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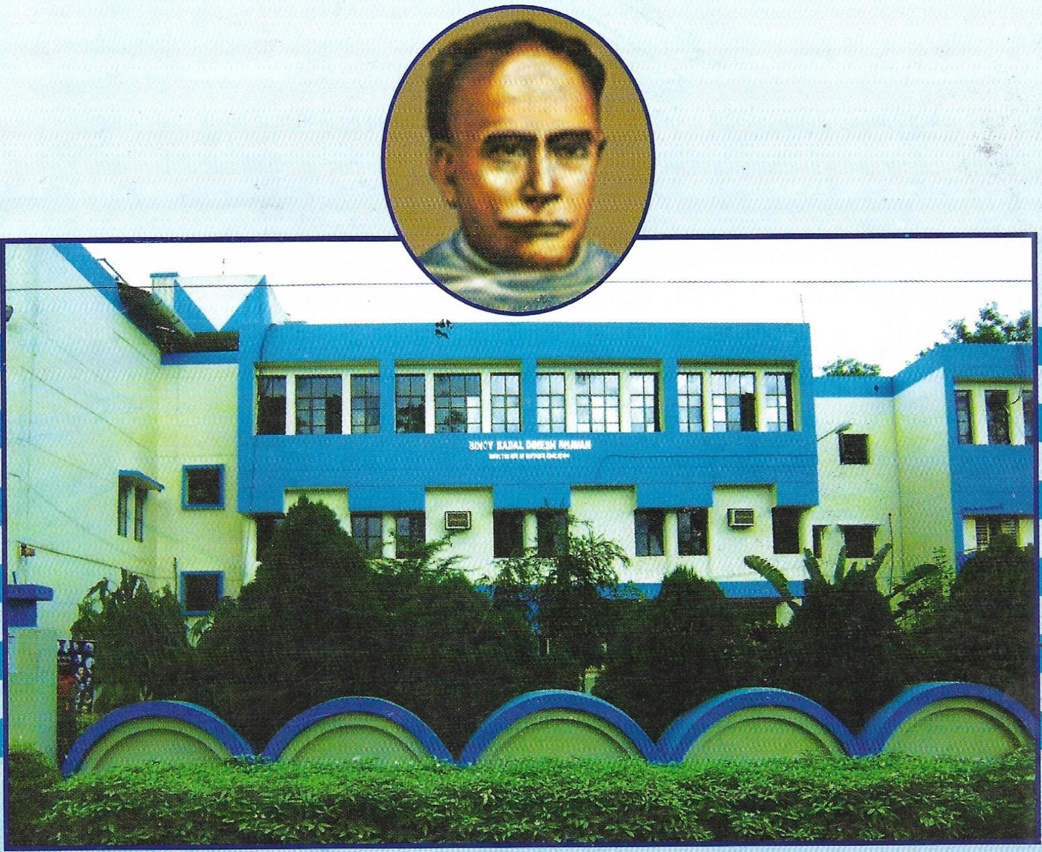


VIDYASAGAR UNIVERSITY
MIDNAPORE-721 102

M. Sc. in Mathematics
Part - II
Paper - VII
Module No. 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84

CONTENTS

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper-VII Group-B

Module No. - 73

Electromagnetic Theory

Module Structure :

❑ 73.1. Introduction :

Electromagnetic theory is related to the microscopic study of the behaviour of charges in motion and their interaction with matter. In this subject, the aim is to summarise various laws of electrostatics and magnetostatics along with a few mathematical concepts.

Vector analysis is a powerful mathematical machinery devised to handle the actual physical processes easily and rapidly. The equations of electrodynamics become more concise if written in vector notation and physical content becomes more clear.

The fundamental problem, electromagnetic theory hopes to solve is this : we have some electric charges $q_1, q_2, q_3 ...$ (let us call them **source charges**); what force do they exert on another charge Q (let us call this as **test** charge)? This positions of the source charges are given (as a function of time); the trajectory of the test particle is to be calculated. The solution to this problem is facilitated by the **principle of superposition,** which states that the interaction between any to charges is completely unaffected by the presence of others. Force on a test charge can be found using **Coulomb's law.** Idea of potential and field play an important role in electromagnetic theory.

Electric field $\vec{E}$ is a very special kind of vector function, whose curl is always zero. As $\nabla \times \vec{E} = 0$, the line integral of $\vec{E}$ around any closed loop is zero (that follows from Stoke's theorem). Because the line integral is independent of path, we can define a function

$$\phi(r) = -\int_0^r \vec{E} \cdot \overline{dl}$$

Here, ϕ depends only or $\vec{r}$ and is called the **electric potential.**

The relation between $\vec{E}$ and ϕ is

$$\vec{E} = -\nabla\phi$$

Like force, potential obeys the superposition principle.

The electrostatic problems can be solved by the following methods :

(i) The method of electrical images.

(ii) The method involving superposition of fictitious charge.

(iii) The method of separation of variables (Laplace and Poisson's equations).

❍ Poisson's equation :

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

❍ Laplace's equation :

$$\nabla^2 \phi = 0$$

❑ 73.2. Objectives :

After going through this module we shall be able to know the following mattters

❍ Potentials and conservative force.

❍ Potential and field due to a dipole; dipole-dipole interaction.

❍ Method of images.

❍ Maxwell's stress tensor.

❑ 73.3. Some Basic Equations :

♦ VECTOR DERIVATIVES

❍ **Cartesian.** $\vec{dl} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \quad d\tau = dx\,dy\,dz$

$$\textit{Gradient}: \ \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}$$

$$\textit{Divergence}: \ \nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\textit{Curl}: \ \nabla\times\vec{A} = \left(\frac{\partial A_z}{\partial y} + \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} + \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y}\right)\hat{k}$$

$$\textit{Laplacian}: \ \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

❍ **Spherical.** $\vec{dl} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\textit{Gradient}: \ \nabla\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\phi}\hat{\phi}$$

$$\textit{Divergence}: \ \nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\text{Curl}: \nabla\times\vec{A}=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_{\phi}\right)-\frac{\partial A_{\theta}}{\partial\phi}\right]\hat{r}$$

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}\left(rA_{\phi}\right)\right]\hat{\phi}+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_{\theta}\right)-\frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

$$\text{Laplacian}: \nabla^2\phi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\phi^2}$$

❍ **Cylindrical.** $\overline{dl}=ds\,\hat{s}+s\,d\phi\,\hat{\phi}+dz\,\hat{z};\ d\tau=s\,ds\,d\phi\,dz$

$$\text{Gradient}: \nabla\phi=\frac{\partial\phi}{\partial s}\hat{s}+\frac{1}{s}\frac{\partial\phi}{\partial\phi}\hat{\phi}+\frac{\partial\phi}{\partial z}\hat{z}$$

$$\text{Divergence}: \nabla\cdot\vec{A}=\frac{1}{s}\frac{\partial}{\partial s}(sA_s)+\frac{1}{s}\frac{\partial A_{\phi}}{\partial\phi}+\frac{\partial A_z}{\partial z}$$

$$\text{Curl}: \nabla\cdot\vec{A}=\left[\frac{1}{s}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_{\phi}}{\partial z}\right]\hat{s}+\left[\frac{\partial A_s}{\partial z}-\frac{\partial A_z}{\partial s}\right]\hat{\phi}+\frac{1}{s}\left[\frac{\partial}{\partial s}\left(sA_{\phi}\right)-\frac{\partial A_s}{\partial\phi}\right]\hat{z}$$

$$\text{Laplacian}: \nabla^2\phi=\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\phi}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2\phi}{\partial\phi^2}+\frac{\partial^2\phi}{\partial z^2}$$

♦ VECTOR IDENTITIES

❍ **Triple Products**

(1) $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\vec{B}\cdot\left(\vec{C}\times\vec{A}\right)=\vec{C}\left(\vec{A}\times\vec{B}\right)$

(2) $\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right)$

❍ **Product Rules**

(3) $\nabla(fg)=f(\nabla g)+g(\nabla f)$

(4) $\nabla\left(\vec{A}\cdot\vec{B}\right)=\vec{A}\times\left(\nabla\times\vec{B}\right)+\vec{B}\times\left(\nabla\times\vec{A}\right)+\left(\vec{A}\cdot\nabla\right)\vec{B}+\left(\vec{B}\cdot\nabla\right)\vec{A}$

(5) $\nabla\cdot\left(f\vec{A}\right)=f\left(\nabla\cdot\vec{A}\right)+\vec{A}\cdot(\nabla f)$

(6) $\nabla\cdot\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot\left(\nabla\times\vec{A}\right)-\vec{A}\cdot\left(\nabla\times\vec{B}\right)$

(7) $\nabla\times(f\vec{A})=f(\nabla\times\vec{A})-\vec{A}\times(\nabla f)$

(8) $\nabla\times(\vec{A}\times\vec{B})=(\vec{B}\cdot\nabla)\vec{A}-(\vec{A}\cdot\nabla)\vec{B}+\vec{A}(\nabla\cdot\vec{B})-\vec{B}(\nabla\cdot\vec{A})$

❍ **Second Derivatives**

(9) $\nabla\cdot(\nabla\times\vec{A})=0$

(10) $\nabla\times(\nabla f)=0$

(11) $\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$

♦ FUNDAMENTAL THEOREMS

❍ **Gradient Theorem :** $\int_a^b(\nabla f)\cdot\overline{dl}=f(b)-f(a)$

❍ **Divergence Theorem (Gauss's Theorem) :** $\int(\nabla\cdot\vec{A})d\tau=\oint\vec{A}\cdot\overline{ds}$.

❍ **Curl Theorem (Stoke's Theorem) :** $\int(\nabla\times\vec{A})\cdot\overline{ds}=\oint\vec{A}\cdot\overline{dl}$

♦ BASIC EQUATIONS OF ELECTRODYNAMICS

❍ **Maxwell's Equations**

In general

$$\begin{cases}\nabla\cdot\vec{E}=\dfrac{1}{\epsilon_0}\rho\\ \nabla\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}\\ \nabla\cdot\vec{B}=0\\ \nabla\times\vec{B}=\mu_0\vec{J}+\mu_0\epsilon_0\dfrac{\partial\vec{E}}{\partial t}\end{cases}$$

In matter :

$$\begin{cases}\nabla\cdot\vec{D}=\rho f\\ \nabla\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}\\ \nabla\cdot\vec{B}=0\\ \nabla\times\vec{H}=\vec{J}_c+\dfrac{\partial\vec{D}}{\partial t}\end{cases}$$

❍ **Auxiliary Fields**

Definitions :

$$\begin{cases}\vec{D}=\epsilon_0\vec{E}+\vec{P}\\ \vec{H}=\dfrac{1}{\mu_0}\vec{B}-\vec{M}\end{cases}$$

Linear media :

$$\begin{cases}\vec{P}=\epsilon_0\chi_e\vec{E},\ \vec{D}=\epsilon\vec{E}\\ \vec{M}=\chi_m\vec{H},\vec{H}=\dfrac{1}{\mu}\vec{B}\end{cases}$$

○ **Potentials** $\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$, $\vec{B} = \nabla\times\vec{A}$

○ **Lorentz force law** $\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$

○ **Energy, Momentum, and Power**

Energy : $U = \frac{1}{2}\int\left(\epsilon_0\, E^2 + \frac{1}{\mu_0}B^2\right)d\tau$

Momentum : $\vec{P} = \epsilon_0\int\left(\vec{E}\times\vec{B}\right)d\tau$

Poynting vector : $\vec{N} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)$

Larmor formula : $W = \frac{\mu_0}{6\pi c}q^2 a^2$

♦ **FUNDAMENTAL CONSTANTS**

$\epsilon_0 = 8.85\times10^{-12}C^2/Nm^2$	(permitivity of of free space)
$\mu_0 = 4\pi\times10^{-7}N/A^2$	(permeability of free space)
$c = 3.00\times10^{8}m/s$	(speed of light in vacuum)
$e = 1.60\times10^{-19}C$	(charge of the electron)
$m = 9.11\times10^{-31}kg$	(mass of the electron)

♦ **SPHERICAL AND CYLINDRICAL COORDINATES**

○ **Spherical**

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\cos\phi \\ z = r\cos\theta \end{cases}$$

$$\begin{cases} \hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} \\ \hat{y} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} - \cos\phi\,\hat{\phi} \\ \hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2 + y^2/z}\right) \\ \phi = \tan^{-1}(y/x) \end{cases}$$

$$\begin{cases} \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \\ \hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \\ \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{cases}$$

❍ **Cylindrical**

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{x} = \cos\phi\,\hat{s} - \sin\phi\,\hat{\phi} \\ \hat{y} = \sin\phi\,\hat{s} - \cos\phi\,\hat{\phi} \\ \hat{z} = \hat{z} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{s} = \cos\phi\,\hat{x} - \sin\phi\,\hat{y} \\ \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \\ \hat{z} = \hat{z} \end{cases}$$

❑ 73.4. Conservative Force

A force is said to be conservative if the work done by it in moving a charged particle from one point to another point depends only on these points and not on the path followed.

The region in which a charged particle experiences a conservative force is called a **conservative force field.** The **central force** is the example of a conservative force. It is mentioned that, if a force act on a charged particle in such a way that it is always directed towards or away from a point and its magnitude depends only upon the distance (r) from the point, then the force is called a **central force.** Gravitational, electrostatic, elastic forces are central forces and hence conservative forces.

(i) **For conservative forces, the work done around any closed path is zero.**

$$\oint \vec{F}\cdot\vec{dr} = 0$$

(ii) **The conservative force is the negative gradient of potential energy.**

$$\vec{F} = -\nabla U \quad U = \text{Potential energy.}$$

Potential energy (U) of a particle at a point $\vec{r}$ is defined as the amount of work done by an applied force in moving the charged particle from infinity to that point.

(iii) **The curl of a conservative force vanishes**

$$\text{Curl}\,\vec{F} = 0$$

(iv) The total energy of a particle remains constant in a conservative force field.

◆ **Conservation Laws :**

Conservation laws are very powerful tools for solving the problems in physical sciences. A member of conservation laws are known to us; familiar examples are the laws of conservation of energy, linear momentum, angular momentum, charge etc. These laws are usually the consequence of some underlying symmetry in the universe.

The advantages of the conservation laws are :

(i) Conservation laws do not depend on the details of the trajectory of a particle and often, on the details of the force involved. Hence, these laws enable us to state very useful and general consequences of equation of motion.

(ii) These laws have been used in the science of elementary particles (electron, proton, meson etc.) even when the forces involved are not known. On the basis of these laws, some new elementary particles have also been predicted.

(iii) Conservation laws are the most striking physical facts which have been used in the tackling of new and not understood problems.

(iv) Some times conservation laws predict with certainty that a particular phenomena will not occur :

Example : γ-ray photon cannot create 'electron-positron pair' in vacuum. A γ-ray photon of energy $h\gamma$ possesses a momentum $\left(\frac{h\gamma}{c}\right)$. In case, if the pair production in possible, then the momentum $\frac{h\gamma}{c}$ must be shared in equal amounts by electron and positron. Now the total energy of the pair is $2\sqrt{m_0^2c^4 + p^2c^2} = 2\sqrt{m_0^2c^2 + \left(\frac{h\gamma}{2c}\right)^2 c^2}$

$= \sqrt{4m_0^2c^4 + (h\gamma)^2}$ which is greater than $h\gamma$. This violates the law of conservation of energy and hence pair production by a γ-ray photon is impossible in vacuum.

(v) A conservation law may be used conveniently in solving the problems for the motion of a particle, even if we know the force involved exactly.

For solving a problem, first we use the conservation laws and if there remains anything to the problem, we make use of other methods of calculations.

