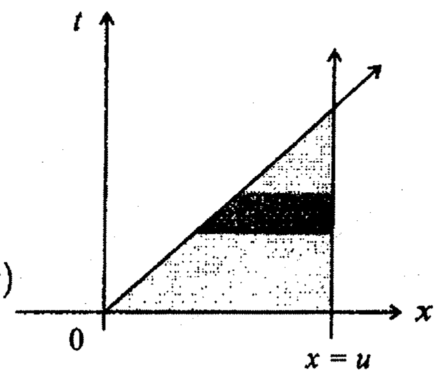
$\Rightarrow x=u-v\,(u-t)$

$x-t=(u-t)-v(u-t)$

$=(u-t)\,(1-v)$

when $x=t,\ v=1$

$x=u,\ v=0$



t : 0 to u

x : t to u

$$=B(\alpha,\ 1-\alpha)=\frac{\Gamma(\alpha)\,\Gamma(1-\alpha)}{\Gamma(\alpha+1-\alpha)}=\frac{\Gamma(\alpha)\,\Gamma(1-\alpha)}{\Gamma(1)}=\frac{\pi}{\sin(\alpha\pi)},\quad 0<\alpha<1$$

$\therefore$ From (2), $\int\limits_{0}^{u}\frac{f(x)}{(u-x)^{1-\alpha}}\,dx=\int\limits_{t=0}^{u}\frac{\pi}{\sin(\alpha\pi)}y(t)\,dt=\frac{\pi}{\sin(\alpha\pi)}\int\limits_{0}^{u}y(t)\,dt$

$$\Rightarrow\int\limits_{0}^{u}y(t)\,dx=\frac{\sin(\alpha\pi)}{\pi}\int\limits_{0}^{u}\frac{f(x)}{(u-x)^{1-\alpha}}\,dx$$

Differentiating with respect to u

$$\frac{d}{du}\int_0^u y(t)\,dx = \frac{\sin(\alpha\pi)}{\pi}\frac{d}{du}\int_0^u \frac{f(x)}{(u-x)^{1-\alpha}}\,dx$$

$$y(x) = \frac{\sin(\alpha\pi)}{\pi}\frac{d}{du}\int_0^u \frac{f(x)}{(u-x)^{1-\alpha}}\,dx$$

change u to t, in above, $\Rightarrow y(t) = \frac{\sin(\alpha\pi)}{\pi}\frac{d}{dt}\int_0^t \left[f(x)/(t-x)^{1-\alpha}\right]dx$ is the required solution.

4.12 Solution of Non homogeneous Fredholm Integral Equation of 2nd kind having its Kernel Separable:

Let us consider non homogeneous FIE of 2nd kind with separable Kernel

$$y(x) = f(x) + \lambda\int_a^b k(x, t)\,y(t)\,dt \qquad (1)$$

$$\text{where } k(x, t) = \sum_{i=1}^{n} a_i(x)\,b_i(t) \qquad (2)$$

$$\text{so } y(x) = f(x) + \lambda\int_a^b \left(\sum_{i=1}^{n} a_i(x)\,b_i(t)\right)y(t)\,dt$$

$$= f(x) + \lambda\sum_{i=1}^{n} a_i(x)\int_a^b b_i(t)\,y(t)\,dt$$

$$= f(x) + \lambda\sum_i C_i\,a_i(x) \qquad (3)$$

$$\text{where } C_i = \int_a^b b_i(t)\,y(t)\,dt \qquad (4)$$

Multiplying (3) by $b_i(x)$ and integrating with respect to x between the limit a and b,

$$\int_a^b b_i(x)\,y(x)\,dx = \int_a^b b_i(x)\,f(x)\,dx + \lambda\sum_k C_k\int_a^b a_k(x)\,b_i(x)\,dx \qquad (5)$$

$$\text{Denote } \int_a^b b_i(x)\,f(x)\,dx = f_i \qquad (6)$$

and $\int_a^b b_i(x)\, a_k(x)\, dx = a_{ik}$ (7)

Using (4), (6) and (7) we get from (5),

$C_i = f_i + \lambda \sum_k a_{ik}\, C_k, \quad i = 1, 2, \ldots, n$

Ex.: 21. Solve the FIE $y(x) = x + \lambda \int_0^1 \left(xt^2 + x^2 t\right) y(t)\, dt$

Solu.: Given that $y(x) = x + \lambda \int_0^1 \left(xt^2 + x^2 t\right) y(t)\, dt$ (1)

Comparing with $y(x) = f(x) + \lambda \int_0^1 k(x, t)\, y(t)\, dt$, we get

$f(x) = x,\ k(x, t) = xt^2 + x^2 t = \sum_i a_i(x)\, b_i(t)$

$\therefore\ a_1(x) = x,\ a_2(x) = x^2,\ b_1(t) = t^2,\ b_2(t) = t$

Also we know that

$a_{ik} = \int_a^b b_i(x)\, a_k(x)\, dx$

$a_{11} = \int_0^1 b_1(x)\, a_1(x)\, dx = \int_0^1 x^2 x\, dx = 1/4$

$a_{12} = \int_0^1 b_1(x)\, a_2(x)\, dx = \int_0^1 x^2 x^2\, dx = 1/5$

$a_{21} = \int_0^1 b_2(x)\, a_1(x)\, dx = \int_0^1 xx\, dx = 1/3$

$a_{22} = \int_0^1 b_2(x)\, a_2(x)\, dx = \int_0^1 x.x^2\, dx = 1/4$

Also we know that $C_i = f_i + \lambda \sum_k a_{ik}\, C_k$

where $C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}$, $F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \vdots & \cdots \\ \cdots & \cdots & \vdots & \cdots \\ a_{n_1} & a_{n_2} & \cdots & a_{nn} \end{bmatrix}$

Now $f_i = \int_a^b b_i(x) f(x)\, dx$

so $f_1 = \int_0^1 b_1(x) f(x)\, dx = \int_0^1 x^2 x\, dx = 1/4$

$f_2 = \int_0^1 b_2(x) f(x)\, dx = \int_0^1 x.x\, dx = 1/3$

$\therefore\ [I - \lambda A]\, C = F$

$\Rightarrow (1 - \lambda a_{11})\, C_1 - \lambda a_{12} C_2 = f_1$

and $-\lambda a_{22} C_1 + (1 - \lambda a_{22})\, C_2 = f_2$

$$\therefore \left.\begin{aligned} &(1 - \lambda/4)\, C_1 - (\lambda/5)\, C_2 = 1/4 \\ &(-\lambda/3)\, C_1 + (1 - \lambda/4)\, C_2 = 1/3 \end{aligned}\right\}$$

Solving above we get, $C_1 = \dfrac{60 + \lambda}{240 - 120\lambda - \lambda^2}$, $C_2 = \dfrac{80}{240 - 120\lambda - \lambda^2}$

Hence the solution can be expressed in the form

$y(x) = f(x) + \lambda \sum C_i\, a_i(x)$

$= x + \lambda[C_1 a_1(x) + C_2 a_2(x)]$

$$y(x) = x + \lambda \frac{(60 + \lambda)x}{240 - 120\lambda - \lambda^2} + \lambda \frac{80x^2}{240 - 120\lambda - \lambda^2}$$

which is the required solution

Homogeneous FIE with Separable Kernel :

Let us consider homogeneous FIE with separable kernel

$$y(x) = \lambda \int_a^b k(x, t) y(t)\, dt \qquad (1)$$

where $\quad k(x, t) = \sum_i a_i(x)\, b_i(t) \qquad (2)$

Using (2) in (1), we have $y(x) = \lambda \int_a^b \left(\sum_i a_i(x)\, b_i(t) \right) y(t)\, dt$

$$= \lambda \sum_i a_i(x) \int_a^b b_i(t)\, y(t)\, dt$$

$$= \lambda \sum C_i\, a_i(x) \qquad (3)$$

where $\quad C_i = \int_a^b b_i(t)\, y(t)\, dt \qquad (4)$

Multiplying (3) by $b_i(x)$ and then integrating *w. r.* to x between a to b

$$\int_a^b b_i(x)\, y(x)\, dx = \lambda \int_a^b b_i(x) \sum_k C_k\, a_k(x) dx$$

$$\Rightarrow C_i = \lambda \sum_k C_k \int_a^b b_i(x)\, a_k(x)\, dx = \lambda \sum_k C_k\, a_{ik}, \text{ where } \quad a_{ik} = \int_a^b b_i(x)\, a_k(x)\, dx$$

$$\Rightarrow (I - \lambda A)\, C = O.$$

where I is the identity matrix of order n,

$$C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \vdots & \cdots \\ \cdots & \cdots & \vdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

From the theory of system of linear equation, we can conclude that if $|I - \lambda A| \neq 0$ then the only solution to the homogeneous equation (5) is the trivial solution i.e., $C = O$ and therefore by using (3) we get the solution to the homogeneous FIE (1) as the trivial solution $y(x) = 0$.

On the otherhand when $|I - \lambda A| = 0$, then (5) and hence the integral equation (1) have infinite number of solution.

4.13 FREDHOLM ALTERNATIVE :

It is homogeneous IE $y(x)=\lambda\int_a^b k(x, t)y(t)\,dt$ (1) has only the trivial solution $y(x)=0$, then the corresponding non homogeneous IE $y(x)=f(x)+\lambda\int_a^b k(x, t)y(t)\,dt$ (2) always has one and only one solution.

On the contrary, if the homogeneous IE (1) has some non-trivial $[i.e., |I-\lambda A|=0]$ solution then the non homogeneous IE (2) has either no solution or an infinity of solution depending upon the function $f(x)$.

4.14 EIGEN VALUE AND EIGEN VECTOR :

For the homogeneous FIE with degenerate Kernel $k(x, t)$ is given by,

$y(x)=\lambda\int_a^b k(x, t)y(t)\,dt$ (1) the parameter $\lambda\ (\neq 0)$ for which (1) does not have a trivial solution is called the eigen value or characteristic value of the IE (1). The non trivial solution $y(x)\neq 0$, corresponding to the eigen value is called the eigen function of the IE (1).

Ex.: 22. Solve the Integral Equation $y(x)=\lambda\int_0^\pi(\cos^2 x\cos 2t+\cos 3x\cos^3 t)y(t)\,dt$

Solu.: The given equation is $y(x)=\lambda\int_0^\pi(\cos^2 x\cos 2t+\cos 3x\cos^3 t)\,dt$

In this case $k(x, t)=\cos^2 x\cos 2t+\cos 3x\cos^3 t=\sum_{i=1}^{2}a_i(x)\,b_i(t)$

$\therefore\ a_1(x)=\cos^2 x,\ a_2(x)=\cos 3x,\ b_1(t)=\cos 2t,\ b_2(t)=\cos^3 t$

$$a_{11}=\int_0^\pi b_1(x)\,a_1(x)\,dx=\int_0^\pi\cos 2x\cos^2 x\,dx=\pi/4$$

$$a_{12}=\int_0^\pi b_1(x)\,a_2(x)\,dx=\int_0^\pi\cos^3 x\cos^2 x\,dx=0$$

$$a_{21}=\int_0^\pi b_2(x)\,a_1(x)\,dx=\int_0^\pi\cos^3 x\cos^2 x\,dx=0$$

$$a_{22}=\int_0^\pi b_2(x)\,a_2(x)\,dx=\int_0^\pi\cos 3x\cos^2 x\,dx=\pi/8$$

The solution is $y(x)=\lambda[C_1a_1(x)+C_2a_2(x)]$

$$=\lambda[C_1\cos^2 x+C_2\cos 3x]$$

$\therefore\ |I-\lambda A|=O$

$$\Rightarrow\begin{vmatrix}1-\lambda\pi/4 & 0\\ 0 & 1-\lambda\pi/8\end{vmatrix}=0\Rightarrow\lambda=4/\pi,\ 8/\pi$$

$\therefore\ \lambda_1=4/\pi,\ \lambda_2=8/\pi$

Now $\begin{vmatrix}1-\lambda\pi/4 & 0\\ 0 & 1-\lambda\pi/8\end{vmatrix}\begin{bmatrix}C_1\\ C_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

$\Rightarrow (1-\lambda\pi/4)\,C_1=0$ (2)

and $(1-\lambda\pi/8)\,C_2=0$ (3)

Case I : $\lambda=\lambda_1(=4/\pi)$

(2) gives, $0.\,C_1=0$

$\Rightarrow C_1=C$ (say)

(3) gives, $\frac{1}{2}C_2=0$

$\Rightarrow C_2=0$ for $\lambda=\lambda_1=4/\pi$

$\therefore$ for $\lambda=\lambda_1$, $y=y(x)=\lambda_1\,C_1\cos^2 x=\lambda_1\,C\cos^2 x$

$y(x)=A\cos^2 x$ for $A=\lambda_1 C$

Case II : $\lambda=\lambda_2=8/\pi$

(2) gives $\Rightarrow (-1)\,C_1=0$

$\Rightarrow C_1=0$

(3) gives, $(1-8/\pi\cdot\pi/8)\,C_2=0$

$\Rightarrow C_2=C$ (say) for $\lambda=\lambda_2=8/\pi$

$\therefore\ y=y(x)=\lambda_2 C\cos 3x=B\cos 3x$ for $B=\lambda_2 C$.

Ex.: 23 Show that the integral equation $g(s)=f(s)+1/\pi\int_0^{2\pi}\sin(s+t)\,g(t)dt$ possesses no solution for $f(s)=s$, but that it possesses infinitely many solutions when $f(s)=1$.

Solu.: Given $g(s)=f(s)+\frac{1}{\pi}\int_0^{2\pi}\sin(s+t)\,g(t)\,dt$

$$g(s)=f(s)+\tfrac{1}{\pi}\int_0^{2\pi}(\sin s\ \cos t+\cos s\ \sin t)\,g(t)\,dt$$

$$\Rightarrow\ g(s)=f(s)+(\tfrac{1}{\pi})\sin s\int_0^{2\pi}\cos t\ g(t)\,dt+\frac{\cos s}{\pi}\int_0^{2\pi}\sin t\ g(t)\,dt \qquad (1)$$

$$\Rightarrow\ g(s)=f(s)+\frac{C_1\sin s}{\pi}+\frac{C_2\cos s}{\pi} \qquad (2)$$

where $C_1=\int_0^{2\pi}\cos t\ g(t)\,dt$ (3)

& $C_2=\int_0^{2\pi}\sin t\ g(t)\,dt$ (4)

We now discuss two particular cases as mentioned in the problem

Case 1 : Let $f(s)=s$. Then (2) reduces to

$$g(s)=s+\frac{C_1\sin s}{\pi}+\frac{C_2\cos s}{\pi} \qquad (5)$$

From (5), $g(t)=t+\dfrac{C_1\sin t}{\pi}+\dfrac{C_2\cos t}{\pi}$ (6)

Using (6), (3) becomes, $C_1=\int_0^{2\pi}\cos t\left(t+\dfrac{C_1\sin t}{\pi}+\dfrac{C_2\cos t}{\pi}\right)dt$

$$\Rightarrow\ C_1=\int_0^{2\pi}t\ \cos t\,dt+\frac{C_1}{2\pi}\int_0^{2\pi}2\sin t\ \cos t\,dt+\frac{C_2}{2\pi}\int_0^{2\pi}2\cos^2 t\,dt$$

$$\Rightarrow\ C_1=[t\ \sin t]_0^{2\pi}-\int_0^{2\pi}\sin t\,dt+\frac{C_1}{2\pi}\left[\frac{-\cos 2t}{2}\right]_0^{2\pi}+\frac{C_2}{2\pi}\left[t+\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$\Rightarrow\ C_1=+[\cos t]_0^{2\pi}+\frac{C_1}{2\pi}\left[-\frac{1}{2}+\frac{1}{2}\right]+\frac{C_2}{2\pi}[2\pi+0-0]$$

$$\Rightarrow\ C_1=C_2$$

$$\Rightarrow C_1 - C_2 = 0 \qquad (7)$$

Again using (6), (4) becomes

$$C_2 = \int_0^{2\pi} \sin t\left(t + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi}\right) dt$$

$$C_2 = \int_0^{2\pi} t \sin t\, dt + \frac{C_1}{2\pi}\int_0^{2\pi} 2\sin^2 t\, dt + \frac{C_2}{2\pi}\int_0^{2\pi} 2\sin t \cos t\, dt$$

$$\Rightarrow C_2 = -2\pi + C_1 + 0$$

$$\Rightarrow C_1 - C_2 = 2\pi \qquad (8)$$

The system of equations (7) & (8) is inconsistent and so it possesses no solution.

Hence C_1 & C_2 cannot be determined and so (5) shows that the given integral equation possesses no solution when $f(s) = s$.

Case II : Let $f(s) = 1$. Then (2) reduces to

$$g(s) = 1 + \frac{C_1 \sin s}{\pi} + \frac{C_2 \cos s}{\pi} \qquad (9)$$

from $$g(t) = 1 + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi} \qquad (10)$$

Using (10), (3) becomes

$$C_1 = \int_0^{2\pi} \cos t\left(1 + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi}\right) dt$$

$$= \int_0^{2\pi} \cos t\, dt + \frac{C_1}{2\pi}\int_0^{2\pi} 2 \cos t \sin t\, dt + \frac{C_2}{2\pi}\int_0^{2\pi} 2 \cos^2 t\, dt$$

$$= [\sin t]_0^{2\pi} + \frac{C_1}{2\pi}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= 0 + 0 + \frac{C_2}{2\pi}[2\pi + 0]$$

$$\Rightarrow C_1 - C_2 = 0 \qquad (11)$$

Again using (10), (4) becomes

$$C_2 = \int_0^{2\pi} \sin t\left(1 + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi}\right) dt$$

$$\Rightarrow C_1 = C_2 \qquad (12)$$

From (11) & (12) we see that $C_1 = C_2 = C'$ (say). Here C' is an arbitany constant. Thus the system (11) - (12) has infinite number of solutions $C_1 = C'$ and $C_2 = C'$. Putting these values in (9), the required solution of given integral equation is $g(s) = 1 + \frac{C'}{\pi}(\sin s + \cos s) \Rightarrow g(s) = 1 + C(\sin s + \cos s)$ where $C = C'/\pi$ is another arbitany constant. Since C is an arbitary constant. We have infinitely many solutions of (1) when $f(s) = 1$.

4.15 Unit Summary : The gist of the unit is depicted as follows :

(I) After converting an initial value or a boundary value problem to an integral equation, it can be solve by shorter methods of solving integral equation.

(II) Most of the difficult problems in Mechanics and Mathematical Physics can be converted in an integral equation and obtained the solution in very lucid way.

4.16 Exercises :

1. Reduce the problems to an Integral Equation :

$$\frac{d^2y}{dx^2} + \lambda\, y(x) = f(x),\ y'(0) = 0,\ y(0) = 1$$

2. Reduce the problems to an Integral Equation :

$$\frac{d^2y}{dx^2} + y = f(x),\ 0 < x < \pi,\ \text{ with } y(0) = 0,\ y(\pi) = 0$$

3. Solve by Resolvent Kernel Method

$$y(x) = x - \int_0^x (x-t)\, y(t)\, dt$$

4. Consider $y(x) = 1 + \lambda \int_0^1 (1 - 3xt)\, y(t)\, dt$

Evaluate the Resolvent Kernel. For what values of λ the solution does not exist. Obtain the solution of the given Integral Equation.

5. Solve the following Integral Equation $y(x) = 1 + \lambda \int_0^1 (x+t)\, y(t)\, dt$ by the method of successive approximation to third order with initial approximation $y_0(x) = 1$.

6. Solve the Integral equations by Laplace transform technique

$$y(x) = x + 2\int_0^t \cos(x-t)\,y(t)\,dt$$

7. Solve the integral equation,

$$f(x) = \int_a^x \frac{y(t)\,dt}{(\cos t - \cos x)^{1/2}}, \quad 0 \le a < x < b < \pi.$$

Answers :

1. $y(x) = F(x) + \lambda\int_0^x k(x;\, t)\,y(t)\,dt$, where, $F(x) = 1 + \int_0^x (x-t)f(t)\,dt$

   & $k(x,\, t) = t - x$

2. $y(x) = F(x) + \lambda\int_0^\pi k(x,\, t)\,y(t)\,dt$, where $F(x) = \int_0^x (x-t)\,y(t)\,dt - \frac{x}{\pi}\int_0^\pi (x-t)f(t)\,dt$

   & $k(x,\, t) = \begin{cases} (t/\pi)(\pi - x), & t < x \\ (x/\pi)(\pi - t), & t > x \end{cases}$

3. $y(x) = \sin x$

4. $R(x,\, t;\, \lambda) = \frac{4}{4-\lambda^2}\left[1 + \lambda - \frac{3}{2}x\lambda - 3t\left(x + \frac{\lambda}{2} - \lambda x\right)\right]$

   for $|\lambda| > 2$, the solution will not exist, and

   the solution, $y(x) = \frac{4 + 2\lambda(2-3x)}{4-\lambda^2}$, for $|\lambda| > 2$.

5. $y_3(x) = 1 + \lambda(x + 1/2) + \lambda^2(x + 7/12) + \lambda^3\left(\frac{13}{12}x + 5/8\right)$

6. $y(x) = 2e^x(x-1) + 2 + x$

7. $y(t) = \left(1/\pi\right)\frac{d}{dt}\left[\int_a^t \frac{\sin u\, f(u)\,du}{(\cos u - \cos t)^{1/2}}\right], \quad a < t < b$

4.17 References / Suggested further Readings :

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M.Sc. Course in
Applied Mathematics with Oceanology
and
Computer Programming

Part - II

Paper - VIII **Group - A**

Module No. 89

5 : Ordinary Differential Equations of Sturm Liouville type and Green's Function

Structure

5.1 Introduction :

In this module, we have consider two different type of problems consisting of ordinary differential equation of sturm liouville type and Green's function and divided into two sections. In the first section, we shall consider a special kind of boundary value problem known as sturm-liouville problem and in the last section, we shall discuss how to find Green's function involving ordinary differential equation subject to boundary conditions or initial conditions.

5.2 Objectives :

The problem of an ordinary differential equations of sturm liouville type have introduced us to several important concepts including characteristic (eigen) value and characteristic (eigen) function and orthogonality. These concepts are frequently employed in the applications of differential equations to Applied Mathematics and Engineering. Also the Green's function are useful in the application of differential equation to Applied Mathematics and Engineering.

5.3 Key words :

Sturm Liouville Type, Orthoganality, Characteristic Function, Green's Function.

5.3 Ordinary Differential Equations of Sturm Liouville Type :

5.3.1 Some Definitions :

Definition 1 :

We consider a boundary-value problem which consists of

1. a second-order homogeneous linear differential equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda r(x)\right]y=0 \qquad (1)$$

where p, q and r are real functions such that p has a continuous derivative, q and r are continuous, and $p(x)>0$ and $r(x)>0$, for all x on a real interval $a \le x \le b$, and λ is a parameter independent of x; and

2. two supplementary conditions

$$\left.\begin{aligned} A_1 y(a)+A_2 y'(a)=0 \\ B_1 y(b)+B_2 y'(b)=0 \end{aligned}\right\} \qquad (2)$$

where A_1, A_2, B_1 and B_2 are real constants such that A_1 and A_2 are not both zero and B_1 and B_2 are not both zero.

This type of boundary value problem is called a *Sturm-Liouville problem* or *Sturm-Liouville System* or *Regular Sturm-Liouville System.*

Definition 2 :

We consider a boundary-value problem which consists of

1. a second-order homogeneous linear differential equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda r(x)\right]y=0 \qquad (3)$$

where p, q and r are real functions such that p has a continuous derivative, q and r are continuous, and $p(x)>0$ and $r(x)>0$, for all x on a real interval $a \le x \le b$ and λ is a parameter independent of x; and

2. three supplementary conditions with periodic end conditions,

$$\left.\begin{array}{l} p(a)=p(b) \\ y(a)=y(b) \\ y'(a)=y'(b) \end{array}\right\} \qquad (4)$$

This type of boundary-value problem is called a *Periodic Sturm-Liouville problem.*

Example 1 :

The boundary-value problem

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\left[2x^2+\lambda x^3\right]y=0 \qquad (5)$$

with conditions

$$\left.\begin{array}{l} 3y(1)+4y'(1)=0 \\ 5y(2)-3y'(2)=0 \end{array}\right\} \qquad (6)$$

is a sturm-liouville problem (Regular). The differential equation (5) is of the form (1), where $p(x) = x$, $q(x)=2x^2$ and $r(x)=x^3$. The conditions (6) are of the form (2), where $a=1$, $b=2$, $A_1=3$, $A_2=4$, $B_1=5$ and $B_2=-3$.

Example 2 :

The boundary-value problem

$$\frac{d^2y}{dx^2}+\lambda y=0 \;\; or, \;\; \frac{d}{dx}\left[1.\frac{dy}{dx}\right]+\lambda y=0 \qquad (7)$$

with conditions

$$\left.\begin{array}{l} p(-\pi)=p(\pi), \text{ since } b(x)=1 \\ y(-\pi)=y(\pi) \\ y'(-\pi)=y'(\pi) \end{array}\right\} \qquad (8)$$

is a sturm-liouville problem (Periodic). The differential equation (7) is of the form (3), where $p(x)=1$, $q(x)=0$ and $r(x)=1$. The conditions (8) are of the form (4), where $a=-\pi$, $b=\pi$.

Characteristic (Eigen) Values and Characteristic (Eigen) Functions

Definition 3 : Consider the Sturm-Liouville problem (Regular & Periodic of the differential equation [(1) and same as (3)] and the supplementary conditions [(2) or (3)]. The values of the parameter λ in (1) for which there exist nontrivial solutions of the problem are called the characteristic (eigen) values of the problem. The corresponding nontrivial solutions themselves are called the characteristic (eigen) functions of the problem.

Important Note :

The boundary value problem is said to be homogeneous if both the differential equation and the boundary conditions are homogeneous [i.e., the right side of equations are vanish]. Otherwise the problem is non homogeneous.

5.3.2 Properties of Sturm Liouville Type :

Property 1 :

The eigen values of a sturm-liouville problem are all real and nonnegative.

Property 2 :

The eigen values of a sturm-liouville problem can be arranged to form a strictly increasing infinite sequence that is, $0 \le \lambda_1 < \lambda_2 < \lambda_3 < \ldots$, Furthermore, $\lambda_n \to \infty$ as $n \to \infty$

Property 3 :

For each eigen value of a sturm-liouville problem, there exists one and only one linearly independent eigen function.

Property 4 :

The set of eigen functions $\{\phi_1(x), \phi_2(x), \ldots\}$ of a sturm-liouville problem satisfies the relation

$\int_a^b r(x)\,\phi_n(x)\,\phi_m(x)dx = 0$ for $n \ne m$, where $r(x)$ is continuous and positive on $[a, b]$.

Example 3 :

Find the eigen values and eigen functions of

$y'' + \lambda y = 0,\ y(0) = 0,\ y(l) = 0.$

Show that the above boundary value problem is a sturm-liouville problem. Also verify that the four properties for the sturm-liouville problem.

Solu : Let $y = ce^{mx}\ (c \ne 0)$ be the trial solution of the given diff. equation. Then the arbitary equation is $m^2 + \lambda = 0$. We consider the cases $\lambda = 0,\ \lambda < 0$ and $\lambda > 0$ separately, since they lead to different solutions.

Case 1 : $\lambda = 0$: The solution is $y = c_1 + c_2 x$. Applying the boundary conditions, we obtain $c_1 = c_2 = 0$, which results in the trivial solution.

Case 2 : $\lambda < 0$: The solution is $y = c_1 e^{\sqrt{-\lambda x}} + c_2 e^{-\sqrt{-\lambda x}}$, where $-\lambda$ and $\sqrt{-\lambda}$ are positive. Applying the boundary conditions, we obtain $c_1 + c_2 = 0,\ c_1 e^{\sqrt{-\lambda}} + c_2 e^{-\sqrt{-\lambda}} = 0$

Here $\begin{vmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}} & e^{-\sqrt{-\lambda}} \end{vmatrix} = e^{-\sqrt{-\lambda}} - e^{\sqrt{-\lambda}}$

which is never zero for any value of $\lambda < 0$. Hence, $c_1 = c_2 = 0$ and $y = 0$ which is also trivial.

Case 3 : $\lambda > 0$: The solution is $c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x$. Applying the boundary conditions, we obtain $c_2 = 0$ and $c_1 \sin \sqrt{\lambda} = 0$. Note that $\sin \theta = 0$ if and only if $\theta = n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$. Further more, if $\theta > 0$, then n must be positive. To satisfy the boundary conditions, $c_2 = 0$ and either $c_1 = 0$ or $\sin \sqrt{\lambda} = 0$. This last equation is equivalent to $\sqrt{\lambda} = n\pi$ where $n = 1,2,3, \ldots$, the choice $c_1 = 0$ results in the trivial solution, the choice $\sqrt{\lambda} = n\pi$ results in the nontrivial solution $y_n = c_n \sin(n\pi x)$. Here the notation c_n signifies that the arbitary constant c_n can be different for different values of n.

Collecting the result of all three cases, we conclude that the eigen values are $\lambda_n = n^2\pi^2$ and the corresponding eigen functions are $y_n = c_n \sin(n\pi x)$, for $n = 1, 2, 3, \ldots$.

The given differential equation can be expressed in the form

$\frac{d}{dx}\left[1.\frac{dy}{dx}\right] + [0 + 1\lambda]y = 0$ which has the same form of (1) i.e., sturm-liouville problem. Also $p(x) = 1,\ q(x) = 0$ and $r(x) = 1$. Here both $p(x)$ and $r(x)$ are positive and continuous everywhere, in particular on [0, 1].

Hence the given boundary-value problem is a sturm-liouville problem.

Verification of Properties :

We have that the eigen values are $\lambda_n = n^2\pi^2$ and the corresponding eigen functions are $y_n(x) = c_n \sin(n\pi x)$, for $n = 1,2,3, \ldots$. The eigen values are obviously *real* and *nonnegative*, and they can be ordered as $\lambda_1 = \pi^2 < \lambda_2 = 4\pi^2 < \lambda_3 = 9\pi^2 < \ldots$, i.e., eigenvalues form a strictly *increasing infinite sequence*. Each eigen value has a single *linearly independent eigen function* $\phi_n(x) = \sin n\pi x$ associated with it. Finally, since $\sin(n\pi x) \sin(m\pi x) = \frac{1}{2}\cos(n-m)\pi x - \frac{1}{2}\cos(n+m)\pi$ we have for $n \neq m$ and $r(x) = 1$.

then $\int_a^b r(x)\,\phi_n(x)\,\phi_m(x)dx = \int_0^1 \sin(n\pi x)\,\sin(m\pi x)\,dx$

$$= \int_0^1 \left[\frac{1}{2}\cos(n-m)\pi x - \frac{1}{2}\cos(n+m)\pi x\right]dx$$

$$= \left[\frac{1}{2(n-m)\pi}\sin(n-m)\pi x - \frac{1}{2(n+m)\pi}\sin(n+m)\pi x\right]_0^1$$

$$= 0 .$$

5.3.3 Orthogonality of Characteristic Functions and Some Theorems :

Orthogonality of Characteristic Functions :

Definition 4 :

Let $\{\phi_n\}$, $n = 1, 2, 3, ...$, be an infinite set of functions defined on the interval $a \le x \le b$. The set $\{\phi_n\}$ is called an orthogonal system with respect to the weight function r on $a \le x \le b$ if every two distinct functions of the set are orthogonal with respect to r on $a \le x \le b$. That is, set $\{\phi_n\}$ is orthogonal with respect to r on $a \le x \le b$, if $\int_a^b \phi_m(x)\,\phi_n(x)\,r(x)\,dx = 0$, for $m \ne n$.

Example 4 :

Consider the infinite set of functions $\{\phi_n\}$, where $\phi_n(x) = \sin nx,\ (n = 1, 2, 3, ...)$, on the interval $0 \le x \le \pi$. The set $\{\phi_n\}$ is an orthogonal system with respect to the weight function having the constant value 1 on the interval $0 \le x \le \pi$, for

$$\int_0^\pi (\sin mx)(\sin nx)dx = \left[\left[\frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}\right]_0^\pi\right]$$

$$= 0,\ for\ m \ne n .$$

Theorem 1 :

Let the coefficients $p(x)$, $q(x)$ and $r(x)$ in the sturm-liouville system be continuous in [a, b]. Let the eigen functions ϕ_j and ϕ_k corresponding to λ_j and λ_k be continuously differentable. Then ϕ_j and ϕ_k are orthogonal with respect to the weight function $r(x)$ in $[a, b]$.

Proof :

We have the sturm-liouville system

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda r(x)\right]y=0 \qquad (1)$$

Since ϕ_j and ϕ_k are the eigen functions of (1) corresponding to the eigen values λ_j and λ_k respectively, thus we have

$$\frac{d}{dx}\left[p\frac{d\phi_j}{dx}\right]+\left[q+\lambda_j r\right]\phi_j=0 \qquad (2)$$

$$\& \quad \frac{d}{dx}\left[p\frac{d\phi_k}{dx}\right]+\left[q+\lambda_k r\right]\phi_k=0 \qquad (3)$$

Multiplying to the equation (2) by ϕ_k and to the equation (3) by ϕ_j and then subtracting we get,

$$\phi_k\frac{d}{dx}\left[p\,\phi'_j\right]+\left[q+r\lambda_j\right]\phi_j\phi_k-\phi_j\frac{d}{dx}\left[p\,\phi'_k\right]-\left[q+r\lambda_k\right]\phi_j\phi_k=0$$

$$\Rightarrow r\left(\lambda_j-\lambda_k\right)\phi_j\phi_k=\phi_j\frac{d}{dx}\left[p\phi'_k\right]-\phi_k\frac{d}{dx}\left[p\phi'_j\right]$$

$$=\frac{d}{dx}\left[\left(p\phi_j\phi'_k\right)-\left(p\phi'_j\phi_k\right)\right]$$

Now integrating to the above with respect to x within the limits a to b

$$\left(\lambda_j-\lambda_k\right)\int_a^b\phi_j\;\phi_k r\,dx=\left[p\;\phi_j\phi'_k-p\phi'_j\phi_k\right]_a^b$$

$$\left(\lambda_j-\lambda_k\right)\int_a^b\phi_j\;\phi_k r\,dx=p(b)\left[\phi_j(b)\;\phi'_k(b)-\phi'_j(b)\;\phi_k(b)\right]$$

$$-p(a)\left[\phi_j(a)\;\phi'_k(a)-\phi'_j(a)\;\phi_k(b)\right] \qquad (4)$$

Now the supplementary conditions of SL system are

$$A_1\phi_j(a)+A_2\phi'_j(a)=0 \qquad (5)$$

$$B_1\phi_j(b)+B_2\phi'_j(b)=0 \qquad (6)$$

& $\quad A_1\phi_k(a)+A_2\phi_k'(a)=0 \qquad (7)$

$B_1\phi_k(b)+B_2\phi_k'(b)=0 \qquad (8)$

Multiplying to the equation (6) by $\phi_k(b)$ and to the equation (8) by $\phi_j(b)$ and then subtracting, assuming $B_2 \neq 0$, we have

$$B_2\left[\phi_j'(b)\,\phi_k(b)-\phi_k'(b)\,\phi_j(b)\right]=0$$

$$\Rightarrow \phi_j'(b)\,\phi_k(b)-\phi_k'(b)\,\phi_j(b)=0 \qquad (9)$$

Similarly from (5) & (7) multiply by $\phi_k(a)$ & $\phi_j(a)$ respectively and then subtracting, assuming $A_2 \neq 0$, we have

$$A_2\left[\phi_j'(a)\,\phi_k(a)-\phi_k'(b)\,\phi_j(a)\right]=0$$

$$\Rightarrow \left[\phi_j'(a)\,\phi_k(a)-\phi_k'(b)\,\phi_j(a)\right]=0 \qquad (10)$$

Using (9) and (10) in (4) we have

$$\left(\lambda_j-\lambda_k\right)\int_a^b \phi_j\phi_k r dx=0 \qquad (11)$$

Since λ_j and λ_k are distinct eigen values, their difference $\lambda_j-\lambda_k \neq 0$, therefore from (11) we get

$$\int_a^b \phi_j\phi_k r dx=0 \Rightarrow \int_a^b \phi_j(x)\,\phi_k(x)\,r(x)dx=0$$

which shows that ϕ_j and ϕ_k are orthogonal with respect to the weight function $r(x)$.

Hence the theorem.

Theorem 2 :

If $\phi_1(x)$ and $\phi_2(x)$ are any two solutions of the sturm-liouville equation on $[a, b]$, then $p(x)W[x;\phi_1,\phi_2]=$ Constant, where W is the wronskian and $W(x;\phi_1,\phi_2)=\begin{vmatrix}\phi_1 & \phi_2\\ \phi_1' & \phi_2'\end{vmatrix}$.

Proof :

Since ϕ_1 and ϕ_2 are the solutions of SL equation then

$$\frac{d}{dx}\left[p(x)\,\frac{d\phi_1}{dx}\right]+[q+\lambda r]\,\phi_1=0 \qquad (1)$$

$$\& \quad \frac{d}{dx}\left[p(x)\ \frac{d\phi_2}{dx}\right]+[q+\lambda r]\,\phi_2=0 \qquad (2)$$

Multiplying to the equation (1) by ϕ_2 and that of equation (2) by ϕ_1 and then subtracting, we get,

$$\Rightarrow \phi_1\frac{d}{dx}\left[p\frac{d\phi_2}{dx}\right] - \phi_2\frac{d}{dx}\left[p\frac{d\phi_1}{dx}\right]=0$$

$$\Rightarrow \frac{d}{dx}\left[(p\phi_2')\phi_1-(p\phi_1')\phi_2\right]=0.$$

Integrating above *w.r.* to x to the limits a to x, we get,

$$\int_a^x d\left[(p\phi_2')\phi_1-(p\phi_1')\phi_2\right]=0$$

$$\Rightarrow p(x)\,\phi_2'(x)\,\phi_1(x)-p(x)\,\phi_1'(x)\,\phi_2(x)=p(a)\,\phi_2'(a)\,\phi_1(a)-p(a)\,\phi_1'(a)\,\phi_2(a)$$

$$\Rightarrow p(x)\left[\phi_2'(x)\,\phi_1(x)-\phi_1'(x)\,\phi_2(x)\right]=p(a)\left[\phi_2'(a)\,\phi_1(a)-\phi_1'(a)\,\phi_2(a)\right]$$

$$\Rightarrow p(x)\,W\left[x;\,\phi_1,\,\phi_2\right]=\text{Constant}.$$

Theorem 3 :

The eigen function of the poriodic sturm-liouville system in $[a, b]$ are orthogonal with respect to the weight function $r(x)$ in $[a, b]$.

Proof : We have the periodic sturm-liouville system

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda r(x)\right]y=0 \qquad (1)$$

Since ϕ_j and ϕ_k are the eigen functions of (1) corresponding to the eigen values λ_j and λ_k respectively, thus we have

$$\frac{d}{dx}\left[p\frac{d\phi_j}{dx}\right]+\left[q+\lambda_j r\right]\phi_j=0 \qquad (2)$$

$$\& \quad \frac{d}{dx}\left[p\frac{d\phi_k}{dx}\right]+\left[q+\lambda_k r\right]\phi_k=0 \qquad (3)$$

Multiplying to the equation (2) by ϕ_k and to the equation (3) by ϕ_j and then subtracting we get,

$$\phi_k \frac{d}{dx}[p\phi_j'] + [q + r\lambda_j]\,\phi_j\phi_k - \phi_j \frac{d}{dx}[p\phi_k'] - [q + r\lambda_k]\,\phi_j\phi_k = 0$$

$$\Rightarrow\ r(\lambda_j - \lambda_k)\,\phi_j\phi_k = \phi_j \frac{d}{dx}[p\phi_k'] - \phi_k \frac{d}{dx}[p\phi_j']$$

$$= \frac{d}{dx}[(p\phi_j\,\phi_k') - (p\phi_j'\,\phi_k)]$$

Now integrating to the above with respect to x within the limits a to b

$$(\lambda_j - \lambda_k)\int_a^b \phi_j\phi_k r\,dx = [p\phi_j\phi_k' - p\phi_j'\phi_k]_a^b$$

$$(\lambda_j - \lambda_k)\int_a^b \phi_j\phi_k r\,dx = p(b)[\phi_j(b)\phi_k'(b) - \phi_j'(b)\phi_k(b)]$$

$$-p(a)[\phi_j(a)\phi_k'(a) - \phi_j'(a)\phi_k(a)] \qquad (4)$$

Now the supplementany conditions of periodic SL system are

$$p(a) = p(b) \qquad (5)$$

and $\left.\begin{array}{l}\phi_j(a) = \phi_j(b)\\ \phi_j'(a) = \phi_j'(b)\end{array}\right\}$ (6)

and $\left.\begin{array}{l}\phi_k(a) = \phi_k(b)\\ \phi_k'(a) = \phi_k'(b)\end{array}\right\}$ (7)

Using (5), (6) & (7) in (4), we get

$$(\lambda_j - \lambda_k)\int_a^b \phi_j\phi_k r(x)\,dx = 0$$

Since λ_j and λ_k are distinct characteristic values, their difference $\lambda_j - \lambda_k \neq 0$. Therefore we must have

$$\int_a^b \phi_j\phi_k r(x)\,dx = 0$$

Hence ϕ_j and ϕ_k are orthogonal with respect to $r(x)$ on $a \le x \le b$.

Theorem 4 :

All the eigen values of a regular sturm-liouville system with $r(x) > 0$, are real.

Proof :

We have the sturm-liouville system

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda r(x)\right]y=0 \qquad (1)$$

Since ϕ_j and ϕ_k are the eigen functions of (1) corresponding to the eigen values λ_j and λ_k respectively, thus we have

$$\frac{d}{dx}\left[p\frac{d\phi_j}{dx}\right]+\left[q+\lambda_j r\right]\phi_j=0 \qquad (2)$$

$$\& \quad \frac{d}{dx}\left[p\frac{d\phi_k}{dx}\right]+\left[q+\lambda_k r\right]\phi_k=0 \qquad (3)$$

Multiplying to the equation (2) by ϕ_k and to the equation (3) by ϕ_j and then subtracting we get.

$$\phi_k\frac{d}{dx}\left[p\phi_j'\right]+\left[q+r\lambda_j\right]\phi_j\phi_k-\phi_j\frac{d}{dx}\left[p\phi_k'\right]-\left[q+r\lambda_k\right]\phi_j\phi_k=0$$

$$\Rightarrow \; r\left(\lambda_j-\lambda_k\right)\phi_j\phi_k=\phi_j\frac{d}{dx}\left[p\phi_k'\right]-\phi_k\frac{d}{dx}\left[p\phi_j'\right]$$

$$=\frac{d}{dx}\left[\left(p\phi_j\phi_k'\right)-\left(p\phi_j'\phi_k\right)\right]$$

Now integrating to the above with respect to x within the limits a to b

$$\left(\lambda_j-\lambda_k\right)\int_a^b\phi_j\phi_k r\,dx=\left[p\phi_j\phi_k'-p\phi_j'\phi_k\right]_a^b$$

$$\left(\lambda_j-\lambda_k\right)\int_a^b\phi_j\phi_k r\,dx=p(b)\left[\phi_j(a)\phi_k'(b)-\phi_j'(b)\phi_k(b)\right]$$

$$-p(a)\left[\phi_j(a)\phi_k'(a)-\phi_j'(a)\phi_k(b)\right] \qquad (4)$$

Now the supplementary conditions of SL system are

$$A_1\phi_j(a)+A_2\phi_j'(a)=0 \qquad (5)$$

$$B_1\phi_j(b)+B_2\phi_j'(b)=0 \quad (6)$$

& $$A_1\phi_k(a)+A_2\phi_k'(a)=0 \quad (7)$$

$$B_1\phi_k(b)+B_2\phi_k'(b)=0 \quad (8)$$

Multiplying to the equation (6) by $\phi_k(b)$ and to the equation (8) by $\phi_j(b)$ and then subtracting assuming $B_2 \neq 0$, we have

$$B_2[\phi_j'(b)\,\phi_k(b)-\phi_k'(b)\,\phi_j(b)]=0$$

$$\Rightarrow \phi_j'(b)\,\phi_k(b)-\phi_k'(b)\,\phi_j(b)=0 \quad (9)$$

Again multiplying to the equation (5) by $\phi_k(a)$ and by $\phi_j(a)$ to the equation (7) respectively and then subtracting and assuming $A_2 \neq 0$, we have

$$A_2[\phi_j'(a)\,\phi_k(a)-\phi_k'(b)\,\phi_j(a)]=0$$

$$\Rightarrow \phi_j'(a)\,\phi_k(a)-\phi_k'(b)\,\phi_j(a)=0 \quad (10)$$

Using (9) & (10) in (4) we have,

$$\left(\lambda_j-\lambda_k\right)\int_a^b \phi_j\phi_k r dx=0 \quad (11)$$

Let us assume that $\lambda_j=\alpha+i\beta$ corresponding to $\phi_j=u+iv$. Then as the coefficients of SL equation are real, the complex conjugate of λ_j is also an eigen value.

Thus, there exists an eigen function $\phi_k=u-iv=\bar{\phi}_j$ corresponding to the eigen value $\lambda_k=\alpha-i\beta=\bar{\lambda}_j$.

Using above conditions, in equation (11) we get

$$[(\alpha+i\beta)-(\alpha-i\beta)]\int_a^b (u+iv)\,(u-iv)\,r(x)dx=0$$

$$\Rightarrow 2i\beta\int_a^b\left(u^2+v^2\right)r(x)dx=0$$

Since $r(x)$ is positive and u^2+v^2 is positive.

Therefore β must be equal to zero.

Hence eigen values of regular SL system are real.

5.3.4 Exercises

Ex-1.: Find the characteristic values and characteristic functions of the sturm-liouville problem

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\frac{\lambda}{x}y=0;\quad y'(1)=0,\ y'\left(e^{2\pi}\right)=0.$$

Also verify that the four properties for the sturm-liouville problem.

Ex-2.: Find the eigen values and eigen functions of

$$y''+\lambda y=0,\ y(0)=0,\ y'(\pi)=0.$$

Also verify that the four properties for the sturm-liouville problem.

Ex-3.: Verify the properties for following sturm-liouville problem.

$$(xy')'+\left[x^2+1+\lambda e^x\right]y=0;\ y(1)+2y'(1)=0,$$
$$y(2)-3y'(2)=0.$$

Answers :

Ex. 1: Characteristic values : $0,\frac{1}{4},1,\frac{9}{4},4,\frac{25}{4},\ldots,\frac{n^2}{4},\ldots,$

Characteristic function : $c_0, c_1\cos\left(\frac{\ln x}{2}\right), c_2\cos(\ln x), c_3\cos\left(\frac{3}{2}\ln x\right),\ldots,$

Ex. 2 : Characteristic values : $\lambda_n=\left(n-\frac{1}{2}\right)^2, n=1,2,3,\ldots.$

Characteristic functions : $y_n=C_n\, S_n\left(n-\frac{1}{2}\right)x, n=1,2,3,\ldots$

5.3.5 References :

1. Shepley L. Ross; Differential Equations, 3rd Edition, 2004, New York.
2. Richard Bronson; Differential Equations, 2nd Edition, 1994, New York.
3. Birkhoff, G and G.C. Rota, Ordinary Differential Equations, 3rd Edition, 1978, New York.

5.4 GREEN'S FUNCTION

5.4.1 Definition :

Suppose we have a differential equation of order n (linear homogeneous) :

$$L[y]\equiv p_0(x)y^n+p_1(x)y^{n-1}+\ \ldots\ +p_n(x)y=0 \qquad (1)$$

where the functions $p_0(x), p_1(x), ..., p_n(x)$ are continuous on $[a, b]$, $p_0(x) \neq 0$ on $[a, b]$ and the boundary conditions are $V_k(y) = 0,\ k = 1, 2, ..., n$ (2)

$$\text{where } V_k(y) = \alpha_k y(a) + \alpha_k^{(1)} y'(a) + \ ... \ + \alpha_k^{(n-1)} y^{(n-1)}(a) + \beta_k y(b)$$
$$+ \beta_k^{(1)} y'(b) + \ ... \ + \beta_k^{(n-1)} y^{(n-1)}(b) \qquad (2.1)$$

where the linear forms $V_1, V_2, ..., V_n$ in $y(a), y'(a), ..., y^{(n-1)}(a), y(b), y'(b), ..., y^{(n-1)}(b)$ are linearly independent.

We assume that the homogeneous boundary value problem (1) – (2.1) has only a trivial solution $y(x) = 0$.

Definition :

Green's function of the boundary-value problem (1) – (2.1) is the function $G(x, \zeta)$ constructed for any point ζ, $a < \zeta < b$, and having the following four properties :

1. $G(x, \zeta)$ is continuous and has continuous derivatives with respect to x up to order $(n-2)$ inclusive for $a \leq x < b$.

2. Its $(n-1)$th derivative with respect to x at the point $x = \zeta$ has a discontinuity of the first kind, the jump being equal to $\frac{1}{p_0(x)}$, i.e.,

$$\left[\frac{\partial^{n-1} G(x,\zeta)}{\partial x^{n-1}}\right]_{x=\zeta+0} - \left[\frac{\partial^{n-1} G(x,\zeta)}{\partial x^{n-1}}\right]_{x=\zeta-0} = \frac{1}{p_0(\zeta)} \qquad (3)$$

3. In each of the intervals $[a, \zeta)$ and $(\zeta, b]$ the function $G(x, \zeta)$, considered as a function of x, is a solution of equation (1) : $L(G) = 0$ (4)

4. $G(x, \zeta)$ satisfies the boundary conditions (2) :

$$V_k(G) = 0,\ (k = 1, 2, ..., n) \qquad (5)$$

5.4.2 Some theorems and Examples :

Theorem 1 :

If the boundary - value problem (1) – (2.1) has only the trivial solution $y(x) = 0$, then the operation L has one and only one Green's function $G(x, \zeta)$.

Proof : Let $y_1(x), y_2(x), \ldots, y_n(x)$ be linearly independent solutions of the equation $L(y) = 0$. Then by virtue of property (3), the unknown function $G(x, \zeta)$ must have the following representation on the intervals $[a, \zeta)$ and $(\zeta, b]$:

$$G(x, \zeta) = \begin{cases} a_1y_1(x) + a_2y_2(x) + \ldots + a_ny_n(x), & \text{for } a \le x < \zeta \\ b_1y_1(x) + b_2y_2(x) + \ldots + b_ny_n(x), & \text{for } \zeta < x \le b \end{cases}$$

where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are some functions of ζ.

Again by virtue of property (1), the continuity of the function $G(x, \zeta)$ and of its first $(n-2)$ derivatives with respect to x at the point $x = \zeta$ yields the relation

$$\left[b_1y_1(\zeta) + b_2y_2(\zeta) + \ldots + b_ny_n(\zeta)\right] - \left[a_1y_1(\zeta) + a_2y_2(\zeta) + \ldots + a_ny_n(\zeta)\right] = 0$$

$$\left[b_1y_1'(\zeta) + b_2y_2'(\zeta) + \ldots + b_ny_n'(\zeta)\right] - \left[a_1y_1'(\zeta) + a_2y_2'(\zeta) + \ldots + a_ny_n'(\zeta)\right] = 0$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\left[b_1y_1^{(n-2)}(\zeta) + b_2y_2^{(n-2)}(\zeta) + \ldots + b_ny_n^{(n-2)}(\zeta)\right] - \left[a_1y_1^{(n-2)}(\zeta) + a_2y_2^{(n-2)}(\zeta) + \ldots + a_ny_n^{(n-2)}(\zeta)\right] = 0$$

and by the property (2), the condition (3) takes the form

$$\left[b_1y_1^{(n-1)}(\zeta) + b_2y_2^{(n-1)}(\zeta) + \ldots + b_ny_n^{(n-1)}(\zeta)\right] - \left[a_1y_1^{(n-1)}(\zeta) + a_2y_2^{(n-1)}(\zeta) + \ldots + a_ny_n^{(n-1)}(\zeta)\right] = \frac{1}{p_0(\zeta)}$$

Let us put $C_k(\zeta) = b_k(\zeta) - a_k(\zeta)$, $k = 1, 2, \ldots, n$; then we get a system of linear equations in $c_k(\zeta)$:

$$\left.\begin{aligned} &c_1y_1(\zeta) + c_2y_2(\zeta) + \ldots + c_ny_n(\zeta) = 0 \\ &c_1y_1'(\zeta) + c_2y_2'(\zeta) + \ldots + c_ny_n'(\zeta) = 0 \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ &c_1y_1^{(n-2)}(\zeta) + c_2y_2^{(n-2)}(\zeta) + \ldots + c_ny_n^{(n-2)}(\zeta) = 0 \\ &c_1y_1^{(n-1)}(\zeta) + c_2y_2^{(n-1)}(\zeta) + \ldots + c_ny_n^{(n-1)}(\zeta) = \frac{1}{p_0(\zeta)} \end{aligned}\right\} \qquad (6)$$

The determinant of system of linear equation (6) is equal to the value of the Wronskiam $W(y_1, y_2, \ldots, y_n)$ at the point $x = \zeta$ and is non zero, because $y_1, y_2, \ldots, y_n$ are linearly independent, and it follows that

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1(\zeta) & y_2(\zeta) & \dots & y_n(\zeta) \\ y_1'(\zeta) & y_2'(\zeta) & \dots & y_n'(\zeta) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-2)}(\zeta) & y_2^{(n-2)}(\zeta) & \dots & y_n^{(n-2)}(\zeta) \\ y_1^{(n-1)}(\zeta) & y_2^{(n-1)}(\zeta) & \dots & y_n^{(n-1)}(\zeta) \end{vmatrix} \neq 0$$

For this reason, the system (6) possess a unique solution for c_k, $k = 1, 2, \dots, n$. To determine the functions $a_k(\zeta)$ and $b_k(\zeta)$, let us take the property (2) of Green's function. We write $V_k(y)$ in the form

$$V_k(y) = A_k(y) + B_k(y) \tag{7}$$

where $A_k(y) = \alpha_k y(a) + \alpha_k^{(1)} y'(a) + \dots + \alpha_k^{(n-1)} y^{(n-1)}(a)$

& $B_k(y) = \beta_k y(b) + \beta_k^{(1)} y'(b) + \dots + \beta_k^{(n-1)} y^{(n-1)}(b)$.

Then by the property (iv) and by the conditions (5), we get

$$V_k(G) = a_1 A_k(y_1) + a_2 A_k(y_2) + \dots + a_n A_k(y_n) + b_1 B_k(y_1) + b_2 B_k(y_2) + \dots + b_n B_k(y_n) = 0,$$

$$k = 1, 2, \dots, n \tag{7.1}$$

since we have $a_k = b_k - c_k$, $k = 1, 2, \dots, n$ (7.2)

Using (7.2) in (7.1) we have

$$(b_1 - c_1)A_k(y_1) + (b_2 - c_2)A_k(y_2) + \dots + (b_n - c_n)A_k(y_n) + b_1 B_k(y_1) + b_2 B_k(y_2) + \dots$$
$$+ b_n B_k(y_n) = 0, \ k = 1, 2, \dots, n.$$

Again using (7) in above, we get,

$$b_1 V_k(y_1) + b_2 V_k(y_2) + \dots + b_n V_k(y_n) = c_1 A_k(y_1) + c_2 A_k(y_2) + \dots + c_n A_k(y_n) \tag{8}$$

$$k = 1, 2, \dots, n$$

Here equation (8) is a linear system of n equations for determination of n quantities $b_1, b_2, \dots, b_n$. Also since $V_1, V_2, \dots, V_n$ are linearly independent then the determinant of the system (8) is non zero, i.e.,

$$\begin{vmatrix} V_1(y_1) & V_1(y_2) & \cdots & V_1(y_n) \\ V_2(y_1) & V_2(y_2) & \cdots & V_2(y_n) \\ \cdots & \cdots & \cdots & \cdots \\ V_n(y_1) & V_n(y_2) & \cdots & V_n(y_n) \end{vmatrix} \neq 0 \qquad (9)$$

By virtue of (9), we see that the system of equations obtained from (8) possesses a unique solution in $b_1(\zeta), b_2(\zeta), \ldots b_n(\zeta)$ and since $a_k(\zeta) = b_k(\zeta) - c_k(\zeta)$, it follows that the quantities $a_k(\zeta)$, $k = 1, 2, \ldots, n$ are defined uniquely, as $c_1, c_2, \ldots, c_n$ are uniquely determined earlier.

Thus the existence and uniqueness of Green's function $G(x, \zeta)$ are established and a method has been described for constructing the Green's function.

Note 1 :

If the boundary value problem (1) – (2.1) is self adjoint, then Green's function is symmetric, i.e., $G(x, \zeta) = G(\zeta, x)$. The converse is true as well.

Note 2 :

If at one of the extremities of an interval $[a, b]$ the coefficient of the highest derivative vanishes, for example $p_0(a) = 0$, then the natural boundary conditions for boundedness of the solution at $x = a$ is imposed, and at the other extremity the ordinary boundary condition is specified.

An Important Special Case :

Let us consider for construction of Green's funotion for a 2nd order linear homogeneous differential equation of the form

$$(p(x)y')' + q(x)y = 0$$

$$\text{or} \quad p(x)\frac{d^2y}{dx^2} + p'(x)\frac{dy}{dx} + q(x)y = 0,\ p(x) \neq 0 \ on\ [a, b] \qquad (1)$$

with boundary conditions $y(a) = y(b) = 0$ (2)

Suppose that $y_1(x)$ is a solution of equation (10) defined by the initial conditions

$$y_1(a) = 0,\ y_1'(a) = \alpha \neq 0. \qquad (3)$$

Generally speaking, this solution need not necessarily satisfy the second boundary condition. We will therefore assume that $y_1(b) \neq 0$. But functions of the form $c_1 y_1(x)$, where c_1 is an arbitary constant, are obviously solutions of equation (1) and satisfy the boundary condition $y(a)=0$.

Similarly, we find the nonzero solution $y_2(x)$ of equation (1), such that it should satisfy the second boundary condition i.e., $y_2(b)=0$ (4)

This same condition will be satisfied by all solutions of the family $c_2 y_2(x)$, where c_2 is an arbitary constant.

We now seek Green's function for the problem (1) – (2) in the form

$$G(x,\ \zeta)=\begin{cases} c_1 y_1(x), & a \le x \le \zeta \\ c_2 y_2(x), & \zeta \le x \le b \end{cases} \qquad (5)$$

and we shall choose the constants c_1 and c_2 so that the properties of Green's function (1) & (2) are fulfilled, i.e., so that the function $G(x,\ \zeta)$ is continuous in x for fixed ζ, in particular, continuous at the point $x=\zeta$; then $c_1 y_1(\zeta)=c_2 y_2(\zeta)$

and so that $G_x'(x,\ \zeta)$ has a jump, at the point $x=\zeta$, equal to $\dfrac{1}{p(\zeta)}$:

therefore $c_2 y'(\zeta)-c_1 y'(\zeta)=\dfrac{1}{p(\zeta)}$

Rewriting the last two equations we have

$$\left.\begin{aligned} -c_1 y_1(\zeta)+c_2 y_2(\zeta)&=0 \\ -c_2 y_1'(\zeta)+c_2 y_2'(\zeta)&=\frac{1}{p(\zeta)} \end{aligned}\right\} \qquad (6)$$

The determinant of the above system (6) is the wrorskian $W[y_1(x),\ y_2(x)]=W(x)$ computed at the point $x=\zeta$ for linearly independent solutions $y_1(x)$ and $y_2(x)$ of equation (1) and hence it is non zero, so $W(\zeta)\neq 0$;

On solving the system equation (6) we have $c_1=\dfrac{y_2(\zeta)}{p(\zeta)\ W(\zeta)}$, $c_2=\dfrac{y_1(\zeta)}{p(\zeta)\ W(\zeta)}$ substituting the values of c_1 & c_2 in (5), we have

$$G(x,\zeta)=\begin{cases}\dfrac{y_1(x)\,y_2(\zeta)}{p(\zeta)\,W(\zeta)}, & a\le x\le\zeta\\[2mm] \dfrac{y_1(\zeta)\,y_2(x)}{p(\zeta)\,W(\zeta)}, & \zeta\le x\le b\end{cases} \qquad (7)$$

Ex. 1 : Construct Green's function for the homogeneous boundary value problem

$$\frac{d^4y}{dx^4}=0,\ y(0)=y'(0)=y'(1)=0=y(1)$$

Solu. : 1st Step : Given that the differential equation $\dfrac{d^4y}{dx^4}=0$ (1) with boundary conditions

$$y(0)=0,\ y'(0)=0,\ y(1)=0,\ y'(1)=0 \qquad (2)$$

The auxilliary equation of (1) is $m^4=0, \Rightarrow m=0,0,0,0$.

Hence the general solution of (1) is $y(x)=A+Bx+Cx^2+Dx^3$, (3)

where A, B, C, D are arbitary constants. Indeed, the fundamental system of solutions for equation (1) is $y_1(x)=1,\ y_2(x)=x,\ y_3(x)=x^2,\ y_4(x)=x^3$ (4)

Now we shall first show that the boundary value problem (1) – (2) has only a trivial solution.

Using the boundary conditions (2), the equation (3) give us four relations for determing A, B, C & D.

$$y(0)=A=0$$

$$y'(0)=B=0$$

$$y(1)=A+B+C+D=0$$

$$y'(1)=A+2C+3D=0$$

This gives, $A=B=C=D=0$.

Thus, the problem (1) – (2) has only a trivial solution $y(x)=0$ and, hence for it we can construct a Green's function (unique) $G(x,\zeta)$.

2nd Step : We now construct Green's function. Utilizing the fundamental system of solutions (4), represent the unknown Green's function $G(x,\zeta)$ in the form

$$G(x,\zeta)=\begin{cases}a_1.1+a_2.x+a_3.x^2+a_4.x^3, & 0\le x<\zeta\\ b_1.1+b_2.x+b_3.x^2+b_4.x^3, & \zeta<x\le 1\end{cases} \qquad (5)$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are as yet unknown function of ζ. The continuity of the function $G(x, \zeta)$ and of its first (4–2) derivatives with respect to x at the point $x = \zeta$ yields the relations :

(1) $G(x, \zeta), \frac{\delta G}{\delta x}; \frac{\partial^2 G}{\partial x^2}$ are continuous ar x = ζ, thus we have

$$b_1 + b_2\zeta + b_3\zeta^2 + b_4\zeta^3 = a_1 + a_2\zeta + a_3\zeta^2 + a_4\zeta^3 \quad (6)$$

$$b_2 + 2b_3\zeta + 3b_4\zeta^2 = a_2 + 2a_3\zeta + 3a_4\zeta^2 \quad (7)$$

$$2b_3 + 6b_4\zeta = 2a_3 + 6a_4\zeta \quad (8)$$

(2) The derivative $\frac{\partial^3 G}{\partial x^3}$ of G has a discontinuity of magnitude $+ 1/p_0(\zeta)$ at the point $x = \zeta$, where $p_0(x)$ = coefficient of the highest order derivative in (1) = 1.

thus $\left(\frac{\delta^3 G}{\delta x^3}\right)_{x=\zeta+0} - \left(\frac{\delta^3 G}{\delta x^3}\right)_{x=\zeta-0} = 1$

$$\Rightarrow 6b_4 - 6a_4 = 1 \quad (9)$$

(3) Green's function must satisfy the boundary conditions (2), thus, we have

$G(0, \zeta) = 0$ so that $a_1 = 0$ (10)

$G'(0, \zeta) = 0$ so that $a_2 = 0$ (11)

$G(1, \zeta) = 0$ so that $b_1 + b_2 + b_3 + b_4 = 0$ (12)

$G'(1, \zeta) = 0$ so that $b_2 + 2b_3 + 3b_4 = 0$ (13)

Put $c_k(\zeta) = b_k(\zeta) - a_k(\zeta)$, k = 1, 2, 3, 4. (14)

Then equations (6), (7), (8) & (9) may be written as

$$\left.\begin{aligned} c_1 + c_2\zeta + c_3\zeta^2 + c_4\zeta^3 &= 0 \\ c_2 + 2c_3\zeta + 3c_4\zeta^2 &= 0 \\ 2c_3 + 6c_4\zeta &= 0 \\ 6c_4 &= 1 \end{aligned}\right\} \quad (15)$$

Solving the system of equations (15) we have,

$$c_1 = -\tfrac{1}{6}\zeta^3,\ c_2 = +\tfrac{1}{2}\zeta^2,\ c_3 = -\tfrac{1}{2}\zeta,\ c_4 = \tfrac{1}{6} \qquad (16)$$

Using (16), equations (14) becomes,

$$\left.\begin{aligned} b_1 - a_1 &= -\tfrac{\zeta^3}{6} \\ b_2 - a_2 &= +\tfrac{\zeta^2}{2} \\ b_3 - a_3 &= -\tfrac{\zeta}{2} \\ b_4 - a_4 &= \tfrac{1}{6} \end{aligned}\right\} \qquad (17)$$

Using (10), (11), (12) & (13) in the system of equations (17), we have

$$\left.\begin{aligned} &a_1 = 0,\ b_1 = -\zeta^3/6,\ a_2 = 0,\ b_2 = +\zeta^2/2,\ b_3 = +\zeta^3/2 - \zeta^2,\ b_4 = +\zeta^2/2 - \zeta^3/3 \\ &a_3 = \tfrac{\zeta}{2} - \zeta^2 + \tfrac{1}{2}\zeta^3,\ a_4 = -\tfrac{1}{6} + \tfrac{1}{2}\zeta^2 - \tfrac{1}{3}\zeta^3 \end{aligned}\right\} \qquad (18)$$

Putting the values of the coefficients $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ & b_4 from (18) in (5), we obtain the desired Green's function

$$G(x,\zeta) = \begin{cases} \left(\frac{\zeta}{2} - \zeta^2 + \frac{\zeta^3}{2}\right)x^2 - \left(\frac{1}{6} - \frac{1}{2}\zeta^2 + \frac{1}{3}\zeta^3\right)x^3, & 0 < x < \zeta. \\ -\frac{\zeta^3}{6} + \frac{\zeta^2}{2}x + \left(\frac{1}{2}\zeta^3 - \zeta^2\right)x^2 + \left(\frac{1}{2}\zeta^2 - \frac{1}{3}\zeta^3\right)x^3, & \zeta < x < 1. \end{cases}$$

$$\text{or}\quad G(x,\zeta) = \begin{cases} \left(\frac{\zeta}{2} - \zeta^2 + \frac{\zeta^3}{2}\right)x^2 + \left(-\frac{1}{6} + \frac{1}{2}\zeta^2 - \frac{1}{3}\zeta^3\right)x^3, & 0 \le x < \zeta. \\ \left(\frac{x}{2} - x^2 + \frac{1}{2}x^3\right)\zeta^2 + \left(-\frac{1}{6} + \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\zeta^3, & \zeta < x \le 1. \end{cases} \qquad \text{(Answer)}$$

Ex-2 : Find the Green's function for the boundary value problem

$$y''(x) + k^2 y = 0,\ y(0) = y(1) = 0.$$

Solu. : Given boundary value problem is $y''(x) + k^2 y = 0$ (1)

with the boundary conditions $y(0) = 0,\ y(1) = 0$ (2)

The auxillary equation of (1) is $m^2 + k^2 = 0 \Rightarrow m = \pm ik$.

Hence the general solution of (1) is $y(x) = A\cos kx + B\sin kx$ (3)

Putting $x = 0$ in (3) and using boundary conditions (2), $y(0) = 0$ we get $A = 0$.

Again putting $x = 1$, in (3) and using boundary conditions $y(1) = 0$ we get $B = 0$.

Hence the equation (3) reduces $y(x) = 0$. Therefore equation (3) yields the trivial solution $y(x) = 0$ for the given boundary value problem. So, the Green's function exists and is given by

$$G(x, \zeta) = \begin{cases} a_1 \cos kx + a_2 \sin kx, & 0 \le x < \zeta \\ b_1 \cos kx + b_2 \sin kx, & \zeta < x \le 1 \end{cases} \quad (4)$$

where a_1, a_2, b_1, b_2 are as yet unknown function of ζ.

Now the Green's function must also satisfy the following three properties (1) $G(x, \zeta)$ is continuous at $x = \zeta$, that is

$$b_1 \cos k\zeta + b_2 \sin k\zeta = a_1 \cos k\zeta + a_2 \sin k\zeta$$

$$\Rightarrow (b_1 - a_1)\cos k\zeta + (b_2 - a_2)\sin k\zeta = 0 \quad (5)$$

(2) The derivative of G has a discontinuity of magnitude $+ 1/p_0(\zeta)$ at the point $x = \zeta$, where $p_0(x)$ = coefficient of highest order derivative in (1) = 1.

$$\therefore \left(\frac{\partial G}{\partial x}\right)_{x=\zeta+0} - \left(\frac{\partial G}{\partial x}\right)_{x=\zeta-0} = 1$$

$$\Rightarrow k[-b_1 \sin k\zeta + b_2 \cos k\zeta] - k[-a_1 \sin k\zeta + a_2 \cos k\zeta] = 1$$

$$\Rightarrow -(b_1 - a_1) \sin k\zeta + (b_2 - a_2) \cos k\zeta = + 1/\zeta \quad (6)$$

(3) $G(x, \zeta)$ must satisfy the boundary conditions (2) i.e.,

$G(0, \zeta) = 0$ so that $a_1 = 0$ (7)

$G(1, \zeta) = 0$ so that $b_1 \cos k + b_2 \sin k = 0$ (8)

Put $c_j = b_j - a_j,\ j = 1, 2$. (9)

Then equation (5) & (6) becomes,

$$c_1 \cos k\zeta + c_2 \sin k\zeta = 0 \quad (10)$$

$$\& \quad -c_1 \sin k\zeta + c_2 \cos k\zeta = 1/k \quad (11)$$

solving these we get

$$\frac{c_1}{\sin k\zeta(-1/k)} = \frac{c_2}{\cos k\zeta(1/k)} = \frac{1}{\cos^2 k\zeta + \sin^2 k\zeta}$$

$$\Rightarrow c_1 = -\frac{\sin k\zeta}{k},\ c_2 = \frac{\cos k\zeta}{k}$$

$$b_1 - a_1 = -\frac{\sin k\zeta}{k} \qquad (12)$$

$$b_2 - a_2 = \frac{\cos k\zeta}{k} \qquad (13)$$

Again from (7), $a_1 = 0,\ b_1 = -\frac{\sin k\zeta}{k}$

Using the value of b_1 from above in (8), we get,

$$b_2 \sin k = -b_1 \cos k = +\frac{\sin k\zeta}{k}\cos k$$

$$\Rightarrow b_2 = \frac{\sin k\zeta \cos k}{k \sin k}$$

Putting the values of b_2 in (13) we get

$$a_2 = b_2 - \frac{\cos k\zeta}{k} = \frac{\sin k\zeta \cos k}{k\sin k} - \frac{\cos k\zeta}{k}$$

$$= \frac{\sin k\zeta \cos k - \cos k\zeta \sin k}{k\sin k}$$

$$= \frac{\sin k(\zeta - 1)}{k\sin k}$$

Now $a_1 \cos kx + a_2 \sin kx = \sin kx \frac{\sin k(\zeta-1)}{k\sin k}$.

$$b_1 \cos kx + b_2 \sin kx = -\frac{\sin k\zeta}{k}\cos kx + \frac{\sin k\zeta \cos k}{k \sin k}\sin kx$$

$$= -\frac{\sin k\zeta}{k}\left[\frac{\cos kx \sin k - \cos k \sin kx}{\sin k}\right]$$

$$= -\frac{\sin k\zeta}{k}\frac{\sin (k - kx)}{\sin k} = \frac{\sin k\zeta \sin k(x-1)}{k\sin k}$$

Substituting the above values in (4), we get the required Green's function is given by

$$G(x, \zeta) = \begin{cases} \dfrac{\sin kx \sin k(\zeta - 1)}{k \sin k}, & 0 \le x < \zeta \\ \dfrac{\sin k\zeta \sin k(x-1)}{k \sin k}, & \zeta < x \le 1 \end{cases}$$

Also seen that Green's function is symmetric i.e., $G(x, \zeta) = G(\zeta, x)$.

Ex-3 : Construct Green's function for the differential equation $xy'' + y' = 0$ for the following boundary conditions : $y(x)$ is bounded as $x \to 0$, $y(1) = \alpha y'(1)$, $\alpha \neq 0$.

Solu. : Given boundary value problem is $xy'' + y' = 0$ (1) with the boundary conditions $y(x)$ is bounded as $x \to 0$ (2) and $y(1) = \alpha y'(1)$, $\alpha \neq 0$ (3). Since equation (1) is linear homogeneous differential equation. To solve it put $x = e^z$ then given diff. equation (1) reduces to $[\delta(\delta - 1) + \delta]y = 0 \Rightarrow \delta^2 y = 0$ where $\delta \equiv \frac{d}{dz}$. The general solution of above is $y = A + Bz = A + B\log x$ (4), since $\log x = z$ and A & B are constants.

Since the $B.C$ (2) follows $y(x)$ is bounded as $x \to 0$ then $B = 0$.

Again $y'(x) = \frac{B}{x}$ (5) & put $x = 1$ in (4) & (5) we have

$y(1) = A$ & $y'(1) = B/1 = B$.

Putting these values from above in (3) we get $A = \alpha B \Rightarrow A = 0$ *since* $B = 0$. Thus $A = B = 0$. Hence equation (4) yields only the trivial solution $y(x) = 0$. Therefore Green's function exists and and is given by

$$G(x, \zeta) = \begin{cases} a_1 \log x + a_2, & 0 \le x < \zeta \\ b_1 \log x + b_2, & \zeta < x < 1 \end{cases} \qquad (6)$$

Now the Green's function must also satisfy the following three properties (1) $G(x, \zeta)$ is continuous at $x = \zeta$, that is

$$a_1 \log\zeta + a_2 = b_1 \log\zeta + b_2$$

$$\Rightarrow (b_1 - a_1)\log\zeta + (b_2 - a_2) = 0 \qquad (7)$$

(2) The derivative of G has discontinuity of magnitude $1/p_0(\zeta)$ at the point $x = \zeta$, where $p_0(x) =$ coefficient of highest order derivative in (1) $= x$.

$$\therefore \left(\frac{\partial G}{\partial x}\right)_{x=\zeta+0} - \left(\frac{\partial G}{\partial x}\right)_{x=\zeta-0} = \frac{1}{\zeta}$$

$$\Rightarrow \frac{b_1}{\zeta} - \frac{a_1}{\zeta} = \frac{1}{\zeta} \Rightarrow (b_1 - a_1) = 1 \qquad (8)$$

(3) $G(x, \zeta)$ must satisfy the boundary condition (2) & (3) i.e., $G(x, \zeta)$ must be bounded as $x \to 0$ i.e., $a_1 \log x + a_2$ must be bounded as $x \to 0$ which is possible only take $a_1 = 0$ (9)

Again we have $G(1, \zeta) = \alpha G'(1, \zeta)$

$$\Rightarrow b_1 \log 1 + b_2 = \alpha\left[\frac{b_1}{1}\right] = \alpha b_1$$

$$\Rightarrow b_2 = \alpha b_1 \qquad (10)$$

Solving (7), (8), (9) & (10) we have, $a_1 = 0,\ b_1 = 1,\ b_2 = \alpha,\ a_2 = \log\zeta - \alpha$.

Substituting two above values in (6) we get

$$G(x, \zeta) = \begin{cases} \log\zeta - \alpha, & 0 \le x < \zeta \\ \log x - \alpha, & \zeta < x \le 1 \end{cases}$$

or $$G(x, \zeta) = \begin{cases} -\alpha + \log\zeta, & 0 \le x < \zeta \\ -\alpha + \log x, & \zeta < x \le 1 \end{cases} \qquad \text{(Ans)}$$

Using Green's function in the solution of Boundary value problems :
(Nonhomogeneous Differential Equation)

Let there be given a non homogeneous differential equation

$L(y) = p_0(x)\, y^{(n)}(x) + p_1(x)\, y^{(n-1)}x + \ldots + p_n(x)\, y(x) = f(x)$ (1) and the boundary conditions $V_1(y) = 0,\ V_2(y) = 0,\ \ldots,\ V_n(y) = 0$ (2)

Let us consider that the linear forms $V_1, V_2, \ldots, V_n$ in $y(a), y'(a), \ldots, y^{n-1}(a)$, $y(b), y'(b), \ldots, y^{n-1}(b)$, are linearly independent.

Theorem : If $G(x, \zeta)$ is Green's function of the homogeneous boundary-value problem

$$L(y) = 0,\ V_k(y) = 0,\ k = 1, 2, \ldots, n$$

then the solution of the boundary-value problem (1) & (2) is given by the formula

$$y(x) = \int_a^b G(x, \zeta) f(\zeta) d\zeta \qquad (3)$$

The Proof of above theorem is not included here.

Ex. 4 : Using Green's function, solve the boundary value problem

$$y'' - y = x, \; y(0) = y(1) = 0$$

Solu.: The given boundary value-problem

$$y'' - y = x \qquad (1) \text{ with } y(0) = y(1) = 0 \qquad (2)$$

Let us first find out whether Green's function, exists for the corresponding homogeneous boundary value problem

$$y'' - y = 0 \qquad (3) \text{ with } y(0) = y(1) = 0 \qquad (4)$$

The auxillary equation of (3) is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

Hence the general solution of (3) is $y(x) = A \cosh x + B \sinh x$ (5)

Putting $x = 0$ and $x = 1$ in (5) and using boundary conditions (4) we get,

$$A = 0, 0 = A\cosh 1 + B\sinh 1 \Rightarrow B\sinh 1 = 0 \Rightarrow B = 0 \text{ since } \sinh 1 \neq 0$$

So $A = B = 0$. Therefore equation (5) yields only the trivial solution $y(x) = 0$ for the boundary value problem (3) and (4). Therefore, Green's function exists and is given by

$$G(x, \zeta) = \begin{cases} a_1 \cosh x + a_2 \sinh x, & 0 \leq x < \zeta \\ b_1 \cosh x + b_2 \sinh x, & \zeta < x \leq 1 \end{cases} \qquad (6)$$

Now the Green's function must satisfy the following three properties :

(1) $G(x, \zeta)$ is continuous at $x = \zeta$, that is,

$$b_1 \cosh \zeta + b_2 \sinh \zeta = a_1 \cosh \zeta + a_2 \sinh \zeta$$

$$\Rightarrow (b_1 - a_1)\cosh \zeta + (b_2 - a_2)\sinh \zeta = 0 \qquad (7)$$

(2) the derivative of G has a discontinuity of magnitude $\frac{1}{p_0(\zeta)}$ at the point $x = \zeta$, where $p_0(x) =$ coefficient of the highest order derivative in (1) = 1, that is

$$\left(\frac{\partial G}{\partial x}\right)_{x=\zeta+0} - \left(\frac{\partial G}{\partial x}\right)_{x=\zeta-0} = 1$$

$$\Rightarrow (b_1 \sinh\zeta + b_2 \cosh\zeta) - (a_1 \sinh\zeta + a_2 \cosh\zeta) = 1$$

$$\Rightarrow (b_1 - a_1)\sinh(\zeta) + (b_2 - a_2)\cosh(\zeta) = 1 \qquad (8)$$

(3) $G(x, \zeta)$ must satisfy the boundary conditions (4) i.e.,

$$G(0, \zeta) = 0 \text{ so that } a_1 = 0 \qquad (9)$$

$$\& \quad G(1, \zeta) = 0 \text{ so that } b_1 \cosh(1) + b_2 \sinh(1) = 0 \qquad (10)$$

Let $C_K = b_K - a_K,\ K = 1,\ 2$ (11). Then equation (7) & (8) becomes

$$\left.\begin{aligned} C_1 \cosh(\zeta) + C_2 \sinh(\zeta) = 0 \\ C_1 \sinh(\zeta) + C_2 \cosh(\zeta) = 1 \end{aligned}\right\} \text{ solving these we have } C_1 = -\sin h\zeta,\ C_2 = \cos h\zeta$$

$$\therefore\ b_1 - a_1 = -\sinh(\zeta)\ \&\ b_2 - a_2 = \cosh(\zeta), \quad \text{[Using (11)]}$$

From (9), $a_1 = 0 \ \therefore\ b_1 = -\sinh(\zeta)$,

From (10), $b_2 = \dfrac{-b_1 \cosh(1)}{\sinh(1)} = \dfrac{\sinh(\zeta)\cosh(1)}{\sinh(1)}$

then $$a_2 = b_2 - \cosh(\zeta) = \frac{\sinh(\zeta)\cosh(1)}{\sinh(1)} - \cosh(\zeta)$$

$$= \frac{\sinh(\zeta)\cosh(1) - \cosh(\zeta)\sinh(1)}{\sinh(1)}$$

$$= \frac{\sinh(\zeta - 1)}{\sinh(1)}$$

Now $$a_1 \cosh x + a_2 \sinh x = 0 + \frac{\sinh(\zeta - 1)}{\sinh(1)} \sinh x$$

$$b_1 \cosh x + b_2 \sinh x = -\sinh(\zeta)\cosh(\zeta) + \frac{\sinh(\zeta)\cosh(1)}{\sinh(1)} \sinh x$$

$$= + \frac{\sinh(\zeta)}{\sinh(1)} [-\sinh(1)\cosh x + \sinh x \cosh(1)]$$

$$= \frac{\sinh(\zeta)}{\sinh(1)} [\sinh(x - 1)]$$

Substituting that above values in (6), the Green's function of the boundary value problem (3) - (4) is given by

$$G(x,\zeta)=\begin{cases}\dfrac{\sinh x\ \sinh(\zeta-1)}{\sin h1}, & 0\le x<\zeta\\ \dfrac{\sinh(\zeta)\ \sinh(x-1)}{\sinh(1)}, & \zeta<x\le 1\end{cases}\qquad(12)$$

Hence the solution of the given boundary value problem $y''-y-x=0,\ y(0)=y(1)=0$ is given by

$y(x)=\int_0^1 G(x,\zeta)f(\zeta)d\zeta,$ where $G(x,\zeta)$ is defined by formula (12) and $f(\zeta)$ is given by (1)

$$=\int_0^x G(x,\zeta)f(\zeta)d\zeta+\int_x^1 G(x,\zeta)f(\zeta)\,d\zeta$$

$$=\int_{\zeta=0}^{x}\frac{\sinh(\zeta)\ \sinh(x-1)}{\sinh(1)}.\zeta d\zeta+\int_{\zeta=x}^{1}\frac{\sinh x\ \sinh(\zeta-1)}{\sinh(1)}.\ \zeta d\zeta$$

$$=\frac{\sinh(x-1)}{\sinh(1)}\int_{\zeta=0}^{x}\zeta\sinh\ \zeta d\zeta+\frac{\sinh\ x}{\sinh(1)}\int_{\zeta=x}^{1}\zeta\sinh\ (\zeta-1)\,d\zeta$$

$$=\frac{\sinh(x-1)}{\sinh(1)}\left[\zeta\cosh(\zeta)-\sinh(\zeta)\right]_0^x+\frac{\sinh x}{\sinh(1)}\left[\zeta\cosh(\zeta-1)-\sin\ h(\zeta-1)\right]_x^1$$

$$=\frac{\sinh(x-1)}{\sinh(1)}\left[x\cosh x-\sinh x\right]+\frac{\sinh x}{\sinh(1)}\left[1-x\cosh(x-1)+\sinh(x-1)\right]$$

$$=\frac{\sinh x}{\sinh(1)}-x$$

which is the required solution of the given boundary value problem.