❑ 73.5. Potential Energy of a System of Charges :

The potential energy of a system of two particles is usually defined as the amount of workdone by an external agent to assemble the system, starting from infinite separation. Thus if there are two charges q_1 and q_2 at a distance r_{12} apart, then the electrostatic potential energy of the system is,

$$U_{12} = -\int_{\alpha}^{r_{12}} \vec{F} \cdot \overline{dr} = -\int_{\alpha}^{r_{12}} \frac{q_1 q_2}{4\pi \in_0 r^2} dr = \frac{q_1 q_2}{4\pi \in_0 r_{12}} \left[\because F = \frac{q_1 q_2}{4\pi \in_0 r_2} = \text{Coulomb (conservative) force.} \right]$$

Now, let we bring a third charge q_3 from infinity at a position P, whose distance from the charge q_1 is r_{31} and from the charge q_2 is r_{32}. The amount of work done i.e. the increase in the P.E. of the system is

$$= -\int_{\alpha}^{P} \vec{F}_3 \cdot \overline{dr} \text{ where } \vec{F}_3 \text{ is the Coulomb-force on } q_3.$$

As the electrical interactions are additive, so we can write,

$$\vec{F}_3 = F_{31} + F_{32}$$

or. $$-\int_{\alpha}^{P} \vec{F} \cdot \overline{dr} = -\int \vec{F}_{31} \cdot \overline{dr} - \int \vec{F}_{32} \cdot \overline{dr}$$

$$= -\int_{\alpha}^{r_{31}} \frac{q_1 q_2}{4\pi \in_0 r^2} dr - \int_{\alpha}^{r_{31}} \frac{q_1 q_2}{4\pi \in_0 r} dr$$

$$= \frac{q_1 q_3}{4\pi \in_0 r_{31}} + \frac{q_2 q_3}{4\pi \in_0 r_{32}}$$

$$= U_{13} + U_{23}$$

∴ P.E. of the system of three charges.

$$U = \frac{q_1 q_2}{4\pi \in_0 r_{12}} + \frac{q_1 q_3}{4\pi \in_0 r_{13}} + \frac{q_2 q_3}{4\pi \in_0 r_{23}}.$$

$$= U_{12} + U_{13} + U_{23}$$

∴ For a system of n charges

$$U = \sum_{r \neq j} \frac{q_i q_j}{4\pi \in_0 r_{ij}}$$

or, $$U=\sum_{i=1}^{i=n}\sum_{j=1}^{j=n-1}\frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$$

or, $$U=\frac{1}{2}\sum_{i=1}^{i=n}\sum_{\substack{j=1\\ j\neq i}}^{j=n}\frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}.$$

❑ 73.6. Electric Potential

The potential of an electric field at a point is defined as the amount of work done in moving a unit charge from infinity to that point. In other words, *the electric potential is the potential energy of a unit charge placed at that point.* Potential is assumed to be zero at infinity.

$$\therefore \text{ Potential. } \phi=\frac{U(r)}{q}=-\int_{\alpha}^{r}\vec{E}\cdot\vec{dr}=\int_{r}^{\alpha}\vec{E}\cdot\vec{dr}$$

where $U(r)$ is the potential energy of charge q at the concerned point and $\vec{E}$ the intensity of electric field at the same point.

The intensity at a distance $\vec{r}$ from a charge Q is defined as

$$\vec{E}=\frac{q\vec{r}}{4\pi\epsilon_0 r^3}$$

$$\therefore \phi=-\int_{\alpha}^{r}\frac{q\vec{r}}{4\pi\epsilon_0 r^3}\cdot\vec{dr}=\frac{q}{4\pi\epsilon_0 r}.$$

This is the expression of the potential at a point r due to a point charge q.

Invoking the superposition principle, the potential of a collection of charges is,

$$\phi(r)=\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$$

For a continuous distribution,

$$\phi(r)=\frac{1}{4\pi\epsilon_0}\int\frac{dq}{r}$$

In particular, for a volume charge, it is,

$$\phi(r)=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(r')d\tau'}{r}$$

Here, ρ = volume density of charge.

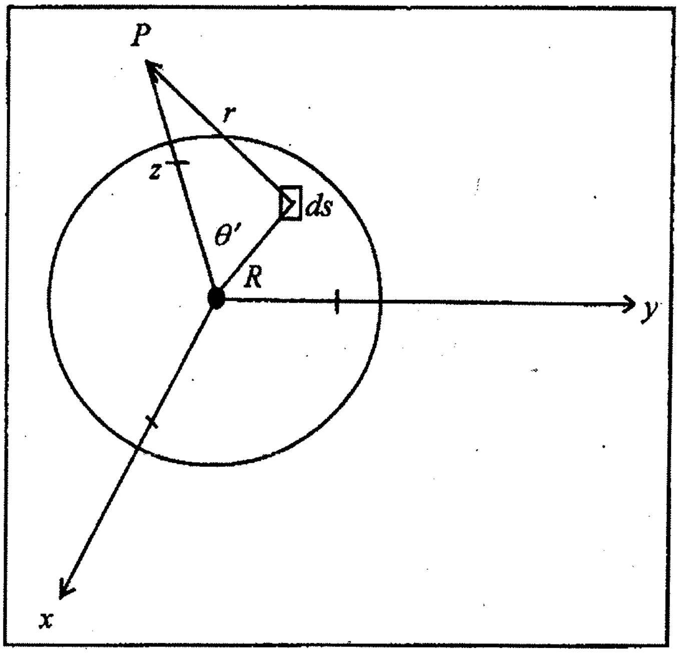
Field $\vec{E}(r) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(r')}{r^3} \vec{r}\, d\tau'$

The potentials of line and surface charges are

$$\frac{1}{4\pi \epsilon_0} \int \frac{\lambda(r')dl'}{r} \text{ and } \frac{1}{4\pi \epsilon_0} \int \frac{(r')dsl}{r}$$

λ-line charge density; σ-surface density of charges.

Example : Find the potential of a uniformly charged spherical shell of radius *R*.



Solution : We know, potential

$$\phi(r) = \frac{1}{4\pi \epsilon_0} \int \frac{\sigma ds}{r}$$

where σ=surface density of charge.

Let us set the point *r* on the *z*-axis and use the law of cosines to express *r* in terms of the polar angle θ';

$$r^2 = R^2 + Z^2 - 2Rz\cos\theta'$$

An element of surface area on this sphere is $R^2 \sin\theta' d\theta' d\phi'$, so

$$4\pi \epsilon_0 \phi(z) = \sigma \int \frac{R^2 \sin q' d\theta' d\phi'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}$$

$$= 4\pi R^2 \sigma \int_0^\pi \frac{\sin\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}$$

$$= 2\pi R^2 \sigma \left(\frac{1}{RZ} \sqrt{R^2 + z^2 - 2Rz\cos\theta'} \right) \Bigg|_0^\pi$$

$$=\frac{2\pi R\sigma}{z}\left(\sqrt{R^2+z^2+2Rz}-\sqrt{R^2+z^2-2Rz}\right)$$

$$=\frac{2\pi R\sigma}{z}\left[\sqrt{(R+z)^2}-\sqrt{(R-z)^2}\right]$$

For points outside the sphere z is greater than R and hence $\sqrt{(R-z)^2}=(z-R)$; for points inside the sphere, $\sqrt{(R-z)^2}=(R-z)$.

Thus

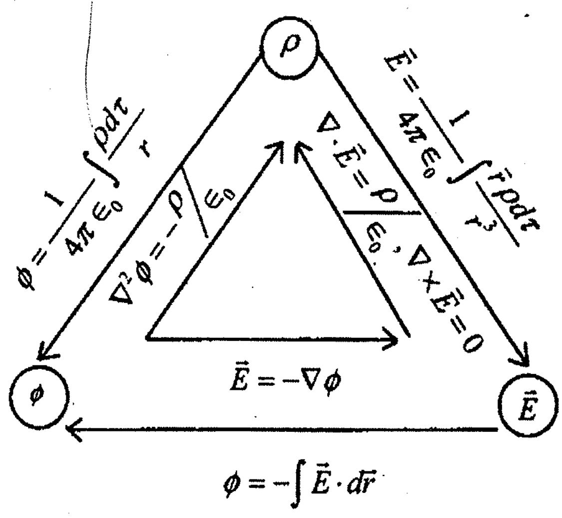
$$\phi(z)=\frac{R\sigma}{2\epsilon_0 z}\left[(R+z)-(z-R)\right]=\frac{R^2\sigma}{\epsilon_0 z}\text{ : outside.}$$

$$\phi(z)=\frac{R\sigma}{2\epsilon_0 z}\left[(R+z)-(R-z)\right]=\frac{R\sigma}{\epsilon_0}\text{ : inside.}$$

In terms of the total charge on the shell, $q=4\pi R^2\sigma;\phi(z)=\frac{q}{4\pi\epsilon_0 z}$ (or, in general $\phi(r)=\frac{q}{4\pi\epsilon_0 r}$ for points outside the sphere and $\frac{q}{4\pi\epsilon_0 R}$ for points inside.)

❑ 73.7. Electrostatic Boundary Conditions :

There are three fundamental quantities of electrostatics : $\rho, \vec{E}$ and ϕ. The formulas interrelating the above mentioned quantities are shown in the following fig.



The boundary conditions in electrostatics are :

1. *The tangential component of $\vec{E}$ is always continuous across the interface. i.e.*

$$E_{t_1} = E_{t_2}$$

The normal component of $\vec{E}$ is in discontinuous by an amount $\frac{\sigma}{\epsilon_0}$ at any boundary.

2. The potential is continuous across any boundary i.e.

$$\phi_1 = \phi_2$$

However the gradient of ϕ inherits the discontinuity in $\vec{E}$; since $\vec{E} = -\nabla\phi$

$\therefore E_{n_1} - E_{n_2} = \frac{\sigma}{\epsilon_0}\hat{n}$; $\hat{n}$ is the unit vector perpendicular to the surface.

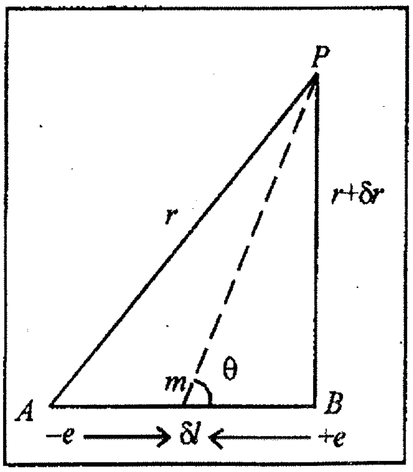
or, $\nabla\phi_1 - \nabla\phi_2 = -\frac{\sigma\hat{n}}{\epsilon_0}$

or, $\frac{\partial\phi_1}{\partial n} - \frac{\partial\phi_2}{\partial n} = -\frac{\sigma}{\epsilon_0}$

where $\frac{\partial\phi}{\partial n} = \phi\cdot\hat{n} \rightarrow$ denotes the normal derivative of ϕ, that is, the rate of change in the direction perpendicular to the surface.

❑ 73.8. Electric Dipole or Electric Doublet :

♦ **The electric dipole** (*or electric doubet or electric bipole*) : *A combination of two equal and opposite point charges −e and +e, such that distance δl between them is infinitely small and e infinitely great such that e. δl is finite, is called an electric doublet.*



♦ **Strength of a doublet :** *The strength of an electric doublet is defined as the limiting value of the product e δl such that when $e \rightarrow \infty$, e. δl is finite.*

The strength of a doublet is usually called the moment of the doublet and is denoted by $\vec{m}$. In fact the moment of a dipole is vector $\vec{m}$ whose magnitude is m i.e. $m = e\,\delta l$. The axis of the doublet is in the sense along the line joining the negative charge to the positive charge.

❑ 73.8.1. Potential of the field produced by a small doublet :

Let us consider an electric doublet AB composed of two point charges $-e$ at A and $+e$ at B where $AB = \delta l$ and $\lim_{\delta l \to 0} e\,\overrightarrow{\delta l} = \vec{m}$. Take P as origin and $\overrightarrow{PA} = \vec{r}$; then the scalar potential at P due to $-e$ at A and $+e$ at B.

$$\phi_P = -\frac{1}{4\pi \epsilon_0}\left[\frac{e}{PA} + \frac{e}{PB}\right] = \frac{e}{4\pi \epsilon_0}\left(\frac{1}{PB} - \frac{1}{PA}\right)$$

$$\phi_P = \frac{1}{4\pi \epsilon_0}\left[-\frac{e}{PA} + \frac{e}{PB}\right] = \frac{e}{4\pi \epsilon_0}\left(\frac{1}{PB} - \frac{1}{PA}\right)$$

$$= \frac{e}{4\pi \epsilon_0}\left[\text{increase in } \frac{1}{r} \text{ from } A \text{ to } B\right]$$

$$= \frac{e}{4\pi \epsilon_0}\left[\text{differential of } \frac{1}{r} \text{ w.r.t. the coordinate of } A \text{ keeping } P \text{ as fixed}\right]$$

$$= \frac{e}{4\pi \epsilon_0}\left[\overline{\delta l} \cdot \text{grad}_A\left(\frac{1}{r}\right)\right] \qquad (\because d\phi = dr.\text{grad}\,\phi)$$

$$= \frac{e}{4\pi \epsilon_0}\overline{\delta l} \cdot \nabla_A\left(\frac{1}{r}\right)$$

$$= \frac{e}{4\pi \epsilon_0}\overline{\delta l} \cdot \nabla_P\left(\frac{1}{r}\right) \qquad \left[\because \nabla_A\left(\frac{1}{r}\right) = -\nabla_P\left(\frac{1}{r}\right)\right].$$

Proceeding to limit as $\delta l \to 0$, we have

$$\phi_P = \lim_{\delta l \to 0} \frac{e}{4\pi \epsilon_0}\overline{\delta l} \cdot \nabla_P\left(\frac{1}{r}\right)$$

$$= -\frac{\vec{m}}{4\pi \epsilon_0} \cdot \nabla_P\left(\frac{1}{r}\right)$$

$$= -\frac{\vec{m}}{4\pi \epsilon_0} \cdot \left(-r^{-1-2}r\right) \qquad \left(\because \nabla r^m = m r^{m-2} r\right)$$

$$= -\frac{\vec{m}\cdot\vec{r}}{4\pi\in_0 r^3}\cdot\left(-r^{-1-2}r\right) \qquad \text{where } r = \overline{PA}$$

$$= \frac{m\cos\theta}{4\pi\in_0 r^2}.$$

Note. If there are a number of doublets then the poetntial at P due to all these doublets is

$$\phi_P = \sum_{i=1}^{n}\frac{\vec{m}_i\cdot\vec{r}_i}{4\pi\in_0 r_i^3}$$

Example : *Define an electric dipole. Show that the two electric dipoles of moments m_1, m_2 centred at the same point are equivalent to a single dipole at this point of moment m_1+m_2.*

Hint. Since with the vector law of addition

$$\frac{\vec{m}_1\cdot\vec{r}}{4\pi\in_0 r^3}+\frac{\vec{m}_2\cdot\vec{r}}{4\pi\in_0 r^3}=\frac{(\vec{m}_1+\vec{m}_2)\cdot\vec{r}}{4\pi\in_0 r^3},$$

hence the result.

❑ **73.8.2. Electric field due to a doublet.** Let the electric doublet be situated at the origin; then the electric field vector E at P is given by :

$$\vec{E} = -\text{grad}_P\,\phi$$

$$= -\nabla_P\left[\frac{1}{4\pi\in_0 r^3}(\vec{m}\cdot\vec{r})\right]$$

$$= \frac{1}{4\pi\in_0 r^3}\nabla_P(\vec{m}\cdot\vec{r})-(\vec{m}\cdot\vec{r})\nabla_P\left(\frac{1}{r^3}\right)$$

$$= \frac{1}{4\pi\in_0}\left[-\frac{\vec{m}}{r^3}-(\vec{m}\cdot\vec{r})\left(-3r^{-5}r\right)\right]$$

$$\left[\nabla_P(\vec{m}\cdot\vec{r})=\vec{m}\right],$$

$$\because (\vec{m}\cdot\vec{r}) = (\vec{r}\cdot\nabla)\vec{m}+(\vec{m}\cdot\nabla)\vec{r}+\vec{r}\times(\nabla\times\vec{m})+\vec{m}\times(\nabla\times\vec{r})$$

$$= 0+\left\{m\vec{k}\cdot\nabla\right\}\vec{r}+0+0$$ (suppose the axis of doublet is along z-axis; $\because \vec{m} = m\vec{k}$)

$$= m\frac{\partial}{\partial z}\left(z\vec{k}\right)$$

$= m\vec{k} = \vec{m}]$.

Therefore $\vec{E} - \frac{1}{4\pi \in_0}\left[\frac{m}{r^3} + \frac{3(m \cdot r)}{r^5} r \cdot\right]$

Component of electric field along r and θ. Let E_r and E_0 be the component of electric field along the radius vector and in the direction of θ increasing; then

$$E_r = -\frac{\partial \phi}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{m\cos\theta}{4\pi \in_0 r^2}\right) = \frac{2m\cos\theta}{4\pi \in_0 r^3}$$

$$E_\theta = -\frac{\partial \phi}{r\,\partial g} = -\frac{\partial}{r\,\partial \theta}\left(\frac{m\cos\theta}{4\pi \in_0 r^2}\right) = \frac{m\sin\theta}{4\pi \in_0 r^3}.$$

The resultant field is $= \sqrt{\left[(E_r)^2 + (E_\theta)^2\right]}$

$$= \frac{m}{4\pi \in_0 r^3}\sqrt{\left(1 + 3\cos^2\theta\right)}.$$

The direction of the resultant with radius vector

$$\left(\frac{E_\theta}{E_r}\right) = \tan^{-1}\left(\frac{1}{2}\tan\theta\right).$$

The differential equations of lines of force are given by

$$\frac{dr}{E_r} = \frac{r\,d\theta}{E_\theta}$$

i.e. $$\frac{dr}{r} = \frac{2\cos\theta\, d\theta}{\sin\theta}$$

which on integration yields

$$\log r = 2\log\sin\theta + \log c$$

c is arbitrary constant.

$$r = c\sin^2\theta.$$

❑ 73.8.3. Potential energy of an electric doublet in a given field :

Let the doublet be placed in a given electrostatic field E; if ϕ is the potential of the field at A, then the potential at B must be $\phi + d\phi$.

or; $\phi + d\phi = \phi + \overrightarrow{AB}\cdot(\nabla_A \phi)$

where the potential energy of a doublet is the amount of work done against the field in placing the doublet in the assigned position.

But work done = change in to potential. Therefore the poetntial energy must be given by :

$$W = -e\phi + e\left(\phi + \overrightarrow{AB}\cdot\nabla_A\phi\right)$$

$$= e\overrightarrow{AB}\cdot\nabla_A\phi$$

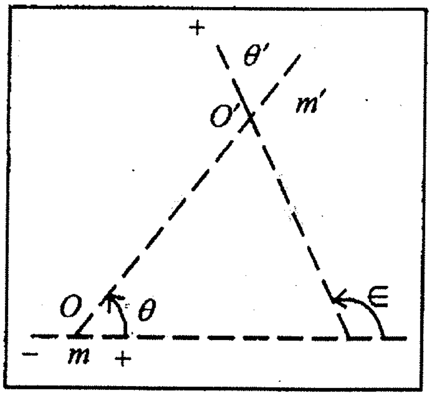
$$= \vec{m}\cdot\nabla_A\phi. \quad \left(\because \overrightarrow{AB} = \vec{m}\right)$$

$$= -\vec{m}\cdot\vec{E}, \quad \left(\because \vec{E} = -\text{grad}\,\phi\right)$$

E being the intensity of the field at A.

❑ 73.8.4. Mutual potential energy of two doubltets (when the doublets are non-coplanar) : Dipole-Dipole interaction :

Let $\vec{m}$ and $\vec{m}'$ be the moments of two doublets and r the distance between their cenres O and O', let θ and θ' be the angles which the line joining their centres makes with positive direction of axes of doublets and $\in$ the angle between the axes of doublets.



The potential energy of the doublet m',

$$W = -m'\cdot(\text{the field due to } \vec{m} \text{ at } O')$$

$$= -\vec{m}\cdot\frac{1}{4\pi\in_0}\left\{-\frac{\vec{m}}{r^3} + \frac{3(\vec{m}\cdot\vec{r})}{r^5}r\right\}$$

$$= \frac{1}{4\pi\in_0}\left[\frac{\vec{m}\cdot\vec{m}'}{r^3} - \frac{3(\vec{m}\cdot\vec{r})(\vec{m}'\cdot\vec{r})}{r^5}\right] \qquad \text{...............(1)}$$

$$= \frac{1}{4\pi\in_0}\left[\frac{mm'\cos\in}{r^3} - \frac{3(mr\cos\theta)(m'r\cos\theta')}{r^5}\right]$$

$$= \frac{mm'}{4\pi\in_0 r^3}\cos\in - \frac{3mm'}{r^3}\cos\theta\cos\theta'$$

$$= \frac{mm'}{4\pi\in_0 r^3}(\cos\in - 3\cos\theta\cos\theta') \qquad \text{...............(2)}$$

Note. *The potential energy of $\vec{m}$ when placed in the field of $\vec{m}'$ is obtained by interchanging $\vec{m}$ and m, we see that the result remains unaltered.*]

When the doublets are coplanar, *i.e.* when the axes of the two doublets lie in the same plane,

$\in = (\theta' - \theta)$ from the given figure.

We then have $W = \dfrac{mm'}{4\pi \in_0 r^3}\left[\cos(\theta'-\theta) - 3\cos\theta\cos\theta'\right]$

$$= \frac{mm'}{4\pi \in_0 r^3}\left[\sin\theta\sin\theta' - 2\cos\theta\cos\theta'\right] \qquad \text{...............(3)}$$

❑ **73.8.5. Couple exerted by one doublet on another doublet.**

Let us first find a couple in a field E due to a doublet. If $\vec{E}$ be the intensity of field at A, then $\vec{E} + \overrightarrow{dE}$ will be the intensity of the field at B.

The force on the charge $-e = -e\vec{E}$.

The force on the charge $+e = +e\left(\vec{E} + \delta\vec{E}\right)$.

$$= e\left[\vec{E} + \overrightarrow{AB}\cdot(\nabla_A E)\right].$$

Therefore the resultant force on the doublet is

$$\vec{F} = -e\vec{E} + e\left[\vec{E} + \left\{\overrightarrow{AB}\cdot(\nabla_A E)\right\}\right]$$

$$= e\left[\overrightarrow{AB}\cdot(\nabla_A E)\right].$$

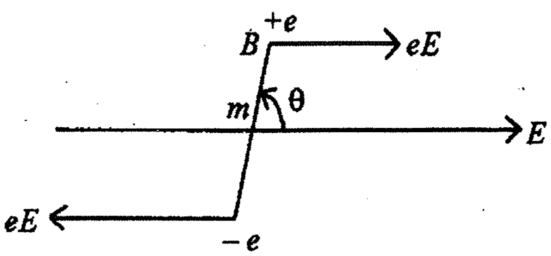
If the doublet is free to rotate in the field $\vec{E}$, the forces $-e\vec{E}$ at A and $+eE$ at B constitute a couple G of moment = (force) (perpendicular distance between forces) = $(AB\sin\theta)(eE)$

$$= \left(\overrightarrow{AB}\right)\times\left(e\vec{E}\right)$$

$$= e\overrightarrow{AB}\times\vec{E}$$

$$= \vec{m}\times\vec{E},$$

whose axis is perpendicular to $\vec{m}$ and $\vec{E}$ both.



Therefore the couple exerted by $\vec{m}'$ on $\vec{m}$

$$= \vec{m}\times(\text{field due to } \vec{m}')$$

$$= \vec{m}\times\left[\frac{1}{4\pi\in_0}\left\{\frac{\vec{m}'}{r^3} + \frac{3(\vec{m}\cdot\vec{r})}{r^5}\vec{r}\right\}\right]$$

$$= \vec{m} \times \frac{1}{4\pi \in_0} \left[\frac{3(\vec{m} \cdot \vec{r})}{r^5} \vec{r} - \frac{\vec{m}'}{r^3} \right]$$

$$= \frac{1}{4\pi \in_0} \left[\frac{3(\vec{m} \times \vec{r})(\vec{m}' \cdot \vec{r})}{r^5} - \frac{(\vec{m} \times \vec{m}')}{r^3} \right]$$

Similarly couple exerted by $\vec{m}$ on $\vec{m}'$

$$= \vec{m} \times \frac{1}{4\pi \in_0} \left[\frac{3(\vec{m} \cdot \vec{r})}{r^5} \vec{r} - \frac{\vec{m}'}{r^3} \right]$$

$$= \frac{1}{4\pi \in_0} \left[\frac{3(\vec{m} \times \vec{r})(\vec{m}' \cdot \vec{r})}{r^5} - \frac{(\vec{m} \times \vec{m}')}{r^3} \right]$$

❑ 73.8.6. Force exerted by a doublet on another doublet :

Here we shall calculate force exerted by doublet $\vec{m}$ on $\vec{m}'$. Let $\vec{F}$ be the required force. If $\vec{m}'$ undergoes a small displacement $\overline{\delta r}$ in which the axis of doublet remains parallel to itself, then the work done in this displacement must be equal to the increase in the potential energy of $\vec{m}'$.

Hence $\vec{F} \cdot \overline{\delta r} = -\delta W = -\delta \frac{1}{4\pi \in_0} \left[\frac{\vec{m} \cdot \vec{m}'}{r^3} - 3(\vec{m} \cdot \vec{r}) \frac{(\vec{m} \cdot \vec{r})}{r^5} \right]$

Since $\vec{m}$ and $\vec{m}'$ are constant vectors, we have

$$\vec{F} \cdot \overline{\delta r} = \frac{3(\vec{m} \cdot \vec{m}')}{4\pi \in_0 r^5} \delta r + \frac{3(\vec{m} \cdot \overline{\delta r})(\vec{m}' \cdot \vec{r}) + 3(\vec{m} \cdot \vec{r})(\vec{m}' \cdot \overline{\delta r})}{4\pi \in_0 r^5} + \frac{15}{4\pi \in_0 r^6} (\vec{m} \cdot \vec{r})(\vec{m}' \cdot \vec{r}) \overline{\delta r}$$

$$= \frac{1}{4\pi \in_0} \left[\frac{3\vec{m} \cdot \vec{m}'}{r^5} \vec{r} - \frac{15}{r^6} \cdot \frac{(\vec{m}' \cdot \vec{r})(\vec{m} \cdot \vec{r})}{r} \vec{r} \frac{3\vec{m}(\vec{m}' \cdot \vec{r})}{r^5} + \frac{3\vec{m}'(\vec{m} \cdot \vec{r})}{r^5} \right] \cdot \overline{\delta r},$$

$$(\because \vec{r} \cdot \overline{\delta r} = r\, dr).$$

Since $\overline{\delta r}$ is arbitrary, we have

$$\vec{F} = \frac{1}{4\pi \in_0} \left[\frac{3(\vec{m} \cdot \vec{m}')}{r^5} - \frac{15(\vec{m} \cdot \vec{r})(\vec{m}' \cdot \vec{r})}{r^7} \right] \vec{r} + \frac{3(\vec{m}' \cdot \vec{r})}{r^5} \vec{m} + \frac{3(\vec{m} \cdot \vec{r})}{r^5} \vec{m}'$$

The above expression shows that $\vec{F}$ has three components along $\vec{r}, \vec{m}$ and $\vec{m}'$ which are

$$\frac{1}{4\pi \in_0}\left[\frac{3(\vec{m}\cdot\vec{m}')}{r^5}-\frac{15(\vec{m}\cdot\vec{r})(\vec{m}'\cdot\vec{r})}{r^7}\right]\vec{r};\frac{3(\vec{m}'\cdot\vec{r})\vec{m}}{4\pi \in_0 r^5};\frac{3(\vec{m}\cdot\vec{r})\vec{m}'}{4\pi \in_0 r^5}$$

respectively.

Example. ***If the law of force between the charges e_1 and e_2 was e_1e_2/r^n, show that the potential due to charge e would be $e/4\pi \in_0 \{(n-1)r^{n-1}\}$ and the potential due to an electric dipole of moment $\vec{m}$ would be $(\vec{m}\times\vec{r}_s)/4\pi \in_0 r^{n+1}$***

Solution : The law of force between two charges e_1 and e_2

$$F=\frac{e_1e_2}{4\pi \in_0 r^n}$$

$\therefore$ The law of force between the charge e and a unit charge will be

$$F=\frac{e}{4\pi \in_0 r^n}.$$

Now if ϕ is the required potential,

$$F=-\frac{\partial\phi}{\partial r}=4\pi \in_0 \frac{e}{r^n}.$$

$$\phi=\int_r^\infty -\frac{e}{4\pi \in_0 r^n}dr.$$

$$=\frac{e}{4\pi \in_0 (n-1)r^{n-1}}.$$

We know that when the law of force is $1/r^2$, the potential at any point due to a doublet $\vec{m}$

$$=\frac{\vec{m}\cdot\vec{r}}{4\pi \in_0 r^3}$$

$$=\frac{\vec{m}}{4\pi \in_0 r^2}\cdot\left(\frac{\vec{r}}{r}\right)$$

Therefore when the law of force is $1/r^n$, the potential at any point due to a doublet must be

$$=\left(\frac{\vec{m}}{4\pi \in_0 r^n}\right)\cdot\left(\frac{\vec{r}}{r}\right)$$

$$= \frac{(m \cdot r)}{4\pi \in_0 r^{n+1}}$$

❑ 73.9. Method of Images :

The method of electrical images was first explained by **Lord Kelvin**. This method gives a convenient mathematical device for solving three dimensional conductor problems involving planes, spheres, ellipsoids.

❍ Definition of the electrical image :

An electrical image is a point charge or a set of point charges on one side of a conducting surface which would produce on the other side of the surface the same electric field as produced by the actual electrification of the surface.

Green's theorem of the equivalent stratum is of great importance for the solution of image problems. Often we need geometrical method for evaluating an induced charge. Here the idea of the method can be obtained from an optical analogy. The illumination at a point in front of say a plane mirror is the combined effect of light directly received from the source and that reflected from the mirror. The latter we may imaine to have proceeded from an imaginary source behind the mirror which is the virtual image of the source. Let us now take a point charge e in front of a conductor which, let us suppose, is earthed. The electric field in front of the conductor is due to e and the induced charge on the conductor. If a single point charge e' or a set of charges $\sum e'$ placed behind the conductor produces a field identical in all respects to that produced by the induced charge on the conductor, then e' or $\sum e'$ from the optical analogy may be called the electrical image or images of e.

The electrical image method is generally applied for solution of electrostatic problems.

In order to solve a particular problem by the method of electrical images the following procedure is adopted.

(i) Without violating the boundary conditions, the magnitude and position of the image charge are determined by inspection.

(ii) The potential and intensity at any point are calculated using the given charge and image charge by ignoring the presence of the conductor.

(iii) The normal component of the intensity at the surface of the conductor is determined from the intensity and using Coulomb's law, $E_n = -\frac{\sigma}{\epsilon_0}$. From this value, the surface density of induced charge (σ) can be easily calculated.