Direct Verification convinces us that the function

$y(x)=\dfrac{\sinh x}{\sinh(1)}-x$ satisfies equation (1) and the boundary condition (2).

Ex.- 5 : Reduce to an integral equation the following boundary value problem for the nonlinear differantial equation

$y''=f[x,y(x)]$ (1) with boundary conditions, $y(0)=y(1)=0$ (2)

Solu.: Constructing Green's function for the problem $y''=0$ (3) with boundary conditions (2).

And we find

$$G(x,\zeta)=\begin{cases}(\zeta-1)x, & 0\le x<\zeta\\ (x-1)\zeta, & \zeta<x\le 1\end{cases}$$

Regarding the right hand side of equation (1) as the known function we get

$$y(x)=\int_0^1 G(x,\zeta)f(\zeta,y(\zeta))\,d\zeta \qquad (4)$$

Thus the solution of the boundary value problem (1) - (2) reduces to the solution of a nonlinear integral equation of the special type (Hammerstein-type), the kernel of which is Green's function for the problem (2) - (3). The significance of the Hammerstein type integral equations lies precisely in the fact that the solution of many boundary - value problems for non linear differential equations reduces to the solution of integral equations of this type.

5.4.3 Exercise :

Solve the following boundary value problems using Green's function

1. $\dfrac{d^3y}{dx^3}=0$, with boundary conditions, $y(0)=y(1)=0,\ y'(0)=y'(1)$.

2. $\dfrac{d^2y}{dx^2}=0$, with boundary conditions, $y(0)=y'(1),\ y'(0)=y(1)$

3. $x^2y''+xy'-y=0$, $y(x)$ is bounded as $x\to 0,\ y(1)=0$.

4. $y''+y=x;$ with boundary conditions $y(0)=y(\pi/2)=0$

5. $y''+\pi^2y=\cos\pi x$, with boundary conditions, $y(0)=y(1),\ y'(0)=y'(1)$.

6. $y''-y=-2e^x$, with boundary conditions, $y(0)=y'(0),\ y(l)+y'(l)=0$

Answers :

1. Green's function $G(x,\zeta)=\begin{cases}\dfrac{x(\zeta-1)}{2}(x-x\zeta+2\zeta), & 0\le x<\zeta\\ \dfrac{\zeta}{2}[(2-x)\,(\zeta-2)x+\zeta], & \zeta<x\le 1\end{cases}$

2. Green's function $G(x,\zeta)=\begin{cases}(\zeta-1)+(\zeta-2)x, & 0\le x<\zeta\\ (\zeta-1)\,x-1, & \zeta<x\le 1\end{cases}$

3. Green's function $G(x, \zeta) = \begin{cases} \frac{x}{2}\left(1 - \frac{1}{\zeta^2}\right), & 0 \le x < \zeta \\ \frac{1}{2}\left(x - \frac{1}{x}\right), & \zeta < x \le 1 \end{cases}$

4. The solution is $y = x - \left(\frac{\pi}{2}\right)\sin x$

5. The solution is $y = \frac{1}{4\pi}(2x - 1)\ \sin\ \pi x$

6. The solution is $y = \ \sinh x + (l - x)\ e^x$

5.4.5 References :

1. George Yankovsky, Problems and Exercises in Integral Equations; Mir Publishers, Moscow, 1971.
2. M. Birkhoff and G. C. Rota, Ordinary Differential Equations; Wiley Publishers, 3rd Edition, NY 1978.

5.5 Unit Summary :

At the end of discussion, the gist of this module are depicted as follows :

1. Using Sturm Liouville boundary value problems we have obtained characteristic (eigen) value & characteristic (eigen) function for the Ordinary Differential equation which has an important role in Applied Mathematics and Engineering.
2. Using Green's function technique, we have generated the solution for ordinary differential equations.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

Part - II

Paper - VIII **Group - A**

Module No. 90
Generalised Functions

STRUCTURE

6.1. Introduction :

In this unit, mainly deals with such functions having special properties and shall describe them in the light of differentiation, integration and the transform concept also. Before going to elaborate discussion of Generalised functions first we intorduced the concept of Good functions, Fairly Good functions and Regular sequcces which are very useful in this context.

6.2 Objectives :

It is enough to say that usual functions such as e^x, $\sin x$, $\cos x$ etc., guaranted the existence of differentiation, Integration and transformation like as Fourier and Laplace transform. But some special functions such as Dirac Delta function, signum function and Heaviside unit functions still have no differentiation and others in ordinary sense. Therefore the introduction of Generalised function is very much essential to remove this difficulty. Now in this Unit, the main objective is to show that the existence of differentiation, Integration and others are applicable to the special functions described earlier.

Keywords :

Good Function, Fairly Good Function, Regular Sequence, Generalised Function.

6.3 Good Function :

A function $\gamma(x): R \to C$ will be called a good function if it satisfy the following properties :

(i) it is infinitely differentiable often on R.

(ii) $$\lim_{|x|\to\infty} x^r \frac{d^k\gamma(x)}{dx^k} = 0$$

for every integer $r \geq 0$ and for every integer $k \geq 0$.

where R is the set of Real Number and C is the set of Complex Number. Generally we denote the set of good function by $\hat{G}$.

Example : e^{-x^2}, $(1-i)e^{-x^2}$, $x^n e^{-x^2}$, (n is a positive integer) all are good functions but e^x is not a good function because it does not behave properly at infinity.

6.4 Properties of Good Function :

(i) If $\gamma_1(x)$ and $\gamma_2(x) \in \hat{G}$ then $\gamma_1(x)+\gamma_2(x) \in \hat{G}$ and $\gamma_1(x).\gamma_2(x) \in \hat{G}$.

(ii) If $\gamma(x) \in \hat{G}$ then $\gamma_1(x) = \gamma(ax+b) \in \hat{G}$, where $a, b \in R$

(iii) If $\gamma(x) \in \hat{G}$ then its derivatives $\frac{d\gamma(x)}{dx} \in \hat{G}$.

(iv) If $\gamma(x) \in \hat{G}$ and $x_1, x_2 \in R$ then $|\gamma(x_1)-\gamma(x_2)| \leq |x_1 - x_2| M$, where M is a constant independent of x_1 and x_2.

Important Property : The integral of a good function is not necessarily a good function. It can be shown that by taking an example as follows :

Let $\gamma(x) = e^{-x^2}$ is a good function and assume that $f(x) = \int_{-\infty}^{x} \gamma(t)dt$ then $f'(x) = \gamma(x) \in G$ and

$$\lim_{|x|\to\infty} x^r \frac{d^k f(x)}{dx^k} = 0$$

Now choose $r = 0, k = 0$, then from the Left hand side of above equation, we have

$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} \int_{-\infty}^{x} \gamma(t)\,dt = \int_{-\infty}^{\infty} e^{-x^2}\,dt = \sqrt{\pi} \neq 0.$$

Hence the integral of a good function is not necessarily a good function.

The necessary and sufficient condition for $f(x) = \int_{-\infty}^{x} \gamma(t)\,dt$ to be of good function.

Theorem 1 : Let $\gamma(x) \in \hat{G}$ and $f(x) = \int_{-\infty}^{x} \gamma(t)dt$, then $f(x) \in \hat{G}$ if and only if $\int_{-\infty}^{\infty} \gamma(t)dt = 0.$

Proof : Necessary Part : Given that $f(x) = \int_{-\infty}^{x} \gamma(t)dt$, then $f'(x) = \gamma(x) \in \hat{G}$. Thus $f(x)$ is infinitely differentiable. Assume that $\int_{-\infty}^{\infty} \gamma(t)dt = 0.$

To prove that $f(x) \in \hat{G}$ i.e., to show that $\gamma(x)\psi(x) \in \hat{F}$

$$\lim_{|x|\to\infty} x^r \frac{d^k f(x)}{dx^k} = 0$$

Now left hand side of above for $k = 0$

$$\lim_{|x|\to\infty} x^r f(x) = \lim_{|x|\to\infty} \frac{f(x)}{x^{-r}} = \lim_{|x|\to\infty} \frac{f'(x)}{-rx^{-r-1}} = \lim_{|x|\to\infty} \gamma(x)x^{r+1}\left(-\frac{1}{r}\right) = 0.$$

Since $\gamma(x) \in \hat{G}$. Therefore $f(x) \in \hat{G}$.

Sufficient Part :

Let $f(x) \in \hat{G}$. Now we are to show that $\int_{-\infty}^{\infty} \gamma(t)dt = 0.$

We have

$$\lim_{|x|\to\infty} x^r \frac{d^k f(x)}{dx^k} = 0$$

Now choose r = 0, k = 0, then

$$\lim_{|x|\to\infty} f(x) = 0$$

that is $\int_{-\infty}^{\infty} \gamma(t)dt = 0.$

Hence the theorem.

6.5 Fairly Good Function :

A function $\psi(x): R \to C$ will be called a fairly good function if it satisfy the following properties :

(i) it is infinitely differentiable everywhere on R.

(ii) $$\lim_{|x|\to\infty} x^{-N} \frac{d^k\psi(x)}{dx^k} = 0$$

for every integer $k \geq 0$ and N is positive integer.

where R is the set of Real Number and C is the set of Complex Number. Generally we denote the set of fairly good function by $\hat{F}$.

Example : e^{ix} and any polynomial are fairly good functions but e^x is not a fairly good function because it does not behave properly at infinity.

6.6 Properties of Fairly Good Function :

(i) If $\psi_1(x)$ and $\psi_2(x) \in \hat{F}$ then $\psi_1(x) + \psi_2(x) \in \hat{F}$ and $\psi_1(x) . \psi_2(x) \in \hat{F}$.

(ii) If $\psi_1(x) \in \hat{F}$ then $\psi_1(x) = \psi(ax+b) \in \hat{F}$, where $a, b \in R$

(iii) If $\psi(x) \in \hat{F}$ then its derivatives $\frac{d\psi(x)}{dx} \in \hat{F}$.

Theorem 2 : Let $\gamma(x) \in \hat{G}$ and $\psi(x) \in \hat{F}$ then $\gamma(x).\psi(x) \in \hat{G}$.

Proof : Since $\gamma(x) \in \hat{G}$, This implies that $\gamma(x)$ is infinitely differentiable. Again since $\psi(x) \in \hat{F}$, This implies that $\psi(x)$ is infinitely differentiable. This indicates that $\gamma(x)\psi(x)$ is also infinitely differentiable. Now we are to show that

$$\lim_{|x|\to\infty} x^r \frac{d^k\{\gamma(x)\psi(x)\}}{dx^k} = 0$$

for $r \geq 0,\ k \geq 0$

We have

$$\frac{d^k\{\gamma(x)\psi(x)\}}{dx^k} = \sum_{j=0}^{k} {}^{k}c_j \left\{\frac{d^{k-j}\gamma(x)}{dx^{k-j}}\right\}\left\{\frac{d^j\psi(x)}{dx^j}\right\}$$

by help of Leibnitz Theorem on Successive Differentiation.

or,

$$x^r \frac{d^k\{\gamma(x)\psi(x)\}}{dx^k} = \sum_{j=0}^{k} {}^{k}c_j \left\{x^r \frac{d^{k-j}\gamma(x)}{dx^{k-j}}\right\}\left\{\frac{d^j\psi(x)}{dx^j}\right\}$$

or,

$$x^r \frac{d^k\{\gamma(x)\psi(x)\}}{dx^k} = \sum_{j=0}^{k} k_{c_j} \left\{ x^{N+r} \frac{d^{k-j}\gamma(x)}{dx^{k-j}} \right\} \left\{ x^{-N} \frac{d^j\psi(x)}{dx^j} \right\}$$

or,

$$\lim_{|x|\to\infty} x^r \frac{d^k\{\gamma(x)\psi(x)\}}{dx^k} = \lim_{|x|\to\infty} \sum_{j=0}^{k} k_{c_j} \left\{ x^{N+r} \frac{d^{k-j}\gamma(x)}{dx^{k-j}} \right\} \left\{ x^{-N} \frac{d^j\psi(x)}{dx^j} \right\}$$

or,

$$\lim_{|x|\to\infty} x^r \frac{d^k\{\gamma(x)\psi(x)\}}{dx^k} = \sum_{j=0}^{k} k_{c_j} \lim_{|x|\to\infty} \left\{ x^{N+r} \frac{d^{k-j}\gamma(x)}{dx^{k-j}} \right\} \lim_{|x|\to\infty} \left\{ x^{-N} \frac{d^j\psi(x)}{dx^j} \right\} = 0$$

Since $\gamma(x) \in \hat{G}$ then

$$\lim_{|x|\to\infty} x^{N+r} \frac{d^k\gamma(x)}{dx^k} = 0$$

and $\psi(x) \in \hat{F}$ then

$$\lim_{|x|\to\infty} x^{-N} \frac{d^k\psi(x)}{dx^k} = 0$$

Therefore $\gamma(x).\psi(x) \in \hat{G}$.

Hence the theorem.

6.7 Regular Sequence : A sequence $\{\gamma_n\}$ of good function is said to be regular if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(x)\, \gamma(x) dx$$

exists finitely $\forall \gamma(x) \in \hat{G}$.

6.8 Equivalence of Regular sequences : Two regular sequences which gives the same limit are said to be equivalence. Thus if $\{\gamma_n^1\}$ and $\{\gamma_n^2\}$ be two regular sequences, they are equivalence i.e., $\{\gamma_n^1\} \sim \{\gamma_n^2\}$ if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n^1(x)\gamma(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n^2(x)\gamma(x)dx, \quad \forall \gamma(x) \in \hat{G}.$$

Example 1 : Show that the sequence $\left\{ e^{-\frac{x^2}{2n}} \right\}$ is regular.

Solution : To do this problem we have to show that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\gamma_n(x)\gamma(x)dx$$

exists finitely $\forall\gamma(x)\in\hat{G}$.

To show this use the inequality $\left|e^{-\frac{x^2}{2n}}-1\right|=\left|\int_0^x 2xe^{-\frac{x^2}{2n}}\frac{dx}{n}\right|\le\int_0^x 2x\frac{dx}{n}\le\frac{x^2}{n}$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2n}}\gamma(x)dx=\lim_{n\to\infty}\int_{-\infty}^{\infty}\left(e^{-\frac{x^2}{2n}}-1\right)\gamma(x)dx+\int_{-\infty}^{\infty}\gamma(x)dx.$$

and

$$\lim_{n\to\infty}\left|\int_{-\infty}^{\infty}\left(e^{-\frac{x^2}{2n}}-1\right)\gamma(x)dx\right|<\lim_{n\to\infty}\int_{-\infty}^{\infty}x^2\gamma(x)\frac{dx}{n}=0.$$

Since

$$\int_{-\infty}^{\infty}x^2|\gamma(x)|\frac{dx}{n}<\infty.$$

because γ is good

Hence

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2n}}\gamma(x)dx=\int_{-\infty}^{\infty}\gamma(x)dx.$$

So that the limit exists and is finite for every good γ; Therefore the given sequence is regular.

6.9 Generalised Function : An equivalence class of regular sequences is said to be generalised function.

If $g(x)$ is the generalised function associated with the equivalence class of which $\{\gamma_n\}$ is a member, then we define the product of $g(x)$ with $\gamma(x)$ by the following relation :

$$\int_{-\infty}^{\infty}g(x)\gamma(x)dx=\lim_{n\to\infty}\int_{-\infty}^{\infty}\gamma_n(x)\gamma(x)dx,\quad\forall\ \gamma(x)\in\hat{G}.$$

the arrow implies that limiting process involved and left hand side is not an ordinary integral.

Let $\{\gamma_n^1(x)\}$ and $\{\gamma_n^2(x)\}$ be two regular sequences, defining the generalised functions and $g^2(x)$ respectively.

Then we have

$$\int_{-\infty}^{\infty}g^1(x)\gamma(x)dx=\lim_{n\to\infty}\int_{-\infty}^{\infty}\gamma_n^1(x)\gamma(x)dx,\quad\forall\ \gamma(x)\in\hat{G}.$$

and

$$\int_{-\infty}^{\infty}g^2(x)\gamma(x)dx=\lim_{n\to\infty}\int_{-\infty}^{\infty}\gamma_n^2(x)\gamma(x)dx,\ \forall\ \gamma(x)\in\hat{G}.$$

If they are equivalence i.e., $\{\gamma_n^1\} \sim \{\gamma_n^2\}$ then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n^1(x)\gamma(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n^2(x)\gamma(x)dx, \ \forall \gamma(x) \in \hat{G}.$$

Hence $g^1(x) \sim g^2(x)$ iff $\{\gamma_n^1\} \sim \{\gamma_n^2\}$.

Example 2 : Show that the sequence $\left\{e^{-\frac{x^2}{n}}\right\}$ is regular and defines a generalised function $I(x)$ such that

$$\int_{-\infty}^{\infty} I(x)\gamma(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2}{n}}\gamma(x)dx, \ \forall \gamma(x) \in \hat{G}.$$

Solution : First part is already shown in Example 1 and we get

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2n}}\gamma(x)dx = \int_{-\infty}^{\infty} \gamma(x)dx. \qquad (1)$$

Let $I(x)$ be generalised function associated with the regular sequence $\left\{e^{-\frac{x^2}{n}}\right\}$ so that

$$\int_{-\infty}^{\infty} I(x)\gamma(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2}{n}}\gamma(x)dx, \ \forall \gamma(x) \in \hat{G}. \qquad (2)$$

From equation (1) and (2) we have

$$\int_{-\infty}^{\infty} I(x)\gamma(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2}{n}}\gamma(x)dx = \int_{-\infty}^{\infty} \gamma(x)dx, \ \forall \gamma(x) \in \hat{G}.$$

This implies $I(x) = 1$.

Definition : The sequence $\left\{\sqrt{\frac{n}{\pi}}e^{-nx^2}\right\}$ is regular and defines a generalised function, denoted by $\delta(x)$, such that

$$\int_{-\infty}^{\infty} \delta(x)\gamma(x)dx = \gamma(0).$$

Example 3 : Show that the sequence $\left\{\sqrt{\frac{n}{\pi}}e^{-nx^2}\right\}$ is regular sequence and defines a generalised function, denoted by $\delta(x)$, such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}}e^{-nx^2}\gamma(x)dx = \int_{-\infty}^{\infty} \delta(x)\gamma(x)dx = \gamma(0).$$

Solution : Let $\gamma(x)$ be a good function i.e., $\gamma(x) \in \hat{G}$ then $|\gamma(x) - \gamma(0)| = \left|\int_0^x \frac{d\gamma(x)}{dx}\right| \le |x - 0|M$, where M is a finite constant independent of x. Since $\frac{d\gamma(x)}{dx}$ being good function and is bounded.

$$\left|\int_{-\infty}^{\infty}\sqrt{\tfrac{n}{\pi}}e^{-nx^2}|\gamma(x)-\gamma(0)|\,dx\right| \le M\int_{-\infty}^{\infty}\sqrt{\tfrac{n}{\pi}}e^{-nx^2}|x|dx$$

$$\le M\int_{-\infty}^{0}\sqrt{\tfrac{n}{\pi}}\,e^{-nx^2}(-x)\,dx + M\int_{0}^{\infty}\sqrt{\tfrac{n}{\pi}}\,e^{-nx^2}x\,dx$$

$$\le 2M\int_{0}^{\infty}\sqrt{\tfrac{n}{\pi}}\,e^{-nx^2}x\,dx$$

$$\le \tfrac{2M}{\pi\sqrt{n}} \to 0 \text{ as } n\to\infty$$

So that $lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{\tfrac{n}{\pi}}\,e^{-nx^2}\gamma(x)\,dx = lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{\tfrac{n}{\pi}}\,e^{-nx^2}\gamma(0)\,dx$

$$=\gamma(0)\lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{\tfrac{n}{\pi}}e^{-nx^2}dx$$

$$=\gamma(0)\sqrt{n/\pi}\,\sqrt{\pi/n} \qquad =\gamma(0)$$

Therefore

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{\tfrac{n}{\pi}}e^{-nx^2}\gamma(x)dx=\gamma(0) \tag{3}$$

which shows that the sequence $\left\{\sqrt{\tfrac{n}{\pi}}e^{-nx^2}\right\}$ is regular sequence. Let the sequence $\left\{\sqrt{\tfrac{n}{\pi}}e^{-nx^2}\right\}$ defines a generalised function, denoted by $\delta(x)$, such that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{\tfrac{n}{\pi}}e^{-nx^2}\gamma(x)dx = ⨍_{-\infty}^{\infty}\delta(x)\gamma(x)dx, \quad \forall\ \gamma(x)\in\hat{G}. \tag{4}$$

From the equation (3) and (4) we get

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{\tfrac{n}{\pi}}e^{-nx^2}\gamma(x)dx = ⨍_{-\infty}^{\infty}\delta(x)\gamma(x)dx=\gamma(0).$$

6.10 Properties of Generalised Functions :

Property 1 : If sequence $\{\gamma_n(x)\}$ is regular sequence and defines a generalised function $g(x)$, then $\{\gamma_n(ax+b)\}$, $a\ne 0,\ b\in R$ is also regular sequence such that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\gamma_n(ax+b)\gamma(x)dx = ⨍_{-\infty}^{\infty}g(x)\gamma\left(\frac{x-b}{a}\right)\frac{dx}{|a|}, \quad \forall\ \gamma(x)\in\hat{G}.$$

Proof : Let $ax + b = t$, then

$$\int_{-\infty}^{\infty} \gamma_n(ax+b)\gamma(x)dx = \int_{-\infty}^{\infty} \gamma_n(t)\gamma\left(\frac{t-b}{a}\right)\frac{dt}{|a|}, \quad \forall\ \gamma(x) \in \hat{G},$$

$$\Rightarrow \lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(ax+b)\gamma(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(t)\gamma\left(\frac{t-b}{a}\right)\frac{dt}{|a|}, \quad \forall\ \gamma(x) \in \hat{G}.$$

$$\Rightarrow \lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(ax+b)\gamma(x)dx = \fint_{-\infty}^{\infty} g(x)\gamma\left(\frac{x-b}{a}\right)\frac{dx}{|a|}, \quad \forall\ \gamma(x) \in \hat{G}.$$

Devinition : If $g(x)$ is a generalised function defined by regular sequence $\{\gamma_n(x)\}, a \neq 0, b \in R$ we define $g(ax+b)$ to be the generalised function defined by the regular sequence $\{\gamma_n(ax+b)\}$.

Mathematically, if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(x)\gamma(x)\,dx = \fint_{-\infty}^{\infty} g(x)\gamma(x)dx, \quad \forall\ \gamma(x) \in \hat{G}.$$

then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(ax+b)\gamma(x)\,dx = \fint_{-\infty}^{\infty} g(ax+b)\gamma(x)\,dx = \fint_{-\infty}^{\infty} g(x)\gamma\left(\tfrac{x-b}{a}\right)\tfrac{dx}{|a|}, \quad \forall\ \gamma(x) \in \hat{G}.$$

Therefore,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(ax+b)\gamma(x)dx = \fint_{-\infty}^{\infty} g(x)\gamma\left(\tfrac{x-b}{a}\right)\tfrac{dx}{|a|}, \quad \forall\ \gamma(x) \in \hat{G}.$$

Example 4 : Show that if $g_1(x) = g_2(x)$, then $g_1(ax+b) = g_2(ax+b)$, $a \neq 0, b \in R$.

Solution : We have

$$\fint_{-\infty}^{\infty} g_1(ax+b)\gamma(x)dx = \fint_{-\infty}^{\infty} g_1(x)\gamma\left(\tfrac{x-b}{a}\right)\tfrac{dx}{|a|}, \quad \forall\ \gamma(x) \in \hat{G}.$$

$$\Rightarrow \fint_{-\infty}^{\infty} g_1(ax+b)\gamma(x)dx = \fint_{-\infty}^{\infty} g_2(x)\gamma\left(\tfrac{x-b}{a}\right)\tfrac{dx}{|a|}, \quad \text{Since } g_1(x) = g_2(x)$$

Finally,

$$\fint_{-\infty}^{\infty} g_1(ax+b)\gamma(x)dx = \fint_{-\infty}^{\infty} g_2(ax+b)\gamma(x)dx.$$

Therefore $g_1(ax+b) = g_2(ax+b)$.

Example 5 : Show that

$$\fint_{-\infty}^{\infty} \delta(ax-b)\gamma(x)dx = \frac{1}{|a|}\gamma(b/a), \quad a \neq 0, b \in R.$$

Solution : Let $\delta(x)$ be generalised function associated with the regular sequence $\{\gamma_n(x)\}$ so that

$$\int_{-\infty}^{\infty} \delta(x)\gamma(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(x)\gamma(x)dx, \ \forall \gamma(x) \in \hat{G}. \tag{5}$$

$$\int_{-\infty}^{\infty} \delta(ax-b)\gamma(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(ax-b)\gamma(x)dx, \ \forall \gamma(x) \in \hat{G}. \tag{6}$$

Now let $ax-b=t$ and $a>0$ then the right side of equation (6) becomes

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(ax-b)\gamma(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(t)\,\gamma\left(\tfrac{t+b}{a}\right)\tfrac{dt}{a}, \ \forall \gamma(x) \in \hat{G}.$$

and $a<0$ then the right side of equation (6) becomes

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(ax-b)\gamma(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(t)\,\gamma\left(\tfrac{t+b}{a}\right)\tfrac{dt}{-a}, \ \forall \gamma(x) \in \hat{G}.$$

Combining above we get

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(ax-b)\gamma(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(t)\,\gamma\left(\tfrac{t+b}{a}\right)\tfrac{dt}{|a|}, \ \forall \gamma(x) \in \hat{G}.$$

since $\gamma(x) \in \hat{G}$, this implies that $\gamma\left(\frac{b+t}{a}\right) \in \hat{G}$. Therefore,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(ax-b)\gamma(x)dx = \int_{-\infty}^{\infty} \delta(t)\gamma\left(\frac{t+b}{a}\right)\frac{dt}{|a|}$$

Since we know that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(x)\gamma(x)dx = \int_{-\infty}^{\infty} \delta(x)\gamma(x)dx = \gamma(0).$$

Hence we get

$$\int_{-\infty}^{\infty} \delta(x)\gamma(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(ax-b)\gamma(x)dx = \int_{-\infty}^{\infty} \delta(t)\gamma\left(\frac{t+b}{a}\right)\frac{dt}{|a|} = \frac{\gamma(b/a)}{|a|}.$$

Property 2 : If $\psi(x)$ is a fairly good function and $\{\gamma_n(x)\}$ is regular sequence then $\{\psi(x)\gamma_n(x)\}$ is regular sequence and if $g(x)$ be the generalised function defined by $\{\gamma_n(x)\}$, then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \{\psi(x)\gamma_n(x)\}\gamma(x)dx = \int_{-\infty}^{\infty} g(x)\{\psi(x)\gamma(x)\}dx, \ \forall \gamma(x) \in \hat{G}, \psi(x) \in \hat{F}.$$

Property 3 : If $\{\gamma_n(x)\}$ is regular sequence defining the generalised function $g(x)$ and $\psi(x)$ is a fairly good function then $\{\psi(x)g(x)\}$ will be defined to be the generalised function determined by the regular sequence $\{\psi(x)\gamma_n(x)\}$, i.e., if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(x)\gamma(x)dx = \int_{-\infty}^{\infty} g(x)\gamma(x)dx, \ \forall \gamma(x) \in \hat{G}.$$

then

$$\int_{-\infty}^{\infty}\{\psi(x)g(x)\}\gamma(x)dx = \int_{-\infty}^{\infty} g(x)\{\psi(x)\gamma(x)\}dx, \ \forall\ \gamma(x)\in\hat{G},\ \psi(x)\in\hat{F}.$$

Example 6 : Show that if $\psi(x)$ is a fairly good function then $\psi(x)\delta(x)=\psi(0)\delta(x)$.

Solution : Let $\{\gamma_n(x)\}$ be the regular sequence associated with the generalised function denoted by $\delta(x)$, then

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\gamma_n(x)\gamma(x)dx = \int_{-\infty}^{\infty}\delta(x)\gamma(x)dx = \gamma(0),\ \forall\ \gamma(x)\in\hat{G}. \quad (7)$$

Since $\psi(x)$ is a fairly good function and $\{\gamma_n(x)\}$ is regular sequence therefore $\{\psi(x)\gamma_n(x)\}$ is also a regular sequence associated with the generalised function $\{\psi(x)\delta(x)\}$.

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\{\psi(x)\gamma_n(x)\}\gamma(x)dx = \int_{-\infty}^{\infty}\{\psi(x)\delta(x)\}\gamma(x)dx,\ \ \forall\gamma(x)\in\hat{G},\psi(x)\in\hat{F} \quad (8)$$

Now the left hand side of above i.e.,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\{\psi(x)\gamma_n(x)\}\gamma(x)dx = \lim_{n\to\infty}\int_{-\infty}^{\infty}\gamma_n(x)\{\psi(x)\gamma(x)\}dx;\ \forall\ \gamma(x)\in\hat{G}$$

$$= \int_{-\infty}^{\infty}\delta(x)\Gamma(x)dx,$$

where $\Gamma(x)=\psi(x)\gamma(x)$

$$=\Gamma(0)=\psi(0)\gamma(0)$$

by (7),

Using above the equation (8) becomes

$$\int_{-\infty}^{\infty}\{\psi(x)\delta(x)\}\gamma(x)dx = \psi(0)\gamma(0)$$

$$=\psi(0)\int_{-\infty}^{\infty}\delta(x)\gamma(x)dx, \text{ using definition}$$

$$=\int_{-\infty}^{\infty}\{\psi(0)\delta(x)\}\gamma(x)dx$$

Therefore, $\psi(x)\delta(x)=\psi(0)\delta(x)$.

6.11 Derivatives of a Generalised Function :

Theorem 3 : If $\{\gamma_n(x)\}$ is regular sequence defining the generalised function $g(x)$, then $\left\{\frac{d\gamma_n(x)}{dx}\right\}$ is also a regular sequence such that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{d\gamma_n(x)}{dx}\gamma(x)\,dx = \fint_{-\infty}^{\infty} g(x)\frac{d\gamma(x)}{dx}dx\,,\ \forall\ \gamma(x)\in\hat{G}.$$

Proof :

$$\int_{-\infty}^{\infty}\frac{d\gamma_n(x)}{dx}\,\gamma(x)dx = \left[\gamma_n(x)\,\gamma(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\gamma_n(x)\frac{d\gamma(x)}{dx}dx$$

$$= -\int_{-\infty}^{\infty}\gamma_n(x)\frac{d\gamma(x)}{dx}dx$$

by the condition of good function.

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{d\gamma_n(x)}{dx}\gamma(x)dx = -\lim_{n\to\infty}\int_{-\infty}^{\infty}\gamma_n(x)\frac{d\gamma(x)}{dx}dx = -\fint_{-\infty}^{\infty} g(x)\frac{d\gamma(x)}{dx}dx$$

since

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\gamma_n(x)\gamma(x)\,dx = \fint_{-\infty}^{\infty} g(x)\gamma(x)dx,\ \forall\ \gamma(x)\in\hat{G}.$$

Hence $\left\{\frac{d\gamma_n(x)}{dx}\right\}$ is also regular sequence.

Definition : If $\{\gamma_n(x)\}$ is a regular sequence defining the generalised function $g(x)$. We define $g'(x)$ to be the generalised function defining by the regular sequence $\left\{\frac{d\gamma_n(x)}{dx}\right\}$ and we call it the derivative if the generalised function of $g(x)$. Thus,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{d\gamma_n(x)}{dx}\,\gamma(x)\,dx = \fint_{-\infty}^{\infty} g'(x)\gamma(x)dx,\ \ \forall\ \gamma(x)\in\hat{G} \tag{9}$$

Also we know that,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{d\gamma_n(x)}{dx}\gamma(x)dx = -\fint_{-\infty}^{\infty} g(x)\frac{d\gamma(x)}{dx}dx \tag{10}$$

From the above two we get,

$$\fint_{-\infty}^{\infty} g'(x)\gamma(x)dx = -\fint_{-\infty}^{\infty} g(x)\frac{d\gamma(x)}{dx}dx$$

Again

$$\fint_{-\infty}^{\infty} g''(x)\gamma(x)dx = -\fint_{-\infty}^{\infty} g'(x)\frac{d\gamma(x)}{dx}dx = (-1)^2\fint_{-\infty}^{\infty} g(x)\frac{d}{dx}\left(\frac{d\gamma(x)}{dx}\right)dx$$

$$= (-1)^2\fint_{-\infty}^{\infty} g(x)\frac{d^2\gamma(x)}{dx^2}dx$$

Generalising the above idea we have

$$\int_{-\infty}^{\infty} g^r(x)\gamma(x)dx = (-1)^r \int_{-\infty}^{\infty} g(x)\frac{d^r\gamma(x)}{dx^r}dx$$

6.12 Ordinary Function as Generalised Function : If $f(x)$ be an ordinary function, then it can be regarded a generalised function if

$f \in K'(R)$

$$\int_{-\infty}^{\infty} \frac{|f(x)|}{(1+x^2)^N}dx < \infty$$

where $k^p(R) : 1 \le p < \infty$ is the class of all complex lebesgue measurable function on R for which

$$\frac{|f(x)|^p}{(1+x^2)^N} \in L'(R)$$

for some $N \ge 0$. $L'(R)$ is the collection of all lebesgue measurable function for which

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty$$

Example 7 : Show that $H'(x)$ exists and $H'(x) = \delta(x)$, where $H(x)$ is Heaviside function and the prime denotes the generalised derivative.

Solution : Since $H(x)$ is Heaviside function and define as follows

$$H(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \le 0. \end{cases}$$

Now to show that $H(x) \in K'(R)$ i.e.,

$$\int_{-\infty}^{\infty} \frac{|H(x)|}{(1+x^2)^N}dx < \infty$$

for some $N \ge 0$

Choose N = 1

$$\int_{-\infty}^{\infty} \frac{|H(x)|}{(1+x^2)^N}dx = \int_{-\infty}^{\infty} \frac{H(x)}{(1+x^2)}dx = \int_{0}^{\infty} \frac{1}{(1+x^2)}dx = \pi/2 < \infty$$

i.e., $H(x) \in K'(R)$ so that it can be regarded as a generalised function and $H'(x)$ exists as a generalised function such that

$$\underset{-\infty}{\overset{\infty}{\not\int}} H'(x)\, \gamma(x)dx = -\underset{-\infty}{\overset{\infty}{\not\int}} H(x)\frac{d\gamma(x)}{dx}dx = -\int_{-\infty}^{\infty} H(x)\frac{d\gamma(x)}{dx}dx$$

since $H(x)$ is an ordinary function according to the definition.

$$= -\int_{-\infty}^{0} 0.\frac{d\gamma(x)}{dx}dx - \int_{0}^{\infty} 1.\frac{d\gamma(x)}{dx}dx = -[\gamma(x)]_0^{\infty} = \gamma(0)$$

Therefore

$$\underset{-\infty}{\overset{\infty}{\not\int}} H'(x)\, \gamma(x)dx = \underset{-\infty}{\overset{\infty}{\not\int}} \delta(x)\gamma(x)dx$$

Hence $H'(x) = \delta(x)$.

Example 8 : Show that $(sgn(x))'$ exists and $(sgn(x))' = 2\delta(x)$, where $(sgn(x))$ is Signum function and the prime denotes the generalised derivative.

Solution : Since $(sgn(x))$ is signum function and define as follows

$$sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Now to show that $sgn(x) \in K'(R)$ i.e.,

$$\int_{-\infty}^{\infty} \frac{|sgn(x)|}{(1+x^2)^N} dx < \infty$$

for some $N \geq 0$

Choose $N = 1$

$$\int_{-\infty}^{\infty} \frac{|sgn(x)|}{(1+x^2)} dx = \int_{-\infty}^{0} \frac{sgn(x)}{(1+x^2)} dx + \int_{0}^{\infty} \frac{sgn(x)}{(1+x^2)} dx$$

$$= \int_{-\infty}^{0} \frac{-1}{(1+x^2)} dx + \int_{0}^{\infty} \frac{1}{(1+x^2)} dx = \pi < \infty$$

i.e., $sgn(x) \in K'(R)$ so that it can be regarded as a generalised function and $(sgn(x))'$ exists as a generalised function such that

$$\underset{-\infty}{\overset{\infty}{\not\int}} (sgn(x))'\gamma(x)\,dx = -\underset{-\infty}{\overset{\infty}{\not\int}} sgn(x)\frac{d\gamma(x)}{dx}dx = -\int_{-\infty}^{\infty} sgn(x)\frac{d\gamma(x)}{dx}dx$$

since $sgn(x)$ is an ordinary function according to the definition.

$$= -\int_{-\infty}^{0} -1.\frac{d\gamma(x)}{dx}\,dx - \int_{0}^{\infty} 1.\frac{d\gamma(x)}{dx}\,dx = [\gamma(x)]_{-\infty}^{0} - [\gamma(x)]_{-\infty}^{0} = 2\gamma(0)$$

Therefore

$$\int_{-\infty}^{\infty} (sgn(x))'\gamma(x)dx = \int_{-\infty}^{\infty} 2\delta(x)\gamma(x)dx$$

Hence $(sgn(x))' = 2\delta(x)$.

Example 9 : Show that $|x|'$ exists and $|x|' = sgn(x)$, where a prime denotes the generalised derivative.

Solution : Since $|x|$ is mod function and define as follows

$$|x| = \begin{cases} x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Now to show that $|x| \in K'(R)$ i.e.,

$$\int_{-\infty}^{\infty} \frac{|x|}{(1+x^2)^N}\,dx < \infty$$

for some $N \geq 0$

Choose $N = 1$

$$\int_{-\infty}^{\infty} \frac{|x|}{(1+x^2)}\,dx = \int_{-\infty}^{0} \frac{|x|}{(1+x^2)}\,dx + \int_{0}^{\infty} \frac{|x|}{(1+x^2)}\,dx$$

$$= \int_{-\infty}^{0} \frac{-x}{(1+x^2)}\,dx + \int_{0}^{\infty} \frac{x}{(1+x^2)}\,dx = 2\int_{0}^{\infty} \frac{x}{(1+x^2)}\,dx < \infty$$

i.e., $|x| \in K'(R)$ so that it can be regarded as a generalised function and $(|x|)'$ exists as a generalised function such that

$$\int_{-\infty}^{\infty} (|x|)'\gamma(x)dx = -\int_{-\infty}^{\infty} |x|\frac{d\gamma(x)}{dx}\,dx = -\int_{-\infty}^{\infty} |x|\frac{d\gamma(x)}{dx}\,dx$$

since $|x|$ is an ordinary function according to the definition.

$$= -\int_{-\infty}^{0} -x.\frac{d\gamma(x)}{dx}\,dx - \int_{0}^{\infty} x.\frac{d\gamma(x)}{dx}\,dx = \int_{-\infty}^{0} x\,d\gamma(x) - \int_{0}^{\infty} x\,d\gamma(x)$$

$$= \{x\gamma(x)\}_{-\infty}^{0} - \int_{-\infty}^{0} 1.\gamma(x)\,dx - \{x\gamma(x)\}_{0}^{\infty} + \int_{0}^{\infty} 1.\gamma(x)\,dx$$

$$= \int_{-\infty}^{0} (-1).\gamma(x)dx + \int_{0}^{\infty} (1)\,\gamma(x)\,dx$$

$$= \int_{-\infty}^{\infty} sgn(x)\gamma(x)dx = \int_{-\infty}^{\infty} sgn(x)\gamma(x)dx$$

Hence $(|x|)' = sgn(x)$.

Example 10 : Suppose that $f(x)$ is continuous function with derivative $\frac{df}{dx}$ such that $f(x), \frac{df}{dx} \in K'(R)$ then show that $[f(x)\,H(x)]' = \frac{df}{dx}H(x) + f(0)\,\delta(x)$, where a prime denotes the generalised derivative and $H(x)$ is the Heaviside function.

Solution : Given that $f(x), \frac{df}{dx} \in K'(R)$ then $[f(x)H(x)]$ and $\frac{df}{dx}H(x) \in K'(R)$.

Hence $[f(x)\,H(x)]$ can be regarded as a generalised function and $[f(x)\,H(x)]'$ exists as a generalised function such that

$$\int_{-\infty}^{\infty} [f(x)H(x)]'\gamma(x)dx = -\int_{-\infty}^{\infty} [f(x)H(x)]\,\frac{d\gamma(x)}{dx}dx = -\int_{-\infty}^{\infty} [f(x)\,H(x)]\,\frac{d\gamma(x)}{dx}dx$$

since $[f(x)H(x)]$ is an ordinary function according to the definition.

$$= -\int_{0}^{\infty} f(x)\frac{d\gamma(x)}{dx}\,dx = -[f(x)\,\gamma(x)]_{0}^{\infty} + \int_{0}^{\infty} \gamma(x)\,\frac{df}{dx}dx$$

$$= [f(0)\,\gamma(0)] + \int_{0}^{\infty} \gamma(x)\frac{df}{dx}dx$$

$$= f(0)\int_{-\infty}^{\infty} \delta(x)\gamma(x)\,dx + \int_{-\infty}^{\infty} H(x)\frac{df}{dx}\gamma(x)dx$$

$$= \int_{-\infty}^{\infty}\left[f(0)\delta(x) + H(x)\frac{df}{dx}\right]\gamma(x)dx$$

Therefore,

$$[f(x)\, H(x)]' = \left[f(0)\, \delta(x) + H(x)\frac{df}{dx}\right]$$

Example 11 : Suppose that $\psi(x)$ is fairly good function then show that $[\psi(x)\, g(x)]' = \frac{d\psi(x)}{dx} g(x) + \psi(x)\, g'(x)$, where a prime denotes the generalised derivative.

Solution : Let $\gamma(x)$ be the good function and $g(x)$ be the associated generalised function then we have

$$\int_{-\infty}^{\infty} g'(x)\gamma(x)dx = -\int_{-\infty}^{\infty} g(x)\frac{d\gamma(x)}{dx}dx$$

Now $\psi(x)$ is fairly good function then $\psi(x)\,\gamma(x)$ is a generaised function by the property.

So its derivative $[\psi(x)\, g(x)]'$ exists as a generalised function such that

$$\int_{-\infty}^{\infty} (\psi(x)g(x))'\,\gamma(x)dx = -\int_{-\infty}^{\infty} (\psi(x)\, g(x))\frac{d\gamma(x)}{dx}dx$$

$$= -\int_{-\infty}^{\infty} g(x)\left(\psi(x)\frac{d\gamma(x)}{dx}\right)dx = -\int_{-\infty}^{\infty} g(x)\left[\frac{d(\psi(x)\gamma(x))}{dx} - \gamma(x)\frac{d\gamma(x)}{dx}\right]dx$$

$$= -\int_{-\infty}^{\infty} g(x)\left[\frac{d(\psi(x)\gamma(x))}{dx}\right]dx + \int_{-\infty}^{\infty} g(x)\left[\frac{d(\psi(x))}{dx}\right]\gamma(x)dx$$

$$= \int_{-\infty}^{\infty} g'(x)[\psi(x)\ \gamma(x)]dx + \int_{-\infty}^{\infty} g(x)\left[\frac{d(\psi(x))}{dx}\gamma(x)\right]dx$$

$$= \int_{-\infty}^{\infty}\left[g'(x)\ \ \psi(x) + g(x)\frac{d\psi(x)}{dx}\right]\gamma\,(x)dx$$

Hence $[\psi(x)\, g(x)]' = \dfrac{d\psi(x)}{dx} g(x) + \psi(x)\, g'(x).$

6.13. Fourier Transform of Generalised Function :

Let $\gamma(x)$ be a good function then the fourier transform of $\gamma(x)$ is denoted by $\Gamma(\alpha)$ and defined as

$$\Gamma(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\gamma(x)e^{i\alpha x}dx$$

Theorm 4 : If $\{\gamma_n(x)\}$ is a regular sequence defining a generalised function $g(x)$ then $\Gamma(\alpha)$ is a regular sequence such that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\Gamma_n(\alpha)\Gamma(\alpha)d\alpha = \fint_{-\infty}^{\infty} g(x)\gamma(-x)dx,\ \forall\gamma(x)\in\hat{G}.$$

Proof: Let the fourier transform of $f(x)$ is $F(\alpha)$ and the fourier transform of $g(x)$ is $G(\alpha)$ then by convolution theorem of fourier transform we have

$$\int_{-\infty}^{\infty} F(\alpha)\,G(\alpha)\,e^{-it\alpha}d\alpha = \int_{-\infty}^{\infty} f(x)\,g(t-x)\,dx.$$

Let the fourier transform of $\gamma_n(x)$ is $\Gamma_n(\alpha)$ and the fourier transform of $\gamma(x)$ is $\Gamma(\alpha)$ then by help of convolution theorem of fourier transform we have

$$\int_{-\infty}^{\infty}\Gamma_n(\alpha)\Gamma(\alpha)\,e^{-it\alpha}d\alpha = \int_{-\infty}^{\infty}\gamma_n(x)\,\gamma(t-x)\,dx.$$

Now put $t = 0$ in above we get

$$\int_{-\infty}^{\infty}\Gamma_n(\alpha)\Gamma(\alpha)\,d\alpha = \int_{-\infty}^{\infty}\gamma_n(x)\,\gamma(-x)\,dx$$

or,

$$\lim_{n\to\alpha}\int_{-\infty}^{\infty}\Gamma_n(\alpha)\Gamma(\alpha)\,d\alpha = \lim_{n\to\alpha}\int_{-\infty}^{\infty}\gamma_n(x)\,\gamma(-x)\,dx$$

$$= \fint_{-n}^{\infty} g(x)\gamma(-x)dx,\ \forall\,\gamma(x)\in\hat{G},\ \gamma(-x)\in\hat{G}$$

since $\{\gamma_n(x)\}$ is a regular sequence defining a generalised function $g(x)$.

Definition : If $\{\gamma_n(x)\}$ is a regular sequence defining a generalised function $g(x)$ then we define $G(\alpha)$ to be the generalised function defined by the sequence $\{\gamma_n(\alpha)\}$ and as follows

$$G(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\gamma(x)\,e^{i\alpha x}dx$$

then

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\Gamma_n(\alpha)\,\Gamma(\alpha)d\alpha = \int_{-\infty}^{\infty}\Gamma(\alpha)G(\alpha)d\alpha = \fint_{-\infty}^{\infty} g(x)\gamma(-x)\,dx$$

Example 12 : If $\gamma(x)$ is a good function with its fourier transform $\Gamma(\alpha)$ then prove that the Fourier transform of $\gamma(ax+b)$ is $\frac{1}{|a|}e^{-ib\alpha/a}\Gamma(\alpha/a.)$

Solution: Since $\Gamma(\alpha)$ is the fourier transform of $\gamma(x)$ then by definition of fourier transform we have

$$G(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\gamma(x)\,e^{i\alpha x}dx$$

Now the fourier transform of $ax+b$ i.e.,

$$F[\gamma(ax+b)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\gamma(ax+b)\,e^{i\alpha x}dx$$

put $ax+b=t$

$$=\frac{1}{|a|}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\gamma(t)\,e^{i\alpha\left(\frac{t-b}{a}\right)}dt=\frac{1}{|a|}\frac{1}{\sqrt{2\pi}}e^{\left(-i\frac{b}{a}\alpha\right)}\int_{-\infty}^{\infty}\gamma(t)\,e^{\left(i\frac{\alpha}{a}t\right)}dt$$

$$=\frac{1}{|a|}e^{\left(-i\frac{b}{a}\alpha\right)}\Gamma(\alpha/a).$$

Example 13 : Show that the fourier transform of $\delta(x)$ is $\frac{1}{\sqrt{2\pi}}$.

Solution : According to the definition, the regular sequence $\left\{\sqrt{\frac{n}{\pi}}e^{-nx^2}\right\}$ defines a generalised function $\delta(x)$. By the definition of fourier transform we can write,

$$F\left[\sqrt{\frac{n}{\pi}}e^{-nx^2}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-nx^2}e^{i\alpha x}dx=\left(e^{-\alpha^2/4n}\right)/\sqrt{2\pi}.$$

As we know that $\delta(x)$ is not a function in the classical sense, it can be approximated by a sequence of ordinary functions and we write $\delta(x)=\lim_{n\to\infty}\sqrt{\frac{n}{\pi}}e^{-nx^2}$. Again the regular sequence $\left\{e^{-\alpha^2/4n}\right\}$ defines the generalised function 1 (which shown in earlier Example 3).

$$F[\delta(x)]=\frac{1}{\sqrt{2\pi}}\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta(x)\,e^{i\alpha x}dx=\frac{1}{\sqrt{2\pi}}\lim_{n\to\infty}e^{-\alpha^2/4n}=\frac{1}{\sqrt{2\pi}}.$$

Hence the fourier transform of $\delta(x)$ is $\frac{1}{\sqrt{2\pi}}$.

6.14 Unit Summary : At the end of discussion, the whole unit is summarised as follows :

1. The differentiation can be made to the special functions such as Dirac Delta function, signum function, Heaviside step function and others.
2. The Integration can be made to the special functions such as Dirac Delta function, signum function, Haviside step function and others.
3. Fourier transform are applied to the special functions.

6.15 Exercises / Self Assessment Questions :

1. Show that the sequences $\left\{e^{-\frac{x^2}{2n}}\right\}$ and $\left\{e^{-\frac{x^2}{4n}}\right\}$ are regular sequences and they are equivalent to the regular sequence $\left\{e^{-\frac{x^2}{n}}\right\}$.

2. Show that (i) $e^{i\alpha x}\delta(x)=\delta(x)$ (ii) $P_n(x)\delta(x)=a_0\delta(x)$

   (iii) $x\delta(x)=0$, where $P_n(x)=a_0+a_1x+a_2x^2+...$

3. Show that

$$\int_{-\infty}^{\infty}\delta_r(x)\,\gamma(x)\,dx=(-1)^r\left[\frac{d^r\gamma(x)}{dx^r}\right]_{x=0},\quad r=1,\,2,\,3...$$

4. Find the fourier transform of the generalised function, $\theta(x)$ which can be identified with the unit stip function.

5. Give the definition of the derivative of a generalised function. Show that $\frac{d\theta(x)}{dx}=\delta(x)$, where $\theta(x)$ is the unit step function.

6. If $G(\alpha)$ is the fourier transform of $g(x)$ then prove that the fourier transform of $g(ax+b)$ is $\frac{1}{|a|}e^{\left(i\frac{b}{a}\alpha\right)}G(\alpha/a)$, $a\neq 0,\ b\in R$ where $g(x)$ is the generalised function.

7. Prove that the fourier transform of $\frac{dg(x)}{dx}$ is $i\alpha G(\alpha)$, where $g(x)$ is the generalised function.

8. Prove that the fourier transform of $\delta\,(ax-b)$ is $\frac{1}{\sqrt{2\pi}}\frac{1}{|a|}e^{\left(-i\frac{b}{a}\alpha\right)}$, $a\neq 0,\ b\in R$.

6.16 References / Suggested Further Readings :

1. D. S. Jones, Generalised Functions, Mc Graw-Hill Publisling Company Limited, London, 1966.
2. M. J. Lighthill, Introduction To Fourier Series and Generalised Functions, At The University Press, Cambridge, 1958.

M.Sc. Course in Applied Mathematics with Oceanology and Computer Programming

Part - II

Paper - VIII **Group - B**

Module No. 91
Dynamical Oceanography

Structure

1. Introduction :

Oceanography is the study of the ocean making use of the various basic sciences, physics, chemistry, biology and geology, with mathematics being used as an aid to parts of all of these studies. Particular attention is paid to the ocean as an environment both for the organisms with inhabit it naturally and in relation to man's activities and also to its interaction with the atmosphere, the environment in which man lives.

The physicist's contribution is to study the distribution of properties such as temperature, salinity, density, transparency, etc., which distinguish one water mass from another, and to study and understand the motions of the ocean in response to the forces acting on it.

Physical studies are carried out both by direct observation of the properties and movements and also by applying the basic physical principles of mechanics and thermodynamics to determine the motions. The observational approach is called descriptive or synoptic oceangraphy because the physicist tries to reduce his observations to a simple synopsis or summary. The essential feature of the second approach, dynamical oceanography, is to use physical laws to endeavour to obtain mathematical relations between the forces acting on the ocean waters and their consequent motions. In either case, the ultimate objective is to learn enough about the structure and motions of the ocean to be able to predict its future state.

To achieve this objective the dynamical approach is most likely to be successful because it should result in analytical expressions which can be used for prediction into future, whereas the synoptic approach simply describes what has happened in the past. But in practice, it turns out that some characteristics of the oceans do not change much.with time, or they repeat themselves with recognized periods, so that a good description of the present state may be applicagle for some time into the future. However, some features do change and the amplitudes of cyclic variations may alter, so that a quantitative understanding of the relations between the causative forces and the reaction of the ocean is desirable. Therefore, the preliminary quantitative description of the ocean and its movements prepared by the synoptic oceanographer is used by the dynamical oceanographer to suggest what kinds of motion he may expect and what forces may be cuasing them, so helping him to start his theoretical study. Also, if he meets mathematics difficulties in his analysis, as often happens, the available observations may suggest what mathematical approximations may be made while keeping the investigation physically realistic. When the dynamical oceanographer has made a preliminary analysis, it will probably suggest the need for more extensive or sophisticated observations; when these have been made he may refine his analysis. Systematic physical observations of the ocean have been made for a century or so, the rate of accumulation of data having increased enormously during the last 25 years.

2. Objectives :

The main objective of this module is to learn enough about the structure and motions of the sea water and how the general thermodynamics plays an important role.

3. Keywords :

Navier-Stokes equation, Salinity, Gibb's relation, Gibb's-Duhem relation, Adiabatic temperature gradient, Potential temperature, Co-efficient of thermal expansion and isothermal compression, adiabatic compression.

4. Navier Stokes Equation :

The constitutive equation for isotropic homogeneous linearly viscous fluid are

$$T_{ij} = -p\delta_{ij} + \lambda\vec{\nabla}\cdot\vec{V}\delta_{ij} + 2\mu d_{ij}$$

$$= -\left(p + \frac{2\mu}{3}\vec{\nabla}\cdot\vec{V}\right)\delta_{ij} + 2\mu d_{ij} \text{ .... (1)}$$

where $d_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

$\lambda = -\frac{2}{3}\mu, \mu$ is the coefficient of viscosity.

The equation of motion of continuum is

$$T_{ij,j} + \rho F_i = \rho\frac{du_i}{dt} \text{ .... (2)}$$

Now, from (1) we have

$$T_{ij,j} = -p_{,j}\delta_{ij} - \frac{2}{3}\mu\delta_{ij}\left(\vec{\nabla}\cdot\vec{V}\right)_{,j} + \mu\left(u_{i,jj} + u_{j,ij}\right)$$

$$= -p_{,i} - \frac{2}{3}\mu\delta_{ij}\left(\vec{\nabla}\cdot\vec{V}\right)_{,j} + \mu\nabla^2 u_i + \mu\left(u_{j,j}\right)_{,i}$$

$$= -p_{,i} - \frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)_{,i} + \mu\nabla^2 u_i + \mu\left(\vec{\nabla}\cdot\vec{V}\right)_{,i}$$

$$\therefore T_{ij,j} = -p_{,i} + \frac{\mu}{3}\left(\vec{\nabla}\cdot\vec{V}\right)_{,i} + \mu\nabla^2 u_i \text{ ....(3)}$$

Using (3) into (2), we get

$$\frac{du_i}{dt} = F_i - \frac{1}{\rho}p_{,i} + \frac{\gamma}{3}\left(\vec{\nabla}\cdot\vec{V}\right)_{,i} + \gamma\nabla^2 u_i \text{ .... (4)}$$

where $\gamma = \frac{\mu}{\rho}$ is the kinematical coefficient of viscosity.

Equation (4) can be written as in vector notation

$$\frac{d\vec{V}}{dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p + \frac{\gamma}{3}\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right) + \gamma\nabla^2\vec{V} \text{ .... (5)}$$

This is known as Navier-Stoke's equation of motion for compressible viscous fluid. Also equation (5) can be written as

$$\frac{\partial\vec{V}}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p + \frac{\gamma}{3}\left(\vec{\nabla}\cdot\vec{V}\right) + \gamma\nabla^2\vec{V} \text{ .... (6)}$$

$$\left(\because \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla}\right).$$

Note :

For incompressible fluid, we must have

$$u_{j,j} = 0$$

i.e., $\vec{\nabla}\cdot\vec{V}=0.$

Then equation (5) and (6) takes the form

$$\frac{d\vec{V}}{dt}=\vec{F}-\frac{1}{\rho}\vec{\nabla}p+\gamma\nabla^2\vec{V}$$

and, $$\frac{\partial\vec{V}}{\partial t}+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}=\vec{F}-\frac{1}{\rho}\vec{\nabla}p+\gamma\nabla^2\vec{V}$$

These are called Navier-Stokes equation of motion for incompressible fluid.

5. Subject Matter : Actually the subject matter of the dynamical oceanography is concerned with the study of waves and currents (or, large scale motion of water in the ocean) kindred phenomena on the basis of well-known principles of hydrodynamics and thermodynamics of fluid taking into account the rotation and sphericity of the earth. The results of the study are used in various disciplines such as meteorology, geology and marine biology etc.

6. Sea water as a Two-Components Solution :

The principal substances, dissolved in sea water, are strong electrolytes and they are practically dissociated into ions. The basic components of the mixture are chloride ions (Cl^-), sodium ions (Na^+), sulphate ions $\left(SO_4^{2-}\right)$, magnesium ions $\left(Mg^{2+}\right)$, Calcium ions $\left(Ca^{2+}\right)$, Potassium ions $\left(K^+\right)$ and hydro carbonate ions $\left(HCO_3^-\right)$.

The ion composition of sea water (according to Sverdrup *et al.* (p.120, Chapter-VI]) are

Na^+ – 30.61% Cl^- – 55.04%

Mg^{2+} – 3.69% SO_4^{2-} – 7.68%

Ca^{2+} – 1.16% HCO_3^- – 0.41%

K^+ – 1.10%

Next we shall discuss about the salinity.

7. Salinity :

Consider a volume V and let $m_1, m_2, \ldots, m_{n-1}$ be the masses of the components of the admixture (salt) in solution and m_n the mass of pure water. Then salinity.

$$s = \frac{\sum_{k=1}^{n-1} m_k}{m}$$

where m is the mass of the volume V, i.e., $m = \sum_{k=1}^{n} m_k$.

Defined in this manner, salinity turns out to be a non-dimensional quantity, usually expressed in *parts per thousand* and denoted by S. Thus

$$S = 1000s.$$

The following important fact is experimental in origin : Far away from ocean shores, the composition of principal ions in sea water is constant. In other words, the salinity of sea water changes because of addition of pure water or its disappearance (precipitation, evaporation, formation and thawing of ice), but the composition of the salt of sea water remains unchanged. As a consequence of the constancy of its salt composition, sea water may be considered as a two component mixture: *pure water (solvent) and salt (solute).*

Note 1 :

The physical properties of pure water relevant to fluid dynamics studies are functions of pressure and temperature while those of sea water are functions of *pressure, temperature and salinity.*

Note 2 :

The salinity of sea water is essentially a measure of the mass of dissolved salts in one kilogram of sea-water.

Note 3 :

An average value for the oceans is about 35 grams per kilogram (34.7 parts per thousand). It is not practical to measure this quantity directly, and for a long time it was determined indirectly by measuring the halide content (Cl, mostly chloride) by a silver nitrate titration and using, on the basis of previous careful analyses, a linear relation

$$s = 1.80655 \times \text{Cl}$$

between total dissolved salts and halide. However, it has been demonstrated that there is a close relationship between the salinity and the electrical conductivity of sea water and, as this latter property can now be measured easily and precisely, it has for some time been used to estimate salinity.

Note 4 :

In the laboratory, salinity is determined from measurements of the electrical conductivity and temperature (because of constant pressure, conductivity $= f(S, T)$.

Concentration :

The *relative concentration* of the salt component among themselves are defined as

$$\lambda_k = \frac{m_k}{\sum_{k=1}^{n-1} m_k} = \text{constant}$$

as per observation mentioned above.

The *absolute concentration* in a mass $m = \sum_{k=1}^{n} m_k$ of sea water is

$$c_k = \frac{m_k}{m} = \frac{m_k}{\sum_{k=1}^{n-1} m_k} \times \frac{\sum_{k=1}^{n-1} m_k}{m}$$

$$= \lambda_k s, \; k = 1, 2, ..., n-1.$$

The concentration of pure water is

$$c_w = \frac{m - \sum_{k=1}^{n-1} m_k}{m} = 1 - \frac{\sum_{k=1}^{n-1} m_k}{m}$$

$$= 1 - s$$

If salt is taken as a whole, then we have

$$\sum_{k=1}^{n-1} \lambda_k = \frac{\sum_{k=1}^{n-1} m_k}{\sum_{k=1}^{n-1} m_k} = 1.$$

Q. Show that $s + c_w = 1$.

9. Basic Thermodynamics :

The thermodynamical state of a single (or one component) homogeneous (or one phase) chemically inert gas of a constant mass is described by certain state of variables such as

i) the pressure (p);

ii) the specific volume (Υ) i.e., the volume per unit mass $= \frac{1}{\rho}$;

iii) absolute temperature (T);

iv) the specific internal energy (E);

v) the specific entropy (η);

vi) the specific enthalpy (χ) etc.

9.1 Hypothesis :

For all gases in any type of motion, there exists an internal energy function E and an entropy function η which depend only on the variables of state p, ρ, T and are each a differentiable function of these variables.

9.2 First law of thermodynamics :

The first law of thermodynamics state that when a small quantity of heat δQ is communicated to a unit mass of gas, a change of state take place such that

$$\delta Q = dE + pd\Upsilon,$$

i.e., quantity of heat = increase in specific internal energy + work done by gas pressure in expanding the volume.

9.3 Second law of thermodynamics :

The second law of thermodynamics state that when a small quantity of heat δQ is communicated to a unit mass of gas, a change of state take place such that

$$\delta Q = Td\eta$$

and the entropy of an isolated system can never decrease $(d\eta \geq 0)$.

Thus for a single component of gas, we have, by combining first and second law of thermodynamics,

$$T\,d\eta = dE + pd\Upsilon$$

9.4 Specific Heat :

The concept of the heat capacity of a system as the amount of heat δQ which must be added to a closed system in order to increase its temperature T by one centigrade. Since the quantity δQ is not a differential of any function of state and depends on the type of process, one may introduce heat capacity for constant pressure

$$C_p = \left(\frac{\partial Q}{\partial T}\right)_{p=\text{constant.}}$$

and for constant volume

$$C_v = \left(\frac{\partial Q}{\partial T}\right)_{v=\frac{1}{\Upsilon}=\text{constant.}}$$

respectively.

Clearly, $C_p > C_v > 0$ for in C_p, heat supplied increase the temperature as well as volume while in C_v only temperature is increased.

In order to express C_p and C_v in terms of derivatives of entropy η, it is convenient to change over from the independent variables E, v, c_j to the independent variables T, p, c_j and T, v, c_j respectively. Then

$$c_p = T\left(\frac{\partial \eta}{\partial T}\right)_{p,c_j} \quad \text{and} \quad c_v = T\left(\frac{\partial \eta}{\partial T}\right)_{v,c_j}$$

9.5 Change of State :

Isothermal change of state in which

$\delta T = 0$

i.e., T= constant.

Adiabatic change of state in which

$\delta\eta = 0$

i.e., η= constant.

Note :

The thermodynamic parameters of a system are sub-divided into *extensive* and *intensive parameters.* Extensive parameters change by a factor M as the masses m_j of all components of the system change by a factor M; intensive parameters do not change for such a change of the masses m_j. In other words, *extensive parameters are homogeneous functions of* m_j *of order one, while intensive parameters are homogeneous functions of* m_j *of order zero,* and therefore depend only on the concentration $c_j = \frac{m_j}{m}$.

Note 2.

Chemical potential

$$\mu_j = -T\left(\frac{\partial \eta}{\partial m_j}\right)_{E,v,m_i(i\neq j)} \quad , j = 1,2,\ldots,n$$

of a component in a mixture depends on the mass of the other component and is an intensive quantity.

Note 3.

Entropy of a mixture of two ideal gases is extensive.

10. Gibb's General Thermodynamical Relation for Sea Water :

The thermodynamic state of a multicomponent isotropic equilibrium system of mass m without phase transitions is completely determined, if its specific internal energy E, its specific volume Υ and its concentration $c_k = \frac{m_k}{m}$, $k = 1, 2, \ldots, n$ of its different components of mass m_k are specified. Assuming that there exists single valued specific entropy function $\eta = \eta(E, \Upsilon, c_k)$ for such a system.

The concept of entropy of an equilibrium thermodynamic system permit to introduce into the consideration such parameters as the absolute temperature of the system T and the chemical potentials of the separate components of the system μ_j :

$$\frac{1}{T} = \left(\frac{\partial \eta}{\partial E}\right)_{\Upsilon,c_j} \quad , j = 1,2,\ldots,n \quad \ldots\ldots\ldots\ldots (1)$$

$$\mu_j = -T\left(\frac{\partial \eta}{\partial c_j}\right)_{E,\Upsilon,c_k,j\neq k} \quad \ldots\ldots\ldots\ldots (2)$$

The parameter used as a subscripts are to kept constants during the differentiation and indicates which are (in terms specified) the independent variable.

In an adiabatic reversible or equilibrium process leading to a change of state, heat and mass exchanges with the surrounding medium are negligible, and so a change in E occurs only. On account of work δA performed by the external forces namely the pressure p acting on the surface Σ enclosing the volume Υ. Then

$$\delta A = (-p\,d\Sigma)$$
$$= -p\,d\Upsilon.$$

Also for adiabatic process

$$dE = -p\,d\Upsilon$$

which gives, in general,

$$\left(\frac{\partial E}{\partial \Upsilon}\right)_{\eta,c_j} = -p$$

Using the property of Jacobians, we can finds

$$\left(\frac{\partial \eta}{\partial \Upsilon}\right)_{E,c_j} = \left\{\frac{\partial(\eta,E)}{\partial(\Upsilon,E)}\right\}_{c_j} \quad j = 1, 2, \ldots, n$$

$$= \left\{\frac{\partial(\eta,E)}{\partial(\Upsilon,\eta)} \cdot \frac{\partial(\Upsilon,\eta)}{\partial(\Upsilon,E)}\right\}_{c_j}$$

$$= -\left(\frac{\partial E}{\partial \Upsilon}\right)_{\eta,c_j} \cdot \left(\frac{\partial \eta}{\partial E}\right)_{\Upsilon,c_j}$$

$$\therefore \left(\frac{\partial \eta}{\partial \Upsilon}\right)_{E,c_j} = \frac{p}{T} \text{ (using (1) and (3))} \ldots\ldots\ldots\ldots\ldots (4)$$

Since $\eta = \eta\left(E, \Upsilon, c_j\right), j = 1, 2, \ldots, n$

$$\therefore d\eta = \left(\frac{\partial \eta}{\partial E}\right)_{\Upsilon,c_j} dE + \left(\frac{\partial \eta}{\partial \Upsilon}\right)_{E,c_j} d\Upsilon + \sum_{j=1}^{n}\left(\frac{\partial \eta}{\partial c_j}\right)_{E,\Upsilon} dc_j$$

$$d\eta = \frac{1}{T}dE + \frac{p}{T}d\Upsilon + \sum_{j=1}^{n}\left(-\frac{\mu_j}{T}\right)dc_j \text{ (using (1), (2), (4))}$$

or, $$T\,d\eta = dE + p\,d\Upsilon - \sum_{j=1}^{n}\mu_j dc_j$$

This formula, called Gibb's relation, is one of the basic relations of thermodynamic for an open system, i.e., one which exchange mass with surrounding mixture or system of variable composition.

10.1 Deduction : Gibb's relation for sea-water as a Two Components :

Let us assume that sea water as a two components of mixture of salt and pure water. We denote μ_s and μ_w as their respective chemical potential and c_s and c_w as their respective concentration in a mixture of mass *m*.

Then

$$c_s = \frac{m_s}{m} = s,$$

and $$c_w = \frac{m_w}{m} = \frac{m - m_s}{m} = 1 - \frac{m_s}{m} = 1 - s,$$

where m_s and m_w are masses of salt and pure waer, and

$$m = m_s + m_w.$$

Therefore,

$$\mu_s dc_s + \mu_w dc_w = \mu_s ds + \mu_w d(1-s)$$

$$= \mu_s ds - \mu_w ds$$

$$= (\mu_s - \mu_w) ds.$$

$$= \mu\, ds, \text{ say,}$$

where $\mu = \mu_s - \mu_w$.

Hence the Gibb's relation for sea water takes the form

$$T\, d\eta = dE + pd\Upsilon - (\mu_s dc_s + \mu_w dc_w)$$

$$\therefore T\, d\eta = dE + pd\Upsilon - \mu ds$$

Important Result :

If c_j be the cencentration of various components of masses m_j $(j = 1, 2, \ldots, n-1)$ of the salt, then

$$c_j = \frac{m_j}{m} = \frac{m_j}{\sum_{k=1}^{n-1} m_k} \cdot \frac{\sum_{k=1}^{n-1} m_k}{m}$$

$$= \lambda_j \cdot s \text{ where } \lambda_j = \frac{m_j}{\sum_{k=1}^{n-1} m_k} = \text{constant (by observation)}$$

i.e., $c_j = \lambda_j s$

$$\therefore \mu_s ds = \sum_{k=1}^{n-1} \mu_k dc_k$$

$$= \sum_{k=1}^{n-1} \mu_k \lambda_k ds$$

$$\Rightarrow \mu_s = \sum_{k=1}^{n-1} \lambda_k \mu_k .$$

10.2 Significance of Gibb's Relation :

By Gibb's relation, the significance of the entropy of a system as a function of E, Υ, c_j permits to determine all the basic thermodynamic parameters. Thus we may finally get :

i) T, absolute temperature, from the relation : $\frac{1}{T} = \left(\frac{\partial \eta}{\partial E} \right)_{\Upsilon, c_j}$

ii) μ_j, chemical potential, from the relation : $\mu_j = -T \left(\frac{\partial \eta}{\partial c_j} \right)_{E, \Upsilon, c_k (j \neq k)}$

iii) p, pressure, from the relation : $p = T \left(\frac{\partial \eta}{\partial \Upsilon} \right)_{E, c_j}$

10.3 Equation of State :

Any relation of the form $f\left(p, \rho, T, c_j\right)$ is called the equation of state.

Eliminating E between $T\left(E, \Upsilon, c_j\right) = T$ and $p = p\left(E, \Upsilon, c_j\right)$ we find the equation of state as

$f\left(p, \rho, T, c_j\right) = 0$ where $\rho = \frac{1}{\Upsilon}$.

Likewise, from the known entropy $\eta = \eta\left(E, \Upsilon, c_j\right)$ we can determine all other parameters such as :

iv) the specific heats :

$$C_p = T \left(\frac{\partial \eta}{\partial T} \right)_{p, c_j} ; C_v = T \left(\frac{\partial \eta}{\partial T} \right)_{\Upsilon, c_j}$$

v) From Gibb's relation

$$T\, d\eta = dE + p\, d\Upsilon - \sum_{j=1}^{n} \mu_j dc_j$$

we have

$$d(E+p\Upsilon)=d\chi=T\,d\eta+\Upsilon dp+\sum_{j=1}^{n}\mu_j dc_j,$$

$$d(E-T\eta)=d\psi=-\eta dT-pd\Upsilon+\sum_{j=1}^{n}\mu_j dc_j,$$

and $d(E+p\Upsilon-T\eta)=dJ=-\eta dT+\Upsilon dp+\sum_{j=1}^{n}\mu_j dc_j$

where, the function

$$\chi=E+p\Upsilon$$

is called the *specific enthalpy* (or *heat content function*) of the system;

the function

$$\psi=E-T\eta$$

is called the *free energy function* (or *Helmholtz function*) of the system;

the function

$$J=E+p\Upsilon-T\eta$$

is called the *Gibb's potential function* (or *Gibb's function*) of the system.

For this reason, the entropy is called a thermodynamical potential w.r.t. the variables E,Υ,c_j. However rewriting Gibb's relation with the differential of either E, or χ or ψ or J expressed in terms of differentials of the independent variables $(\eta,\Upsilon,c_j),(\eta,p,c_j),(T,\Upsilon,c_j),(T,p,c_j)$ respectively.

10.4 Result : Gibb's relation for sea water in (T,p,s) variables :

The Gibb's function is

$$J=E+p\Upsilon-T\eta$$

$$\therefore dJ=dE+p\,d\Upsilon-Td\eta+\Upsilon\,dp-\eta\,dT$$

or, $dJ=\Upsilon dp-\eta\,dT+\mu\,ds$

$$(\because T\,d\eta=dE+pd\Upsilon-\mu\,ds)$$

which shows that R.H.S. of above relation must be an exact as L.H.S. is so. Therefore from the conditions of exactness we have

$$-\frac{\partial \eta}{\partial p}=\frac{\partial \Upsilon}{\partial T}, \frac{\partial \Upsilon}{\partial s}=\frac{\partial \mu}{\partial p}, \frac{\partial \mu}{\partial T}=-\frac{\partial \eta}{\partial s}.$$

If $\eta=\eta(T,p,s)$ then

$$d\eta=\left(\frac{\partial \eta}{\partial T}\right)_{p,s} dT+\left(\frac{\partial \eta}{\partial p}\right)_{T,s} dp+\left(\frac{\partial \eta}{\partial s}\right)_{T,p} ds$$

or, $$d\eta=\frac{C_p}{T}dT-\frac{\partial \Upsilon}{\partial T}dp-\frac{\partial \mu}{\partial T}ds$$

(using $C_p=T\left(\frac{\partial \eta}{\partial T}\right)_{p,s}$ and above conditions of exactness)

which is the Gibb's relation for sea water as a two components in T, p, s variables.

11. Gibb's - Duhem Relation :

The Gibb's function w.r.t. T, p, c_j is

$$J=E+p\Upsilon-T\eta$$

$$\therefore dJ=dE+pd\Upsilon-Td\eta+\Upsilon dp-\eta dT$$

or, $$dJ=\Upsilon dp-\eta dT+\sum_{j=1}^{n}\mu_j dc_j \left(\because T d\eta=dE+pd\Upsilon-\sum_{j=1}^{n}\mu_j dc_j\right)$$

$$\therefore \left(\frac{\partial J}{\partial c_j}\right)_{T,p,c_k(k\neq j)}=\mu_j$$

So, $$\sum_{j=1}^{n}\mu_j c_j=\sum_{j=1}^{n}\left(\frac{\partial J}{\partial c_j}\right)c_j=J(p,T,c_k)$$

(by Euler's formula for homogeneous function of first order).

Here J is a homogeneous functions of degree one in c_j because J is an extensive thermodynamic parameter, i.e., if the masses of different components are multiplied by m', then $J=E+p\Upsilon-T\eta\left(E,\Upsilon,c_j\right)$ is also multiplied by m'.

Hence accordingly Euler's identity, we have

$$\sum_{j=1}^{n}\mu_j c_j = E + p\Upsilon - T\eta.$$

Differentiating both sides, we get

$$\sum_{j=1}^{n}\mu_j dc_j + \sum_{j=1}^{n}c_j d\mu_j = dE + pd\Upsilon - T\,d\eta + \Upsilon dp - \eta\,dT$$

Using Gibb's relation, we get

$$\sum_{j=1}^{n}c_j d\mu_j = \Upsilon dp - \eta\,dT$$

or, $\eta\,dT - \Upsilon\,dp + \sum_{j=1}^{n}c_j d\mu_j = 0$

This formula is known as the Gibbs-Duhem relation.

11.1 Gibb's-Duhem Relation for Two Components Sea Water :

Let μ_s and μ_w be the chemical potentials of salt and pure water; c_s and c_w being their corresponding concentrations then we have from their definitions

$$c_s = \frac{m_s}{m} = s, \text{ (salinity)}$$

and $c_w = \dfrac{m_w}{m} = \dfrac{m - m_s}{m} = 1 - s$

where m_s and m_w are the masses of salt and pure water.

such that $m_s + m_w = m$, mass of the sea water.

Therefore,

$$\begin{aligned} c_s d\mu_s + c_w d\mu_w &= s d\mu_s + (1-s)d\mu_w \\ &= s\,d(\mu_s - \mu_w) + d\mu_w \\ &= s\,d\mu + d\mu_w, \text{ say } \ldots\ldots\ldots\ldots(1) \end{aligned}$$

where $\mu = \mu_s - \mu_w$.

Hence, the Gibbs-Duhem relation for sea-water

$$\eta\, dT - \Upsilon\, dp + \sum_{k=1}^{n} c_k\, d\mu_k = 0$$

becomes

$$\eta\, dT - \Upsilon\, dp + c_w d\mu_w + c_s d\mu_s = 0$$

or, $\eta\, dT - \Upsilon\, dp + s\, d\mu + d\mu_w = 0$ [using (1)].

Also (1) can be written as

$$\begin{aligned} s\,d\mu + d\mu_w &= s\,d(\mu_s - \mu_w) + d\mu_w \\ &= s\,d\mu_s + (1-s)\,d\mu_w \\ &= s\,d\mu_s + (1-s)\,d\mu_w - (1-s)\,d\mu_s + (1-s)\,d\mu_s \\ &= -(1-s)\,d(\mu_s - \mu_w) + (1-s)\,d\mu_s + s\,d\mu_s \\ &= -(1-s)\,d\mu + d\mu_s. \end{aligned}$$

So, Gibbs-Duhem relation may also be written as

$$\eta\, dT - \Upsilon\, dp - (1-s)\,d\mu + d\mu_s = 0.$$

Note :

At constant temperature and constant pressure we must have

$dT = 0$ and $dp = 0$.

Then Gibbs-Duhem relation becomes

$$-(1-s)\,d\mu + d\mu_s = 0$$

or, $-(1-s)\,d(\mu_s - \mu_w) + d\mu_s = 0$

or, $(1-s)\,d\mu_w + s\,d\mu_s = 0$

or, $s\left(\dfrac{\partial \mu_s}{\partial s}\right) + (1-s)\left(\dfrac{\partial \mu_w}{\partial s}\right) = 0.$

which shows that at constant temperature and constant pressure μ_w increases if μ_s decreases and vice-versa.

Result :

$$c_v = c_p + T\frac{\left(\frac{\partial \Upsilon}{\partial T}\right)^2}{\left(\frac{\partial \Upsilon}{\partial p}\right)}, c_p > c_v > 0.$$

Proof. From the definitions of specific heats we have

$$c_p = \left(\frac{\partial Q}{\partial T}\right)_{p,s} = T\left(\frac{\partial \eta}{\partial T}\right)_{p,s} \quad \text{..............(i)}$$

and $c_v = \left(\frac{\partial Q}{\partial T}\right)_{T,s} = T\left(\frac{\partial \eta}{\partial T}\right)_{\Upsilon,s}$ ....(ii)

$$\therefore c_v = T\left(\frac{\partial \eta}{\partial T}\right)_{\Upsilon,s} = T\frac{\partial(\eta,\Upsilon)}{\partial(T,\Upsilon)} \text{ [using the properties of Jacobians]}$$

$$= T\frac{\partial(\eta,\Upsilon)}{\partial(T,p)}\cdot\frac{\partial(T,p)}{\partial(T,\Upsilon)} \text{ [using the properties of Jacobians]}$$

$$= T\frac{\partial(\eta,\Upsilon)}{\partial(T,p)}\cdot\frac{1}{\frac{\partial(T,\Upsilon)}{\partial(T,p)}} \text{ [using the properties of Jacobians]}$$

$$= T\left[\frac{\partial \eta}{\partial T}\cdot\frac{\partial \Upsilon}{\partial p} - \frac{\partial \eta}{\partial p}\cdot\frac{\partial \Upsilon}{\partial T}\right] \div \left(\frac{\partial \Upsilon}{\partial p}\right)$$

$$= T\left(\frac{\partial \eta}{\partial T}\right)_p - T\frac{\partial \Upsilon}{\partial T}\cdot\frac{\left(\frac{\partial \eta}{\partial p}\right)}{\left(\frac{\partial \Upsilon}{\partial p}\right)} \quad \text{................(iii)}$$

Now Gibb's function is

$$J = E + p\Upsilon - T\eta$$

$$\therefore dJ = dE + pd\Upsilon - Td\eta + \Upsilon dp - \eta dT$$

$$= \mu\, ds + \Upsilon\, dp - \eta\, dT \quad \text{...................(iv)}$$

$[\because Td\eta = dE + pd\Upsilon - \mu\, ds,$ for two components sea water]

Above relation shows that R.H.S. is an perfect (Exact) differential, as the L.H.S. is an perfect (exact) differential in T, p, s variables. So, by the conditions of exactness we have

$$\frac{\partial \mu}{\partial p} = \frac{\partial \Upsilon}{\partial s}, \frac{\partial \mu}{\partial T} = -\frac{\partial \eta}{\partial s}, \frac{\partial \Upsilon}{\partial T} = -\frac{\partial \eta}{\partial p} \quad \text{..............(v)}$$

Using (v) and (i) in (iii) we get

$$c_v = c_p + T\frac{\left(\frac{\partial \Upsilon}{\partial T}\right)^2}{\left(\frac{\partial \Upsilon}{\partial p}\right)}, \left(c_p > c_v > 0\right).$$

The above formula (or relation) permits to determine c_v from c_p for a known equation of state of sea water. Since

$$\frac{\partial \Upsilon}{\partial p} = \frac{\partial \left(\frac{1}{\rho}\right)}{\partial p} < 0,$$

so, one has $c_p > c_v$.