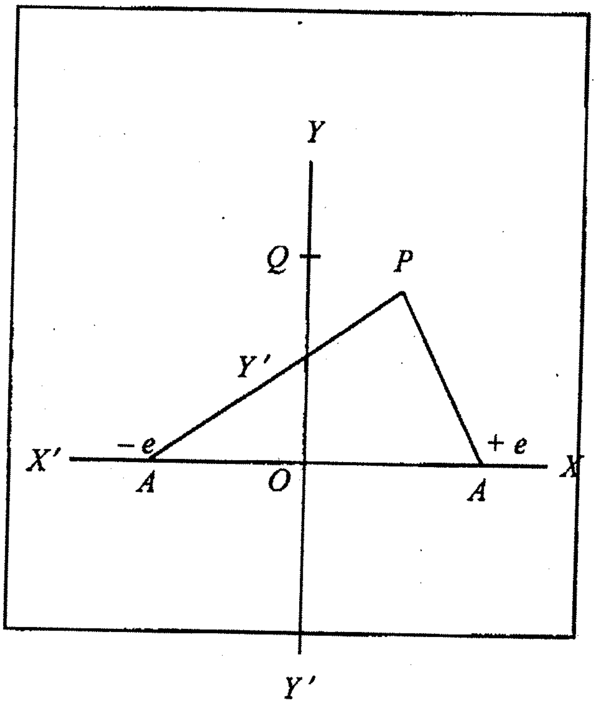
(iv) The total induced charge on the conductor can be obtained by using $\int_S \sigma ds$.

(v) The force between the conductor and the point charge is calculated by using Coulomb's law, using point charge and the image charge.

❑ 73.9.1. Image of a single point charge *e* at a distance '*f*' from a conducting infinite plane sheet kept at zero potnetial.

Let the conducting plane coincide with the plane coincide withe plane $x = 0$ of a system of rectangular axes. Let a point charge e be placed at the point A (0, 0, f). Let a point charge $-e$ be also placed at the point $A'(0, 0, -f)$. The potential ϕ at any point P is

$$\phi = \frac{1}{4\pi \epsilon_0}\left[\frac{e}{r} - \frac{e}{r'}\right] \qquad \text{......} (1)$$



Q

It is quite evident that $\phi = 0$ when $r = r'$.

The potential ϕ of the field must satisfy the following conditions :

(i) It must satisfy Laplace's equation $\nabla^2\phi = 0$ everywhere except at A.

(ii) $\phi = 0$ over the plane $z = 0(YOY')$.

(iii) $\phi = 0$ at infinity.

(iv) $\phi - \frac{e}{r} \rightarrow a$ finite quantity as $P \rightarrow A$, (*i.e.* when $r \rightarrow 0$).

Hence, by the *uniqueness theorem* ϕ given by (1) is the only potential which satisfies the conditions of the problem.

It follows that the field on the right of the plane $z = 0$ is identical with that given by charges e and $-e$ at A and A' respectively.

Thus according to our definition the charge $-e$ at A' is the electrical image of charge e at A.

Therefore, the image of the charge e in the infinite plane is an equal and opposite charge placed at the optical point.

❍ **Surface density :** If we want to calculate the surface density on the plate, then

$$\frac{\sigma}{\epsilon_0} = -\left(\frac{\partial\phi}{\partial x}\right)_{x=0}$$

But (1) gives

$$x(x,y) = \frac{e}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{\{(f-x)^2+y^2\}}} - \frac{1}{\sqrt{\{(x+f)^2+y^2\}}}\right]$$

where $P(x, y)$ is any point in space.

$$\text{Hence } \frac{\sigma}{\epsilon_0} = \frac{e}{4\pi\epsilon_0}\left[\frac{x-f}{\{(f-x)^2+y^2\}^{3/2}} - \frac{1}{\{(x+f)^2+y^2\}^{3/2}}\right]_{x=0}$$

$$= -\frac{2fe}{4\pi\epsilon_0\left(f^2+y^2\right)^{3/2}}.$$

$$\text{or, } \therefore \sigma = \frac{-2ef}{4\pi\left(f^2+y^2\right)^{3/2}}$$

$$\sigma = \frac{-ef}{2\pi(AQ)^3} \text{ where } AQ = \left(f^2+y^2\right)^{3/2}$$

Note. Obviously all the tubes of force leave 'e' at A and end on the infinite plane where the total charge induced is '$-e$' and it can be proved analytically. Clearly the charge induced on a ring on the plane bounded by circles of radii y and $y + dy$ with centre O is

$$2\pi\sigma y\,dy = 2\pi\left(\frac{-ef}{2\pi(AQ)^3}\right)y\,dy = \frac{-efy}{\left(f^2+y\right)^{3/2}}dy.$$

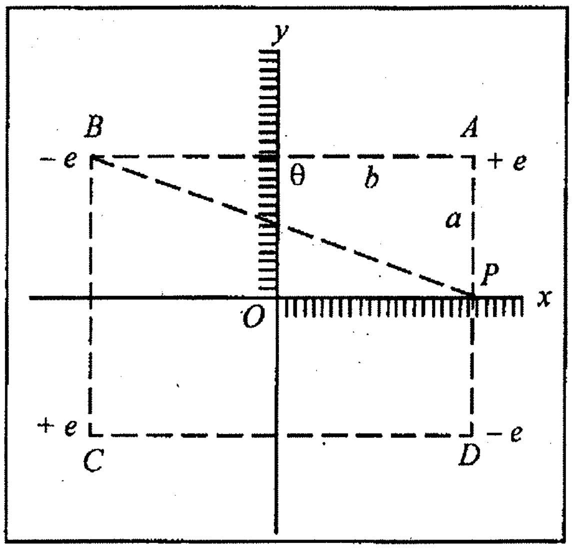
where $AQ = \sqrt{\left(f^2+y^2\right)}$.

Hence the total induced charge

$$= -ef\int_0^{\infty}\frac{y}{\left(f^2+y^2\right)^{3/2}}dy = ef\left[\frac{-1}{\sqrt{f^2+y^2}}\right]_0^{\infty} = -e.$$

Example.

A positive charge e is placed at a distances a and b from two semi infinite planes at zero potential intersecting at right angles. Find the surface densities of electrification on the nearest point of each plane.



Solution : The adjoining figure shows the cross section of the planes by the plane of paper. This section contain a point charge e at A. The image system will be as shown in the figure.

The surface density of induced charge at P due to e at A and $-e$ at D is

$$\sigma_1 = -\frac{ea}{2\pi AP^3} = -\frac{e}{2\pi a^2} \quad \text{............ (1)}$$

Also, surface density of induced charge at P due to $-e$ at B and $+e$ at C is

$$\sigma_1 = -\frac{(-ea)}{2\pi(BP)^3} = \frac{ea}{2\pi\left(a^2+4b^2\right)^{3/2}}$$

Therefore the surface density of induced charge at P is given by

$$\sigma = \sigma_1 + \sigma_2 = -\frac{e}{2\pi a^2} + \frac{ea}{2\pi\left(a^2+4b^2\right)^{3/2}}$$

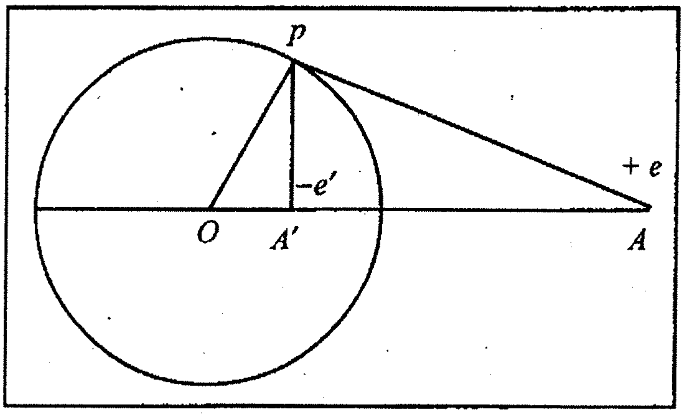
$$= -\frac{e}{2\pi a^2}\left[1 - \frac{a^3}{\left(a^2+4b^2\right)^{3/2}}\right] \qquad \text{..................(2)}$$

Interchanging a and b in the above result (2), we can show similarly that the surface density of induced charge at Q is given by

$$\sigma = -\frac{e}{2\pi b^2}\left[1 - \frac{b^3}{\left(b^2+4a^2\right)^{3/2}}\right] \qquad \text{..................(3)}$$

73.9.2. Image of an external point charge in a sphere :

To investigate the image system and electric field outside a conducting sphere of radius a, when a point charge e is placed at a point outside the sphere at a distance f from the centre of sphere; and to determine the surface density of the induced charge.



♦ ***Case I. When the sphere is at zero potential.***

To solve the problem we have to determine the magnitude the position of a point charge e' inside the sphere, so that the sphere will be at zero potential for the charge, e and e'

Let a point charge e' be placed at A' the inverse point of A w.r.t. the sphere, *i.e.* the point on OA is such that

$$OA' \cdot OA = a^2$$

or, $$\frac{OA}{a} = \frac{a}{OA'}$$

But the triangles OPA and OPA' are similar.

Hence $$\frac{OA}{OP} = \frac{OP}{OA'} = \frac{AP}{A'P} = \frac{r}{r'} = \frac{f}{a}. \qquad \text{where } A'P = r'$$

or, $$\frac{OA}{OP}\cdot\frac{OP}{OA'} = \frac{r^2}{r'^2}, \qquad \text{............(1)}$$

i.e. $\frac{OA}{OA'} = \frac{r^2}{r'^2}$.

Multiplying f/a on both the sides of above equation, we have

$$\frac{f}{a}\frac{OA}{OA'} = \frac{r^2}{r'^2}\cdot\frac{f}{a} = \frac{r^3}{r'^3}, \text{ from (1)},$$

Hence we have $\frac{OA}{OA'} = \frac{r^3}{r'^3}\cdot\frac{a}{f}$. .... (2)

The potential at any arbitrary point P will be zero if

$$\frac{1}{4\pi \in_0}\left[\frac{e}{AP} + \frac{e'}{A'P}\right] = 0,$$

i.e. if $e' = -\frac{er'}{r} = -\frac{ea}{f}$. .... (3)

Thus the image charge is $-\frac{ea}{f}$ at the inverse point of A.

If Q be any point outside the sphere, then the potential at this point is

$$\phi = \frac{1}{4\pi \in_0}\left[\frac{e}{AQ} + \frac{-ea/f}{A'Q}\right]$$ .... (4)

($\because$ induced charge on the surface is the charge of image)

Surface density of charge at *P*. Let σ be the surface density of charge at P. By definiton of electric intensity

$$\vec{E} = \frac{e}{4\pi \in_0 r^3}\vec{r}.$$

Also by Coulomb's law, $\frac{\sigma n}{\in_0} = \vec{E} = \frac{1}{4\pi \in_0}\left[\frac{e}{r^3}\vec{r} - \frac{ea}{f}\frac{\vec{r}}{r'^3}\right]$. .... (5)

n being a positive unit normal to the sphere at P.

Now $\vec{r} = \overrightarrow{AO} + \overrightarrow{OP} = \overrightarrow{AO} + a\vec{n}$

and $\vec{r}' = \overrightarrow{A'O} + \overrightarrow{OP} = \overrightarrow{A'O} + a\vec{n}$.

Therefore from (4), we get

$$\vec{E} = \frac{\sigma\vec{n}}{\in_0} = \frac{1}{4\pi \in_0}\left[\frac{e}{r^3}\left(\overrightarrow{OA} + am\right) - \frac{ea}{f}\left(\frac{\overrightarrow{A'O} + am}{r^3}\right)\right]$$

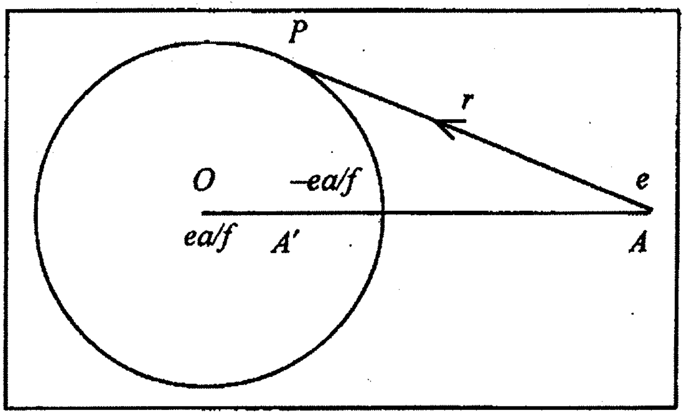
$$\frac{\sigma n}{\epsilon_0}=\frac{1}{4\pi\epsilon_0}\left[\left(\frac{1}{f}\frac{\overline{A'O}}{r'^3}+\frac{a\vec{n}}{r^3}\right)-\frac{ea}{fr'^3}\overline{A'O}-\frac{ea}{fr'^3}a\vec{n}\right]$$

$$=\frac{1}{4\pi\epsilon_0}\left(\frac{ea}{r^3}-\frac{ea^2}{fr'^3}\right)\vec{n} \qquad \therefore\left[\frac{AO}{r^3}=\frac{a}{f}\frac{A'O}{r'^3}(by (2)\right]$$

$$=\frac{ea}{4\pi\epsilon_0 r^3}\left[1-\frac{a}{f}\frac{r^3}{r'^3}\right]\vec{n}=4\pi\epsilon_0\frac{ea}{r^3}\vec{n} \text{ by (1)}$$

$$\sigma=-\frac{e\left(f^2-a^2\right)}{4\pi ar^3}$$

◆ **Case II.** ***When the sphere is insulated and uncharged.***



In this case the sphere is to be insulated without charge. Hence the total induced charge is zero, the effect of induction being to separate equal quantities of positive and negative electricity. The sphere is not at zero potential but must have a constant potential. So all the conditions of the problem are obviously satisfied by distributing a charge $+\frac{ea}{f}$ uniformly over the surface of sphere in case (I), for this will make the total charge zero and leave the sphere an equipotential surface.

It is quite obvious that the external field due to a uniform spherical charge is the same as if the charge were collected at the centre of the sphere; we may now say that the external fields is due to e at $A-\frac{ea}{f}$ at A' and $\frac{ea}{f}$ at O.

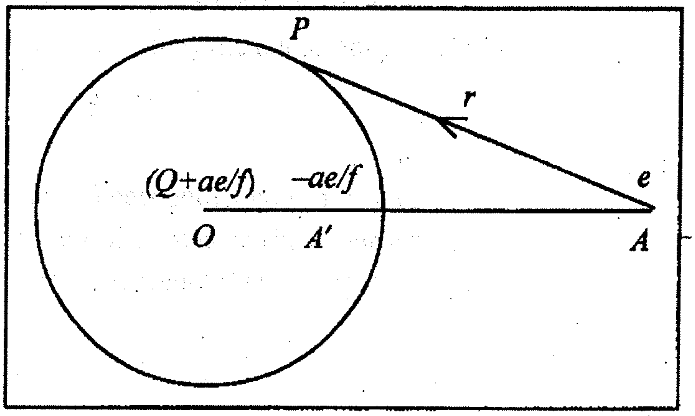
❍ **Surface density at *P*.** The uniformly distributed charge $\frac{ea}{f}$ over the surface of the sphere gives rise to the surface density $\frac{ea}{f\cdot 4\pi a^2}$. Add this surface density to the surface density of the field in case I, so that in this case

$$\sigma=-\frac{e\left(f^2-a^2\right)}{4\pi ar^2}+\frac{e}{4\pi af}$$

The line of no-electrification on the surface of the sphere is given by

$$\sigma = 0 \quad \text{or} \quad r^3 = f\left(f^2 - a^2\right)$$

◆ **Case III.** ***When the sphere is kept insulated and carries a total charge Q.***



This case may be derived easily from case II by distributing a charge Q uniformly over the sphere. But it is desirable to derive it from the fundamental case I.

In this case we add to the field of case I, the field due to a charge $\left(Q + \frac{ae}{f}\right)$ uniformly distributed over the sphere. This result in the total charge on the sphere to be Q and leaves the sphere at an equipotential surface.

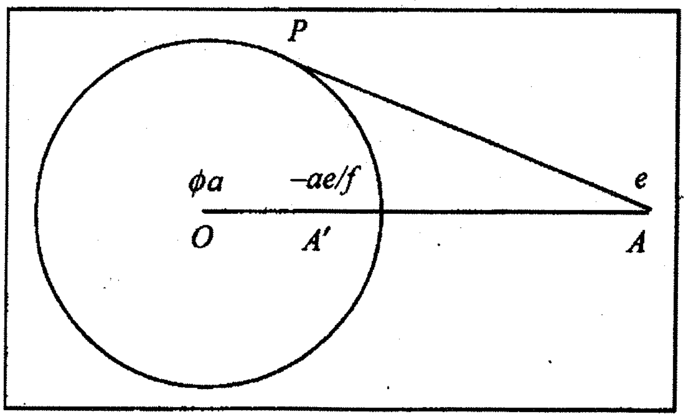
The external field is due to e at A, $-\frac{ea}{f}$ at A' and $\left(\frac{ea}{f} + Q\right)$

at O (becasue the external field due to a uniform spherical charge distribution is identical with the charge collected at the centre of the sphere) which is the required image system.

Surface density at *P*. The surface density of charge is given by

$$\sigma = -\frac{e\left(f^2 - a^2\right)}{4\pi a r^3} + \frac{e}{4\pi a f} + \frac{Q}{4\pi a^2}.$$

◆ **Case IV.** ***When the sphere is kept at a given constant potential ϕ.***



To keep the sphere at a constant potential ϕ it is necessary to add a field due to a charge $a\phi$ to the field of case I.

The external field is simply due to e at $A, -\frac{ea}{f}$

at A' and $a\phi$ at O (because the external field due to a uniform spherical charge collected at the centre of the sphere).

○ **Surface density at P.** The surface density of charge is given by

$$\sigma = -\frac{e\left(f^2-a^2\right)}{4\pi a r^3}+\frac{\phi}{4\pi a}.$$

❑ **To find the force on the point charge e in all the above cases.**

The force on the charge e in case I is that of attraction of magnitude

$$F_1 = \frac{e\left(\frac{ea}{f}\right)}{4\pi\epsilon_0\left(f-\frac{a^2}{f}\right)^2} = \frac{e^2 af}{4\pi\epsilon_0\left(f^2-a^2\right)^2} \text{ by inverse square law.}$$

The force on the charge e in second case is given by

$$F_2 = \frac{e\left(\frac{ea}{f}\right)}{4\pi\epsilon_0\left(f-\frac{a^2}{f}\right)^2} - \frac{e\cdot\frac{ea}{f}}{4\pi\epsilon_0 f^2}$$

$$= \frac{e^2a^2\left(2f^2-a^2\right)}{4\pi\epsilon_0 f^3\left(f^2-a^2\right)^2},$$

= positive; hence that of attraction.

The force on the charge in third case is given by

$$F_3 = \frac{e\left(Q+\frac{ae}{f}\right)}{4\pi\epsilon_0 f^2} - \frac{e\cdot\frac{ae}{f}}{4\pi\epsilon_0\left(f-\frac{a^2}{f}\right)^2}$$

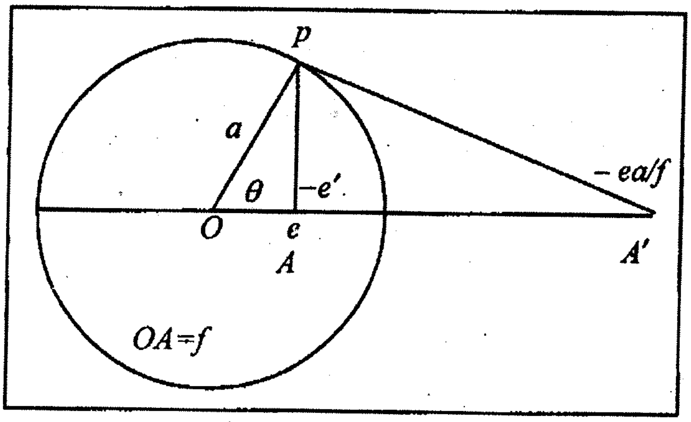
$$F_3 = \frac{eQ}{4\pi\epsilon_0 f^2}+\frac{e^2a}{4\pi\epsilon_0 f^3}-\frac{e^2af}{4\pi\epsilon_0\left(f^2-a^2\right)^2}$$

the attraction or repulsion depends upon the relative values.

♦ Images of an internal point charge in a sphere.

To investigate the image system and electric field outside a conducting sphere of radius a, when a, point charge e is placed at a point inside the sphere at a distance f from the centre; and to determine the surface density of induced charge.

To solve the problem we have to determine the magnitude and position of a point charge e' outside the sphere, so that the sphere may be at zero potential for the charges e and e'.



Let a point charge e' be placed at A', the inverse point of A w.r.t. the sphere *i.e.* the point on OA produced is such that

$$OA \cdot OA' = a^2$$

or, $$\frac{OA'}{a} = \frac{a}{OA}.$$

But the trainingles OPA and OPA' are similar,

Hence $$\frac{OA'}{OP} = \frac{OP}{OA} = \frac{A'P}{AP} \; i.e. \frac{a^2/f}{a} = \frac{a}{f} = \frac{r'}{r}.$$

The potential at any arbitrary point P will be zero, if

$$\frac{1}{4\pi \in_0}\left[\frac{e}{AP} + \frac{e'}{A'P}\right] = 0 \; i.e. \text{if } e' = -e\frac{A'P}{AP} = -\frac{ea}{f}$$

Thus the image charge is $\frac{-ea}{f}$ at the inverse of point A.

If Q be any point outside the sphere then the potential at this point is given by

$$\phi = \frac{e}{4\pi \in_o AQ} + \frac{-ea/f}{4\pi \in_o AQ}$$

($\because$ induced charge on the surface is the charge of image).

Surface density of charge at P. Let σ be the surface density of charge at P. Since he potential at the point P is

$$\phi(P) = \frac{1}{4\pi\epsilon_o}\left[\frac{e}{AP} - \frac{ea/f}{A'P}\right] = \frac{1}{4\pi\epsilon_o}\left[\frac{e}{\sqrt{(r^2+f^2-2rf\cos\theta)}} - \frac{ea}{f}\frac{1}{\sqrt{(f'^2+r^2-2rf'\cos\theta)}}\right]$$

where $f' = a^2/f$.

By Coulomb's law

$$\frac{\sigma}{\epsilon_0} = E_n = -\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = \frac{1}{4\pi\epsilon_0}\frac{e(a-f\cos\theta)}{(a^2+f^2-2af\cos\theta)^{3/2}} - \frac{ea}{4\pi\epsilon_0}\frac{\left(a-\frac{a^2}{f}\cos\theta\right)}{f(a^2+f^2-2af\cos\theta)^{3/2}}$$

$$= \frac{e(a-f\cos\theta)}{4\pi\epsilon_0 AP^3} - \frac{ef}{4\pi\epsilon_0 a}\frac{(f-a\cos\theta)}{AP^3}$$

$$= \frac{e(a^2-f^2)}{4\pi\epsilon_0 aAP^3}$$

$$\therefore \sigma = \frac{e(a^2-f^2)}{4\pi a AP^3}$$

In this case electric field is radially inward. Hence we have

$$\sigma = -\frac{e(a^2-f^2)}{4\pi a\cdot AP^3}$$

The force of attraction on the charge e at A is

$$\frac{e\cdot\frac{ea}{f}}{4\pi\epsilon_0\left(\frac{a^2}{f}-f\right)} = \frac{e^2af}{4\pi\epsilon_0(a^2-f^2)^2},$$

❑ 73.10. Complex Potential :

Potential problems in electromagnetics can be dealt with the theory of complex variables. The basis of this method is that the real and imaginary parts of any analytic function satisfy Laplace's equation under suitable boundary conditions. In electrostatics the boundary conditions are :

(i) The potential ϕ satisfies Laplace's equation $\nabla^2\phi = 0$ every where except the points on the surface of discontinuity.

(ii) The potential ϕ is constant at each and every point on the conductor (solid or hollow).

(iii) The potential and normal components of displacement vector are continuous at the surface of separation of the two media (uncharged)

In mathematical form :

$\phi_1 = \phi_2$ and $D_{n_1} = D_{n_2}$ or $\in_1 E_{n_1} = \in_2 E_{n_2}$ or $\in_1 \frac{\partial \phi_1}{\partial n} = \in_2 \frac{\partial \phi_2}{\partial n}$ where $\left(\frac{\partial}{\partial n}\right)$ denotes the differentiation along the normal to the surface of spearation.

❍ **Use of complex variable :**

If $w = f(z)$, where $z = x + iy$, then w may be written in the form

$$w = \phi(x, y) + i\psi(x, y).$$

If w is analytic in Z, it satisfies **Laplacesequation.**

$$\nabla^2 w = 0$$

i.e. $\nabla^2(\phi + i\psi) = 0$

or, $\nabla^2\phi + i\nabla^2\psi = 0$ .... (1)

Equation (1) gives

$\nabla^2\phi = 0$ and $\nabla^2\psi = 0$ .... (2)

Thus the real and imaginary part ϕ and ψ of an analytic function are solutions of Laplace's equation in two dimensions.

We know **Cauchy-Riemann equations namely**

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}; \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad \text{............... (3)}$$

These two statements may be summed up in one statement that if S_1 and S_2 are perpendicular directions related in the anti-clockwise fashion, then,

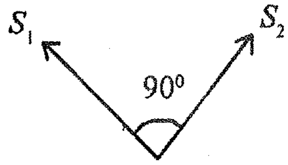
$$\frac{\partial \phi}{\partial S_1} = \frac{\partial \psi}{\partial S_2} \qquad \text{............... (4)}$$

In particular, if S_1 is taken along a conductor ϕ = constant,

$\frac{\partial \phi}{\partial S_1} = 0$, so that $\frac{\partial \psi}{\partial S_2} = 0$

showing that ψ is constant along the S_2 direction. This means that the curves ϕ = constant and ψ=constant intersect each other orthogonally whereever they intersect.

Conveniently ϕ is taken as the **potential function** and ψ is called the **stream function.** The function ϕ and ψ are called **conjugate function.** The lines ϕ = constant are **equipotential lines** and lines ψ = constant are **lines of force.**



Example : Let us consider a function.

$f(z) = z^2$ so that

$w = f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy = \phi + i\psi$

$\therefore \phi(x,y) = x^2 - y^2$ and $\psi(x,y) = 2xy$ .... (5)

From equation (5) we conclude that equipotential lines are $y = \pm x$ and a family of rectangular hyperbolas $x^2 - y^2$ = constant, having the two lines $y = \pm x$ as asymptotes. Along $y = \pm x, \phi = 0$ and along other hyperbolas of the family ϕ has a non-zero value. The family of hyperbolas xy = constant (including the lines $x = 0, y = 0$) represent the orthogonal family of curves to ϕ = constant and are the lines of force.

If one of the equipotentials is made a conductor, the surface charge density $\left(\frac{\sigma}{\epsilon_0}\right)$ is given by $-\frac{\partial \phi}{\partial n}$.

But $-\frac{\partial \phi}{\partial n} = \frac{d\psi}{ds}$

$\therefore \frac{\sigma}{\epsilon_0} = \frac{\partial \psi}{ds}$ .... (6)

$\therefore$ total charge $q = \int \sigma ds = \int \epsilon_0 \frac{\partial \psi}{\partial s} ds = \epsilon_0 [\psi]$. .... (7)

We can also calculate $\vec{E}$ from equation (3)

$E_x = -\frac{\partial \phi}{\partial x} = -$ real part of $f'(z)$

$E_y = -\frac{\partial \phi}{\partial y}$ = imaginary part of $f'(z)$

$\therefore E = \sqrt{E_x^2 + E_y^2}$

$$= \sqrt{\left[\left(real\ part\ of\ \frac{dw}{dz}\right)^2 + \left(imagenary\ part\ of\ \frac{dw}{dz}\right)^2\right]}$$

$$= \left|\left(\frac{dw}{dz}\right)\right|$$

From the above, it is clear that, it is easier to work directly interms of w rather than interms of ϕ and ψ. This (w) is called the **complex potential** of the problem.

❑ 73.11. Maxwell's Stress Tensor :

Maxwell states that the electrostatic field is simply the stress transmitting medium.

Let us consider a given volume in electrostatic field bounded by a surface. If the force acting on this volume is transmitting outside through the surface enclosing the volume, then transmitting force can be expressed in the form of space stress tensor T. The stress will be present even in the absence of the medium. Actually this is simply a physical concept and the only reality is the action at a distance between charges as given by Coulomb's law. The transmitting force is just a medium to explain the action at a distance. The transmitting force expressed in terms of stress tensor T gives more vivid picture of the phenomenon of the action at a distance. Let us consider a pq^{th} component of stress tensor T as T_{pq}. Let dF_p be the component of the force dF transmitted across an elemental surface ds and ds_q the component of the surface in q^{th} direction; the dF_p is given in terms of stress tensor as below:

$$dF_p = \sum_{q=1}^{3} T_{pq} ds_q. \qquad \text{.............. (1)}$$

The tensor T is of symmetric type. The equation (1) may be written as

$$dF_p = T_{pq} ds_q. \qquad \text{.............. (2)}$$

or, $F_p = \int T_{pq} ds_q.$

Now, this force is say, the p^{th} component of the volume force, then

where F_{up} represents the p^{th} component of the volume force. Using Gauss' divergence theorem in thensor form, as given below:

$$\int \frac{\partial \tau_{pq}}{\partial x_q} dv = \int T_{pq} ds_q,$$

we get the p^{th} component of volume force in terms of stress tensor *i.e.,*

$$F_{vp} = \frac{\partial T_{pq}}{\partial x_q} \quad \text{...............} (3)$$

If a volume force is expressed as a tensor divergence of a quantity T as by equation (3), then the quantity T represents the surface stress tensor as shown by equation (2). The tensor corresponding to volume force F_v is not unique and an additional tensor having very small divergence can be added to it.

The volume force equation is absence of dielectric is given by

$$\vec{F}_v = \rho\vec{E} = \left(\nabla \cdot \vec{D}\right)\vec{E}.$$

The equation in tensor notation is

$$F_{vp} = \varepsilon_0 E_p \frac{\partial E_q}{\partial x_q} = \varepsilon_0 \left[\frac{\partial}{\partial x_q}\left(E_p E_q\right) - E_q \frac{\partial E_p}{\partial x_q}\right] \quad \text{...............} (4)$$

To change second term to the same form, we know that for electrostatic fields $\nabla \times \vec{E} = 0$ *i.e.,*

$$\frac{\partial E_p}{\partial x_q} = \frac{\partial E_q}{\partial x_p}$$

$$\therefore F_q \frac{\partial E_p}{\partial x_q} = E_q \frac{\partial E_q}{\partial x_p} = \frac{1}{2}\cdot\frac{\partial}{\partial x_p}\left(E_q E_q\right) = \frac{1}{2}\cdot\frac{\partial\left(E^2\right)}{\partial x_p}. \quad \text{...............} (5)$$

where magnitude of E is E. This equation can be written as :

$$E_q \frac{\partial E_p}{\partial x_q} = \frac{1}{2}\delta_{pq}\frac{\partial}{\partial x_p}\left(E^2\right) = \frac{\partial}{\partial x_q}\left(\frac{1}{2}\delta_{pq}E^2\right).$$

Hence equation (4) becomes :

$$F_{vp} = \varepsilon_0\left[\frac{\partial}{\partial x_q}\left(E_p E_q\right) - \frac{\partial}{\partial x_q}\left(\frac{1}{2}\delta_{pq}E^2\right)\right]. \quad \text{...............} (6)$$

Comparing equations (3) and (6), we get

$$T_{pq} = \varepsilon_0\left(E_p E_q - \frac{1}{2}\delta_{pq}E^2\right) \quad \text{...............} (7)$$

The matrix corresponding to this tensor is given below. This is known as Maxwell electric stress tensor in the absence of dielectrics :

$$T=\varepsilon_0\begin{bmatrix}\frac{1}{2}\left(E_x^2-E_y^2-E_z^2\right) & E_xE_y & E_xE_z\\ E_yE_x & \frac{1}{2}\left(E_y^2-E_z^2-E_x^2\right) & E_yE_z\\ E_zE_x & E_yE_z & \frac{1}{2}\left(E_z^2-E_x^2-E_y^2\right)\end{bmatrix} \quad \text{.............. (8)}$$

The tensor will possess additional terms for irrotational field for which $\nabla\times E\neq 0$.

where δ_{pq} gives coefficients of transformation as

$$\delta_{pq}=\begin{vmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{vmatrix}$$

This is symmetric tensor of second rank. It can therefore be reduced to three components only by transformation to three principal axes. Thus in terms of principal co-ordinates it becomes :

$$T=\frac{\varepsilon_0}{2}\begin{pmatrix}E^2 & 0 & 0\\ 0 & -E^2 & 0\\ 0 & 0 & -E^2\end{pmatrix} \quad \text{.............. (9)}$$

The principal values of the matrix have been determined by solving the secular determinant : .

$$\left|T_{pq}-\delta_{pq}\lambda\right|=0. \quad \text{.............. (10)}$$

The principal values are :

$$\lambda_1=\frac{\varepsilon_0}{2}E^2,\ \lambda_2=\lambda_3=-\frac{\varepsilon_0}{2}E^2. \quad \text{.............. (11)}$$

The electric field transmits a tension $\varepsilon_0\,E^2/2$ parallel to field and a transverse pressure of magnitude $\varepsilon_0\,E^2/2$ transverse to the field. The axes orientation is such that λ_1 is parallel to $\vec{E}$ and λ_2,λ_3 perpendicular to $\vec{E}$.