The physical significance is as follows : If one adds to a system a definite quantity of heat, then for constant volume it heats up more than for constant pressure, since in the second case the system will perform work on account of part of the heat. Note that for sea water the ratio $\frac{c_p}{c_v}$ is very close to unity.

12. A diabatic Temperature Gradient :

The change in temperature of a system during an equilibrium adiabatic process in which entropy η and salinity s remains constant is expressed by the parameters

$$\Gamma = \left(\frac{\partial T}{\partial p}\right)_{\eta, s}$$

Γ is called the adiabatic temperature gradient of the system.

Result : $\Gamma = \frac{T}{c_p}\left(\frac{\partial \Upsilon}{\partial T}\right).$

Proof. According to the definition of the adiabatic temperature gradient we have

$$\Gamma = \left(\frac{\partial T}{\partial p}\right)_{\eta,s} = \frac{\partial(T,\eta)}{\partial(p,\eta)} \text{ [using properties of Jacobians]}$$

$$= \frac{\partial(T,\eta)}{\partial(p,T)} \cdot \frac{\partial(p,T)}{\partial(p,\eta)} \text{ [using properties of Jacobians]}$$

$$= \frac{\partial(T,\eta)}{\partial(p,T)} \cdot \frac{1}{\frac{\partial(p,\eta)}{\partial(p,T)}} \text{ [using properties of Jacobians]}$$

$$\therefore \Gamma = -\frac{\left(\frac{\partial \eta}{\partial p}\right)_{T,s}}{\left(\frac{\partial \eta}{\partial T}\right)_{p,s}} \qquad \text{.................. (vi)}$$

But from the Gibb's function

$$J = E + p\Upsilon - T\eta$$

we get $dJ = dE + pd\Upsilon - Td\eta + \Upsilon dp - \eta dT$

$$= \mu\, ds + \Upsilon\, dp - \eta\, dT$$

[since, for the sea water as two components we know that $Td\eta = dE + pd\Upsilon - \mu\, ds$]

As L.H.S. of above expression is an exact differential, so R.H.S. is. Hence, we have

$$\frac{\partial \mu}{\partial p} = \frac{\partial \Upsilon}{\partial s}, \frac{\partial \mu}{\partial T} = -\frac{\partial \eta}{\partial s}, \frac{\partial \Upsilon}{\partial T} = -\frac{\partial \eta}{\partial p}.$$

Using above result in (vi), we get .

$$\therefore \Gamma = \frac{\frac{\partial \Upsilon}{\partial T}}{\left(\frac{\partial \eta}{\partial T}\right)_{p,s}} = \frac{T}{c_p} \cdot \frac{\partial \Upsilon}{\partial T} \quad \left[\because c_p = T\left(\frac{\partial \eta}{\partial T}\right)_{p,s}\right].$$

An idea of the numerical values of Γ can be gained from the following table according to Montgomery.

Dependence of Γ on the pressure (difference from atmospheric pressure) for $T = 0°C$ and $S = 35‰$:

p (abar) :	0	2000	4000	6000	8000	10,000
Γ (0°C/1000 abar) :	0.035	0.072	0.104	0.133	0.159	0.181

13. Potential Temperature and Potential Density :

The potential temperature Θ of a system is the temperature which a system acquires during an equilibrium adiabatic transition from pressure p to atmospheric pressure p_a. And the corresponding density for such a transition is termed potential density and is denoted by ρ_{pot}.

From definitions we have

$$\Theta(\eta,s)=T(\eta,p_a,s)$$

$$=T(\eta,p,s)+\int_p^{p_a}\left(\frac{\partial T}{\partial p}\right)_{\eta,s}dp$$

$$\therefore \Theta(\eta,s)=T(\eta,p,s)-\int_{p_a}^{p}\Gamma(\eta,p,s)\,dp$$

and, similarly,

$$\rho_{\text{pot}}(\eta,s)=\rho(\eta,p_a,s)$$

$$=\rho(\eta,p,s)-\int_{p_a}^{p}\left(\frac{\partial \rho}{\partial p}\right)_{\eta,s}dp$$

$$=\rho(\Theta,s,p_a).$$

The quantities Θ and ρ_{pot} described the effect of removal of pressure influence on the temperature and density of sea-water.

14. Some Important Quantities :

i) The Co-efficient of thermal expansion (volumetry) in T, p, s variables is defined as

$$\alpha=\rho\left[\frac{\partial\left(\frac{1}{\rho}\right)}{\partial T}\right]_{p,s}=\frac{1}{\Upsilon}\left(\frac{\partial \Upsilon}{\partial T}\right)_{p,s}$$

ii) The co-efficient of isothermal compression in T, p, s variables is defined as

$$K_T=\frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_{T,s}=-\frac{1}{\Upsilon}\left(\frac{\partial \Upsilon}{\partial p}\right)_{T,s}$$

iii) The coefficient of adiabatic compression in T, p, s variables is defined as

$$K_\eta = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_{\eta,s} = -\frac{1}{\Upsilon}\left(\frac{\partial \Upsilon}{\partial p}\right)_{\eta,s}$$

iv) The velocity of the sound in sea-water is defined as

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_{\eta,s}}$$

Result :

$K_\eta = K_T - \Gamma\alpha$. Hence deduce $K_T > K_\eta$.

Proof. From the definition of K_η, we have

$$K_\eta = -\frac{1}{\Upsilon}\left(\frac{\partial \Upsilon}{\partial p}\right)_{\eta,s}$$

$$= -\frac{1}{\Upsilon}\frac{\partial(\Upsilon,\eta)}{\partial(p,\eta)}$$

$$= -\frac{1}{\Upsilon}\frac{\partial(\Upsilon,\eta)}{\partial(p,T)}\cdot\frac{\partial(p,T)}{\partial(p,\eta)}$$

$$= -\frac{1}{\Upsilon}\left[\frac{\partial \Upsilon}{\partial p}\cdot\frac{\partial \eta}{\partial T} - \frac{\partial \Upsilon}{\partial T}\cdot\frac{\partial \eta}{\partial p}\right]\cdot\frac{1}{\dfrac{\partial(p,\eta)}{\partial(p,T)}}$$

$$= \frac{-\dfrac{1}{\Upsilon}\left[\dfrac{\partial \Upsilon}{\partial p}\cdot\dfrac{\partial \eta}{\partial T} - \dfrac{\partial \Upsilon}{\partial T}\cdot\dfrac{\partial \eta}{\partial p}\right]}{\left(\dfrac{\partial \eta}{\partial T}\right)}$$

$$= -\frac{1}{\Upsilon}\left(\frac{\partial \Upsilon}{\partial p}\right)_{T,s} - \frac{1}{\Upsilon}\left(\frac{\partial \Upsilon}{\partial T}\right)^2\cdot\frac{1}{\dfrac{c_p}{T}} \quad \left[\text{since } \frac{\partial \eta}{\partial p} = -\frac{\partial \Upsilon}{\partial T} \text{ and } c_p = T\left(\frac{\partial \eta}{\partial T}\right)_{p,s}\right]$$

$$\therefore K_\eta = K_T - \frac{T}{\Upsilon c_p}\cdot\left(\frac{\partial \Upsilon}{\partial T}\right)^2$$

Hence, $K_\eta = K_T - \Gamma \cdot \alpha$ $\left[\because \alpha = \frac{1}{\Upsilon}\left(\frac{\partial \Upsilon}{\partial T}\right)_{p,s} \text{ and } \Gamma = \frac{T}{c_p}\cdot\frac{\partial \Upsilon}{\partial T}\right]$

Again,

$$K_\eta = K_T + \frac{K_T \cdot T}{c_p}\cdot\frac{1}{\left(\frac{\partial \Upsilon}{\partial p}\right)}\cdot\left(\frac{\partial \Upsilon}{\partial T}\right)^2$$

$$= K_T\left[1 + \frac{T}{c_p}\cdot\frac{\left(\frac{\partial \Upsilon}{\partial T}\right)^2}{\left(\frac{\partial \Upsilon}{\partial p}\right)}\right]$$

$$= K_T \cdot \frac{c_v}{c_p}\left(\because c_v = c_p + T\frac{\left(\frac{\partial \Upsilon}{\partial T}\right)^2}{\left(\frac{\partial \Upsilon}{\partial p}\right)^2}\right)$$

$$\therefore \frac{K_T}{K_\eta} = \frac{c_p}{c_v} > 1 \ (\because c_p > c_v)$$

Hence $\quad K_T > K_\eta$.

Result : $\quad c^2 = \frac{1}{\rho K_\eta}$.

Proof. From the definition of the sound in sea-water we have

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_{\eta,s}}$$

and $K_\eta = -\frac{1}{\Upsilon}\left(\frac{\partial \Upsilon}{\partial p}\right)_{\eta,s}$

with $\Upsilon = \frac{1}{\rho}$

$$\therefore \log \Upsilon = -\log \rho$$

$$\Rightarrow \frac{d\Upsilon}{\Upsilon} = -\frac{d\rho}{\rho}$$

$$\Rightarrow \frac{d\Upsilon}{\partial \rho} = -\frac{\Upsilon}{\rho}.$$

So, $$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\eta,s}$$

$$= \left[\frac{\partial p}{\partial \Upsilon}\cdot\frac{\partial \Upsilon}{\partial \rho}\right]_{\eta,s}$$

$$= \left[-\frac{\Upsilon}{\rho}\frac{\partial p}{\partial \Upsilon}\right]_{\eta,s}$$

$$= \frac{1}{\rho K_\eta}\left[\because K_\eta = -\frac{1}{\Upsilon}\left(\frac{\partial \Upsilon}{\partial p}\right)_{\eta,s}\right].$$

15. Summary :

This module has been discussed the basic idea about sea-waer, and how the basic thermodynamics holds good in case of sea-water.

16. Self Assessment Questions :

1. Establish the Navier-Stokes equation of motion for compressible fluid. Hence obtain the equation motion for incompressible fluid.
2. What are the general ion composition of sea-water? What do you mean by salinity?
3. Define salinity and concentration. Show that $s + c_w = 1$.
4. Define 1st and 2nd laws of thermodynamics, specific heat, Chemical potential.
5. Obtain the Gibb's general thermodynamical relation for sea-water. Hence deduce Gibb's-Duhem relation.
6. Obtain the relations : $c_u = c_p + T\left(\frac{\partial \Upsilon}{\partial T}\right)^2 \Big/ \left(\frac{\partial \Upsilon}{\partial p}\right); \Gamma = \frac{T}{c_p}\left(\frac{\partial \Upsilon}{\partial T}\right); k_\eta = k_T - \Gamma\alpha; c^2 \rho k_\eta = 1.$
7. Define potential temp. and potential density of sea-water.
8. Show that $\eta dT - \Upsilon dp - (1-s)\,d\mu + d\mu_s = 0.$

9. Obtained the Gibb's relation for sea-water in T, p, s variables.

17. Further Suggested Readings :

1. Fundamentals of Ocean Dynamics : V.M. Kamenkovich translated by R. Radok.
2. The Dynamic Method in Oceanography : L.M. Fomin.
3. Boundary layer problem in Applied Mechanics: G.F. Carrier.

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M.Sc. Course in Applied Mathematics with Oceanology and Computer Programming

Part - II

Paper - VIII **Group - B**

Module No. 92
Dynamical Oceanography

Structure

1. Introduction :

This module is actually continuation of Module 91. This module will help to learn how the sea water form the thermodynamic equilibrium, mechanical equilibrium and equation of motion of the sea water etc.

2. Keywords :

Linear momentum, angular momentum, Pressure equation, stratified fluid, Brunt-väisälä frequency, conservation of mass, boundary condition, equation of motion, etc.

3. Objectives :

The main objectives of this module is to learn enough about the structure of equation of motion of sea-water referred to a frame rotating with the earth.

Conditions of Equilibrium (Thermodynamically) of Finite Volume Sea-Water :

Let us consider, V the volume of mass m of sea water in a state of thermodynamic equilibrium located in a field of stationary conservative forces with specific potential $U(P)$, where P is a point of the medium; in addition, also assume that separate parts of the fluid may undergo macroscopic motion with velocity $\vec{q}(P)$. For entropy of such system, we decompose the volume V into separate particles, which are somewhat small and such that within their limits the fields U and $\vec{q}$ may be assumed to be homogeneous, but at the same time sufficiently large so that the statistical concept of entropy may make sense for them. Then the entropy of each particle will depend only on its internal energy E, volume V, and composition; the presence of the fields U and $\vec{q}$ will not affect the magnitude of the entropy of separate particles.

Thus, by definition, the entropy of a finite volume V of the liquid will be

$$\eta' = \int_V \eta(P)\rho(P)\,dV,$$

where the specific entropy $\eta(P)$ is a function of the specific internal energy $E(P)$, the density of the medium ρ and the concentration of the admixture s.

The conditions of thermodynamic equilibrium will now be derived. Assume that the system under consideration is isolated; then its entropy must be a maximum and simultaneously the following laws of conservation must be fulfilled.

(i) The total impulse of the system must be constant (linear momentum), i.e.

$$\int_V \rho\vec{q}\,dV = \text{constant};$$

(ii) The total moment of momentum (angular momentum) of the system must be constant, i.e.,

$$\int_V \vec{r} \times (\rho\vec{q}\, dV) = \text{constant, where } OP = \vec{r};$$

(iii) The total energy of the system (sum of kinetic energy of macroscopic motions with velocity $\vec{q}$, potential energy U and internal energy E) must be constant, i..e.,

$$\int_V \rho\left[E + \frac{q^2}{2} + U\right] dV = \text{constant};$$

(iv) The mass of each component of the system must be constant, i.e.,

$$\int_V \rho_s\, dV = \text{constant},$$

and $$\int_V \rho_w\, dV = \text{constant}.$$

The relations (conditions) (i) to (iv), in essence, yield an exact formulation of what be understood by the condition of isolation of a system. Also we have seen that the density of the entropy $\rho\eta$ may be assumed to be a function of $\rho E, \rho_s, \rho_w$. However, the entropy η' of a finite volume V, by the constraint (iii), will, in general, depend not only on the fields $\rho E, \rho_s, \rho_w$, but also on the field $\rho\vec{q}$.

[Here note that the entropy of separate particles also does not depend on the field $\rho\vec{q}$; since the field U is stationary, it has not been included in the list of functions on which η' depends; for example, for the gravity field, $U = -gz$ with g the gravitational acceleration and z the downward vertical co-ordinate].

Thus, the determination of the conditions of equilibrium of a system has been reduced to an analysis of the extrema of the functional $\eta'(\rho E, \rho_s, \rho_w, \rho\vec{q})$ for the constraints (i) to (iv), imposed on the possibel functions $\rho E, \rho_s, \rho_w, \rho\vec{q}$.

Applying Lagrange's method of undetermined multipliers to find the extremum η', we introduce the auxiliary function L, as

$$L = \int_V \eta\rho\, dV + \lambda_s \int_V \rho_s dV + \lambda_w \int_V \rho_w dV + \lambda \int_V \rho\left[E + \frac{q^2}{2} + U\right] dV$$

$$+\vec{a}\cdot\int_V \rho\vec{q}\,dV+\vec{b}\cdot\int_V \vec{r}\times\rho\vec{q}\,dV, \quad \text{.......................... (1)}$$

where $\lambda_s, \lambda_w, \lambda, \vec{a}, \vec{b}$ are constant numbers and vectors.

Taking variation of L, then we have

$$\delta L=\int_V \delta(\rho\eta)dV+\lambda_s+\int_V \delta\rho_s\,dV+\lambda_w\int_V \delta\rho_w dV+\lambda\int_V \delta\left[\rho\left(E+\frac{q^2}{2}+U\right)\right]dV$$

$$+\vec{a}\cdot\int_V \delta(\rho\vec{q})dV+\vec{b}\cdot\int_V \delta(\vec{r}\times\rho\vec{q})dV \quad \text{.......................... (2)}$$

Again from Gibb's relation we have

$$T\,d\eta=dE+pd\Upsilon-\mu ds$$

$$=dE+pd\Upsilon-(\mu_s-\mu_w)ds \quad (\because \mu=\mu_s-\mu_w)$$

$$=dE+pd\Upsilon-\mu_s dc_s-\mu_w dc_w \quad (\because c_s=s, c_w=1-s)$$

Replacing specific parameters by their no specific values without changing the relation. Then we have

$$Td(\rho\eta)=d(\rho E)+pd(\rho\Upsilon)-\mu_s d(\rho c_s)-\mu_w d(\rho c_w)$$

$$=d(\rho E)-(\mu_s d\rho_s+\mu_w d\rho_w)\left[\because \Upsilon=\frac{1}{\rho}, c_s=\frac{m_s}{m}=\frac{\rho_s V}{\rho V}=\frac{\rho_s}{\rho}, c_w=\frac{m_w}{m}=\frac{\rho_w V}{\rho V}=\frac{\rho_w}{\rho}\right]$$

.... (3)

Also $\delta\left(\frac{1}{2}\rho q^2\right)=\frac{q^2}{2}\delta\rho+\frac{1}{2}\rho\delta q^2$

$$=\frac{q^2}{2}\delta\rho+\rho\vec{q}\cdot\delta\vec{q}+\vec{q}\cdot\vec{q}\,\delta\rho-q^2\delta\rho$$

$$=\frac{q^2}{2}\delta\rho+\vec{q}\cdot\delta(\rho\vec{q}) \quad \text{.................... (4)}$$

Using (3), (4) in (2), then we have

$$\delta L=\int_V \frac{1}{T}\left[\delta(\rho E)-\mu_s\delta\rho_s-\mu_w\delta\rho_w\right]dV+\lambda_s\int_V \delta\rho_s dV+\lambda_w\int_V \delta\rho_w dV$$

$$+\lambda\iint_V\left[\delta(\rho E)+\delta(\rho_s+\rho_w)\left(-\frac{q^2}{2}+U\right)+\vec{q}\cdot\delta(\rho\vec{q})dV\right]$$

$$+\vec{a}\cdot\int_V\delta(\rho\vec{q})dV+\vec{b}\cdot\int\vec{r}\times\delta(\rho\vec{q})dV$$

or, $$\delta L=\iint_V\left[\left(\frac{1}{T}+\lambda\right)\delta(\rho E)+\left\{-\frac{\mu_s}{T}+\lambda_s+\lambda\left(-\frac{q^2}{2}+U\right)\right\}\delta\rho_s\right.$$

$$+\left\{-\frac{\mu_w}{T}+\lambda_w+\lambda\left(-\frac{q^2}{2}+U\right)\right\}\delta\rho_w$$

$$\left.+\left\{\lambda\vec{q}+\vec{a}+\vec{b}\times\vec{r}\right\}\delta(\rho\vec{q})\right]dV$$

For extremum, we must have, $\delta L=0$ and so coefficient of the above integrand equal to zero and therefore the coefficients of variation of each independent variables $\rho E, \rho_s, \rho_w, \rho\vec{q}$ are equal to zero.

Hence,

$$\frac{1}{T}+\lambda=0 \qquad \Rightarrow T=-\frac{1}{\lambda},$$

$$-\frac{\mu_s}{T}+\lambda_s+\lambda\left(-\frac{q^2}{2}+U\right)=0 \qquad \Rightarrow \mu_s=T\lambda_s+T\lambda\left(-\frac{q^2}{2}+U\right),$$

$$-\frac{\mu_w}{T}+\lambda_w+\lambda\left(-\frac{q^2}{2}+U\right)=0 \qquad \Rightarrow \mu_w=T\lambda_w+T\lambda\left(-\frac{q^2}{2}+U\right),$$

and $$\lambda\vec{q}+\vec{a}+\vec{b}\times\vec{r}=\vec{0} \qquad \Rightarrow \vec{q}=-\frac{\vec{a}}{\lambda}-\frac{1}{\lambda}\left(\vec{b}\times\vec{r}\right).$$

which are equivalent to

$$T=-\frac{1}{\lambda},$$

$$\mu_s=-U-\frac{\lambda_s}{\lambda}+\frac{q^2}{2},$$

$$\mu_w=-U-\frac{\lambda_w}{\lambda}+\frac{q^2}{2},$$

$$\vec{q} = -\frac{\vec{a}}{\lambda} - \frac{\vec{b} \times \vec{r}}{\lambda}.$$

These are also the necessary conditions of thermodynamic equilibrium of a finite volume of sea water. Thus, in an equilibirum state :

I. The temperature T is constant throughout the entire volume of fluid.

II. The chemical potentials μ_s and μ_w differ only by a constant amount.

III. The volume of fluid may move only like a rigid body with velocities of translation $-\frac{\vec{a}}{\lambda}$ and of rotation $-\frac{\vec{b}}{\lambda}$.

Deduction :

If the fluid is at rest, then $\vec{q} = 0$.

Also in thermodynamical equilibrium, we have

$$T = \text{constant}, \ \mu = \text{constant}.$$

Then Gibb's-Duhem relation

$$\eta dT - \Upsilon dp + s d\mu + d\mu_w = 0$$

becomes

$$\Upsilon dp = d\mu_w$$

$$= d\left[-U - \frac{\lambda_w}{\lambda} + \frac{q^2}{2}\right]$$

$$= -dU$$

$$\therefore \frac{dp}{\rho} = X\,dx + Y\,dy + Z\,dz,$$

where $\Upsilon = \frac{1}{\rho}$ and $X = -\frac{\partial U}{\partial x}, Y = -\frac{\partial U}{\partial y}, Z = -\frac{\partial U}{\partial z}$ are force components per unit mass along the co-ordinate axes.

This is well-known differential equation for hydrostatic pressure.

It is seen that in the presence of mass forces X, Y, Z the pressure p cannot be a constant quantity. However, then also the salinity cannot be constant. In fact, since $T, \mu =$ constant, then

$$\frac{\partial\mu}{\partial p}dp+\frac{\partial\mu}{\partial s}ds=0 \quad \left(\because \mu=\mu(T,p,s),T=\text{constant}\right)$$

or, $$\frac{\partial\mu}{\partial p}\nabla_{\alpha}p+\frac{\partial\mu}{\partial s}\nabla_{\alpha}s=0, \text{ in } x^{\alpha} \text{ co-ordinates.}$$

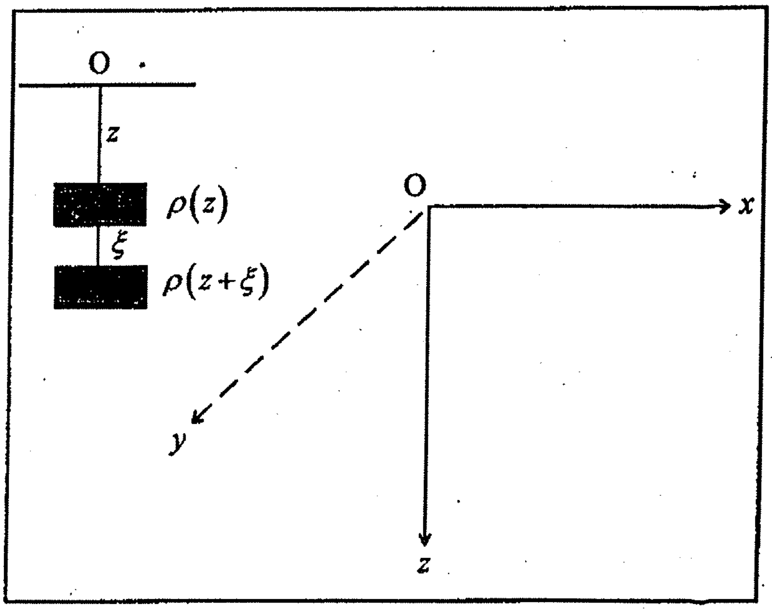
which is useful in estimating equilibrium of vertical salinity gradient $\frac{\partial s}{\partial z}$ in a gravitational force field.

Since $s\frac{\partial\mu}{\partial s}\simeq 7.5\times10^{8} erg\cdot g^{-1}\left|\frac{\partial\mu}{\partial p}\right|\simeq 1 erg\cdot g^{-1}\left(dyne.cm^{2}\right)^{-1}$,

then $\frac{\partial s}{\partial z}\simeq 1.3\times10^{-4}\cdot s\left(m^{-1}\right)$, which differs essentially from what is observed in the oceans.

5. Condition for the Absence of Convection : Väisälä Frequencey

Assume that the temperature of sea water T, the salinity s and the density ρ, and also all other thermodynamical parameters depend only on the vertical co-ordinate z (increasing downwards). Such a fluid is said to be stratified.



Since the temperature T of the fluid is not constant, i.e., $T\neq$ constant, then it cannot find itself in a state of thermodynamic equilibrium.

However, let it be assumed that the fluid is in a state of mechanical equilibrium and study the condition of stability of such an equilibrium (the condition of absence of convection).

Let a particle of fluid, located at the level z, move adiabatically to the nearby level $z+\xi$. Assume that at each instant of time the thermodynamic state of the particle may be assumed to be an equilibrium state. The density of the particle and of the surrounding medium at the level z is $\rho(z)$. But at a depth $z+\xi$, the density of the particle is

$$[\rho(z+\xi)]_{\eta=\text{constant}, s=\text{constant}}$$

while the density of the surrounding medium is $\rho(z+\xi)$.

Then, net vertical upward force (by Archimedis) per unit volume on the particle is

(weight of the displaced fluid – weight of the particle) / (unit volume)

$$=g\left\{\rho(z+\xi)-[\rho(z+\xi)]_{\eta=\text{constant}, s=\text{constant}}\right\}$$

$$=g\left[\left\{\rho(z)+\xi\frac{d\rho}{dz}+\ldots\right\}-\left\{\rho(z)+\xi\left(\frac{d\rho}{dz}\right)_{\eta,s}+\ldots\right\}\right] \text{ [Taylor expansion]}$$

$$=g\left[\rho(z)+\xi\frac{d\rho}{dz}-\rho(z)-\xi\left(\frac{d\rho}{dz}\right)_{\eta,s}\right] \text{ (neglecting higher order of small quantities)}$$

$$=g\left[\frac{d\rho}{dz}-\left(\frac{d\rho}{dz}\right)_{\eta,s}\right]\xi$$

Clearly, this force will tend to return the particle to its former (original) level z only if

$$g\left[\frac{d\rho}{dz}-\left(\frac{d\rho}{dz}\right)_{\eta,s}\right]>0$$

for stability of equilibrium,

$$\text{i.e., } \frac{d\rho}{dz}>\left(\frac{d\rho}{dz}\right)_{\eta,s} \quad \ldots\ldots\ldots\ldots\text{(1)}$$

This is the well known condition of stability of the equilibrium of a stratified fluid (the condition of absence of convection).

We supposed that this condition (1) is fulfilled. Then the equation of the small vertical motion of the particle is

$$\rho\frac{d^2}{dt^2}(z+\xi)=-g\left[\frac{d\rho}{dz}-\left(\frac{d\rho}{dz}\right)_{\eta,s}\right]\xi$$

$$\text{or, } \frac{d^2\xi}{dt^2}=-\frac{g}{\rho}\left[\frac{d\rho}{dz}-\left(\frac{d\rho}{dz}\right)_{\eta,s}\right]\xi$$

or, $\ddot{\xi} = -N^2\xi$, say, ... (2)

where $N^2 = \frac{g}{\rho}\left[\frac{d\rho}{dz} - \left(\frac{d\rho}{dz}\right)_{\eta,s}\right]$ .... (3)

The equation (2) is the equation of S.H.M. about $\xi = 0$ as mean position with N its circular frequency.

Thus, if a stratified fluid is stable, mechanical equilibrium of a fluid particle when slightly displaced from its equilibrium position will perform small oscillation about this mean position with circular frequency N. This frequency, an important parameter of a stratified medium, is referred to as Brunt-Väisälä frequency (this name not being used universally).

Note :

If the fluid is incompressible, then

$$\left(\frac{d\rho}{dz}\right)_{\eta,s} = 0 \quad (\because \text{ volume of an element remains unchange})$$

and hence condition of stability would be simply

$$\frac{d\rho}{dz} > 0$$

and $N^2 = \frac{g}{\rho}\cdot\frac{d\rho}{dz} > 0.$

5.1 Different Forms of N² :

a) **Interms of c²:**

The speed of the sound in the sea-water is

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\eta,s}.$$

Now $\left(\frac{d\rho}{dz}\right)_{\eta,s} = \left(\frac{\partial \rho}{\partial z}\right)_{\eta,s}\cdot\frac{dp}{dz} = \frac{g\rho}{c^2} \quad \left(\because \frac{dp}{dz} = g\rho\right)$

Hence, $N^2 = \frac{g}{\rho}\left[\frac{d\rho}{dz} - \left(\frac{d\rho}{dz}\right)_{\eta,s}\right]$

$$= \frac{g}{\rho}\left[\frac{d\rho}{dz} - \frac{g\rho}{c^2}\right]$$

$$= \frac{g}{\rho}\frac{d\rho}{dz} - \frac{g^2}{c^2} \quad \ldots\ldots\ldots\ldots\ldots (4)$$

In the upper layers of the sea (outside a homogeneous layer), the first term of the R.H.S. of (4) significantly exceeds the second term, i.e., second term is negligible compared to the first, and in this case

$$N^2 = \frac{g}{\rho}\frac{d\rho}{dz}$$

and stability condition is $\frac{d\rho}{dz} > 0$.

b) **Interms of T:**

We assume that T, p, s are independent variable and ρ be a function of T, p, s i.e., $\rho = \rho(T, p, s)$, equation of state, then

$$\frac{d\rho}{dz} = \left(\frac{\partial\rho}{\partial T}\right)_{p,s}\frac{dT}{dz} + \left(\frac{\partial\rho}{\partial p}\right)_{T,s}\frac{dp}{dz} + \left(\frac{\partial\rho}{\partial s}\right)_{T,p}\cdot\frac{ds}{dz}.$$

But we have

$$N^2 = \frac{g}{\rho}\left[\frac{d\rho}{dz} - \left(\frac{d\rho}{dz}\right)_{\eta,s}\right]$$

So, $$N^2 = \frac{g}{\rho}\left[\left(\frac{\partial\rho}{\partial T}\right)_{p,s}\cdot\frac{dT}{dz} + \left(\frac{\partial\rho}{\partial p}\right)_{T,s}\frac{dp}{dz} + \left(\frac{\partial\rho}{\partial s}\right)_{T,p}\frac{ds}{dz} - \left(\frac{d\rho}{dz}\right)_{\eta,s}\right]$$

$$= \frac{g}{\rho}\left[\left(\frac{\partial\rho}{\partial T}\right)_{p,s}\frac{dT}{dz} + \left(\frac{\partial\rho}{\partial p}\right)_{T,s}\frac{dp}{dz} + \left(\frac{\partial\rho}{\partial s}\right)_{T,p}\frac{ds}{dz} - \left(\frac{\partial\rho}{\partial p}\right)_{\eta,s}\frac{dp}{dz}\right]$$

$$\left(\because \left(\frac{d\rho}{dz}\right)_{\eta,s} = \left(\frac{\partial\rho}{\partial p}\right)_{\eta,s}\cdot\frac{dp}{dz}\right)$$

$$= \frac{g}{\rho}\left[\left(\frac{\partial\rho}{\partial T}\right)_{p,s}\cdot\frac{dT}{dz} + \left(\frac{\partial\rho}{\partial s}\right)_{T,p}\cdot\frac{ds}{dz} + \frac{dp}{dz}\left\{\left(\frac{\partial\rho}{\partial p}\right)_{T,s} - \left(\frac{\partial\rho}{\partial p}\right)_{\eta,s}\right\}\right]$$

$$= \frac{g}{\rho}\left[\left(\frac{\partial\rho}{\partial T}\right)_{p,s}\frac{dT}{dz} + \left(\frac{\partial\rho}{\partial s}\right)_{T,p}\frac{ds}{dz} + g\rho\left\{-\frac{\rho}{\Upsilon}\left(\frac{\partial\Upsilon}{\partial p}\right)_{T,s} + \frac{\rho}{\Upsilon}\left(\frac{\partial\Upsilon}{\partial p}\right)_{\eta,s}\right\}\right]$$

$$\left[\because \rho=\frac{1}{\Upsilon} \text{and} \frac{dp}{dz}=\rho g, \frac{\partial\rho}{\rho}=-\frac{d\Upsilon}{\Upsilon}\right.$$

$$\left.\Rightarrow\left(\frac{\partial\rho}{\partial p}\right)_{T,s}=\left[\frac{\partial\rho}{\partial\Upsilon}\cdot\frac{\partial\Upsilon}{\partial p}\right]_{T,s}=-\frac{\rho}{\Upsilon}\left(\frac{\partial\Upsilon}{\partial p}\right)_{T,s}\right]$$

$$\therefore N^2=\frac{g}{\rho}\left[\left(\frac{\partial\rho}{\partial T}\right)_{p,s}\frac{dT}{dz}+\left(\frac{\partial\rho}{\partial s}\right)_{T,p}\cdot\frac{ds}{dz}+g\rho^2\left(k_T-k_\eta\right)\right]$$

$$\left[\because k_T=-\frac{1}{\Upsilon}\left(\frac{\partial\Upsilon}{\partial p}\right)_{T,s}, k_{\eta}=-\frac{1}{\Upsilon}\left(\frac{\partial\Upsilon}{\partial p}\right)_{\eta,s}\right]$$

$$=\frac{g}{\rho}\left[\left(\frac{\partial\rho}{\partial T}\right)_{p,s}\frac{dT}{dz}+\left(\frac{\partial\rho}{\partial s}\right)_{T,p}\cdot\frac{ds}{dz}+g\rho^2\Gamma\alpha\right]\text{......................} (5)$$

$$\left[\because k_T-k_\eta=\Gamma\alpha\right]$$

Here k_T and k_η are the co-efficients of isothermal compression and adiabatic compression, Γ is adiabatic temperature, α is the co-efficient of thermal expansion.

$$\alpha=\frac{1}{\Upsilon}\frac{\partial\Upsilon}{\partial\rho}\cdot\left(\frac{\partial\rho}{\partial T}\right)_{p,s}=-\frac{1}{\Upsilon}\cdot\frac{\Upsilon}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{p,s}=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{p,s}$$

Using above in (5), we get,

$$N^2=\frac{g}{\rho}\left[\left(\frac{\partial\rho}{\partial T}\right)_{p,s}\frac{dT}{dz}+\left(\frac{\partial\rho}{\partial s}\right)_{T,p}\frac{ds}{dz}-g\rho\Gamma\left(\frac{\partial\rho}{\partial T}\right)_{p,s}\right]$$

$$=\frac{g}{\rho}\left[\left(\frac{\partial\rho}{\partial T}\right)_{p,s}\left(\frac{dT}{dz}-g\rho\Gamma\right)+\left(\frac{\partial\rho}{\partial s}\right)_{T,p}\cdot\frac{ds}{dz}\right]\text{..............} (6)$$

In the upper layer of sea-water $g\rho\Gamma\sim10^{-4}\ ^\circ C$ per meter i.e., there is a change of 0.01°C per hundred meters of depth. If further the salinity gradient is not large, we have then

$$N^2=\frac{g}{\rho}\left[\left(\frac{\partial\rho}{\partial T}\right)_{p,s}\cdot\frac{dT}{dz}\right]\text{................} (7)$$

for characteristic temperature in the upper layer, the stability condition is that $\frac{dT}{dz} > 0$, since $\left(\frac{\partial \rho}{\partial T}\right)_{p,s} < 0$ for such temperature.

In the layers where temperature and salinity variations with depth are large, we may omit the term containing $-g\rho\Gamma$ in (6) and take the resulting expression as the value of N^2, i.e.,

c) **Interms of C_p and C_v:**

Again we assume that T, p, s are independent variables and $\rho = \rho(T, p, s)$ then we have

$$\frac{d\rho}{dz} = \left(\frac{\partial \rho}{\partial T}\right)_{p,s}\frac{dT}{dz} + \left(\frac{\partial \rho}{\partial p}\right)_{T,s}\frac{dp}{dz} + \left(\frac{\partial \rho}{\partial s}\right)_{T,p}\frac{ds}{dz}$$

$$= \left[\left(\frac{d\rho}{dz}\right)_p + \left(\frac{\partial \rho}{\partial p}\right)_{T,s}\rho g\right] \quad \left[\because \frac{dp}{dz} = \rho g\right]$$

where $\left(\frac{d\rho}{dz}\right)_p = \left[\left(\frac{\partial \rho}{\partial T}\right)_{p,s}\frac{dT}{dz} + \left(\frac{\partial \rho}{\partial s}\right)_{T,p}\frac{ds}{dz}\right]$

$$\therefore N^2 = \frac{g}{\rho}\left[\frac{d\rho}{dz} - \left(\frac{d\rho}{dz}\right)_{\eta,s}\right]$$

$$= \frac{g}{\rho}\left[\left(\frac{d\rho}{dz}\right)_p + \left(\frac{\partial \rho}{\partial p}\right)_{T,s}\rho g - \frac{g\rho}{c^2}\right] \left(\because \left(\frac{d\rho}{dz}\right)_{\eta,s} = \frac{g\rho}{c^2}\right)$$

$$= \frac{g}{\rho}\left[\left(\frac{d\rho}{dz}\right)_p + \left(\frac{\partial \rho}{\partial p}\right)_{T,s}\rho g - \left(\frac{\partial \rho}{\partial p}\right)_{\eta,s}\cdot \rho g\right] \quad \left[\because c^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\eta,s}\right]$$

$$= \frac{g}{\rho}\left[\left(\frac{d\rho}{dz}\right)_p + \frac{\rho g}{\left(\frac{\partial p}{\partial \rho}\right)_{\eta,s}}\left\{\frac{\left(\frac{\partial \rho}{\partial p}\right)_{T,s}}{\left(\frac{\partial \rho}{\partial p}\right)_{\eta,s}} - 1\right\}\right]$$

$$= \frac{g}{\rho}\left[\left(\frac{d\rho}{dz}\right)_p + \frac{\rho g}{c^2}\left\{\frac{-\frac{\rho}{\Upsilon}\left(\frac{\partial \Upsilon}{\partial p}\right)_{T,s}}{-\frac{\rho}{\Upsilon}\left(\frac{\partial \Upsilon}{\partial p}\right)_{\eta,s}} - 1\right\}\right]$$

$$= \frac{g}{\rho}\left[\left(\frac{d\rho}{dz}\right)_p + \frac{\rho g}{c^2}\left(\frac{k_T}{k_\eta} - 1\right)\right]$$

$$= \frac{g}{\rho}\left[\left(\frac{d\rho}{dz}\right)_p + \frac{\rho g}{c^2}\left(\frac{c_p}{c_v} - 1\right)\right] \quad \ldots\ldots\ldots\ldots\ldots (8)$$

This formula is convenient for obtaining an estimate N within the limits of homogeneous layers, where the temperature and salinity vary little with depth. On the boundary in those layers of the ocean where $\frac{dT}{dz}$ and $\frac{ds}{dz}$ are large, the second term in (8) may be omitted (since $c_p \simeq c_v$). Therefore, in those layers where T and s varies significantly with depth, then we have

$$N^2 = \frac{g}{\rho}\left(\frac{d\rho}{dz}\right)_p.$$

Again, since the density of sea-water varied very little from 1 gm/c.c., it is convenient to use the parameter

$$\sigma_t(T, p, s) = 10^3\left[\rho(T, p, s) - 1\,\text{gm/c.c.}\right]$$

Assuming that $\left(\frac{d\rho}{dz}\right) \simeq 10^{-3}\left(\frac{d\sigma_t}{dz}\right)$, one obtains an approximate formula for the [illegible]ion of N in those layer where T and s vary essentially with depth:

In conclusion, it should be pointed out that in the ocean the smallest value of N is of the order of $10^{-3} \div 10^{-4}\ \text{sec}^{-1}$ (corresponding to periods of $1.7 - 17$ hours), while the largest value of N, usually attained in the seasonal thermocline, is of an order of $10^{-2}\ \text{sec}^{-1}$ (corresponding to a period of ~ 10 minutes).

6. Thermodynamic Parameters in a Non-equilibrium State :

Hitherto, only states of thermodynamic equilibrium have been considered when the internal state of a system is characterized completely by such parameters as, E, Υ, c_s and c_w. As it has been seen, for an equilibrium state, one may introduce entropy η as a function E, Υ, c_s and c_w. Further, changes of the function of state have been studied for transition from one equilibrium state to another (equilibrium processes) and conditions of thermodynamic equilibrium of a finite fluid volume have been derived.

Now, non-equilibrium processes of transition in a fluid medium will be studied.

Assume that the characteristic relaxation time of the system (time of transition to an equilibrium state) decreases as the dimensions of the system decrease. Therefore sub-divide this system into a set of somewhat small particles (containing, however, a large number of molecules) in order that the relaxation time of each particle will be significantly shorter than the characteristic time scale of the process under consideration. Then it may be assumed in approximation that at any instant of time any particle finds itself in a state of thermodynamic equilibrium, and for each particle the entropy may be determined as an equilibrium function of its internal energy E, volume V and composition, i.e., $\eta = \eta(E, \Upsilon, c_k)$. After this, temperature, pressure, chemical potentials, etc. may be determined by ordinary means. In this manner, Gibb's relation proves to be valid for each particle and, consequently, also all formulae of equilibrium thermodynamics.

In the sequel, entropy of a non-equilibrium system will be understood to be the sum of the entropies of all equilibrium particles into which the original system has been decomposed. By strength of the extensiveness of the entropy of an equilibrium system, further decomposition of equilibrium particles into small parts does not affect the magnitude of the entropy of a non-equilibrium system. Thus, the entropy of a finite volume V of a fluid volume is defined by

$$\eta'(t) = \int_V \rho(P,t)\eta(P,t)dV$$

where $\rho(P,t), \eta(P,t)$ are density and specific entropy at any point P of the medium.

Similarly, the other extensive thermodynamic parameters for a finite volume of a continuous medium may be defined as temperature $T(P, t)$, pressure $p(P, t)$, specific entropy $\eta(P,t)$, specific internal energy $E(P, t)$, etc. and assume Gibb's relation for specific quantities to be true at each point of the medium.

It should be emphasized that, in contrast to the specific entropy η, the entropy η' of a finite volume V of a fluid medium depends, of course, not only on the internal energy E of this volume, the magnitude of the volume V and its composition c_k, but also on a number of other parameters (naturally, this statement does not relate to the case when this volume is in an equilibrium state). Thus approximation is referred to as the approximation of local thermodynamic equilibrium. It is clear intuitively that for systems with not very large gradients in the basic parameters the approximation above must be true. In what follows, consideration will be restricted to just such systems.

The basic physical laws for continuous fluid media will now be formulated.

Equation of Conservation of Mass :

Assume that each component of a mixture (sea-water as two components, salt and purewater) may be considered as a continuous medium with its own velocity field.

Let $\rho_s, \rho_w, \vec{q}_s, \vec{q}_w$, resapectively, be the densities and velocities of the salt component and pure water. Suppose there is no chemical reaction taking place between the two components, we apply the principle of conservation of mass to each of the two moving deforming indivisual volume V_s and V_w, each of which always consists of one and the same particles of salt and pure water. This gives

$$\frac{D}{Dt}\int_{V_s}\rho_s\, dV = 0 \quad\text{.................}\ (1)$$

$$\frac{D}{Dt}\int_{V_w}\rho_w\, dV = 0 \quad\text{.................}\ (2)$$

Now, $\frac{D}{Dt}\int_{V_s}\rho_s\, dV$ = rate of change of salt mass within volume V_s with change of time only +

{efflux – inlfux} of salt mass across the surface Σ,

$$= \frac{\partial}{\partial t}\int_{V_s}\rho_s dV + \int_{\Sigma_{exit}}\rho_s\vec{q}_s\cdot\hat{n}\, d\Sigma - \int_{\Sigma_{entry}}\rho_s\vec{q}_s\cdot\hat{n}\, d\Sigma$$

where $\hat{n}$ is the outward drawn unit normal to Σ.

$$\therefore \frac{\partial}{\partial t}\int_{V_s}\rho_s dV + \int_{\Sigma_{exit\text{-}entry}}\rho_s\vec{q}_s\cdot\hat{n}\, d\Sigma = 0$$

or, $\int_{V_s}\frac{\partial\rho_s}{\partial t}dV + \int_{V_s}\vec{\nabla}\cdot(\rho_s\vec{q}_s)dV = 0$ (by Gauss div. theorem)

or, $\int_{V_s}\left[\frac{\partial\rho_s}{\partial t} + \vec{\nabla}\cdot(\rho_s\vec{q}_s)\right]dV = 0$

Since the volume is arbitrary, so we must have

$$\frac{\partial \rho_s}{\partial t}+\vec{\nabla}\cdot(\rho_s\vec{q}_s)=0 \quad \text{.................. (3)}$$

By similar manner, from (2), we get

$$\frac{\partial \rho_w}{\partial t}+\vec{\nabla}\cdot(\rho_w\vec{q}_w)=0 \quad \text{.................. (4)}$$

Let $\vec{q}$ be the velocity of the centre of inertia of a particle of salt and a particle of pure water, then

$$\vec{q}=\frac{\rho_s\vec{q}_s+\rho_w\vec{q}_w}{\rho_s+\rho_w}$$

$$=\frac{\rho_s\vec{q}_s+\rho_w\vec{q}_w}{\rho}, \text{ where } \rho=\rho_s+\rho_w = \text{density of sea-water.}$$

Adding (3) and (4) we get

$$\frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot(\rho \underline{q})=0 \quad \text{..................... (5)}$$

or, $\frac{\partial \rho}{\partial t}+(\vec{q}\cdot\vec{\nabla})\rho+\rho\vec{\nabla}\cdot\vec{q}=0 \quad \left(\because \vec{\nabla}\cdot\phi\vec{A}=\vec{A}\cdot\vec{\nabla}\phi+\phi\vec{\nabla}\cdot\vec{A}\right)$

or, $\frac{D\rho}{Dt}+\rho\vec{\nabla}\cdot\vec{q}=0$

$\left[\because \frac{D}{Dt}\equiv\frac{\partial}{\partial t}+\vec{q}\cdot\vec{\nabla}\right]$

Again, we introduce

$$\left.\begin{aligned} \vec{I}_s&=\rho_s(\vec{q}_s-\vec{q}) \\ \text{and } \vec{I}_w&=\rho_w(\vec{q}_w-\vec{q}) \end{aligned}\right\} \quad \text{..................... (7)}$$

Then (3) becomes,

$$\frac{\partial \rho_s}{\partial t}+\vec{\nabla}\cdot(\vec{I}_s+\rho_s\vec{q})=0 \quad \text{................. (8)}$$

Also, $\frac{\rho_s}{\rho}=\frac{\rho_s V}{\rho V}=\frac{m_s}{m}=s$, salinity

$\therefore \rho_s=\rho s$

Hence, (8) becomes

$$\frac{\partial}{\partial t}(\rho s)+\vec{\nabla}\cdot(s\rho\vec{q})=-\vec{\nabla}\cdot\vec{I}_s$$

or, $$\rho\frac{\partial s}{\partial t}+\rho\vec{q}\cdot\vec{\nabla}s+s\frac{\partial \rho}{\partial t}+s\vec{\nabla}\cdot(\rho\vec{q})=-\vec{\nabla}\cdot\vec{I}_s$$

or, $$\rho\left(\frac{\partial s}{\partial t}+\vec{q}\cdot\vec{\nabla}s\right)+s\left(\frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{q})\right)=-\vec{\nabla}\cdot\vec{I}_s$$

or, $$\rho\frac{Ds}{Dt}+s\cdot 0=-\vec{\nabla}\cdot\vec{I}_s \text{ (using (5))}$$

or, $$\rho\frac{Ds}{Dt}=-\vec{\nabla}\cdot\vec{I}_s \quad \text{.................. (9)}$$

Equations (1) and (2) or (5) or (6) and (9) may be taken as the two equations of conservation of mass. We note that, equation (5) is identical with the equation of continuity.

Note :

(i) The vectors $\vec{I}_s$ and $\vec{I}_w$ are normally referred as diffusive transport of mass of salt and pure water.

(ii) Equation (8) is normally referred as diffusion equation for salt.

(iii) The vectors $\rho\vec{q}, \rho_s\vec{q}$ and $\rho_w\vec{q}$ characterize advective transport of mass (caused by macroscopic motion of sea water particles with velocity $\vec{q}$).

(iv) The vectors $\rho\vec{q}, \rho_s\vec{q}+\vec{I}_s, \rho_w\vec{q}+\vec{I}_w$ are normally referred as vectors of density of mass fluxes of sea-water, salt and pure water respectively.

(v) Adding two equation in (7) and using $(\rho_s+\rho_s)\vec{q}=\rho_s\vec{q}_s+\rho_w\vec{q}_w$ we see that $\vec{I}_s+\vec{I}_w=\vec{0}$.

8. Boundary Condition at the Free Ocean Surface :

Let $F(\vec{r},t)=0$ be the equation of the free ocean surface. Let us investigate the kinematic (boundary) conditions to be satisfied at the surface, when mass exchange across it is taken into accounts, the sea-water being supposed to be a two components of mixture of salt and pure water. Such mass exchange may be in the form of evaporation, precipitation, formation and thawing of ice. The total effect of these processes may be described by specification of the flux of pure water b in unit time per unit area. Then one has that at the ocean surface

$$F(\vec{r},t)=0$$

$$\rho_w(\vec{q}_w-\vec{q}_F)\cdot\hat{n}=b \quad \text{............. (1)}$$

$$\rho_s(\vec{q}_s - \vec{q}_F)\cdot\hat{n} = O \text{ .... (2)}$$

where $\vec{q}_F = \dfrac{d\vec{r}}{dt}$ is the velocity of the motion of points of the surface F; $\vec{q}_s, \vec{q}_w$ are velocities of salt, pure water particle at $\vec{r}; \hat{n} = \dfrac{\vec{\nabla}F}{|\vec{\nabla}F|}$ is the normal to this surface, and it has been assumed in writing down (1) and (2) that

$b > 0$, if $\vec{\nabla}F$ is directed into the ocean

and $b < 0$, if $\vec{\nabla}F$ is directed out of the ocean.

To find $\vec{q}_F$ we differentiate the equation $F(\vec{r}, t) = 0$ w.r.t. 't' then we have

$$\frac{\partial F}{\partial t} + \vec{\nabla}F \cdot \frac{d\vec{r}}{dt} = 0 \text{ .... (3)}$$

Now, $\vec{q}_F \cdot \hat{n} = \dfrac{d\vec{r}}{dt}\cdot\hat{n}$

$$= \frac{d\vec{r}}{dt}\cdot\frac{\vec{\nabla}F}{|\vec{\nabla}F|}$$

$$= \left(\frac{d\vec{r}}{dt}\cdot\vec{\nabla}F\right)\frac{1}{|\vec{\nabla}F|}$$

$$= \frac{1}{|\vec{\nabla}F|}\frac{\partial F}{\partial t} \text{ [using (3)] .... (4)}$$

Again adding (1) and (2), then we get

$$\{(\rho_w\vec{q}_w + \rho_s\vec{q}_s) - (\rho_w + \rho_s)\vec{q}_F\}\cdot\hat{n} = b$$

or, $(\rho\vec{q} - \rho\vec{q}_F)\cdot\hat{n} = b \quad \left[\because \vec{q} = \dfrac{\rho_w\vec{q}_w + \rho_s\vec{q}_s}{\rho_w + \rho_s}, \rho = \rho_s + \rho_w\right]$

or, $\vec{q}\cdot\hat{n} - \vec{q}_F\cdot\hat{n} = \dfrac{b}{\rho}$

or, $\vec{q}\cdot\hat{n} = \vec{q}_F\cdot\hat{n} = \dfrac{b}{\rho}$ .... (5)

or, $\vec{q}\cdot\hat{n} = -\frac{1}{|\vec{\nabla}F|}\frac{\partial F}{\partial t} = \frac{b}{\rho}$ [using (4)]

or, $\frac{\partial F}{\partial t} + \vec{q}\cdot\vec{\nabla}F = \frac{b}{\rho}|\vec{\nabla}F|$

or, $\frac{DF}{Dt} = \frac{b}{\rho}|\vec{\nabla}F|$ .... (6) $\left[\because \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{q}\cdot\vec{\nabla}\right]$

Since, $\vec{I}_s = \rho_s(\vec{q}_s - \vec{q})$

so, from (2) we get

$$\left(\vec{I}_s + \rho_s\vec{q} - \rho_s\vec{q}_F\right)\cdot\hat{n} = 0$$

or, $\rho_s(\vec{q} - \vec{q}_F)\cdot\hat{n} = -\vec{I}_s\cdot\hat{n}$ .... (7)

Also, $\rho_s V = m_s = \frac{m_s}{m}\cdot m = s\rho V$

i.e., $\rho_s = \rho s$ .... (8)

Using (5) and (8) in (7), then we get

$$s\rho\frac{b}{\rho} = -\vec{I}_s\cdot\frac{\vec{\nabla}F}{|\vec{\nabla}F|}$$

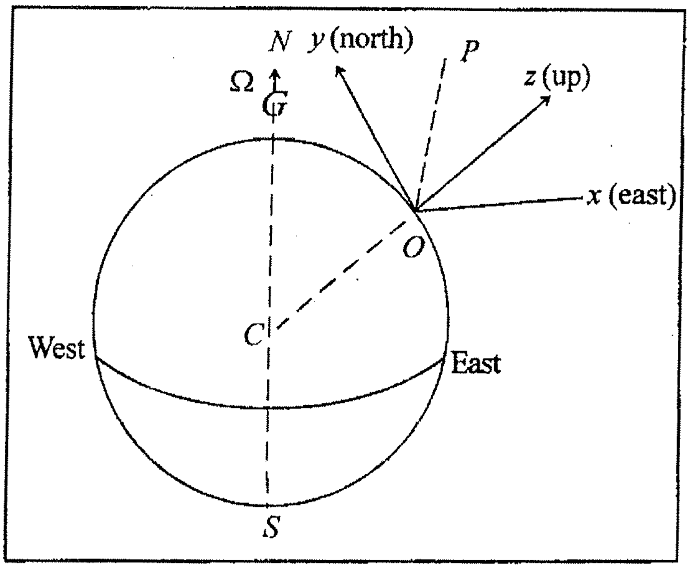
or, $sb = -\vec{I}_s\cdot\frac{\vec{\nabla}F}{|\vec{\nabla}F|}$

or, $\vec{I}_s\cdot\vec{\nabla}F = -sb|\vec{\nabla}F|$ .... (9)

Equations (6) and (9) are the required final form of the boundary conditions at the free ocean surface $F(\vec{r},t) = 0$.

9. Motion of a Particle Referred to a Frame Rotating with the Earth :

As a rule, it may be assumed that the Earth has the shape of a sphere and rotates with constant angular velocity Ω : $\Omega = 7.29\times10^{-5}$ rad./sec. about its north-south axis from west to east. It is natural to consider motion of sea water from the point of view of an earthbound observer. However, then one is forced to operate with a non-inertial reference system and to take in the equations of motion inertia forces into account, i.e., centripetal and Coriolis forces.



Let C be the centre of the earth, O the observer and P the particle at any time t such that $\overline{OP} = \vec{r}$, NS being the earth's axis of rotation, Oz vertically upwards, Oy horizontal line perpendicular to Oz in the meridian plane pointing towards north, Ox the horizontal line perpendicular to the meridian plane pointing towards east. Thus $Oxyz$ is a rectangular right handed system rotating with the earth.

Let $\frac{d}{dt}$ and $\frac{d'}{dt}$ refer to the rate of change with respect to the rotating frame $Oxyz$ and fixed frame with which $Oxyz$ coincides instanteniously at time t. Then absolute velocity of P is

$\frac{d'\vec{r}}{dt}$ = velocity of P relative to O + velocity of O (both w.r.t. the fixed frame)

= velocity of P relative to the rotating frame at O

+ velocity of P due to rotating of $Oxyz$

+ absolute velocity of O.

$$\text{or,}\quad \frac{d'\vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\Omega}\times\vec{r} + \vec{v}_o$$

The absolute acceleration of P is

$$\frac{d'^2\vec{r}}{dt^2} = \frac{d'}{dt}\left(\frac{d'\vec{r}}{dt}\right)$$

$$= \frac{d'}{dt^2}\left[\frac{d\vec{r}}{dt} + \vec{\Omega}\times\vec{r} + \vec{v}_0\right]$$

$$= \frac{d'}{dt}\left[\frac{d\vec{r}}{dt} + \vec{\Omega}\times\vec{r}\right] + \vec{f}_0$$

$$= \left(\frac{d}{dt} + \vec{\Omega}\right)\left(\frac{d\vec{r}}{dt} + \vec{\Omega}\times\vec{r}\right) + \vec{f}_0$$

$$=\frac{d^2\vec{r}}{dt^2}+2\vec{\Omega}\times\frac{d\vec{r}}{dt}+\vec{\Omega}\times(\vec{\Omega}\times\vec{r})+\vec{f}_0$$

Equation of absolute motion of P is

$$\frac{d^2\vec{r}}{dt^2}=\vec{A}+\vec{F}$$

where $\vec{A}$ = earth's force of attraction,

and $\vec{F}$ = all forces other than $\vec{A}$ per unit mass.

Thus the equation may be written as

$$\frac{d^2\vec{r}}{dt^2}=\vec{A}+\vec{\Omega}\times(\vec{\Omega}\times\vec{r})+2\frac{d\vec{r}}{dt}\times\vec{\Omega}+\vec{F} \quad \text{..................} (1)$$

Now, $\vec{\Omega}\times\vec{r}$ = a velocity vector perpendicular to the plance CNP (C, O, P approximately co-linear) subject to $\vec{\Omega},\vec{r},\vec{\Omega}\times\vec{r}$ are right handed,

$\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$ = an acceleration vector perpendicular to $\vec{\Omega}$ in the meridian plane CNP subject to $\vec{\Omega},\vec{\Omega}\times\vec{r},\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$ are a right handed,

= centrifugal acceleration of P,

$\vec{A}-\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$ = resultant of earth's attraction and centrifugal acceleration of P,

= an acceleration having a line of action very near $\vec{A},\vec{\Omega}$ is very small

= $\vec{g}$

= acceleration due to gravity.

Here note that the term $2\frac{d\vec{r}}{dt}\times\vec{\Omega}$ in (1) is known as the Coriolis acceleration.

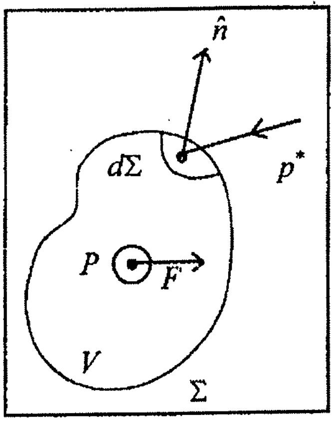
Thus the equation of relative motion of P is

$$\frac{d^2\vec{r}}{dt^2}=\vec{g}+\vec{F}+2\times\frac{d\vec{r}}{dt}\times\vec{\Omega}$$

which is identical with the equation of motion of the earth $(\vec{\Omega}=\vec{0})$ and an additional mass force equal to $2m\ \vec{q}\times\vec{r}$ where imposed on P, $\vec{q}$ being the velocity of P relative to O.

10. Equation of Motion of Sea-water :

Let us consider the motion of an arbitrary moving volume consisting of one and the same particles of sea-water. Let $p^*, \rho, \vec{q}$ be the pressure, density and velocity at any point P enclosed by an arbitrary volume dV. Let $\vec{F}$ be the external force per unit mass at P and $2\vec{q}\times\vec{\Omega}$ denotes the Coriolis force per unit mass at $P, \vec{\Omega}$ denotes the earth's angular velocity; the motion may be referred to a frame rotating with the earth, while the earth is assumed as fixed, provided this force is induced amongst the massforce acting at P.



Also the fluid surrounding V exerts on its surface Σ. A force $\vec{\varphi}(\hat{n})$ per unit area acts on each surface element $d\Sigma$ and $\vec{\varphi}$ depends on $\hat{n}$.

Now, the reverse effective force at P

$$= -\rho\frac{D\vec{q}}{Dt}dV,$$

and the external body force at P

$$= \left(\vec{F} + 2\vec{q}\times\vec{\Omega}\right)\rho\, dV,$$

and the surface force on the boundary Σ

$$= \sum_{\Sigma}\vec{\varphi}(\hat{n})d\Sigma.$$

By D'Alembert's principle, the reverse effective force and the external force acting at a point P together with similar forces acting at every other point of V and the surface forces on Σ form a system of equilibrium.

The conditions of equilibrium, then gives,

(i) $$\iint_V\left(-\frac{D\vec{q}}{Dt} + \vec{F} + 2\vec{q}\times\vec{\Omega}\right)\rho\, dV + \int_\Sigma \vec{\varphi}(\hat{n})d\Sigma = \vec{O} \quad\text{............ (1)}$$

(ii) $$\int_V \vec{r}\times\left(-\frac{D\vec{q}}{Dt} + \vec{F} + 2\vec{q}\times\vec{\Omega}\right)\rho\, dV + \int_\Sigma \vec{r}\times\vec{\varphi}(\hat{n})d\Sigma = \vec{O} \quad\text{............ (2)}$$

where $OP = \vec{r}$.

Using a Cartesian orthogonal co-rodinate system with base vectors e_1, e_2, e_3 and letting the volume of the tetrabedron with vertex at P and edges along the co-ordinate axes vanish, then one obtains

$$\vec{\varphi}(\hat{n})=\vec{\varphi}(e_1)n_1+\vec{\varphi}(e_2)n_2+\vec{\varphi}(e_3)n_3$$

where $\hat{n}=n_i e_i$, $i=1,2,3$.

and $\varphi(e_j)=\varphi_{ji}e_i$ .... (3)

Again, by divergence theorem, we have

$$\int_{\Sigma}\vec{\Phi}(\hat{n})\,d\Sigma=\int_{V}\left(\Phi_{ji}e_i\right)_{,j}dV \text{ .... (4)}$$

Using (4) in (1), we get

$$\int_{V}\left[\rho\left(-\frac{D\vec{q}}{Dt}+\vec{F}+2\vec{q}\times\vec{\Omega}\right)+\left(\Phi_{ji}e_i\right)_{,j}\right]dV=0$$

Since the volume V is an arbitrary, the integrand vanishes at any point of continuity and hence we have

$$\rho\left(-\frac{D\vec{q}}{Dt}+\vec{F}+2\vec{q}\times\vec{\Omega}\right)+\left(\Phi_{ji}e_i\right)_{,j}=0$$

or, $$\frac{D\vec{q}}{Dt}=\vec{F}+2\vec{q}\times\vec{\Omega}+\frac{1}{\rho}\left(\Phi_{ji}e_i\right)_{,j} \text{ ....(5)}$$

For an isotropic body, the stress-strain rate relations are

$$\Phi_{ji}=\left(-p^*+\lambda\Theta\right)\delta_{ij}+\mu e_{ij}$$

where p^* = mean pressure,

Θ = rate of cubical dilation,

$\lambda=-\frac{2}{3}\mu$, μ is the coefficient of viscosity,

and u_i = component of $\vec{q}$.

Then,

$$\Phi_{ji,j}=\sum_j\left(-\frac{\partial p^*}{\partial x_j}+\lambda\frac{\partial\Theta}{\partial x_j}\right)\delta_{ij}+\mu\left(\frac{\partial^2 u_i}{\partial x_j^2}+\frac{\partial^2 u_j}{\partial x_i\partial x_j}\right)$$

$$=-\frac{\partial p^*}{\partial x_i}-\frac{2}{3}\mu\frac{\partial\Theta}{\partial x_i}+\mu\nabla^2 u_i+\mu\frac{\partial\Theta}{\partial x_i}\left[\because u_{i,j}=0\right]$$

$$= -\frac{\partial p^*}{\partial x_i} - \frac{\mu}{3}\frac{\partial \Theta}{\partial x_i} + \mu \nabla^2 u_i$$

Hence, $$\left(\Phi_{ji} e_i\right)_{,j} = -\frac{\partial p^*}{\partial x_i} + \mu\left[\frac{1}{3}\frac{\partial \Theta}{\partial x_i} + \nabla^2 \mu_i\right] e_i$$

$$= \vec{\nabla} p^* + \mu\left[\frac{1}{3}\vec{\nabla}\Theta + \vec{\nabla}\vec{q}\right] \quad \text{................ (6)}$$

Using (6) in (5), then we get

$$\frac{D\vec{q}}{Dt} = \vec{F} + 2\vec{q} \times \vec{\Omega} - \frac{1}{\rho}\vec{\nabla} p^* + \gamma\left[\frac{1}{3}\vec{\nabla}\Theta + \vec{\nabla}^2 \vec{q}\right] \quad \text{.............. (7)}$$

where $\gamma = \frac{\mu}{\rho}$, the kinematic co-efficeint of viscosity.

Equation (7) is called the equation of motion of sea-water.

Another form :

Equation (7) can be written as following form.

$$\frac{D\vec{q}}{Dt} = \vec{F} + 2\vec{q} \times \vec{\Omega} - \frac{1}{\rho}\vec{\nabla} p^* + \frac{\mu}{\rho}\cdot\frac{1}{3}\vec{\nabla}\Theta + \lambda \nabla^2 \vec{q}$$

$$= \vec{F} + 2\vec{q} \times \vec{\Omega} - \frac{1}{\rho}\vec{\nabla} p^* + \frac{1}{\rho}\left(1 - \frac{2}{3}\right)\mu \vec{\nabla}\Theta + \gamma \nabla^2 \vec{q}$$

$$= \vec{F} + 2\vec{q} \times \vec{\Omega} - \frac{1}{\rho}\vec{\nabla} p^* + \frac{1}{\rho}\left(\mu - \frac{2}{3}\mu\right)\vec{\nabla}\Theta + \gamma \nabla^2 \vec{q}$$

$$= \vec{F} + 2\vec{q} \times \vec{\Omega} - \frac{1}{\rho}\vec{\nabla} p^* + \frac{1}{\rho}(\lambda + \mu)\vec{\nabla}\Theta + \gamma \nabla^2 \vec{q}$$

where $\lambda = -\frac{2}{3}\mu$.

10.1 Symmetric Stress Tensor :

Condition (ii) is similar to the condition of equilibrium obtained by deformable body which expresses the vanishing of the moment of the forces about the origin.

This condition thus leads analogously to the result that the stress tensor σ_{ji} is symmetric.

For instance let

$$\vec{G} = \rho\left(-\frac{D\vec{q}}{Dt} + \vec{F} + 2\vec{q}\times\vec{\Omega}\right).$$

Then the first term of L.H.S. of (2) is

$$= \int_V \vec{r}\times\vec{G}\,dV$$

$$= \int_V x_j e_j \times G_k e_k dV$$

$$= \int_V e_j \times e_k x_j G_k dV$$

$$= \int_V e_{ijk} e_i x_j G_k dV$$

Similarly, the second term of L.H.S. of (2) is

$$= \int_V \vec{r}\times\vec{\sigma}(\hat{n})\,d\Sigma$$

$$= \int_\Sigma e_{ijk} e_i x_j \sigma_{lk} x_i\, d\Sigma$$

$$= \int_V \left(e_{ijk} e_i x_j x_i \sigma_{lk}\right)_{,l} dV \quad \text{[using Gauss div. theorem]}$$

$$= \int_V \left\{e_{ijk} e_i \left(x_j \sigma_{lk,l} + \sigma_{lk}\delta_{il}\right)\right\} dV$$

and hence condition (ii) becomes

$$= \int_V e_{ijk} e_i \left\{x_j\left(G_k + \sigma_{lk,l}\right) + \sigma_{lk}\delta_{il}\right\} dV = 0$$

or, $= \int_V e_{ijk} e_i \sigma_{lk}\delta_{il} dV = 0$ [by the equation of motion $G_k = -\sigma_{lk,l}$]

Since the volume is arbitrary, therefore

$$e_{ijk} = 0.$$

Putting i=1, j=2, k=3, then we get

$$\sigma_{23} = \sigma_{32}$$

and similarly from other components of σ_{jk} and hence σ_{jk} is symmetric stress thensor.

11. Summary :

In this module we have discussed equilibrium state and non-equilibrium state of sea water in simple way which has been used to deduct the question of conservatic~ of mass, condition (boundary) at the free ocean surface and last of all equation of motion of sea water w.r.t. rotatins earth. We hope that student has learn how the different types of forces involved in case of motion of sea-water.

12. Self Assessment Questions :

1. Show that the necessary conditions of thermodynamical equilibrium of a finite volume of sea-water are

$$T = -\frac{1}{\lambda}, \mu_s = -U - \frac{\lambda_s}{\lambda} + \frac{q^2}{2}, \mu_w = -U - \frac{\lambda_w}{\lambda} + \frac{q^2}{2}, \vec{q} = -\frac{\vec{a}}{\lambda} - \frac{\vec{b} \times \vec{r}}{\lambda}.$$

2. Define stratified fluid. Discuss Brunt-Väisälä frequency.

   Express this frequency interms of c; T; c_p and c_v.

3. Express the principle of conservation of mass in the form of

$$\frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{q} = 0, \rho \frac{Ds}{Dt} = -\vec{\nabla} \cdot \vec{I}_s$$

4. Obtain the boundary conditions at the free-ocean surface $F(\vec{r}, t) = 0,$ in terms of

$$\frac{DF}{Dt} = \frac{b}{\rho} \left| \vec{\nabla} F \right|, \vec{I}_s \cdot \frac{\vec{\nabla} F}{\left| \vec{\nabla} F \right|} = -b\,s.$$

5. Obtain the equation of motion of sea water in the form

$$\frac{D\vec{q}}{Dt} = \vec{F} + 2\vec{q} \times \vec{\Omega} - \frac{1}{\rho} \vec{\nabla} p + \frac{\gamma}{3} \vec{\nabla} \left(\vec{\nabla} \cdot \vec{q} \right) + \gamma \vec{\nabla} \vec{q}.$$

   Hence

13. Further suggested Readings :

1. Fundamentals of Ocean Dynamics : V.M. Kamenkovich translated by R. Radok.
2. Dynamic Method in Oceanography : L.M. Fomin.
3. Boundry Layer Problem of Applied Mechanics : G.F. Carrier.

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M.Sc. Course in
Applied Mathematics with Oceanology
and
Computer Programming

Part - II

Paper - VIII **Group - B**

Module No. 93
Dynamical Oceanography

Structure

1. Introduction :

This Module is the continuation of previous model which is also last part of the Dynamical Oceanography.

2. Objectives :

The main objective of the this module is to learn about linearised equation of small amplitude oceanian wave motion on a rotating earth, and some approximation which is very helpful for higher study.

3. Keywords :

Wave motion on a rotating earth, β-phase approximation, Boussinesq approximation, conservation of energy etc.

4. Oscilation of Layer of Liquid :

As usual, during an analysis of wave motions, dissipative processes (friction, heat conduction, diffusion) will be neglected. The starting point will be the equation of motion, of conservation of mass of sea water, of diffusion of salt and of evolution of entropy. In the case under consideration, the number of these equations may be reduced.

If the velocity of sound is assumed a known function of the pressure and density of the medium, then the three equations of motion (without friction), the equation of conservation of mass of sea water and the equation

$$\frac{d\rho}{dt}=\left(\frac{\partial\rho}{\partial p}\right)_{\eta,s}\frac{dp}{dt}$$

contain only the five unknown function $\vec{q}(u,v,w), p, \rho$. In other words, a closed system of equations has been obtained.

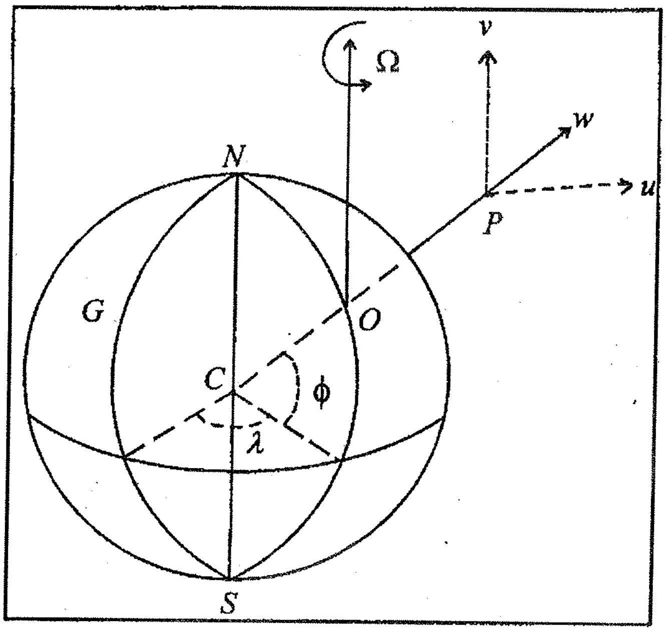
In the sequel, wave motions in the ocean will be considered as small oscillations of layer of liquid of constant depth H in the gravity field.

5. Linearised Equations of Small Amplitude Oceanian Wave Motion on a Rotating Earth :

Let us consider the motion of a layer of fluid of constant depth H in a gravity field on a rotating earth (assume spherical).

Let z be the upward vertical distance of a point P about the undisturbed surface. Let ϕ and λ be the geographical latitude and longitude of P, then $r = a+z, \frac{\pi}{2}-\varphi, \lambda$ where $0<\lambda<2\pi, -\frac{\pi}{2}<\varphi<\frac{\pi}{2}$ are spherical polar co-ordinates of P, relative to earth centre C as pole and with earth axis CN and Greenwich meridian G as the initial line.

Let w, v, u denotes the velocity components (relative to earth) in the direction of increasing of z, the direction perpendicular to the radius vector $\overline{OP}$ in the meridian plane in φ increasing sence and the direction perpendicular to the meridian plane in λ increasing sence (i.e., vertical, meridian and zonal components of velocity) respectively. When dissipative forces (destructive) such as viscosity, heat conduction, and diffusion are neglected but the Coriolis force and the apparent gravity are taken into account, the equation of motion referred to the rotating frame of reference takes the form



$$\frac{\partial \vec{q}}{\partial t}+\left(\vec{q}\cdot\vec{\nabla}\right)\vec{q}\equiv\frac{D\vec{q}}{Dt}=-g\hat{k}_1+2\vec{q}\times\vec{\Omega}-\frac{1}{\rho}\vec{\nabla}p \text{ ....(1)}$$

where $\hat{k}_1$ is the unit vector along z-direction.

If the motion be so small that squares and products of velocities and their derivatives are negligible. So, the second term on the extreme left hand side i.e., $\left(\vec{q}\cdot\vec{\nabla}\right)\vec{q}$ may be omitted. Then equation of small motion is

$$\frac{\partial \vec{q}}{\partial t}=-g\hat{k}_1+2\left(u\hat{i}_1+v\hat{j}_1+w\hat{k}_1\right)\times\left(0\hat{i}_1+\Omega\cos\phi\,\hat{j}_1+\Omega\sin\phi\,\hat{k}_1\right)$$

$$-\frac{1}{\rho}\left(\frac{1}{r\cos\varphi}\frac{\partial p}{\partial \lambda}\hat{i}_1+\frac{1}{r}\frac{\partial p}{\partial \varphi}\hat{j}_1+\frac{\partial p}{\partial r}\hat{k}_1\right)$$

where $\hat{i}_1, \hat{j}_1, \hat{k}_1$ are the unit vectors along $-x$, $-y$, $-z$ axes, respectively.

Moreover, the traditional approximation to the coriolis force is to neglect the components $2\Omega u\cos\varphi$ and $2\Omega w\cos\varphi$ (since $2\Omega u\cos\varphi$ and $-2\Omega w\cos\varphi$ containing the vertical components of $2\vec{q}\times\vec{\Omega}$ and vertical velocity of w) as so y is very small compared with their horizontal components.

Also, $z << a$

$\therefore r \simeq a$ .... (3)

Further, if we assume that the motion in a small perturbation (disturbance) from the equilibrium state, the density and pressure may be written as

$$\left.\begin{aligned} p &= p_o(z) + p' \\ \rho &= \rho_o(z) + \rho' \end{aligned}\right\} \text{ .... (4)}$$

where ρ_o is the equilibrium density and

$$p_o(z) = p_a - \int_o^z \rho_o(z)\, g\, dz$$

is the hydrostatic pressure and p', ρ' are small quantities of first order.

Now, $$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{1}{\rho}\frac{\partial p_o}{\partial z} + \frac{1}{\rho}\frac{\partial p'}{\partial z}$$

$$= \frac{1}{\rho_o + \rho'}\left[(-g\rho_o) + \frac{\partial p'}{\partial z}\right] (\because \rho = \rho_o + \rho')$$

$$= \frac{1}{\rho_o\left(1 + \frac{\rho'}{\rho_o}\right)}\left[-g\rho_o + \frac{\partial p'}{\partial z}\right]$$

$$= \left(1 + \frac{\rho'}{\rho_o}\right)^{-1}\left[-g + \frac{1}{\rho_o}\cdot\frac{\partial p'}{\partial z}\right]$$

$$= \left(1 - \frac{\rho'}{\rho_o}\right)\left[-g + \frac{1}{\rho_o}\cdot\frac{\partial p'}{\partial z}\right] \text{ (neglecting all other small quantities)}$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial p}{\partial r} = -g + g\frac{\rho'}{\rho_o} + \frac{1}{\rho_o}\frac{\partial p'}{\partial z} \text{ .... (5)}$$

(neglecting the term containing ρ_o^2)

Thus, the equation of motion (2) takes the following form :

$$\frac{\partial \bar{q}}{\partial t}=-g\,\hat{k}_1+2\Omega\left(-u\sin\varphi\,\hat{j}_1+v\sin\varphi\,\hat{i}_1\right)-\frac{1}{\rho_o}\left(\frac{1}{a\cos\varphi}\frac{\partial p'}{\partial\lambda}\hat{i}_1+\frac{1}{a}\frac{\partial p'}{\partial\varphi}\hat{j}_1+\frac{\partial p'}{\partial z}\hat{k}_1\right)$$

$$+g\hat{k}_1-\frac{g\rho'}{\rho_o}\hat{k}_1\ \ (\because r=a)$$

or, $$\frac{\partial \bar{q}}{\partial t}=2\Omega\left(-u\sin\varphi\,\hat{j}_1+v\sin\varphi\,\hat{i}_1\right)$$

$$-\frac{1}{\rho_o}\left(\frac{1}{a\cos\varphi}\frac{\partial p'}{\partial\lambda}\hat{i}_1+\frac{1}{a}\frac{\partial p'}{\partial\varphi}\hat{j}_1+\frac{\partial p'}{\partial z}\hat{k}_1\right)-\frac{g\rho'}{\rho_o}\hat{k}_1 \text{..........} (6)$$

This is the linearised form of the equation of small amplitude waves. In scalar notation the equation of motion in λ,φ,z directions are :

$$\frac{\partial u}{\partial t}=2\Omega v\sin\varphi-\frac{1}{\rho_o}\frac{1}{a\cos\varphi}\frac{\partial p'}{\partial\lambda} \text{...............} (7)$$

$$\frac{\partial v}{\partial t}=-2\Omega u\sin\varphi-\frac{1}{\rho_o}\cdot\frac{1}{a}\frac{\partial p'}{\partial\varphi} \text{...............} (8)$$

$$\frac{\partial w}{\partial t}=-g\frac{\rho'}{\rho_o}-\frac{1}{\rho_o}\frac{\partial p'}{\partial z} \text{...............} (9)$$

Next, the equations expressing conservation of mass are

$$\frac{D\rho}{Dt}+\rho\vec{\nabla}\cdot\vec{q}=0$$

and $$\rho\frac{Ds}{Dt}=-\vec{\nabla}\cdot\vec{I}_s=0\ \left(\because \vec{I}_s=0\right)$$

Equation of entropy transfer is

$$\frac{D\eta}{Dt}=0\left[\because \vec{Q}=\vec{O},\vec{I}_s=\vec{O},\varphi=O^o,\frac{Ds}{Dt}=O\right]$$

From (3) and (4), the form of these equation becomes :

$$\left(\frac{\partial}{\partial t}+u\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}+v\cdot\frac{1}{a}\frac{\partial}{\partial\varphi}+w\frac{\partial}{\partial z}\right)(\rho_o+\rho')+(\rho_o+\rho')\vec{\nabla}\cdot\vec{q}=O$$

or, $\dfrac{\partial\rho'}{\partial t}+w\dfrac{\partial\rho_o}{\partial z}+\rho_o\vec{\nabla}\cdot\vec{q}=0$

Again considering ρ as a function of η, p, s, then we have

$$\frac{D\rho}{Dt}=\left(\frac{\partial\rho}{\partial p}\right)_{\eta,s}\frac{Dp}{Dt}=\frac{1}{c^2}\frac{Dp}{Dt} \text{ where } c^2=\left(\frac{\partial p}{\partial\rho}\right)_{\eta,s}.$$

By the approximation used to derive (10), the above equation becomes

$$\frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}+w\frac{\partial\rho_o}{\partial z}=\frac{1}{c^2}\frac{Dp}{Dt}$$

$$=\frac{1}{c^2}\left(\frac{\partial p'}{\partial t}+w\frac{\partial p_o}{\partial z}\right)$$

$$=\frac{1}{c^2}\left(\frac{\partial p'}{\partial t}-g\rho_o w\right)\text{............ (11)}$$

$$\left(\because\frac{\partial p_o}{\partial z}=-\rho_o g\right)$$

Hence, $\dfrac{1}{c^2}\left(\dfrac{\partial p'}{\partial t}-g\rho_o w\right)+\rho_o\vec{\nabla}\cdot\vec{q}=0$ [using (11) in (10)]

or, $\dfrac{\partial p'}{\partial t}-g\rho_o w+\rho_o c^2\vec{\nabla}\cdot\vec{q}=0$ .... (12)

Therefore, the equations (7) to (9) (10) and (12) are the five linearised governing equations for determination of five unknows u, v, w, p', ρ' in small amplitude wave motion, i.e.,

$$\frac{\partial u}{\partial t}-2\Omega v\sin\varphi=-\frac{1}{\rho_o}\cdot\frac{1}{a\cos\varphi}\frac{\partial p'}{\partial\lambda},$$

$$\frac{\partial v}{\partial t}+2\Omega u\sin\varphi=-\frac{1}{\rho_o}\frac{1}{a}\frac{\partial p'}{\partial\varphi},$$

$$\frac{\partial w}{\partial t}=-\frac{1}{\rho_o}\frac{\partial p'}{\partial z}-g\frac{\rho'}{\rho_o},$$

$$\frac{\partial\rho'}{\partial t}+w\frac{\partial\rho_o}{\partial z}+\rho_o\,div\,\vec{q}=0,$$

and $\dfrac{\partial p'}{\partial t}-w\rho_o g+\rho_o c^2\text{div}\,\vec{q}=0.$

Note 1.

The coriolis force in above equations has been presented in the so called 'traditional' approximation; its complete expression is $2\Omega v(-2\Omega v\sin\varphi + 2\Omega w\cos\varphi, 2\Omega u\sin\varphi, -2\Omega u\cos\varphi)$. The accuracy of this approximation is not always clear. As a rule, it is based on the smallness of vertical velocities compared with horizontal ones, but this condition is only true for long waves and, besides, at the equator $(\varphi = 0^\circ)$ the neglected terms $2\Omega w\cos\varphi, -2\Omega u\cos\varphi$ may turn out to be significant. However, the problem would be considerably more complicated, if the complete expressions of the Coriolis forces were retained (since it would preclude the ensuing employment of the method of separation of variables).

Note 2. Linearization of Boundary Conditions :

Let the equation of the free ocean surface be

$z = \tau(\lambda, \varphi, t)$, (where τ is the sea level) and the wave amplitude be << wave length.

The dynamic free surface condition is that pressure should be continuous accorss the surface.

$$\therefore p_o + p' = p_a \text{ on } z = \tau = 0.$$

Linearization of the dynamic condition for small τ yields

$$p_a = p_o(\tau) + p'(\tau)$$

$$= p_o(0) + \tau\left(\frac{dp_o}{dz}\right)_{z=0} + p'(0) + \ldots\ldots + \ldots.$$

$$= p_a - g\rho_o(0)\tau + p'(0) \quad \text{(1st order approximation and } p_o = p_a\text{)}$$

$$\therefore p'(0) = \rho_o(0)g\tau. \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

The kinematic free surface condition (when evaporation, precipitation, thawing and formation of ice are absent) is

$$\frac{D}{Dt}(z - \tau) = 0 \quad \text{on } z = \tau$$

or, $$\left\{\frac{\partial}{\partial t} + \left(\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\varphi} + w\frac{\partial}{\partial z}\right)\right\}(z - \tau) = O \text{ on } z = \tau$$

or, $$w - \frac{\partial\tau}{\partial t} = 0 \text{ on } z = \tau$$

i.e., $w = \frac{\partial \tau}{\partial t}$ on $z = \tau$ .... (14)

Thus, the boundary conditions at the free ocean surface may be written in the form :

$$p' = g\rho_o \tau, w = \frac{\partial \tau}{\partial t} \quad \text{for } z = 0;$$

and, also obviously, the boundary condition at the sea floor i.e., at the bottom of the ocean is

$$w = 0 \;\; \text{for } z = -H.$$

which are the required boundary conditions for linearized wave motion.

Note 3.

Linearizing wave motion for incompressible fluid :

For incompressible fluid, we must have

$$\frac{D\rho}{Dt} = 0$$

which gives

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_o}{\partial z} = 0 \text{ .... } (15)$$

and the equation of continuity becomes

$$\vec{\nabla} \cdot \vec{q} = 0 \text{ .... } (16)$$

Hence equations (7), (8), (9), (10), (12) and (15), (16) are the governing equations of incompressible fluid for small amplitude ocean wave having the same boundary conditions in above Note 2.

6. Some Approximations and their Analysis :

Under conditions characteristic for the ocean, energies of different types of wave motions may differ strongly from each other. For example, as a rule, the energy of acoustic waves in the ocean is negligibly small compared with that of other types of waves. Therefore there arises often the problem of filtration of those and other waves. In this context, it is expedient to discuss "filtering" properties of approximations which are employed in the theory of wave motion.

6.1 Boussinesq's Approximation :

In the Boussinesq's approximation we assume that :

i) The gradient of the basic parameters T, p, s, q_j namely $\nabla_i T, \nabla_i p, \nabla_i s. \nabla_i q_j$ are small as also the mach number $\frac{q}{c}$ of the flow is very small.

ii) The actual density distribution differ only slightly from the mean ocean surface density ρ_o, the density at the free surface in the equilibrium state or any other reference state, while the vertical scale of the motion is small compared with the local scale height $\frac{c^2}{g}$.

iii) Gravity is the only external force.

The governing equations of motion are

$$\frac{D\rho}{Dt}+\rho\vec{\nabla}\cdot\vec{q}=O \quad \text{.............(1)}$$

$$\rho\frac{Ds}{Dt}=-\vec{\nabla}\cdot\vec{I}_s \quad \text{.................. (2)]}$$

$$\frac{D\vec{q}}{Dt}=\vec{F}+2\vec{q}\times\vec{\Omega}-\frac{1}{\rho}\vec{\nabla}p+\gamma\left(\frac{1}{3}\vec{\nabla}\Theta+\nabla^2\vec{q}\right) \quad \text{............. (3)}$$

$$\rho\left[\frac{DE}{Dt}+p\frac{D\Upsilon}{Dt}\right]=-\vec{\nabla}\cdot\vec{Q}-\vec{\nabla}\left(\frac{\partial I}{\partial s}\right)_{T,p}\cdot\vec{I}_s+\varphi \quad \text{............. (4)}$$

where, the heat flux density $\vec{Q}$, the diffusive salt flux density $\vec{I}_s$ and dissipation function φ for sea water (assumed isotropic) are given by.

$$\vec{Q}=-K\vec{\nabla}T+K_T\left(\frac{\partial\mu}{\partial s}\right)_{T,p}\vec{I}_s \quad \text{................. (5)}$$

$$\vec{I}_s=-\rho D\left(\vec{\nabla}s+K_T\frac{\vec{\nabla}T}{T}+\frac{K_p}{p}\vec{\nabla}p\right) \quad \text{............. (6)}$$

and $\varphi=\lambda\Theta^2+\mu\left(u_{j,i}+u_{i,j}\right)$.

The dissipation function φ involves squares and products of derivatives of velocity components and is therefore a small quantity of second order by hypothesis (*i*).

By the same hypothesis we see from (5) and (6) that both $\left|\vec{I}_s\right|$ and $\left|\vec{Q}\right|$ are small quantity of second order and so we may replace ρ in equation (2) by ρ_o (as per condition (ii)), to get the approximate equation

$$\rho_o \frac{Ds}{Dt} = -\vec{\nabla}\cdot\vec{I}_s \text{ .... (7)}$$

But Gibb's function, $J = E + p\Upsilon - T\eta$ gives

$$dJ = dE + pd\Upsilon - Td\eta + \Upsilon dp - \eta dT$$

$$= \mu\, ds + \Upsilon\, dp - \eta\, dT \;\left(\because dE = Td\eta - pd\Upsilon + \mu ds\right)$$

= exact differential

so, $\dfrac{\partial\mu}{\partial p} = \dfrac{\partial\Upsilon}{\partial s}, \dfrac{\partial\Upsilon}{\partial T} = -\dfrac{\partial\eta}{\partial p}, \dfrac{\partial\mu}{\partial T} = -\dfrac{\partial\eta}{\partial s}.$

Again, $\eta = \eta(p, T, s)$

$$\therefore d\eta = \frac{\partial\eta}{\partial p}dp + \frac{\partial\eta}{\partial T}dT + \frac{\partial\eta}{\partial s}ds$$

$$= -\frac{\partial\Upsilon}{\partial T}dp + \frac{1}{T}\frac{\partial Q}{\partial T}dT - \frac{\partial\mu}{\partial T}ds \;\left(\because \delta Q = T\,d\eta\right)$$

$$= -\frac{\partial\Upsilon}{\partial T}dp + \frac{c_p}{T}dT - \frac{\partial\mu}{\partial T}ds \left(\because c_p = \frac{\partial Q}{\partial T} = T\frac{\partial\eta}{\partial T}\right)$$

Hence, in T, p, s variables, we have

$$d\eta = -\frac{\partial\Upsilon}{\partial T}dp + \frac{c_p}{T}dT - \frac{\partial\mu}{\partial T}ds \text{ ....(8)}$$

Since dT, dp, ds are small, so by the hypothesis (i), it is clear that

$\left|\vec{\nabla}\eta\right|$ is a small quantity of first order. .... (9)

Again the heat content function (specific enthalpy) I,

$$I = E + p\Upsilon$$

gives $dI = dE + pd\Upsilon + \Upsilon dp$

$$= Td\eta + \mu ds + \Upsilon dp \;\left(\because Td\eta = dE + pd\Upsilon - \mu ds\right)$$

which implies that $\left|\vec{\nabla}I\right|$ is a small quantity of first order.

Hence $\vec{\nabla}\left(\dfrac{\partial I}{\partial s}\right)_{T,p}\cdot\vec{I}_s$ is a small quantity of second order.

Therefore, the equation (4) becomes after replacing ρ by ρ_o and neglecting the last two terms on R.H.S.,

$$\rho_o\left[\frac{DE}{Dt}+p\frac{D\Upsilon}{Dt}\right]=-\vec{\nabla}\cdot\vec{Q}$$

Further using Gibb's relation

$$Td\eta = dE + pd\Upsilon - \mu ds$$

we get, $\rho_o\left[T\frac{D\eta}{Dt}+\mu\frac{Ds}{Dt}\right]=-\vec{\nabla}\cdot\vec{Q}$ .... (10)

Since, $\mu=-T\left(\frac{\partial\eta}{\partial s}\right)_{E,\Upsilon}$ so μ is the first order of smallness. Hence $\mu\frac{Ds}{Dt}$ is the second order of smallness.

Therefore, from (8), we have

$$T\frac{\partial\eta}{DT}=c_p\frac{DT}{Dt}-T\frac{Dp}{Dt}\cdot\frac{\partial\Upsilon}{\partial T}-T\frac{\partial\mu}{\partial T}\cdot\frac{Ds}{Dt}$$

$$\simeq c_p\left[\frac{DT}{Dt}-\frac{T}{c_p}\frac{\partial\Upsilon}{\partial T}\cdot\frac{Dp}{Dt}\right].$$

$$=c_p\left[\frac{DT}{Dt}-\Gamma\frac{Dp}{Dt}\right]\left(\text{since }\Gamma=\frac{T}{c_p}\frac{\partial\Upsilon}{\partial T}\right)$$

As $T\frac{\partial\mu}{\partial T}\cdot\frac{Ds}{Dt}$ is a small quantity of second order and $\Gamma=\frac{T}{c_p}\frac{\partial\Upsilon}{\partial T}$, adiabatic temperature gradient, and hence equation (10) becomes

$$\rho_o c_p\left(\frac{Dt}{Dt}-\Gamma\frac{Dp}{Dt}\right)=-\vec{\nabla}\cdot\vec{Q} \text{ .... (11)}$$

$(\because \mu\frac{Ds}{Dt}$ is small).

It is estimated that on lowering particles from the ocean surface to a depth of $2000m$, $\Gamma dp \sim \frac{1}{2}^{\circ}C$, at the same time temperature gradient $dT \sim 10^{\circ}C$ for horizontal displacement the contribution of the term Γdp is quite insignificant. Therefore, in the upper layer of the ocean the second term of (11) on L.H.S. may be omitted.

Therefore, the equation of energy becomes,

$$K_o \frac{DT}{Dt} = -\vec{\nabla} \cdot \vec{Q} \text{ .... (12)}$$

where $K_o = \rho_o c_p$.

Next, from the equation (1), we have

$$\frac{\partial \rho}{\partial t} + \vec{q} \cdot \vec{\nabla} \rho + \rho \vec{\nabla} \cdot \vec{q} = 0$$

So, $$\frac{\vec{q} \cdot \vec{\nabla} \rho}{\rho \vec{\nabla} \cdot \vec{q}} \sim \frac{u \frac{\partial \rho}{\partial x}}{\rho \frac{\partial u}{\partial x}} \sim \frac{\partial \rho}{\rho} << 1.$$

Hence, (1) is approximately.

$$\frac{\partial \rho}{\partial t} + \rho \vec{\nabla} \cdot \vec{q} = 0 \text{ .... (13)}$$

Also, $$\frac{\partial \rho}{\partial t} = \left(\frac{\partial \rho}{\partial p} \right)_\eta \frac{\partial p}{\partial t} = \frac{1}{c^2} \frac{\partial p}{\partial t}.$$

Since $$\frac{\partial \rho}{\partial t} dT \sim \frac{\partial}{\partial T} \left(\frac{1}{\Upsilon} \right) dT \left(\because \Upsilon = \frac{1}{\rho} \right)$$

$$= -\frac{1}{\Upsilon^2} \frac{\partial \Upsilon}{\partial T} dT$$

$$= -\frac{\rho}{\Upsilon} \frac{\partial \Upsilon}{\partial T} dT$$

$$= -\frac{\rho}{\Upsilon} \left(\frac{T}{c_p} \frac{\partial \Upsilon}{\partial T} \right) \left(\frac{c_p}{T} dT \right)$$

$$= -\frac{\rho}{\Upsilon} \Gamma c_p \frac{dT}{T}$$

= second order small quantity $\left(\because \rho g \Gamma \sim 10^{-4^\circ} C/m \right)$.

= a small quantity.

Suppose the pressure drop is δp mainly due to vertical displacement. Therefore,

$$\frac{\partial p}{\partial t} \approx \frac{\partial p}{\partial z}\cdot\frac{\partial z}{\partial t} = -\rho g w \text{ where } \frac{\partial z}{\partial t} = w.$$

$$\text{So, } \frac{\frac{\partial p}{\partial t}}{\rho \vec{\nabla}\cdot\vec{q}} \sim \frac{-\rho g w}{\rho c^2 \frac{\partial w}{\partial z}} \sim \frac{\partial z}{\frac{c^2}{g}} \sim \frac{\text{vertical scale height}}{\text{local scale height}} << 1.$$

Hence, equation (13) becomes,

$$\vec{\nabla}\cdot\vec{q} = 0 \text{ .... (14)}$$

In the euqation of motion (3), we substitute $\rho = \rho_o + \rho'$ and $p = p_o + p'$ where $dp_o = -g\rho_o dz$ and $\rho', p' << 1$. Then we have

$$\frac{1}{\rho}\vec{\nabla}p - g\hat{k}_1 = -\frac{1}{(\rho_o + \rho')}\vec{\nabla}(p_o + p') - g\hat{k}_1$$

$$= -\frac{1}{\rho_o}\left(1 + \frac{\rho'}{\rho_o}\right)^{-1}\left(\vec{\nabla}p_o + \vec{\nabla}p'\right) - g\hat{k}_1$$

$$= -\frac{1}{\rho_o}\left[1 - \frac{\rho'}{\rho_o} + ...\right]\left(\vec{\nabla}p_o + \vec{\nabla}p'\right) - g\hat{k}_1$$

$$= -\frac{\vec{\nabla}p_o}{\rho_o} + \frac{\rho'}{\rho_o}\left(\frac{\vec{\nabla}p_o}{\rho_o}\right) - \frac{\vec{\nabla}p'}{\rho_o} + \frac{\rho'}{\rho_o}\cdot\frac{\vec{\nabla}p'}{\rho_o} - g\hat{k}_1$$

$$= -\frac{\vec{\nabla}p_o + \vec{\nabla}p'}{\rho_o} + \frac{\rho'}{\rho_o}\left(-g\hat{k}_1\right) - g\hat{k}_1$$

$$= -\frac{\vec{\nabla}(p_o + p')}{\rho_o} + \left(\frac{\rho' + \rho_o}{\rho_o}\right)\left(-g\hat{k}_1\right)$$

$$= -\frac{\vec{\nabla}p}{\rho_o} + \frac{\rho}{\rho_o}\left(-g\hat{k}_1\right)$$

$$= -\frac{\vec{\nabla}p}{\rho_o} - \frac{\rho}{\rho_o}g\hat{k}_1$$

Replacing ρ by ρ_o in other term of (3), we have finally the equation of motion as

$$\frac{D\vec{q}}{Dt} = -\frac{\rho}{\rho_o} g\hat{k}_1 - \frac{1}{\rho_o}\vec{\nabla}p + 2\vec{q}\times\vec{\Omega} + \gamma\left(\frac{1}{3}\vec{\nabla}\Theta + \nabla^2\vec{q}\right)$$

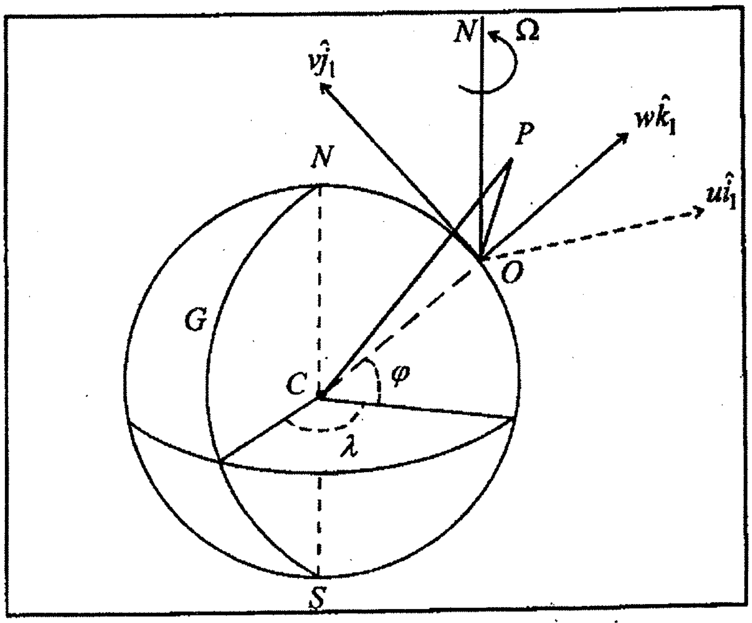
where $\gamma = \frac{\mu}{\rho_o}$.

Approximation leading to (14) and (15) and likewise replacement of ρ by ρ_o in all the remaining equations we referred as Boussinesq approximation. Under these approximation, equations (7), (12), (14) and (15) i.e.,

$$\left.\begin{aligned} &\rho_o \frac{Ds}{Dt} = -\vec{\nabla}\cdot\vec{I}_s \\ &K_o \frac{DT}{Dt} = -\vec{\nabla}\cdot\vec{Q}, K_o = \rho_o c_p \\ &\vec{\nabla}\cdot\vec{q} = 0 \\ \text{and } &\frac{D\vec{q}}{Dt} = -\frac{\rho}{\rho_o} g\hat{k}_1 - \frac{1}{\rho_o}\vec{\nabla}p + 2\vec{q}\times\vec{\Omega} + \gamma\left(\frac{1}{3}\vec{\nabla}\Theta + \nabla^2\vec{q}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots 16$$

are governing equations of motion.

6.2 The Beta Plane (β-plane) Approximation :



Let 'a' be the radius of the earth, Ω the angular velocity, λ the longitude of P, φ the latitude of P and z the vertical upwards distance of P above the undisturbed surface of O.

Let w, v, u be the components of velocity of P in the directions ($\overline{OP}$, perpendicular to $\overline{OP}$ in the meridian plane, φ increasing sence and perpendicular to the meridian plane in the λ increasing sence) parallel to the unit vectors $\hat{k}_1, \hat{j}_1, \hat{i}_1$ respectively.

Let x, y, z be a set of curvilinear co-ordinates defined by

$$\left.\begin{aligned} x &= \lambda a \cos\varphi_o \\ y &= a(\varphi - \varphi_o) \\ z &= r - a \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (1)$$

where φ_o is reference latitude suitable for the region under consideration.

The origina of the right handed system lies on theundisturbed surface z=0 at latitude $\varphi = 0°$ and longitude $\lambda = 0°$. The directions of the corresponding velocity components are taken as eastward, northward and vertically upward direction respectively.

Let L and H denote characteristic horizontal and vertical length parameters while U and $\frac{H}{L}U$ denotes characteristic horizontal and vertical velocity parameters.

In the β-plane approximation, the following assumptions are made :

$$\left.\begin{aligned} &\frac{H}{a} << 1, \\ &\left(\frac{1}{a}\right)^2 << 1, \\ \text{and} \quad &\frac{L}{a}\tan\varphi_o << 1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

These assumptions reduce the spherical metric to the cartesian metric defined by (1) as

$$ds^2 = dr^2 + r^2 d\varphi^2 + r^2\cos^2\varphi\, d\lambda^2$$

$$\simeq dz^2 + a^2 d\varphi^2 + a^2\cos^2\varphi_o d\lambda^2$$

$$\because r \simeq a, \cos\varphi = \cos\left(\varphi_o + \frac{y}{a}\right) = \cos\varphi_o \cos\frac{y}{a} - \sin\varphi_o \sin\frac{y}{a} \simeq \cos\varphi_o - \frac{y}{a}\sin\varphi_o,$$

$$\Rightarrow \cos^2\varphi = \cos^2\varphi_o - 2\frac{y}{a}\sin\varphi_o\cos\varphi_o$$

i.e. $\cos^2\varphi d\lambda^2 \simeq \cos^2\varphi_o\lambda^2$.

Hence, $ds^2 = dz^2 + dy^2 + dx^2$.

Neglecting viscosity, the governing equations of motionof sea-water (asusmed isotropic) are

$$\frac{D\rho}{Dt}+\rho\vec{\nabla}\cdot\vec{q}=0 \;\ldots\ldots\ldots\;(3)$$

$$\rho\frac{Ds}{Dt}=-\vec{\nabla}\cdot\vec{I}_s \;\ldots\ldots\ldots\;(4)$$

$$\frac{D\vec{q}}{Dt}=\vec{F}+2\vec{q}\times\vec{\Omega}-\frac{1}{\rho}\vec{\nabla}p \;\ldots\ldots\ldots\;(5)$$

$$\rho\left[\frac{DE}{Dt}+p\frac{D\Upsilon}{Dt}\right]=-\vec{\nabla}\cdot\vec{Q}+\rho\frac{Ds}{Dt}\cdot\left(\frac{\partial I}{\partial s}\right)_{T,p}-\vec{I}_s\cdot\vec{\nabla}\left(\frac{\partial I}{\partial s}\right)_{T,p} \;\ldots\ldots\ldots\;(6)$$

where $\vec{Q}$ and $\vec{I}_s$ are given by

$$\vec{Q}=-K\vec{\nabla}T+K_T\left(\frac{\partial\mu}{\partial s}\right)_{T,p}\vec{I}_s \;\ldots\ldots\ldots\;(7)$$

$$\vec{I}_s=-\rho D\left(\vec{\nabla}s+K_T\frac{\vec{\nabla}T}{T}+K_p\frac{\vec{\nabla}p}{p}\right) \;\ldots\ldots\ldots\;(8)$$

I. In (λ,φ,r) co-ordinates, we have

$$\vec{\nabla}\equiv\hat{i}_1\frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda}+\hat{j}_1\frac{1}{r}\frac{\partial}{\partial\varphi}+\hat{k}_1\frac{\partial}{\partial r}$$

or, $$\vec{\nabla}\sim\hat{i}_1\frac{\partial}{\partial x}+\hat{j}_1\frac{\partial}{\partial y}+\hat{k}_1\frac{\partial}{\partial z}.$$

$$\therefore\vec{q}\cdot\vec{\nabla}=u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$$

So, $$\frac{Du}{Dt}=\left(\frac{\partial}{\partial t}+\vec{q}\cdot\vec{\nabla}\right)u=\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)u$$

$$=\frac{D'u}{Dt},\text{ say.}$$

II. If $\hat{i},\hat{j},\hat{k}$ denotes the unit vectors parallel to $Cx\left(line\ \lambda=\varphi=o^\circ\right)$, Cy (line perpendicular to Cx), Cz the we must have

$$\hat{k}_1 = \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}$$

$$= \hat{i}\cos\varphi\cos\lambda + \hat{j}\cos\varphi\sin\lambda + \hat{k}\sin\varphi,$$

$$\hat{j}_1 = -\hat{i}\sin\varphi\cos\lambda - \hat{j}\sin\varphi\sin\lambda + \hat{k}\cos\varphi \text{ (replacing } \varphi \to \frac{\pi}{2} + \varphi, \hat{k}_1 \to \hat{j}_1)$$

and $\hat{i}_1 = -\hat{i}\sin\lambda + \hat{j}\cos\lambda$. ($\hat{k}_1 \to \hat{i}_1$ if $\varphi = 0^\circ, \lambda \to \frac{\pi}{2} + \lambda$)

$$\therefore \frac{\partial \hat{i}_1}{\partial \lambda} = -\hat{i}\cos\lambda - \hat{j}\sin\lambda = \hat{j}_1\sin\varphi - \hat{k}_1\cos\varphi,$$

$$\frac{\partial \hat{j}_1}{\partial \lambda} = \hat{i}\sin\varphi\sin\lambda - \hat{j}\sin\varphi\cos\lambda = -\hat{i}_1\sin\varphi,$$

$$\frac{\partial \hat{j}_1}{\partial \varphi} = -\hat{i}\cos\varphi\cos\lambda - \hat{j}\cos\varphi\sin\lambda - \hat{k}\sin\varphi = -\hat{k}_1,$$

$$\frac{\partial \hat{k}_1}{\partial \lambda} = -\hat{i}\cos\varphi\cos\lambda + \hat{j}\cos\varphi\cos\lambda = \hat{i}_1\cos\varphi,$$

$$\frac{\partial \hat{k}_1}{\partial \varphi} = -\hat{i}\sin\varphi\cos\lambda - \hat{j}\sin\varphi\sin\lambda + \hat{k}\cos\varphi = \hat{j}_1.$$

So, $$(\vec{q}\cdot\vec{\nabla})(\hat{i}_1, \hat{j}_1, \hat{k}_1) = \left(\frac{u}{r\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\varphi} + w\frac{\partial}{\partial r}\right)(\hat{i}_1, \hat{j}_1, \hat{k}_1)$$

$$= \left[\frac{u}{r\cos\varphi}(\hat{j}_1\sin\varphi - \hat{k}_1\cos\varphi), -\frac{u}{r}\tan\varphi\,\hat{i}_1 - \frac{v}{r}\hat{k}_1, \frac{u}{r}\hat{i}_1 + \frac{v}{r}\hat{j}_1\right].$$

$$\therefore u(\vec{q}\cdot\vec{\nabla})\vec{i}_1 + v(\vec{q}\cdot\vec{\nabla}) + w(\vec{q}\cdot\vec{\nabla})\hat{k}_1$$

$$= \left[\frac{u^2}{r\cos\varphi}(\hat{j}_1\sin\varphi - \hat{k}_1\cos\varphi), +v\left(-\frac{u}{r}\tan\varphi\,\hat{i}_1 - \frac{v}{r}\hat{k}_1\right) + w\left(\frac{u}{r}\hat{i}_1 + \frac{v}{r}\hat{j}_1\right)\right].$$

$$\simeq -\frac{u^2 + v^2}{a^2}\hat{k}_1 \text{ .... (9)}$$

$$\left(\because r \sim a, \frac{H}{a} << 1, \frac{HU}{L}\tan\varphi << 1\right).$$

Again, since,

$$W = O\left(\frac{HU}{L}\right), \frac{H}{L} \text{ is small and } \frac{L}{a}\tan\varphi << 1,$$

so that

$$\frac{WH}{a} = O\left(\frac{HU}{L}\right)\cdot\frac{U}{a}$$

= a small quantity of second order.

Now,

$$\frac{D\vec{q}}{Dt} = \frac{D}{Dt}\left(\hat{i}_1 u + \hat{j}_1 v + \hat{k}_1 w\right)$$

$$= \left(\hat{i}_1\frac{Du}{Dt} + \hat{j}_1\frac{Dv}{Dt} + \hat{k}_1\frac{Dw}{Dt}\right) + \left\{u\left(\vec{q}\cdot\vec{\nabla}\right)\vec{i}_1 + v\left(\vec{q}\cdot\vec{\nabla}\right)\hat{j}_1 + w\left(\vec{q}\cdot\vec{\nabla}\right)\hat{k}_1\right\}$$

$$\therefore \frac{D\vec{q}}{Dt} = \hat{i}_1\frac{D'u}{Dt} + \hat{j}_1\frac{D'v}{Dt} + \hat{k}_1\left(\frac{D'w}{Dt} - \frac{u^2+v^2}{a^2}\right) \text{............ (10)}$$

III. Next,

$$2\vec{q}\times\vec{\Omega} = 2\begin{vmatrix} \hat{i}_1 & \hat{j}_1 & \hat{k}_1 \\ u & v & w \\ 0 & \Omega\cos\varphi & \Omega\sin\varphi \end{vmatrix}$$

$$\therefore 2\vec{q}\times\vec{\Omega} = 2\Omega\left[\hat{i}_1\left(v\sin\varphi - w\cos\varphi\right) - \hat{j}_1 u\sin\varphi + \hat{k}_1 u\cos\varphi\right].$$

Now the local vertical component of $2\vec{\Omega}$ may be approximated as

$$f = 2\Omega\sin\varphi$$

$$= 2\Omega\sin\left(\varphi_o + \frac{y}{a}\right)$$

$$= 2\Omega\left(\sin\varphi_o + \frac{y}{a}\cos\varphi_o\right)$$

$$= 2\Omega\sin\varphi_o + \frac{2\Omega}{a}y\cos\varphi_o$$

$$\therefore f = f_o + \beta y, \text{ say .... (12)}$$

where $f_o = 2\Omega \sin \varphi_o$

and $\beta = \frac{2\Omega}{a} \cos \varphi_o$.

Also local horizontal component of $2\vec{\Omega}$ may be approximated as

$$\bar{f} = 2\Omega \cos \varphi$$

$$= 2\Omega \cos\left(\varphi_o + \frac{y}{a}\right)$$

$$\approx 2\Omega \cos \varphi_o \text{ .... } (13)$$

Hence, $2\vec{q} \times \vec{\Omega} = \hat{i}_1\left(fv - \bar{f}w\right) - \hat{j}_1 fu + \hat{k}_1 \bar{f} u$ .... (14)

Next,

$$\vec{\nabla} \cdot \vec{q} = \left(\hat{i}_1 \frac{\partial}{r \cos \partial \lambda} + \hat{j}_1 \frac{1}{r} \frac{\partial}{\partial \varphi} + \hat{k}_1 \frac{\partial}{\partial r}\right) \cdot \left(\hat{i}_1 u + \hat{j}_1 v + \hat{k}_1 w\right)$$

$$= \frac{1}{r \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{r} \frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial r} + \hat{i}_1 \frac{\partial \hat{j}_1}{\partial \lambda} \frac{v}{r \cos \varphi} + \hat{i}_1 \frac{\partial \hat{k}_1}{\partial \lambda} \frac{w}{r \cos \varphi} + \hat{j}_1 \frac{\partial \hat{k}_1}{\partial \varphi} \frac{w}{r}$$

$$= \left(\frac{1}{r \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{r} \frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial r}\right) - \frac{v}{r} \tan \varphi + \frac{2w}{r}$$

$$\approx \frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{a} \frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} - \frac{v}{a} \tan\left(\varphi_o + \frac{y}{a}\right) + \frac{2w}{a}$$

$$= \frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{a} \frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} - \frac{v}{L} \cdot \frac{L \tan \varphi_o}{a}$$

($\because \frac{2w}{a}$ is a small quantity of second order)

$$\therefore \vec{\nabla} \cdot \vec{q} \approx \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \text{ .... } (15)$$

($\therefore$ last term $\frac{v}{L} \cdot \frac{L \tan \varphi_o}{a}$ << the remaining terms)

Using (9), (10), (14) and (15), we reduce the equations (3) to (6) into the following form:

$$\frac{D'\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \text{ .... } (16)$$

$$\rho\frac{D's}{Dt}+\left(\frac{\partial I_{s\lambda}}{\partial x}+\frac{\partial I_{s\varphi}}{\partial y}+\frac{\partial I_{sr}}{\partial z}\right)=0 \quad \text{..............} (17)$$

$$\frac{D's}{Dt}=X+\left(fv-\bar{f}w\right)-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{.................} (18)$$

$$\frac{D'v}{Dt}=Y-fu-\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \text{............} (19)$$

$$\frac{D'w}{Dt}=Z+\bar{f}u-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{u^2+v^2}{a^2} \quad \text{...............} (20)$$

$$\rho\left(\frac{D'E}{Dt}+p\frac{D'\Upsilon}{Dt}\right)=-\left(\frac{\partial Q_\lambda}{\partial x}+\frac{\partial Q_\varphi}{\partial y}+\frac{\partial Q_z}{\partial z}\right)+\rho\frac{D's}{Dt}\left(\frac{\partial I}{\partial s}\right)_{T,p}$$

$$-\left(I_{s\lambda}\frac{\partial I'}{\partial x}+I_{s\varphi}\frac{\partial I'}{\partial y}+I_{sz}\frac{\partial I'}{\partial z}\right) \quad \text{...........} (21)$$

where, $\vec{Q}=\hat{i}_1Q_\lambda+\hat{j}_1Q_\varphi+\hat{k}_1Q_z$ .... (22)

$\vec{I}_s=\hat{i}_1I_{s\lambda}+\hat{j}_1I_{s\varphi}+\hat{k}_1I_{sz}$ .... (23)

$I'=\left(\frac{\partial I}{\partial s}\right)_{T,p}$ .... (24)

Equations (16) to (21) are the β-plane equations.

Note 1. The quantity $f=2\Omega\sin\varphi=f_o+\beta y$ is known as the *Coriolis parameter.*

At mid latitude i.e., for $\varphi=\pm45^\circ$

$f_o=\pm10^{-4}$ rad./sec.

$\beta=1.6\times\pm10^{-11}\,\text{m}^{-1}$.rad./sec.

2. Although x and y are curvilinear co-ordinates lying on the spherical surface z=0, they behave as ordinary Cartesian co-ordinates.

Approximation $\frac{H}{a}<<1$ means that only thin layer of fluid is conserved.

Approximation $\left(\frac{H}{a}\right)^2 << 1$ means that the horizontal length of the motion is small compared with earth's radius although it is not so small as the vertical length.

Since $a = 6.4 \times 10^4 \text{m}$,

$$H = O\left(5 \times 10^3 \text{m}\right) \text{ for the deepest ocean.}$$

$\therefore$ these two conditions are well satisfied if

$$L \le O\left(10^6 \text{m}\right).$$

Again, since

$$\left(\frac{L}{a}\right) = O\left(10^{-1}\right)$$

and $\frac{L}{a}\tan\varphi_o << 1$

means that corresponding to mid latitude or low latitude $\tan\varphi_o < 1$.

7. Equation of Conservation of Energy for Linearised Wave Motion :

The linearised equations for small amplitude wave motion in spherical co-ordinates λ, φ, z relative to the rotating earth are

$$\frac{\partial u}{\partial t} = 2\Omega v \sin\varphi - \frac{1}{\rho_o}\frac{1}{a\cos\varphi}\frac{\partial p'}{\partial \lambda} \quad \text{.............. (1)}$$

$$\frac{\partial v}{\partial t} = -2\Omega u \sin\varphi - \frac{1}{\rho_o}\frac{1}{a}\frac{\partial p'}{\partial \varphi} \quad \text{................ (2)}$$

$$\frac{\partial w}{\partial t} = -g\frac{\rho'}{\rho_o} - \frac{1}{\rho_o}\frac{\partial p'}{\partial z} \quad \text{................. (3)}$$

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_o}{\partial z} + \rho_o \vec{\nabla}\cdot\vec{q} = 0 \quad \text{............. (4)}$$

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_o}{\partial z} = \frac{1}{c^2}\left(\frac{\partial p'}{\partial t} - g\rho_o w\right) \quad \text{........... (5)}$$

where u, v, w are the zonal, meridian and vertically upwards components of velocity $\vec{q}$, p' and ρ' are the deviation of pressure and density from their undisturbed values $p_o(z)$ and $\rho_o(z)$.

The z-axis being vertically upwards, the Väisälä frequency N is given by

$$N^2 = -\left(\frac{g}{\rho_o}\frac{d\rho_o}{dz} + \frac{g^2}{c^2}\right) \quad \text{..............(6)}$$

Now from (4) and (5) we have

$$-\rho_o \vec{\nabla}\cdot\vec{q} = \frac{1}{c^2}\left(\frac{\partial p'}{\partial t} - g\rho_o w\right)$$

or, $$\frac{1}{\rho_o c^2}\frac{\partial p'}{\partial t} = \frac{gw}{c^2} - \vec{\nabla}\cdot\vec{q} \quad \text{............ (7)}$$

Again, rewriting (5) by (6), we get

$$\frac{\partial}{\partial t}\left(\rho' - \frac{p'}{c^2}\right) = -w\left(\frac{d\rho_o}{dz} + \frac{g\rho_o}{c^2}\right)$$

$$= \frac{\rho_o N^2 w}{g}$$

or, $$\frac{g}{\rho_o N^2}\frac{\partial}{\partial t}\left(\rho' - \frac{p'}{c^2}\right) = w \quad \text{.............. (8)}$$

Now multiplying (1), (2), (3) by $u\rho_0, v\rho_o, w\rho_o$ respectively and adding them, then we get

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_o q^2\right) = -\left(\frac{u}{a\cos\varphi}\frac{\partial p'}{\partial\lambda} + \frac{v}{a}\frac{\partial p'}{\partial\varphi} + w\frac{\partial p'}{\partial z}\right) - g\rho' w$$

or, $$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_o q^2\right) = -\vec{q}\cdot\vec{\nabla}p' - g\rho' w \quad \text{..................(9)}$$

Again multiplying (7), (8) by $p', g\left(\rho' - \frac{p'}{c^2}\right)$ respectively and adding them then we get

$$\frac{\partial}{\partial t}\left[\frac{p'}{2\rho_o c^2} + \frac{g^2}{2\rho_o N^2}\left(\rho' - \frac{p'}{c^2}\right)^2\right] = \frac{gp'w}{c^2} - p'\vec{\nabla}\cdot\vec{q} + gw\left(\rho' - \frac{p'}{c^2}\right) \quad \text{.............. (10)}$$

Next adding (9) and (10) then we get

$$\frac{\partial}{\partial t}\left[\rho_o\frac{q^2}{2} + \frac{p'^2}{2\rho_o c^2} + \frac{g^2}{2\rho_o N^2}\left(\rho' - \frac{p'}{c^2}\right)^2\right] = -\vec{q}\cdot\vec{\nabla}\cdot p' - p'\vec{\nabla}\cdot\vec{q} = -\vec{\nabla}(p'\vec{q}) \quad \text{........... (11)}$$

Now we compare (11) with the energy transport equation

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{q^2}{2}+U+E\right)\right]=-\vec{\nabla}\cdot\left[\rho\left(\frac{q^2}{2}+U+E\right)+p'\right]\vec{q}$$

and denote by the quantity $\in$,

$$\in=\frac{p'^2}{2\rho_o c^2}+\frac{g^2}{2\rho_o N^2}\left(\rho'-\frac{p'}{c^2}\right)^2 \quad \text{..................} (12)$$

Since the sea-water is only slightly compressible, the contribution of internal energy E may be ignored in the energy transport equation.

Therefore, in analogy with a concept introduced by Lorenz for a definite class of atmospheric motions, the quantity $\in$ may be called available potential energy (per unit volume) for wave motions of small amplitude in a stably stratified fluid. Generally speaking, the quantity $\in$ contains internal as well as potential energy contributions; however, by strength of the small compressibility of sea water, the contribution of the internal energy is not large, and therefore, for the sake of brevity, the quantity $\in$ will be referred to as available potential energy.

Hence, we can regard equation (11) as a form of the equation of conservation of energy for small amplitude wave motions in a stratified fluid.

Here note that the term other than p' in the square bracket on the R.H.S. of the energy transport equation are negligible or zero, i.e.,

$$\rho\left(\frac{q^2}{2}+U+E\right)+p_o \simeq \rho_o U+p_o$$

$$=0.$$

8. Summary :

In this module we have discussed linearised equation of small amplitude ocenian wave motion on a rotating earth, Boussinesq approximation, β-plane approximation and equation of conservation of energy for linearised wave motion in simple and straight forward way.

9. Self Assessment Questions :

1. What do you mean by oscillation of layer of liquid?
2. Establish the equations of small amplitude oceanian wave motion on a rotating earth.
3. Establish the governing equations of incompressible fluid for small amplitude ocean wave.

4. State clearly the assumption of Boussinesq's approximation. Hence derive field equations approximately according to Boussinesq's approximation.
5. What are β-plene approximations? Derive the field equations approximately according to β-plane approximation.
6. Obtained the equation of conservation of energy for small linearised wave motion in a stratified fluid.

10. Further Suggested Reading :

1. Fundamentals of Ocean Dynamics : V.M. Kamenkovich translated by R. Radok.
2. Dynamic Method in Oceanography : L.M. Fomin.
3. Boundary Layer Problem in Applied Mechanics : G.F. Carrier.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper- Group–

Module No. - 94

THERMODYNAMICS IN METEOROLOGY

Contents:

1.0 Introduction

Heat is a form of energy, which may be converted into other forms of energy. The study of the dynamical aspect of heat is known as thermodynamics. One of the most enlightening applications of thermodynamics is to study more deeply real gases like water vapor because it is the fuel of the atmosphere. Also the atmosphere is actually a mix of different gases and one should know under which conditions this mix may be treated as perfect or real gas. The atmosphere is a huge thermodynamic engine, driven by the energy received from the sun. The general circulation over the earth, all winds, storms and clouds result from the differences in the amount and utilization of this energy. Since the radiant energy appears principally as heat, the student of meteorology must understand how the air reacts to heat changes; in other words, he/she must understand the thermodynamics of the atmosphere.

2.0 Objectives

In this module, the following topics are discussed

* heat balance in the atmosphere,
* atmospheric air composition,
* equatino of state for dry air and moist air,
* internal energy, first law of thermodynamics, change in internal energy in the atmosphere,
* adiabatic process,
* hydrostatic equation,
* variation of pressure with altitude,
* potential temperature,
* stability of dry air,
* humidity variables

3.0 Key Words and Study guides

heat balance, internal energy, adiabatic process, lapse rate, humidity

4.0 Main Discussion

4.1 The Heat Balance in the Atmosphere

Since there is no evidence of any significant increase or decrease in the mean temperature of the earth's atmosphere over a period of years, it is evident that the radiant energy received from the sun and that emitted by the earth and its atmosphere to outer space must be equal. The total amount of the heat reaching the earth during a year may be computed without difficulty by using the value of the solar constant, 1.94 *cal per* cm^2 per min, and remembering that the area intercepting solar radiation at any instant is the cross-sectional area of the earth. A simple computation shows that the earth receives energy from the sun at the rate of 130×10^{22} *cal per annum*.

The various transformations of this energy before it is radiated again to outer space are of interest and it is discussed in the following given by Landsberg, Baur, Philips and Moller. The unit of energy used is 10^{22} *cal per annum*. Of the 130 units received by the earth from the sum, clouds reflects 39 units and the atmosphere scatters 12 units back to outer space, the atmosphere absorbs 19 units, and 35 units reach the earth's surface in the direct solar beam, and 25 units reach it as diffuse sky radiation. There is an exchange of heat between the earth and its atmosphere by terrestrial radiation, by condensation and evaporation process, and by turbulence. Heat is absorbed at the earth's surface in the process of evaporating water, the water vapor is carried upward, and when it condenses, this heat is released to the atmosphere. There is thus a transfer of heat from the surface to the atmosphere.

The various amounts of heat transferred by these processes are given in the following table. The letter (S) indicates that the radiation is of the short wave or solar type, and (L) shows that it is of the long wave or terrestrial type. The upward and downward fluxes of energy through a surface at the outer limit of the atmosphere and through a surface at mean sea level are given in the upper portion of the table. At the bottom the gain and loss of heat in the atmosphere between these two surface are indicated. The value given in the table are only approximate, but they indicate the order of magnitude of the several processes involved.

	The Downward Flux from the	The Upward Flux from the
At the outer limit of the atmosphere	Sun 130 ------- 130	Clouds (S) 30 Atmosphere (S) 12 Atmosphere(L) 65 Earth(L) 14 ----- 130
At the earth's surface	Sun (direct) (S) 35 Sun (diffuse) (S) 25 Atmosphere By radiation (L) 125 By turbulence 5 ----- 190	Earth to outer space (L) 14 Earth to the atmosphere By radiation (L) 146 By evaporation 30 ------ 190
	Gains by	Loses by
The atmosphere	The absorption of radiation from the Sun (S) 19 Earth (L) 146 Condensation 30 ----- 195	The emission of radiation to The earth (L) 125 Outer space (L) 65 Turbulence 5 ----- 195

4.1.1 Atmospheric Air Composition

We may consider the atmospheric air as composed of (i) a mixture of gases, (ii) water substance in any of its three physical states, and (iii) solid or liquid particles of very small size. Water substance is a very important component for the processes in the atmosphere. The solid and liquid particles in suspension (other than that of water substance) constitute what is called the **atmospheric aerosol**. It is very important for cloud and precipitation physics, etc. But, it is not significant for atmospheric thermodynamics. The mixture of gases is what we call **dry air**. The dry air constitutes of four main components: nitrogen, argon and carbon dioxide.

4.1.2 The Equation of State for Dry air

Two basic relationships of thermodynamics are the laws of Boyle and Charles. **Boyle's law states:** The pressure of a given mass of a gas at constant temperature varies inversely as the volume. **Charles' law is:** The temperature in degrees Absolute of a given mass of gas at constant pressure varies directly as the volume. These two laws may be combined in the following manner. First assume that a unit mass of gas has a pressure p_1, a volume v_1, and a temperature T_1. Let p_1 and v_1 vary at constant temperature T_1, until a pressure p_2 and a volume v are obtained. From Boyle's law it follow that

$$p_1 v_1 = p_2 v \text{ and } v = \frac{p_1 v_1}{p_2} \qquad (1.2.1)$$

Now let the temperature T_1 changes to T_2, the pressure remaining constant at p_2. From Charles' law, then we have

$$\frac{v}{v_2} = \frac{T_1}{T_2} \text{ and } v = \frac{v_2 T_1}{T_2} \qquad (1.2.2)$$

where v_2 is the resulting temperature.

Combining (1.2.1) and (1.2.2) we have

$$\frac{p_1 v_1}{p_2 v_2} = \frac{T_1}{T_2} \qquad (1.2.3)$$

Now the equation (1.2.3) indicates that when a unit mass of perfect gas is undergoing variations in pressure, volume and temperature, then

$$pv \propto T \qquad (1.2.4)$$

$$\text{i.e. } pv = \frac{R^*}{M}T \qquad (1.2.5)$$

where R^* (=83.14 × 10^6 *ergs per degree*) is called the **Universal gas constant** and M is the molecular weight of the gas.

This equation may be used for any perfect gas, when its molecular weight is inserted. Of the gases comprising the atmosphere, only water vapor condenses at the temperatures and presures encountered in the atmosphere, so that absolutely dry air may be considered as a perfect gas. Although the constitution of the atmosphere is subject to variations, these are small and 28.97 may **be taken as the molecular weight of dry air.** Therefore, the value of the gas constant for dry air, R_d, is given by

$$R_d = \frac{R^*}{M_d} = \frac{83.14\times10}{28.97} = 2.87\times10^6 ergs \deg^{-1} = 2.87\times10^6 cm^2 \sec^{-2} \deg^{-1}.$$

The specific volume α of a gas is defined as the volume occupied by unit mass of that gas, so that $\alpha = \frac{1}{\rho}$ where ρ is the density of the gas. Hence the **equation of state for dry air** may be given as

$$p\alpha = RT \ \ or \ \ p = \rho RT \qquad (1.2.6)$$

4.1.3 The Concept of Internal Energy:

The kinetic theory of gases relates the temperature of a gas to the rate at which the molecules are moving about, showing that the temperature is proportional to the mean kinetic energy of the moving molecules. The kinetic theory shows that the kinetic energy of the translatory motion of the molecules is proportional to the temperature. If no heat is added but the gas is mechanically compressed, the kinetic energy will also be increased by the compression, and the temperature of the gas will rise. Therefore we define the **internal energy** of a given mass of a gas as the total energies of all the molecules in that mass and note that it is proportional to the temperature.

4.1.4 The First Law of Thermodynamics:

The first law of thermodynamics states that an increase in the internal energy can be brought about by the addition of heat or by performing work on the gas, or by combination of both. If we represent the change in internal energy by dE, positive for an increase, and the heat added as $+dQ$, and if dW represents the work done on the gas, negative for work done by the gas, then we have the following relationship.

$$dE = dQ + dW \qquad (1.4.1)$$

Therefore, when a gas is compressed without adding heat, we have $dE = dW$. In other words, the increase in internal energy would be caused by work (compression) performed on the gas. In case of direct heating where no work is donw, $dE = dQ$.

4.1.5 Work-done by External Forces:

In considering the work done on a gas, we are mainly concerned with pressure. Pressure is defined as force per unit area. Now, we consider the following figure, which represents a parcel of air in the atmosphere. Any pressure p, which may be applied in the atmosphere, would act equally in all directions. Therefore this pressure would be exerted on the parcel from all sides, causing it to be compressed to a small volume by an amount

$$-dV = Adn \qquad (1.5.1)$$

where dn is the distance between the outer surface of the parcel in the two positions and A is the area of this surface, assumed to be the same in both cases because dn is taken as infinitesimal.

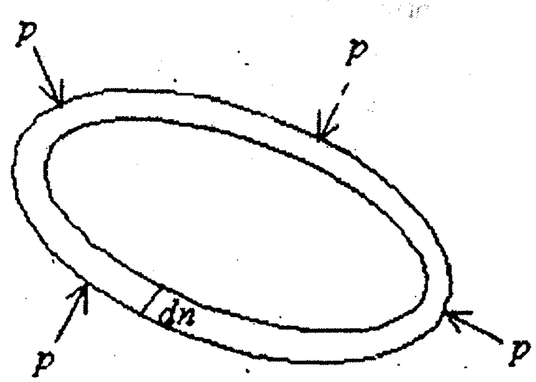
The work done on the parcel in this compression is given by the product of the force acting on the parcel times the distance through which the force acts, measured in the direction of the force, so we have

$$dW = Fdn$$

$$= pAdn, \text{ since } p = \frac{F}{A}$$

$$= pdV \text{ by } (1.5.1)$$

Therefore, $dW = -pdV$ (1.5.2)



4.1.6 Changes in Internal Energy:

The temperature may be changed by holding the volume constant and varying the pressure, or by keeping the pressure constant and varying the volume. This may be seen in the following figure, where the equation of state is represented in a pV diagram, with temperatures shown in the curved lines (rectangular hyperbolas). It is apparent that following along a harizontal line p = constant, or along a vertical line V = constant change the temperature. Now we shall find the change in internal energy by considering these two processes.

Now from the first law of thermodynamics using (1.5.2) we have

$dE = dQ - pdV$ (1.6.1)

If we consider a constant-volume process, then we have $pdV = 0$. The heat added dQ, corresponding to a change in temperature dT, will be given by the mass times the heat required to raise unit mass 1°C at constant volume i.e., the specific heat, multiplied by dT i.e.,

$$dE = mC_v dT \qquad (1.6.2)$$

where m is the mass of the air parcel, C_v is the specific heat constant.

The equation (1.6.2) is taken as the definition of the change in initial energy and in this no external work is involved. The heat added all goes toward changing the internal energy.

If heat is added at constant pressure, $dV \neq 0$ and the gas performs work against the environment through expansion as it is heated. We get two different results depending upon which of these two conditions is imposed during the heating. The amount of heat required to raise the temperature of a given mass of the substance by one degree will be different in two cases. That is there are two specific heats of a gas, the specific heat at constant volume C_v and the specific heat at constant pressure C_p, both constant.

To find the difference between these two specific heat constants, we consider a constant-pressure process involving the same change in internal energy dE as in the constant-volume in (1.6.2). So using (1.6.2) from the first law of thermodynamics, we have

$$mC_v dT = dQ - pdV \qquad (1.6.3)$$

Now in this process, we have $dQ = mC_p dT$ where C_p is the specific heat at constant pressure. So from the equation (1.6.3), we have

$$mC_v dT = mC_p dT - pdV \qquad (1.6.4)$$

i.e. $m\left(C_p - C_v\right)dT = pdV$ (1.6.5)

Now the equation of state is as follows:

$pV = nRT$ where n is the mole of the gas.

i.e., $pdV + Vdp = nRdT$ (1.6.6)

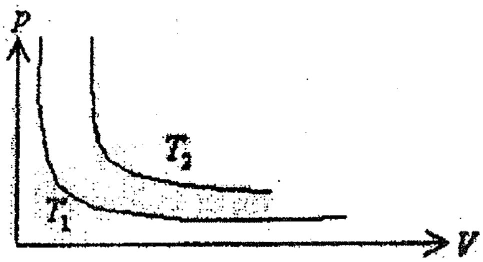
i.e., $pdV = nRdT$ since at constant-pressure, $dp = 0$ (1.6.7)

Using the equation (1.6.7) in (1.6.5), we have

i.e., $m\left(C_p - C_v\right) dT = nRdT$

i.e., $C_p - C_v = \frac{nR}{m} = \frac{R}{M}$

The difference $C_p - C_v = \frac{R}{M}$ is sometimes called the molecular heat difference because it is inversely proportional to the molecular weight M.



Application to the Atmosphere: In the atmosphere, volume measurements are difficult and it is preferable to use the following form after dividing the equation (1.6.3) by *m* as

$mC_v dT = dQ - pdV$

i.e., $C_v dT = \frac{dQ}{m} - p\frac{dV}{m}$

i.e., $C_v dT = dq - pd\alpha$ where $\frac{dQ}{m} = dq$ as $\frac{dV}{m} = d\alpha$

i.e., $dq = C_v dT + pd\alpha$ (1.6.8)

where dq is the heat added to unit mass and $d\alpha$ is the change in specific volume.

Now the equation of state in differential form, from (1.6.6), is

$$pdV + Vdp = nRdT$$

$$\text{i.e., } p\frac{dV}{m} + \frac{V}{m}dp = \frac{n}{m}RdT$$

$$\text{i.e., } pd\alpha + \alpha dp = \frac{R}{M}dT$$

$$\text{i.e., } pd\alpha = \frac{R}{M}dT - \alpha dp \qquad (1.6.9)$$

Using (1.6.9) in (1.6.8) we have the following

$$dq = C_v dT + \frac{R}{M}dt - \alpha dp \qquad (1.6.10)$$

But since $C_p - C_v = \frac{R}{M}$, hence we have $C_p = \frac{R}{M} + C_v$. Using this in (1.6.10) we have

$$dq = \left(C_v + \frac{R}{M}\right)dT - \alpha dp$$

$$\text{i.e. } dq = C_p dT - \frac{R}{M}T\frac{dp}{p} \qquad (1.6.11)$$

since $pV = nRT$ i.e. $p\frac{V}{m} = \frac{n}{m}RT$ i.e. $p\alpha = \frac{R}{M}T$ (1.6.12)

The equation (1.6.11) is highly important absic equation used to describe the properties, motions and transformations of the atmospheric medium. It is statement of the first law of thermodynamics in terms of the easily measured atmospheric variables, temperature and pressure, and requires no consideration of measured or controlled volumes or masses.

4.1.7 Adiabatic Process:

If a mass of a substance such as air undergoes variations in state in such a manner that there is no exchange of heat between the substance and its environment, the process by which this occurs is said to be adiabatic process. So in this process we have $dq = 0$. In an adiabatic process the change in internal energy of the gas world, therefore, be due entirely to the work performed on it, particularly by the forces compressing the gas, or by the work done by the gas in the form of an expansion. In the atmosphere both adiabatic and non-adiabatic processes are occurring. The addition of heat to the atmosphere near the ground when the surface is warmer than the overlying air and the removal of heat in the same manner when the surface is colder is continuous **non-adiabatic processes**. In the free atmosphere where there is no solid or liquid surface to give off or remove heat and where the amount of energy absorbed from the sun's rays or lost by direct radiation is insignificant by comparison, we are justified in assuming that all short-period processes are essentially **adiabatic**. Since in the atmosphere the pressure decreases rapidly with elevation, adiabatic temperature changes occur most readily when portions of the air undergo motions having a vertical component. Therefore, a parcel of air carried upward cools adiabatically with the decreasing pressure that it experiences, and conversely, a descent with increasing pressure causes an increase in the temperature.

From the first law of thermodynamics it is possible to obtain an equation by means of which, if given the temperature and pressure of a parcel at the beginning of an adiabatic process, its temperature at any known pressure on during that process can be calculated as follows. So for an adiabatic process, from the equation (1.6.11) we have

$$C_p dT - \frac{R}{M} T \frac{dp}{p} = 0 \text{ since } dq = 0$$

i.e., $C_p \frac{dT}{T} = \frac{R}{M}\frac{dp}{p}$

Now we say that in an adiabatic process with the temperature T_0 and the pressure p_0, we wish to find the temperature T at some other pressure p. To do this, we integrate the above equation from T_0 to T and from p_0 to p as follows:

$$\int_{T_0}^{T} C_p \frac{dT}{T} = \int_{p_0}^{p} \frac{R}{M}\frac{dp}{p}$$

$$\text{i.e., } C_p \ln T - C_p \ln T_0 = \frac{R}{M} \ln p - \frac{R}{M} \ln p_0$$

$$\text{i.e., } C_p \ln\left(\frac{T}{T_0}\right) = \frac{R}{M} \ln\left(\frac{p}{p_0}\right)$$

$$\text{i.e., } \left(\frac{T}{T_0}\right)^{C_p} = \left(\frac{p}{p_0}\right)^{\frac{R}{M}}$$

$$\text{i.e., } \frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{R}{MC_p}} \qquad (1.7.1)$$

This equation is known as **Poisson's equation** and $\frac{R}{MC_p}$ is usually equal to $\frac{2}{7} = 0.286$.

Similarly a relation in terms of temperature and specific volume can be obtained from the equation (1.6.8) for the adiabatic process as follows:

$C_v dT + p d\alpha = 0$ since $dq = 0$

i.e., $C_v dT = -p d\alpha$

i.e., $C_v \frac{dT}{T} = -\frac{p}{T} d\alpha$

i.e., $C_v \frac{dT}{T} = -\frac{R}{M}\frac{d\alpha}{\alpha}$ by the eqaution (1.6.12)

i.e., $C_v \int_{T_0}^{T} \frac{dT}{T} = -\frac{R}{M}\int_{\alpha_0}^{\alpha} \frac{d\alpha}{\alpha}$

i.e., $C_v \text{ in } \frac{T}{t_0} = \frac{R}{M} \text{ in } \frac{\alpha_0}{\alpha}$

i.e., $\frac{T}{T_0} = \left(\frac{\alpha_0}{\alpha}\right)^{\frac{R}{C_v M}}$

which is the required relationship between temperature and specific volume during adiabatic process.

4.1.8 Hydrostatic Equation:

The only force acting vertically on the atmosphere is that force produced by the acceleration of gravity. The force produced by the acceleration of gravity g on a mass m is $F = mg$. This is the definition of weight. The pressure at any point in the atmosphere would be equivalent to the weight of an air column of 1 cm^2 cross section extending to the top of the atmosphere. In a state of equilibrium, the pressure forces, whose resultant is opposite tot he force of gravity, everywhere balance the force of gravity. Now we consider a layer of thickness δz in a column of unit area as in the figure. A pressure force p acts on its base, directed upwards while adownward pressure force $p + \delta p$ acts on teh top of the layer.

Therefore a net force $-\delta p$ is acting upwards on a mass $\rho\delta z$ of this layer, where ρ is the density of that layer. Now the force of gravity is $g\rho\delta z$. Due to vertical equilibrium of the atmosphere, we have

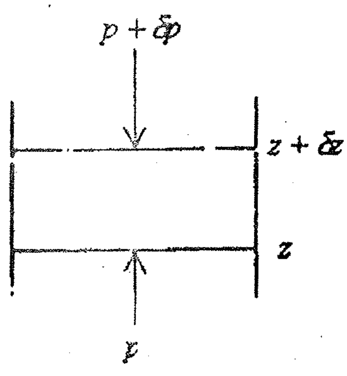
$$-\delta p - g\rho\delta z = 0$$

i.e., $\delta p = -g\rho\delta z$

i.e., $\frac{\delta p}{\delta z} - g\rho$

i.e., $\frac{\partial p}{\partial z} = -g\rho \text{ as } \delta z \to 0$ **(1.8.1)**

This is the **hydrostatic equation**. The partial derivative is used because the pressure may change also in the horizontal direction and with time. The minus sign indicates that z is measured upward in the direction of decreasing pressure.



The total differentiation of the pressure takes the following form

$$dp = \frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$$

where t is time, x, y are the two horizontal distances, and z is vertical. In an atmospheric column at rest, with pressure changing only with height, $\frac{\partial p}{\partial t}$, $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$ are zero. Therefore we have

$$dp = \frac{\partial p}{\partial z}dz - g\rho dz \text{ using (1.8.1)}$$

$$\text{i.e., } dp = -g\rho dz \qquad (1.8.2)$$

We are now in a position to extend the three basic relationships (equation of state, first law of thermodynamics and hydrostatic equation) to a variety of problems of thermodynamics and statics in the atmosphere.

4.1.9 Adiabatic Temperature Change with height:

The adiabatic rate of change with pressure in equation $C_p \frac{dT}{T} = \frac{R}{M}\frac{dp}{p}$ can be substituted in the hydrostatic equation (1.8.2) to obtain the adiabatic rate of temperature change with height. It is assumed that the atmosphere is in complete hydrostatic equilibrium. So we have

$$dp = -g\rho dz$$

Using this in above equation we have $C_p \frac{dT}{T} = (-g\rho)\frac{R}{M}\frac{dz}{p}$ and then using the equation of state $p = \rho\frac{RT}{M}$ we get the following

$$\frac{dT}{dz} = -\frac{g}{C_p} \qquad (1.9.1)$$

where negative sign indicates that the temperature of a sample of air world decrease with ascent in the temperature and $-\frac{g}{C_p}$, having the dimensions of Kelvin per centimeter in cgs units, is the dry adiabatic rate of temperature change.

This rate of decrease of temperature with height of adiabatically ascending dry air is known as the **dry adiabatic lapse rate** Γ_d and is given by $\Gamma_d = \frac{g}{C_p} = 9.8^{\circ} C km^{-1}$. So we have defined here a process rate for a sample of air moving up or down through an atmosphere at rest and in hydrostatic equilibrium.

4.1.10 Adiabatic Chart:

In physics and engineering, the thermodynamic processes of gases, such as isothermal processes and adiabatic processes, are represented on the so-called pV-diagram, having pressure and volume as coordinates. This diagram has the property that the work involved in any closed cycle, made up of any series or combination of processes bringing the gas back to its starting pressure and volume, is directly proportional to the area enclosed by the plot of the cycle on the diagram. It is seen that the work done on the gas in compression to change its volume by an amount $-dV$ would be $dW = -pdV$. So the total work done in a cyclic process would be then

$$W = -\oint_C pdV \qquad (1.11.1)$$

From this equation it is seen that the total work done is the area enclosed by the closed path C on the pV–diagram. The negative sign means work done on the gas and the positive sign would denote work done by the gas.

In the atmosphere, where we are not dealing with controlled volumes, it is more convenient to represent the processes in terms of pressure are temperature. It is desirable to retain, however, the work-measuring property of pV integral. This can be done in terms of p and T with a suitable transformation. Since in the atmosphere it is more convenient to deal with unit mass and specific volume (α), **we measure the work on a unit mass,**

$$dw = -p\,d\alpha$$

i.e., $dw = \alpha dp - \frac{R^*}{M} dT$ by (1.6.9)

so, $w = \oint_c \alpha dp - \oint_c \frac{R^*}{M} dT$ (1.11.2)

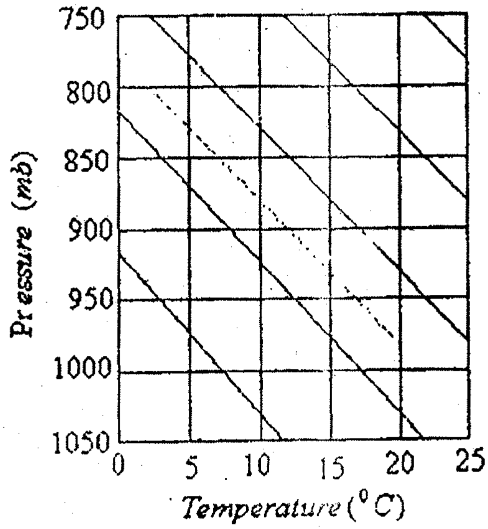
i.e. $w = \oint_c \alpha dp$ since $\oint_c \frac{R^*}{M} dT = 0$

i.e., $w = \frac{R^*}{M} \oint_c T \frac{dp}{p}$ by means of the equation of state

i.e., $w = \frac{R^*}{M} \oint_c T d(In\, p)$ (1.12.3)

Hence, a diagram with T *and* $\ln p$ as coordinates **will have the desired property of representing work by an area.**

This is particularly, fortunate for atmospheric studies, since **height is very nearly a function of the logarithm of pressure,** and a diagram with temperature as abscissa and logarithm of pressure, decreasing upward, as ordinate. Another fortunate circumstances is that **the exponent in the Poisson's equation has such a value (0.286)** that the differential of p to that power is proportional to $d(\ln p)$ at atmospheric values and the ordinate in the diagram can therefore be p to the power 0.286 with close approximation. This transformation produces adiabats-lines representing the adiabatic process as straight lines. As adiabatic chart in which the slopping lines are the adiabats, is given below:



$$T_0 = 293\ K\ (20^0C)$$

$$T = T_0\left(\frac{p}{p_0}\right)^{0.286}$$

$$T = 293\left(\frac{800}{980}\right)^{0.286}$$

$$T = 276.5\ K\ (3.5\ ^0C)$$

Example: The Poisson's equation can be graphically with this diagram. We suppose that we have a sample of air at a temperature of 200C ($T_0 = 283$K) and a pressure p_0 of 980 mb. If we want to find the temperature after the sample has been carried through an adiabatic process in which it has been brought to a pressure p of 800 mb, the required temperature T (=3.5°C) can be read from the adiabatic chart.

4.1.11 The Variation of Pressure with Altitude:

The pressure at any height in the atmosphere is, by definition, the weight of the vertical column of air of unit cross section which extends from that height to the top of the atmosphere. It follows directly then that pressure decreases with increasing height in the atmosphere. The difference in pressure between height z and height $z + dz$ may be found in the following manner. Since unit cross section is being considered, the volume of the element of air is dz. Since weight is given by the product of the mass and the acceleration of gravity, and since p decreases with increasing z, then

$$dp = -g\rho dz \tag{1.11.1}$$

Using $p = \rho RT$ in (1.11.1), we have

$$dp = -\frac{gp}{RT} dz \tag{1.11.2}$$

This is the **basic statistical relationship of meteorology**. Now if T can be expressed as a function of z, then the differential equation may be integrated. Here **two cases** are of fundamental importance, which are as follows:

Case - 1: T is constant with height so the atmosphere is isothermal

In this case the portion of the atmosphere under consideration is an isothermal layer. Therefore the equation (1.11.2) may be expressed as

$$\frac{dp}{p} = -\frac{g}{RT} dz \tag{1.11.3}$$

Integrating this we have

$$\ln \frac{p}{p_0} = -\frac{g}{RT}(z - z_0)$$

i.e., $p = p_0 e^{\frac{g}{RT}(z-z_0)}$ (1.11.4)

where p_0 is the pressure at height z_0.

Case - 2: T decreases at a constant rate with increasing z.

The second case is that in which T decreases at a constant rate with increasing z. Then the temperature at any height is given by

$$T = T_0 - \alpha z \tag{1.11.5}$$

where T_0 is the temperature at z_0, and α is known as **the lapse rate of temperature.**

Differentiating this we have $dz = -\frac{dT}{\alpha}$, then using this in (1.11.3) we have

$$\frac{dp}{p} = -\frac{g}{R\alpha}\frac{dz}{T}$$

Integrating this we have

$$p = p_0\left(\frac{T - \alpha z}{T_0}\right)^{\frac{g}{R\alpha}} \qquad (1.11.6)$$

If, as a first approximation, it is assumed that the troposphere is composed of **dry air** having a **constant lapse rate of temperature,** then (1.11.6) gives the variation of pressure in the troposphere under those conditions.

4.1.12 Effects of Water Vapor:

All the gases in the atmosphere including the water vapor exert their own partial pressures. Concerning this, we have Dalton's law, which sates that the total pressure of a gas mixture is equal to the sum of the partial pressures. In meteorology, we consider the mixture of "dry gases" forming on "gas" (molecular weight being 28.9) and the water vapor (molecular weight being 18) another.

The equation of state can be applied to a mixture of gases provided that the relative quantities of the different gases in the mixture do not vary. Otherwise, the molecular weight M would not be a constant. Water vapor is the only important gas in the atmosphere that varies in significant amounts. **Since water vapor has a lesser molecular weight thatn that of dry air, its presence at a given temperature and pressure would to lower the density of the mixture.** Now we may write the equation of state for water vapor only as

$$e = \rho_w R_w T \qquad (1.12.1)$$

where ρ_w and R_w be the partial pressure, density and gas constant of the water vapor.

Again the equation of sate for dry air is given by

$$p_d = \rho_d R_d T \tag{1.12.2}$$

where p_d, ρ_d and R_d be the partial pressure, density and gas constant of the dry air only.

Now the density of the air (moist air) is the sum of the density of water vapor and of dry air. Therefore, if ρ be the total density of the air (moist air), then we have

$$\rho = \rho_d + \rho_w$$

i.e., $\rho = \dfrac{p_d}{R_d T} + \dfrac{e}{R_w T}$ by the equations (1.12.1) and (1.12.2).

$$\text{i.e., } \rho = \frac{p-e}{R_d T} + \frac{e}{R_w T} \tag{1.12.3}$$

where $p(= p_d + e)$ is the total pressure of the air by Dalton's law.

Now we know that $R^* = M_d R_d = M_w R_w$. So $R_w = \dfrac{M_d}{M_w} R_d = \dfrac{1}{\varepsilon} R_d$ where $\varepsilon = \dfrac{M_w}{M_d} = \dfrac{18.016}{28.97} = 0.622$.

Using this in the equation (1.12.3) we have

$$\rho = \frac{p-e}{R_d T} + \frac{\varepsilon e}{R_d T}$$

$$\text{i.e., } \rho = \frac{p}{R_d T}\left(1 - (1-\varepsilon)\frac{e}{p}\right) \tag{1.12.4}$$

The above equation shows that a given volume of moist air is lighter than an equal volume of dry air at the same pressure and temperature. So the equation of state for moist air, unsaturated air is given by

$$\rho = \frac{p}{R_d T}\left(1-(1-\varepsilon)\frac{e}{p}\right) = \frac{p}{R_m T} \text{ where } R_m = \frac{R_d}{1-(1-\varepsilon)\frac{e}{p}} \quad (1.12.5)$$

$$\text{i.e., } \rho = \frac{p}{R_m} \text{ i.e., } p = \rho R_m T \quad (1.12.6)$$

But from the equation (1.12.5) we get that R_m is not constant,for all amounts of water vapor in the air, but varies with vapor pressure e and that total pressure p. So R_m can not treated as a gas constant for moist air. The form of (1.12.6) is disregarded as a equation of state for moist air. Rather than this, it is often more conventent to use a different value of the temperature, known as the virtual temperature T_v. So the equation of state the moist, unsaturated air then becomes

$$\rho = \frac{p}{R_d T}\left(1-(1-\varepsilon)\frac{e}{p}\right) = \frac{\rho}{R_d T_w}$$

$$\text{i.e., } \rho = \frac{p}{R_d T_w} \text{ i.e., } p = \rho R_d T_w \quad (1.12.7)$$

where $T_v = \dfrac{\varepsilon T}{1-(1-\varepsilon)\dfrac{e}{p}}$ is known as the virtual temperature.

Therefore, the virtual temperature T_v of moist, unsaturated air may be defined as the temperature at which dry air of the same pressure would have the same density as the moist air.

4.1.13 Determination of Altitude:

On the adiabatic chart, altitudes can be represented conveniently because all three factors, pressure, temperature and humidity, determine the altitude. **The relationship between pressure and height at a given temperature** may be determined from the hydrostatic equation

$$dp = -\rho g dz$$

i.e., $dp = -\frac{pMg}{RT_v}dz$ by the equation of sate

where T_v is the virtual temperature.

This equation is then integrated from the sea level, where the pressure is p_0, to the height z, having the pressure p, as follows:

$$\int_{p_0}^{p}\frac{dp}{p} = -\frac{Mg}{R}\int_0^z \frac{dz}{T_v}$$

i.e., $\ln\frac{p}{p_0} = -\frac{Mg}{R}\int_0^z \frac{dz}{T_v}$

i.e., $p = p_0 e^{-\frac{Mg}{R}\int_0^z \frac{dz}{T_v}}$ (1.13.1)

It is convenient and feasible to divide the atmosphere into layers of about **100 mb** in thickness, using the mean virtual temperature of each layer to determine its depth. The various depths are then added together to obtain the total height. Hence we have for a layer between z_1 and z_2, having the mean virtual temperature $\overline{T}_v$,

$$\int_{p_1}^{p_2}\frac{dp}{p} = -\frac{Mg}{R\overline{T}_v}\int_{z_1}^{z_2} dz$$

i.e., $\ln p_2 - \ln p_1 = -\frac{Mg}{R\overline{T}_v}(z_2 - z_1)$

$$\text{i.e., } z_2 - z_1 = \frac{R\overline{T}_v}{Mg}\left(\text{In } p_1 - \text{In } p_2\right) \qquad (1.13.2)$$

This equation is known as the **hypsometric formula** and the **total height** is the sum of the thickness of the layers.

4.1.14 Geo-potential Altitude:

The earth is not quite a perfect sphere since its equatorial radius 6378 km, whereas its polar radius is 6357 km. Practically in all meteorological problems this divergence from a true spherical shape is of no significance, and a value of 6370 km may be regarded.

A horizontal surface on the earth or in the atmosphere is one that everywhere parallels the surface of teh sea, or in other words, surface that is everywhere at the same distance above (or below) mean sea level. From elementary physics, we know that a particle is straight steady frictionless horizontal motion neither performs nor requires work in its motion. In the gravitational field of the earth, the work (W) performed in a displacement of a mass M from a height z_1 to a greater height z_2 is given by

$$W = \int_{z_1}^{z_2} Mg\,dz$$

$$\text{i.e., } W = \int_{z_1}^{z_2} g\,dz \text{ for unit mass} \qquad (1.14.1)$$

If the acceleration of gravity were constant in a horizontal plane all over the earth, then a surface of z = constant, in other words, a **horizontal surface would be one along which air particles could move in straight frictionless flow without work**. Such a surface would then be one of constant potential energy in the earth's gravitation field (Mgz = constant). Such a surface is called **equipotential surface or geopotential surface** or **level surface.**

However, we know that g is not constant at any given height over the earth. It varies especially with latitude, being maximum near the pole and minimum near the equator. Therefore, horizontal surfaces are not level surfaces. So for an air particle to move in a horizontal surface over the earth, work must be performed.

The concept of geopotential has been introduced into meteorology in order to make allowance for the variation of g with height when that is necessary. The geopotential Φ at a height z is defined as the potential energy of unit mass at that height. The potential energy at a point is by definition, the work required to raise unit mass from some standard level, usually mean sea level, to that point. So, if we start with zero at sea level, we define a geopotential given by

$$\Phi = \int_0^z g dz \qquad (1.14.2)$$

i.e., $d\Phi = gdz$ (1.14.3)

In meteorology, heights or depths are given in terms of this geopotential under the name **geopotential height or depth.** If z is expressed in meters, the geopotential height in meters (**geopotential meter**) is given by

$$\Phi = \frac{1}{9.8}\int_0^z g dz \qquad (1.14.4)$$

Using the geopotential in hydrostatic equation (1.8.2) we have

$$dp = -g\rho dz = -\rho d\Phi = -\frac{pM}{RT} d\Phi$$

i.e., $\int_{p_1}^{p_2} \frac{dp}{p} = -\frac{M}{R}\int_{\Phi_1}^{\Phi_2} \frac{d\Phi}{T^*}$ (1.14.5)

where the virtual temperature T^* is introduced to provide a more accurate measurement. In practice using mean virtual temperature $\bar{T}^*$ we have from (1.14.5)

$$\int_{p_1}^{p_2} \frac{dp}{p} = -\frac{M}{R}\int_{\Phi_1}^{\Phi_2} \frac{d\Phi}{\overline{T}^*}$$

$$\text{i.e. In } p_2 - \text{In } p_1 = -\frac{M}{\overline{T}^* R}(\Phi_2 - \Phi_1)$$

$$\text{i.e., } \Phi_2 - \Phi_1 = -\frac{\overline{T}^* R}{M}(\text{In } p_2 - \text{In } p_1) \qquad (1.14.6)$$

which is the required geopotential thickness i.e., the height between the two pressures.

4.1.16 Potential Temperature:

It is desirable frequently to have some property of dry air, which is invariant during the adiabatic processes. Such a property is the potential temperature, designated by θ, defined as the temperature, which the air would attain when brought dry-adiabatically to a standard pressure, usually 1000 mb. So from the Poisson's equation (1.7.1) replacing T_0 by θ and p_0 by 1000 mb we have

$$\theta = T\left(\frac{1000}{p}\right)^{\frac{R^*}{MC_p}} \qquad (1.15.1)$$

The fact that θ is invariant for adiabatic processes may be readily demonstrated. We assume that initially the air at pressure p_0 has a temperature T_0. According to (1.15.1), its potential θ_0 is given by

$$\theta_0 = T_0\left(\frac{1000}{p}\right)^{\frac{R^*}{MC_p}} \qquad (1.15.2)$$

If as a result of an adiabatic process, its pressure becomes p_1, then, according to (1.7.1), its temperature T_1 is given by

$$T_1 = T_0\left(\frac{p_1}{p_0}\right)^{\frac{R^*}{MC_p}} \qquad (1.15.3)$$

Now the potential temperature (θ_1) of the air given by

$$\theta_1 = T_1\left(\frac{1000}{p_1}\right)^{\frac{R^*}{MC_p}} \qquad (1.15.4)$$

By substituting for T_1 from (1.15.3) in (1.15.4), it follows that

$$\theta_1 = T_0\left(\frac{p_1}{p_0}\right)^{\frac{R^*}{MC_p}}\left(\frac{1000}{p_1}\right)^{\frac{R^*}{MC_p}} = T_0\left(\frac{1000}{p_0}\right)^{\frac{R^*}{MC_p}} = \theta_0 \qquad \text{by (1.15.2)}$$

Therefore, no matter how many adiabatic processes occur, the potential temperature of the air does not change. So, a parcel of dry air moving adiabatically will conserve its potential temperature.

Note: In the atmosphere, no processes are completely adiabatic, since there is some mixing between a given mass of air and the surrounding air, and there may be loss or gain of heat by radiation. However, these are secondary effects, and they may usually be neglected.

Adiabatic Lapse Rate in terms of potential temperature: A relationship between the lapse rate of temperature (i.e., the rate of decrease of temperature with respect to height) and the rate of change of potential temperature with respect to height can be obtained by taking logarithm of (1.15.1) and differentiating with respect to height, we find that

$$\frac{\partial}{\partial z}(\ln\theta) = \frac{\partial}{\partial z}(\ln T) + \frac{\partial}{\partial z}\left[\frac{R}{C_p}(\ln 1000 - \ln p)\right] \text{ where is gas constant}$$

i.e., $\frac{T}{\theta}\frac{\partial \theta}{\partial z} = \frac{\partial T}{\partial z} + \frac{g}{C_p}$

For an atmosphere in which the potential temperature is constant with respect to height the lapse rate is thus

$$-\frac{dT}{dz} = \frac{g}{C_p} = \Gamma_d$$

Hence, the dry adiabatic lapse rate is approximately constant throughout the lower atmosphere. If potential temperature is function of height the actual lapse rate $\Gamma \equiv -\frac{\partial T}{\partial z}$ will differ from the adiabatic lapse arte and we have

$$\frac{T}{\theta}\frac{\partial \theta}{\partial z} = \Gamma_d - \Gamma \qquad (1.15.5)$$

4.1.16 Stability of Dry air:

Stability is sometimes defined as that condition in the atmosphere in which vertical motions are absent or definitely restricted, and conversely, **instability** is defined as that wherein vertical movement is prevalent. The effects are demonstrated from a consideration of the temperature distribution. By noting at any given level the difference in temperature between an upward moving element and the surrounding atmosphere, definite conclusions can be drawn as to the stability or instability. The surrounding atmosphere is described as stable or unstable depending on whether its temperature lapse rate brings about a decrease or an increase of the buoyancy forces on an upward moving element. In the normal case, the rising air has at a corresponding level a lower temperature than the surroundings. This represents stability. We illustrate taking following consideration.

We suppose that an unsaturated layer in the atmosphere is 200 meter thick and has the prevailing lapse rate represented by the curve AA' of the following figure. It is seen that it is less than the dry-adiabatic lapse rate. We suppose that an element of air at O, midway between the top and bottom of the layer is displaced upward, encountering steadily lowering pressures and therefore cooling at the adiabatic rate. Its course through the layer in terms of temperature and pressure is shown by the upper half of the broken line passing through O. It will be noted that along its path, as soon as it leaves the original level, its temperature is less than that of its surroundings (represented by AA') and that this temperature difference between moving element and surrounding atmosphere becomes larger the greater the displacement. Knowing from the equation of state that at the same pressure and, therefore, at the same height in the atmosphere the density of the air depends inversely on the temperature, we recognize that the moving element becomes heavier than its surroundings; and as the temperature is lowered more and more by expansion, this difference in density will become greater. Therefore, unless a strong mechanical force pushes it strongly upward, it will sink back to the original level. Similarly, if the element moves downward from O, represented by the lower part of the broken line, it will become increasingly warmer than its surroundings and therefore lighter, so that the buoyancy forces will tend to return it to the original position. Therefore, we say that the curve AA' represents a stable lapse rate or the atmosphere in this case is stable.

The lapse rate BB' is characteristics of an unstable atmosphere. This is seen by again moving an element from O through this unstable, is, for now it is continually becoming lighter than its surroundings as it moves upward and heavier (older) as it sinks. Given an impetus (momentum) upward from O, this air will continue to rise of its own accord, and if pressed downward, it will continue to sink an accelerated rate.

When the lapse rate of the surrounding atmosphere is dry adiabatic, then the rising or descending element would always have the temperature of its surroundings. So the atmosphere would offer no resistance to vertical motion.