❑ 73.12. Self Assessment Questions :

1. What do you mean by 'conservative force'? Write down its characteristics.
2. Define electric potential. Find the expression for the potential of a uniformly charged spherical shell.
3. Write down the electrostatic boundary conditions.
4. What is an electric dipole? Find the potential and field at an external point due to a dipole.

5. What do you mean by potential energy of an electric dipole in a given fied? Deduce the expression for the mutual potential energy of two dipoles placed in a plane.
6. What is electrical image? Write the general procedures which are adopted to solve a problem by the method of electrical image.
7. Find the image of a single point charge a placed at a distance from a conducting infinite plane sheet kept at zero potential. Find the express in for surface-density of induced charge in this case.
8. Investigate the image system and electric field outside a conducting sphere or radius a, when a point charge e is placed at a point outside a sphere at a distance 'f' from the centre of sphere. Also determine the surface density of the induced charge.
9. Using method of image find the expression for the surface density of charge whent he sphere as mentioned in question no. 8, is kept insulated and carries a total charge Q. Find the force on the external point charge in this case.
10. Write notes on :
    (a) Complex potential;
    (b) Maxwell's stress tensor.

❑ 73.13. References :

1. Electro and Magneto Statics : K. Bhattacharya. Hindustan Publishing concern. Kolkata.
2. Electromagnetic Theory : P.P. Gupta. Krishna Prakashan Mandir. Meerut.
3. Introduction of Electrodynamics : David J. Griffiths. Prentice-Hall of India Pvt. Ltd.
4. Electrodynamics : J.C. Upadhyaya. Ram Prasad and Sons. Agra.
5. Classical Electrodynamics : J.D. Jackson John Wiley & Sons Inc.
6. Methods of Mathematical Physics. H. Jeffreys *et al.* Cambridge University Press.
7. Mathematical Theory of Electricity and Magnetism : J.H. Jeans. Cambridge University Press.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper-VII

Group-A

Module No. - 74

(Electromagnetic Theory)

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Module Structure :

❑ 74.1 Introduction

There is an interdependence of the field quantities. Hence we have to consider the general concept of an electromagnetic field. The time dependent electromagnetic field equations are called Maxwell's field equations. These equations are mathematical abstractions of experimental results.

In this module we seek to establish the formation of the field equations, to show that their solutions are unique, to discuss the scalar and vector potentials of the field and to consider the law of conservation of charge and energy.

❑ 74.2 Objective :

After going through this module we shall be able to know about the following topics :

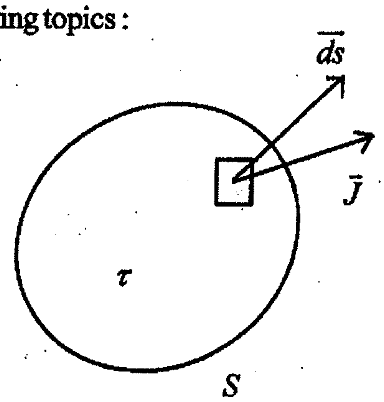
(i) Maxwell's field equations;

(ii) Idea of displacement current density;

(iii) Poynting's theorem and idea of Poynting's vector;

(iv) Electromagnetic potentials;

and (v) Gauge Transformation.



❑ 74.3 Equation of Continuity :

Equation of continuity is based on the law of conservation of charge.

Let us consider a closed surface S enclosing a volume τ.

Let ρ = volume density of charge

$\therefore$ Total charge within volume $\tau = \int_\tau \rho\, d\tau$

Since electric charge can neither be created nor destroyed (i.e. the charge is conserved : law of conservation of charge), it follows that the net flow of the charge, out of this volume must be equal to the rate of decrease of the total charge inside the volume. The net charge that passes through the surface of unit area (normal to the direction of charge flow) in unit time is defined as the current density $(\vec{J})$. Hence the total current flowing through the surface S is

$$I = \int_S \vec{J} \cdot \overrightarrow{ds}$$

If there is a net outward flow of current through a closed surface, then the charge that is contained within the volume which is bounded by the surface must decrease. Hence we can write,

$$\int_S \vec{J}\cdot\overrightarrow{ds} = -\frac{dq}{dt}$$

or, $\int_S \vec{J}\cdot\overrightarrow{ds} = -\frac{d}{dt}\int_\tau \rho d\tau$ [$\therefore$ Total charge $q = \int_\tau \rho d\tau$]

If we hold the surface fixed in space, the time variation of the volume integral must be due to solely to the time variation of ρ; or we can say operators are commutative in character.

So, $\int_S \vec{J}\cdot\overrightarrow{ds} = -\int_\tau \frac{\partial\rho}{\partial t}d\tau.$

or, $\int_\tau (\nabla\cdot\vec{J})d\tau = -\int_\tau \frac{\partial\rho}{\partial t}d\tau$

or, $\int_\tau \left(\nabla\cdot\vec{J}+\frac{\partial\rho}{\partial t}\right)d\tau = 0$

Since the above equation is true for any arbitrary finite volume, the integrant must be zero. So,

$$\nabla\cdot\vec{J}+\frac{\partial\rho}{\partial t} = 0$$

This equation is called the **equation of continuity** and is an expression of the experimental fact that electrical charge is conserved.

❑ 74.4. Decay of Free Charge :

According to equation of continuity,

$$\nabla\cdot\vec{J}+\frac{\partial\rho}{\partial t} = 0$$

and according to Ohm's **law,**

$$\vec{J} = \sigma\vec{E}$$

$\therefore \nabla\cdot(\sigma\cdot\vec{E})+\frac{\partial\rho}{\partial t} = 0$

or, $\sigma(\nabla\cdot\vec{E})+\frac{\partial\rho}{\partial t} = 0$

or, $\frac{\sigma}{\epsilon}\left(\nabla\cdot\epsilon\,\vec{E}\right)+\frac{\partial\rho}{\partial t}=0$

or, $\frac{\sigma}{\epsilon}\left(\nabla\cdot\vec{D}\right)+\frac{\partial\rho}{\partial t}=0 \quad \left[\because \vec{D}=\epsilon\,\vec{E}\right]$

or, $\frac{\sigma}{\epsilon}\rho+\frac{\partial\rho}{\partial t}=0 \quad \left[\because \nabla\cdot\vec{D}=\rho\right]$ ....(i)

Let us solve this differential equation :

It is seen that the unit of $\left(\frac{\sigma}{\epsilon}\right)$ is second (i.e. time).

so, $\frac{\sigma}{\epsilon}=\Upsilon(\text{say})$; Υ is known as **relaxation time.**

So, equation (*i*) can be written as

$$\frac{\partial\rho}{\partial t}=-\frac{1}{\Upsilon}\rho$$

or, $$\frac{\partial\rho}{\rho}=-\frac{\partial t}{\Upsilon}$$

Integrating and putting the condition at $t=0$, $\rho=\rho_0$ (say)

We get the solution as

$$\rho=\rho_0 e^{-t/\Upsilon} \quad \text{..................}(ii)$$

This relation shows that any original distribution of charge decays exponentially at a rate which is independent of any other electromagnetic disturbances that may be taking place.

74.5. Displacement Current :

We shall now see how Maxwell changed the definition of total current density to adapt the equation of continuity to time dependent field.

We know that Ampere's circuital law in its most general form is given by

$$\oint_C \vec{H}\cdot\overline{dl}=\int_S \vec{J}\cdot\overline{ds}$$

i.e. $\int \nabla\times\vec{H}\cdot\overline{ds}=\int \vec{J}\cdot\overline{ds}$

or, $\nabla\times\vec{H}=\vec{J}$. .... (*i*)

Let us now examine the validity of this equation in the event that the fields are allowed to vary with time. If we take the divergence of both sides of equation (*i*) then

$$\nabla\cdot\left(\nabla\times\vec{H}\right)=\nabla\cdot\vec{J} \qquad \text{..................... } (ii)$$

Now as *div* of *curl* of any vector is zero, we get from equation (*ii*)

$$\nabla\cdot\vec{J}=0 \qquad \text{..................... } (iii)$$

Now the continuity equation in general states

$$\nabla\cdot\vec{J}=-\frac{\partial\rho}{\partial t} \qquad \text{..................... } (iv)$$

and will therefore vanish only in the special case that the charge density is static. Consequently we must conclude that Amere's law as sated in equation (*i*) is valid only for steady state conditions and is insufficient for the case of time-dependent fields. Because of this Maxwell assumed that equation (*i*) is not complete but should have something to it. Let this 'something' be denoted by $\vec{J}_d$, then equation (*i*) can be rewritten as

$$\nabla\times\vec{H}=\vec{J}+\vec{J}_d. \qquad \text{..................... } (v)$$

In order to identify J_d, we calculate the divergence of equation (5) and get

$$\nabla\cdot\nabla\times\vec{H}=\nabla\cdot\left(\vec{J}+\vec{J}_d\right)$$

i.e. $\nabla\cdot\left(\vec{J}+\vec{J}_d\right)=0$ (as div curl $\vec{H}=0$)

or, $\nabla\cdot\vec{J}+\nabla\cdot\vec{J}_d=0$

or, $\nabla\cdot\vec{J}_d=-\nabla\cdot\vec{J}$

i.e. $\nabla\cdot\vec{J}_d=\dfrac{\partial\rho}{\partial t}$ [from equation (*iv*)]

i.e. $\nabla\cdot\vec{J}_d=\dfrac{\partial}{\partial t}\left(\nabla\cdot\vec{D}\right)$ [as div $D=\rho$]

i.e. $$\nabla\cdot\left(\vec{J}_d-\frac{\partial\vec{D}}{\partial t}\right)=0 \qquad \text{..................... } (vi)$$

As equation (vi) is true for any arbitrary volume

$$\vec{J}_d=\frac{\partial\vec{D}}{\partial t}. \qquad \text{..................... } (A)$$

And so the modified from of Ampere's law becomes

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \qquad \text{..................} (B)$$

The term which Maxwell added to Ampere's law *viz.* $\left(\partial \vec{D} / \partial t\right)$ is called the *displacement current* to distinguish it from $\vec{J}$, the *conduction current*. By adding this term to Ampere's law, Maxwell assumed that a time rate of change of displacement produces a magnetic field, just as a conduction current does.

Regarding displacement current it is worthy to note that :

(i) *Displacement current is a current only in the sense that it produces a magnetic field. It has none of the other properties of current. For example displacement current can have a finite value in a perfect vacum where there are no charges of any type.*

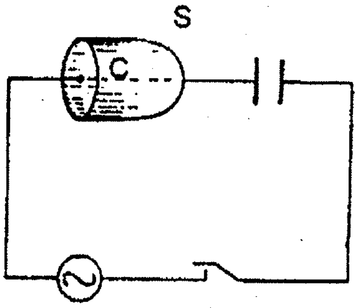
(ii) *The magnitude of the displacement current is equal to the time rate of change of electric displacement vector $\vec{D}$.*

(iii) *Displacement current serves to make the total current continuous across discontinuties in conduction current.*

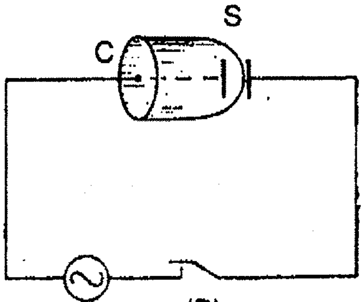
(iv) *The displacement current in a good conductor is negligible compared to the conduction current at any frequency lower than the opticald frequencies ($\approx 10^{15}$ hertz).*

It must be emphasized here that the ultimate justification for Maxwell's assumption of displacement current is in the experimental verification. Indeed the effects of the displacement current are difficult to observe directly except at very high frequencies. However indirect verification is afforded by prediction of many effects particularly in electromagnetic theory of light which are confirmed by experiments. We may therefore consider that Maxwell's form of Ampere's law has been subjected to experimental tests and has been found to be generally valid.

❍ **Example 1.** (a) *Apply Ampere's law to the closed curve C bounded by surface S in fig. (A) and (B); (b) Show that the circuit (A) and (B) yields the same result if we include the displacement current in Ampere's law.*



(A)



(B)

Solution :

(a) According to Ampere's circuital law

$$\oint_C \vec{H}\cdot\vec{dl} = \int_S \vec{J}\cdot\vec{ds}\cdot$$

So for the circuit (*A*)

$$\oint_C \vec{H}\cdot\vec{dl} = \vec{J}\cdot\vec{s} = I\cdot$$

And for the circuit (*B*)

$$\oint_C \vec{H}\cdot\vec{dl} = \int_S 0\cdot ds = 0.$$

(b) If in Ampere's Law we include displacement current, the law becomes

$$\oint_C \vec{H}\cdot dl = \int_S\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)\cdot\vec{ds}\cdot$$

And for the circuit (*A*) as $\vec{D}=0$ it yields

$$\oint_C \vec{H}\cdot\vec{dl} = \int\vec{J}\cdot\vec{ds} = I. \qquad \text{.......................... (1)}$$

While for the circuit (*B*) as $\vec{J}=0$ it yields

$$\oint_C \vec{H}\cdot\vec{dl} = \int_S \frac{\partial\vec{D}}{\partial t}\cdot\vec{ds}$$

i.e. $\oint_C \vec{H}\cdot\vec{dl} = \frac{\partial}{\partial t}\int_S \vec{D}\cdot\vec{ds}$

or, $\oint_C \vec{H}\cdot\vec{dl} = \frac{\partial}{\partial t}\int_\tau \nabla\cdot\vec{D}\,d\tau$

or, $\oint_C \vec{H}\cdot\vec{dl} = \frac{\partial}{\partial t}\int_\tau \rho\,d\tau$ (as div $D=\rho$)

or, $\oint_C \vec{H}\cdot\vec{dl} = \frac{\partial q}{\partial t} = I\left(\text{as}\int_\tau \rho\,d\tau = q\right)$ .... (2)

From equations (1) and (2) it is clear that both yields the same result.