Hence if we represent the environmental lapse rate by Γ and the dry adiabatic lapse rate Γ_d, then we may write the following conditions:

$\Gamma < \Gamma_d$ $\Rightarrow$ Stable

$\Gamma = \Gamma_d$ $\Rightarrow$ Neutral

$\Gamma > \Gamma_d$ $\Rightarrow$ Unstable (1.16.1)

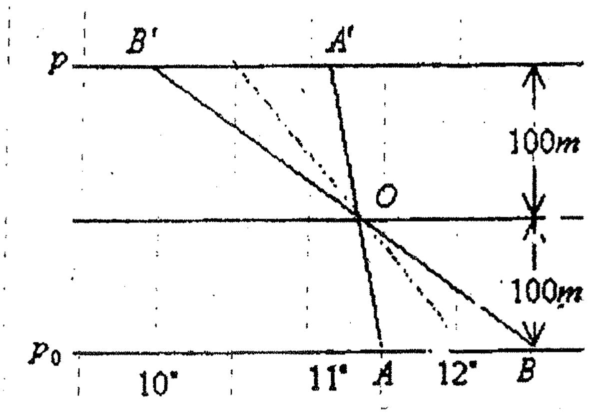
Also, since a dry-adiabatic line is a line of constant potential temperature θ, so we may write the following also:

$\frac{\partial \theta}{\partial z} < 0$ $\Rightarrow$ Stable

$\frac{\partial \theta}{\partial z} = 0$ $\Rightarrow$ Neutral

$\frac{\partial \theta}{\partial z} < 0$ $\Rightarrow$ Unstable

The requirements for stability and instability under conditions of water vapor saturation, that is, with 100 percent relative humidity, are quite different from these.



4.1.17 Stability criteria for dry air in terms of potential temperatre:

Static Stability considers only buoyancy to estimate flow stability. Flow stability indicates whether the atmosphere will develop turbulence or growing waves. Unstable air becomes turbulent. Stable air becomes laminar (non-turbulent). Dynamic stability considers both wind shears and buoyancy.

To determine static stability, a small portion of the environmental air can be conceptually displaced from its starting point and tacked as an air parcel. When conceptually moving this air parcel, follow a dry adiabat while the air is unsaturated (i.e., below its LCL), otherwise follow a moist adiabat. At any height, buoyant force is based on the temperature difference between the parcel and the environment.

From the equation (1.15.5) and (1.16.1) we can say that if $\Gamma < \Gamma_d$ so that θ increases with height, an air parcel which undergoes an adiabatic displacement from its equilibrium level will be positively (negatively) buoyant when displaced vertically downward (upward) so that it will tend to return to its equilibrium level and the atmosphere is said to be statistically stable or stably satisfied.

Adiabatic oscillations of a fluid parcel about its equilibrium level in a stably stratified atmosphere are referred to as **buoyancy oscillations**. The characteristic frequency of such oscillations can be derived by considering a parcel which is displaced vertically a small distance ∂z withot disturbing its environment. If the environment is in hydrostatic balance we have

$$-\frac{d\bar{p}}{dz} = \bar{\rho} g,$$

where $\bar{p}$ and $\bar{\rho}$ are the pressure and density of the environment. The vertical acceleration of the parcel will be

$$\frac{dw}{dz} = \frac{d^2}{dt^2}(\delta z) = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (1.17.1)$$

where p and ρ are the pressure and density of the parcel, respectively. In the parcel method it is assumed that the pressure of the parcel instantaneously adjusts to the environmental pressure during the displacement: $p = \bar{p}$. This condition must be true if the parcel is to leave the environment undisturbed. Thus with the aid of hydrostatic relationship pressure can be eliminated in (1.17.1) to give the following:

$$\frac{d^2}{dt^2}(\delta z) = -g + \frac{1}{\rho}\bar{\rho}g$$

$$\text{i.e., } \frac{d^2}{dt^2}(\delta z) = g\left(\frac{\bar{\rho}-\rho}{\rho}\right)$$

$$\text{i.e., } \frac{d^2}{dt^2}(\delta z) = g\left(\frac{\theta-\bar{\theta}}{\bar{\theta}}\right) \qquad (1.17.2)$$

where the Poisson's equation and the ideal gas law have been used to express the buoyancy force in terms of potential temperature. If the parcel is initially at level $z=0$ where the potential temperatre is θ_0, then for small displacement δz we can represent the environmental potential temperatre as

$$\bar{\theta}(\delta z) \cong \theta_0 + \frac{d\bar{\theta}}{dz}\delta z$$

If the parcel displacement is adiabatic, the potential temperatre of the parcel is conserved. So we have $\theta(\delta z) = \theta_0$. Therefore, the equation (1.17.2) becomes

$$\frac{d^2}{dt^2}(\delta z) = -\frac{g}{\bar{\theta}}\frac{d\bar{\theta}}{dz}\delta z$$

$$\frac{d^2}{dt^2}(\delta z) = -N^2\delta z \quad (1.17.3)$$

where $N^2 = \frac{g}{\bar{\theta}}\frac{d\bar{\theta}}{dz}$ is a measre of the static stability of the environment. The equation (1.17.3) has a general solution of the form $\delta z = Ae^{iNt}$.

Case-1: Therefore, if $N > 0$, then the parcel will oscillate about its initial level with a period $\tau = \frac{2\pi}{N}$.

The corresponding frequency N is the **buoyancy frequency.**

Case-2: In the case of $N = 0$, the equation (1.17.3) indicates that no accelerating force will exist and the parcel will be in neutral equilibrium at its new level.

Case-3: On the other hand, if $N < 0$ (potential temperature decreasing with height) the displacement will increase exponentially in time.

Therefore, we arrive the static stability criteria for dry air as follows:

$$\frac{d\theta}{dz} = \begin{cases} > 0 & stable \\ = 0 & neutral \\ < 0 & unstable \end{cases} \qquad (1.17.4)$$

4.2 Humidity Variables:

The amount of water vapor in the atmosphere may be expressed in any one of four ways, which are as follows:

a) Mixing Ratio

b) Specific Humidity

c) Absolute Humidity

d) Relative Humidity

e = vapor pressure

p = total pressure of the moist air

$p - e$ = pressure of dry air

m_v, m_d = molecular weight of vapor pressure and dry air

R_v, R_d = gas constants of vapor pressure and dry air

R^* = universal gas constant, V = Volume of the moist air

Note: The state of air:

$pV = nR^*T$ where n = number of moles of the gas

i.e., $pV = \frac{M}{m}R^*T$ where M = mass of the gas and m = **molecular weight of the gas**

$$= M\frac{R^*}{m}T = MRT \text{ where } R = \frac{R^*}{m}$$

i.e., $p = \frac{M}{V}RT$, i.e., $p = \rho RT$ where $\rho\frac{M}{V}$ = density of the gas

(a) Mixing Ratio: The ratio of mass of water vapor to mass of dry air is called the mixing ratio (r). It is given by

$$r = \frac{m_v}{m_d} = \frac{\rho_v V}{\rho_d V} = \frac{\rho_v}{\rho_d}$$

$$= \frac{e}{R_v T} \bigg/ \frac{R_d T}{(p-e)} \text{ since for water vapor, } e = \rho_v R_v T \text{ and for dry air, } p - e = \rho_d R_d T$$

$$= \frac{e}{p-e}\frac{R_d}{R_v} = \frac{\varepsilon e}{p-e} \text{ since } R^* = m_v R_v = m_d R_d \Rightarrow \frac{R_d}{R_v} = \frac{m_v}{m_d} = \frac{18.016}{28.97} = 0.622 = \varepsilon$$

Therefore, $\frac{\varepsilon e}{p-e}$ (2.1.1)

Since p is of the order of one hundred times as great as e, for most practical purposes the latter may be neglected in comparison with the former. So in this case we have

$$r \cong \frac{\varepsilon e}{p} \quad (2.1.2)$$

The equation (1) says that r is proportional to the ratio of partial pressure of water vapor (e) to the partial pressure of the remaining gases in the air ($p - e$). The **saturated mixing ratio r_s** is given by (1) except with e_s in place of e.

Unit: Although units of mixing ratio are g/g (i.e., grams of water vapor per gram of dry air), it is usually presented as g/kg (i.e., grams of water vapor per kilogram of dry air) by multiplying the g/g value by 1000. Also **note that**

g/g does not cancel to become 1, because the numerator and denominator represent mass of different substances.

(b) Specific Humidity: The ratio of mass of water vapor to mass of total (moist) air is called the specific humidity (q) i.e., it is given by

$$q = \frac{m_v}{m_v + m_d} = \frac{m_v}{m} = \frac{\rho_v V}{\rho V} = \frac{\rho_v}{\rho} = \frac{\dfrac{e}{R_v T}}{\dfrac{\rho}{R_d T}\left[1-(1-\varepsilon)\dfrac{e}{p}\right]} = \frac{R_d}{R_v}\frac{e}{\left[p-(1-\varepsilon)e\right]}$$

i.e., $q = \dfrac{\varepsilon e}{p-(1-\varepsilon)e}$ where (mass of air) (moist) $m = m_d + m_v$ (2.1.3)

Since p is of the order of one hundred times as great as e, for most practical purposes the latter may be neglected in comparison with the former. So the good approximation is given by

$$q = \frac{\varepsilon e}{p} \tag{2.1.4}$$

Or for saturated air

$$q_s = \frac{\varepsilon e_s}{p}$$

It has units of g/g or g/kg.

Note: the density of air is given by

$$\rho = \rho_d + \rho_v = \frac{p-e}{R_d T} + \frac{e}{R_v T} = \frac{p-e}{R_d T} + \frac{e}{\dfrac{R_d}{\varepsilon}T} = \frac{p-e}{R_d T} + \frac{\varepsilon e}{R_d T} = \frac{p}{R_d T}\left[\frac{p-e}{p} + \frac{\varepsilon e}{p}\right]$$

$$= \frac{P}{R_d T}\left[1 - \frac{e}{p} + \frac{\varepsilon e}{p}\right] = \frac{p}{R_d T}\left[1-(1-\varepsilon)\frac{e}{p}\right]$$

(c) Absolute Humidity: The number of grams of water vapor in unit volume or the density of water vapor is known as absolute humidity (ρ_v) i.e., $\rho_v = \dfrac{e}{R_v T} = \dfrac{\varepsilon e}{R_d T}$

It has unit g/m^3.

(d) Relative Humidity: It is defined as the ratio of the actual amount of water vapor in the air compared to the equilibrium (saturation) amount at that temperature. So we have

$$R_H = \frac{m_v}{m_s} = \frac{\rho_v V}{\rho_s V} = \frac{\rho_v}{\rho_s}$$

$$= \frac{\dfrac{e}{R_v T}}{\dfrac{e_s}{R_v T}} = \frac{e}{e_s}$$

i.e., $R_H = \dfrac{e}{e_s}$

So relative humidity indicates the amount of net evaporation that is possible into the air, regardless of the temperature. At R_H = 100% no net evaporation occurs because the air is already saturated.

Note: If the relative humidity is to be given as a percentage, then it is customary in practice that the right hand side of above equation must be multiplied by 100.

Exercise: Prove that

(i) $q = \dfrac{r}{1+r}$, (ii) $e = \dfrac{r}{\varepsilon + r}$, (iii) $q = \dfrac{\rho_v}{\rho_d + \rho_v}$

4.2.2 The equation of state for moist unsaturated air:

For most purposes, the equation of state for dry air may be used without modification for moist unsaturated air. On some occasions, however, a more accurate relationship in necessary. The appropriate gas constant R_m for moist unsaturated air may then be used instead of the gas constant for dry air R_d. It is given by as follows:

Now the density of moist unsaturated air (ρ) is given by

$$\rho = \rho_d + \rho_v = \frac{p-e}{R_d T} + \frac{e}{R_v T} = \frac{p-e}{R_d T} + \frac{e}{\frac{R_d}{\varepsilon} T} = \frac{p-e}{R_d T} + \frac{\varepsilon e}{R_d T} = \frac{p}{R_d T}\left[\frac{p-e}{p} + \frac{\varepsilon e}{p}\right]$$

$$= \frac{p}{R_d T}\left[1 - \frac{e}{p} + \frac{\varepsilon e}{p}\right] = \frac{p}{R_d T}\left[1 - (1-\varepsilon)\frac{e}{p}\right]$$

So the equation of state for moist unsaturated air is given by

$$p = \rho R_m T$$

i.e., $\frac{p}{R_m T} = \rho$

i.e., $\frac{p}{R_m T} = \frac{p}{R_d T}\left[1 - (1-\varepsilon)\frac{e}{p}\right]$

i.e., $R_m = \frac{R_d}{1 - (1-\varepsilon)\frac{e}{p}}$

So R_m is not constant for all amounts of water vapor in the air, but varies with the vapor pressure e and the total pressure p. Using the relation $q = \frac{\varepsilon e}{p}$ we have

$$R_m = \frac{R_d}{1 - (1-\varepsilon)\frac{e}{p}} = \frac{\varepsilon R_d}{\varepsilon - q(1-\varepsilon)} \qquad (2.2.1)$$

This equation shows that R_m is a function of the specific humidity alone (approx.)

It is often more convenient to use a different value of the temperature, known as virtual temperature T_{vir} rather than a different value for the gas constant. So the equation of state for moist unsaturated air then becomes

$$\text{i.e., } \rho = \frac{p}{R_d T}\left[1-(1-\varepsilon)\frac{e}{p}\right]$$

$$\text{i.e., } \rho = \frac{p}{R_d T_{vir}} \text{ where } T_{vir} = \frac{T}{1-(1-\varepsilon)\frac{e}{p}}$$

$$\text{i.e., } p = \rho R_d T_{vir}$$

Therefore the virtual temperature of the moist unsaturated air may be defined as the temperature at which dry air of the same pressure would have the same density as the moist air.

Exercise: Prove that $T_{vir} = T(1+0.61r)$ for moist unsaturated air.

Now we know that

$$T_{vir} = \frac{T}{1-(1-\varepsilon)\frac{e}{p}} \text{ and } r = \frac{\varepsilon e}{p-e}$$

Note: The equation of state for dry air is given by

$p_d = \rho_d R_d T$ where $R_d = 287.053\, J\, K^{-1}\, kg^{-1}$ is the gas constant for dry air.

4.2.3 The Adiabatic Lapse Rate for Moist, Unsaturated Air:

The humidity-mixing ratio of a mass of air is given by the mass of the water vapor per unit mass of dry air. From this method of expressing the mixing ratio, it follows that it is invariant during ascent or descent, as long as condensation and evaporation processes do not occur and if mixing with the surrounding atmosphere is so small as to be negligible. **Now it will be shown that the lapse rate for moist, unsaturated air is a function of the mixing ratio.**

Now the specific heat at constant pressure of 1 gm of moist, unsaturated air is given by

$$C_{pm} = \frac{C_p + rC'_p}{1+r} \qquad (2.3.1)$$

where C'_p is the specific heat for water vapor at constant pressure. By using this value for the specific heat C_{pm} and the gas constant R_m for moist, unsaturated air in (1.7) for adiabatic process, we have

$$\frac{dT}{T} = \frac{R_m}{C_{pm}} \frac{dp}{p} \qquad (2.3.2)$$

where p is now the total pressure. Substituting for R_m from (2.2.1) and R_m from (2.3.1) in 2.3.2), we have

$$\frac{dT}{T} = \frac{1+r}{C_p + rC'_p} \frac{\varepsilon R_d}{\varepsilon - (1-\varepsilon)q} \frac{dp}{p} = \frac{1+r}{C_p + rC'p} \frac{\varepsilon R_d}{\varepsilon - (1-\varepsilon)r} \frac{dp}{p}$$ using (2.1.2) and (2.1.4)

i.e., $\frac{dT}{T} = \frac{R_d}{bC_p} \frac{dp}{p}$ (2.3.3)

where $b = \frac{\left(1 + r\frac{C'_p}{C_p}\right)\left[\varepsilon - (1-\varepsilon)r\right]}{\varepsilon(1+r)} = \frac{1+1.95r}{1+1.61r} = 1 + 0.34r$ by putting

$C'_p = 0.465$ cal gm^{-1} deg^{-1}, $C_p = 0.239$ cal gm^{-1} deg^{-1} and $\varepsilon = 0.622$ and neglecting

r^2 in comparison with r.

Integrating (2.3.3) we have

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{R_d}{bC_p}} \qquad (2.3.4)$$

If the moisture content of the environment is taken into account, from hydrostatic equation using equation of state for moist, unsaturated air, we have

$$\frac{dp}{p} = -\frac{g}{R_d \overline{T}_v} dz \qquad (2.3.5)$$

where $\overline{T}_v$ is the virtual temperature of the environment at the pressure under consideration. Combining (2.3.3) and (2.3.5) we have

$$-\frac{dT}{T} = \frac{g}{bC_p} \frac{T}{\overline{T}_v}$$

Since $\overline{T}_v$ is very little different in value from T, their ratio may be taken as unity. So we have

$$-\frac{dT}{T} = \frac{g}{C_p(1+0.34r)} \qquad (2.3.6)$$

which the required adiabatic lapse rate for moist, unsaturated air.

5.0 Unit Summary

In this module, heat balance in the atmosphere, atmospheric air composition, equation of state for dry air and moist air, internal energy, first law of thermodynamics, change in internal energy in the atmosphere, adiabatic process, hydrostatic equation, variation of pressrue with altitude, potential temperature, stability of dry air, humidity variables are discussed.

6.0 Self Assessment Questions

Q1 (a) Derive the following expression for the vertical distribution of the density when the lapse rate of temperature is constant

$$\rho = \rho_0 \left(\frac{T_0 - \alpha z}{T_0} \right)^{\frac{g}{Rm} - 1}$$

Assume that the air is dry and that g is constant.

(b) For what value of the temperature lapse rate is the density constant with height?

Q2. Calculate at what height in a dry atmosphere the pressure is one-half of that at the surface when the surface temperature is 10^0C, and the lapse rate is (a) 6^0C, per km, (b) zero. Assume that g is constant.

Q3. Expressing the first law of thermodynamics in terms of dT and $d\alpha$ (where α is the specific volume), obtain Poisson's equation in terms of T, T_0, α, α_0.

Q4. Find the rate of change of circulation in the atmosphere.

Q5. Derive the thermal wind equation.

Q6. What is the purpose of the Aerological diagram. Mention all characteristics to draw an aerological diagram.

Q7. Define mixing ratio, specific humidity and relative humidity, absolute humidity. establish the relation between these.

Q8. What is the virtual temperature? Find the adiabatic lapse rate for moist unsaturated air.

Q9. Describe the temperature distribution in the atmosphere.

Q10. Show that the dry adiabatic lapse rate is approximately constant throughout the lower atmosphere.

Q11. What are different kinds of wind that may exist in the atmosphere. Obtain the governing equation of one such wind.

Q12. Obtain the atmosphere energy equation stating clearly the assumptions you have made. Interpret each term of your equation.

Q13. Define potential temperature. Obtain the relation $S = C_p \ln \theta + \text{constant}$ where S is the specific entropy and θ is the potential temperature for a parcel of dry air.

Q14. What are the general characteristics of the atmosphere? State the first law of thermodynamics.

Q15. Derive the expression for the vorticity of an air parcel.

Q16. What is geodynamic paradox?

Q17. Write down the basic assumption made in determining the stability criteria for the vertical motions of an individual parcel of air. Show that the parcel of air will be stable, neutral and unstable according as $\Gamma_d >, =, < \gamma$.

Q18. Write down the equation of motion of an atmosphere. Obtain Gradient wind equation, stating clearly the assumptions you have made.

Q19. What do you mean by an adiabatic process and isobaric process in the atmosphere?

Q20. Derive the area equivalence of the emagram and tephigram. Discuss the important features of these.

Q21. The temperature at a point 50 km north is 3^{0}C cooler than at the station. If the wind is blowing from NE at 32 ms^{-1} and the air is being heated by radiation at the rate of 1^{0}C, what is the local temperature change at the station?

Q22. Show that as the pressure gradient approaches zero the gradient wind reduces to the geostrophic wind for a normal anticyclone and to the inertia circle for an anomalous anticyclone.

Q23. Show that the geostrophic wind is independent of height in a barotropic atmosphere.

Q24. If wind rotates as a solid body about the centre of a low pressure system, and the tangenital velocity is 10m/s at radius 300 km, find the relative vorticity.

7.0 Suggested further Readings

1. Brunt, D., Physical and Dynamical Meteorology, London, Cambridge University Press, 1939.

2. Hewson, W.E., and Longley, W.R., Meteorology Theoretical and Applied, John Wiley & Sons, INC., Chapman & Hall, LTD., London.

3. Byers, H.R., General Meteorology, McGraw-Hall., Godson, W.L. and Iribarne, J.V., Atmospheric Thermodynamics, D. Reidel Publishing Company.

4. Holton, J.R., An Introduction to Dynamic Meteorology, Academic Press, New York.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper- Group–

Module No. - 95
DYNAMICAL METEOROLOGY

Contents:

1.0 Introduction

Thermodynamics, literally speaking, concerns itself with the flow of heat and conversion of heat into work. Heat, as we know now, is a form of energy and the science of thermodynamics now governs not only the transformation of heat into work and vice versa but also includes all types of interconversion of one kind of energy into another, e.g., electric energy into work, chemical energy into electrical energy etc. It is based on three simple but empirically derived laws of nature, acquired from long experience. The thermodynamics of the atmosphere is very important on dynamic meteorology.

2.0 Objectives

In this module, the followings are discussed

* some important Aerological diagrams, graphical computation on the diagrams, meteorological Conventions, atmospheric structure,
* atmospheric motion, fundamental atmospheric forces,
* inertial and non-inertial frame of reference,
* the education of motion of an air parçel,
* atmospheric motion under balanced forces,
* the geostrophic wind equation

3.0 Key Words and Study guides

Aerological diagrams, fundamental atmospheric forces, geostrophic wind equation.

4.0 Main Discussion

4.1 Aerological or thermodynamic diagram:

It is a graphical display of the different isopleths such as isobars, isotherms, equi-saturated curves or vapor lines (for which $R_H = 1$), curves of equal potential temperature or isentropics for dry air, or dry adiabats, etc. which are represented by different thermodynamic processes in the atmosphere. We may plot on such a diagram the observed state of any set of air parcels and then we may evaluate graphically the effects of any of these processes.

Purposes of aerological diagram: The importance of the diagram lies in the large amount of information that can be rapidly obtained from them. Some of this information is as follows:

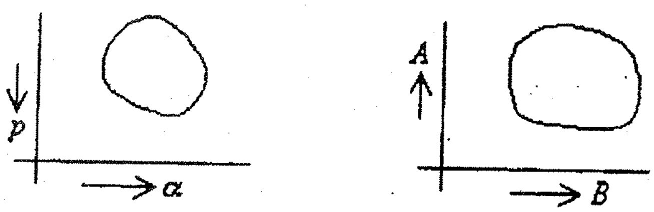
- They allow the study of the vertical stability of the atmosphere.
- The thickness of layers between two given values of the pressure is readily computed from them.
- A number of atmospheric processes can be conveniently studied with their aid.
- This diagram can measure a number of properties of the atmosphere above a certain location.

Criteria for aerological diagram: There are certain criteria, which are taken into account in the selection of the diagram, which are as follows:

- The first desirable characteristic of such a diagram is that the area enclosed by the lines representing any cyclic process be proportional to the change in energy or the work done during the process. In other words, if energy integrals (such as, for instance, the work performed by a parcel of moist air along a cycle) can be determined by measuring areas on the diagram. Such diagrams are sometimes called equivalent, or area preserving.
- The second important property of thermodynamic diagram is that which and how many isopleths are straight lines. Straight lines facilitate the use of the diagram for plotting as well as analysis and representation.
- The third property is the angle between isotherms and adiabats. The larger this angle is, the more sensitive is the diagram to variations in the rate of change of temperature with pressure along the vertical. This is important for the analysis of stability.
- Some advantage may be obtained if one of the main isopleths is congruent with respect to a displacement along one of the coordinates.
- If the ordinate varies monotonically with height, roughly proportional to it, the atmosphere can be more conveniently imagined or visualized over the diagram.

rea Equivalence: The fundamental expression of specific work for an element is $dw = -pd\alpha$. It suggests that 'e may use p and α as the co-ordinates in order to satisfy the first criteria. However, the angle between the isotherms and adiabats of a $(\alpha, -p)$ diagram is quite small; such a diagram does not satisfy the third property. Therefore we have to transform this $(\alpha, -p)$ diagram into a suitable diagram in which co-ordinates are two functions of thermodynamic variables, subject to the restriction that area enclosed by any cycle in this new thermodynamic diagram must be equal to the area enclosed by cycle on $(\alpha, -p)$ diagram. Such a diagram is called 'equal area' transformation of the $(\alpha, -p)$ diagram.

Now we consider two variables A and B of which is a function of one or more thermodynamic variables. The representation of a cycle on a $(\alpha, -p)$ diagram and its equal area transformation to a (B, A) diagram is shown in the following figure.



Therefore, if (B, A) is the thermodynamic diagram, then for a cyclic process, we have

$$-\oint p\,d\alpha = \oint A\,dB$$

i.e., $\oint (p\,d\alpha + A\,dB) = 0$

Since the integral vanishes along a closed path, hence the integrand must be an exact differential, dS saying. So we have

$$\oint (p\,d\alpha + A\,dB) = dS \qquad (3.1)$$

where S is assumed to be a function of α and B.

Since S is a function α and B, hence we have

$$dS = \left(\frac{\partial S}{\partial \alpha}\right)_B d\alpha + \left(\frac{\partial S}{\partial B}\right)_\alpha dB \qquad (3.2)$$

Using (3.2) in (3.1), we have

$$\oint (p\,d\alpha + A\,dB) = \left(\frac{\partial S}{\partial \alpha}\right)_B d\alpha + \left(\frac{\partial S}{\partial B}\right)_\alpha dB$$

From this relation, we get

$$p = \left(\frac{\partial S}{\partial \alpha}\right)_B \text{ and } A = \left(\frac{\partial S}{\partial B}\right)_\alpha \tag{3.3}$$

which is the sufficient condition for an equal area transformation. Also from (3.3) we get

$$\left(\frac{\partial p}{\partial B}\right)_\alpha = \frac{\partial^2 S}{\partial \alpha \partial B} \text{ and } \left(\frac{\partial A}{\partial \alpha}\right)_B = \frac{\partial^2 S}{\partial B \partial \alpha}$$

$$\text{i.e., } \left(\frac{\partial A}{\partial \alpha}\right)_B = \left(\frac{\partial p}{\partial B}\right)_\alpha \tag{3.4}$$

If the condition (3.4) is satisfied then the areas will be equal on two diagrams.

4.1.1 Some important diagrams:

Many different diagrams have been developed for general and for special purposes. Among all these, we discuss only Tephigram, Emagram.

Tephigram: This derives its name from its co-ordinates of temperature and entropy (T, ϕ). Sir Napier Shaw introduced it. A line of constant entropy is a line of constant potential temperature. So, in this diagram, the co-ordinates are T, In ϕ. The pressure lines are diagonals with the lowest values in the upper left. Again the moisture lines are the straight, dashed lines and pseudoadiabats (saturated lines) are curved.

Equivalent property: We say that $B = T$. So from (3.4) we have

$$\left(\frac{\partial A}{\partial \alpha}\right)_T = \left(\frac{\partial p}{\partial T}\right)_\alpha \tag{3.1.1}$$

Now the equation of state of air is given by

$$p\alpha = RT$$

$$\text{i.e., } \alpha\left(\frac{\partial p}{\partial T}\right)_\alpha = R$$

$$\text{i.e., } \left(\frac{\partial p}{\partial T}\right)_\alpha = \frac{R}{\alpha} \tag{3.1.2}$$

Thus (3.1.1) reduces to the following form

$$\left(\frac{\partial A}{\partial \alpha}\right)_T = \frac{R}{\alpha}$$

$$\text{i.e.,} \left(\frac{\partial A}{\partial \alpha}\right)_T d\alpha = \frac{R}{\alpha} d\alpha$$

$$\text{i.e.,} A = R \ln \alpha + F(T) \tag{3.1.3}$$

where $F(T)$ is a unspecified function of T which can be chosen arbitrary. Now we introduce the potential temperature θ by the Poisson's equation of the following form

$$\left(\frac{T}{\theta}\right) = \left(\frac{p}{1000}\right)^{\frac{R}{C_p}}$$

$$\text{i.e.,} \left(\frac{T}{\theta}\right) = \left(\frac{RT}{1000\alpha}\right)^{\frac{R}{C_p}} \quad \text{since } p\alpha = RT$$

$$\text{i.e.,} \ln T - \ln \theta = \frac{R}{C_p}\{\ln R + \ln T - \ln 1000 - \ln \alpha\}$$

$$\text{i.e.,} \ R \ln \alpha = C_p \ln \theta - C_p \ln T + R \ln R + R \ln T - R \ln 1000$$

$$\text{i.e.,} \ R \ln \alpha = C_p \ln \theta + G(T) \tag{3.1.4}$$

where $G(T) = -C_p \ln T + R \ln R + R \ln T - R \ln 1000$. Using (3.1.4) in (3.1.3) we have

$$A = C_p \ln \theta + G(T) + F(T) \tag{3.1.5}$$

Now we choose the function $F(T)$ such that $F(T) = -G(T)$. So the equation (3.1.5) reduces to

$$A = C_p \ln \theta \tag{3.1.6}$$

Therefore in Tephigram, we have the co-ordinates

$$\left.\begin{aligned} A &= C_p \ln \theta \\ B &= T \end{aligned}\right\} \tag{3.1.7}$$

As C_p In θ is equal to the entropy apart from an additive constant, Sir Napier who introduced this diagram called it the T – θ diagram or Tephigram.

Case - 1: Equations of isotherms (i.e., T = constant)

$\therefore B$ = constant

So, these are straight lines parallel to ordinate.

Case - 2: Equations of dry-adiabats (i.e., θ = constant).

$\therefore A$ = constant

So, these are straight lines parallel to abscissa.

Case - 3: Equations of isobars (i.e., p = constant)

Now the Poisson's equation may be written as

$$\frac{\theta}{T} = \left(\frac{1000}{p}\right)^{\frac{R}{C_p}}$$

$$\text{i.e., } \ln\theta - \ln T = \frac{R}{C_p}(\ln 1000 - \ln p)$$

$$\text{i.e., } C_p \ln\theta = C_p \ln T + R(\ln 1000 - \ln p)$$

$$\text{i.e., } A = C_p \ln T + R(\ln 1000 - \ln p) \text{ since } A = C_p \ln\theta$$

$$\text{i.e., } y = C_p \ln x + R(\ln 1000 - \ln p) \text{ taking } y = A \text{ and } x = T$$

Since one co-ordinate of this diagram is C_p In θ but the other is a linear function of T, hence the isobars are logarithmic curves which slope upwards to the right and decrease in slope with increasing temperature. For meteorological purpose the isobars are nearly straight lines.

Case-4: Equations of constant saturation mixing ratio (i.e., r_s = constant)

Now the saturation-mixing ratio is defined by

$$r_s = \frac{\varepsilon e_s}{p}$$

i.e., $\text{In}\ r_s = \text{In}\ \varepsilon + \text{In}\ e_s - \text{In}\ p$ **(3.1.8)**

Again from the Poisson's equation we have

$$\frac{\theta}{T} = \left(\frac{1000}{p}\right)^{\frac{R}{C_p}}$$

i.e., $\text{In}\ \theta - \text{In}\ T = \frac{R}{C_p}(\text{In}\ 1000 - \text{In}\ p)$

i.e., $C_p\ \text{In}\ \theta = C_p\ \text{In}\ T + R\ \text{In}\ 1000 + R\{\text{In}\ r_s - \text{In}\ \varepsilon - \text{In}\ \varepsilon_s\}$ by (3.1.8) (3.1.9)

Now from the Clausius - Clapeyron equation we have

$$\text{In}\ \frac{e_s}{6.11} = \frac{ML}{R^*}\left(\frac{1}{273} - \frac{1}{T}\right)$$

i.e., $\text{In}\ e_s = \text{In}\ 6.11 + \frac{ML}{R^*}\left(\frac{1}{273} - \frac{1}{T}\right)$

Using this relation in (3.1.9) we have

$$C_p\ \text{In}\ \theta = C_p\ \text{In}\ T + R\ \text{In}\ 1000 + R\ \text{In}\ r_s - R\ \text{In}\ \varepsilon - R\ \text{In}\ 6.11 - L\left(\frac{1}{273} - \frac{1}{T}\right), \therefore R^* = RM$$

i.e., $y = C_p\ \text{In}\ x + \frac{L}{x} + R\{\text{In}\ 1000 + \text{In}\ r_s - \text{In}\ \varepsilon - \text{In}\ 6.11\} - \frac{L}{273}$

In general it is of curved nature, but within meteorological range, this represents nearly straight line.

Case-5: Equation of Pseudoadaiabatic process (i.e., θ_e = constant)

The equation of pseudoadiabatic process may be written in the following form

$$\theta_e = \theta e^{\frac{L r_s}{C_p T}}$$

i.e., $\text{In}\,\theta_e = \text{In}\,\theta + \frac{Lr_s}{C_p T}$

i.e., $C_p \,\text{In}\,\theta_e = C_p \,\text{In}\,\theta + \frac{Lr_s}{T}$

i.e., $C_p \,\text{In}\,\theta = C_p \,\text{In}\,\theta_e - \frac{Lr_s}{T}$

i.e., $y = -Lr_s \frac{1}{x} + C_p \,\text{In}\,\theta_e$

So these are also curved lines.

Hence the tephigram has the following properties:

* The angle between isotherms and adiabats is exactly 90^0.
* It is area equivalent.
* Four sets of lines, which are exactly or nearly straight, and one set which is quite curved (for pseudoadiatic).
* From the equation In $\theta = \text{In}\,T - \frac{R}{C_p}\,\text{In}\,p + \text{constant}$, it is obvious that isobars are congruent with respect to a displacement along the ordinate.

Emagram: We consider now the diagram with coordinates T and $-\text{In}\,p$. It is usually called the Emagram.

Equivalent property: Now we consider $B = T$, where $B = T$ the temperature is. Similarly as in the case of tephigram, from (3.1.3) we have

$$A = R\,\text{In}\,\alpha + F(T) \qquad (3.1.10)$$

where $F(T)$ is a unspecified function of T which can be chosen arbitrary. Now the equation of state is

$$p\alpha = RT$$

i.e., $\ln \alpha + \ln p = \ln R + \ln T$

i.e., $\ln \alpha = -\ln p + \ln R + \ln T$ (3.1.11)

Now from the equations (3.1.10) and (3.1.11), we have

$$A = R\left[-\ln p + \ln R + \ln T\right] + F(T)$$

i.e., $A = -R \ln p + \left[R \ln R + R \ln T + F(T)\right]$

Now we chose $F(T)$ in such a way that the expression within the square bracket vanish. So with such choice, we have

$$A = -R \ln p$$

Therefore, we have

$$\left.\begin{array}{l} A = -R \ln p \\ B = T \end{array}\right\} \qquad (3.1.12)$$

Case-1: Equations of isobars (i.e., p = constant)

In this case, A = constant and the lines are parallel to abscissa and also are at right angles to the isotherms.

Case-2: Equations of isotherms (i.e., T = constant)

In this case, B = constant and the lines are parallel to the ordinate and also are at right angles to the isobers.

Case-3: Equations of dry adiabats (i.e., θ = constant)

Now the Poisson's equation gives

$$\frac{\theta}{T} = \left(\frac{1000}{p}\right)^{\frac{R}{C_p}}$$

i.e., $\ln \theta = \ln T + \dfrac{R}{C_p}(\ln 1000 - \ln p)$ (3.1.13)

i.e., $\dfrac{R}{C_p}\ln p = \ln T + \dfrac{R}{C_p}\ln 1000 - \ln \theta$

i.e., $R \ln p = C_p \ln T + R \ln 1000 - C_p \ln \theta$

i.e., $A = -C_p \ln T - R \ln 1000 + C_p \ln \theta$

i.e., $y = -C_p \ln x + \text{constant for } \theta = \text{constant}$

Therefore for dry adiabats these are logarithmic curves.

Case-4: Equations of saturation vapor pressure line (i.e., $e_s = \text{constant}$)

Now the Clausius-Clapeyron equation can be written as

$$\ln \frac{e_s}{6.11} = \frac{ML_{eva}}{R^*}\left(\frac{1}{273} - \frac{1}{T}\right)$$ where L_{eva} is the latent heat of evaporation.

i.e., $$\frac{R^*}{ML_{eva}} \ln \frac{e_s}{6.11} = \left(\frac{1}{273} - \frac{1}{T}\right)$$

i.e., $$\frac{1}{T} = \frac{1}{273} - \frac{R^*}{ML_{eva}} \ln \frac{e_s}{6.11}$$

i.e., $$\therefore T = \frac{1}{\frac{1}{273} - \frac{R^*}{ML_{eva}} \ln \frac{e_s}{6.11}}$$

So, for constant e_s we get the lines, which are parallel to ordinate.

Case-5: Equation of Pseudoadiabatic process (i.e., $\theta_e = \text{constant}$)

The equation of pseudoadiabatic process may be written in the following form

$$\theta_e = \theta e^{\frac{Lr_s}{C_p T}}$$

i.e., $$\ln \theta_e = \ln \theta + \frac{Lr_s}{C_p T}$$

i.e., $$\ln \theta_e = \ln T + \frac{R}{C_p}(\ln 1000 - \ln p) + \frac{Lr_s}{C_p T}$$ by (3.1.13)

$$\text{i.e., } R \ln p = C_p \ln T + R \ln 1000 + \frac{Lr_s}{T} - C_p \ln \theta_e \tag{3.1.14}$$

Again from the Clausius-Clapeyron equation we have

$$\ln \frac{e_s}{6.11} = \frac{ML}{R^*}\left(\frac{1}{273} - \frac{1}{T}\right)$$

$$\text{i.e., } e_s = 6.11 e^{\frac{ML}{R^*}\left(\frac{1}{273} - \frac{1}{T}\right)}$$

$$\text{i.e., } \frac{pr_s}{\varepsilon} = 6.11 e^{\frac{ML}{R^*}\left(\frac{1}{273} - \frac{1}{T}\right)} \quad \text{since } r_s = \frac{\varepsilon e_s}{p}$$

$$\text{i.e., } r_s = 6.11 \frac{\varepsilon}{p} e^{\frac{ML}{R^*}\left(\frac{1}{273} - \frac{1}{T}\right)} \tag{3.1.15}$$

Using (3.1.15) the equation (3.1.14) represents a curve.

Hence the Emagram has the following properties:

- Area is proportional to energy
- Four sets of lines, which are exactly or nearly straight lines, and one set which is curved.
- An adequately good angle between adiabats and isotherms.

Hence it is convenient thermodynamic diagram and it is used widely.

Graphical computations on the Diagrams:

In practical meteorological work the thermodynamic diagrams are used for computing a variety of quantities from the given data. Radiosondes provide measurements of pressure, temperature, and relative humidity. The soundings are plotted on the diagram, and from the plot other variables can be read off or plotted on related curves. Some of the data obtainable in this manner are as follows:

* **Potential temperature** is obtained by following the dry adiabat to 1000 mb and reading the temperature there, or, more easily, by noting the values of potential temperature printed on the dry-adiabatic lines.

* **Saturation mixing ratio** is read immediately at any point on the temperature-pressure plot by interpolation from the mixing ratio lines.

* **Mixing ratio** is obtained by multiplying the saturation-mixing ratio by the measured relative humidity.

* **Dew point temperature** is obtained as the temperature at the point where the actual mixing ratio line or interpolated line interests the observed pressure line. It is useful to plot this for every significant level of the sounding to produce a curve that is a plot of both the mixing ratio and the dew point temperature. From such a curve, working in a reverse sense, one can get the relative humidity from the ratio of the mixing ratio to the saturation-mixing ratio.

* **Condensation pressure, temperature (isentropic)** is found at the intersection of the potential temperature line and the mixing ratio line, which corresponds to the value at the starting point. For certain purposes it is convenient to plot this quantity for the various significant levels.

* **Wet-bulb temperature and wet-bulb potential temperature** are obtained by following the pseudo-adiabatic line from the condensation point to the original pressure and to 1000 mb, respectively.

* **Equivalent temperature and equivalent potential temperature** are found by determining the potential temperature line, which is approached asymptotically by the pseudo-adiabat that passed through the condensation point. This potential temperature line has the value of the equivalent potential temperature. By following it back to the original pressure, the equivalent temperature is read on the temperature scale.

/et-Bulb Temperature: The wet-bulb temperature of a given mass of all may be defined as the lowest temperature which that air may be cooled by the evaporating water into it. In practice it is measured directly by means of a :ntilated wet-bulb thermometer. It is denoetd by T_w. It has been shown by Normand that, to a high degree of approximation, the dry adiabat through the dry-bulb temperature, saturated adiabat through the wet-bulb temperature, and the saturation mixing ratio line through the dew-point temperature all meet at a point (**Given before**). This relationship is shown on the tephigram given in the following figure, in which *A* represents the dry-bulb, *D* represents the wet-bulb and *B* represents the dew-point temperature. The three lines intersect at *C*, the condensation level. Thus if any two of these quantities and the pressure are known, the other two may be obtained directly with the aid of an adiabatic chart such as the tephigram.

This relationship is useful in many ways. For instance, **if the temperature, pressure, and relative humidity of a mass of air are known, the wet-bulb temperature may be obtained readily from the tephigram as follow.** The saturation-mixing ratio for the air is given by the saturation-mixing ratio for the air is given by the saturation mixing ratio line through the dry-bulb temperature *A* on the tephigram. According to the formula, $R_H = 100\frac{r}{r_s}$, the actual mixing ratio is found with sufficient accuracy by multiplying the saturation mixing ratio by the relative humidity expressed as a fraction. Proceed horizontally along the dry adiabat from *A* until the mixing ratio line corresponding to the actual mixing ratio is reached at *C*. Follow the saturated adiabat through this point until it meets the pressure line through *A* at *D*. The temperature at *D* is then the wet-bulb temperature of the air. Because of the approximation in Normand's treatment, the value found in this manner is not exactly the same as the actual reading of a wet-bulb thermometer in the air. However, the difference is so small that it may be considered negligible for all practical purposes.

Example: If the particle of air has pressure 950 mb, temperature 3°C, and relative humidity 60 percent, then the dew point is –4°C, and the wet-bulb temperature is 0°C.

Wet-Bulb Potential Temperature: The wet-bulb potential temperature of an air mass, denoted by θ_w, is defined as the wet-bulb temperature of that air when brought adiabatically to a standard pressure, usually 1000 mb. It was known that the wet-bulb temperature of ascending or descending air changes at the saturated adiabatic lapse rate.

The wet-bulb potential temperature is therefore obtained by noting the temperature at the point of intersection of the saturated adiabat through the wet-bulb temperature with the 100 mb line.

Example: It may be found with the aid of teh tephigram that air with $p = 700$ mb, $T = 3^0$C, and $R_H = 44$ percent has a wet-bulb potential temperature of 14°C.

Note: It follows by definition that wet-bulb potential temperature is conservative for processes of ascent and descent, even if condensation occurs during ascent. Like the wet-bulb temperature, it is also conservative for evaporation and condensation. It is not conservative for processes of radiation and transfer of heat and water vapor by turbulent mixing.

Equivalent Temperature: The equivalent temperature, denoted by T_e, may be defined as the temperature attained by a mass of air, which ascends until all moisture in the air condenses and is precipitated, and which then descends dry adiabatically to the original pressure. Therefore, in the following figure, air with dry-bulb temperature A and wet-bulb temperature D ascends to the condensation level C. The air continues to ascend to F, the moisture condensing and being precipitated during ascent. All the water vapor has been removed by the time the air reaches F, and it then descends dry adiabatically to the original pressure at E. The temperature at this latter point is then the equivalent temperature.

Example: We consider air at 900 mb pressure, with dry-bulb temperature 4°C, and relative humidity 80 percent. With the aid of tephigram, the value obtained is $T_e = 16^0$C approximately.

Equivalent Potential Temperature: The equivalent potential temperature, denoted by θ_e of an air mass may be defined as the equivalent temperature of that air if brought adiabatically to a standard pressure 1000 mb. There are several ways in which this property may be defined. The first defines it as the temperature attained if the air is brought dry adiabatically to a pressure of 1000 mb, then all the water vapor in the air is condensed, and the latent heat released in the process is added to the air, raising its temperature. So from the Poisson's equation, it can be seen that the equivalent potential temperature may also be given by the following equation

$$\theta_e = T_e\left(\frac{1000}{p}\right)^{0.286}$$

The value of θ_e obtained will depend on which value of T_e is used. A value of θ_e may readily be found with the aid of the tephigram. It is obtained by descending dry adiabatically from E to the 1000 mb line at G.

Example: In the above example, using $T_e = 16^0$C, the value of θ_e is 25^0C.

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4.2 Meteorology

It is classical Newtonian physics applied to the atmosphere. Motions obey Newton's second law. Heat satisfies the laws of thermodynamics. Air mass and moisture are conserved. When applied to a fluid such as air, these physical processes describe fluid mechanics. Meteorology is fluid mechanics applied to the atmosphere.

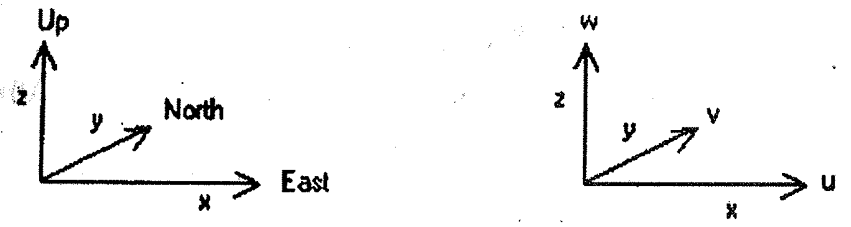
Dynamic Meteorology:

Dynamical meteorology is the study of those motions of the atmosphere, which are associated with weather and climate. For all such motions the discrete molecular nature of the atmosphere can be ignored, and the atmosphere can be regarded as a continuous fluid medium or continuum. The various physical quantities that characterize the state of the atmosphere-**pressure, density, temperature and velocity**-are assumed to have unique values at each point in the atmospheric continuum. Moreover, these field variables and their derivatives are assumed to be continuous functions of space and time. The fundamental laws of fluid mechanics and thermodynamics that govern the motions of the atmosphere may then be expressed in terms of partial differential equations involving the field variables.

4.2.1 Meteorological Conventions:

Although the earth is approximately spherical, we need not always use spherical coordinates. For the weather at a point or in a small region such as a town, state, or province, we can use local right-hand Cartesian (rectangular) coordinates (Figure-1). Usually, this coordinate system is aligned with x **pointing east,** y **pointing north, and** z **pointing up.** Other orientations are sometimes used.

Velocity components u, v, and w correspond to motion in the x, y, and z directions. For example, a positive value of u is a velocity component from west to east, while negative is from east to west. Similarly, v is positive northward, and w is positive upward.



Local Cartesian coordinates and velocity components
Figure-1

In polar coordinates, horizontal velocities can be expressed as a direction (α), and speed or magnitude (M). Historically, horizontal wind directions are based on the compass, with 0^0 to the north (the positive y direction), and with degrees increasing in a clockwise direction through 360^0. Negative angles are not usually used. Unfortunately, this differs from the usual mathematical convention of 0^0 in the x direction, increasing counter-clockwise through 360^0.

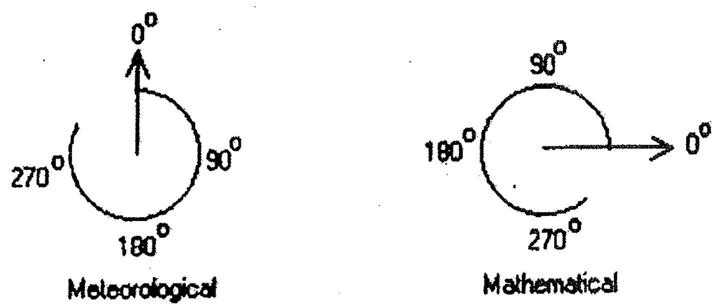


Figure-2

Also, historically winds are named by the direction they come from. Thus, **west wind** is a wind from the west, namely **from** 270^0. It corresponds to a positive value of u, with air moving in the positive *x* direction.

4.2.2 Atmospheric structure:

Atmospheric structure refers to the state of the air at different heights. The true vertical structure of the atmosphere varies with time and location due to changing weather conditions and solar activity.

Layers of the Atmosphere:

4.3 Atmospheric motion:

The motion of the atmosphere is governed by the distributions of high (H) pressure and low (L) pressures with respect to their positions and intensifications. The atmospheric motions are defined by the basic field-variables pressure (p), temperature (T), wind speeds and direction $(\vec{V})$ and density of the air (ρ). All these field-variables $(p, T, \vec{V}, \rho)$ are functions of space and time. Now the change of one of these field variables is discussed below (The Total Derivative).

Note: Air is a mixture of several gases while wind is the flow of the atmosphere.

The Total Derivative:

Now we think about a little balloon moving in the air and of which we want to study how its temperature T changes. That temperature will be a function of position and time, $T(x, y, z, t)$. If the balloon moves to the point with coordinates $(x+\Delta x, y+\Delta y, z+\Delta z)$ in the time Δt, its temperature will change by an amount

$$\Delta t = \left(\frac{\partial T}{\partial x}\right)\Delta x + \left(\frac{\partial T}{\partial y}\right)\Delta y + \left(\frac{\partial T}{\partial z}\right)\Delta z + \left(\frac{\partial T}{\partial t}\right)\Delta t \qquad (1)$$

Dividing by Δt we get for the derivative

$$\frac{DT}{Dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} + \frac{\partial T}{\partial t} \qquad (2)$$

In a Cartesian system (x, y, z) the velocity components along the axis are $u = \frac{dx}{dt}, v = \frac{dy}{dt}, w = \frac{dz}{dt}$. So the equation (2) becomes, keeping in mind the definition of gradient and scalar product

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{V}.\nabla T \qquad (3)$$

The total derivative expresses the rate of change of temperature in a coordinate system that moves with the air parcel (in this case the balloon). The total derivative is made up of two parts; the first expresses the **local derivative** $\frac{\partial T}{\partial t}$ and $\vec{V}.\nabla T$ expresses the **convective derivative.** If the system used to make the measurements is fixed with respect to the surface, we call it an Eulerian reference frame. If the system moves with the air parcel, we call it a Lagrangian. Normally we use the first one. The second term on the right is called **advective** and is one of the most important in meteorology.

Example: An example is considered to clarify the concept. We assume that a cold front has passed over a meteorological station where the temperature is 10^0C and falls at a rate of 3^0C per hour. The wind arrives from the north at 40 km hr^{-1} and the vertical component is zero. Atanother station 100 km to the north the temperature is -2^0C. What is the rate of change of the temperature for the masses following the cold front?

The first thing that interests us is the advection, which can be easily calculated starting from the temperature gradient $\frac{(-2-10)^0 C}{100} = -\frac{(2+10)^0 C}{100}$ km directed from south to north. The wind velocity is directed from north to south and is then negative. So the rate of change of temperature is given by

$$\frac{DT}{dt} = \frac{-3^0 C}{hr} + \frac{-40km}{hr} \times \frac{-(2+10)^0 C}{100km} = 18^0 C\ hr^{-1}$$

which represents heating. In practice, what is happening is that the advection warms the air mass by about 4.8^0C hr^{-1} and this heating largely compensates for the local cooling at the station.

4.3.1 Fundamental Atmospheric forces:

The fundamental physical laws of conservation of mass, momentum and energy govern the motions of the atmosphere. For atmospheric motions, the forces, which are of primary concern, are **the pressure gradient force, the gravitational force and friction.** Now, Newton's second law of motion states that the rate of change of momentum of an object referred to coordinates **fixed in space** equals the sum of all the forces acting. But if the motion is referred to a coordinate system rotating with the earth, then Newton's second law may still be applied provided that certain apparent forces, the centrifugal force and the Coriolis force, are included among the forces acting. So the fundamental atmospheric forces are

i) Pressure gradient force

ii) Gravitational force

iii) Frictional force

iv) Centrifugal force

v) Coriolis force

(i) Pressure Gradient Force:

We consider an infinitesimal volume element of air, $\delta V = \delta x \delta y \delta z$ with center at the point (x_0, y_0, z_0) along a rectangular system of coordinates. Due to random molecular motions, momentum is continually imparted to the walls of the volume element by the surrounding air. This momentum transfer per unit time per unit area is just the pressure exerted on the walls of the volume element by the surrounding air. If the pressure at the center (x_0, y_0, z_0) of the volume element is designated by p_0, then the pressure P_A on the wall labeled A at the point A at a distance $\frac{\delta x}{2}$ from the center along the x-axis in the following figure, is given by

$$P_A = p\left(x_0 + \frac{\delta x}{2}, y_0, z_0\right) = p(x_0, y_0, z_0) + \frac{\partial p}{\partial x}\frac{\delta x}{2} + \text{higher order of small quantity } (\delta x/2)$$

by taylor series expansion

$$= p_0 + \frac{\partial p}{\partial x}\frac{\delta x}{2} \text{ neglecting the higher order terms in this expansion}$$

Therefore, the force acting on this face of the volume element is given by

$F_{Ax} = -\left(p_0 + \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y \delta z$ directed along the negative direction of x axis and where $\delta y \delta z$ is the area of wall A.

Similarly, the force acting on the face at B, along the positive direction of x-axis is given by

$$F_{Ax} = +\left(p_0 - \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y \delta z$$

Therefore, the net pressure force acting on the volume element along the x-axis is

$$F_x = F_{Ax} + F_{Bx} = +\left(p_0 + \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y \delta + \left(p_0 - \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y \delta z = -\frac{\partial p}{\partial x}\delta x \delta y \delta z$$

$$= -\frac{\partial p}{\partial x}\delta V = -\frac{1}{\rho}\frac{\partial p}{\partial x}\rho \delta V = -\frac{1}{\rho}\frac{\partial p}{\partial x}\delta m$$

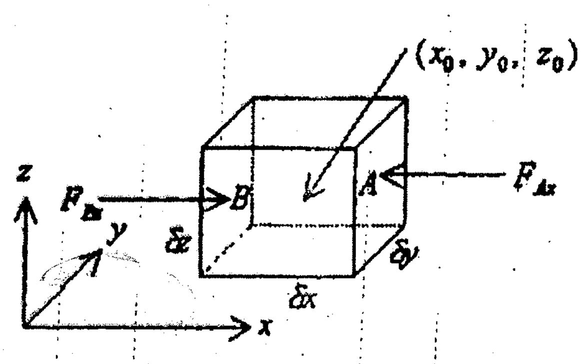
where the δm and ρ be the mass and the density of the volume element. .

So the force per unit mass in the x-direction is $\frac{F_x}{\delta m} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$. Similarly, it can be shown that y and z components of the pressure gradient force per unit mass are given by

$$\frac{F_y}{\delta m} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \text{ and } \frac{F_z}{\delta m} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

So the total pressure gradient force per unit mass in vector form is

$$\frac{F}{\delta m} = -\frac{1}{\rho}\nabla p$$



(ii) Gravitational Force:

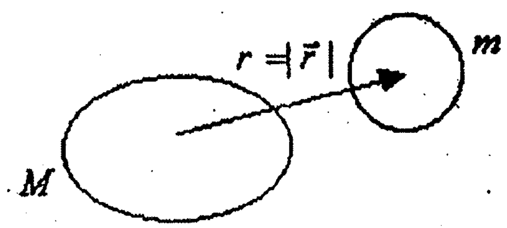
Newton's law of universal gravitation states that any two elements of mass in the universe attract each other with a force proportional to their masses and inversely proportional to the square of the distance separating them. Therefore, if two mass elements M and m are separated by a distance $r = |\vec{r}|$ (with the vector $\vec{r}$ directed toward m), then the force exerted by the mass M on the mass m due to gravitation is

$$\vec{F}_g = -\frac{GMm}{r^2}\left(\frac{\vec{r}}{r}\right) \text{ directed away from mass } M$$

where G is a universal gravitational constant.

Therefore, the earth is denoted as mass M and m is a mass of the atmosphere, then the acceleration of mass m is

$$\vec{g} = \frac{\vec{F}_g}{m} = -\frac{GM}{r^2}\left(\frac{\vec{r}}{r}\right).$$



(iii) Frictional force or viscosity force:

Now we consider a layer of incompressible fluid confined between two horizontal plates separated by distance l. The lower plate is fixed and the upper plate is moving in the x direction at a speed u_0. The fluid particles in the layer in contact with the plate must move at the velocity of the plate. So the fluid velocities at the top are $u(z = l) = u_0$ and at the bottom $u(z = 0) = 0$. The force F tangential to the upper plate required to keep it in uniform motion is as follows

$$F \propto \begin{cases} A, A \text{ is the area of the plate} \\ u_0 \\ \frac{1}{l} \text{ i.e., the inverse of the distance separating the plates} \end{cases}$$

Therefore, we have

$$F \infty \frac{Au_0}{l}$$

i.e., $F = \mu \frac{Au_0}{l}$

where μ is a constant of proportionality, known as the dynamic viscosity coefficient.

This force must just equal the force exerted by the upper plate on the fluid immediately below it. For a state of uniform motion, every horizontal layer of fluid must exert the same force on the fluid below. Therefore, taking the limit as the fluid layer depth goes to zero, we may write the viscous force or friction force per unit area, also known as shearing stress as

$$\tau_{zx} = \mu \frac{\partial u}{\partial z}$$

where τ_{zx} indicates the component of the shearing stress in the x direction due to vertical shear of the x velocity component.

The effect of shearing stress is the download transport of momentum. **For more general**, we consider non-steady two-dimension shear flow in an incompressible fluid. We may calculate the net viscous force by considering as elementary volume element of fluid centered at (x, y, z) with sides $\delta x, \delta y, \delta z$. If τ_{zx} be the shearing stress in the x-direction through the center of the elementary volume, then shearing stress on the top layer (acting across the upper boundary) in the x-direction is given by

$$\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \frac{\delta z}{2}$$

Similarly shearing stress on the bottom layer (acting across the lower boundary) in the x-direction is

$$\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z} \frac{\delta z}{2}$$

Therefore the net viscous force or friction force on the elementary volume in the x-direction is

$$\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta y\delta x-\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta y\delta x$$

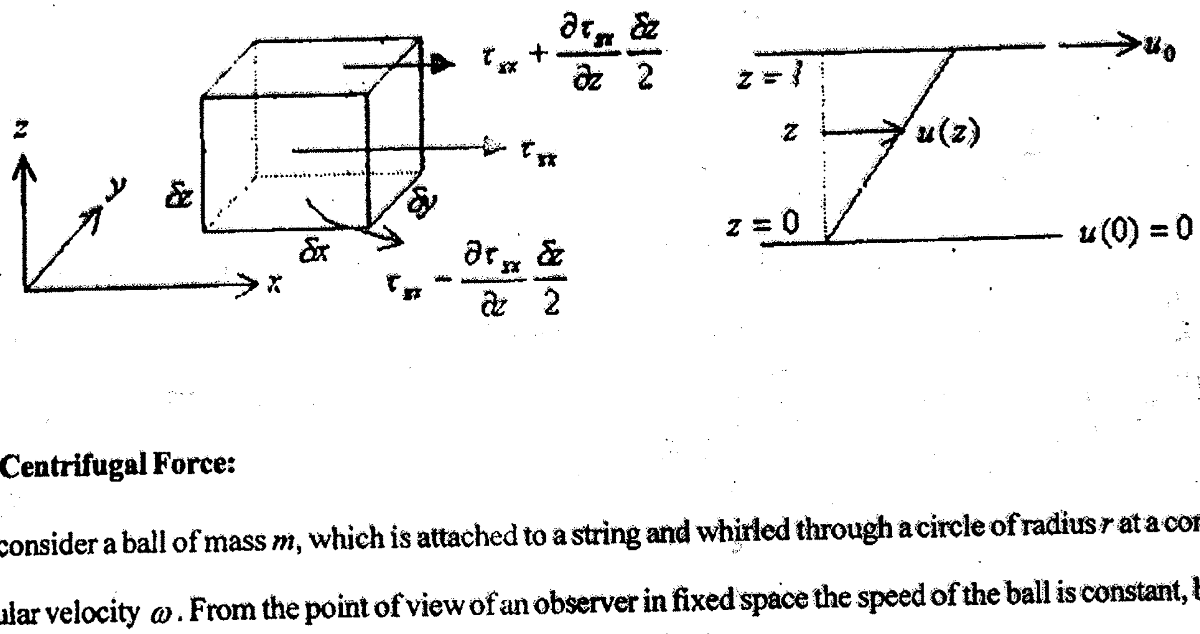
$$=\frac{\partial\tau_{zx}}{\partial z}\delta y\delta x\delta z=\frac{1}{\rho}\frac{\partial\tau_{zx}}{\partial z}\rho\delta V=\frac{1}{\rho}\frac{\partial\tau_{zx}}{\partial z}\delta m$$

Therefore, the viscous force per unit mass due to vertical shear of the component of motion in the x-direction is

$$\frac{1}{\rho}\frac{\partial\tau_{zx}}{\partial z}=\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2}(\text{for constant }\mu)=\nu\frac{\partial^2 u}{\partial z^2}\text{ where }\nu=\frac{\mu}{\rho}$$

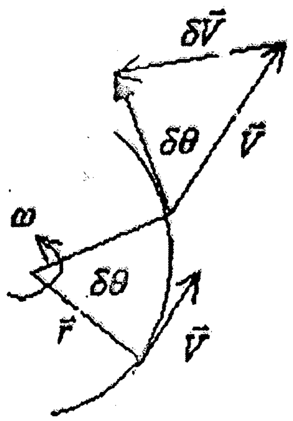
is called the kinemetic viscosity coefficient.

Note: For the atmosphere below 100 km, ν is so small that molecular viscosity is negligible except in a thin layer within a few centimeters of the earth's surface where the vertical shear is very large.



(iv) Centrifugal Force:

We consider a ball of mass m, which is attached to a string and whirled through a circle of radius r at a constant angular velocity ω. From the point of view of an observer in fixed space the speed of the ball is constant, but its direction of travel is continuously changing so that its velocity is not constant. To compute the acceleration we consider the change in velocity $\delta\vec{V}$, which occurs for a time increment δt during which the ball rotates through an angle $\delta\theta$ as in the following figure.



Since $\delta\theta$ is also the angle between the vectors $\vec{V}$ and $\vec{V}+\delta\vec{V}$, the magnitude of $\delta\vec{V}$ is just $|\delta\vec{V}| = |\vec{V}|\delta\theta$. If we divide by δt and in the limit $\delta t \to \infty$, $\delta\vec{V}$ is directed toward the axis of rotation, then we obtain

$$\frac{\delta\vec{V}}{dt} = |\vec{V}|\frac{d\theta}{dt}\left(-\frac{\vec{r}}{r}\right)$$

But, $|\vec{V}| = \omega r$ and $\frac{d\theta}{dt} = \omega$ so that $\frac{\delta\vec{V}}{dt} = -\omega^2\vec{r}$

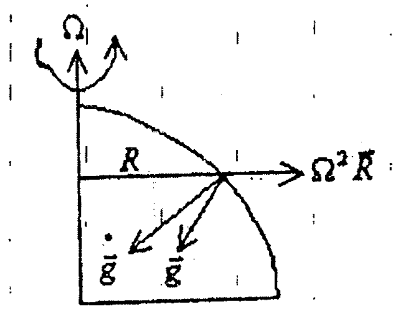
This acceleration is called the centripetal acceleration. The force of string pulling the ball causes it. Now we suppose that we observe the motion in a coordinate system rotating with the ball. In this rotating system, the ball is stationary. But there is still force acting on the ball, namely the pull of the string. Therefore, in order to apply Newton's second law to describe the motion relative to this rotating coordinate system we must include an additional apparent force known as the centrifugal force, which just balance the force of the string on the ball. Therefore the centrifugal force is just equal and opposite to the centripetal force.

(v) Gravity Force:

A particle of unit mass at rest on the surface of the earth, observed in a reference frame rotating with the earth, is subject to a centrifugal force $\Omega^2\vec{R}$, where Ω is the angular speed of rotation of earth and $\vec{R}$ the position vector from the axis of rotation to the particle. Therefore, the weight of a particle of mass m at rest on the earth's surface, which is just the reaction force of the earth on the particle, will generally, be less than the gravitational force $m\vec{g}$

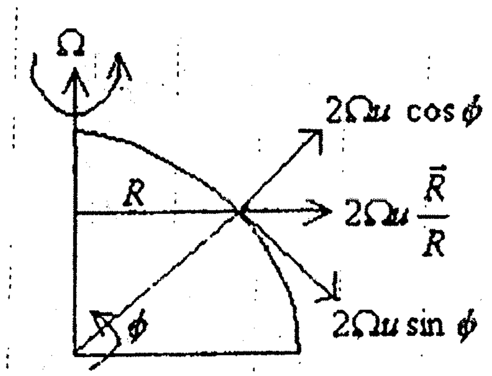
because the centrifugal force partly balances the gravitational force. So, it is convenient to combine the effects of teh gravitational force and centrifugal force by defining a gravity force $\vec{g}$ such that

$$\vec{g} = \dot{\vec{g}} + \Omega^2 \vec{R}$$



(vi) Coriolis Force:

The apparent forces generated due to rotation of earth about its axis when viewed from non-inertial frame of reference on the surface of earth and deflected to the northern hemisphere and left in the southern hemisphere is the coriolis force i.e., always towards the equator.



Now we consider any object (particle) on the earth's surface at latitude ϕ and radius vector $\vec{R}$ from the axis of rotation. Again we suppose that the particle is set in motion in the eastward direction by an impulsive force and u be its eastward velocity. Since the particle is now rotating faster than the earth, the centrifugal force on the particle will be increased. We say that Ω be the magnitude of the angular velocity of the earth, $\vec{R}$ the position vector from

Example of Coriolis force (Deflecting force):

The portion of the earth's surface near the poles may be considered as a plane, which rotates about the polar axis with angular velocity of the earth (Ω). Only forces acting in the horizontally will be considered so that the force of gravity which acts vertically does not have taken into account. It is assumed that frictional forces are absent.

A man stationed at the North Pole throws a ball horizontally. To an observer in space the ball moves in a straight line with uniform velocity while the earth rotates beneath it. To the man at the pole, however, the ball appears to curve to the right. Referring to the following figure, to the observer in space the ball thrown from *O* travels in a straight path and reaches *P* after time *t*. When the man at *O* throws the ball, he is facing the point *P*. During the time *t*, however, he rotates with the earth through an angle Ωt, where Ω is the angular velocity of the earth, so that with respect to the space frame of reference he faced the point P_1 at the end of time *t*. He has not changed his position on the earth, and still faced in the same direction, since *P* and P_1 represents the same point on the earth's surface. To him, then, the ball has been deflected to the right, following the curved path shown in the figure. He naturally attributes this curved motion to a horizontally deflecting force. But no true force has been acting during the time *t*; it only appears to the observer at the pole that a force has been acting. Just as the ball leaves *O*, its velocity is the same to both the observer in space and to the man at *O*. The time scale is the same in both frames of reference, so that to both observers it reaches point *P* after the time interval *t*. In the fixed frame of reference, the ball has travelled to *P* in a straight path with a uniform velocity in conformity with the Newtonian laws, since no external forces are acting. But, to the man at *O* the ball has traversed the longer curved path in the same time, so that to him it has undergone acceleration in the direction of motion, which he attributes to force acting in that direction. It is clear, however, that no true force has been acting. This impression results from the operation along the direction of motion of a component of the two fictitious forces (coriolis force and centrifugal force).

the axis of rotation to the particle and u is the eastward speed of the particle relative to the ground. So the total centrifugal force is

$$\left(\Omega+\frac{u}{R}\right)^2 \vec{R}=\Omega^2\vec{R}+\frac{2\Omega\mu}{R}\vec{R}+\frac{u^2}{R^2}\vec{R}$$

The **first term** on the right is just the centrifugal force due to the rotation of earth. This is included in gravity. The **other two terms** represent deflecting forces, which act outward along the vector $\vec{R}$ i.e., perpendicular to the axis of rotation. For synaptic scale motions $u << \Omega R$ and the last may be neglected in a first approximation. The remaining term $\frac{2\Omega\mu}{R}\vec{R}$ is the **Coriolis force** due to relative motion parallel to a latitude circle.

This Coriolis force can be divided into components in the vertical and meridian directions. Therefore relative motion along the east-west coordinate will produce acceleration in the north-south direction given by

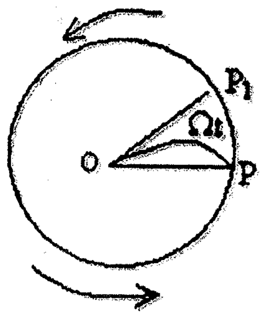
$$\left(\frac{dv}{dt}\right)_{\infty} = -2\Omega u\sin\phi \text{ in northern direction}$$

$$\left(\frac{dw}{dt}\right)_{\infty} = -2\Omega u\cos\phi \text{ in vertical direction}$$

where (u, v, w) denotes the eastward, northward and upward velocity components respectively and ϕ is the latitude of the place.

Note: So a particle moving eastward in the horizontal plane in the northern hemisphere will be deflected southward by the coriolis force, whereas a westward moving particle will be deflected northward. But in both cases, the deflection is to the right of the direction of motion.

Note: The vertical component of the coriolis force is ordinarily much smaller than the gravitation force so that its only effect is to cause a very minor change in the apparent weight of an object depending on whether the object is moving eastward or westward.



4.3.2 Inertial frame and Non-inertial frame:

A system relative to which the motion of any object can be described is called a **frame of reference.** There are in general two frames of reference - (i) Inertial and (ii) Non-inertial. The frame with respect to which the body, not acted upon by external force, is unaccelerated, i.e., either at rest or moving with uniform velocity is called **inertial frames.** For example, we consider that a moving body has coordinates (x, y, z) relative to any co-ordinate system. If no external force acts on it, then we have

$$m\frac{d^2x}{dt^2} = 0, m\frac{d^2y}{dt^2} = 0, m\frac{d^2z}{dt^2} = 0$$

$$\text{or, } \frac{d^2x}{dt^2} = \frac{d^2x}{dt^2} = \frac{d^2x}{dt^2} = 0$$

$$\text{or, } \frac{dx}{dt} = \text{constant}, \frac{dy}{dt} = \text{constant}, \frac{dz}{dt} = \text{constant}$$

In other words, in the absence of any external force, a body in motion continues to move in a straight line with constant velocity, which is Newton's First Law. As a consequence we may say **"An inertial frame is one in which the law of inertia or Newton's First Law is valid".**

Conversely speaking, the frame of reference with respect to which the body, not acted by external force, is accelerated is called a **non-inertial frame.** Any frame of reference moving with constant velocity relative to an inertial frame is also an inertial frame. Since the acceleration of the body in both frames is zero, while the velocity of the body is different but uniform.

A frame of reference fixed in stars is an inertial frame. A co-ordinate system fixed in the earth is not an inertial frame, since the earth rotates round the sun and also about its own axis. A body on the earth is neither at rest nor moving in a straight line with constant velocity through no forces act on it.

Relation between inertial and non-inertial Cartesian co-ordinate system:

(a) Frame of reference with translation motion: We consider S be a inertial frame and S' a non-inertial frame of reference which has a translation motion relative to S. A particle A has position vectors $\vec{r} = \vec{OA}$ and $\vec{r} = \vec{O'A}$ with respect the origins O and O' of two frames as in the figure. Then we have

$$\vec{r} = \vec{r}_0 + \vec{r}' \text{ where } \vec{r}_0 = \vec{OO'} \quad (1)$$

Differentiating with respect to t

$$\vec{v} = \vec{v}_0 + \vec{v}' \quad (2)$$

Differentiating again

$$\vec{f} = \vec{f}_0 + \vec{f}' \quad (3)$$

where $\vec{f}$ and $\vec{f}'$ are the acceleration of the particle in a fixed and moving systems respectively and $\vec{f}_0$ is the acceleration of the moving system relative to the fixed system.

$$m\vec{f} = m\vec{f}_0 + m\vec{f}' \quad (4)$$

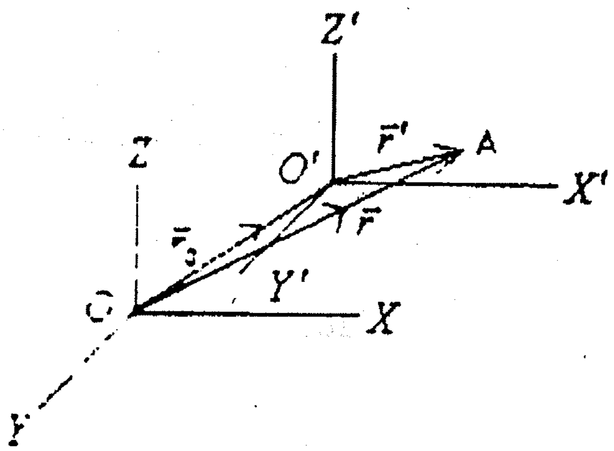
or, $\vec{F} = m\vec{f}_0 + \vec{F}'$

or, $\vec{F}' = \vec{F} - m\vec{f}_0$

where $\vec{F}$ and $\vec{F}'$ are the forces acting on the particle in the fixed and moving systems respectively.

Therefore, from the equation (5), the force $\vec{F}$ acting on the particle in fixed frame has been transformed into $\vec{F}-m\vec{f}_0$ in the moving frame; the fictitious form $m\vec{f}_0$ appearing is due to the acceleration of the moving system.

Example: When a car suddenly starts or stops, the force experienced by a passenger is this fictitious force.



(b) Frame of reference with rotational motion:

S be an inertial frame and S' a frame of reference which is rotating with an angular velocity ω with respect to S.

We let $\vec{A}$ be an arbitrary vector whose Cartesian components in an inertial frame S are given by

$$\vec{A}=\vec{i}A_x+\vec{j}A_y+\vec{k}A_z \tag{6}$$

where $\vec{i},\vec{j},\vec{k}$ are unit vectors along OX, OY, OZ respectively. Similarly if A'_x, A'_y, A'_z are the components of $\vec{A}$ along $O'X', O'Y', O'Z'$ respectively and $\vec{i}', \vec{j}', \vec{k}'$ are unit vectors along these directions, then we have

$$\vec{A}=\vec{i}'A'_x+\vec{j}'A'_y+\vec{k}'A'_z \tag{7}$$

Differentiating the equation (6) with respect to t and letting $\frac{d_a\vec{A}}{dt}$ be the total derivative of $\vec{A}$ in the inertial frame, we have

$$\frac{d_a\vec{A}}{dt} = \vec{i}\frac{dA_x}{dt} + \vec{j}\frac{dA_y}{dt} + \vec{k}\frac{dA_z}{dt} \quad (8)$$

since the unit vectors $\vec{i}, \vec{j}, \vec{k}$ remain unchanged in the fixed system. But in the rotating frame these unit vectors change with time, so we have from (7)

$$\frac{d_a\vec{A}}{dt} = \vec{i}'\frac{dA'_x}{dt} + \vec{j}'\frac{dA'_y}{dt} + \vec{k}'\frac{dA'_z}{dt} + \frac{d\vec{i}'}{dt}A'_x + \frac{d\vec{j}'}{dt}A'_y + \frac{d\vec{k}'}{dt}A'_z \quad (9)$$

But, now letting $\frac{d\vec{A}}{dt}$ as the total deerivative in the non-inertial frame $\frac{d\vec{A}}{dt} = \vec{i}'\frac{dA'_x}{dt} + \vec{j}'\frac{dA'_y}{dt} + \vec{k}'\frac{dA'_z}{dt}$ is the total derivative of $\vec{A}$ as viewed in the rotating co-ordinate system, that is, the rate of change of $\vec{A}$ following the relative motion: Hence from (9) we have

$$\frac{d_a\vec{A}}{dt} = \frac{d\vec{A}}{dt} + \frac{d\vec{i}'}{dt}A'_x + \frac{d\vec{j}'}{dt}A'_y + \frac{d\vec{k}'}{dt}A'_z \quad (10)$$

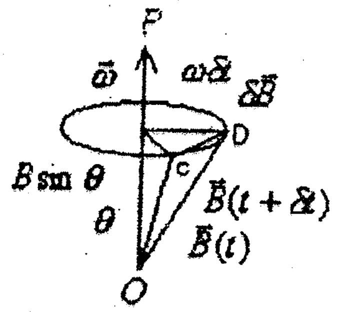
Next we prove that the rate of change of a vector $\vec{B}$ with respect time fixed in the rotating frame with respect to fixed frame is given by

$$\frac{d\vec{B}}{dt} = \vec{\omega} \times \vec{B} \quad (11)$$

Let the rotating frame be revolving about the axis OP and the angle between $\vec{B}$ and $\vec{\omega}$ is θ. In the interval δt, the change in $\vec{B}$ is $\vec{CD}$. Now $CD = B\sin\theta(\omega\delta t) = \delta B$. Hence the rate of change of $B = \frac{dB}{dt} = \underset{\delta t \to 0}{Lt}\frac{\delta B}{\delta t} = B\sin\theta\omega$.

So in the vector notation,

$$\frac{d\vec{B}}{dt} = B\sin\theta\omega\vec{n},\ \vec{n} \text{ is unit vector along } \vec{CD} = \vec{\omega} \times \vec{B}$$



Now applying the equation (11) in (10), we have

$$\frac{d_a\vec{A}}{dt}=\frac{d\vec{A}}{dt}+\left(\vec{\omega}\times\vec{i}\right)A'_x+\left(\vec{\omega}\times\vec{j}'\right)A'_y+\left(\vec{\omega}\times\vec{k}'\right)A'_z$$

$$\frac{d\vec{A}}{dt}+\vec{\omega}\times\left(\vec{i}'A'_x+\vec{j}'A'_y+\vec{k}'A'_z\right)$$

$$\frac{d\vec{A}}{dt}+\vec{\omega}\times\vec{A}$$

Hence, $\dfrac{d_a\vec{A}}{dt}=\dfrac{d\vec{A}}{dt}+\vec{\omega}\times\vec{A}$ (12)

4.3.3 The equation of motion of an air parcel in inertial system:

Taking into account all the forces acting on a parcel of air (pressure, viscous and gravitational), in an inertial frame the Newton's second law of motion is written as

$$\frac{d_a\vec{V}_a}{dt}=\sum\vec{F} \qquad (13)$$

where $\vec{V}_a$ is the velocity in the inertial reference frame i.e., **absolute velocity.** The forces on the right are all real forces and these are all the external forces and do not include apparent forces like coriolis and centrifugal. So the right-hand side represents the sum of the real forces acting per unit mass.

Now, we, observers, are in a non-inertial system. So we need to find a relation between our relative velocity $\left(\vec{V}\right)$ i.e., the velocity relative to the rotating system and the absolute velocity. This relationship is obtained by applying (12) to the position vector $\vec{r}$ for an air parcel on the rotating earth. So we have

$$\frac{d_a\vec{r}}{dt}=\frac{d\vec{r}}{dt}+\vec{\Omega}\times\vec{r} \tag{14}$$

where $\vec{\Omega}$ is the angular velocity of the earth.

But, $\frac{d_a\vec{r}}{dt}\equiv\vec{V}_a$ and $\frac{d\vec{r}}{dt}\equiv\vec{V}$. So (14) may be written as

$$\vec{V}_a=\vec{V}+\vec{\Omega}\times\vec{r} \tag{15}$$

which state simply that the absolute velocity of an object on the rotating earth is equal to its velocity to the earth plus the velocity due to the rotation of the earth.

Next we apply (12) to the velocity vector $\vec{V}_a$ and we obtain

$$\frac{d_a\vec{V}_a}{dt}=\frac{d\vec{V}_a}{dt}+\vec{\Omega}\times\vec{V}_a$$

$$=\frac{d}{dt}\left(\vec{V}+\vec{\Omega}\times\vec{r}\right)+\Omega\times\left(\vec{V}+\vec{\Omega}\times\vec{r}\right)$$

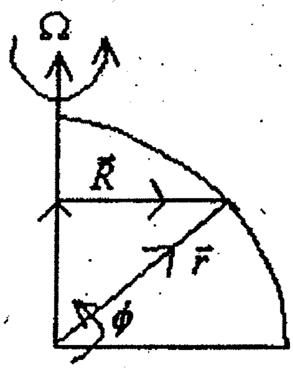
$$=\frac{d\vec{V}}{dt}+\vec{\Omega}\times\frac{d\vec{r}}{dt}+\frac{d\vec{\Omega}}{dt}\times\vec{r}+\vec{\Omega}\times\vec{V}+\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)$$

$$=\frac{d\vec{V}}{dt}+2\vec{\Omega}\times\vec{V}+\frac{d\vec{\Omega}}{dt}\times\vec{r}+\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)\text{ where }\frac{d\vec{r}}{dt}=\vec{V}$$

$$=\frac{d\vec{V}}{dt}+2\vec{\Omega}\times\vec{V}+\frac{d\vec{\Omega}}{dt}\times\vec{r}+\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)$$

i.e., $$\frac{d_a\vec{V}_a}{dt}=\frac{d\vec{V}}{dt}+2\vec{\Omega}\times\vec{V}+\frac{d\vec{\Omega}}{dt}\times\vec{r}+\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) \tag{16}$$

where $\frac{d\vec{V}}{dt}$ is the acceleration of the air parcel in the rotating frame; $\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)$ is the centripetal acceleration because $\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)=\vec{\Omega}\times\left(\vec{\Omega}\times\vec{R}\right)=-\Omega^2\vec{R}$, $\vec{R}$ being a vector perpendicular to the axis of rotation, with magnitude equal to the distance to the axis of rotation and $2\vec{\Omega}\times\vec{V}$ is called the coriolis acceleration. Since the angular velocity $(\vec{\Omega})$ of rotation of the frame (the earth) is constant, hence $\frac{d\vec{\Omega}}{dt}\times\vec{r}=0$.



So the equation (16) reduces to

$$\text{i.e., } \frac{d_a\vec{V}_a}{dt}=\frac{d\vec{V}}{dt}+2\vec{\Omega}\times\vec{V}-\Omega^2\vec{R} \tag{17}$$

Therefore, the equation (17) states that the acceleration following the motion in an inertial system equals the acceleration following the relative motion in a rotating system plus the coriolis acceleration plus the centripetal acceleration.

If we assume that the only real forces acting on the atmosphere are the pressure gradient force, gravitation and friction, so from equations (13) and (17) we have

$$\text{i.e., } \frac{d\vec{V}}{dt}=-2\vec{\Omega}\times\vec{V}-\frac{1}{\rho}\nabla p+\vec{g}+\vec{F} \tag{18}$$

where $\vec{F}$, denotes the friction force and the centrifugal force has been combined with gravitation in the gravity term $\vec{g}$.

So the equation (18) states that the acceleration following the relative motion in the rotating frame equals the sum of the Chriolis force, the pressure gradient force, effective gravity and friction. It is the momentum equation of an air parcel..

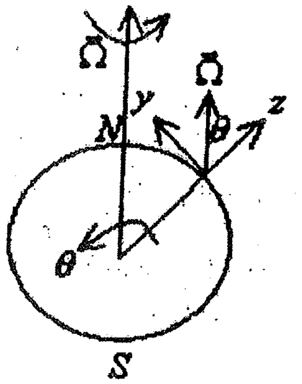
Now the Cartesian component form can be obtained from the equation (18) as follows

$$\vec{i}\frac{du}{dt}+\vec{j}\frac{dv}{dt}+\vec{k}\frac{dw}{dt}=-2\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \Omega_x & \Omega_y & \Omega_z\\ u & v & w\end{vmatrix}-\left(\vec{i}\frac{1}{\rho}\frac{\partial p}{\partial x}+\vec{j}\frac{1}{\rho}\frac{\partial p}{\partial y}+\vec{k}\frac{1}{\rho}\frac{\partial p}{\partial z}\right)$$

$$+\left(\vec{i}\,0+\vec{j}\,0-\vec{k}g\right)+\left(\vec{i}F_x+\vec{j}F_y-\vec{k}F_z\right)$$

$$\text{Now } -2\vec{\Omega}\times\vec{V}=-2\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \Omega_x & \Omega_y & \Omega_z\\ u & v & w\end{vmatrix}=-2\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & \Omega\cos\phi & \Omega\sin\phi\\ u & v & w\end{vmatrix}$$

$$=-\left(2\Omega w\cos\phi-2\Omega v\sin\phi\right)\vec{i}-2\Omega u\sin\phi\vec{j}+2\Omega u\cos\phi\vec{k}$$



Here, $\theta = 90^0 - \phi, \phi$ is the latitude of the place

$$\frac{du}{dt}=-\left(2\Omega w\cos\phi-2\Omega v\sin\phi\right)-\frac{1}{\rho}\frac{\partial p}{\partial x}+F_x \tag{19}$$

$$\frac{dv}{dt}=-2\Omega u\sin\phi-\frac{1}{\rho}\frac{\partial p}{\partial y}+F_y \tag{20}$$

$$\frac{dw}{dt}=2\Omega u\cos\phi-\frac{1}{\rho}\frac{\partial p}{\partial z}-g+F_z \tag{21}$$

4.3.5 Atmospheric motion under balanced forces:

Balance of Forces:

Now the forces acting on a unit mass in the atmosphere are (i) the coriolis force $(2\Omega v\sin\phi, -2\Omega u\sin\phi, 2\Omega u\cos\phi)$, the pressure gradient force $\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}, -\frac{1}{\rho}\frac{\partial p}{\partial y}, -\frac{1}{\rho}\frac{\partial p}{\partial z}\right)$, the gravity (0,0,–g) and the frictional force (F_x, F_y, F_z). Then all forces will be in balance when

$$2\Omega v\sin\phi - \frac{1}{\rho}\frac{\partial p}{\partial x} + F_x = 0 \tag{22}$$

$$-2\Omega u\sin\phi - \frac{1}{\rho}\frac{\partial p}{\partial y} + F_y = 0 \tag{23}$$

$$2\Omega u\cos\phi - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z = 0 \tag{24}$$

We consider that there is no frictional force in the atmosphere. Now first we dispose of the **vertical component.**

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = 2\Omega u\cos\phi - g$$

If we substitute approximate values occurring in the atmosphere, we have $10^3\,cm^3gm^{-1}$ for $\frac{1}{\rho}$ (i.e., α, the specific volume) and 1 dyne $cm^{-2}cm^{-1}$ for $\frac{\partial p}{\partial z}$, giving 10^3 dyne gm^{-1} for the term on the left. The value of g is approximately 980 dyne. For 2Ω we have 1.4×10^{-4} sec^{-1} and taking u to be 10^3 cm sec^{-1} and $\cos\phi$ as $1(\phi = 0)$, we get 0.14 cm sec^{-2} for the first term on the right. Thus we neglect the northward component of the coriolis force and we have the vertical balance in the atmosphere as

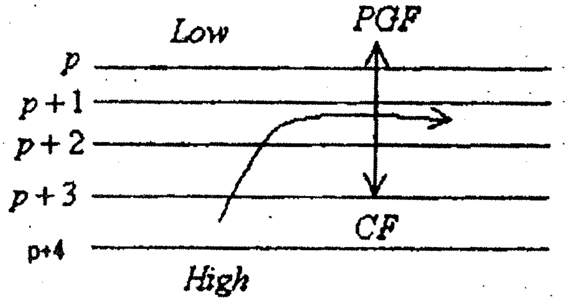
$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$$

which is known as hydrostatic equation.

In the horizontal plane, two forces in addition to the coriolis and pressure gradient forces are important, namely (1) the **frictional force** including frictional stresses at the surface of the earth and internal stresses within the atmosphere, and (2) a **centripetal force** which arises when the air is moving in a curved path relative to the earth.

4.3.5.1 The Geostrophic Wind Equation:

The first of the atmospheric motions to be considered is that resulting when the pressure gradient force just balances the deflecting force (coriolis force). Only the pressure gradient force acts upon a mass of air at rest in a pressure field. As soon as it commences to move as a result of that force, however, the deflecting force starts to operate and the more rapid the motion, the greater the corresponding deflecting force becomes. As the parcel of air picks up the speed across the isobars the Coriolis force will begin at right angle to the direction of motion. It is clear, therefore, that air does not flow perpendicular to the isobars, from high to low pressure, but that it is subjected to a force, which continually acts to deflect it to the right. As this action goes on, the Coriolis force increases since the speed increases and ultimately the two forces may balance is that where the air is moving parallel to isobars, which are themselves **straight**, and **parallel**. The resulting wind is known as the **geostrophic wind.** This situation described above is known as **geodynamic Paradox**.



For frictionless flow in a straight path on a horizontal plane, the coriolis force and pressure gradient force are in balance. This is known as the geostrophic balance. It is represented by the equations that can be obtained from (22) and (23)

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = 2\Omega v \sin\phi \qquad (25)$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -2\Omega u \sin\phi \qquad (26)$$

The velocity u and v are the components of the geostrophic wind. If the gradient is measured directly to determine $-\frac{\partial p}{\partial n}$ where n is the distance measured in a direction normal to the isobars, then we have

$$-\frac{1}{\rho}\frac{\partial p}{\partial n} = 2\Omega c \sin\phi$$

where c is the total geostrophic wind velocity, having the magnitude $\sqrt{u^2 + v^2}$ and a direction always specified as 90° toward the right of the pressure gradient in the northern hemisphere and toward the left in the southern hemisphere.

It is assumed in the above treatment that no frictional forces are present, or if present, they are so small as to be negligible. This is an accurate statement of conditions in the atmosphere at heights greater than 2000 or 3000 ft. above the surface. At lower levels, the frictional drag of the irregularities at the earth's surface on the moving air is appreciable and leads to a decrease in velocity. The pressure gradient force is not affected thereby, but the deflecting force is reduced. This leads to air motion across isobars, from high to low pressure. Steady motion results, with the deflecting and frictional forces balancing the pressure gradient force. The flow across isobars may be noted on any surface weather map.

5.0 Unit Summary

In this module, some important Aerological diagrams, graphical computation on the diagrams, meteorological Conventions, atmospheric structure, atmospheric motion, fundamental atmospheric forces, inertial and non-inertial frame of reference, the equation of motion of an air parcel, atmospheric motion under balanced forces, and the geostrophic wind equation are discussed.

6.0 Self Assessment Questions

Q1(a): Derive the following expression for the vertical distribution of the density when the lapse rate of temperature is constant

$$\rho = \rho_0 \left(\frac{T_0 - \alpha z}{T_0} \right)^{\frac{g}{R\alpha} - 1}$$

Assume that the air is dry and that g is constant.

(b): For what value of the temperature lapse rate is the density constant with height?

Q2: Calculate at what height in a dry atmosphere the pressure is one-half of that at the surface when the surface temperature is 10°C, and the lapse rate is (a) 6°C per km, (b) zero. Assume that g is constant.

Q3: Expressing the first law of thermodynamics in terms of dT and $d\alpha$ (where α is the specific volume), obtain Poisson's equation in terms of T, T_0, α, α_0.

Q4: Find the rate of change of circulation in the atmosphere.

Q5: Derive the thermal wind equation.

Q6. What is the purpose of the Aerological diagram. Mention all characteristics to draw an aerological diagram.

Q7: Define mixing ratio, specific humidity and relative humidity, absolute humidity. Establish the relation between these.

Q8: What is the virtual temperature? Find the adiabatic lapse rate for moist unsaturated air.

Q9: Discuss the temperature distribution in the atmosphere.

Q10: Show that the dry adiabatic lapse rate is approximately constant throughout the lower atmosphere.

Q11: What are different kinds of wind that may exist in the atmosphere? Obtain the governing equation of one such wind.

Q12: Obtain the atmosphere energy equation stating clearly the assumptions you have made. Interpret each term of your equation.

Q13: Define potential temperature. Obtain the relation $S = C_p \ln \theta +$ constant where S is the specific entropy and θ is the potential temperature for a parcel of dry air.

Q14: What are the general characteristics of the atmosphere? State the first law of thermodynamics.

Q15: Derive the expression for the vorticity of an air parcel.

Q16: What is geodynamic paradox?

Q17: Write down the basic assumption made in determining the stability criteria for the vertical motions of an individual parcel of air. Show that the parcel of air will be stable, neutral and unstable according as $\Gamma_d >, =, < \gamma$.

Q18: Write down the equation of motion of an atmosphere. Obtain Gradient wind equation, stating clearly the assumptions you have made.

Q19: What do you mean by an adiabatic process and isobaric process in the atmosphere?

Q20: Derive the area equivalence of the emagram and tephigram. Discuss the important features of these.

Q21: The temperature at a point 50 km north is 3°C cooler than at the station. If the wind is blowing from NE at 32ms^{-1} and the air is being heated by radiatino at the rate of 1°C, what is the local temperature change at the station?

Q22: Show that as the pressure gradient approaches zero the gradient wind reduces to the geostrophic wind for a normal anticyclone and to the inertia circle for an anomalous anticyclone.

Q23: Show that the geostrophic wind is independent of height in a barotropic atmosphere.

Q24: If wind rotates as a solid body about the center of a low pressure system, and the tangenital velocity is 10m/s at radius 300 km, find the relative vorticity.

7.0 Suggested further Readings

1. Brunt, D., Physical and Dynamical Meteorology, London, Cambridge University Press, 1939.

2. Hewson, W.E., and Longley, W.R., Meteorology Theoretical and Applied, John Wiley & Sons, INC., Chapman & Hall, LTD., London.

3. Byers, H.R., General Meteorology, McGraw-Hill Godson, W.L. and Iribarne, J.V., Atmospheric Thermodynamics, D. Reidel Publishing Company.

4. Holton, J.R., An Introduction to Dynamic Meteorology, Academic Press, New York.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper- Group–

Module No. - 96
DYNAMICAL METEOROLOGY

Contents:

1.0 Introduction

The earth's atmosphere is teh gaseous envelop surrounding the planet. Like other planetary atmospheres, the earth's atmosphere figures centrally in transfers of energy between the sun and the planet's surface and from one region of the globe to another; these transfers maintain thermal equilibrium and determine the planet's climate. However, the earth's atmosphere is unique in that it is related closely to the oceans and to surface processes, which together with the atmosphere form the basis for life. Because it is a fluid system, the atmosphere is capable of supporting a wide spectrum of motions, ranging from turbulent eddies of a few meters to circulations having dimensions of the earth itself.

2.0 Objectives

In this unit, the followings are discussed:

* the gradient wind, the thermal wind, cyclostrophic wind,
* characteristics of fluid flow applied to the atmosphere,
* the atmospheric energy equation,
* rate of change of circulation,
* surface of discontinuity, cyclogenesis
* sea level pressure tendency

3.0 Key Words and Study guides

Gradient wind, thermal wind, cyclostrophic wind, absolute vorticity, relative vorticity, energy equation, circulation, surface of discontinuity, cyclogenesis, and pressure tendency.

4.0 Main Discussion

4.1 The Gradient Wind Equation:

When isobars are curved, the motion of the air can no longer be linear, and the centrifugal force must also be considered. The centrifugal force per unit mass has the magnitude $\frac{c^2}{r}$, where c is the velocity of the air, and r the radius of curvature of its motion: it acts in the direction of increasing r. The air motion corresponding to a balance among the pressure gradient force, the deflecting force and the centrifugal force is known as the **gradient wind.** If the motion is to be steady, then the radius of curvature of the motion must be constant. So the wind blows in circular paths.

We consider a region of low pressure, having circular isobars. Since the pressure gradient force $-\frac{1}{\rho}\frac{\partial p}{\partial r}$ always acts at right angles to the isobars, while the deflecting force and centrifugal force always act at right angles to the direction of motion, it follows that the only direction of motion which permits a balance of forces is that parallel to the isobars. Theoretically either clockwise or counter-clockwise motion is possible. If **clockwise motion is assumed,** the centrifugal force must balance the sum of the pressure gradient and deflecting forces. The centrifugal force is, however, considerable smaller than either of the other under ordinary meteorological conditions. Only when r is small ($r < 100$), does the centrifugal force approach the sum of the other two in magnitude. It is clear than that the hypothesis of clockwise motion around a low-pressure area must be discarded. The **counterclockwise motion** means that the pressure gradient force just balances the sum of the deflecting and centrifugal forces. No limitations of the above kind are present and the balanced condition may hold for any value of the radius of curvature. This result is confirmed by an inspection of weather maps, which show counterclockwise winds around low-pressure areas.

So in frictionless flow in a curved path the balance of forces is

$$-\frac{1}{\rho}\frac{\partial p}{\partial n} + fc \pm \frac{c^2}{r} = 0$$

The two alternative signs in the last term indicate that, with respect to the direction n, the centripetal force will be positive or negative depending on whether the rotation is counterclockwise or clockwise. Since the centripetal force $\left(\frac{c^2}{r}\right)$ acts inward, it will act along the direction of coriolis force in an anticyclone and opposite to the coriolis force in cyclone.

Now **for the** ***cyclonic case*** it is given by

$$-\frac{1}{\rho}\frac{\partial p}{\partial r}+fc+\frac{c^2}{r}=0 \qquad (27)$$

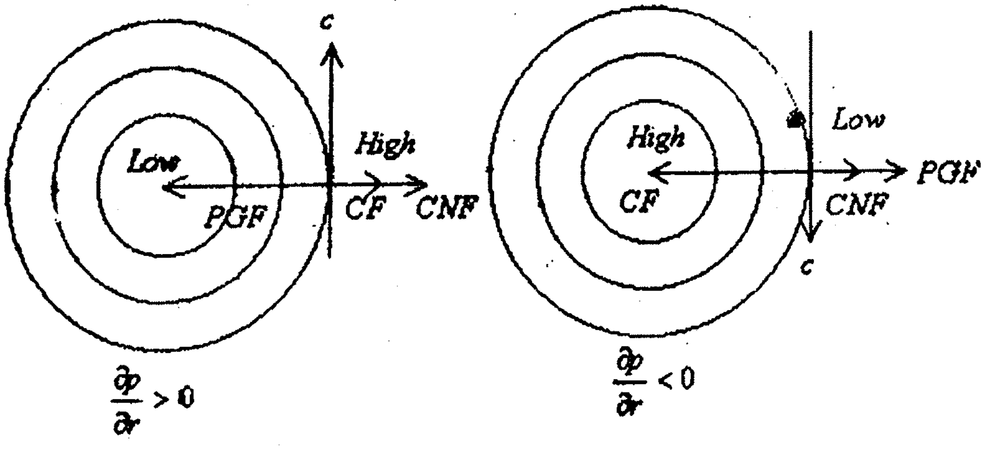
where $f=2\Omega\sin\phi$ is the coriolis parameter and c is the velocity.

Solving the equation (27), we have

$$c=-\frac{fr}{2}\pm\sqrt{\frac{f^2}{4}r^2+\frac{r}{\rho}\frac{\partial p}{\partial r}}$$

Case-I: when $c=-\frac{fr}{2}+\sqrt{\frac{f^2}{4}r^2+\frac{r}{\rho}\frac{\partial p}{\partial r}}$

As $\frac{\partial p}{\partial r}\to 0$, then $c\to 0$. This describes the normal observed case of cyclinic flow around a low-pressure centre. When $\frac{\partial p}{\partial r}>0$ (Low pressure centre) the value of c is positive. Again when $\frac{\partial p}{\partial r}<0$ (High pressure centre), the value of c is negative.



Case-II: when $c=-\frac{fr}{2}-\sqrt{\frac{f^2}{4}r^2+\frac{r}{\rho}\frac{\partial p}{\partial r}}$

This is rejected because if it is retained, $c\neq 0$ when $\frac{\partial p}{\partial r}=0$, a result having no physical significance.

Note: It follows from the equations (25) and (27) that the gradient wind in cyclones is less than the geostrophic wind for the same pressure gradient.

Again **for the anti-*cyclonic case*,** the balance of forces may be expressed as

$$-\frac{1}{\rho}\frac{\partial p}{\partial r}+fc-\frac{c^2}{r}=0 \tag{28}$$

The solution of this quadratic equation is given by

$$c=\frac{fr}{2}-\sqrt{\frac{f^2}{r}r^2-\frac{r}{\rho}\frac{\partial p}{\partial r}}$$

Here the positive sign is rejected, since if $\frac{\partial p}{\partial r}=0, c\neq 0$ and the equation loss its physical signficance.

The maximum possible velocity in an anticyclone occurs if

$$\frac{f^2}{4}r^2-\frac{r}{\rho}\frac{\partial p}{\partial r}=0$$

$$\text{i.e., } \frac{1}{\rho}\frac{\partial p}{\partial r}=\frac{f^2}{4}r \text{ i.e., } \frac{\partial p}{\partial r}=\frac{\rho f^2}{4}$$

Since this maximum cannot be exceeded, hence it is necessary that the following conditions be obeyed in the anticyclone.

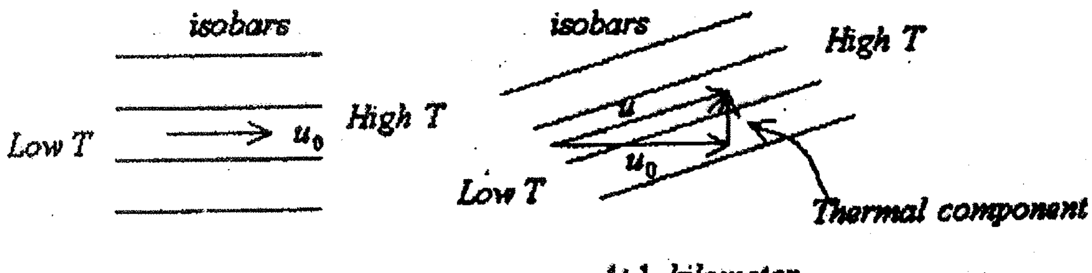
$$\frac{\partial p}{\partial r}\leq\frac{\rho f^2}{4}$$

Therefore the pressure gradient over an anticyclone is subject to certain restrictions, if motion under balanced forces is to occur.

Note: The equations (25) and (28) show that the gradient wind in an anticyclone is greater than the geostrophic wind for the same pressure gradient.

4.2 The Thermal Wind Equation:

The equation $dp = -g\rho dz$ shows that the rate of change of pressure with height is a function of the density only, if the small variations in the acceleration of gravity are neglected. It can be seen from this equation that the pressure decreases more rapidly with height in cold air than in warm air $\left(dp = -g\frac{p}{RT}dz \right)$. We consider two points at the earth's surface, which have equal pressure, but with colder air above one than above the other. At a higher of 1 km, say, the pressure in the colder air will therefore be less than that in the warm air. An isobar may join the two points at the surface, but can not join the two points 1 km above these, since the pressures there are not the same. It follows that when a horizontal temperature gradient is present in the atmosphere, the pressure gradient and hence the geostrophic wind must vary with height. This variation may be a change in direction, in speed, or both. The geostrophic wind at 1 km may then be thought of as the resultant of two components, one the surface geostrophic wind, and other component resulting from the horizontal temperature gradient. The latter component is known as the **thermal wind component.** The magnitude of this component is obviously proportional to the horizontal tempera



At the surface   *At 1 kilometer*

The typical variation of geostrophic wind with height is shown above. The pressure distribution and the geostrophic wind u_0 at the surface are given in the figure. The air to the west is colder than that to the east. Conditions at 1 km are indicated in the figure. The thermal component, blowing from the south, is added vectorially to u_0, given the geostrophic wind u at 1 km. If colder air were to the north, the geostrophic wind would increase with height without undergoind any change in direction. **The above considerations apply equally well to any levels in the atmosphere.** If the geostrophic wind at 2 km is known, that at 3 km, or 4 km, can be found in this qualitative fashion, provided that the approximate direction and magnitude of the horizontal temperature gradient are known. **Equations for this variation of geostrophic wind with height may be derived.** Using the equation $p = \rho RT$, the statistical equation $\frac{\partial p}{\partial z} = -\rho g$ may be expressed as

$$\frac{1}{p}\frac{\partial p}{\partial z} = -\frac{g}{RT} \tag{1}$$

Differentiating (1) with respect to x and changing the order of differentiation we get

$$\frac{\partial}{\partial x}\left(\frac{1}{p}\frac{\partial p}{\partial z}\right) = \frac{\partial}{\partial x}\left(-\frac{g}{RT}\right)$$

$$\text{i.e.,} \frac{\partial}{\partial z}\left(\frac{1}{p}\frac{\partial p}{\partial z}\right) = \frac{g}{RT^2}\frac{\partial^2 T}{\partial x^2} \tag{2}$$

Similarly differentiating (1) with respect to y gives

$$\frac{\partial}{\partial z}\left(\frac{1}{p}\frac{\partial p}{\partial y}\right) = \frac{g}{RT^2}\frac{\partial^2 T}{\partial y^2} \tag{3}$$

Now the equations of geostrophic wind are

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = fv \tag{4}$$

and $$\frac{1}{\rho}\frac{\partial p}{\partial y} = -fu \tag{5}$$

where $f = 2\Omega \sin\phi$ is the coriolis parameter.

Using $p = \rho RT$ in (4) and (5), the equations for the geostrophic wind components are

$$\frac{v}{T}=\frac{R}{f}\frac{1}{p}\frac{\partial p}{\partial x} \quad (6)$$

$$\frac{u}{T}=\frac{R}{f}\frac{1}{p}\frac{\partial p}{\partial y} \quad (7)$$

Differentiating (6) and (7) with respect to z, we have

$$\frac{\partial}{\partial z}\left(\frac{v}{T}\right)=\frac{R}{f}\frac{\partial}{\partial z}\left(\frac{1}{p}\frac{\partial p}{\partial x}\right) \quad (8)$$

$$\frac{\partial}{\partial z}\left(\frac{u}{T}\right)=-\frac{R}{f}\frac{\partial}{\partial z}\left(\frac{1}{p}\frac{\partial p}{\partial y}\right) \quad (9)$$

Using (2) and (3) in (8) and (9) respectively, we have

$$\frac{\partial}{\partial z}\left(\frac{v}{T}\right)=\frac{g}{f}\frac{1}{T^2}\frac{\partial p}{\partial x} \quad (10)$$

$$\frac{\partial}{\partial z}\left(\frac{u}{T}\right)=-\frac{g}{f}\frac{1}{T^2}\frac{\partial p}{\partial y} \quad (11)$$

Integrating between the levels z_0 and z leads to the expressions

$$\frac{v}{T}=\frac{v_0}{T_0}+\frac{g}{f}\int_{z_0}^{z}\frac{1}{T_2}\frac{\partial p}{\partial x}dz \quad (12)$$

$$\frac{u}{T}=\frac{u_0}{T_0}-\frac{g}{f}\int_{z_0}^{z}\frac{1}{T_2}\frac{\partial p}{\partial y}dz \quad (13)$$

where u_0 and v_0 are the geostrophic wind components and T_0 the temperature at the height z_0. Rearranging (12) and (13) we get

$$v=\frac{T}{T_0}v_0+\frac{gT}{f}\int_{z_0}^{z}\frac{1}{T_2}\frac{\partial p}{\partial x}dz \quad (14)$$

$$u = \frac{T}{T_0}u_0 - \frac{gT}{f}\int_{z_0}^{z}\frac{1}{T_2}\frac{\partial p}{\partial y}dz \tag{15}$$

Assumption: Since T and T_0 are not likely to defer widely and they and they are in degrees Absolute, the ratio $\frac{T}{T_0}$ may be taken as unity. In addition, T may be taken as the mean temperature of the layer from z_0 to z, and $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial y}$ may be assumed to be constant with respect to z. Under these assumptions, the equations (14) and (15) reduce to

$$v = v_0 + \frac{g}{fT}\frac{\partial p}{\partial x}(z - z_0) \tag{16}$$

$$u = u_0 + \frac{g}{fT}\frac{\partial p}{\partial y}(z - z_0) \tag{17}$$

The right-hand terms on the right-hand sides of (16) and (17) are the **thermal wind components** in the y and x directions and v and u are the geostrophic wind components at height z, expressed in terms of the geostrophic wind components v_0 and u_0 at the height z_0 and the appropriate thermal components.

4.3 Cyclostrophic Wind:

If the **horizontal scale of a disturbance is mall enough,** the Coriolis force may be neglected in the Gradient wind equation $-\frac{1}{\rho}\frac{\partial p}{\partial r} + fc + \frac{c^2}{r} = 0$, compared to the pressure gradient force and the centrifugal force. The force balance normal to the direction of flow is then

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{c^2}{r}$$

If this equation is solved for c, we obtain the **speed of the cyclostrophic wind**

$$c = \left(-\frac{R}{\rho}\frac{\partial p}{\partial r}\right)^{\frac{1}{2}} \tag{18}$$

The cyclostrophic wind may be either cyclonic or anti-cyclonic. In both cases the pressure gradient force is directed toward the center of curvature and the centrifugal force away from the center of curvature. The cyclostrophic balance approximation is valid provided that the ratio of the centrifugal force to the Coriolis force is large.

4.4 Characteristics of Fluid Flow Applied to the Atmosphere:

An imbalance, however small and difficult it may be detect, must exist either continuously or sporadically (sporadic-scattered, occurring here and there or now and then). Predicting whether non-equilibrium conditions will or will not develop is one of the chief problems of weather forecasting. Characteristics of fluid flow, which are well known from classical hydrodynamics, can be applied to the atmosphere to help in solving these problems.

Vorticity:

When the equation representing the relation between linear velocity (c) and angular velocity (ω) $c = \omega r$ is differentiated with respect to r, the result is $\frac{\partial c}{\partial r} = \omega$. **For solid rotation,** ω at any given instant is constant throughout and the linear speed must increase at a fixed rate ω with radial distance. **For fluids,** since they are not constrained to rotate as solids, may turn at varying speeds both in time and in space, and not all the particles of the fluid may have the same center of rotation or the same radius of curvature.

To obtain a quantitative understanding of these fluid motions, we first consider fluid elements, which are small enough so that the change of ω in the infinitesimal distances used is zero. It is convenient to express the motion in the Cartesian coordinates x and y. The rotation is considered for two infinitesimal lines of fluid particles represented initially in dx and dy on the x and y axes as Figure-1. The v component represents the linear velocity of the rotation at the x-axis and the u component at the y-axis.

The angular velocity does not vary in the distance *dx* or *dy* but it may be different on two axes. If rotation is considered positive in a counterclockwise sense, then

$$v = \omega_1 x \qquad -u = \omega_2 y$$

$$\frac{\partial v}{\partial x} = \omega_1 \qquad -\frac{\partial u}{\partial y} = \omega_2$$

The average of the two angular velocities is

$$\frac{1}{2}(\omega_1 + \omega_2) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (1)$$

The quantity enclosed in the second parentheses is called the **vorticity** (ζ) i.e., $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ which is 2ω (twice the local average angular velocity). Since only the x, y plane has been considered, this is only one component (vertical) of the vorticity, but in the atmosphere the other two components are seldom taken into account. So the relative vorticity is defined as follows.

Relative Vorticity: Relative vorticity (ζ) is a measure of the rotation of fluids about a vertical axis relative to the earth's surface. It is defined as positive in the counterclockwise direction. The unit of measurement of vorticity is inverse seconds.

$$\zeta = \frac{\Delta v}{\Delta x} - \frac{\Delta u}{\Delta y}$$

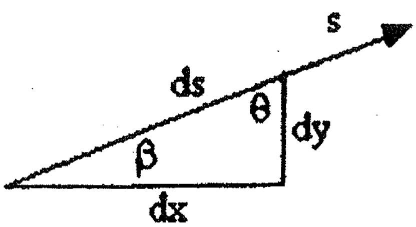
Physical Interpretation:

Absolute vorticity: We know that the **earth has its own** counterclockwise vertical component of vorticity given by $f = 2\Omega \sin\phi$, also twice the local angular velocity. We then define the absolute vorticity as the sum of two vorticities $(f + \zeta)$ i.e., measured with respect the

Circulatino and Vorticity:

In the following figure, fluid is considered as flowing along the straight path s, infinitesimal part of which is represented by ds, with components dx and dy. It is seen that

$$\frac{dx}{ds} = \cos\beta \qquad \frac{dy}{ds} = \cos\theta \qquad (1)$$



These may be designated as the directional cosines l and m of s.

$$dx = l ds \qquad dy = m ds$$

Now the equation $udx + vdy = 0$ (***) becomes

$$udx + vdy = 0$$

$$\text{i.e., } (ul + vm)ds = 0 \qquad (2)$$

Also $u = \frac{dx}{dt} = l\frac{ds}{dt} = lc$ (3)

$$v = \frac{dy}{dt} = m\frac{ds}{dt} = mc \qquad (3)$$

So $ul + vm = \left(l^2 + m^2\right)c$ (4)

Now $m = \cos\theta = \sin\beta$ so $\sin^2\beta + \cos^2\beta = 1$

Therefore $ul + vm = c$ but since $l = \frac{dx}{ds}$ and $m = \frac{dy}{ds}$

Hence $u\frac{dx}{ds} + v\frac{dy}{ds} = c$ i.e., $udx + vdy = cds$ (5)

where c is the actual velocity ds/dt along s.

The value of the integral

$$\int_{s_1}^{s_2} udx + vdy = \int_{s_1}^{s_2} cds \tag{6}$$

is the **flow of the fluid from** s_1 to s_2. It has the dimensions cm^2 per sec or the product of speed and distance.

When the flow is not of the simple form described in the figure, then u and v may each be a function of both x and y in the fluid. This can be a curved line, a line with angular turns in it, or a closed circuit. The flow around a closed circuit around a closed path is called the **circulation.**

Barotropic and Baroclinic Atmosphere:

A portion of the atmosphere in which the surfaces of pressure, specific volume (or density), temperature or potential temperature are all parallel is called barotropic. In other words, barotropic atmosphere is one without solenoids. **The atmosphere is approximately barotropic in large regions in the tropics.** As a first approximation for the solution of certain problems in meteorology, the atmosphere sometimes is assumed to be barotropic when it really is not. Such simplifications are common in theoretical studies in order to handle otherwise intractable (unmanageable, obstinate) problems and thus to gain incomplete yet important knowledge about intricate (entangled) processes.

The opposite of barotropic is **baroclinic** i.e., characterized by the presence of a solenoidal field. **The natural atmosphere is mainly baroclinic**, since horizontal temperature gradients are the rule and often have a direction opposite to that of the pressure gradients.

Continuity and divergence:

A statement of the law of **conservation of matter** is the form of the **"equation of continuity"** is useful in meteorology. This equation indicates that, in a continuous fluid or gaseous medium, the mass of the fluid material passing into a given volume must be equal to that coming out unless a density change has occurred in the volume.

Now we consider a small volume in the form of a box with sides $\delta x, \delta y$ and δz as in the following figure. Initially the mass of the air in the box is $\delta M = \rho\delta V = \rho(\delta x\delta y\delta z)$. We examine first the motion in and out of the pair of faces $\delta y\delta z$ as indicated by the arrow. The contribution of the flow in this component to the net accumulation of mass would be

Inflow at x_0 − outflow at x_1

$$\delta y\delta z\left[\rho u-\left(\rho u+\frac{\partial \rho u}{\partial x}\delta x\right)\right]=\frac{\partial \rho u}{\partial x}\delta x\delta y\delta z \quad (1)$$

If we go though the same reasoning for the other pairs of sides, we obtain the total net inflow of mass in the time *dt* as

$$-\left(\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial x}+\frac{\partial \rho w}{\partial x}\right)\delta V dt \quad (2)$$

This net inflow, or accumulation, must give rise to a local increase in the mass, and if δV is to remain constant, the increase in mass in time *dt* is therefore

$$\left(\rho+\frac{\partial \rho}{\partial t}dt\right)\delta V-\rho\delta V=\frac{\partial \rho}{\partial t}\delta V dt \quad (3)$$

The principal of conservation of mass requires that the two expressions (2) and (30) be the same. So we have

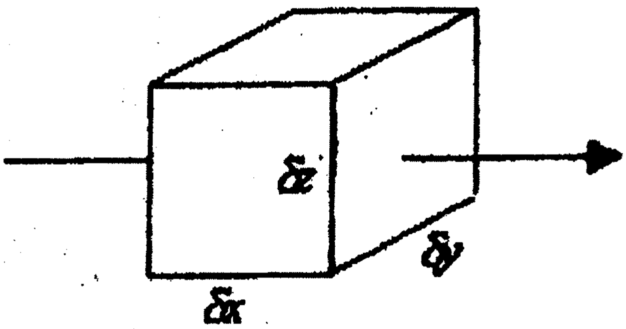
$$\frac{\partial \rho}{\partial t}=-\left(\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}\right) \quad (4)$$

i.e., $\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}+\rho\frac{\partial u}{\partial x}+\rho\frac{\partial v}{\partial y}+\rho\frac{\partial w}{\partial z}=0$

i.e., $\frac{\partial \rho}{\partial t}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$ where $\frac{\partial \rho}{\partial t}=\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}$

i.e., $\frac{1}{\rho}\frac{\partial \rho}{\partial t}+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$ (5)

which is the equation of continuity and it is often used in meteorology.



The partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$ express the divergence of the air. This may be seen by considering the speed u to be greater at the x_1 face than at the x_0 face of the figure. Under this condition, $\frac{\partial u}{\partial x}$ would be positive. The air particles would be pulling or stretching further apart. If the speed decreased between x_0 and x_1 would be negative, and the air particles would be pushing closer together. In the first case we would have divergence $\left(\frac{\partial u}{\partial x}>0\right)$ and in the second case, convergence $\left(\frac{\partial u}{\partial x}<0\right)$ in the direction of the x axis. The sum of the three partial derivatives gives the total divergence in the box. If there is divergence, these terms are positive and we have

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=-\frac{1}{\rho}\frac{\partial \rho}{\partial t} \qquad (6)$$

showing that if we have divergence so that more comes out than goes in, the density within the volume has to decrease. Conversely, if convergence occurs, the term on the left of (6) is negative and $\frac{\partial \rho}{\partial t} > 0$. For the case of incompressibility, or **when compression does not occur,** the divergence is zero, or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{7}$$

In meteorology we are often concerned with the horizontal divergence; so in applying this last expression we write

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z} \tag{8}$$

which emphasizes that horizontal divergence must be compensated by vertical shrinking or vertical convergence, while horizontal convergence must be accompanied by vertical stretching or vertical divergence $\left(\frac{\partial w}{\partial z} > 0\right)$. Any wind component that in increasing downstream is divergent in that component but may be convergent in one or both of the other two components.

The origin of Pressure changes:

If it is assumed that the atmosphere is in statical equilibrium, a valid assumption except near cumulonimbus clouds where marked vertical accelerations often occur, it is possible to determine the several factors which cause pressure changes at any level. The pressure p at any height z may be found by integrating $dp = -g\rho dz$ from z to ∞. Therefore we have

$$p = \int_z^\infty g\rho dz \tag{9}$$

Differentiating partially with respect to time and assuming the acceleration of gravity g to be constant, it follows that the tendency at any level z is

$$\frac{\partial p}{\partial t} = g\int_z^\infty \frac{\partial \rho}{\partial t}\,dz \tag{10}$$

Substituting for $\frac{\partial \rho}{\partial t}$ from the equation of continuity (4), the equation (10) becomes

$$\frac{\partial p}{\partial t} = -g\int_z^\infty\left(\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}\right)dz = -g\int_z^\infty\left(\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}\right)dz - g\int_z^\infty\left(\frac{\partial \rho w}{\partial z}\right)dz \tag{11}$$

Then by carrying out the differentiation in the first term and rearranging, and integrating the second term, (11) becomes

$$\frac{\partial p}{\partial t} = -g\int_z^\infty \rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz - g\int_z^\infty\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}\right)dz + g\left(\rho w\right)_z \tag{12}$$

since $\rho = 0$ at $z = \infty$

This equation shows that a pressure variation at height z can arise from the operation of one or more of three distinct processes. The first term represents the effect of horizontal divergence or convergence at all heights greater than z. The second gives the effect of horizontal advection of air of different density at heights greater than z. The last term represents the effect on p of vertical motion at the height z.

At the surface of the earth $w = 0$, so that the variation of pressure p_0 with time at any fixed point at the surface, i.e., the tendency at the surface, is

$$\frac{\partial p_0}{\partial t} = -g\int_z^\infty\left(\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}\right)dz$$

$$\text{i.e., } \frac{\partial p_0}{\partial t} = -g\int_z^\infty \rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz - g\int_z^\infty\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}\right)dz \qquad (13)$$

Thus the pressure at the surface can vary only as a result of horizontal divergence or convergence or of advection of air of different density at higher levels.

When the air motion at all levels in the atmosphere is geostrophic, the tendency at the surface must be zero. Now the equation of the geostrophic wind are in the form

$$\rho v = +\frac{1}{f}\frac{\partial p}{\partial x} \qquad (14)$$

$$\rho u = -\frac{1}{f}\frac{\partial p}{\partial y} \qquad (15)$$

Differentiating (14) and (15) partially with respect to y and x respectively, we have

$$\frac{\partial \rho v}{\partial y} = \frac{1}{f}\frac{\partial^2 p}{\partial x \partial y} \quad (16)$$

$$\frac{\partial \rho u}{\partial x} = -\frac{1}{f}\frac{\partial^2 p}{\partial x \partial y} \quad (17)$$

Using (16) and (17) in (13), we have

$$\frac{\partial p_0}{\partial t} = 0$$

Since no changes in surface pressure are possible in a geostrophic wind field, it follows that variations in the distribution of pressure at the surface can occur only with departures from geostrophic motion.

Note: Although the equations (12) and (13) are comparatively simple, the determination of the actual divergence or convergence and advection in the troposphere and atmosphere requires a quantity accuracy of the upper air data which have not yet been attained.

4.5 The atmospheric energy equation:

The equations of motion in the atmosphere in its component form are given by

$$\frac{du}{dt} = 2\Omega v \sin\theta - 2\Omega w \cos\theta - \frac{1}{\rho}\frac{\partial p}{\partial x} + F_x \tag{18}$$

$$\frac{dv}{dt} = -2\Omega u \sin\theta - \frac{1}{\rho}\frac{\partial p}{\partial y} + F_y \tag{19}$$

$$\frac{dw}{dt} = 2\Omega u \cos\theta - \frac{1}{\rho}\frac{\partial p}{\partial z} + F_z \tag{20}$$

where (u, v, w) be the velocity of the air parcel, $\alpha = \frac{1}{\rho}$ be the specific volume, Ω is the angular velocity of the earth, (F_x, F_y, F_z) be the frictional force in the atmosphere.

Multiplying the equations (18), (19) and (20) by u, v, w respectively and then adding, we have

$$u\frac{du}{dt} + v\frac{dv}{dt} + w\frac{dw}{dt} = -\alpha\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) + \left(uF_x + vF_y + wF_z\right) - gw$$

i.e., $$\frac{d}{dt}\left(\frac{u^2+v^2+w^2}{2}\right) = -\alpha\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) + \left(uF_x + vF_y + wF_z\right) - g\frac{dz}{dt} \quad \text{since } w = \frac{dz}{dt}$$

i.e., $$\frac{d}{dt}\left(\frac{|\vec{V}|^2}{2} + gz\right) = -\alpha\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) + \left(uF_x + vF_y + wF_z\right) \tag{21}$$

where $|\vec{V}| = \sqrt{u^2+v^2+w^2}$

In the equation (21), $u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}$ is the work done by the pressure gradient and $uF_x + vF_y + wF_z$ is the energy dissipated (wasted) by the frictional force. Now the first law of thermodynamics is

$$dQ = dU + p\,d\alpha = C_V dT + p\,d\alpha$$

i.e., $$\frac{dQ}{dt} = C_V\frac{dT}{dt} + p\frac{d\alpha}{dt} \tag{22}$$

Now adding (21) by $\frac{d}{dt}(C_V T + \alpha p)$, we have

$$\frac{d}{dt}\left(\frac{|\vec{V}|^2}{2} + gz\right) + \frac{d}{dt}(C_V T + \alpha p) = \frac{d}{dt}(C_V T + \alpha p) - \alpha\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right)$$

$$+\left(uF_x + vF_y + wF_z\right)$$

i.e., $$\frac{d}{dt}\left(\frac{|\vec{V}|^2}{2} + gz + C_V T + \alpha p\right) = C_V\frac{dT}{dt} + p\frac{d\alpha}{dt} + \alpha\frac{dp}{dt} - \alpha\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right)$$

$$+\left(uF_x + vF_y + wF_z\right)$$

i.e., $$\frac{d}{dt}\left(\frac{|\vec{V}|^2}{2} + gz + C_V T + \alpha p\right) = \frac{dQ}{dt} + \alpha\frac{dp}{dt} - \alpha\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right)$$

$$+\left(uF_x + vF_y + wF_z\right) \text{ by (22)}$$

i.e., $$\frac{d}{dt}\left(\frac{|\vec{V}|^2}{2} + gz + C_V T + RT\right) = \frac{dQ}{dt} + \alpha\frac{dp}{dt} + \left(uF_x + vF_y + wF_z\right)$$, since

$\frac{dp}{dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}$ and the equation of state being $p\alpha = RT$.

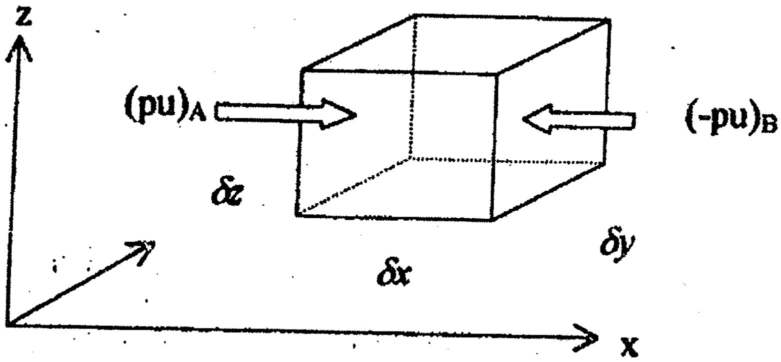
i.e., $$\frac{d}{dt}\left(\frac{|\vec{V}|^2}{2} + gz + C_p T\right) = \frac{dQ}{dt} + \alpha\frac{\partial p}{\partial t} + \left(uF_x + vF_y + wF_z\right), \text{ since } C_P - C_V = R \qquad (23)$$

Assumption: Now if we assume that the motion of the air parcel is (i) steady $\left(\frac{\partial p}{\partial t} = 0\right)$ (ii) adiabatic $\left(\frac{dQ}{dt} = 0\right)$, and (iii) frictionless, then the equation (23) reduces to i.e., $\frac{d}{dt}\left(\frac{|\vec{V}|^2}{2}+gz+C_pT\right)=0$

i.e., $\frac{|\vec{V}|^2}{2}+gz+C_pT=$ constant (24)

The equation (24) is known as the atmospheric energy equation, which implies that the sum of kinetic energy, potential energy and enthalpy of an air parcel is constant in a steady, adiabatic, frictionless flow.

Note: The rate at which work is done on the fluid element by the x component of the pressure force is illustrated in the following figure.



Recalling that pressure is a force per unit area, and that the rate at which a force does work is given by the dot product of the force and velocity vectors, we see that the rate at which the surrounding fluid does work on the element due to the pressure force on the boundary surfaces parallel to the y-z plane is given by

$$(pu)_A\,\delta y\delta z-(pu)_B\,\delta y\delta z$$

(The negative sign is needed before the second term because the work done on the fluid element is positive if u is positive across face B). Now by expanding in a Taylor series we can write

$$(pu)_B = (pu)_A + \left[\frac{\partial}{\partial x}(pu)\right]\delta x + ...$$

Therefore, the net rate of working of the pressure force due to the x-component of motion is

$$\left[(pu)_A - (pu)_B\right]\delta y \delta z = -\left[\frac{\partial}{\partial x}(pu)\right]_A \delta V \text{ where } \delta V = \delta x \delta y \delta z$$

Similarly, we can show that the rates of working by the perssure due to the y and z components of motion are $-\left[\frac{\partial}{\partial y}(pv)\right]\delta V$ and $-\left[\frac{\partial}{\partial z}(pw)\right]\delta V$ respectively. Thus the total rate of working by the pressure force is simply $-\nabla\left(p\vec{V}\right)\delta V$ where $\vec{V}$ is the velocity of the fluid element.

4.6 Rate of Change of Circulation:

Kelvin's circulation theorem, which gives the rate of change of circulation with time, may be derived in the following manner. The circulation around a closed curve is, by definition, the line integral of velocity around the curve. Defined mathematically, it is

$$C = \oint(udx + vdy + wdz) \tag{3.10.1}$$

where (u, v, w) be the velocity of the air. If (3.10.1) is differentiated with respect to time, then we have

$$\frac{dC}{dt} = \oint\left(\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz\right) + \oint\left[u\frac{d}{dt}(dx) + v\frac{d}{dt}(dy) + w\frac{d}{dt}(dz)\right] \tag{3.10.2}$$

Now since $\frac{dx}{dt} = u, \frac{dy}{dt} = v, \frac{dz}{dt} = w$, hence the second term on the right-hand side of (3.10.2) may be written as follows:

$$\oint\left[u\frac{d}{dt}(dx) + v\frac{d}{dt}(dy) + w\frac{d}{dt}(dz)\right]$$

$$= \oint\left[ud\left(\frac{dx}{dt}\right) + vd\left(\frac{dy}{dt}\right) + wd\left(\frac{dz}{dt}\right)\right]$$

$$= \int [udu + vdv + wdw] = \frac{1}{2}\int d\left(u^2 + v^2 + w^2\right)$$

$= 0$, since the integration is that of a total differentiation carried out around a closed path.

Therefore, the equation (3.10.2) reduces to

$$\frac{dC}{dt} = \oint\left(\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz\right) \tag{3.10.3}$$

By some approximations, the equations of motion of an air become

$$\frac{du}{dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} + X \tag{3.10.4}$$

$$\frac{dv}{dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} + Y \tag{3.10.5}$$

$$\frac{dw}{dt} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + Z \tag{3.10.6}$$

when X, Y, Z represent external forces, such as those resulting from fiction. The equation (3.10.6) gives the vertical acceleration when gravity, pressure gradient, and external forces are not balanced. Now substituting the equation (3.10.4), (3.10.5) and (3.10.6) in the equation (3.10.3) we have

$$\frac{dC}{dt} = -\oint\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right) + \oint(Xdx + Ydy + Zdz)$$

$$\begin{aligned}\frac{dC}{dt} = & -\oint\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right) + \oint(Xdx + Ydy + Zdz) \\ & -\oint gdz - 2\Omega\oint\sin\phi(udy - vdx) \quad \text{where } f = 2\Omega\sin\phi\end{aligned} \tag{3.10.7}$$

Since, when there is no change of pressure at a point, then we have

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$$

So using this in the first integral of (3.10.7), we have,

$$-\oint \frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right) = -\oint \frac{dp}{\rho} \qquad (3.10.8)$$

The second integral may be written as

$$W = \oint (Xdx + Ydy + Zdz) \qquad (3.10.9)$$

where W represents the work done by the external forces.

Since the integration is around a closed path, the third integral is zero. So we have

$$-\oint g dz = 0 \qquad (3.10.10)$$

The fourth integral may be interpreted in the following way. Now we consider a cyclonic circulation at latitude ϕ. At that latitude we take x-axis positive to the east and the y-axis positive to the north and project these axes on the equatorial plane of the earth. The projected axes are denoted by x and x_1. The x and x_1 axes are parallel, so that

$$x = x_1, dx = dx_1 \text{ and } u = u_1$$

The y and y_1 axes are not parallel; the relationship between them may be seen from the following figure, which represents a cross section of the earth through the origin O of axes x and y and through the axes of rotation of the earth. It is apparent from the figure that

$$y_1 = y \sin\phi \text{ and hence, } dy_1 = dy \sin\phi \text{ and } v_1 = v \sin\phi$$

The area F enclosed by the projection of the circulation at latitude ϕ on the equatorial plane is given by

$$F = \oint x_1 dy_1 \qquad (3.10.11)$$

Now the rate of increase of area $\frac{dF}{dt}$ is obtained by differentiating (3.10.11) with respect to time, which is as follows

$$\frac{dF}{dt} = \oint\left(\frac{dx_1}{dt}dy_1 + x_1\frac{d}{dt}(dy_1)\right)$$

i.e., $\frac{dF}{dt} = \oint (u_1 dy_1 + x_1 dv_1)$ (3.10.12)

where $u_1 = \frac{dx_1}{dt}, dv_1 = d\left(\frac{dy_1}{dt}\right)$. Dividing (3.10.12) into two integrals and subtracting $v_1 dx_1$ from the first and adding it to the second lead to

$$\frac{dF}{dt} = \oint (u_1 dy_1 - v_1 dx_1) + \oint (x_1 dv_1 + v_1 dx_1)$$

i.e., $\frac{dF}{dt} = \oint (u_1 dy_1 - v_1 dx_1) + \oint d(x_1 v_1)$

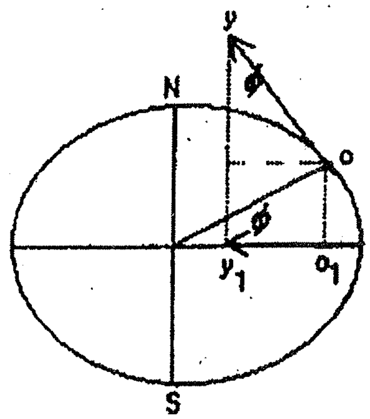
i.e., $\frac{dF}{dt} = \oint (u_1 dy_1 - v_1 dx_1)$ $\quad \because \oint d(x_1 v_1) = 0$

i.e., $\frac{dF}{dt} = \oint \sin\phi (udy - vdx)$, substituting for u_1, v_1, dx_1, dy_1 (3.10.13)

Using all these relations in the equation (3.10.7), finally we have

$$\frac{dC}{dt} = -\oint \frac{dp}{\rho} + W - 2\Omega \frac{dF}{dt} \quad (3.10.14)$$

This equation gives Kelvin's circulatino theorem.



Example: The circulation theorem has a number of applications to meteorology. For example, considering the last term only, if convergence occurs in the air motion over an area of low pressure at latitude ϕ, then in the equatorial plane $\frac{dF}{dt} < 0$ and $\frac{dC}{dt} > 0$. This agrees with the observed fact that the greater the convergence in the lower levels of a depression, the greater the cyclonic motion of the system becomes. In anticyclones, an increase in anti-cyclonic motion accompanies an increase in divergence in the lower levels.

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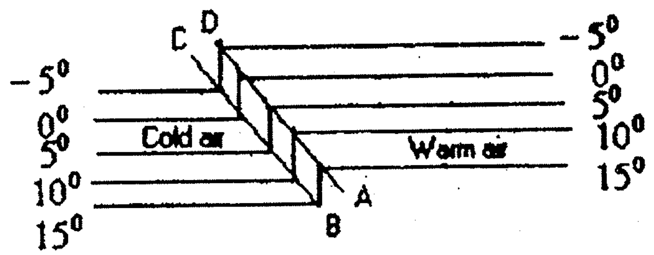
4.7 Surface of discontinuity:

One characteristic of an air mass is its uniformity. Properties of the air in an air mass change little if any in the horizontal direction. When two air masses lie side by side there is rapid change in properties from one air mass to the other. The following figure illustrates this in terms of the isotherms when one air mass is warmer than the other.



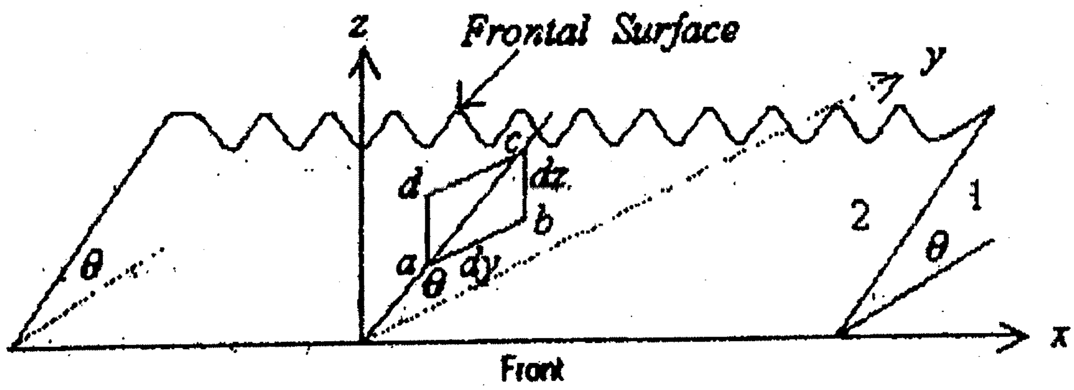
The region of discontinuity between the air masses is called a **frontal surface** or **front.** The term frontal surface is used for the surface of discontinuity between the two air masses when it is desirable to make a distinction between the discontinuities at the earth's surface and the above it. The front refers then to the line of discontinuity at the ground.

As a result of tubulent mixing between the air masses, there is no discontinuity in mathematical sense across a narrow zone of transition. The limits of this zone are indicated by AD and BC in the above figure. The discontinuity is usually one of temperature, but at time the contrast in moisture content across the surface is as marked as that of temperature.

When two liquids of unequal density, such as oil and water, are put into the same vessel, equilibrium is reached when the lighter liquid rests above the heavier. **In the atmosphere there is a tendency for the warmer, and**

therefore lighter, air to lie above the colder and heavier air. But as a result of the rotation of the earth, the equilibrium condition is reached when the frontal surface intersects the ground at a small angle.

The angle between the frontal surface and the ground: To find the angle between the frontal surface and the ground, we consider now a front lying in a line running from west to east, with the frontal surface slopping upward to the north. The situation is illustrated in the following figure, where the x-axis is positive towards the east, the y-axis is positive towards the north, and the z-axis is vertical. The angle, which the frontal surface makes with the horizontal, xy-plane is denoted by θ. We consider the rectangle *abcd*, of length dy and height dz, lying in the yz-plane. The subscript 1 denotes the denser air mass, which lies under the frontal surface and north of the front. The subscript 2 denotes the lighter air, extending from south of the front up to and over the frontal surface.



Since the pressure is continuous at the frontal surface, it is immaterial whether the point a is approached from air mass 1 or air mass 2; the pressure attained on reaching a is the same is both. Similarly at c, so that

$$p_{c_1} - p_{a_1} = p_{c_2} - p_{a_2} \tag{4.11.1}$$

In addition,

$$p_{c_1} - p_{a_1} = \left(p_{b_1} - p_{a_1}\right) + \left(p_{c_1} - p_{b_1}\right) = \frac{\partial p_1}{\partial y}dy + \frac{\partial p_1}{\partial z}dz \tag{4.11.2}$$

and

$$p_{c_2} - p_{a_2} = (p_{b_1} - p_{a_1}) + (p_{c_1} - p_{b_1}) = \frac{\partial p_2}{\partial y} dy + \frac{\partial p_2}{\partial z} dz \qquad (4.11.3)$$

Substituting (4.11.2) and (4.11.3) in (4.11.1) we have

$$\frac{\partial p_1}{\partial y} dy + \frac{\partial p_1}{\partial z} dz = \frac{\partial p_2}{\partial y} dy + \frac{\partial p_2}{\partial z} dz$$

$$\text{i.e.,} \left(\frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y}\right) dy = -\left(\frac{\partial p_1}{\partial z} - \frac{\partial p_2}{\partial z}\right) dz \qquad (4.11.4)$$

Therefore,

$$\frac{dz}{dy} = -\frac{\left(\frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y}\right)}{\left(\frac{\partial p_1}{\partial z} - \frac{\partial p_2}{\partial z}\right)}, \text{ i.e., } \frac{dz}{dy} = \tan\theta = -\frac{\left(\frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y}\right)}{\left(\frac{\partial p_1}{\partial z} - \frac{\partial p_2}{\partial z}\right)} \qquad (4.11.5)$$

According to the hydrostatics equation, we have

$$\frac{\partial p_1}{\partial z} = -g\rho_1 \text{ and } \frac{\partial p_2}{\partial z} = -g\rho_2$$

Using these relationships in (4.11.5) we get

$$\tan\theta = \frac{1}{g} \frac{\left(\frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y}\right)}{(\rho_1 - \rho_2)} \qquad (4.11.6)$$

Another form of the equation for the slope is obtained by substituting for $\frac{\partial p_1}{\partial y}$ and $\frac{\partial p_2}{\partial y}$ from the geostrophic motion: $\frac{1}{\rho}\frac{\partial p}{\partial x} = 2\Omega \sin\phi v$ and $\frac{1}{\rho}\frac{\partial p}{\partial y} = 2\Omega \sin\phi u$.

$$\tan\theta = \frac{2\Omega\sin\phi}{g}\left(\frac{\rho_2 u_2 - \rho_1 u_1}{\rho_1 - \rho_2}\right) \qquad (4.11.7)$$

Here u_2 and u_1 represent the components of the geostrophic wind parallel to the front. Again substituting for ρ from $p = \rho RT$ gives finally as follows

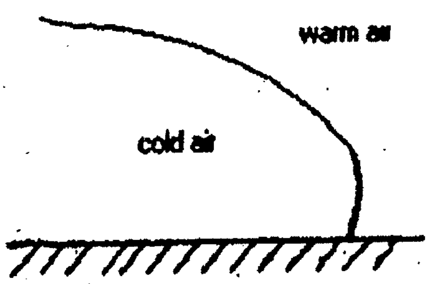
$$\tan\theta = \frac{2\Omega\sin\phi}{g}\left(\frac{T_1 u_2 - T_2 u_1}{T_2 - T_1}\right) \qquad (4.11.8)$$

This gives the angle between the frontal surface and the ground, assumed horizontal, Ω the angular velocity of the earth, ϕ the latitude, g the acceleration of the gravity, T_1, u_1 represent the temperature and the component parallel to the front of the geostrophic wind in the cold air, and T_1, u_1 the corresponding quantities in the warm air.

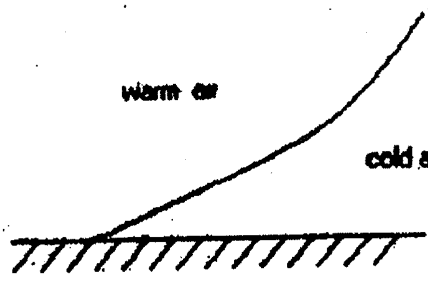
Note: According to (4.11.8), the frontal surface becomes vertical when $T_1 = T_2$. The angle θ increases with increasing difference in velocity, and also increases with decreasing difference in temperature across the frontal surface. Since a front is a boundary between two air masses, it must move when the air masses.

Cold front and warm front: When a front moves toward the warmer air, so that the cold air is occupying territory formerly covered by warm air, it is called a cold front. If it is moving toward the colder air, with warm air occupying territory formerly covered by cold air, it is called a warm front. A front may move in one direction at one portion of its length and another direction at another sector; therefore, it may be partly a warm front and partly a cold front.

In a cold front, the wedge of the cold air is moving actively forward, and the effect of surface friction is to hold back the part near the ground so that it tends to steepen the front. In a warm front, the cold air wedge is receding and the effect of surface is to hold back the front near the ground so that it trails with small slope.

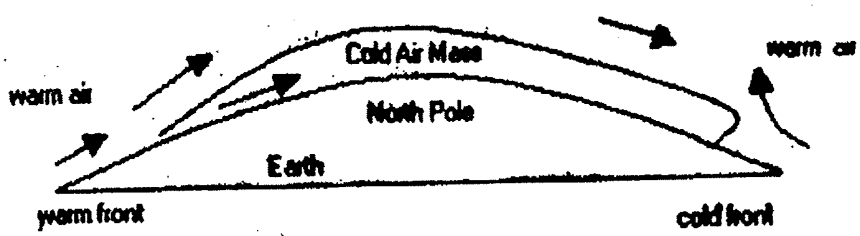


Cold front slope



Warm front slope

Air masses: An air mass is a contiguous, widespread body of air that has remained stagnant (still standing) over a surface for sufficient duration to be modified by the surface as in the following figure. For example, arctic (relating to the North Pole) air masses that develops with cold temperatures over Polar Regions. Air masses are usually classified by their potential temperature and humidity, which by definition are relatively homogeneous within the air mass. Air masses can also be characterized by their visibility, cloudiness, static stability, and turbulence etc.



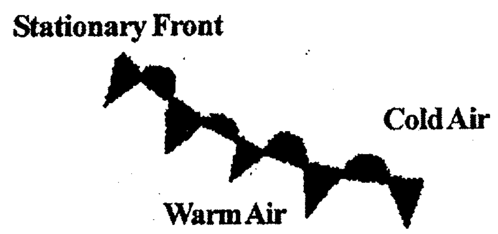
Vertical cross section through an arctic air mass

Front and its classiciation:

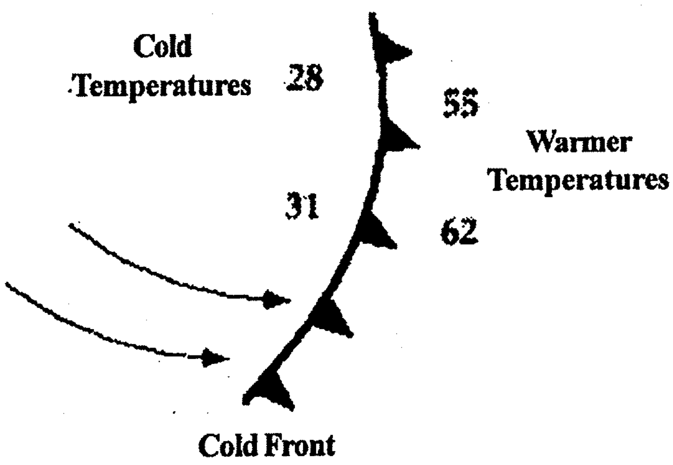
Often, we can observe regions of strong temperature contrasts on weather maps. These regions of contrast are often very deep and extend through the troposphere. Such zones of contrast are marked by lines on the surface weather map called *fronts*.

Polar Front: The *Polar front* is the general name given to the boundary separating the polar air masses from tropical air masses that extends around the world. Symbols are used on surface weather maps to indicate the characteristics or type of front.

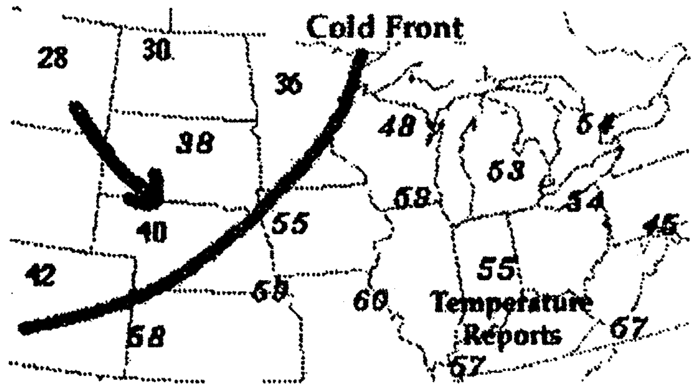
Stationary Front: It is front that is not moving. When a warm or cold front stops moving, it becomes a stationary front. Once this boundary resumes its forward motion, it once again becomes a warm front or cold front or A stationary front is represented by alternating blue and red lines with blue triangles pointing towards the warmer air and red semicircles pointing towards the colder air.



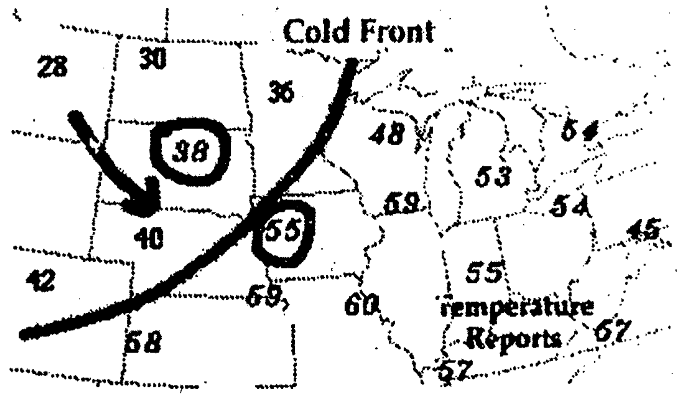
Cold Front: A cold front is defined as the transition zone where a cold air mass is replacing a warmer air mass. Cold fronts generally move from northwest to southeast. The air behind a cold front is noticeably colder and drier than the air ahead of it. When a cold front passes through, temperatures can drop more than 15 degrees within the first hour.



Symbolically, a cold front is represented by a solid line with triangles along the front pointing towards the warmer air and in the direction of movement. On colored weather maps, a cold front is drawn with a solid blue line.

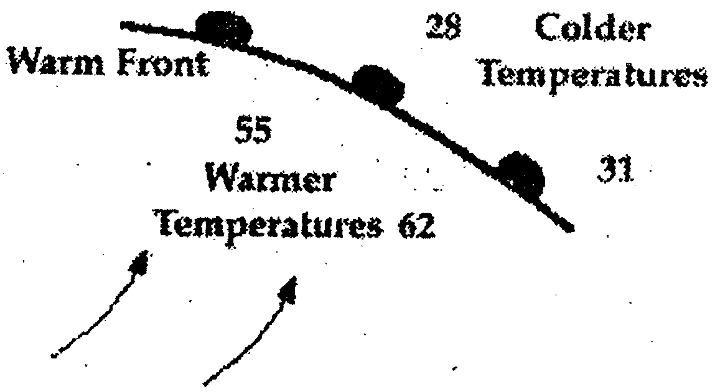


There is typically a noticeable temperature change from one side of a cold front to the other. In the map of surface temperatures below, the station east of the front reported a temperature of 55 degrees Fahrenheit while a short distance behind the front, the temperature decreased to 38 degrees. An abrupt temperature change over a short distance is a good indicator that a front is located somewhere in between

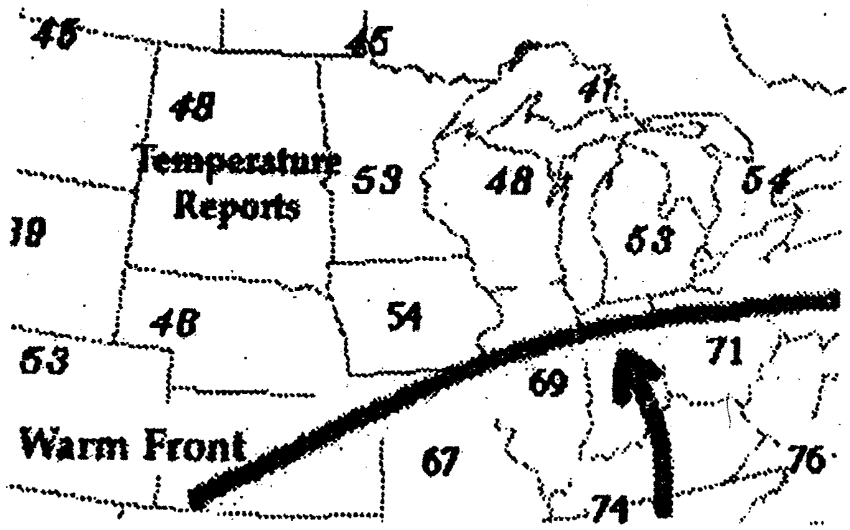


If colder air is replacing warmer air, then the front should be analyzed as a cold front. On the other hand, if warmer air is replacing cold air, then the front should be analyzed as a warm front.

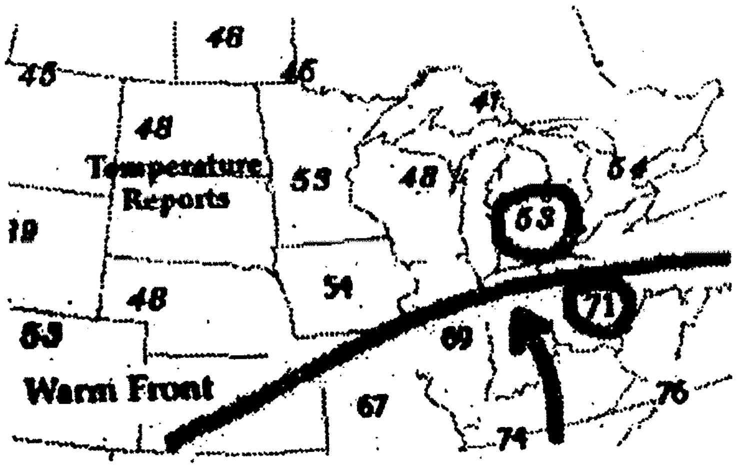
Warm Front: A warm front is defined as the transition zone where a warm air mass is replacing a cold air mass. Warm fronts generally move from southwest to northeast and the air behind a warm front is warmer and more moist than the air ahead of it. When a warm front passes through, the air becomes noticeably warmer and more humid than it was before.



Symbolically, a warm front is represented by a solid line with semicircles pointing towards the colder air and in the direction of movement. On colored weather maps, a warm front is drawn with a solid red line.

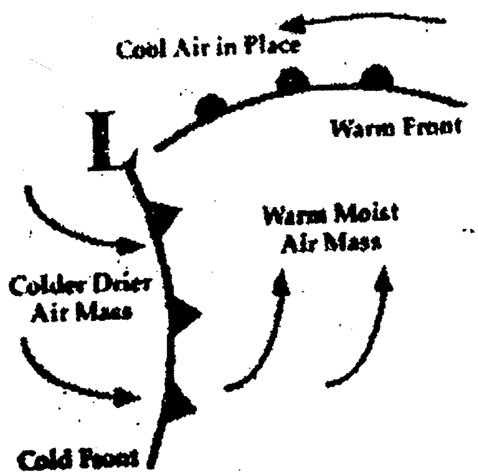


There is typically a noticeable temperature change from one side of the warm front to the other. In the map of surface temperatures below, the station north of the front reported a temperature of 53 degrees Fahrenheit while a short distance behind the front, the temperature increased to 71 degrees. An abrupt temperature change over a short distance is a good indication that a front is located somewhere in between.

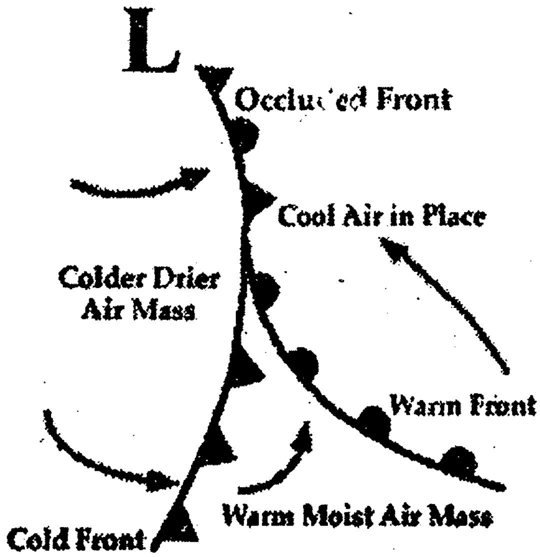


If warmer air is replacing colder air, then the front should be analyzed as a warm front. If colder air is replacing warmer air, then the front should be analyzed as a cold front.

Occluded Front: A developing cyclone typically has a preceding warm front (the leading edge of a warm moist air mass) and a faster moving cold front (the leading edge of a colder drier air mass wrapping around the storm). North of the warm front is a mass of cooler air that was in place before the storm even entered the region.



As the storm intensifies, the cold front rotates around the storm and catches the warm front. This forms an occluded front, which is the boundary that separates the new cold air mass (to the west) from the older cool air mass already in place north of the warm front. Symbolically, an occluded front is represented by a solid line with alternating triangles and circles pointing the direction the front is moving. On colored weather maps, an occluded front is drawn with a solid purple line.



Changes in temperature, dew point temperature, and wind dierction can occur with the passage of an occluded front. In the map below, temperatures ahead (east of) the front were reported in the low 40's while temperatures behind (west of) the front were in the 20's and 30's. The lower dew point temperatures behind the front indicate the persence of drier air.

Anticyclones: High-pressure centers (highs) typically have downward motion (subsidence) in the mid troposphere, and horizontal spreading of air (divergence) near the surface as in the figure-1. Subsidence impedes (to obstruct) cloud development, leading to generally clear skies and fair weather. Winds are also generally light in highs, because gradient-wind dynamics of highs requires weak pressure gradients near the high center.

The diverging air near the surface spirals outward due to the weak pressure-gradient force. Coriolis force causes it to rotate clockwise (anti-cyclinically) around the high-pressure center in the northern hemisphere, and opposite in the southern hemisphere. For this reason, **high pressure centers are called anticyclines.**

4.7.1 Cyclogenesis

Cyclogenesis is the birth and growth of cyclones. Such intensification can be defined by the

* Sea-level pressure decrease,
* Upward-motion increase, and
* Vorticity increase.

These characteristics are not independent; for example, upward motion can reduce surface pressure, which draws in air that begins to rotate due to coriolis force. However, we can gain insight (power of seeing into and understanding things) into the workings of the storm by examining the dynamics and thermodynamics that govern each of these characteristics. Each of these will be explored in details in the following after first examining how mountains can trigger cyclone formation. The following conditions have been found to favor rapid cyclogenesis:

* Strong baroclinicity - a large horizontal temperature gradient
* Weak static stability - temperature decreasing with height faster than the tropospheric standard atmosphere.
* Mid or high-latitude location - earth's contribution to vorticity increase toward the poles.
* Large moisture input - latent heating due to cloud condensation adds energy and reduces static stability.
* Large-amplitude wave in the jet stream - a trough to the west and ridge to the east of the surface low enhance horizontal divergence aloft, which strengthens updrafts
* Terrain elevation decrease toward east - cyclogenesis to the lee of mountains

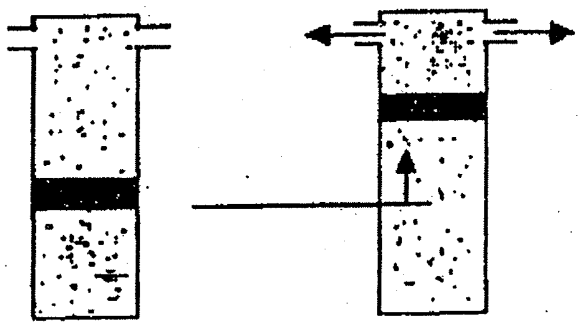
A cyclone that develops extremely fast is called a **cyclone bomb**, and the process is called **explosive cyclogenesis.** To be classified as a bomb, the central pressure of a cyclone must decrease at a rate of 0.1 kPa per hour for at least 24 hours.

The opposite of cyclogenesis is **cyclolysis.** This is literally death of a cyclone. Most cyclones go through a cycle of formation, growth, and death, over a period of about a week at midlatitude. However, lifetimes less than a day and greater than two weeks have been observed.

4.8 Sea-Level-Pressure Tendency:

Sea-level pressure is another measure of cyclone intensity. When pressure drops in an intensifying low center, the low is said to **deepen.** In other words, the low becomes lower. When sea-level pressure increases, the low **fills** (i.e., the low weakens). The pressure change with time is called the **pressure tendency**. The amount of air mass in the column of atmosphere above the low center determines the surface pressure in a hydrostatic environment.

Mass Budget: We will approach this subject heuristically. Picture a small cylinder filled with air as shown in the following figure. Near the middle of the cylinder is a weightless, frictionless piston. Above and below the piston, teh air pressures are identical, and the air densities are identical.



Suppose that some of the air is withdrawn from the top of the cylinder. Pressure in the top of the cylinder will decrease, which will cause the piston to rise until the bottom pressure equals the top pressure. At that point the densities above and below the piston are also identical. Similarly, air mass could have been added to the top of the cylinder, causing the pressure to increase and piston to move down.

Thus, we can use the vertical motion of the piston as a surrogate measure of net mass flow and pressure change. Namely, **upward motion indicates decreases in mass and pressuer, while downward motion indicates increases.**

That seems as a simple enough rule. However, picture what would happen if air were withdrawn or added to the bottom of the cylinder. Upward piston motion would correspond to an increase in mass and pressure, not a decrease as before. Thus we can use vertical piston movement as a surrogate measure of pressure change only if we change the sign of the result depending on whether mass is added at the top or bottom of the column.

We will extend this reasoning to the atmosphere, where the cylinder of the previous example will now be visualized as a column of air from the ground to the top of the atmosphere. A complication is that atmospheric density decreases with height. Recall that pressure is force per unit area, and that force is mass times acceleration. Sea-level pressure p_s is a measure of the total mass m of the air in a column above the surface

$$p_s = \frac{g}{A} m \qquad (4.13.1)$$

where $g = 9.8$ ms^{-2} is gravitational acceleration, and A is the horizontal cross-section area under the column.

Now, change in sea-level pressure with time t are caused by changes in total air mass above the surface:

$$\frac{\Delta p_s}{\Delta t} = \frac{g}{A}\frac{\Delta m}{\Delta t} \qquad (4.13.2)$$

Recall from the definition of density that

$$m = \rho \times volume = \rho A z \qquad (4.13.3)$$

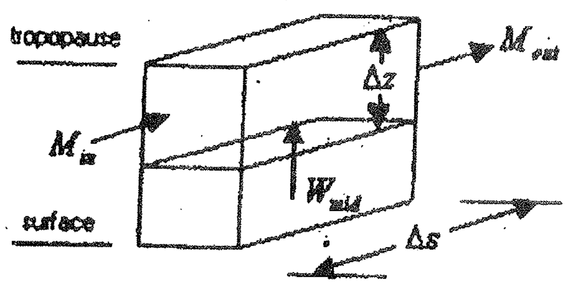
where z is the height of the volume. Define the change of height with time as a surrogate velocity $W_{surrogate}$ analogous to the piston movement. Thus, if the mass flow in or out of the column occurs at a height z where the air density $\rho(z)$ is known, then the equations above can be combined to give the pressure tendency:

$$\frac{\Delta p_s}{\Delta t} = \pm g \times \rho(z) \times W_{surrogate}(z) \qquad (4.13.4)$$

The proper sign for the right-hand-side of the equation must be chosen depending on the cause of the surrogate vertical motion (i.e., whether it is driven at the top or bottom of the troposphere). **There can be more than one mechanism adding or subtracting mass to a column of air, so we will generalized the right-hand-side of the equation (4.13.4) to be a sum of terms. Four mechanisms will be included here in a simplified model for pressure tendency:**

* Upper-level divergence
* Boundary-layer pumping
* Advection
* Diabetic heating

Upper-level divergence: Vertical motion in a cyclone is often driven by changes of horizontal wind speed of the upper-tropospheric jet stream. **Mass continuity requires that horizontal divergence be compensated by vertical convergence, and vice versa.** Namely, air leaving a region horizontally must be balanced by replacement air entering the same region vertically, so as not to leave a vacuum as in the following figure.



The jet stream is near the base of the stratosphere, where strong static stability aloft impedes vertical motion above the jet. Thus, most of the compensating vertical motion occurs in the troposphere, beneath the level of the jet. For the column of air in the above figure, the incompressible continuity equation in two dimensions becomes

$$W_{mid} = \frac{\Delta M}{\Delta s} \Delta z$$

Where $\Delta M = M_{out} - M_{in}$ is the change of horizontal wind speed, Δs is horizontal distance between inflow and outflow, $\frac{\Delta M}{\Delta s}$ is the upper level horizontal divergence, Δs is the upper portion of the troposphere (between the 50 kPa surface and the tropopause), and W_{mid} is the vertical velocity across the 50 kPa surface. We will assume for simplicity that all of the horizontal divergence aloft can be represented by the winds in one direction. So the divergence aloft is a mass removal process. That resulting W_{mid} will be used as one of the forcing terms in the net pressure tendency equation.

Boundary layer pumping: Density at any fixed altitude changes only little with temperature and humidity for most non-violent weather conditions. Therefore, we can neglect mass changes within the volume, compared to the inflow and outflow. The air is said to be incompressible when the density does not change. This approximation fails in strong thunderstorm updrafts and tornadoes.

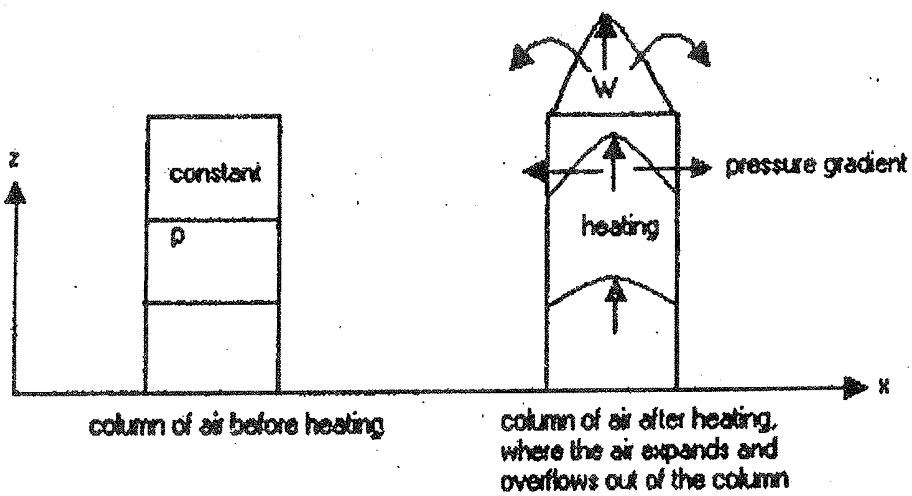
For incompressible flow, the left hand side of continuity equation $\frac{\Delta \rho}{\Delta t} = -\rho\left[\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} + \frac{\Delta w}{\Delta z}\right]$, is zero. This requires inflow to balance outflow:

$$\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} + \frac{\Delta w}{\Delta z} = 0$$

Around low-pressure regions near the surface, turbulent drag causes horizontal inflow (convergence) within the boundary layer. There is no vertical air motion (W=0) at the bottom of the boundary layer because of the ground. Thus, the horizontal inflow must be balanced by vertical outflow from the boundary layer top. This mechanism for creating mean upward motion is called **Ekman pumping** or **boundary layer pumping**. The upward motion carries water vapor, which then condenses in the adiabatically colled air, causing clouds and precipitation.