N.B.: From part (*b*) it is clear that in case of circuit (*A*) $D=0$ the current is of conduction type while in case of circuit (*B*) as $J=0$ the current is of displacement type. Further as the two cases are yielding the same result

$$J = J_d \text{ i.e. } I = I_d$$

i.e. conduction current is equal to displacement current.

Example 2. If alternating field $E = E_0 \cos \omega t$ is applied to a conductor, show that the displacement current is negligible as compared to conduction current at any frequency lower than optical frequencies.

Solution. As the given field is

$$E = E_0 \cos \omega t. \qquad \text{..................(1)}$$

And according to Ohm's law

$$J = \sigma E. \qquad \text{..................(2)}$$

So the conduction current will be given by

$$J = \sigma E_0 \cos \omega t. \qquad \text{..................(3)}$$

However the displacement current will be

$$J_d = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\in E) \text{ [as } D = \in E\text{]}$$

i.e. $J_d = \in_r \in_0 \frac{\partial}{\partial t}(E_0 \cos \omega t)$ [as $\in = \in_r \in_0$]

i.e. $J_d = -\omega \in_r \in_0 E_0 \sin \omega t$

$$\therefore J_d = \omega \in_r \in_0 E_0 \cos\left(\omega t + \frac{\pi}{2}\right). \qquad \text{..................(4)}$$

So from equations (3) and (4) it is clear that

$$\frac{J_d}{J} = \frac{\omega \in_r \in_0 \cos\left(\omega t + \frac{\pi}{2}\right)}{\sigma E_0 \cos \omega t}$$

$$\left|\frac{J_d}{J}\right| = \frac{\omega \in_r \in_0}{\sigma}. \qquad \text{..................(5)}$$

But as for a good conductor

$\in_r \approx 1$ and $\sigma \approx 10^7$ mohs/meter

$$\left|\frac{J_d}{J}\right| \approx \frac{2\pi \times f \times 9 \times 10^{-12}}{10^7} \approx f \times 10^{-17}$$

i.e. the displacement current in a good conductor is completely negligible compared to the conduction current at any frequency lower than optical frequencies ($f_{0p} \approx 10^{15}$ hertz).

74.6. Maxwell's Field Equations :

The differential form of four Maxwell's field equations are :

(i) $\nabla \cdot \vec{D} = \rho$

(ii) $\nabla \cdot \vec{B} = 0$

(iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

(iv) $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$.

$\vec{D}$ = electric displacement in c/m^2;

ρ = volume density of charge c/m^3

$\vec{B}$ = magnetic induction in Wb/m^2.

$\vec{E}$ = electric intensity in V/m.

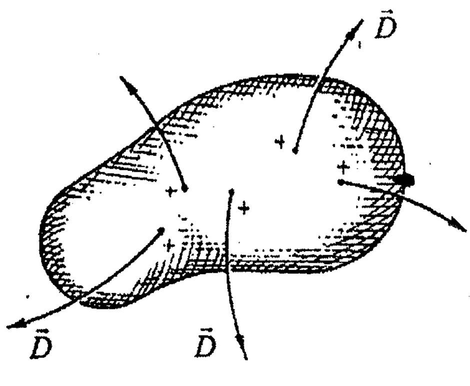
$\vec{H}$ = magnetic field intensity in A/m.

$\vec{J}$ = Current density A/m^2.

❑ 1. $\nabla \cdot D = \rho$

Derivation :

Let us consider a surface 'S' bounding a volume τ within a dielectric. Originally the volume τ contains no net charge but we allow the dielectric to be polarised say by placing it in an electric field. We also delibrately place



so charge on the dielectric body. Thus we have two types of charges–

(a) real charge of density ρ

(b) bound charge density ρ'.

Gauss's law then can be written as

$$\oint_S \vec{E}\cdot\overrightarrow{ds} = \frac{1}{\epsilon_0}\int_\tau (\rho+\rho')d\tau$$

i.e. $\epsilon_0 \oint_S \vec{E}\cdot\overrightarrow{ds} = \int_\tau \rho\, d\tau + \int_\tau \rho'\, d\tau.$ .... (1)

But as the bound charge density ρ' is defined as $\rho' = -\text{div}\ \vec{P}$ and

$$\oint_S \vec{E}\cdot\overrightarrow{ds} = \int_\tau \nabla\cdot\vec{E}\, d\tau$$

So equation (1) becomes

$$\epsilon_0 \int_\tau \nabla\cdot\vec{E}\, d\tau = \int_\tau \rho\, d\tau - \int_\tau div\,\vec{P}\, d\tau \quad [\vec{P} = \text{electric polarization}]$$

i.e. $\int_\tau \nabla\cdot\left(\epsilon_0\vec{E}+\vec{P}\right)d\tau = \int_\tau \rho\, d\tau$

or, $\int \nabla\cdot\vec{D}\, d\tau = \int_\tau \rho\, d\tau; \quad \left(\because \vec{D} = \epsilon_0\vec{E}+\vec{P}\right)$

or, $\int_\tau \left(\nabla\cdot\vec{D}-\rho\right) d\tau = 0.$

Since this equation is true for all volumes, the integrand in it must vanish. Thus we have

$$\nabla\cdot\vec{D} = \rho.$$

When the medium is isotropic the three vectors $\vec{D}, \vec{E}, \vec{P}$ are in the same direction and for small field, $\vec{D}$ is proportional to $\vec{E}$

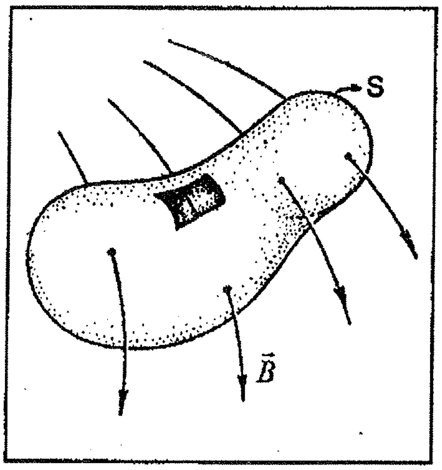
i.e. $\vec{D} = \epsilon\vec{E}.$

where ϵ is called permittivity of the medium.

2. $\nabla\times\vec{B} = 0$

Derivation : Experiments to date have shown that magnetic monopoles do not exist. This in turn implies that the magnetic lines of force are either closed group or go off to infinity. Hence the number of magnetic lines of force entering any arbitrary closed surface is exactly the same leaving it. Therefore the flux of magnetic induction B across any closed surface is always zero *i.e.*

$$\oint_S \vec{B}\cdot\vec{ds} = 0.$$



Transforming this surface integral into volume integral by Gauss' theorem, we get

$$\int_\tau \nabla\cdot\vec{B}\,d\tau = 0$$

But as the surface bounding the volume is quite arbitrary the above equation will be true only when the integrand vanishes *i.e.*

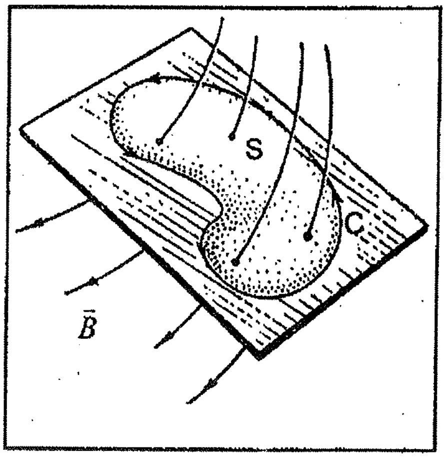
$$\nabla\cdot B = 0.$$

❑ 3. $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

Derivation :

According to Faraday's law of electromagnetic induction we know that the induced e.m.f. is proportional to the rate of change of flux *i.e.*

$$e = -\frac{d\phi_B}{dt}.$$



Now if $\vec{E}$ be the electric intensity at a point, the work done in moving a unit charge through a small distance dl is $\vec{E}\cdot\vec{dl}$. So the work done in moving the unit charge once round the circuit is $\oint_C \vec{E}\cdot\vec{dl}$. Now as e.m.f. is defined as the amount of work done in moving a unit charge round the electric circuit,

$$e = \oint_C \vec{E}\cdot\vec{dl}.$$

So we can write,

$$\oint_C \vec{E}\cdot\vec{dl} = -\frac{d\phi_B}{dt}.$$

But as

$$\phi_B = \int_S \vec{B}\cdot\vec{ds}$$

So

$$\oint_C \vec{E}\cdot\overline{dl} = \frac{d}{dt}\int \vec{B}\cdot\overline{ds}.$$

Transforming the line integral by Stoke's theorem into surface integral we get

$$\int_S \left(\nabla\times\vec{E}\right)\cdot\overline{ds} = -\frac{d}{dt}\int \vec{B}\cdot\overline{ds}.$$

Assuming that surface S is fixed in space and only $\vec{B}$ changes with time above equation yields

$$\int_S \left(\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t}\right)\cdot\overline{ds} = 0$$

As the above integral is true for any arbitrary surface the integrand must vanish.

$$\nabla\times\vec{E} = -\frac{\partial B}{\partial t}.$$

❑ 4. $\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{\Delta}}{\partial t}$

Derivation : From Ampere's circuital law the work done in carring unit magnetic pole once round a closed arbitrary path linked with the current I is expressed by

$$\int_C \vec{H}.dl = I$$

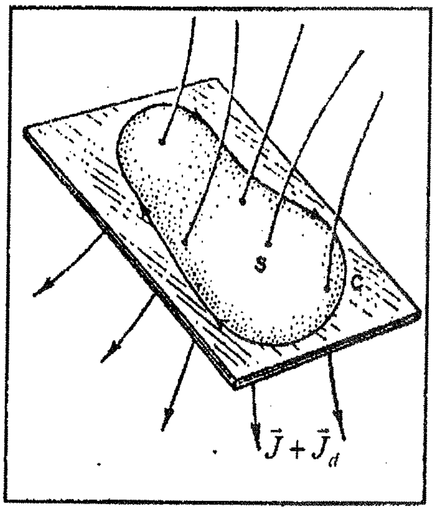
i.e. $\int_C \vec{H}.dl = \int_S \vec{J}\cdot\overline{ds}, \quad \left(\because I = \int \vec{J}\cdot\overline{ds}\right)$

where S is the surface bounded by the closed path C.

Now changing the line integral into surface integral by Stoke's theorem we get

$$\int_S \left(\nabla\times\vec{H}\right)\cdot\overline{ds} = \int_S \vec{J}\cdot\overline{ds}$$

i.e. $\nabla\times\vec{H} = \vec{J}.$



But Maxwell found it to be incomplete for changing electric fields and assumed that a quantity

$$\vec{J}_d = \frac{\partial\vec{D}}{\partial t}$$

called displacement current must also be included in it so that it may satisfy the continuity *i.e.* J must be replaced in the above equation by $\left(\vec{J}+\vec{J}_d\right)$ so that the law becomes

$$\nabla\times\vec{H}=\vec{J}+\vec{J}_d$$

i.e. $$\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}.$$

$\frac{\partial\vec{D}}{\partial t}$ i.e. displacement current density arises when the electric displacement $\vec{D}$ changes with time and is, therefore termed as **displacement current density.** According to Maxwell, it is just as effective as $\vec{J}$ in producing magnetic field.

❑ 74.6.1. Maxwell's Field equations in Integral Form :

(i) Using 1st. equation $\nabla\cdot\vec{D}=\rho,$ and integrating over a volume τ we get,

$$\int_\tau \nabla\cdot\vec{D}\,d\tau=\int_\tau \rho\,d\tau$$

From Gauss-theorem

$$\int_S \vec{D}\cdot\vec{ds}=\int_\tau \rho\,d\tau=q \text{ (net charge in } \tau\text{)}$$

∴ 1st. Maxwell equation signifies that,

the total electric displacement through the surface enclosing a volume is equal to the total charge within the volume.

(ii) Using $\nabla\cdot\vec{B}=0,$ we can write

$$\int_\tau \left(\nabla\cdot\vec{B}\right)d\tau=0$$

or, $$\int_{S} \vec{B}\cdot\vec{ds}=0$$

So, 2nd. equation signifies that the total outward flux of magnetic induction through any closed surface is zero.

(iii) Using 3rd. equation, $\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t},$ we can write

$$\int_S \left(\nabla\times\vec{E}\right)\cdot\vec{ds}=-\int_S \frac{\partial\vec{B}}{\partial t}\cdot\vec{ds}$$

or, $$\oint \vec{E}\cdot\vec{dl}=-\frac{\partial}{\partial t}\int_S \vec{B}\cdot\vec{ds}$$

This signifies :

the electromagnetic force around a closed path is equal to the time derivative of the magnetic flux through any surface found by the path.

(iv) Using 4th equation, $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$, we can write this in integral form as

$$\oint H \cdot \overline{dl} = \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \overline{ds}$$

This signifies :

the magnetomotive force around a closed path is equal to the conduction current plus the time derivative of the electric displacement through any surface bounded by the path.

From the Maxwell's Curl equations we can say that the time varying electric and magnetic fields in empty space are independent, i.e. a **changing electric field being able to generate a magnetic field and vice-versa.** From this we can say that a **time-changing electromagnetic field would propogate energy through empty space with the velocity of light and further, that the light is electro magnetic in nature.**

❑ Maxwells Field Equations in Different Medium

A. Conducting Medium :

In a conducting medium of relative permitivity $\in_r$ and permeability μ_r as

$$\vec{D} = \in \vec{E} = \in_r \in_0 \vec{E}$$

and $\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H}$.

Maxwell's equations reduces to

(i) $\nabla \cdot \vec{E} = \rho / \in_r \in_0$

(ii) $\nabla \cdot \vec{H} = 0$

(iii) $\nabla \times \vec{E} = -\mu_r \mu_0 \frac{\partial \vec{H}}{\partial t}$.

(iv) $\nabla \times \vec{H} = \vec{J} + \in_r \in_0 \frac{\partial \vec{E}}{\partial t}$.

B. **Non-Conducting Medium :**

In a non-conducting media of relative permitivity $\in_r$ and permeability μ_r as

$$\rho = \sigma = 0$$

so $\vec{J} = \sigma\vec{E} = 0$

and hence Maxwell equations becomes

(i) $\nabla \cdot \vec{E} = 0$

(ii) $\nabla \cdot \vec{H} = 0$

(iii) $\nabla \cdot \vec{E} = -\mu_r \mu_0 \dfrac{\partial \vec{H}}{\partial t}$

(iv) $\nabla \cdot \vec{H} = \in_r \in_0 \dfrac{\partial \vec{E}}{\partial t}$.

C. **Free-Space**

In free space as

$$\in_r = \mu_r = 1$$

$$\rho = \sigma = 0$$

Maxwell's equations becomes

(i) $\nabla \cdot \vec{E} = 0$

(ii) $\nabla \cdot \vec{H} = 0$

(iii) $\nabla \times \vec{E} = -\mu_0 \dfrac{\partial \vec{H}}{\partial t}$

(iv) $\nabla \times \vec{H} = \in_0 \dfrac{\partial \vec{E}}{\partial t}$.

❑ **Spl. Note regarding 'Monopole' :**

The asymmetry of electro-magnetism suggests that monopoles (a particle having either north or south magnetic charge) should exist as the concept of magnetic monopoles would bring to electricity and magnetism a symmetry to which nature loves and is lacking in our present picture. **Dirac** has also proved on theoretical grounds that monopoles should exist and predicted their porperties. But so far the magnetic monopoles has frustated all its investigators. The experimenters have failed to find any sign of these. The theorists on the other hand have failed to find any good reason why monopoles should not exist.

❑ 74.7. Energy in Electromagnetic Field :

From Maxwell's equations it is possible to derive very important expression which we shall recognize as the energy principle in an electromagnetic field.

For this let us consider Maxwell's equations Ampere's and Faraday's laws in differential forms i.e.