Advection: Advection describes the movement of cyclones by steering level winds. Namely, surface press can drop at a fixed point on the ground if the horizontal wind blows in an air column having less total mass than column of air that is blowing out.

Diabetic (non adiabatic) heating: When water vapor condenses in clouds, latent heat is released. This heat warms the air in a column, causing it to expand as in the following figure. According to the hypsometric equation, horizontal pressure gradients develop which push the air out of the column. Hence, there is divergence of mass from the column, which lowers the sea-level pressure.



In a simplified point of view, the heated air expands out of the top of the column and overflows. The amount of overflow out of the top in this simple view equals the amount of mass diverged from the sides in a more realistic view. The vertical velocity W at the top of the column is a surrogate measure of the mass loss. If all of the condensation falls as precipitation, then the latent heating rate is related to the rainfall rate RR:

$$\frac{\Delta T_v}{\Delta t} = \frac{1}{\Delta z} \frac{L_v}{C_p} \frac{\rho_{liq}}{\rho_{air}} RR$$

where T_v is the virtual temperature in the column, t is time, L_v is the latent heat of vaporization, C_p is the specific heat at constant pressure for air, ρ_{liq} is the density of liquid water (1025 kg m^{-3}), ρ_{air} is the density of air, Δz is the column depth, and where RR has units of *mm/h*.

The net effect on sea-level pressure can be found by combining the hypsometric equation and the equation above:

$$\frac{\Delta p_s}{\Delta t} = \frac{g}{\overline{T}v}\frac{L_v}{C_p}\rho_{liq} RR$$

where $g = 9.8 \text{ ms}^{-2}$ is gravitational acceleration, and $\overline{T}_v$ is the average virtual temperature in the column $(\cong 300K)$. Because RR is the net precipitation reaching the ground, the equation above represents the net or average effect of latent heating over the whole cyclone depth.

Although this equation depends on absolute temperature, which can vary, to first order we can approximate it as

$$\frac{\Delta p_s}{\Delta t} = -bRR$$

where $b = -\frac{g}{T_v}\frac{L_v}{C_p}\rho_{liq} \cong 0.084 kPa / mm_{rain}$

Net Pressure Tendency: Combining all of the mechanisms just described the following form for the net sea-level pressure tendency equation

$$\underbrace{\frac{\Delta p_s}{\Delta t}}_{\substack{pressure\\tendency}} = \underbrace{-M_c\frac{\Delta p_s}{\Delta s}}_{\substack{horizontal\\advection}} + \underbrace{g\rho_{BL}W_{BL}}_{\substack{boundary-layer\\pumping(in)}} - \underbrace{g\rho_{mid}W_{mid}}_{\substack{upper-level\ horizontal\\divergence(out)}} - \underbrace{bRR}_{\substack{latent\\heating}}$$

where ρ_{BL} and ρ_{mid} are the average air densities at the top of the boundary layer and midpoint of the column, respectively, and the speed of movement of the column is M_c measured positive in the direction of movement, along horizontal path s. A negative value of $\frac{\Delta p_s}{\Delta t}$ corresponds to pressure drop any cyclone intensification.

5.0 Unit Summary

In this module, the gradient wind, the thermal wind, cyclostrophic wind, characteristics of fluid flow applied to the atmosphere, the atmospheric energy equation, and rate of change of circulation, surface of discontinuity, cyclogenesis and sea level pressure tendency are discussed.

6.0 Self Assessment Questions

Q1(a): Derive the following expression for the vertical distribution of the density when the lapse rate of temperature is constant

$$\rho=\rho_0\left(\frac{T_0-\alpha z}{T_0}\right)^{\frac{g}{R\alpha}-1}$$

Assume that the air is dry and that g is constant.

(b): For what value of the temperature lapse rate is the density constant with height?

Q2: Calculate at what height in a dry atmosphere the pressure is one-half of that at the surface when the surface temperature is 10°C, and the lapse rate is (a) 6°C per km, (b) zero. Assume that g is constant.

Q3: Expressing the first law of thermodynamics in terms of dT and $d\alpha$ (where α is the specific volume), obtain Poisson's equation in terms of T, T_0, α, α_0.

Q4: Find the rate of change of circulation in the atmosphere.

Q5. Derive the thermal wind equation.

Q6: What is the purpose of the Aerological diagram. Mention all characteristics to draw an aerological diagram.

Q7. Define mixing ratio, specific humidity and relative humidity, absolute humidity. Establish the relation between these.

Q8: What is the virtual temperature? Find the Adiabatic lapse rate for moist unsaturated air.

Q9: Describe the temperature distribution in the atmosphere.

Q10: Show that the dry adiabatic lapse rate is approximately constant throughout the lower atmosphere.

Q11: What are different kinds of wind that may exist in the atmosphere? Obtain the governing equation of one such wind.

Q12: Obtain the atmosphere energy equation stating clearly the assumptions you have made. Interpert each term of your equation.

Q13: Define potential temperature. Obtain teh relation $S = C_p \ln \theta +$ constant where S is the specific entropy and θ is the potential temperature for a parcel of dry air.

Q14: What are the general characteristics of the atmosphere? State the first law of thermodynamics.

Q15: Derive the expression for the vorticity of an air parcel.

Q16: What is geodynamic paradox?

Q17: Write down the basic assumption made in determining the stability criteria for the vertical motions of an individual parcel of air. Show that the parcel of air will be stable, neutral and unstable according as $\Gamma_d >,=,< \gamma$.

Q18: Write down the equation of motion of an atmosphere. Obtain Gradient wind equation, stating clearly the assumptions you have made.

Q19: What do you mean by an adiabatic process and isobaric process in teh atmosphere?

Q20: Derive the area equivalence of the emagram and tephigram. Discuss the important features of these.

Q21: The temperature at a point 50 km north is 3°C cooler than at the station. If the wind is blowing from NE at 32 ms^{-1} and the air is being heated by radiation at the rate of 1°C, what is the local temperature change at the station?

Q22: Show that as pressure gradient approaches zero the gradient wind reduces to the geostrophic wind for a normal anticyclone and to the inertia circle for an anomalous anticyclone.

Q23: Show that the geostrophic wind is independent of height in a barotropic atmosphere.

Q24: If wind rotates as a solid body about the center of a low pressure system, and the tangenital velocity is 10m/s at radius 300 km, find the relative vorticity.

7.0 Suggested further readings:

1. Brunt, D., Physical and Dynamical Meteorology, London, Cambridge University Press, 1939.
2. Hewson, W.E., and Longley, W.R., Meteorology Theoretical and Applied, John Wiley & Sons, INC., Chapman & Hall, LTD., London.
3. Byers, H.R., General Meteorology, McGraw-Hill, Godson, W.L. and Iribarne, J.V., Atmospheric Thermodynamics, D. Reidel Publishing Company.
4. Holton, J.R., An Introduction to Dynamic Meteorology, Academic Press, New York.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper-IX

Special Paper : Operations Research

Module No. - 97

Advanced Optimization and Operations Research-I

(Theorems of the Alternative)

Module Structure :

97.1 Introduction

In this module, you will learn about the basic theories on optimization problems. You have learnt in under-graduate course about the optimization of linear programming problems. Here we shall discuss about the non-linear programming problems, their optimality criteria, existence theorems, etc.

In the next module, you will learn about the convex sets, convex and concave functions and some fundamental theorems for convex functions. The theories of convex functions have crucial role in optimization of non-linear programming problems.

97.2. Objectives

Go through this module you will learn the following:

* Non-linear programming
* Farkas' theorem
* Optimality criteria of linear programming
* Existence theorems for linear systems
* Theorems of the alternative

97.3. Key words

Non-linear programming, Farkas' theorem, theorems of alternative, convex sets, convex and concave functions.

97.4. The Non-linear Programming Problem

The non-linear programming problem has three fundamental ingredients - a finite number of real variables, a finite number of constraints which the variables must satisfy, and a function of the variables which must be minimized (or maximized).

Mathematically, the problem is

Find specific values $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ if they exist, of the variables $(x_1, x_2, ..., x_n)$ that will satisfy the follwing.

(i) Inequality constraints

$$g_i(x_1, x_2, ..., x_n) \leq 0,\ i = 1, 2, ..., m \qquad ....(1)$$

(ii) The equality constraints

$$h_j(x_1, x_2, ..., x_n) = 0,\ j = 1, 2, ..., k \qquad ....(2)$$

and

(iii) minimize (or maximize) the objective function $\theta(x_1, x_2, ..., x_n)$

over all values of $x_1, x_2, ..., x_n$ satisfying (1) and (2).

Here, g_i, h_j and θ are numerical functions of the variables $x_1, x_2, ..., x_n$, which are defined for all finite values of the variables. The fundamental difference between this problem and that of the classical constrained minimization problem of the ordinary calculus is the presence of the inequalities (1).

Notation.

If $x \geqq 0$, x is said to be non-negative,

if $x \geq 0$, then x is said to be semi-positve (positive or zero)

and, if $x > 0$, then x is said to be positive.

Theorem 97.1. (Farkas' theorem)

Statement. For each $p \times n$ matrix A and each fixed vector b in R^n, either

$Ax \leqq 0, bx > 0$ has a solution $x \in R^n$ .... (A)

or

$A^1 y = b, y \geqq 0$ has a solution $y \in R^p$ .... (B)

but never both.

Geometrical interpretaion of Farkas' Theorem

We rewrite the equations (A) and (B) as follows:

$$A_j x \leqq 0, j = 1, 2, ..., p, bx > 0 \qquad \text{.................... (C)}$$

$$\sum_{j=1}^{p} A_j^1 y_j = \sum_{j=1}^{p} A_j y_j = b, y_j \geqq 0, \quad j = 1, 2, ..., p \qquad \text{.................... (D)}$$

where A'_j denotes the jth columns of A' and A_j the jth row of A. System (D) requires that the vector b be non-negative linear combination of the vectors A_1 to A_p. System (C) requires that we find a vector $x \in R^n$ that makes an obtuse angle with the vector A_1 to A_p and a strictly acute angle with b.

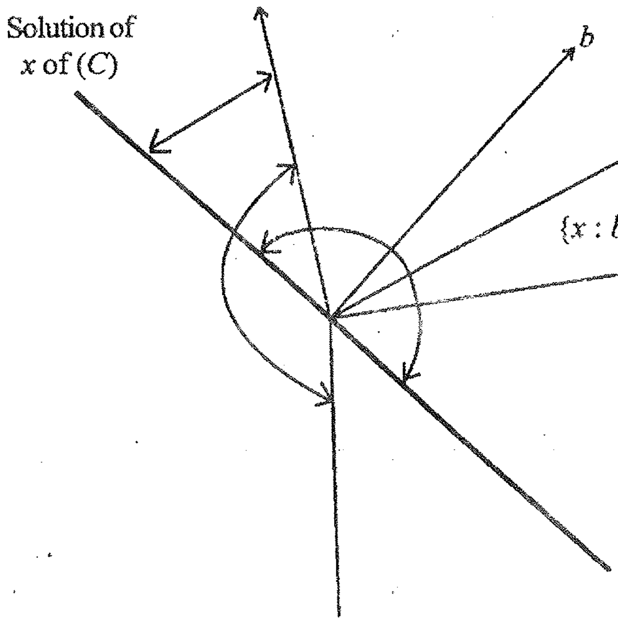


Figure 97.1. Geometrical interpretation of Farkas' theorem : (C) has solution, (D) has no solution.

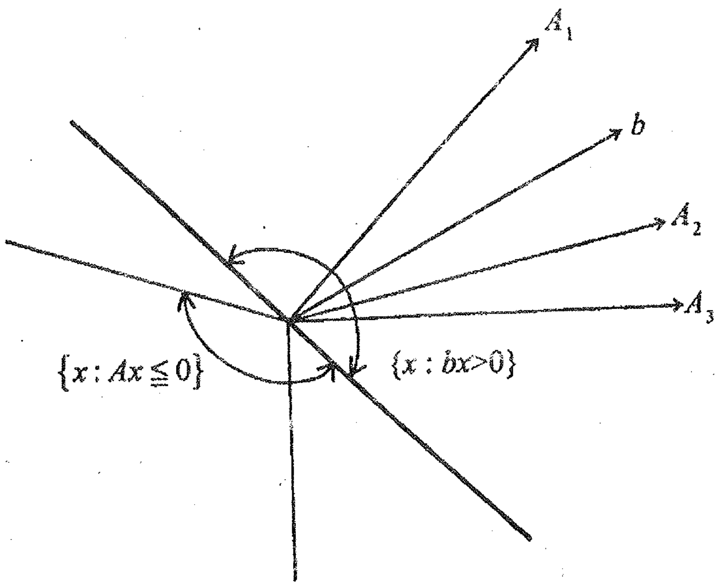


Figure 97.2 : Geometrical interpretation of Farkas' Theorem : (D) has solution, (C) has no solution

Figure 97.2 shows a simple case with $n = 2, p = 3$, in which (D) has a solution, and hence by Farkas' theorem (C) cannot have a solution. Figure 97.1 shows a case with $n = 2, p = 2$, in which (D) has no solution, and hence by Farkas' theorem (C) must have a solution.

97.5. An Application of Farkas' Theorem

Here we cite an application of Farkas' theorem. As a typical example of the use of the theorems of the alternative we shall show how Farkas' theorem can be employed to derive necessary optimality conditions for the following linear programming problem.

Linear programming problem :

Find an $\overline{x}$, if it exists, such that

$$-bx = \min_{x \in x}(-bx), \overline{x} = \{x : x \in R^n, Ax \leqq c\} \qquad \text{.......................} (3)$$

where $\overline{x} \in x$, b and c are given fixed vectors in R^n and R^m respectively, and A is a given fixed mxn matrix.

Optimality criteria of linear programming

Necessary Condition :

Let $\overline{x}$ be a solution of the linear programming problem (3). Then there exists a $\overline{u} \in R^m$ such that $(\overline{x}, \overline{u})$ satisfy

(i) Dual feasibility

$$A\overline{x} \leqq c \qquad \text{.........................} (4)$$

(ii) Primal feasibility

$$A'\overline{u} = b \qquad \text{.........................} (5)$$

$$\overline{u} \geqq 0 \qquad \text{.........................} (6)$$

(iii) Complementarity

$$b\overline{x} = c\overline{u} \qquad \text{.........................} (7)$$

Sufficient Condition

If some $\overline{x} \in R^n$ and some $\overline{u} \in R^m$ satisfy equations (4) to (7), then $\overline{x}$ solves (3).

Proof. The condition is necessary

Let $\overline{x}$ be the solution of (3). Define the index sets P, Q and M as follows:

$$P \cup Q = M = \{1, 2, ..., m\}, p = \{i : A_i \overline{x} = c_i\},$$

$$Q = \{i : A_i \bar{x} < c_i\}$$

and assume that P and Q contain p and q elements, respectively. Then

$$A_P \bar{x} = c_P \qquad \text{.......... (8)}$$

$$A_Q \bar{x} = c_Q \qquad \text{.......... (9)}$$

If $P = \phi$, it follows that

$$A\bar{x} - c < -\delta e.$$

for some real number $\delta > 0$, where e is an m-vector of ones. Then for each $x \in R^n$, we can find a real number $\alpha > 0$ such that

$$A(\bar{x} + \alpha x) - c < -\delta e + \alpha Ax \leqq 0 \ \text{(since } Ax \leqq c\text{)}$$

and hence $\bar{x} + \alpha x \in X$. Since $\bar{x}$ is the minimum solution of (3), we have that for each $x \in R^n$ there exists an $\alpha > 0$ such that

$$-bx \leqq -b(\bar{x} + \alpha x).$$

Hence

$$bx \leqq 0 \ \text{for each } x \in R^n$$

which implies that $b = 0$. By taking $\bar{u} = 0 \in R^m$, the relations (4) to (7) are satisfied because $b = 0$.

If $P \neq \phi$, then we assert that the system

$$A_p x \leqq 0, bx > 0 \qquad \text{.......... (10)}$$

has no solution $x \in R^n$. For, if (10) have a solution $\bar{x}$, say, then αx would also be a solution of (10) for each $\alpha > 0$. Now, consider the point $\bar{x} + \alpha x_1$ where x is a solution of (10) and $\alpha > 0$.

Then

$$-b(\bar{x} + \alpha x) < -b\bar{x} \ \text{for } \alpha > 0 \ \text{[by (10)]} \qquad \text{.......... (11)}$$

$$A_p(\bar{x} + \alpha x) - c_p \leqq 0 \ \text{for } \alpha > 0 \ \text{[by (10)]} \qquad \text{.......... (12)}$$

$$A_Q(\bar{x} + \alpha x) - c_Q \leqq -\delta e + \alpha A_Q x \leqq 0 \ \text{for some } \alpha > 0 \ \text{[by (9)]} \qquad \text{.......... (13)}$$

where in (13) the first inequality follows by defining e as a q-vector on ones and

$$-\delta = \max_{i \in Q}(A_i \bar{x} - c_i) < 0$$

and the second inequality of (13) holds for some $\alpha > 0$ because $-\delta < 0$. But relations (11) to (12) imply that

$\bar{x}+\alpha x \in X$ and $-b(\bar{x}+\alpha x)<-b\bar{x}$, which contradicts the assumption that $\bar{x}$ is a solution of (3). Hence (10) has no solutions $x \in R^n$,

and by Farkas' theorem the system

$$A_P' y = b, y \geqq 0 \qquad \text{.....................(14)}$$

must have a solution $y \in R^P$. If we let $0 \in R^q$, we have then

$$A_P' y + A_Q' 0 = b, y \geqq 0 \qquad \text{.....................(15)}$$

and $$c_P y + c_Q 0 = y c_P = y A_P \bar{x} = b\bar{x} \qquad \text{.....................(16)}$$

[by (10) and (14)]

By defining $\bar{u} \in E^m$ such that $\bar{u}=(\bar{u}_P, \bar{u}_Q)$, where $\bar{u}_P = y \in R^P, \bar{u}_Q = 0 \in R^q$, condition (15) becomes conditions (5) and (6), and condition (16) becomes condition (7). Condition (4) holds because $\bar{x} \in X$. This proofs that the condition is necessary.

The condition is sufficient

Let $\bar{x} \in R^n$ and $\bar{u} \in R^m$ satisfy (4) to (7), and let $x \in X$. By (4) we have that $\bar{x} \in X$. Now,

$$\begin{aligned} -bx-(-b\bar{x}) &= -b(x-\bar{x}) \\ &= -\bar{u}A(x-\bar{x}) \text{ [by (5)]} \\ &= -\bar{u}Ax + c\bar{u} \text{ [by (5) and (7)]} \\ &= -\bar{u}(Ax-c) \\ &\geq 0 \ [by (6), x \in X]. \end{aligned}$$

97.6. Existence Theorems for Linear Systems

We establish now some key thorems for the existence of certain types of solutions for linear systems. We start with a lemma due to Tucker.

Lemma 97.1 (Tucker's lemma)

For any given $p \times n$ matrix A, the systems

$$Ax \geqq 0 \qquad \text{...............(A)}$$

and

$$A'y = 0, y \geqq 0 \qquad \text{................(B)}$$

possess solutions x and y satisfying

$$A_1 x + y_1 > 0.$$

[A_1 is the first row of the matrix A]

Proof. The proof is by induction on p. For $p = 1$, if $A_1 = 0$, take $y_1 = 1, x = 0$; if $A_1 \neq 0$, take $y_1 = 0, x = A_1$.

Now, assume that the lemma is true for a matrix A of p rows and proceed to prove it for a matrix of $p+1$ rows $\bar{A}$, i.e.

$$\bar{A} = \begin{bmatrix} A \\ A_{p+1} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \\ A_{p+1} \end{bmatrix}$$

By applying the lemma to A, we get x, y satisfying

$$Ax \geqq 0, A'y = 0, y \geqq 0, A_1 x + y_1 > 0. \qquad \text{......................(1)}$$

If $A_{p+1} x \geqq 0$, we take $\bar{y} = (y, 0)$. Then

$$\bar{A}x \geqq 0, \bar{A}'\bar{y} = 0, \bar{y} \geqq 0, A_1 x + y_1 > 0 \qquad \text{......................(2)}$$

which extends the lemma to $\bar{A}$.

However, if $A_{p+1} x < 0$, we apply to lemma a second time to the matrix B, where

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_p \end{bmatrix} = \begin{bmatrix} A_1 + \lambda_1 A_{p+1} \\ A_2 + \lambda_2 A_{p+1} \\ \vdots \\ A_p + \lambda_p A_{p+1} \end{bmatrix}$$

$$\text{where } \lambda_j = -\frac{A_j x}{A_{p+1} x} \geqq 0, j = 1, 2, \ldots, p \text{ [by(1)]} \qquad \text{......................(4)}$$

Therefore,

$$B_j x = A_j x + \lambda_j A_{p+1} x = 0$$

$$\text{or, } \quad Bx = 0. \qquad \text{......................(5)}$$

This second use of the lemma yields v, u satisfying

$$Bv \geqq 0, B'u = 0, u \geqq 0, B_1 v + u_1 > 0. \qquad \text{......................} (6)$$

Let $\bar{u} = \left(u, \sum_{j=1}^{p} \lambda_j u_j \right)$. It follows from (4) and (6) that

$$\bar{u} \geqq 0 \qquad \text{......................} (7)$$

$$\bar{A}'\bar{u} = A'u + A'_{p+1} \sum_{j=1}^{p} \lambda_j u_j = \left(B'u - \sum_{j=1}^{p} \lambda_j A'_{p+1} u_j \right) + A'_{p+1} \sum_{j=1}^{p} \lambda_j u_j = 0 \qquad \text{......................} (8)$$

[using (3) and (6)]

Let $$w = v - \frac{A'_{p+1} v}{A_{p+1} x} x \qquad \text{......................} (9)$$

then

$$A_{p+1} w = A_{p+1} v - A_{p+1} v = 0 \qquad \text{..................} (10)$$

and

$$\bar{A}w = \begin{bmatrix} A \\ A_{p+1} \end{bmatrix} w = \begin{bmatrix} Aw \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 w \\ A_2 w \\ \vdots \\ A_p w \\ 0 \end{bmatrix} = \begin{bmatrix} (B_1 - \lambda_1 A_{p+1}) w \\ (B_2 - \lambda_2 A_{p+1}) w \\ \vdots \\ (B_p - \lambda_p A_{p+1}) w \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} Bw \\ 0 \end{bmatrix} = \begin{bmatrix} Bv - \frac{A_{p+1} v}{A_{p+1} x} Bx \\ 0 \end{bmatrix} = \begin{bmatrix} Bv \\ 0 \end{bmatrix} \geqq 0 \qquad \text{..................} (11)$$

where the last inequality follows from (6), the equality before from (5), the equality before from (9), the equality before from (10), and the equality before from (3).

Finally, from (3), (10), (9), (5), and (6) we have

$$A_1 w + u_1 = \left(B_1 - \lambda_1 A_{p+1}\right) w + u_1 = B_1 w + u_1$$

$$= B_1 v - \frac{A_{p+1} v}{A_{p+1} x} Bx + u_1 = B_1 v + u_1 > 0.$$

Relations (8), (2), (11) and (12) extend the lemma to $\bar{A}$.

From Tucker's lemma two important existence theorems follow. These theorems assert the existence of solutions of two linear systems that have a certain positivity property.

Theorem 97.2. (First existence theorem)

For any given $p \times n$ matrix A, the systems

$$Ax \geqq 0 \qquad \text{.....................(A)}$$

and

$$A'y = 0, y \geqq 0$$

possess solution x and y satisfying

$$Ax + y > 0.$$

Proof. In Tueker's lemma the row A_1 played a special role. By renumbering the rows of A, any other row, say, A_i could have played the same role. Hence, by Tucker's lemma 97.1, ehte exists $x^i \in R^n, y^i \in R^p$, $i = 1, 2, 3, ..., p$, such that

$$Ax^i \geqq 0, A'y^i = 0, y^i \geqq 0, A_i x^i + y_i^i > 0, \quad i = 1, 2, ..., p. \qquad \text{.....................(1)}$$

Define

$$x = \sum_{i=1}^{p} x^i, y = \sum_{i=1}^{p} y^i.$$

Hence by (1), we have that

$$Ax = \sum_{i=1}^{p} Ax^i \geqq 0, A'y = \sum_{i=1}^{p} A'y^i = 0, y = \sum_{i=1}^{p} y^i \geqq 0.$$

By(1), $A_i x^i + y_i^i > 0$ and $\sum_{\substack{k=1 \\ k \neq i}}^{p} \left(A_i x^k + y_i^k\right) \geqq 0$.

Hence $A_i x + y_i = A_i x^i + y_i^i + \sum_{\substack{k=1 \\ k \neq i}}^{p} \left(A_i x^k + y_i^k\right) > 0$.

or, $Ax + y = 0$.

Hence the proof.

Theorem 97.3. (Second existence theorem)

Let A and B be given $p^1 \times n$ and $p^2 \times n$ matrices, with A non-vacuous. Then the systems

$$Ax \geqq 0, Bx = 0 \qquad \text{.....................(A)}$$

and

$$A' y_1 + B' y_2 = 0, y_1 \geqq 0 \qquad \text{.....................(B)}$$

possess solutions $x \in R^n, y_1 \in R^{p^1}, y_2 \in R^{p^2}$ satisfying

$$Ax + y_1 > 0.$$

Proof. We apply first existence theorem to the systems

$$\begin{bmatrix} A \\ B \\ -B \end{bmatrix} x \geqq 0$$

and

$$[A', B', -B'] \begin{bmatrix} y_1 \\ z_1 \\ z_2 \end{bmatrix} = 0, \begin{bmatrix} y_1 \\ z_1 \\ z_2 \end{bmatrix} \geqq 0$$

and obtain x, y_1, z_1, z_2 satisfying

$$Ax + y_1 > 0.$$

$$Bx + z_1 > 0$$

$$-Bx + z_2 > 0.$$

Define now, $y_2 = z_1 - z_2$. We have then that x, y_1, y_2 satisfy

$$Ax \geqq 0, Bx = 0$$

$$A'y_1 + B'y_2 = 0, y_1 \geqq 0$$

$$Ax + y_1 > 0.$$

Hence the theorem.

This theorem can be extended for more constraints, which is stated below.

Corollary 97.1. Let A, B, C and D be given $p^1 \times n, p^2 \times n, p^3 \times n$ and $p^4 \times n$ matrices, with A, B or C non-vacuous. Then the systems

$$Ax \geqq 0, Bx \geqq 0, Cx \geqq 0, \ Dx = 0$$

and

$$A'y_1 + B'y_2 + C'y_3 + D'y_4 = 0, y_1 \geqq 0, y_2 \geqq 0, y_3 \geqq 0$$

possess solutions $x \in R^n, y_1 \in R^{p^1}, y_2 \in R^{p^2}, y_3 \in R^{p^3}$, satisfying $Ax + y_1 > 0, Bx + y_2 > 0$ and $Cx + y_3 > 0$.

97.7. Theorems of the Alternative

In this section, you will learn a series of theorems relating to the certain occurrence of one of two mutually exclusive events. The two events, denoted by (A) and (B) will be the existence of solutions of two related systems of linear inequalities and/or equalities. The prototype of the theorem of the alternative can be stated as follows: Either (A) or (B), but never both. We denote $(\bar{A})$ the non-occurrence of (A) and similarly for $(\bar{B})$.

Typical theorem of the alternative

$(A) \Leftrightarrow (\bar{B})$ or equavalently $(\bar{A}) \Leftrightarrow (B)$.

Theorem 97.4. (Slater's theorem of the alternative)

Let A, B, C and D be given matrices, with A and B being non-vacuous. Then either

$Ax > 0, Bx \geq 0, Cx \geqq 0, Dx = 0$ has a solution x ....(A)

or,

$$A'y_1 + B'y_2 + C'y_3 + D'y_4 = 0$$

with $y_1 \geq 0, y_2 > 0, y_3 \geqq 0$ or $y_1 \geqq 0, y_2 > 0, y_3 \geqq 0$ .... (B)

has a solution but never both.

Proof. $(A) \Leftrightarrow (\bar{B})$. We assume (A) holds.

We will now show that if (B) also holds, then there will be a constradiction. If both (A) and (B) hold, then we would have x, y_1, y_2, y_3, y_4 such that

$$x\,A'y_1 + x\,B'y_2 + x\,C'y_3 + x\,D'y_4 > 0$$

because $xD'y_4 = 0, x\,C'y_3 \geqq 0$ and either $x\,B'y_2 \geqq 0$ and $x\,A'y_1 > 0$ or $x\,B'y_2 > 0$ and $x\,A'y_1 \geqq 0$. But, now we have a contradiction to the first equality of (B). Hence (A) and (B) cannot hold simultaneously. Thus,

$(A) \Rightarrow (\bar{B})$

$(\bar{A}) \Rightarrow (B)$. We assume that $\bar{A}$ holds.

$(\bar{A}) \Rightarrow Ax \geqq 0, Bx \geqq 0, Cx \geqq 0, Dx = 0 \Rightarrow Ax \not> 0$ or $Bx = 0$

$\Rightarrow Ax \geqq 0, Bx \geqq 0, C \geqq 0, Dx = 0; A'y_1 + B'y_2 + C'y_1 + D'y_4 = 0;$

$y_1 \geqq 0, y_2 \geqq 0, y_3 \geqq 0$

$\Rightarrow y_1 \geq 0$ or $y_2 > 0$ (by Corollary 97.1)

$\Rightarrow (B)$.

Hence the theorem.

Hheorem 97.5. (Motzkin's theorem of the alternative)

Let A, C and D be given matrices, with A being non-vacuous. Then either

$Ax > 0, Cx \geqq 0, Dx = 0$ has a solution x .... (A)

or,

$A'y_1 + C'y_2 + D'y_4 = 0, y_1 \geq 0, y_3 \geqq 0$ has a solution y_1, y_3, y_4 .... (B)

but never both.

Proof. $(A) \Rightarrow (\bar{B})$. If both A and B hold, then we would have x, y_1, y_3, y_4 such that

$$xA'y_1 + xC'y_3 + xD'y_4 > 0$$

because $xD'y_4 = 0, xC'y_3 \geqq 0$ and $xA'y_1 > 0$. But, now we have a contradiction to the first equality of (B). Hence, (A) and (B) cannot hold simultaneously. Thus

$$(A) \Rightarrow (\bar{B}).$$

$$(\bar{A}) \Rightarrow (B)$$

$$(\bar{A}) \Rightarrow Ax \geqq 0, Cx \geqq 0, Dx = 0 \Rightarrow Ax \not> 0$$

$$\Rightarrow Ax \geqq 0, Cx \geqq 0, Dx = 0; A'y_1 + C'y_3 + D'y_4 = 0; y_1 \geqq 0, y_3 \geqq 0$$

$$\Rightarrow y_1 \geq 0 \text{ [by Corollary 97.1]}$$

$$\Rightarrow (B).$$

Theorem 97.6 (Tucker's theorem of the alternative)

Let B, C and D be given matrices, with B being non-vacuous. Then either

$Bx > 0, Cx \geqq 0, Dx = 0$ has a solution x .... (A)

or,

$B'y_2 + C'y_3 + D'y_4 = 0; y_2 > 0, y_3 \geqq 0,$ has a solution y_2, y_3, y_4 .... (B)

but never both.

The proof is similar to the proof of Motzkin's theorem.

Note : Slater considered his theorem as the one providing the most general system (A) possible, because it involved all the ordering relations $>, \geq, \geqq, =$. Similarly, we can derive another theorem which involves the most general system (B) possible, in which $y_1 > 0, y_2 \geq 0, y_3 \geqq 0$, and y_4 is unrestricted.

Theorem 97.7 (Theorem of the atlernative)

Let A, B, C and D be given matrices, with A and B being non-vecuous. Then either

$Ax \geq 0, Bx \geqq 0, Cx \geqq 0, Dx = 0$ or $Ax \geqq 0, Bx > 0, Cx \geqq 0, Dx = 0$ .... (A)

has a solution x

or,

$$A'y_1 + B'y_2 + C'y_3 + D'y_4 = 0; y_1 > 0, y_2 \geq 0, y_3 \geqq 0 \qquad \text{..................} (B)$$

has a solution y_1, y_2, y_3, y_4

but never both.

Proof. $(A) \Rightarrow (\bar{B})$. If both (*A*) and (*B*) hold, then we would have x, y_1, y_2, y_3, y_4 satisfying

$$xA'y_1 + xB'y_2 + xC'y_3 + xD'y_4 > 0$$

because $xD'y_4 = 0, xC'y_3 \geqq 0$, and either $xB'y_2 \geqq 0$ and $xA'y_1 > 0$, or $xB'y_1 > 0$ and $xA'y_1 \geqq 0$. But, now we have a contradiction to the first equality of (*B*). Hence (*A*) and (*B*) cannot hold simultaneously. Thus

$(A) \Rightarrow (\bar{B})$.

We have to show $(\bar{B}) \Rightarrow (A)$

$$(\bar{B}) \Rightarrow A'y_1 + B'y_2 + C'y_3 + D'y_4 = 0; y_1 \geqq 0, y_2 \geqq 0, y_3 \geqq 0$$

$$\Rightarrow y_1 \not> 0 \text{ or } y_2 = 0.$$

$$\Rightarrow A'y_1 + B'y_2 + C'y_3 + D'y_4 = 0; y_1 \geqq 0, y_2 \geqq 0, y_3 \geqq 0;$$

$$Ax \geqq 0, Bx \geqq 0, Cx \geqq 0, Dx = 0$$

$$\Rightarrow Ax \geq 0 \text{ or } Bx > 0. \text{ [by Corollary 1]}$$

$$\Rightarrow (A).$$

Hence the theorem.

Remark : If either *A* or *B* is vacuous, then we revert to Tucker's theorem or Motzkin's theorem.

Again, in all of the above theorems of the alternative the systems (*A*) are all homogeneous. Hence by defining $z = -x$, the system (*A*) of, say, Slater's theorem can be replaced by

$$Az < 0, Bz \leq 0, Cz \leqq 0, Dz = 0 \text{ has a solution } z.$$

Theorem 97.8 (Farkas' theorem)

For each given *pXn* matrix *A* and each given vector *b* in R^n either

$$Ax \leqq 0, \; bx > 0 \text{ has a solution } x \in R^n$$

or, $\quad A'y = b, y \geqq 0$ has a solution $y \in R^p$

but never both.

Proof. By Motzkin's theorem, either (*A*) holds or $(\bar{B})$

$b\eta - A'y_3 = 0, \eta \geq 0, y_3 \geqq 0$ must have a solution $\eta \in R$ and $y_3 \in R^p$, but not both.

Since $\eta \in R, \eta \geq 0$ means $\eta > 0$. Dividing through by η and letting $y = y_3 / \eta_1$, we have that $(\bar{B})$ is equivalent to (*B*).

97.8. Module Summary

In this module, you have learnt the non-linear programming problem and its mathematical formulation. Some fundamental theorems for linear inequalities (constraints are linear) are presented here. These fundamental theorems also play a crucial role in linear programming. The type of theorem that will concern in this module will involve two systems of linear inequalities and/or equalities, say systems (*A*) and (*B*). A typical theorem of the alternative asserts that either system (*A*) has solution or that system (*B*) has a solution, but never both. The most famous theorem of this type is Farkas' theorem, which is established here. The different types of theorems of the alternative are also discussed in this module.

97.8. Self Assessment Questions

1. Define non-linear programming problem.
2. State and prove Farkas' theorem.
3. State and prove optimality criteria of linear programming problem using Farkas' theorem.
4. (Tucker's Lemma) Prove that, for any given *pXn* matrix *A*, the systems $Ax \geqq 0$ and $A'y = 0, y \geqq 0$ possess solutions *x* and *y* satisfying $A_1x + y_1 > 0$.
5. (First existence theorem). Prove that for, any given *pXn* matrix *A*, the systems $Ax \geqq 0$ and $A'y = 0, y \geqq 0$ possess solutions *x* and *y* satisfying $Ax + y > 0$.
6. (Second existence theorem). Let *A* and *B* be given $p^1 \times n$ and $p^2 \times n$ matrices, with *A* non-vacuous. Then show that the systems $Ax \geqq 0, Bx = 0$ and $A'y_1 + B'y_2 = 0, y_1 \geqq 0$ possess solutions $x \in R^n, y_1 \in R^{p^1}, y_2 \in R^{p^2}$ satisfying $Ax + y_1 > 0$.

7. State and prove Slater's theorem of the alternative.

8. State and prove Motzkin's theorem of the alternative.

9. (Motzkin's theorem of the alternative). Let A, C and D be given matrices, with A being non-vacuous. Then prove that, either

   $Ax > 0, Cx \geqq 0, Dx = 0$ has a solution x

   or, $A'y_1 + C'y_3 + D'y_4 = 0, y_1 \geq 0, y_3 \geqq 0$ has a solution y_1, y_3, y_4, but never both.

10. (Tucker's theorem of the alternative). Let B, C and D be given matrices, with B being non-vacuous. Then prove, either

    $Bx \geq 0, Cx \geqq 0, Dx = 0$ has a solution x

    or, $B'y_2 + C'y_3 + D'y_4 = 0, y_2 > 0, y_3 \geqq 0$ has a solution y_2, y_3, y_4; but never both.

97.9. Reference

1. O.L. Mangasarian, Non-linear Programming, Mc Graw Hill Publishers.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper-IX Special Paper – Operations Research

Module No. - 98

Advanced Optimization and Operations Research-I

(Convex Sets and Convex Functions)

Module Structure

98.1 Introduction

In the previous module, you have learnt about the non-linear programming problems, optimality criteria of linear programming problem, existence theorems and several theorems of the alternative.

In this module you will learn, the fundamental concept of convex sets, properties of these sets, basic separation theorems for convex sets. Also, you will learn convex and concave functions defined on subsets of R^n. The convex and concave functions are extremely important in non-linear programming. They are the only functions for which necessary optimality conditions can be given without linearization. You will learn, some basic properties of convex and concave functions and some fundamental theorems involving these functions.

In next module, you will learn optimality criteria of non-linear programming without differentiability using the results discussed here.

98.2. Objectives

Go through this module you will learn the following:

* Convex sets.
* Separation theorems for convex sets.
* Convex and concave functions and their properties.

98.3. Key words

Convex sets, convex function, concave function, strictly convex and concave functions.

98.4. Convex Sets and their properties

First of all we define line and line segments between two points in R^n.

Line

Let x^1, x^2 be two points in R^n. The line through x^1 and x^2 is defined as the set

$$\{x : x = (1-\lambda)x^1 + \lambda x^2, \lambda \in R\} \qquad \text{......................... (1)}$$

or, equivalently

$$\{x : x = p_1 x^1 + p_2 x^2, p_1, p_2 \in R, p_1 + p_2 = 1\}.$$

The first definition can be written as

$$\{x : x = x^1 + \lambda(x^2 - x^1), \lambda \in R\}.$$

Now, we consider the case $x \in R^2$. Then the vector equation $x = x^1 + \lambda(x^2 - x^1)$ is the parametric equation of elementary analytic geometry of the line through x^1 and x^2

Line segments

Let x^1, x^2 be any two points in R^n. Different types of line segments can be defined joining x^1 and x^2

(i) Closed line segment $[x^1, x^2]$:

$$\{x : x = (1-\lambda)x^1 + \lambda x^2, 0 \leqq \lambda \leqq 1\}. \qquad \text{.............. (2a)}$$

(ii) Open line segment (x^1, x^2)

$$\{x : x = (1-\lambda)x^1 + \lambda x^2, 0 < \lambda < 1\}. \qquad \text{.............. (2b)}$$

(iii) Closed-open line segment $[x^1, x^2)$

$$\{x : x = (1-\lambda)x^1 + \lambda x^2, 0 \leqq \lambda < 1\}. \qquad \text{.............. (2c)}$$

(iv) Open-closed line segment $(x^1, x^2]$

$$\{x : x = (1-\lambda)x^1 + \lambda x^2, 0 < \lambda \leqq 1\}. \qquad \text{.............. (2d)}$$

Convex set

A set $\Gamma \subset R^n$ is a convex set if the closed line segment joining every two points of Γ is in Γ. Equivalently, we have that a set $\Gamma \subset R^n$ if

$$x^1, x^2 \in \Gamma, \lambda \in R, 0 \leqq \lambda \leqq 1 \Rightarrow (1-\lambda)x^1 + \lambda x^2 \in \Gamma. \qquad \text{.............. (3)}$$

The set R^n itself a convex set, the empty set is convex, and all sets consisting each of one point are convex.

Figure 98.1 depicts some convex sets in R^2 and Figure 98.2 some non-convex sets in R^2.

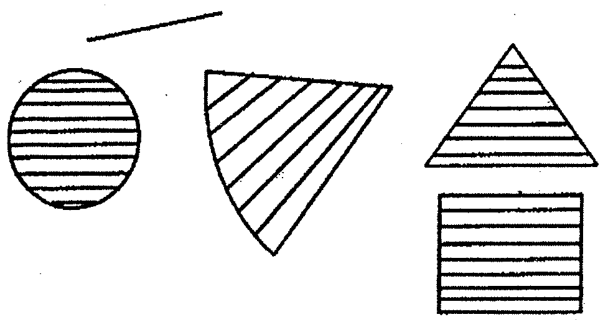


Figure 98.1. : Convex sets.

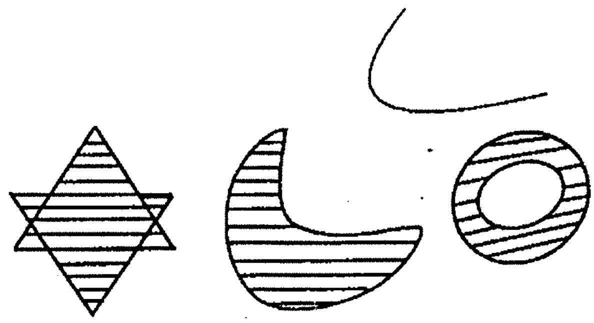


Figure 98.2. : Non -convex sets.

Half space

Let $c \in R^n, c \neq 0$ and $\alpha \in R$. Then the set $\{x : x \in R^n, cd < \alpha\}$ is an open half space in R^n, and the set $\{x : x \in R^n, cx \leqq \alpha\}$ is a closed half space in R^n.

Both half spaces are convex sets.

Plane

Let $C \in R^n, C \neq 0$ and $\alpha \in R$. Then the set $\{x : x \in R^n, Cx = \alpha\}$ is called a plane in R^n.

Each plane in R^n is a convex set.

Subspace

A set $\Gamma \subset R^n$ is a subspace if

$$x^1, x^2 \in \Gamma, p_1, p_2 \in R \Rightarrow p_1x^1 + p_2x^2 \in \Gamma. \qquad \text{.................... (4)}$$

Each subspace of R^n contains the origin and is a convex set.

Vertex

Let Γ be a convex set in R^n. Each $x \in \Gamma$ for which there exist no two distinct points $x^1, x^2 \in \Gamma$ different from x such that $x \in [x^1, x^2]$, is called a vertex of Γ (or an extreme point of Γ).

Theorem 98.1. If $\Gamma_i, i \in I$ is a family (finite or infinite) of convex sets in R^n, then their intersection $\bigcap_{i \in I} \Gamma_i$ is a convex set.

Polytope and Polyhedron

A set in R^n which is the intersection of a finite number of closed half space in R^n is called a polytope. If a polytope is bounded (i.e., for each x in the polytope $\|x\| \leqq \alpha$ for some fixed $\alpha \in R$), it is called a polyhedron.

Polytopes and polyhedra are convex sets.

Convex Combination

A point $b \in R^n$ is said to be a convex combination of the vectors $a^1, a^2, ..., a^m \in R^n$ if there exist m real numbers $p_1, p_2, ..., p_m$ such that $b = p_1a^1 + p_2a^2 + ... + p_ma^m, p_1, p_2, ..., p_m \geqq 0, p_1 + p_2 + ... + p_m = 1$.

Simplex

Let $x^0, x^1, ..., x^m$ be $(m+1)$ distinct points in R^n, with $m \leqq n$. If the vectors $x^1 - x^0, ..., x^m - x^0$ are linearly independent, then the set of all convex combination of $x^0, x^1, ..., x^m$

$$S = \left\{ x : x = \sum_{i=0}^{m} p_i x^i, p_i \in R, p_i \geqq 0, i = 0,1,...,m; \sum_{i=0}^{m} p_i = 1 \right\}$$

is called an m-simplex in R^n with vertices $x^0, x^1, ..., x^m$.

A 0-simplex is a point, a 1-simplex is a closed line segment, a 2-simplex is a triangle and a 3-simplex is a tetrahedron.

Theorem 98.2. A set $\Gamma \subset R^n$ is convex if and only if for each integer $m \geqq 1$, every convex combination of any m points of Γ is in Γ. Equivalently, a necessary and sufficient condition for the set Γ to be convex is that for each integer $m \geqq 1$

$$x^1, x^2, ..., x^m \in \Gamma; p_1, p_2, ..., p_m \geqq 0; p_1 + p_2 + ... + p_m = 1$$
$$\Rightarrow p_1 x^1 + p_2 x^2 + ... + p_m x^m \in \Gamma.$$

Convex hull

Let $\Gamma \subset R^n$. The convex hull of Γ, denoted by $\{\Gamma\}$ is the inter-section of all convex sets in R^n containing Γ.

Obviously, if Γ is convex, then $\Gamma = \{\Gamma\}$.

Theorem 98.3. The convex hull $\{\Gamma\}$ of a set $\Gamma \subset R^n$ is equal to the set of all convex combinations of points of Γ.

Theorem 98.4.

(i) The sum $\Gamma_1 + \Gamma_2$ of two convex sets Γ_1 and Γ_2 in R^n is a convex set,

(ii) The product $\mu\Gamma$ of a convex set Γ in R^n and the real number μ is a convex set.

98.5. Separation Theorems for Convex Sets

Separating Plane

The plane $\{x : x \in R^n, cx = \alpha\}, c \neq 0$, is said to separate (strictly separate) two non-empty sets Γ_1 and Γ_2 in R^n if

$$x \in \Gamma_1 \Rightarrow cx \leqq \alpha \quad (cx < \alpha)$$

$x \in \Gamma_2 \Rightarrow cx \geqq \alpha \quad (cx > \alpha).$

If such a plane exists, the sets Γ_1 and Γ_2 are said to be separable (strictly separable).

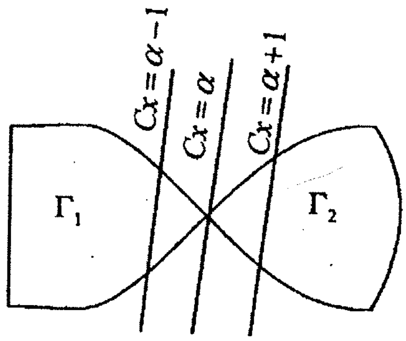


Figure 98.2. Separable but not disjoint sets.

Figure 98.2 gives a simple illustration in R^2 of two sets in R^n which are separable, but which are neither disjoint nor convex. It should be remarked that in general separability does not imply that the sets are disjoint, nor is it true in general that two disjoint sets are separable (Figure 98.3). However, if the sets are non-empty, convex and disjoint then they are separable, and in fact this is a separation theorem we intend to prove.

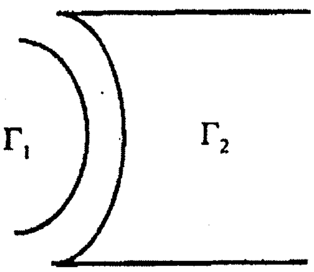


Figure 98.3 : Disjoint but not separable sets.

Theorem 98.5. (Separation theorem). Let Γ_1 and Γ_2 be two non-empty disjoint convex sets in R^n. Then there exists a plane $\{x : x \in R^n, cx = \alpha\}, c \neq 0$, which separates them, that is,

$$x \in \Gamma_1 \Rightarrow cx \leqq \alpha$$
$$x \in \Gamma_2 \Rightarrow cx \geqq \alpha$$

Proof. The set

$$\Gamma_2 - \Gamma_1 = \{x : x = y - z, y \in \Gamma_2, z \in \Gamma_1\}$$

is convex set, and it does not contain the origin 0 because $\Gamma_1 \cap \Gamma_2 = \phi$. There exists a plane $\{x : x \in R^n, cx = 0\}, c \neq 0$, such that

$$x \in \Gamma_2 - \Gamma_1 \Rightarrow cx \geqq 0$$

or

$$y \in \Gamma_2, z \in \Gamma_1 \Rightarrow c(y - z) \geqq 0.$$

Hence $\beta = \underset{y \in \Gamma_2}{\text{infimum}}\, cy \geqq \underset{z \in \Gamma_1}{\text{supremum}}\, cz = \gamma.$

Define $\alpha = \dfrac{\beta + \gamma}{2}$. Then

$$z \in \Gamma_1 \Rightarrow cx \leqq \alpha \text{ and } y \in \Gamma_2 \Rightarrow cy \geqq \alpha.$$

Theorem 98.6. (Strict separation theorem). Let Γ_1 and Γ_2 be two non-empty convex sets in R^n with Γ_1 compact and Γ_2 closed. If Γ_1 and Γ_2 are disjoint, then there exists a plane $\{x : x \in R^n, cx = \alpha\}, c \neq 0$, which strictly separates them, and conversely. In other words

$$\Gamma_1 \cap \Gamma_2 = \phi \Leftrightarrow \exists c \neq 0 \text{ and } x \in \Gamma_1 \Rightarrow cx < \alpha; x \in \Gamma_2 \Rightarrow cx > \alpha.$$

98.6. Convex and Concave Functions

Convex function

A numerical function θ defined on a set $\Gamma \subset R^n$ is said to be convex at $\bar{x} \in \Gamma$ (with respect to Γ) if

$$x \in \Gamma, 0 \leqq \lambda \leqq 1, (1-\lambda)\bar{x} + \lambda x \in \Gamma \qquad \text{.................(5)}$$
$$\Rightarrow (1-\lambda)\theta(\bar{x}) + \lambda\theta(x) \geqq \theta[(1-\lambda)\bar{x} + \lambda x]$$

θ is said to be convex on Γ if it is convex at each $x \in \Gamma$.

This is the more general definition, as

(i) We define convexity at a point first and then convexity, on a set

(ii) We do not require Γ to be a convex set.

It follows immediately from the above definition that a numerical function θ defined on a convex set Γ is convex on Γ if and only if

$$x^1, x^2 \in \Gamma, 0 \leqq \lambda \leqq 1 \Rightarrow (1-\lambda)\theta(x^1) + \lambda\theta(x^2) \geqq \theta[(1-\lambda)x^1 + \lambda x^2].$$

The Figure 98.4 shows two convex functions on convex subsets of $R^n = R$.

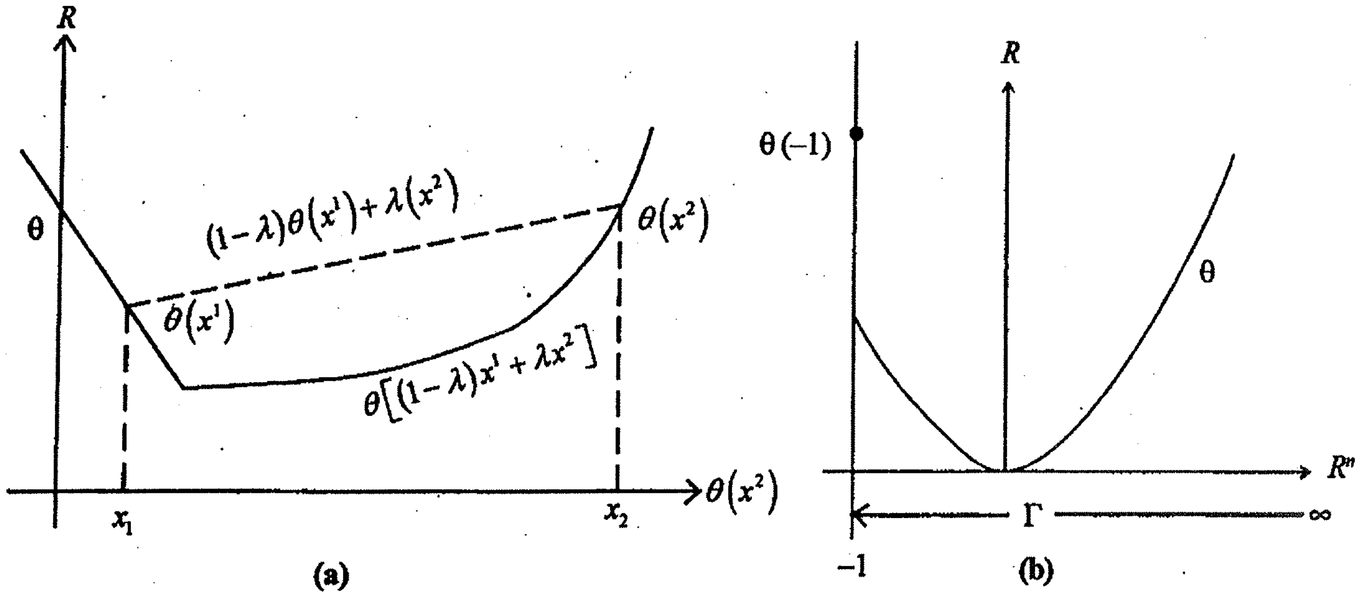


Figure 98.4 : Convex functions on subsets of R^n=R. (a) A convex function θ on R, (*b*) A convex function θ on $\Gamma = \{-1, \infty\}$.

Concave function

A numerical function θ defined on a set $\Gamma \subset R^n$ is said to be concave at $\bar{x} \in \Gamma$ (which respect to Γ) if

$$x \in \Gamma, 0 \leqq \lambda \leqq 1, (1-\lambda)\bar{x} + \lambda x \in \Gamma$$

$$\Rightarrow (1-\lambda)\theta(\bar{x}) + \lambda\theta(x) \leqq \theta[(1-\lambda)\bar{x} + \lambda x] \qquad \text{.................... (6)}$$

θ is said to be concave on Γ if it is concave at each $x \in \Gamma$.

Obviously, θ is concave at $\bar{x} \in \Gamma$ (concave on Γ) if and only if $-\theta$ is convex at $\bar{x}$ (convex on Γ). Results obtained for convex functions can be changed into results for concave functions by the appropriate multiplication by -1, and vice versa.

It follows immediately from the above definition that a numerical function θ defined on a convex set Γ is concave on Γ if and only if

$$x^1, x^2 \in \Gamma, 0 \leqq \lambda \leqq 1 \Rightarrow (1-\lambda)\theta(x^1) + \lambda\theta(x^2) \leqq \theta\left[(1-\lambda)x^1 + \lambda x^2\right].$$

Figure 98.5 shows two concave functions on convex subsets of $R^n = R$.

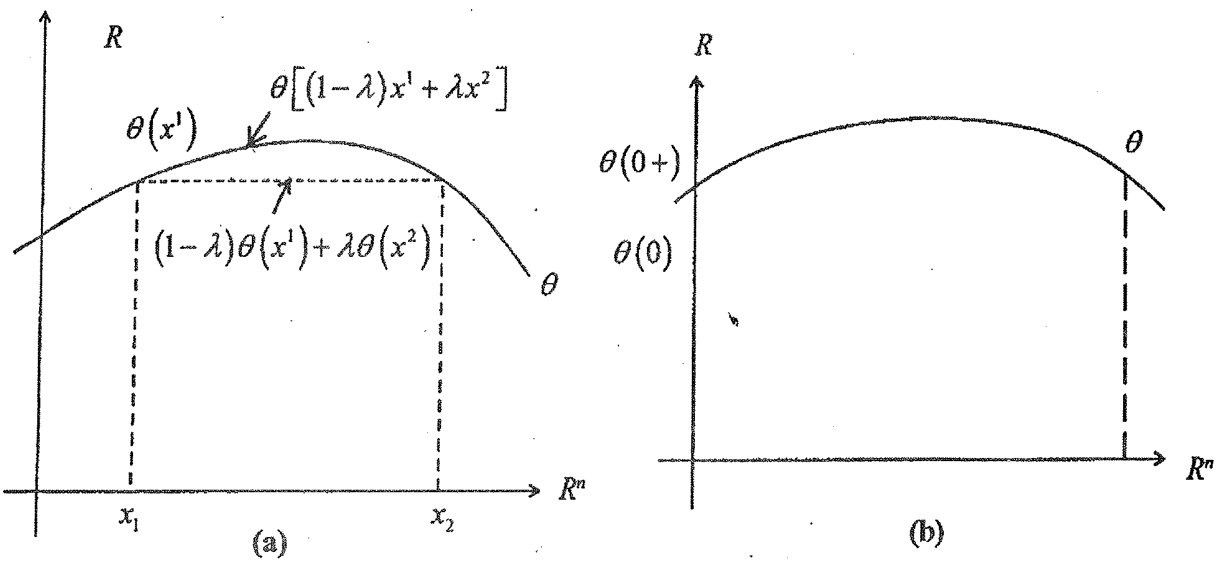


Figure 98.5 : Concave functions on subsets of $R^n = R$. (a) A Concave function θ on R; (b) A concave function θ on $\Gamma = \{0,1\}$.

Example 98.1 : Show that a linear function $\theta(x) = cx - \alpha, x \in R^n$, is both convex and concave, and conversely.

Solution. Let and $x, \bar{x} \in \Gamma$, and $0 \leqq \lambda \leqq 1$.

Then $(1-\lambda)\theta(\bar{x}) + \lambda\theta(x) = (1-\lambda)(c\bar{x} - \alpha) + \lambda(cx - \alpha)$

$= c\left[(1-\lambda)\bar{x} + \lambda x\right] - \alpha$

and $\theta\left[(1-\lambda)\bar{x} + \lambda x\right] = c\left[(1-\lambda)\bar{x} + \lambda x\right] - \alpha.$

Thus, $(1-\lambda)\theta(\bar{x}) + \lambda\theta(x) = \theta\left[(1-\lambda)\bar{x} + \lambda x\right].$

Hence $\theta(x)$ is both convex and concave, and conversely.

Strictly convex function

A numerical function θ defined on a set $\Gamma \subset R^n$ is said to be strictly convex at $\bar{x} \in \Gamma$ (with respect to Γ) if $x \in \Gamma, x \neq \bar{x}, 0 < \lambda < 1, (1-\lambda)\bar{x} + \lambda x \in \Gamma$

$$\Rightarrow (1-\lambda)\theta(\bar{x}) + \lambda\theta(x) > \theta[(1-\lambda)\bar{x} + \lambda x]$$

θ is said to be strictly convex on Γ if it is strictly convex at each $x \in \Gamma$.

Strictly concave function :

A numerical function θ defined on a set $\Gamma \subset R^n$ is said to be strictly concave at $\bar{x} \in \Gamma$ (with respect to Γ) if $x \in \Gamma, x \neq \bar{x}, 0 < \lambda < 1, (1-\lambda)\bar{x} + \lambda x \in \Gamma$

$$\Rightarrow (1-\lambda)\theta(\bar{x}) + \lambda\theta(x) < \theta[(1-\lambda)\bar{x} + \lambda x]$$

θ is said to be strictly concave on Γ if it is strictly concave at each $x \in \Gamma$.

Obviously, a strictly convex (strictly concave) function on a set $\Gamma \subset R^n$ is convex (concave) on Γ, but not conversely. For example a constant function on R^n is both convex and concave on R^n, but neither strictly convex nor strictly concave on R^n. In fact, it can easily be shown that all linear functions $\theta(x) = cx - \alpha$ on R^n are neither strictly convex nor strictly concave on R^n. Hence, because of the linear portion, the function depicted in Figure 98.4(a) is not strictly convex on R, but the function of Figure 98.4(b) is strictly convex on $[-1, \infty)$. Both functions of Figure 98.5 are strictly concave on their domains of definition.

An m-dimensional vector function f defined on a set Γ in R^n is convex at $\bar{x} \in \Gamma$, convex on Γ, etc., if each of its components f_i, $i=1, 2, ..., m$, is convex at $\bar{x} \in \Gamma$, convex on Γ, etc.

Theorem 98.7. Let $f = (f_1, f_2, ..., f_m)$ be an m-dimensional vector function defined on $\Gamma \subset R^n$. If f is convex at $\bar{x} \in \Gamma$ (convex on Γ), then each non-negative linear combination of its components f_i

$$\theta(x) = pf(x), p \geqq 0$$

is convex at $\bar{x}$ (convex on Γ).

Proof. Let $x \in \Gamma, 0 \leq \lambda \leq 1$, and $(1-\lambda)\bar{x} + \lambda x \in \Gamma$. Then

$$\theta[(1-\lambda)\bar{x} + \lambda x] = pf[(1-\lambda)x^2 + \lambda x]$$
$$\leq p[(1-\lambda)f(\bar{x}) + \lambda f(x)]$$

[by convexity of f at $\bar{x}$ and $p \geqq 0$]

$$= (1-\lambda) pf(\bar{x}) + \lambda pf(x)$$

$$= (1-\lambda)\theta(\bar{x}) + \lambda\theta(x).$$

Example 98.2. Let θ be a numerical function defined on a convex set $\Gamma \subset R^n$. Then θ is respective convex, concave, strictly convex, or strictly concave on Γ if and only if for each $x^1, x^2 \in \Gamma$, the numerical function ϕ defined on [0, 1] by

$$\phi(x) = \theta\left[(1-\lambda)x^1 + \lambda x^2\right]$$

is respectively convex, concave, strictly convex, or strictly concave on [0, 1].

Theorem 98.8. For a numerical function θ defined on a convex set $\Gamma \subset R^n$ to be convex on Γ it is necessary and sufficient that its epigraph

$$G_\theta = \{(x, y) : x \in \Gamma, y \in R, \theta(x) \leqq y\} \subset R^{n+1}$$

be a convex set in R^{n+1}

Proof. (Sufficient) Assume that G_θ is convex. Let $x^1, x^2 \in \Gamma$ then $\left[x^1, \theta(x^1) \in G_\theta\right]$ and $\left[x^2, \theta(x^2)\right] \in G_\theta$. By the convexity of G_θ we have that

$$\left[(1-\lambda)x^1 + \lambda x^2, (1-\lambda)\theta(x^1) + \lambda\theta(x^1) \in G_\theta \ for \ 0 \leqq \lambda \leqq 1\right]$$

or, $$\theta\left[(1-\lambda)x^1 + \lambda x^2\right] \leqq (1-\lambda)\theta(x^1) + \lambda\theta(x^2) \text{ for } 0 \leqq \lambda \leqq 1$$

and hence θ is convex on Γ.

(Necessity) Assume that θ is convex on Γ. Let $x^1, y^1 \in G_\theta$ and $x^2, y^2 \in G_\theta$. By the convexity of θ on Γ we have that for $0 \leqq \lambda \leqq 1$

$$\theta\left[(1-\lambda)x^1 + \lambda x^2\right] \leqq (1-\lambda)\theta(x^1) + \lambda\theta(x^2)$$

$$\leqq (1-\lambda)y^1 + \lambda y^2.$$

Hence $\left[(1-\lambda)x^1 + \lambda x^2, (1-\lambda)y^1 + \lambda y^2\right] \in G_\theta$ and G_θ is a convex set on R^{n+1}.

Theorem 98.9. Let θ be a numerical function defined on a convex set $\Gamma \subset R^n$. A necessary but not sufficient condition for θ to be convex on Γ is that the set

$$\Lambda_\alpha = \{x : x \in \Gamma, \theta(x) \leqq \alpha\} \subset \Gamma \subset R^n$$

be convex for each real number α.

Proof. Let θ be convex on Γ and let $x^1, x^2 \in \Lambda_\alpha$.

Then

$$\theta[(1-\lambda)x^1 + \lambda x^2] \leqq (1-\lambda)\theta(x^1) + \lambda\theta(x^2) \text{ (by convexity of } \theta)$$
$$\leqq (1-\lambda)\alpha + \lambda\alpha$$
$$= \alpha. \text{ (because } x^1, x^2 \in \Lambda_\alpha)$$

Hence $(1-\lambda)x^1 + \lambda x^2 \in \Lambda_\alpha$ and Λ_α is convex.

We now show that if Λ_α is convex for each α, it does not follow that θ is a convex function Γ. Consider the function θ on R defined by $\theta(x) = (x)^3$. θ is not convex on R. However, the set

$\Lambda_\alpha = \{x : x \in R, (x)^3 \leqq \alpha\} = \{x : x \in R, x \leqq (x)^{1/3}\}$ is obviously convex for any α.

Conrollary 98.9.1. Let θ be a numerical function defined on the convex set $\Gamma \subset R^n$. A necessary but not sufficient condition for θ to be concave on Γ is that the set $\Lambda_\alpha = \{x : x \in \Gamma, \theta(x) \geqq \alpha\} \subset \Gamma \subset R^n$

be convex for each real number α.

Theorem 98.10. If $(\theta_i)_{i \in I}$ is a family (finite or infinite) of numerical functions which are convex and bounded from above on a convex set $\Gamma \subset R^n$, then the numerical function

$$\theta(x) = \sup_{i \in I} \theta_i(x)$$

is a convex function on Γ.

Proof. Since each θ_i is a convex function on Γ, their epigraphs $G_{\theta_i} = \{(x,y) : x \in \Gamma, y \in R, \theta_i(x) \leqq y\}$ are convex set in R^{n+1}. Hence their intersection

$$\bigcap_{i \in I} G_{\theta_i} = \{(x,y) : x \in \Gamma, y \in R, \theta_i(x) \leqq y, \forall i \in I\}$$
$$= \{(x,y) : x \in \Gamma, y \in R, \theta(x) \leqq y\}$$

is also a convex set in R^{n+1}. But this convex intersection is the epigraph of θ. Hence θ is a convex functions on Γ.

Coronary 98.10.1. If $(\theta_i)_{i\in I}$ is a family (finite or infinite) of numerical functions which are concave and bounded from below on a convex set $\Gamma \subset R^n$, then the numerical function

$$\theta(x) = \inf_{i\in I} \theta_i(x)$$

is a concave function on Γ.

Note : A function θ which is convex on a convex set $\Gamma \subset R^n$ is not necessarily a continuous function. For example on the half line $\Gamma = \{x : x \in R, x \geqq -1\}$, the numerical function

$$\theta(x) = \begin{cases} 2, \text{ for } x = -1 \\ (x)^2 \text{ for } x > -1 \end{cases}$$

is a convex function on Γ, but is obviously not continuous at $x = -1$. However, if Γ is an open convex set, then a convex function θ on Γ is indeed continuous.

Theorem 98.11. Let Γ be an open convex set in R^n. If θ is a convex numerical function on Γ then θ is continuous on Γ.

Proof. Let $x^0 \in \Gamma$, and let α be the distance from x^0 to the closest point in R^n not in Γ $(\alpha = +\infty \text{ if } \Gamma = R^n)$. Let C be an n-cube with center x^0 and side lenght 2δ, that is

$$C = \{x : x \in R^n, -\delta \leqq x_i - x_i^o \leqq \delta, i = 1, 2, ..., n\}.$$

By letting $(n)^{1/2}\delta < \alpha$, we have that $C \subset \Gamma$. Let V denote the set of 2^n vertices of C. Let

$$\beta = \max_{x\in V} \theta(x)$$

The set $\Lambda_\beta = \{x : x \in \Gamma, \theta(x) \leqq \beta\}$ is convex. Sinece C is the convex hull of V and $\Gamma \subset \Lambda_\beta$, it follows that $C \subset \Lambda_\beta$.

Let x be any point such that $0 < \|x - x^0\| < \delta$, and define $x^0 + u, x^0 - u$ on the line through x^0 and x. (see Figure 98.6).

Write x now as a convex combination of x^0 and $x^0 + u$ and x^0 as a convex combination of x and $x^0 - u$. If $\lambda = \|x - x^0\|/\delta$, then

$$x = x^0 + \lambda u = \lambda(x^0 + u) + (1 - \lambda)x^0$$

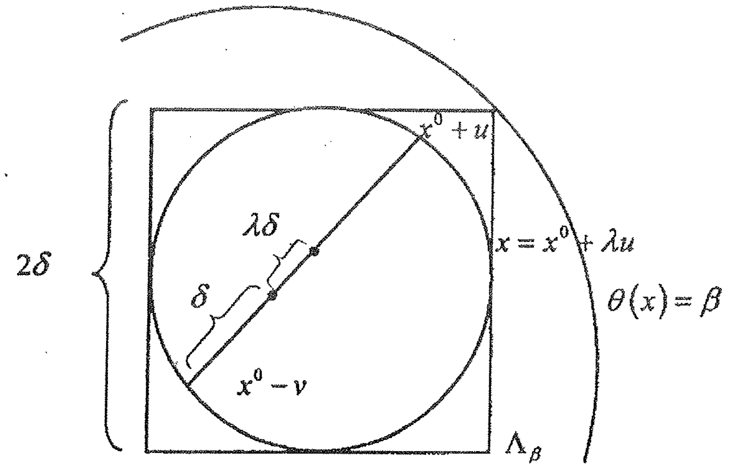


Figure 98.6

or, $\quad x^0 = x - \lambda u = x + \lambda\left(x^0 - u\right) - \lambda x^0$

$$= \frac{1}{1+\lambda}x + \frac{\lambda}{1+\lambda}\left(x^0 - u\right).$$

Since θ is convex on Γ,

$$\theta(x) \leqq \lambda\theta\left(x^0+u\right) + (1-\lambda)\theta\left(x^0\right) \leqq \lambda\beta + (1-\lambda)\theta\left(x^0\right)$$

or, $\quad \theta\left(x^0\right) \leqq \frac{1}{1+\lambda}\theta(x) + \frac{\lambda}{1+\lambda}\theta\left(x^0-u\right) \leqq \frac{\theta(x)+\lambda\beta}{1+\lambda}.$

These inequalities give

$$-\lambda\left[\beta-\theta\left(x^0\right)\right] \leqq \theta(x)-\theta\left(x^0\right) \leqq \lambda\left[\beta-\theta\left(x^0\right)\right]$$

or

$$\left|\theta(x)-\theta\left(x^0\right)\right| \leqq \frac{\beta-\theta\left(x^0\right)}{\delta}\left\|x-x^0\right\|.$$

Thus for any given $\varepsilon > 0$ it follows that $\left|\theta(x)-\theta\left(x^0\right)\right| < \varepsilon$ for all x satisfying $\left[\beta-\theta\left(x^0\right)\right]\left\|x-x^0\right\| < \delta$ and hence $\theta(x)$ is continuous at x^0.

98.7. Some Fundamental Theorems for Convex Functions

In module 97, we have seen that Farkas' theorem of the alternative played a crucial role in deriving the necessary optimality conditions of linear programming. In this section we shall derive what may be considered extensions of theorems of the alternative to convex and concave functions. These theorems will play a similar crucial role in deriving the necessary optimality conditions of non-linear programming in next module.

Theorem 98.12. Let Γ be a non-empty convex set in R^n, let f be an m-dimensional convex vector function on Γ, and let h be a k-dimensional linear vector function on R^n. If

$$f(x)<0, h(x)=0 \text{ has no solution } x\in\Gamma$$

then there exist $p\in R^m$ and $q\in R^k$ such that

$$p\geqq 0, (p,q)\neq 0, pf(x)+qh(x)\geqq 0 \text{ for all } x\in\Gamma.$$

Proof. We define two sets

$$\Lambda(x)=\{(y,z): y\in R^m, z\in R^k, y>f(x), z=h(x)\}, x\in\Gamma$$

and

$$\Lambda=\bigcup_{x\in\Gamma}\Lambda(x).$$

By hypothesis, Λ does not contain the origin $0\in R^{n+1}$. Also, Λ is convex, if (y^1,z^1) and (y^2,z^2) are in Λ, then for $0\leqq\lambda\leqq 1$

$$(1-\lambda)y^1+\lambda y^2>(1-\lambda)f(x^1)+\lambda f(x^2)\geqq f[(1-\lambda)x^1+\lambda x^2].$$

and

$$(1-\lambda)z^1+\lambda z^2>(1-\lambda)h(x^1)+\lambda h(x^2)\geqq h[(1-\lambda)x^1+\lambda x^2].$$

Because Λ is a non-empty convex set not containing the origin, it follows that there exist $p\in R^m, q\in R^k, (p,q)\neq 0$ such that

$$(u,v)\in\Lambda\Rightarrow pu+qv\geqq 0.$$

Since each u_i can be made as large as desired, $p\geqq 0$.

Let $\varepsilon>0, u=f(x)+e\varepsilon, v=h(x), x\in\Gamma$, where e is a vector of ones in R^m. Hence $(u,v)\in\Lambda(x)\subset\Lambda$ and $pu+qv=pf(x)+ep\varepsilon+qh(x)\geqq 0$ for $x\in\Gamma$

or,

$$pf(x)+qh(x)\geqq -\varepsilon pe \text{ for } x\in\Gamma.$$

Now, if

$$\inf_{x\in\Gamma}[pf(x)+qh(x)]=-\delta<0$$

we get, by choosing ε such that $\varepsilon\, pe < \delta$, that

$$\inf_{x\in\Gamma}\left[pf(x)+qh(x)\right] = -\delta < -\varepsilon\, pe$$

which is a contradiction to the fact that

$$pf(x)+qh(x) \geqq -\varepsilon\, pe \text{ for all } x\in\Gamma.$$

Hence $\inf_{x\in\Gamma}\left[pf(x)+qh(x)\right] \geqq 0.$

Theorem 98.13. (Generalized Gordan Theorem). Let f be an m-dimensional convex vector function on the convex set $\Gamma \subset R^n$. Then either

$f(x) < 0$ has a solution $x \in \Gamma$ .... (A)

or

$pf(x) \geq 0$ for all $x \in \Gamma$ for some $p \geq 0, p \in R^m$ .... (B)

but never both.

Proof. Let $(A) \Rightarrow (\bar{B})$. Let $\bar{x} \in \Gamma$ be a solution of $f(x) < 0$.

Then for any $p \geq 0$ in R^m, $pf(\bar{x}) < 0$ and hence (B) cannot hold.

$(\bar{A}) \Rightarrow (B)$. This follows directly from Theorem 98.12 by deleting $h(x) = 0$ from the theorem.

98.8. Modeul Summary

In this module you have learnt the fundamental concept of convex sets, some properties of these sets, the basic separation theorems for convex sets. These separation theorems are the foundations of many optimality conditions of non-linear programming. After that we discussed convex, concave, strictly convex and strictly concave functions. The properties of them are also studied here.

28.9. Self Assessment Questions

1. Define convex set with example.
2. Show that the following open and closed balls

$$B_0(\bar{x}) = \{x : x \in R^n, \|x - \bar{x}\| < \varepsilon\}$$

$$B_C(\bar{x}) = \{x : x \in R^n, \|x - \bar{x}\| \leqq \varepsilon\}$$

around a point $\bar{x} \in R^n$ are convex set.

3. Show that the interior of a convex set is convex.

4. If $(\Gamma_i)_{i \in I}$ is a family (finite or infinite) of convex sets in R^n, then show that their intersection $\bigcap_{i \in I} \Gamma_i$ is a convex set.

5. A set $\Gamma \subset R^n$ is convex then prove that for each integer $m \geqq 1$, every convex combination of any m points of Γ is in Γ.

6. Show that the convex hull $\{\Gamma\}$ of a set $\Gamma \subset R^n$ is equal to the set of all convex combinations of points of Γ.

7. Prove that the sum $\Gamma_1 + \Gamma_2$ of two convex sets Γ_1 and Γ_2 in R^n is a convex set.

8. Let Γ be a convex set in R^n and μ be a real number. Show that $\mu\Gamma$ is a convex set in R^n.

9. Let Γ_1 and Γ_2 be two non-empty disjoint convex sets in R^n. Then show that there exists a plane $\{x : x \in R^n, cx = \alpha\}, c \neq 0$ which separates them, that is,

   $x \in \Gamma_1 \Rightarrow cx \leqq \alpha$ and $x \in \Gamma_2 \Rightarrow cx \leqq \alpha$.

10. Define convex and concave functions with examples.

11. Show that the function $f(x) = ax + b, x \in R^n$ is both convex and concave functions R^n, and conversely.

12. Let $f = (f_1, f_2, \ldots, f_m)$ be an m-dimenstinal vector function defined on $\Gamma \subset R^m$. If f is convex at $\bar{x} \in \Gamma$ (convex on Γ), then each non-negative linear combination of its components f_i

   $$\theta(x) = pf(x), p \geqq 0$$

   is convex at $\bar{x}$ (convex on Γ)

13. For a numerical function θ defined on a convex set $\Gamma \subset R^n$ to be convex on Γ then prove that its epigraph.

   $$G = \{(x, y) : x \in \Gamma, y \in R, \theta(x) \leqq y\} \subset R^{n+1}$$

   be a convex set in R^{n+1}, and conversely.

14. Let θ be a numerical function defined on the convex $\Gamma \subset R^n$. Show that a necessary and sufficient condition for θ to be convex on Γ is that for each integer $m \geqq 1$,

   $x^1, x^2, \ldots, x^m \in \Gamma; p_1, p_2, \ldots, p_m \geqq 0, p_1 + p_2 + \ldots + p_m = 1$

$$\Rightarrow \theta\left(p_1x^1 + p_2x^2 + \ldots + p_mx^m\right) \leqq p_1\theta\left(x^1\right) + p_2\theta\left(x^2\right) + \ldots + p_m\theta\left(x^m\right).$$

15. If $(\theta_i)_{i\in I}$ is a family (finite or infinite) of numerical functions which are convex and bounded from above on a convex set $\Gamma \subset R^n$, then prove that the numerical function

$$\theta(x) = \sup_{i\in I} \theta_i(x)$$

is a convex function on Γ.

98.10. Reference

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper-IX Special Paper – Operations Research

Module No. - 99

Advanced Optimization and Operations Research-I

(Non-linear Programming without Differentiability)

Module Structure :

99.1 Introduction

In previous modules (no. 97 and 98) you have learnt basic tools to development the theories of non-linear programming. The purpose of this module, is to derive the optimility criteria of the saddle point type for non-linear programming problems. In this module, we shall show that the saddle point condition is a sufficient optimality condition without any convexity requirements. We shall develop the optimality criteria without any differentiability assumption on the functions involved.

In next module we shall derive the optimality criteria based on differentiability.

99.2 Objectives

Go through this module you will learn the following

* Minimization problem (MP)
* Local minimization problem (LMP)
* Fritz-John saddlepoint (FJSP)
* Kuhn-Tucker saddlepoint problem (KTSP)
* Optimality Criteria
* Constraint qualification
* Kuhn-Tucker theorem

99.3 Keywords

Saddlepoint, minimizatioin problem, Fritz-John saddlepoint, Kuhn-Tucker problem, Optimality criteria.

99.4 The Minimization and Saddlepoint Problems

The main aim of this module is to derive optimality criteria of the saddlepoint type for non-linear programming problems. Now, we consider an example to explain this type of optimality criteria.

We consider the problem of minimizing the function θ on the set $X = \{x : x \in R, -x+2 \leq 0\}$ where $\theta(x) = (x)^2$. Obviously, the solution is $\bar{x} = 2$ and the minimum is $\theta(\bar{x}) = 4$. The saddle point optimality criterion for this problem is stated below:

A necessary and sufficient condition that $\bar{x}$ be a solution of the minimization problem is that there exists a real number $\bar{u}$ (for this problem $\bar{u} = 4$) such that for all $x \in R$ and all $u \in R, u \geqq 0$

$$\theta(\bar{x}) + u(-\bar{x}+2) \leqq \theta(\bar{x}) + \bar{u}(-\bar{x}+2) \leqq \theta(x) + \bar{u}(-x+2).$$

It is easy to verify that the above inequalities are satisfied for $\bar{x} = 2, \bar{u} = 4$. Hence the function ψ defined on R^2 by

$$\psi(x,u) = \theta(x) + u(-x+2)$$

has a saddle point at $\bar{x}=2, \bar{u}=4$, because it has a minimum at $(\bar{x},\bar{u})$ with respect to x for all real x, and a maximum with respect to u for all real non-negative u.

For the above simple problem, the saddle point criterion happens to be both a necessary and a sufficient optimality criterior for $\bar{x}$ to be a solution of the minimization problem. This is not always the case. We shall show here that the above saddle point conditioin is a sufficient optimality condition without any convexity requirements.

Let X^0 be a subset of R^n. Let θ and g be respectively a numerical function and an m-dimensional vector defined on X^0.

Problem 99.1 The minimization problem (MP)

Find an $\bar{x}$, if it exists, such that

$$\theta(\bar{x}) = \min_{x\in X} \theta(x), \bar{x} \in X = \{x : x \in X^0, g(x) \leqq 0\}. \qquad \text{.............. (1)}$$

The set X is called the feasible region or the constraint set, $\bar{x}$ the minimum solution or solution, and $\theta(\bar{x})$ the minimum. All points x is the feasible region X are called feasible points.

If X is a convex set, and if θ is convex on X, the minimization problem MP is often called a convex programming problem or convex program.

Problem 99.2 The local minimization problem (LMP)

Find an $\bar{x}$ in X, if it exists, such that for some open ball $B_0(\bar{x})$ around $\bar{x}$ with radius $\delta > 0$

$$x \in B_0(\bar{x}) \cap X \Rightarrow \theta(x) \geqq \theta(\bar{x}).$$

Problem 99.3. The Fritz John saddle point problem (FJSP)

Find $\bar{x} \in X^0, \bar{r}_0 \in R, \bar{r} \in R^m, (\bar{r}_0, \bar{r}) \geq 0$ if they exist, such that

$$\left.\begin{array}{l} \phi(\bar{x},\bar{r}_0,r) \leqq \phi(\bar{x},\bar{r}_0,\bar{r}) \leqq \phi(x,\bar{r}_0,\bar{r}) \\ \text{for all } r \geqq 0, r \in R^m, \text{ and all } x \in X^0 \\ \theta(x,r_0,r) = r_0\theta(x) + rg(x). \end{array}\right\} \qquad \text{................ (3)}$$

Problem 99.4. The Kuhn-Tucker saddle point problem (KTSP)

Find $\bar{x} \in X^0, \bar{u} \in R^m, \bar{u} \geqq 0$, if they exist, such that

$$\left.\begin{aligned} &\psi(\bar{x},u) \leqq \psi(\bar{x},\bar{u}) \leqq (x,\bar{u}) \\ &\text{for all } u \geqq 0, u \in R^m, \text{ and all } x \in X^0 \\ &\psi(x,u) = \theta(x) + ug(x). \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

Note. If $(\bar{x},\bar{r}_0,\bar{r})$ is a solution of FJSP and $\bar{r}_0 > 0$ then $(\bar{x},\bar{r}/\bar{r}_0)$ is a solution of KTSP. Conversely, if $(\bar{x},\bar{u})$ is a solution of KTSP, then $(\bar{x},1,\bar{u})$ is a solution of FJSP.

The numerical function $\phi(x,r_0,r)$ and $\psi(x,u)$ defined above are often called Lagrangian functions or simply Lagrangians, and the *m*-dimensional vectors $\bar{r}$ and $\bar{u}$ Lagrange multipliers or dual variables.

99.5 Some Basic Results for Minimization and Local Minimization Problems

Theorem 99.1 Let *X* be a convex set, and let θ be a convex function on *X*. The set of solutions of MP is convex.

Proof. Let x^1 and x^2 be solutions of MP. That is,

$$\theta(x^1) = \theta(x^2) = \min_{x \in X} \theta(x)$$

It follows by the convexity of *X* and θ, that for

$$0 \leqq \lambda \leqq 1, (1-\lambda)x^1 + \lambda x^2 \in X,$$

$$\theta\left[(1-\lambda)x^1 + \lambda x^2\right] \leqq (1-\lambda)\theta(x^1) + \lambda\theta(x^2) = \theta(x^1) = \min_{x \in X} \theta(x).$$

Hence $(1-\lambda)x^1 + \lambda x^2$ is also a solution of MP, and the set of solutions is convex.

Theorem 99.2. (Uniqueness theorem). Let *X* be convex and $\bar{x}$ be a solution of MP. If θ is strictly convex at $\bar{x}$, then $\bar{x}$ is the unique solution of MP.

Proof. Let $\hat{x} \neq \bar{x}$ be another solution of MP, that is $\hat{x} \in X$, and $\theta(\hat{x}) = \theta(\bar{x})$. Since *X* is convex, then $(1-\lambda)\bar{x} + \lambda\hat{x} \in X$ whenever $0 < \lambda < 1$, and by the strict convexity of θ at $\bar{x}$.

$$\theta\left[(1-\lambda)\bar{x} + \lambda\hat{x}\right] < (1-\lambda)\theta(\bar{x}) + \lambda\theta(\hat{x}) = \theta(\bar{x}).$$

This contradicts the assumption that $\theta(\bar{x})$ is a minimum and hence $\hat{x}$ can not be another solution.

Theorem 99.3. Let X be convex, and let θ be a non-constant concave function on X. Then no interior point of X is a solution of MP or equivalently any solution $\bar{x}$ of MP, if it exists, must be a boundary point of X.

Proof. If MP has no solution the theorem is trivially true. Let $\bar{x}$ be a solution of MP. Since θ is not constant on X, there exists a point $x \in X$ such that $\theta(x) > \theta(\bar{x})$. If z is an interior point of X, there exists a point $y \in X$ such that for some $\lambda, 0 \leq \lambda < 1$,

$$z = (1-\lambda)x + \lambda y.$$

Hence

$$\begin{aligned}\theta(z) = \theta[(1-\lambda)x + \lambda y] &\geqq (1-\lambda)\theta(x) + \lambda\theta(y) \\ &> (1-\lambda)\theta(\bar{x}) + \lambda\theta(\bar{x}) \\ &= \theta(\bar{x})\end{aligned}$$

and $\theta(x)$ does not attain its minimum at an interior point z.

Theorem 99.4. If $\bar{x}$ is a solution of MP, then it is also a solution of LMP. The converse is true if X is convex and θ is convex at $\bar{x}$.

Proof. If $\bar{x}$ solves MP, then $\bar{x}$ solves LMP for any $\delta > 0$. To prove the converse now, assume that $\bar{x}$ solves LMP for some $\delta > 0$, and let X be convex and θ be convex and θ be convex at $\bar{x}$. Let y be any point in X distinct from $\bar{x}$. Since X is convex, $(1-\lambda)\bar{x} + \lambda y \in X$ for $0 < \lambda \leqq 1$. By choosing λ small enough, that is, $0 < \lambda < \delta/\|y - \bar{x}\|$ and $\lambda \leqq 1$, we have that

$$\bar{x} + \lambda(y - \bar{x}) = (1-\lambda)\bar{x} + \lambda y \in B_0(\bar{x}) \cap X.$$

Hence $\theta(\bar{x}) \leqq \theta[\bar{x} + \lambda(y - \bar{x})]$ (since $\bar{x}$ solves LMP)

$\leqq (1-\lambda)\theta(\bar{x}) + \lambda\theta(y)$ (by convexity of θ at $\bar{x}$)

from which it follows that $\theta(\bar{x}) \leqq \theta(y)$.

99.6. Sufficient Optimality Criteria

In this section we shall discuss the main sufficient optimality criteria without convexity assumptions on the minimization problem MP.

Theorem 99.5 If $(\bar{x},\bar{u})$ is a solution of KTSP, then $\bar{x}$ is a solution of MP. If $(\bar{x},\bar{r}_0,\bar{r})$ is a solution of FJSP, and $\bar{r}_0 > 0$, then $\bar{x}$ is a solution of MP.

Proof. The second statement of the theorem follows trivially.

Let $(\bar{x},\bar{u})$ be a solution of KTSP. Then for all $u \geqq 0$ in R^n and all x in R^n.

$$\theta(\bar{x})+ug(\bar{x})\leqq\theta(\bar{x})+\bar{u}g(\bar{x})\leqq\theta(x)+\bar{u}g(x).$$

From the first inequality, we have that

$$(x-\bar{u})g(\bar{x})\leqq 0 \text{ for all } u\geqq 0.$$

For any $j, 1\leqq j\leqq m$, let

$$u_i=\bar{u}_1 \text{ for } i=1,2,\ldots,j-1,j+1,\ldots,m; u_j=\bar{u}_j+1.$$

It follows then that $g_j(\bar{x})\leqq 0$. Repeating this for all j, we get that $g(\bar{x})\leqq 0$, and hence $\bar{x}$ is a feasible points, that is, $\bar{x}\in X$.

Now, since $\bar{u}\geqq 0$ and $g(\bar{x})\leqq 0$, we have $\bar{u}\,g(\bar{x})\leqq 0$.

But again from the first inequality of the saddle point problem we have, by setting u=0, that $\bar{u}\,g(\bar{x})\geqq 0$.

Hence $\bar{u}\,g(\bar{x})=0$.

Let x be any point in X, then from the second inequality of the saddle point problem, we get

$$\theta(\bar{x})\leqq\theta(x)+\bar{u}\,g(x) \quad [\bar{u}\,g(\bar{x})=0]$$

$$\leqq\theta(x). \text{ [since } \bar{u}\geqq 0, g(x)\leqq 0]$$

Hence $\bar{x}$ is a solution of MP.

99.7. Necessary Optimality Criteria

The case with respect to necessary criteria is considerably more complicated than the case with respect to sufficient optimality criteria.

Theorem 99.6. (Fritz John saddle point necessary optimality theorem)

Let X^0 be convex set in R^n, and let θ and g be convex on X^0. If $\bar{x}$ is a solution of MP, then $\bar{x}$ and some $\bar{r}_0\in R$, $\bar{r}\in R^m, (\bar{r}_0,\bar{r})\geq 0$ solve FJSP and $\bar{r}\,g(\bar{x})=0$.

Proof. Because $\bar{x}$ solves MP

$\theta(x)-\theta(\bar{x})<0, g(x)\leqq 0$ has no solution $x\in X^0$.

There exist $\bar{r}_0\in R, \bar{r}\in R^m, (\bar{r}_0,\bar{r})\geq 0$ such that

$$\bar{r}_0\left[\theta(x)-\theta(\bar{x})\right]+\bar{r}\,g(x)\geqq 0 \text{ for all } x\in X^0.$$

By letting $x=\bar{x}$ in the above, we get that $\bar{r}\,g(\bar{x})\geqq 0$. But since $\bar{r}\geqq 0$ and $g(\bar{x})\leqq 0$, we also have $\bar{r}\,g(\bar{x})\leqq 0$. Hence $\bar{r}\,g(\bar{x})=0$ and $\bar{r}_0\,\theta(\bar{x})+\bar{r}\,g(\bar{x})\leqq \bar{r}_0\,\theta(x)+\bar{r}\,g(x)$ for all $x\in X^0$, which is the second inequality of FJSP.

We also have, because $g(\bar{x})\leqq 0$, that

$$rg(\bar{x})\leqq 0 \text{ for all } r\geqq 0, r\in R^m$$

and hence, since $\bar{r}\,g(\bar{x})=0$

$$\bar{r}_0\,\theta(\bar{x})+r\,g(\bar{x})\leqq \bar{r}_0\,\theta(\bar{x})+r\,g(\bar{x}) \text{ for all } r\geqq 0, r\in R^m$$

which is the first inequality of FJSP.

Problem 99.5. Slater's Constraint qualification

Let X^0 be a convex set in R^n. The m-dimensional convex vector function g on X^0 which defines the convex feasible region

$$X=\{x : x\in X^0, g(x)\leqq 0\}$$

is said to satisfy Slater's constraint qualificaiton (on X^0) if there exists an $\bar{x}\in X^0$ such that $g(\bar{x})<0$.

Problem 99.6. Karlin's Constraint qualification

Let X^0 be a convex set in R^n. The m-dimensional convex vector function g on X^0 which defines the convex feasible region

$$X=\{x : x\in X^0, g(x)\leqq 0\}$$

is said to satisfy Karlin's constraint qualification (on X^0) if there exists no $p\in R^m, p\geq 0$ such that

$$pg(x)\geqq 0 \text{ for all } x\in X^0.$$

Problem 99.7. The strict constraint qualification

Let X^0 be a convex set in R^n. The m-dimensional convex vector g on X^0 which defines the convex feasible region

$$X = \{x : x \in X^0, g(x) \leq 0\}$$

is said to satisfy the strict constraint qualification (on X^0) if X contains at least two distinct points x^1 and x^2 such that g is strictly convex at x^1.

Lemma 99.1. Slater's constraint qualification and Karlin's constraint qualification are equivalent. The strict constraint qualification implies Slater's and Karlin's Constraint qualifications.

Theorem 99.7. Kuhn-Tucker saddle point necessary optimality theorem

Let X^0 be a convex set in R^n, let θ and g be convex on X^0, and let g satisfy Slater's constraint qualification, Karlin's constraint qualification or the strict constraint qualification on X^0. If $\bar{x}$ is a solution of MP then $\bar{x}$ and some $\bar{u} \in R^m, \bar{u} \geqq 0$, solve KTSP and $\bar{u}g(\bar{x}) = 0$.

Proof. By Fritz John saddle point necessary optimality theorem, $\bar{x}$ and some $\bar{r}_0 \in R, \bar{r} \in R^m, (\bar{r}_0, \bar{r}) \geq 0$ solve FJSP and $\bar{r}\, g(\bar{x}) = 0$.

Again, we known that if $\bar{r}_0 > 0$ and $(\bar{x}, \bar{r}_0, \bar{r})$ is solution of FJSP, then $(\bar{x}, \bar{r} / \bar{r}_0)$ is a solution of KTSP. Conversely, if $(\bar{x}, \bar{u})$ is a solution of KTSP, then $(\bar{x}, 1, \bar{u})$ is a solution of FJSP. Hence the results follows when $\bar{r}_0 > 0$. If $\bar{r}_0 = 0$ then $\bar{r} \geq 0$ and from the second inequality of FJSP

$0 \leqq \bar{r}\, g(x)$ for all $x \in X^0$ (since $\bar{r}_0 = 0$ and $\bar{r}\, g(\bar{x}) = 0$)

which contradicts Karlin's constraint qualification.

Hence $\bar{r}_0 > 0$.

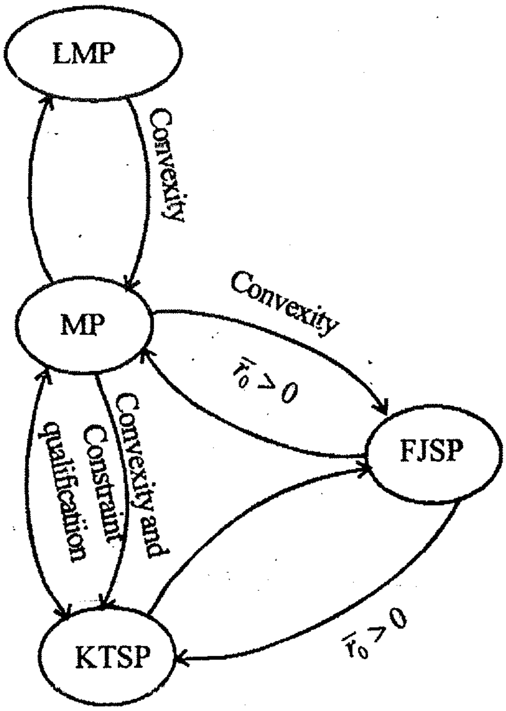

Figure. 99.1 : Relationships between the solutions of the local minimization problem (LMP), the minimization problem (MP), the Fritz John saddle poing problem (FJSP) and the Kuhn-Tucker saddle point problem (KTSP)

Theorem 99.8. Let θ, *g* be respectively a numerical function and an *m*-dimensional vector function which are both convex on R^n. Let *h* be a *k*-dimensional linear vector function on R^n, that is, *h* (*x*), = *Bx–d*, where *B* is a *kXn* matrix, and *d* is a *k*-vector let $\bar{x}$ be a solution of the minimization problem.

$$\theta(\bar{x}) = \min_{x \in x} \theta(x), \bar{x} \in X = \{x : x \in R^n, g(x) \leqq 0, Bx = d\}$$

and let *h* and *h* satisfy any of the constraint qualifications.

(i) (Generalised Slater). $g(x) < 0, Bx = d$ has a solution $x \in R^n$

(ii) (Generalized Karlin). There exists no $p \geq 0, p \in R^m, q \in R^k$ such that

$$pg(x) + q(Bx - d) \geqq 0 \text{ for all } x \in R^n$$

(iii) (Generalized strict). *X* contains at least two distinct points x^1 and x^2 such that *g* is strictly convex at

Then $\bar{x}$ and some $\bar{u} \in R^m, \bar{u} \geqq 0, \bar{v} \in R^k$ satisfy $\bar{u}\, g(\bar{x}) = 0$, and

$$\phi(\bar{x},u,v) \leqq \phi(\bar{x},\bar{u},\bar{v}) \leqq \phi(x,\bar{u},\bar{v})$$

for all $u \geqq 0, u \in R^m$ all $v \in R^k$, and all $x \in R^n$

$$\phi(x,u,v) = \theta(x) + ug(x) + v(Bx - d).$$

Proof. We shall establish the fact that

$$(iii) \Rightarrow (i) \Rightarrow (ii)$$

and then prove the theorem under (*ii*).

To prove $(iii) \Rightarrow (i)$

Since $g(x^1) \leqq 0, g(x^2) \leqq 0, Bx^1 = d, Bx^2 = d$, we have $0 < \lambda < 1$ for that $B\left[(1-\lambda)x^1 + \lambda x^2\right] = d$ and

$$g\left[(1-\lambda)x^1 + \lambda x^2\right] < (1-\lambda)g(x^1) + \lambda g(x^2) \leqq 0.$$

Hence (*i*) holds.

To prove $(i) \Rightarrow (ii)$

If $g(\bar{x}) < 0$ and $B\bar{x} = d$, then for any $p \geq 0, p \in r^m$, and any $q \in R^k, pg(\bar{x}) + q(B\bar{x} - d) < 0$.

Hence (*ii*) holds.

We establish now the theorem under (*ii*). There will be no loss of generality if we assume that the rows $B_1, B_2, ..., B_k$ of B are linearly independent, for suppose that some, B_k say, is linearly dependent on $B_1, B_2, ..., B_{k-1}$, that is $B_k = \sum_{i=1}^{k-1} s_i B_i$, where $s_1, s_2, ..., s_{k-1}$ are fixed real numbers.

Then

$$B_k x - d_k = \sum_{i=1}^{k-1} s_i B_i x - d_k = \sum_{i=1}^{k-1} s_i d_i - d_k$$

for any x satifying $B_i x = d_i, i = 1,2,...,k-1$. But, since $\bar{x} \in X$ and $B_i\bar{x} = d_i, i = 1,2,...,k$, it follows that

$\sum_{i=1}^{k-1} s_i d_i - d_k = 0$ and $B_k x - d_k = 0$ for any x.

satisfying $B_i x = d_i, i = 1,2,...,k-1$.

Hence the equality constraint $B_k x = d_k$ is redundant and can be dropped from the minimization problem without changing the solution $\bar{x}$. Then, once we have established the theorem for the linearly independent rows of B, we can reintroduce the linearly dependent row B_k (without changing the minimization problem) and set $\bar{v}_k = 0$ in the saddle point problem.

Now, there exist $\bar{r}_0 \in R, \bar{r} \in R^m, \bar{s} \in R^k, (\bar{r}_0, \bar{v}) \geqq 0, (\bar{r}_0, \bar{r}, \bar{s}) \neq 0$, which satisfy $\bar{r}\, g(\bar{x}) = 0$ and solve the saddle point problem. If $\bar{r}_0 > 0$, then $\bar{u} = \bar{r}/\bar{r}_0, \bar{v} = \bar{s}/\bar{r}_0$ solved the saddle point problem of the present theorem, and we are done. Suppose $\bar{r}_0 = 0$. Then since $\bar{r}\, g(\bar{x}) = 0$ and $B\bar{x} - d = 0$, we have inequality of the saddle point problem that

$$0 \leqq \bar{r}\, g(x) + \bar{s}(Bx - d) \text{ for all } x \in R^n$$

which contradicts (*ii*) above, if $\bar{r} \geq 0$. Now suppose that $\bar{r} = 0$, then $\bar{s} \neq 0$ and $\bar{s}(Bx - d) \geqq 0$ for all x in R^n. Hence $B'\bar{s} = 0$, which contradicts the assumption that the rows of B are linearly independent. Thus $\bar{r}_0 > 0$.

99.8. Module Summary

In this module, we have defined different types of problems, viz., minimization problem, local minimization problem, Fritz John saddle point problem, Kuhn-Tucker saddle point problem. We also studied some basic results for minimization and local minimization problems. The sufficient and necessary optimality criterian are also studied here. The Slater's, Karlin's and strict constraint qualifications are defined, the most important theorem of this module is Kuhn-Tucker saddle point necessary optimality theorem, which is also proved here.

99.9 Self Assessment Questions

1. Define the following problems:
   (i) The minimization problem (MP).
   (ii) The local minimization problem (LMP).
   (iii) The Fritz John saddle point problem (FJSP).
   (iv) The Kuhn-Tucker saddle point problem (KTSP).
2. Show that the set of solutions of MP is a convex set.
3. Let X be convex and $\bar{x}$ be a solution of MP. If θ is strictly convex at $\bar{x}$, then show that $\bar{x}$ is the unique solution of MP.

4. If $\bar{x}$ is a solution of MP, then prove that it is also a solution of LMP. Also, show that the converse is true if X is convex and θ is convex at $\bar{x}$.

5. If $(\bar{x},\bar{u})$ is a solution of KTSP then show that $\bar{x}$ is a solution of MP. If $(\bar{x},\bar{r}_0,\bar{r})$ is a solution of FJSP and $\bar{r}_0 > 0$ then show that $\bar{x}$ is a solution of MP.

6. Let X^0 be a convex set in R^n and let θ and g be convex on X^0. If $\bar{x}$ is a solution of MP, then prove that $\bar{x}$ and some $\bar{r}_0 \in R, \bar{r} \in R^m, (\bar{r}_0,\bar{r}) \geq 0$ solve FJSP and $\bar{r}\, g(\bar{x}) = 0$.

7. State the following
   (i) Slater's constraint qualification
   (ii) Karlin's constraint qualification.
   (iii) The strict constraint qualificaiton.

8. State and prove Kuhn-Tucker saddle point necessary optimality theorem.

9. State and prove Kuhn-Tucker saddle point necessary optimality theorem in the presence of linear equality constraints.

99.10. Reference

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper-IX Special Paper – Operations Research

Module No. - 100

Advanced Optimization and Operations Research-I
(Optimality Criteria with Differentiability)

Module Structure :

100.1 Introduction

In Module 99, you have learnt the saddle point optimality criteria of non-linear programming problem without differentiability. Many problems involve differentiable functions. It is important then to develop optimality criteria that take advantage of this property. In this module, we shall develop necessary and sufficient optimality criteria. For the sufficient optimality criteria we shall need differentiability and convexity. In this module, we also discuss the duality in non-linear programming like linear programming.

100.2. Objectives

Go through this module you will learn the following:

* Differentiable convex and concave functions
* Twice-differentiable convex and concave functions
* Fritz John and Kuhn-Tucker stationary point problem
* Sufficient optimality criteria
* Duality in non-linear programming
* Weak and Wolfe's duality theorem
* Duality in quadratic programming

100.3 Keywords

Differentiable convex and concave functions, sufficient and necessary optimality criteria, duality, duality in non-linear programming, duality in quadratic programming.

100.4 Differentiable Convex and Concave Functions

Let θ be a numerical function defined on an open set Γ in R^n. If θ is differentiable at $\bar{x} \in \Gamma$, then

$$\bar{x} \in R^n, \bar{x} + x \in \Gamma$$

$$\Rightarrow \theta(\bar{x}+x) = \theta(\bar{x}) + \Delta\theta(\bar{x})x + \alpha(\bar{x},x)\|x\|, \text{ where } \lim_{x\to 0} \alpha(\bar{x},x) = 0,$$

where $\nabla\theta(\bar{x})$ is the n-dimensional gradient vector of θ at $\bar{x}$ where n components are the partial derivatives of θ with respect to $x_1, x_2, \ldots, x_n$ evaluated at $\bar{x}$, and α is a numerical function of x.

Theorem 100.1. Let θ be a numerical function defined on an open set $\Gamma \subset R^n$ and let θ be differentiable at $\bar{x} \in \Gamma$. If θ is convex at $\bar{x} \in \Gamma$, then

$$\theta(x)-\theta(\bar{x}) \geqq \nabla\theta(\bar{x})(x-\bar{x}) \text{ for each } \bar{x} \in \Gamma.$$

If θ is concave at $\bar{x} \in \Gamma$, then

$$\theta(x)-\theta(\bar{x}) \geqq \nabla\theta(\bar{x})(x-\bar{x}) \text{ for each } x \in \Gamma.$$

Proof. Let θ be convex at $\bar{x}$. Since Γ is open, there exists, an open ball $B_0(\bar{x})$ around $\bar{x}$, which contained in Γ. Let $x \in \Gamma$, and let $x \neq \hat{x}$. Then for some μ, such that $0 < \mu < 1$ and $\mu < \delta / \|x-\bar{x}\|$, we have that

$$\hat{x} = \bar{x} + \mu(x-\bar{x}) = (1-\mu)\bar{x} + \mu x \in B_0(\bar{x}) \subset \Gamma.$$

Since θ is convex at $\bar{x}$, it follows, the convexity of $B_0(\bar{x})$, and the fact $\bar{x} \in B_0(\bar{x})$, that for $0 < \lambda \leqq 1$,

$$(1-\lambda)\theta(\bar{x}) + \lambda\theta(\hat{x}) \geqq \theta[(1-\lambda)\bar{x} + \lambda\hat{x}]$$

or

$$\theta(\hat{x})-\theta(\bar{x}) \geqq \frac{\theta[\bar{x}+\lambda(\hat{x}-\bar{x})-\theta(\bar{x})]}{\lambda}$$

$$= \frac{\lambda\nabla\theta(\bar{x})(\hat{x}-\bar{x}) + \alpha[\bar{x},\lambda(\hat{x}-\bar{x})]\lambda\|\hat{x}-\bar{x}\|}{\lambda}$$

$$= \nabla\theta(\bar{x})(\hat{x}-\bar{x}) + \alpha[\bar{x},\lambda(\hat{x}-\bar{x})]\|\hat{x}-\bar{x}\|.$$

Since

$$\lim_{\lambda\to 0} \alpha[\bar{x},\lambda(\hat{x}-\bar{x})] = 0$$

taking the limit of the previous expression as λ approaches zero gives $\theta(\hat{x})-\theta(\bar{x}) \geqq \nabla\theta(\bar{x})(\hat{x}-\bar{x})$.

Since θ is convex at $\bar{x}$, since $\hat{x} \in \Gamma$, and since

$\hat{x} = (1-\mu)\bar{x} + \mu x$ we have

$$\mu[\theta(x)-\theta(\bar{x})] \geqq (\hat{x})-\theta(\bar{x}).$$

But since

$\hat{x}-\bar{x} = \mu(x-\bar{x})$ and $\mu > 0$, the last three relations give

$$\theta(x)-\theta(\bar{x}) \geqq \nabla\theta(\bar{x})(x-\bar{x}).$$

The proof for the concave case follows in a similar way.

Theorem 100.2. Let θ be a numerical differentiable function on an open convex set $\Gamma \subset R^n$. θ is convex on Γ if any only if

$$\theta(x^2)-\theta(x^1) \geqq \nabla\theta(x^1)(x^2-x^1) \text{ for each } x^1, x^2 \in \Gamma$$

θ is concave on Γ if and only if

$$\theta(x^2)-\theta(x^1) \leqq \nabla\theta(x^1)(x^2-x^1) \text{ for each } x^1, x^2 \in \Gamma.$$

Proof. The condition is necessary

Since θ is convex (concave) at each $x^1 \in \Gamma$, this part of the proof follows from Theorem 100.1.

The condition is sufficient

We shall establish the result for the convex case. The concave case follows in a similar way. Let $x^1, x^2 \in \Gamma$ and let $0 \leqq \lambda \leqq 1$. Since Γ is convex, $(1-\lambda)x^1 + \lambda x^2 \in \Gamma$.

We have, then

$$\theta(x^1)-\theta[(1-\lambda)x^1+\lambda x^2] \geqq \lambda\nabla\theta[(1-\lambda)x^1+\lambda x^2](x^1-x^2)$$

and

$$\theta(x^2)-\theta[(1-\lambda)x^1+\lambda x^2] \geqq -(1-\lambda)\nabla\theta[(1-\lambda)x^1+\lambda x^2](x^1-x^2).$$

Multiplying the first inequality by $(1-\lambda)$, the second one by λ, and adding we get

$$(1-\lambda)\theta(x^1)+\lambda\theta(x^2) \geqq \theta[(1-\lambda)x^1+\lambda x^2].$$

Hence the result follows.

Geometrical Interpretation of Theorem 100.2

For a differentiable convex function θ on Γ, the linearization $\theta(\bar{x})+\nabla\theta(\bar{x})(x-\bar{x})$ at $\bar{x}$ never overestimates for any x in Γ (see Figure 100.1).

For a differentiable concave function θ on Γ, the linearization $\theta(\bar{x})+\nabla\theta(\bar{x})(x-\bar{x})$ at $\bar{x}$ never underestimates $\theta(x)$ for any x in Γ (see Figure 100.2).

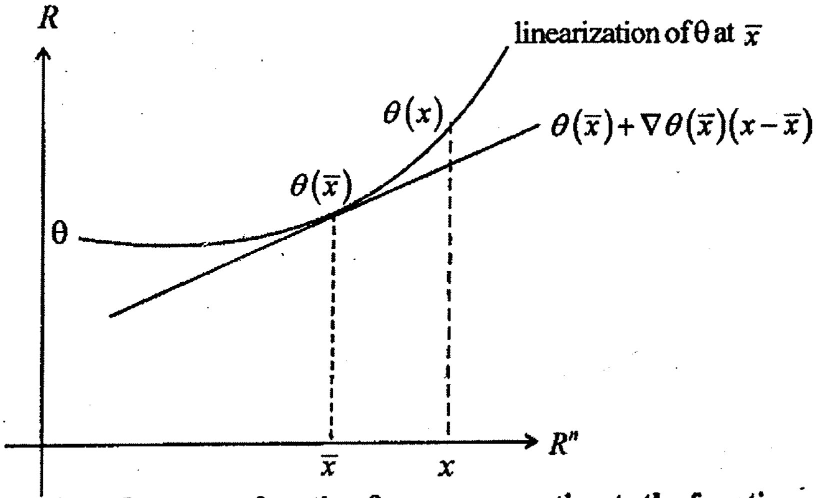


Figure 100.1. Linearization of a convex function θ never overestimate the function.

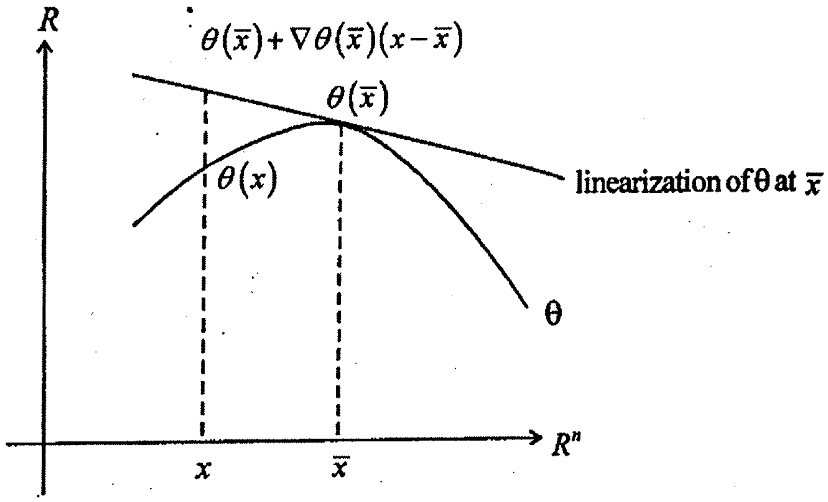


Figure 100.2: Linearization of a concave function θ never underestimates the function.

Theorem 100.3. Let θ be a numerical differentiable function on an open convex set $\Gamma \subset R^n$. A necessary and sufficient condition that θ be convex (concave) on Γ is that for each $x^1, x^2 \in \Gamma$

$$\left[\nabla\theta(x^2)-\nabla\theta(x^1)\right](x^2-x^1) \geqq 0 \quad (\leqq 0).$$

Proof. The condition is necessary

Let θ be convex on Γ and let $x^1, x^2 \in \Gamma$. By theorem 100.2 we have that

$$\theta(x^2)-\theta(x^1)-\nabla(x^1)(x^2-x^1) \geqq 0$$

and $\theta(x^1)-\theta(x^2)-\nabla(x^2)(x^1-x^2) \geqq 0.$

Adding these two inequalities gives

$$\left[\nabla\theta(x^2)-\nabla\theta(x^1)\right](x^2-x^1)\geqq 0.$$

The condition is sufficient

Let $x^1, x^2 \in \Gamma$. Then for $0 \leq \lambda \leq 1, (1-\lambda)x^1 + \lambda x^2 \in \Gamma$.

Now by mean-value theorem, we have for some $\bar{\lambda}, 0 < \bar{\lambda} < 1$,

$$\theta(x^2)-\theta(x^1)=\nabla\theta\left[x^1+\bar{\lambda}(x^2-x^1)\right]\bar{\lambda}(x^2-x^1)\geqq 0$$

or, $\nabla\theta\left[x^1+\bar{\lambda}(x^2-x^1)\right](x^2-x^1)\geqq\nabla\theta(x^1)(x^2-x^1)$.

Hence

$$\theta(x^2)-\theta(x^1)\geqq\nabla\theta(x^1)(x^2-x^1)$$

and by Theorem 100.2, θ is convex on Γ.

The proof is similar for concave function.

Monotone function

If *f* is an *m*-dimensional function on $\Gamma \subset R^n$, and $\left|f(x^2)-f(x^1)\right|(x^2-x^1)\geqq 0$ for all $x^1, x^2 \in \Gamma$, then *f* is said to be monotine on Γ. It is seen from the above theorem that a differentiable numerical function on the open convex set $\Gamma \subset R^n$ is convex if and only if $\nabla\theta$ is monotone on Γ.

100.5 Differentiable Strictly Convex and Concave Functions

The results obtained in the previous section can be extended directly to strictly convex and strictly concave functions by changing the inequalities ≧ and ≦ to strict inequalities > and <.

Theorem 100.4. Let θ be a numerical function defined on an open set $\Gamma \subset R^n$ and let θ be differentiable at $\bar{x} \in \Gamma$. If θ is strictly convex at $\bar{x} \in \Gamma$, then

$$\theta(x)-\theta(\bar{x})>\nabla\theta(\bar{x})(x-\bar{x}) \text{ for each } \bar{x}\in\Gamma, x\neq\bar{x}.$$

If θ is strictly concave at $\bar{x} \in \Gamma$, then

$$\theta(x)-\theta(\bar{x})<\nabla\theta(\bar{x})(x-\bar{x}) \text{ for each } \bar{x}\in\Gamma, x\neq\bar{x}.$$

Proof. Let θ be strictly convex at $\bar{x}$. Then

$$[\theta(1-\lambda)\bar{x}+\lambda x]<(1-\lambda)\theta(\bar{x})+\lambda\theta(x) \text{ for all } x\neq\bar{x},$$

$$0<\lambda<1 \text{ and } (1-\lambda)\bar{x}+\lambda x\in\Gamma. \quad \text{.................(1)}$$

Since θ is convex at $\bar{x}$ it follows from Theorem 100.1 that

$$\theta(x)-\theta(\bar{x})\geqq\nabla\theta(\bar{x})(x-\bar{x}) \text{ for all } x\in\Gamma. \quad \text{.................(2)}$$

we will now show that if equality holds in (2) for some x in Γ which is distinct from $\bar{x}$, then a contradiction ensues. Let equality hold in (2) for $x=\hat{x}, \hat{x}\in\Gamma$ and $\hat{x}\neq\bar{x}$. Then

$$\theta(\hat{x})=\theta(\bar{x})+\nabla\theta(\hat{x}-\bar{x}). \quad \text{.................(3)}$$

From (1) and (3) we have that

$$\theta[(1-\lambda)\bar{x}+\lambda\hat{x}]<(1-\lambda)\theta(\bar{x})+\lambda\theta(\bar{x})+\lambda\nabla\theta(\bar{x})(\hat{x}-\bar{x})$$

for $0<\lambda<1$ and $(1-\lambda)\bar{x}+\lambda\hat{x}\in\Gamma$

or $\theta[(1-\lambda)\bar{x}+\lambda\hat{x}]<\theta(\bar{x})+\lambda\nabla\theta(\bar{x})(\hat{x}-\bar{x})$ ....(4)

for $0<\lambda<1$ and $(1-\lambda)\bar{x}+\lambda\hat{x}\in\Gamma$.

But, by applying Theorem 100.1 again to the points $x(\lambda)=(1-\lambda)\bar{x}+\lambda\hat{x}$, where are in Γ and $\bar{x}$, we obtain that

$$\theta[(1-\lambda)\bar{x}+\lambda\hat{x}]\geqq\theta(\bar{x})+\lambda\nabla\theta(\bar{x})(\hat{x}-\bar{x})$$

for $0<\lambda<1$ and $(1-\lambda)\bar{x}+\lambda\hat{x}\in\Gamma$.

which contradicts (4) since for small λ, $(1-\lambda)\bar{x}+\lambda\hat{x}\in\Gamma$ because Γ is open. Hence equality cannot hold in (2) for any $x\in\Gamma$ which is distinct from $\bar{x}$, and the theorem is proved for convex case. The proof of concave case is similar.

Theorem 100.5. Let θ be a numerical differentiable function on an open convex set $\Gamma\subset R^n$. θ is strictly convex on Γ if and only if

$$\theta(x^2)-\theta(x^1)>\nabla\theta(x^1)(x^2-x^1) \text{ for each } x^1,x^2\in\Gamma, x^1\neq x^2.$$

θ is strictly concave on Γ if and only if

$$\theta(x^2)-\theta(x^1)<\nabla\theta(x^1)(x^2-x^1) \text{ for each } x^1,x^2\in\Gamma, x^1\neq x^2.$$

Theorem 100.6 Let θ be a numerical differentiable function on an open convex set $\Gamma \subset R^n$. A necessary and sufficient condition that θ be strictly convex (concave) on Γ is that for each $x^1, x^2 \in \Gamma, x^1 \neq x^2$.

$$\left[\nabla\theta\left(x^2\right)-\nabla\theta\left(x^1\right)\right]\left(x^2-x^1\right)>0\ (<0).$$

100.6 Twice-differentiable Convex and Concave Functions

Let θ be a numerical function defined on some open set Γ in R^n. If θ is twice differentiable at $\bar{x} \in \Gamma$ then by Taylor's theorem for $x \in R^n, \bar{x}+x \in \Gamma$,

$$\theta(\bar{x}+x)=\theta(\bar{x})+\nabla\theta(\bar{x})x+\frac{1}{2}x\nabla^2\theta(\bar{x})x+\beta(\bar{x},x)(\|x\|)^2,$$

where $\lim_{x\to 0}\beta(\bar{x},x)=0, \nabla\theta(\bar{x})$ is the *n*-dimenstional gradient vector of θ at $\bar{x}$ and $\nabla^2\theta(x)$ is the *nxn* Hessian matrix of θ at $\bar{x}$ whose *ij*th element is $\frac{\partial^2\theta(\bar{x})}{\partial x_i \partial x_j}, i,j=1,2,\ldots,n.$

Theorem 100.7 Let θ be numerical function defined on an open set $\Gamma \subset R^n$ and let θ be twice differentiable at $\bar{x} \in \Gamma$. If θ is convex at $\bar{x}$, then $\nabla^2\theta(\bar{x})$ is positive semidefinite, that is,

$$y\nabla^2\theta(\bar{x})y \geqq 0 \text{ for all } y \in R^n.$$

If θ is concave at $\bar{x}$ then $\nabla^2\theta(\bar{x})$ is negative semidefinite, that is,

$$y\nabla^2\theta(\bar{x})y \leqq 0 \text{ for all } y \in R^n.$$

Proof. Let $y \in R^n$. Because Γ is open, there exists a $\bar{\lambda}>0$ such that $\bar{x}+\lambda y \in \Gamma$ for $0<\lambda<\bar{\lambda}$.

By Theorem 100.1 it follows that

$$\theta(\bar{x}+\lambda y)-\theta(\bar{x})-\lambda\nabla\theta(\bar{x})y \geqq 0 \text{ for } 0<\lambda<\bar{\lambda}.$$

But, since θ is twice differentiable at $\bar{x}$

$$\theta(\bar{x}+\lambda y)-\theta(\bar{x})-\lambda\nabla\theta(\bar{x})y=\frac{(\lambda)^2 y\nabla^2\theta(\bar{x})y}{2}+(\lambda)^2\beta(\bar{x},\lambda y)(\|y\|)^2.$$

Hence

$$\frac{y\nabla^2\theta(\bar{x})y}{2} + \beta(\bar{x},\lambda y)(\|y\|)^2 \geqq 0 \text{ for } 0<\lambda<\bar{\lambda}.$$

Taking the limit as λ approaches zero, and using the result $\lim_{\lambda\to 0}\beta(\bar{x},\lambda y)=0$ we get

$$y\nabla^2\theta(\bar{x})y \geqq 0.$$

The concave case is established in a similar way.

Theorem 100.8. Let θ be a numerical twice-differentiable function on an open convex set $\Gamma \subset R^n$. θ is convex on Γ if and only if $\nabla^2\theta(x)$ is positive semidefinite on Γ, that is, for each $y \in R^n$.

$$y\nabla^2\theta(x)y \geqq 0 \text{ for all } y \in R^n.$$

θ is concave on Γ if and only if $\nabla^2\theta(x)$ is negative semidefinite on Γ, that is, for each $x \in \Gamma$.

$$y\nabla^2\theta(x)y \leqq 0 \text{ for all } y \in R^n.$$

Proof. The condition is necessary

Since θ is convex (concave) at each $x \in \Gamma$, this part of the proof follows from Theorem 100.7.

The condition is sufficient

By Taylor's theorem we have that for any

$$x^1, x^2 \in \Gamma$$

$$\theta(x^2)-\theta(x^1)-\nabla\theta(x^1)(x^2-x^1)=\frac{1}{2}\left[(x^2-x^1)\nabla^2\theta\{x^1+\delta(x^2-x^1)\}(x^2-x^1)\right.$$

for some $\delta, 0<\delta<1$.

But, the right-hand side of the above equality is non-negative (non-positive), because $\nabla^2\theta(x)$ is positive (negative) semidefinite on Γ, and

$$x^1+\delta(x^2-x^1) \in \Gamma.$$

Hence the left-hand side is non-negative (non-positive) and by Theorem 100.2 θ is convex (concave) on Γ.

100.7 The Minimization Problems, the Fritz John and Kuhn-Tucker Stationary-Point Problems

The optimality criteria of this section relate the solutions of a minimization problem, a local minimization problem, and two stationary point problems to each other. The minimization and the local minimization problems considered here will be the same as the corresponding problems of Module 99, with the added differentiability assumption. The Fritz John and Kuhn-Tucker problems of this section follow from the Fritz John and Kuhn-Tucker saddle point problems of Module 99 if differentiability is assumed, and conversely the Fritz John and Kuhn-Tucker saddle point problems of Module 99 follow from the Fritz John and Kuhn-Tucker problems if convexity is assumed.

Let X^0 be an open set in R^n, and let θ and g be respectively a numerical function and an m-dimensional vector function both defined on X^0.

Problem 100.1. The minimization problem (MP)

Find an $\bar{x}$, if it exists, such that

$$\theta(\bar{x}) = \min_{x\in X} \theta(x), \bar{x} \in X = \{x : x \in X^0, g(x) \leqq 0\}.$$

Problem 100.2. The local minimization problem (LMP)

Find an $\bar{x}$ in X, if it exists, such that for some open ball $B_0(\bar{x})$ around $\bar{x}$ with radius $\delta > 0$

$$x \in B_0(\bar{x}) \cap X \Rightarrow \theta(x) \geqq (\bar{x}).$$

Problem 100.3. The Fritz John stationary-point problem (FJR)

Find $\bar{x} \in X^0, \bar{r}_0 \in R, \bar{r} \in R^m$ if they exist, such that

$$\nabla_x \phi(\bar{x}, \bar{r}_0, \bar{r}) = 0, \nabla_r \phi(\bar{x}, \bar{r}_0, \bar{r}) \leqq 0, \bar{r}\, \nabla_r \phi(\bar{x}, \bar{r}_0, \bar{r}) = 0,$$

$$(\bar{r}_0, \bar{r}) \geq 0, \phi(x, r, r_0) = r_0 \theta(x) + rg(x)$$

or equivalently.

$$\bar{r}_0 \nabla \theta(\bar{x}) + \bar{r} \nabla g(\bar{x}) = 0, g(\bar{x}) \leqq 0, \bar{r}\, g(\bar{x}) = 0, (\bar{r}_0, \bar{r}) \geq 0.$$

Here we assume that θ and g are differentable at $\bar{x}$.

Problem 100.4. The Kuhn-Tucker stationary-point problem (KTP)

Let $\bar{x} \in X^0, \bar{u} \in R^m$ if they exist, such that

$$\nabla_x \psi(\bar{x},\bar{u}) = 0, \nabla_u \psi(\bar{x},\bar{u}) \leqq 0, \bar{u}\, \nabla_u \psi(\bar{x})\bar{u} = 0, \bar{u} \geqq 0,$$

$$\psi(x,u) = \theta(x) + ug(x)$$

or equivalently

$$\nabla\theta(\bar{x}) + \bar{u}\nabla g(\bar{x}) = 0, g(\bar{x}) \leqq 0, \bar{u}\, g(\bar{x}) = 0, \bar{u} \geqq 0.$$

It is implicit in the above statement that θ and g are differentiable at $\bar{x}$.

Corollary 100.1

If $(\bar{x},\bar{r}_0,\bar{r})$ is a solution of FJP and $\bar{r}_0 > 0$, then $(\bar{x},\bar{r}/\bar{r}_0)$ is a solution of KTP. Conversely, if $(\bar{x},\bar{u})$ is a solution of KTP, then $(\bar{x},1,\bar{u})$ is a solution of FJP.

100.8. Sufficient Optimality Criteria

Theorem 100.9 (Kuhn-Tucker) Sufficient optimality theorem

Let $\bar{x} \in X^0$, let X^0 be open and let θ and g be differentiable and convex at $\bar{x}$. If $(\bar{x},\bar{u})$ is a solution of KTP then $\bar{x}$ is a solution of MP. If $(\bar{x},\bar{r}_0,\bar{r})$ is a solution of FJP and $\bar{r}_0 > 0$, then $\bar{x}$ is a solution of MP.

Proof. The second statement of the theorem follows trivially from the first statement by Corollary 100.1.

Let $(\bar{x},\bar{u})$ be a solution of KTP. We have for any x in X that

$\theta(x) - \theta(\bar{x}) \geqq \nabla\theta(\bar{x})x(x-\bar{x})$ (by convexity and differentiability of θ at $\bar{x}$)

$= -\bar{u}\nabla g(\bar{x})(x-\bar{x})$ (since $\nabla\theta(\bar{x}) = -\bar{u}\nabla g(\bar{x})$)

$\geqq \bar{u}[g(\bar{x}) - g(x)]$ (by convexity and differentiability of g at $\bar{x}$ and by $\bar{u} \geqq 0$)

$= -\bar{u}\, g(x)$ (since $\bar{u}\, g(\bar{x}) = 0$)

$\geqq 0.$ (since $\bar{u} \geqq 0$ and $g(x) \leqq 0$)

Hence

$\theta(x) \geqq \theta(\bar{x})$ for any $x \in X$.

Since $g(\bar{x}) \leqq 0, \bar{x}$ is in X, and hence

$g(\bar{x}) = \min_{x \in X} \theta(x)$ and $x \in X$.

Example 100.1. Let $\bar{x} \in X^0$, let X^0 be open, let θ and g be differentiable and convex at $\bar{x}$ let B a given *kxn* matrix, and let d be a given k-dimensional vector. Show that if $(\bar{x}, \bar{u}, \bar{v}), \bar{x} \in X^0, \bar{u} \in R^m, \bar{v} \in R^k$ is a solution of the following Kuhn-Tucker problem

$$\nabla\theta(\bar{x}) + \bar{u}\,\nabla g(\bar{x}) + B'\bar{v} = 0$$
$$g(\bar{x}) \leqq 0$$
$$B\bar{x} = d$$
$$\bar{u}\,g(\bar{x}) = 0$$
$$\bar{u} \geqq 0$$

then

$$\theta(\bar{x}) = \min_{x \in X} \theta(x), \bar{x} \in X = \{x : x \in X^0, g(x) \leqq 0, Bx = d\}.$$

Theorem 100.10. Sufficient Optimality Criteria

Let $\bar{x} \in X^0$, X^0 be open, θ be differentiable and convex at $\bar{x}$, and let g be differentiable and strictly convex at $\bar{x}$. If $(\bar{x}, \bar{r}_0, \bar{r})$ is a solution of FJP, then $\bar{x}$ solves MP.

Proof. Let $(\bar{x}, \bar{r}_0, \bar{r})$ solve FJP. Let $I = \{i : g_i(\bar{x}) = 0\}$,

$J = \{i : g_i(\bar{x}) < 0\}, I \cup J = \{1, 2, ..., m\}$.

Since $\bar{r} \geqq 0, g(\bar{x}) \leqq 0$ and $\bar{r}\,g(\bar{x}) = 0$, we have that

$\bar{r}_i g_i(\bar{x}) = 0$ for $i = 1, 2, ..., m$

and hence $\bar{r}_i = 0$ for $i \in J$.

Since $\bar{r}_0 \nabla\theta(\bar{x}) + \bar{r}\,\nabla g(\bar{x}) = 0$ and $(\bar{r}_0, \bar{r}) \geq 0$, we have

that $\bar{r}_0\, \nabla\theta(\bar{x}) + \sum_{i \in J} \bar{r}_i\, \nabla g_i(\bar{x}) = 0, (\bar{r}_0, \bar{r}_i) \geq 0, i \in I.$

It follows then by Gordan's theorem (Module 97) that

$\nabla\theta(\bar{x}) z < 0, \nabla g_I(\bar{x}) z < 0$ has no solution $z \in R^n$. .... (1)

Consequently, $0 > \theta(x) - \theta(\bar{x}), 0 \geqq g_I(x) - g_I(\bar{x})$ has no solution $x \in X^0$ .... (2)

for if it did have a solution $\hat{x} \in X_0$, then $\hat{x} \neq \bar{x}$ and $0 > \theta(\hat{x}) - \theta(\bar{x}) \geqq \nabla\theta(\bar{x})(\hat{x} - \bar{x})$ (by convexity of θ at $\bar{x}$)

$$0 \geqq g_I(\hat{x}) - g_I(\bar{x}) > \nabla g_I(\bar{x})(\hat{x} - \bar{x}) \text{ (by strict convexity of } g \text{ at } \bar{x})$$

which contradicts (1) if we set $z = \hat{x} - \bar{x}$.

Since $g_I(\bar{x}) = 0$, we have from (2) that

$$\theta(\bar{x}) > \theta(x), 0 \geqq g_I(x), 0 \geqq g_J(x) \text{ has no solution } x \in X^0.$$

Since $g(\bar{x}) \leqq 0, \bar{x}$ is in X and hence $\bar{x}$ solve MP.

100.9. Necessary Optimality Criteria

In the necessary optimality conditions to be derived now, convexity does not play any crucial role. The differentiability property of the functions is used to linearize the non-linear programming problem, and then theorems of the alternative are used to obtain the necessary optimality conditions. Again, to derive the more important necessary optimality conditions, constraint qualifications are needed.

Lemma 1001. (Linearization lemma)

Let $\bar{x}$ be a solution of LMP, let X^0 be open, let θ and g be differentiable at $\bar{x}$ let

$$V = \{i : g_i(\bar{x}) = 0, \text{ and } g_i \text{ in concave at } \bar{x}\}$$

and let

$$W = \{i : g_i(\bar{x}) = 0 \text{ and } g_i \text{ is not concave at } \bar{x}\}.$$

Then the system

$$\nabla\theta(\bar{x})z < 0, \nabla g_w(\bar{x})z < 0, \nabla g_v(\bar{x})z \leqq 0 \text{ has no solution } z \text{ in } R^n.$$

Proof. Let $I = V \cup W = \{i : g_i(\bar{x}) = 0\}, J = \{i : g_i(\bar{x}) < 0\}$.

Hence

$$I \cup J = V \cup W \cup J = \{1, 2, ..., m\}.$$

Let $\bar{x}$ be a solution of LMP with $\delta = \bar{\delta}$. We shall show that if z satisfies $\nabla\theta(\bar{x})z < 0, \nabla g_w(\bar{x})z < 0$ and $\nabla g_z(\bar{x})z \leqq 0$ then a contradiction ensues. Let z satisfy these inequalities.

Then, since X^0 is open there exists a $\delta > 0$ such that $\bar{x}+\delta z \in X^0$ for $0<\delta<\bar{\delta}$.

Since θ and g are differentiable at $\bar{x}$, we have that for $0<\delta<\bar{\delta}$

$$\theta(\bar{x}+\delta z)=\theta(\bar{x})+\delta\nabla\theta(\bar{x})+\alpha_0(\bar{x},\delta z)\delta\|z\| \text{ and}$$

$$g_i(\bar{x}+\delta z)=g_i(\bar{x})+\delta\nabla g_i(\bar{x})z+\alpha_i(\bar{x},\delta z)\delta\|z\|, i=1,2,\ldots,m$$

where

$$\lim_{\delta\to 0}\alpha_i(\bar{x},\delta z)=0 \text{ for } i=0,1,\ldots,m.$$

(*i*) If δ is small enough (say $0<\delta<\delta_0$), then

$$\nabla\theta(\bar{x})z+\alpha_0(\bar{x},\delta z)\|z\|<0 \ [\text{since } \nabla\theta(\bar{x})z<0]$$

and hence

$$\theta(\bar{x}+\delta z)-\theta(\bar{x})<0 \text{ for } 0<\delta<\delta_0.$$

(*ii*) Similarly, for $i\in W$ and δ small enough (say, $0<\delta<\delta_0$),

then

$$\nabla g_i(\bar{x})z+\alpha_i(\bar{x},\delta z)\|z\|<0 \ [\text{since } \nabla g_w(\bar{x})z<0]$$

and hence

$$g_i(\bar{x}+\delta z)-g_i(\bar{x})<0 \text{ for } 0<\delta<\delta_i, i\in W.$$

(*iii*) For $i\in V$; we have, since g_i is concave at $\bar{x}$ and $\nabla g_v(\bar{x})z \leqq 0$, that

$$g_i(\bar{x}+\delta z)-g_i(\bar{x})\leqq \delta\nabla g_i(\bar{x})z\leqq 0 \text{ for } 0<\delta<\hat{\delta} \text{ and } i\in V.$$

(*iv*) For $i\in J, g_i(\bar{x})<0$. Hence for small enough δ (say $0<\delta<\delta_i$) we have

$$g_i(\bar{x})+\delta\nabla g_i(\bar{x})z+\alpha_i(\bar{x},\delta z)\delta\|z\|<0 \text{ and hence}$$

$$g_i(\bar{x}+\delta z)<0 \text{ for } 0<\delta<\delta_i, i\in J.$$

Let us call $\bar{\delta}$ the minimum of all the positve number. $\bar{\delta},\hat{\delta},\delta_0,\delta_i\,(i=1,2,\ldots,m)$ define above. Then for any δ in the internal $0<\delta<\bar{\delta}$ we have that

$$\bar{x}+\delta z\in X^0$$

$$\bar{x}+\delta z\in B_\delta(\bar{x})$$

$$\theta(\bar{x}+\delta z)<\theta(\bar{x})$$

$$g_i(\bar{x}+\delta z)\leqq g_i(\bar{x})=0 \text{ for } i\in I$$

$g_i(\bar{x}+\delta z)<0$ for $i \in J$.

Hence for $0<\delta<\bar{\delta}$, we have that $\bar{x}+\delta z \in B_0(\bar{x}) \cap X$ and $\theta(\bar{x}+\delta z)<\theta(\bar{x})$, which contradicts the assumption that $\bar{x}$ is a solution of LMP with $\delta=\bar{\delta}$. Hence there exists no z in R^n satisfying $\nabla\theta(\bar{x})z<0, \nabla g_w(\bar{x})<0$ and $\nabla g_v(\bar{x}) \leq 0$.

Theorem 100.12. (Fritz John stationary-point necessary optimality theorem). Let $\bar{x}$ be solution of LMP or of MP, let X^0 be open and let θ and g be differentiable at $\bar{x}$. Then there exists an $\bar{r}_0 \in R$ and an $\bar{r} \in R^m$ such that $(\bar{x},\bar{r}_0,\bar{r})$ solves JFP and $(\bar{r}_0,\bar{r}_w) \geq 0$, where

$$W=\{i : g_i(\bar{x})=0 \text{ and } g_i \text{ is not concave at } \bar{x}\}.$$

Proof. If $\bar{x}$ solves MP, then $\bar{x}$ solves LMP. Let

$$V=\{i : g_i(\bar{x})=0, \text{ and } g_i \text{ is concave at } \bar{x}\} \text{ and}$$

$$J=\{i : g_i(\bar{x})<0\}.$$

By Lemma 100.1 above we have that

$\nabla\theta(\bar{x})z<0, \nabla g_w(\bar{x})z<0, \nabla g_v(\bar{x})z \leqq 0$ has no solution $z \in R^n$.

Hence by Motzkin's theorem (see in Module 99) there exist $\bar{r}_0,\bar{r}_w,\bar{r}_v$ such that

$$\bar{r}_0\nabla\theta(\bar{x})+\bar{r}_w\nabla g_w(\bar{x})+\bar{r}_v\nabla g_v(\bar{x})=0, (\bar{r}_0,\bar{r}_w) \geq 0, \bar{r}_v \geqq 0.$$

Since $g_w(\bar{x})=0$ and $g_v(\bar{x})=0$, it follows that if we defined $\bar{r}_J=0$ and $\bar{r}=(\bar{r}_w,\bar{r}_v,\bar{r}_J)$ then

$$\bar{r}g(\bar{x})=\bar{r}_w g_w(\bar{x})+\bar{r}_v g_v(\bar{x})+\bar{r}_J g_J(\bar{x})=0,$$

$$\bar{r}_0\nabla\theta(\bar{x})+\bar{r}\nabla g(\bar{x})=0 \text{ and } (\bar{r}_0,\bar{r}) \geq 0.$$

Since $\bar{x}$ is in $X, g(\bar{x}) \leqq 0$. Hence $(\bar{x},\bar{r}_0,\bar{r})$ solves FJP and $(\bar{r}_0,\bar{r}_w) \geq 0$.

Problem 100.5. The Kuhn-Tucker constraint qualification

Let X^0 be an open set in R^n let g be an m-dimensional vector function defined on X^0, and let

$$X = \{x : x \in X^0, g(x) \leqq 0\}$$

g is said to satisfy the Kuhn-Tucker constraint qualification at $\bar{x} \in X$ if g is differentiable at $\bar{x}$ and if

$$y \in R^n, \nabla g_I(\bar{x}) y \leqq 0$$

then there exists an n-dimensional vector function e defined on the interval [0,1] such that

$a. e(0) = \bar{x}$

$b. e(r) \in X$ for $0 \leqq r \leqq 1$

$c. e$ is differentiable at $r = 0$ and $\frac{de(o)}{dr} = \lambda y$ for some $\lambda > 0$,

where

$$I = \{i : g_i(\bar{x}) = 0\}.$$

Problem 100.6. The Arrow-Hurwicz-Uzawa constraint qualification

Let X^0 be an open set in R^n, let g be an m-dimensional vector function defined on X^0, and let

$$X = \{x : x \in X^0, g(x) \leqq 0\}$$

g is said to satisfy the Arrow-Hurwicz-Uzawa constraint qualification at $\bar{x} \in X$ if g is differentiable at $\bar{x}$ and if

$\nabla g_w(\bar{x}) z > 0, \nabla g_v(\bar{x}) z \geqq 0$ has a solution $z \in R^n$

where

$$V = \{i : g_i(\bar{x}) = 0, \text{ and } g_i \text{ is concave at } \bar{x}\}$$

and

$$W = \{i : g_i(\bar{x}) = 0, \text{ and } g_i \text{ is not concave at } \bar{x}\}.$$

Problem 100.7. The reverse convex constraint qualification

Let X^0 be an open set in R^n, let g be an m-dimensional vector function defined on X^0 and let

$$X = \{x : x \in X^0, g(x) \leqq 0\}$$

g is said to satisfy the reverse convex constraint qualification at $\bar{x} \in X$ if g is differentiable at $\bar{x}$, and if for each $i \in I$ either g_i is concave at $\bar{x}$ or g_i is linear on R^n, where

$$I = \{i : g_i(\bar{x}) = 0\}.$$

Lemma 100.2 Let X^0 be an open set in R^n, let g be an m-dimensional vector function defined X^0 and let

$$X = \{x : x \in X^0, g(x) \leqq 0\}$$

(i) If g satisfies the reverse convex constraint qualification (Problem 100.7) at $\bar{x}$, then g satisfies the Arrow-Hurwicz-Uzawa constraint qualification (Problem 100.6) at $\bar{x}$.

(ii) If g satisfies the reverse convex constraint qualification (Problem 100.7) at $\bar{x}$, then g satisfies the Kuhn-Tucker constraint qualification (Problem 100.5) at $\bar{x}$.

(iii) Let X^0 be convex, let g be convex on X^0 and let g be differentiable at $\bar{x}$. If g satisfies Slater's constraint qualification on X^0, Karlin's constraint qualification on X^0, or the strict constraint qualification on X^0, then g satisfies the Arrow-Hurwicz-Uzawa constraint qualificatioin (Problem 100.6) at $\bar{x}$.

Theorem 100.13. Kuhn-Tucker stationary-point necessary optimality theorem

Let X^0 be an open subset of R^n, let θ and g be defined on X^0, let $\bar{x}$ solves LMP or MP, let θ and g be differentiable at $\bar{x}$ and let g satisfy

(i) the Kuhn-Tucker constraint qualification (Problem 100.5) at $\bar{x}$, or

(ii) the Arrow-Hurwicz-Vzawa constraint qualificatioin (Problem 100.6) at $\bar{x}$, or

(iii) the reverse convex constraint qualification (Problem 100.7) at $\bar{x}$, or

(iv) Slater's constraint qualification on X^0, or

(v) Karlin's constraint qualfication on X^0, or

(vi) the strict constraint qualification on X^0.

Then there exists a $\bar{u} \in R^m$ such that $(\bar{x}, \bar{u})$ solves KTP.

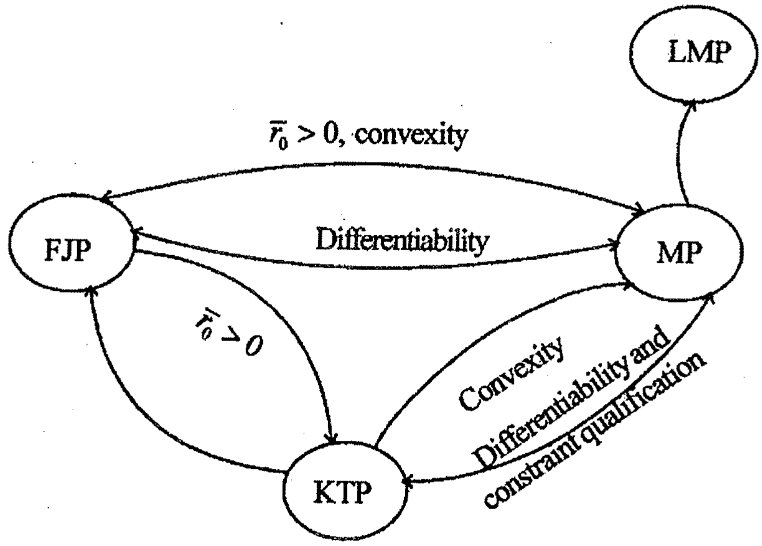
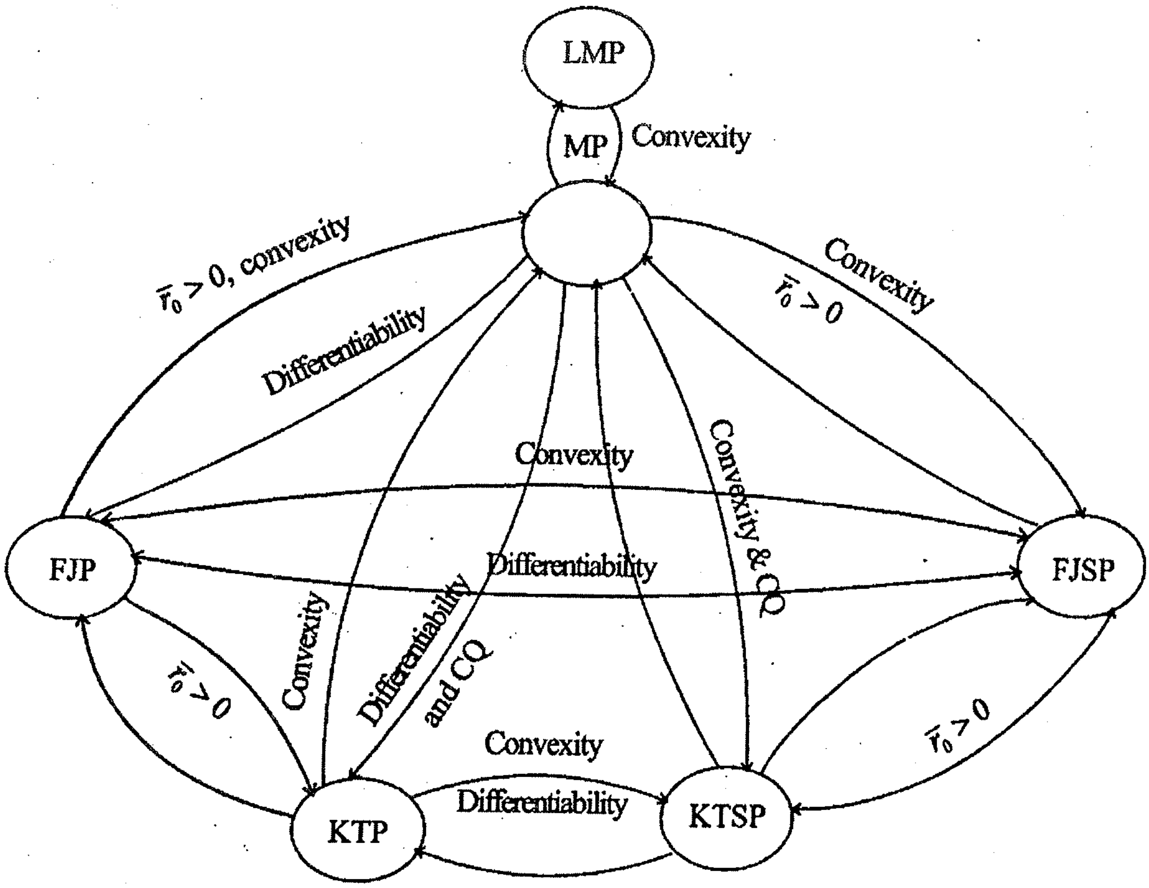


Figure 100.1 : Relationship between the solutions of LMP, MP, FJP and KTSP.



differentiability & convexity —✳— Convexity

Figure 100.2 : Relation between the solutions of the LMP, MP, FJP, KTP, FJSP and KTSP.

100.10. Duality in Non-linear Programming

Let X^0 be an open set in R^n and let θ and g be respectively a numerical function and an m-dimensional vector function, both defined on X^0. We define now the same minimization problem that we have been defined earlier.

Problem 100.8. The (primal) minimization problem (MP)

Find an $\bar{x}$, if it exists, such that

$$\theta(\bar{x}) = \min_{x \in X} \theta(x), \bar{x} \in X = \{x \in X^0, g(x) \leqq 0\}.$$

Problem 100.9. The dual (maximization) problem (DP)

Let θ and g be differentiable on X^0. Find an $\hat{x}$ and a $\hat{u} \in R^m$, if they exist, such that

$$\psi(\hat{x},\hat{u}) = \max_{(x,u) \in Y} \psi(x,u)$$

$$(\hat{x},\hat{u}) \in Y = \{(x,u) : x \in X^0, u \in R^m, \nabla_x \psi(x,u) = 0, u \geqq 0\}$$

$$\psi(x,u) = \theta(x) + ug(x)$$

or equavalently

$$\theta(\hat{x}) + \hat{u}g(\hat{x}) = \max_{(x,u) \in Y} \{\theta(x) + ug(x)\}$$

$$(\hat{x},\hat{u}) \in Y = \{(x,u) : x \in X^0, u \in R^m, \nabla\theta(x) + u\nabla g(x) = 0, u \geqq 0\}$$

The duality results we are about to establish relate solutions $\bar{x}$ of MP and $(\hat{x},\hat{u})$ of DP to each other. They also relate the objective functions θ and ψ to each other.

Theorem 100.14. Weak duality theorem

Let X^0 be open and let θ and g be differentiable on X^0. Then $x^1 \in X, (x^2, u^2) \in Y$, θ and g are convex at X^2 then $\theta(x^1) \geqq \psi(x^2, u^2)$ where X and Y are defined in Problems 100.8 and 100.9.

Proof.

$$\theta(x^1) \geqq \theta(x^2) + \nabla\theta(x^2)(x^1 - x^2)$$

$$= \theta(x^2) - u^2\nabla g(x^2)(x^1 - x^2) \text{ [since } \nabla\theta(x^2) = -u^2\nabla g(x^2)\text{]}$$

$$\geqq \theta(x^2) + u^2\left[g(x^2) - g(x^1)\right] \text{ [since } u^2 \geqq 0\text{]}$$

$$\geqq \theta(x^2) + u^2 g(x^2) \text{ [since } u^2 \geqq 0 \text{ and } g(x^1) \leqq 0\text{]}$$

$$= \psi(x^2, u^2).$$

Theorem 100.15. (Wolfe's duality theorem)

Let X^0 be an open set in R^n, let θ and g be differentiable and convex on X^0, let $\bar{x}$ solve MP and let g satisfy any one of the six constraint qualifications of Kuhn-Tucker stationary-point necessary optimality theorem (Theorem 100.13). Then there exists a $\bar{u} \in R^m$ such that $(\bar{x}, \bar{u})$ solves DP and

$$\theta(\bar{x}) = \psi(\bar{x}, \bar{u}).$$

Proof. By Kuhn-Tucker stationary-point necessary optimality theorem, there exists a $\bar{u} \in R^m$ such that $(\bar{x}, \bar{u})$ satisfies the Kuhn-Tucker conditions

$$\nabla\theta(\bar{x}) + \bar{u}\,\nabla g(\bar{x}) = 0$$

$$\bar{u}\, g(\bar{x}) = 0$$

$$g(\bar{x}) \leqq 0$$

$$\bar{u} \geqq 0.$$

Hence

$$(\bar{x}, \bar{u}) \in Y = \left\{(x, u) : x \in X^0, u \in R^m, \nabla\theta(x) + u\nabla g(x) = 0, u \geqq 0\right\}.$$

Now, let (x, u) be an arbitrary element of the set Y. Then

$$\psi(\bar{x}, \bar{u}) - \psi(x, u) = \theta(\bar{x}) - \theta(x) + \bar{u}\, g(\bar{x}) - u\, g(x)$$

$$\geqq \nabla\theta(x)(\bar{x} - x) - u\, g(x)$$

$$[\bar{u}\, g(\bar{x}) = 0]$$

$$\geqq \nabla\theta(x)(\bar{x}-x)+u[-g(\bar{x})+\nabla g(x)(\bar{x}-x)]$$
$$[u \geqq 0]$$
$$\geqq [\nabla\theta(x)+u\nabla g(x)](\bar{x}-x)-ug(\bar{x})$$
$$= -ug(\bar{x}) \text{ [since } \nabla\theta(x)+u\nabla g(x)=0\text{]}$$
$$\geqq 0 \text{ [since } t \geqq 0 \text{ and } g(\bar{x}) \leqq 0\text{]}$$

Hence

$$\psi(\bar{x},\bar{u}) = \max_{(x,u)\in Y} \psi(x,u), (\bar{x},\bar{u}) \in Y.$$

Since $\bar{u}\, g(\bar{x}) = 0$,

$$\psi(\bar{x},\bar{u}) = \theta(\bar{x}) + \bar{u}\, g(\bar{x}) = \theta(\bar{x}).$$

Theorem 100.16. (Strict converse duality theorem)

Let X^0 be an open set in R^n, let θ and g be differentiable and convex on X^0, let MP have a solution $\bar{x}$ and let g satisfy any one of the six constraint qualifications of Theorem 100.13. If $(\hat{x},\hat{u})$ is a solution of DP and if $\psi(x,u)$ is strictly convex at $\hat{x}$ then $\hat{x}=\bar{x}$, that is, $\hat{x}$ solves MP and $\theta(\bar{x}) = \psi(\hat{x},\hat{u})$.

100.11. Duality in Quadratic Programming

In this section, we consider a particular case of non-linear programming problem, called quadratic programming. In this problem the objective function is quadratic and constraints are linear.

Let b be an n-vector, c an m-vector, C a symmetric $n \times n$ matrix, and A an $m \times n$ matrix.

Problem 100.10. The (primal) quadratic minimization problem (QMP)

Find an $\bar{x}$, if it exists, such that

$$\frac{1}{2}\bar{x}\, c\bar{x} - b\bar{x} = \min_{x\in X}\left(\frac{1}{2}x\,cx - bx\right),$$

$$\bar{x} \in X = \{x : x \in R^n, Ax \leqq c\}.$$

The dual to the above problem is given by

$$\frac{1}{2}\hat{x}C\hat{x} - b\hat{x} + \hat{u}\left(A\hat{x} - \hat{c}\right) = \max_{(x,u)\in Y}\left[\frac{1}{2}x\,cx - bx + u\left(Ac - c\right)\right]$$

$$\left(\hat{x},\hat{u}\right) \in Y = \left\{(x,u) : x \in R^n, u \in R^m, cx - b + A'u = 0, u \geqq 0\right\}.$$

The constraint relation $Cx - b + A'u = 0$ implies that $x\,Cx - bx + u\,Ax = 0$.

Using this equation in the objective function, the dual problem becomes the following.

Problem 100.11. The quadratic dual (maximization) problem (QDP)

Find an $\hat{x} \in R^n$ and a $\hat{u} \in R^m$, if they exist, such that

$$-\frac{1}{2}\hat{x}C\hat{x} - c\hat{u} = \max_{(x,u)\in Y}\left[-\frac{1}{2}x\,Cx - cu\right]$$

where

$$\left(\hat{x},\hat{u}\right) \in Y = \left\{(x,u) : x \in R^n, u \in R^m, Cx + A'u = b, u \geqq 0\right\}.$$

Theorem 100.16. (Weak duality theorem)

Let C be positive semidefinite. Then

$$x^1 \in X, \left(x^2, u^2\right) \in Y \Rightarrow \frac{1}{2}x^1Cx^1 - bx^1 \geqq -\frac{1}{2}x^2Cx^2 - cu^2.$$

Proof. This theorem follows from the weak duality theorem. By observing that $\frac{1}{2}x\,Cx - bx$ is convex on R^n if the matrix C is positive semidefinite.

Theorem 100.17 (Dorn's duality theorem). Let C be positive semidefinite. If $\bar{x}$ solves QMP then $\bar{x}$ and some $\bar{u} \in R^m$ solve QDP and the two extrema are equal.

Proof. The theorem follows from Wolfe's duality theorem (Thorem 100.15) by observing that $\frac{1}{2}x\,Cx - bx$ is convex on R^n if the matrix C is positive semi definite.

Theorem 100.18. (Dorn's converse dualilty theorem).

Let C be positive semidefinite. If $(\hat{x}, \hat{u})$ solves *QDP* then some $\bar{x} \in R^n$ satisfying $c(\bar{x} - \hat{x}) = 0$, solves *QMP* and the two extreme are equal.

Proof. By Kuhn-Tucker stationary-point necessary optimality theorem and the linearity of the constraints $Cx + A'u = b,\ u \geqq 0$, there exists a $\hat{v} \in R^n$ such that $(\hat{x}, \hat{u}, \hat{v})$ satisfies the following Kuhn-Tucker conditions

$$-C\hat{x} + c\hat{v} = 0$$
$$C\hat{x} + A'\hat{v} = b$$
$$-c + A\hat{v} \leqq 0$$
$$-c\,\hat{u} + \hat{u}\,A\,\bar{v} = 0$$
$$\hat{u} \geqq 0.$$

Substitution of the first relation into the second one gives

$$C\hat{v} + A'\hat{u} = b.$$

The last four relations, the assumption that C is positive semidefinite, and Theorems 100.8 and Fritz John stationary-point necessary optimality theorem (Theorem 100.12) imply that $\hat{u}$ solves *QMP*. Hence $\bar{x} = \hat{v}$ solves *QMP* and $C\bar{x} = C\bar{v} = C\hat{x}$.

Now we show that the two extreme are equal.

$$\left(-b\bar{x} + \frac{1}{2}\bar{x}\,C\bar{x}\right) - \left(-c\,\bar{u} - \frac{1}{2}\hat{x}\,C\,\hat{x}\right)$$

$$= -b\hat{v} + \frac{1}{2}\hat{v}\,C\,\hat{v} + c\,\hat{u} + \frac{1}{2}\hat{x}\,C\,\hat{x} \text{ [since } \bar{x} = \hat{v}\text{]}$$

$$= -b\hat{v} + \hat{v}\,C\,\hat{x} + c\hat{u} \text{ [since } C\hat{x} = C\hat{v} \text{ and } C = C'\text{]}$$

$$= -\hat{u}\,A\,\hat{v} + c\hat{u} \text{ [since } C\hat{x} + A'\hat{u} = b\text{]}$$

$$= 0. \text{ [since } -c\hat{u} + \hat{u}\,A\,\hat{v} = 0\text{]}$$

Theorem 100.10. (Unbounded dual theorem)

Let $Y \neq \phi$. Then

$\langle X = \phi \rangle \Rightarrow \langle$*QDP* has an unbounded objective function from above on $Y\rangle$.

Theorem 100.20. (Theorem no primal minimum)

Let C be negative semidefinite and let $X \neq \phi$. Then $\langle Y \neq \phi \rangle \Rightarrow \langle QMP$ has no solution$\rangle$.

100.12. Module Summary

The main aim of this module is to derive optimality criteria in non-linear programming when differentiability involved and duality in non-linear programming. For this purpose, we have studied differentiable convex and concave functions. Two most important problems Fritz John and Kuhn-Tucker stationary-point problems are stated and deduced sufficient and necessary optimality criteria. We also defined and discussed duality in non-linear programming problem like linear programming problem.

100.13. Self Assessment Questions

1. Let θ be a numerical function defined on an open set $\Gamma \subset R^n$ and let θ be differentiable at $\bar{x} \in \Gamma$.

   (i) If θ is convex at $\bar{x} \in \Gamma$ then show that

   $\theta(x) - \theta(\bar{x}) \geqq \nabla\theta(\bar{x})(x - \bar{x})$ for each $x \in \Gamma$.

   (ii) If θ is concave at $\bar{x} \in \Gamma$ then show that

   $\theta(x) - \theta(\bar{x}) \leqq \nabla\theta(\bar{x})(x - \bar{x})$ for each $x \in \Gamma$.

2. Let θ be a numerical differentiable function on an open convex set $\Gamma \subset R^n$. Prove that θ is convex on Γ if and only if

   $\theta(x^2) - \theta(x^1) \geqq \nabla\theta(x^1)(x^2 - x^1)$ for each $x^1, x^2 \in \Gamma$ and θ is concave on Γ if and only if

   $\theta(x^2) - \theta(x^1) \leqq \nabla\theta(x^1)(x^2 - x^1)$ for each $x^1, x^2 \in \Gamma$.

3. Let θ be a numerical differentiable function on an open convex set $\Gamma \subset R^n$. Prove that a necessary and sufficient condition that θ be convex (concave) on Γ is that for each $x^1, x^2 \in \Gamma$.

   $\left[\nabla\theta(x^2) - \nabla\theta(x^1)\right](x^2 - x^1) \geqq 0 \qquad (\leqq 0).$

4. Let θ be a numerical function defined on an open set $\Gamma \subset R^n$ and let θ be differentiable at $\bar{x} \in \Gamma$. If θ is strictly convex at $\bar{x} \in \Gamma$, then show that

   $\theta(x) - \theta(\bar{x}) > \nabla\theta(\bar{x})(x - \bar{x})$ for each $x \in \Gamma, x \neq \bar{x}$.

Again, if θ is strictly concave at $\bar{x} \in \Gamma$, then show that

$$\theta(x) - \theta(\bar{x}) > \nabla\theta(\bar{x})(x - \bar{x}) \text{ for each } x \in \Gamma, x \neq \bar{x}.$$

5. State the following problems:

   (i) The Fritz John stationary-point problem

   (ii) The Kuhn-Tucker stationary-point problem.

6. Let $\bar{x} \in X^0$, X^0 be open and let θ and g be differentiable and convex at $\bar{x}$. If $(\bar{x}, \bar{u})$ is a solution of *KTP*, then $\bar{x}$ is a solution of *MP*. If $(\bar{x}, \bar{r}_0, \bar{r})$ is a solution of *FJP* and $\bar{r}_0 > 0$, then $\bar{x}$ is a solution of *MP*.

7. Let $\bar{x} \in X^0$, let X^0 be open, let θ be differentiable and convex at $\bar{x}$, and let g be differentiable and stricntly convex at $\bar{x}$. If $(\bar{x}, \bar{r}_0, \bar{r})$ is a solution of *FJP*, then $\bar{x}$ solves *MP*.

8. Let $\bar{x}$ be a solution of *LMP* or of *MP*, let X^0 be open and let θ and g be differentiable at $\bar{x}$. Then there exists an $\bar{r}_0 \in R$ and an $\bar{r} \in R^m$ such that $(\bar{x}, \bar{r}_0, \bar{r})$ solves *FJP* and $(\bar{r}_0, \bar{r}_w) \geq 0$, where

   $$W = \{i : g_i(\bar{x}) = 0 \text{ and } g_i \text{ is not concave at } \bar{x}\}.$$

9. State the following :

   (i) The Kuhn-Tucker constraint qualifications.

   (ii) The Arrow-Hurwicz-Uzawa constraint qualification.

   (iii) The reverse convex constraint qualification.

10. State and prove weak duality theorem in non-linear programming problem.

11. State and prove Wolfe's duality theorem in non-linear programming problem.

12. Define primal and its dual quadratic programming problems.

13. State and prove Dorn's duality theorem.

14. Let C be positive semidefinite. If $(\hat{x}, \hat{u})$ solves <u>*QDP*</u> then prove that some $\bar{x} \in R^n$, satisfying $C(\bar{x} - \hat{x}) = 0$ solves <u>*QMP*</u> and the two extreme are equal.

100.14. Reference

1. O.L. Mangasarian, Non-Linear Programming, Mc Graw Hill Publishers.

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"Learner's Feed-back"

After going through the Modules/ Units please answer the following questionnaire.
Cut the portion and send the same to the Directorate.

To
The Director
Directorate of Distance Education,
Vidyasagar University,
Midnapore- 721 102.

1. The modules are : (give ✓ in appropriate box)
   ☐ Easily understandable; ☐ very hard; ☐ partially understandable.
2. Write the number of the Modules/Units which are very difficult to understand :
   ....
   ....
   ....
3. Write the number of Modules/ Units which according to you should be re-written :
   ....
   ....
   ....
4. Which portion/page is not understandable to you? (Mention the page no. And Portion)
   ....
   ....
   ....
   ....
5. Write a short comment about the study material as a learner.
   ....
   ....
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