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \text{................ (1)}$$

and $$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \qquad \text{................ (2)}$$

If we take the scalar product of equation (1) with $\vec{E}$ and of equation (2) with $(\vec{H})$ we get

$$\vec{E} \cdot \nabla \times \vec{H} = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \qquad \text{................ (3)}$$

and $$-\vec{H} \cdot \nabla \times \vec{E} = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t}. \qquad \text{................ (4)}$$

Adding equations (3) and (4) we get

$$-\vec{H} \cdot \nabla \times \vec{E} + \vec{E} \cdot \nabla \times \vec{H} = \vec{J} \cdot \vec{E} + \left[\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right]$$

But by the vector identity

$$\vec{H} \cdot \nabla \times \vec{E} - E \cdot \nabla \times \vec{H} = \nabla \cdot \left(\vec{E} \times \vec{H} \right)$$

above equation reduces to

$$-\nabla \cdot \left(\vec{E} \times \vec{H} \right) = \vec{J} \cdot \vec{E} + \left[\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right] \qquad \text{..................... (5)}$$

Now as $\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \epsilon_r \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \epsilon_r \epsilon_0 \frac{\partial}{\partial t} \left(\vec{E} \cdot \vec{E} \right) = \frac{1}{2} \frac{\partial}{\partial t} \left(\vec{E} \cdot \vec{D} \right)$

and $\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \mu_r \mu_0 \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} = \frac{1}{2} \mu_r \mu_0 \frac{\partial}{\partial t} \left(\vec{H} \cdot \vec{H} \right) = \frac{1}{2} \frac{\partial}{\partial t} \left(\vec{H} \cdot \vec{B} \right)$

So equation (5) reduces to

$$\vec{J}\cdot\vec{E}+\frac{1}{2}\frac{\partial}{\partial t}\left(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B}\right)+\nabla\cdot\left(\vec{E}\times\vec{H}\right)=0. \qquad \text{................ (6)}$$

Each term in the above equation can be given some physical meaning if it is multiplied by an element of volume $d\tau$ and integrated over a volume τ whose enclosing surface is S. Thus the result is,

$$\int_\tau\left(\vec{J}\cdot\vec{E}\right)d\tau+\int_\tau\frac{1}{2}\frac{\partial}{\partial t}\left(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B}\right)d\tau+\int_\tau\nabla\cdot\left(\vec{E}\times\vec{H}\right)d\tau=0$$

But as $\int_\tau\nabla\cdot\left(\vec{E}\times\vec{H}\right)d\tau=\oint_S\left(\vec{E}\times\vec{H}\right)\cdot\overrightarrow{ds}$

so $\int_\tau\left(\vec{J}\cdot\vec{E}\right)d\tau+\int_\tau\frac{1}{2}\frac{\partial}{\partial t}\left(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B}\right)d\tau+\int_S\left(\vec{E}\times\vec{H}\right)\cdot\overrightarrow{ds}=0$ –(A)

To understand what the above equation (A) means, let us now interpret various terms in it –

❍ **Interpretation of** $\int_\tau J\cdot E\,d\tau$ **:**

The current distribution represented by the vector J can be considered as made up of various charges q_i moving with velocity v_i so that

$$\int_\tau\left(\vec{J}\cdot\vec{E}\right)d\tau=\int\left(\vec{J}\,d\tau\right)\cdot\vec{E}$$

i.e. $\int_\tau\left(\vec{J}\cdot\vec{E}\right)d\tau=\int\left(\vec{J}\,ds\,dl\right)\cdot\vec{E}$ (as $d\tau=ds\,dl$)

i.e. $\int_\tau\left(\vec{J}\cdot\vec{E}\right)d\tau=\int\left(J\,ds\right)\overrightarrow{dl}\cdot\vec{E}$

i.e. $\int_\tau\left(\vec{J}\cdot\vec{E}\right)d\tau=\int I\,\overrightarrow{dl}\cdot\vec{E}$ (as $I=J\,ds$)

i.e. $\int_\tau\left(\vec{J}\cdot\vec{E}\right)d\tau=\int\frac{dq}{dt}\,\overrightarrow{dl}\,\vec{E}$ (as $I=dq/dt$)

i.e. $\int_\tau\left(\vec{J}\cdot\vec{E}\right)d\tau=\int dq\,\vec{v}\cdot\vec{E}$ (as $\vec{v}=\overrightarrow{dl}/dt$)

i.e. $\int_\tau\left(\vec{J}\cdot\vec{E}\right)d\tau=\sum_i q_i\left(\vec{v}_i\cdot\vec{E}_i\right)$ .... (7)

where $\vec{E}_i$ denotes the electric field at the position of the charge q_i.

Now electromagnetic force on the ith charged particle is given by the **Lorentz expression**

$$\vec{F}_i=q_i\left(\vec{E}_i+\vec{v}_i\times\vec{B}_i\right).$$

So the work done per unit time on the charge q_i by the field will be

$$\frac{\partial W_i}{\partial t} = \vec{F}_i \cdot \vec{v}_i$$

i.e. $$\frac{\partial W_i}{\partial t} = q_i\left(\vec{E}_i + \vec{v}_i \times \vec{B}_i\right)\cdot \vec{v}_i$$

i.e. $$\frac{\partial W_i}{\partial t} = q_i\vec{v}_i \cdot \vec{E}_i + q_i\vec{v}_i \cdot \left(\vec{v}_i \times \vec{B}_i\right)$$

i.e. $$\frac{\partial W_i}{\partial t} = q_i\vec{v}_i \cdot \vec{E}_i \text{ [as } \vec{v}_i \cdot \left(\vec{v}_i \times \vec{B}_i\right) = \left(\vec{v}_i \times \vec{v}_i\right)\cdot \vec{B}_i = 0]$$

So the rate at which the work is done by the field on the charges

$$\frac{\partial W}{\partial t} = \sum \frac{\partial W_i}{\partial t} = \sum q_i\vec{v}_i \cdot \vec{E}_i. \qquad \text{.......................... (8)}$$

Comparing (7) and (8) we find that *first term i.e.* $\int_\tau (J\cdot E)\,d\tau$ *represents the rate at which the work is done by the field on the charges.*

It is worthy to note here that –

(i) In case of charged particles moving in free space with no external force acting, the work done by the field on the charges appears as kinetic energy of the particles as

$$\frac{\partial W}{\partial t} = \sum \frac{\partial W_i}{\partial t} = \sum \vec{F}_i \cdot \vec{v}_i$$

i.e. $$\frac{\partial W}{\partial t} = \sum m_i \frac{\partial v_i}{\partial t}\cdot v_i \left(\text{as } F_i = m_i \frac{\partial v_i}{\partial t}\right)$$

i.e. $$\frac{\partial W}{\partial t} = \sum \frac{\partial}{\partial t}\left(\frac{1}{2} m_i v_i \cdot v_i\right) = \sum \frac{\partial}{\partial t}\left(\frac{1}{2} m_i v_i^2\right)$$

i.e. $$\frac{\partial W}{\partial t} = \frac{\partial}{\partial t}\sum \frac{1}{2} m_i v_i^2 = \frac{\partial T}{\partial t}.$$

(ii) Inside matter, the work done by the field on the charges *i.e.* the kinetic energy is transferred to random motion, where in described as heat energy or ohmic loss and is given by

$$\frac{\partial W}{\partial t} = \int_\tau \vec{J}\cdot\vec{E}\,d\tau = \int_\tau \frac{J^2}{\sigma}\,d\tau \left(\text{as } E = J/\sigma\right)$$

i.e. $$\frac{\partial W}{\partial t}=\frac{J^2}{\sigma}\times S\times l\ \left(\text{as}\int_\tau d\tau=Sl\right)$$

i.e. $$\frac{\partial W}{\partial t}=\rho\frac{I^2}{S^2}Sl\ \left(\text{as}\,\sigma=l/\rho\text{ and }J=I/S\right)$$

i.e. $$\frac{\partial W}{\partial t}=I^2\rho\frac{l}{S}=I^2R\ \left(\text{as }R=\rho l/S\right).$$

❍ **Ingerpretation** $\int_\tau\frac{1}{2}\frac{\partial}{\partial t}\left(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B}\right)d\tau$

If we allow the volume τ to be arbitrary large, the surface integral in equation (*A*) can be made to vanish by placing the surface *S* Sufficiently far away so that the field cannot propagated to this distance in any finite time i.e. $\int_S\left(\vec{E}\times\vec{H}\right)\cdot ds=0$. So under there circumstances equation (*A*) reduces to

$$\frac{\partial}{\partial t}\int_{all\,space}\frac{1}{2}\left(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B}\right)\tau+\frac{\partial W}{\partial t}=0$$

i.e. $$\frac{\partial}{\partial t}\left[\int_{all\,space}\frac{1}{2}\left(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B}\right)+W\right]=0$$

Thus the quantity in the square bracket is conserved. Now consider a closed system in which the total energy is assumed to be constant. The system consisting of the electromagnetic field and all of the charged particles present in the field. The term *W* represents the total kinetic energy of the particles. We thesefore led to associate the remaining energy term

$$\int_{all}\frac{1}{2}\left(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B}\right)d\tau$$

with the energy of the electro-magnetic field, i.e.

$$U=\int_{all\,space}\frac{1}{2}\left(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B}\right)d\tau. \qquad \text{.................. (9)}$$

The quantity *U* may be considered to be a kind of potential energy. One need not ascribe this potential energy to the charged particles and must consider this term as a field energy. A concept such as energy stored in the field itself rather than residing with the particles is a basic concept of the theory of electromagnetism.

❍ **Interpretation of** $\int_S\left(\vec{E}\times\vec{H}\right)\cdot\overline{ds}$.

Instead of taking the volume integral in equation (*A*) over all space, let us now consider a finite volume. In this case the surface integral of $(\vec{E}\times\vec{H})$ will not in general vanish and so this term must be retained. Let us construct the surface S in such a way that in the interval of time under consideration none of the charged particles will cross this surface. Then for the conservation of energy

$$\frac{\partial}{\partial t}(U+W) = -\oint_S (\vec{E}\times\vec{H})\cdot ds \qquad \text{..................} (10)$$

The left hand side is the time rate of the change of the energy of the field and of the particles contained with in the volume τ. Thus the surface integral $\oint_S (\vec{E}\times\vec{H})\cdot ds$ must be considered as the energy flux flowing out of the volume bounded by the surface S. But by hypothesis no particles are crossing the surface, so the vector $(\vec{E}\times\vec{H})$ is to be interpreted as the **flux of energy of the electromagnetic field and gives the amount of the field energy passing through unit area of the surface in unit time which is normal to the direction of energy flow.** The vector $(E\times H)$ is called Poynting vector and is represented by $\vec{N}$ *i.e.*

$$\vec{N} = (\vec{E}\times\vec{H}). \qquad \text{..................} (11)$$

❍ Interpretation of the Energy Equation

In the light of above all equation (6) in differential form can be written as

$$\vec{J}\cdot\vec{E} + \frac{\partial u}{\partial t} + \nabla\cdot\vec{N} = 0. \qquad \text{..................} (12)$$

In the event that the medium has zero conductivity *i.e.* $\vec{J} = \sigma\vec{E} = 0$, the above equation becomes exactly of the same form as the continuity equation which expresses the law of conservation of charge. We are led by this analogy that the physical meaning of equation 12, 10 or (A) is to represent the law of conservation of energy for electromagnetic phenomena. According equation (10) *the time rate of change of electromagnetic energy with in a certain volume plus the rate at which the work is done by the field on the charges is equal to the energy flowing into the system through the boundary surface of the volume τ per unit time.*

This statement and the concerned equation is known as **Poynting Theorem.**

Poynting vector, $\vec{N} = \vec{E}\times\vec{H}$

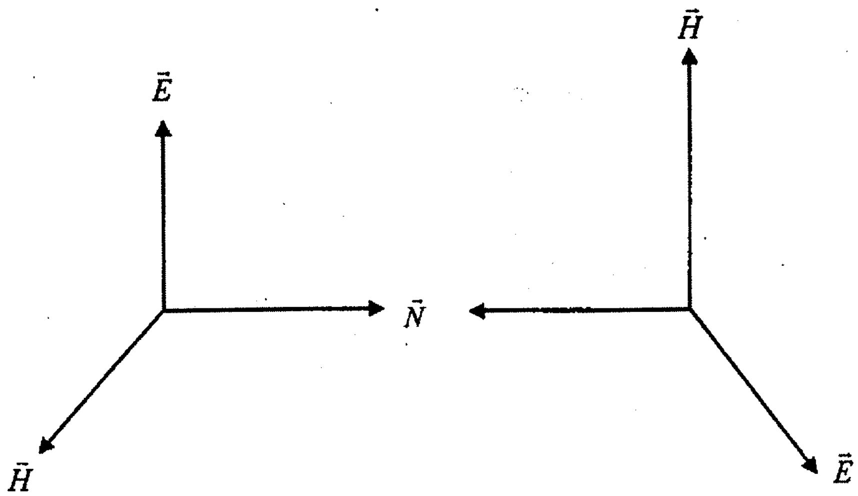
❑ 74.7.1. Poynting Vector

Mathematical form of the Poynting vector,

$$\vec{N} = \vec{E}\times\vec{H}$$

It is interpreted as the amount of field energy passing through unit area of the surface in a direction perpendicular to the plane containing $\vec{E}$ and $\vec{H}$ per unit time.

For example as in a plane electro-magnetic wave E and H are perpendicular to each other and also to the direction of wave propagation, N has a magnitude $EH \sin 90 = EH$ and points in the direction of wave propagation. The dimensions of Poynting vector are (energy/area x time) so the units will be joule/m^2 x second or watt/m^2.



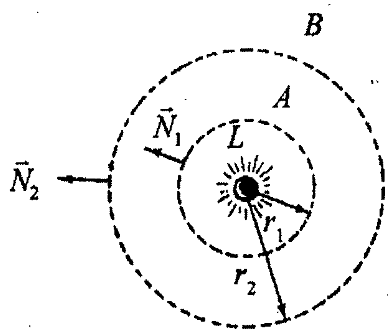
Regarding Poynting vector it is worthy to note that :

(*i*) *Poynting vector at any arbitrary point in the field varies inversely as the square of the distance of the point from the source of radiation.* To understand it consider a source L of electromagnetic radiations which is emitting radiations at the rate of P watts and imagine two concentric spherical surfaces A and B of radii r_1 and r_2 respectively with source being at their common centre. If N_1 and N_2 are the Poynting vector at any point on A and B respectively then

$$N_1 \times 4\pi r_1^2 = N_2 \times 4\pi r_2^2 = P$$

i.e. $$N_1 = \frac{P}{r_1^2}$$

and $$N_2 = \frac{P}{r_2^2}.$$



So in general

$$N \propto \frac{1}{r^2}.$$

(ii) *The definition of Poynting vector is not a mandatory.* Since this vector has been introduced only by way of its divergence, the curl of any arbitrary vector can be added to it without altering the physical facts of the case *i.e.* it is arbitrary to the extent that curl of any vector field can be added to it *i.e.*

$$\vec{N} \equiv \vec{E} \times \vec{H} + \vec{G}$$

where $\vec{G} = \text{curl}\,\vec{M}$

$\vec{M}$ = any arbitrary field vector.

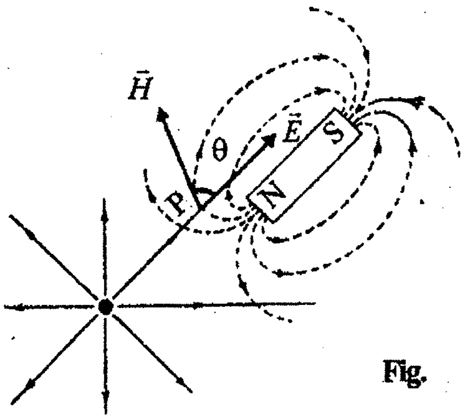
However on various grounds, such as additional term has no physical cosequence, the definition of N is retained as such. This definition is also found to be convenient particularly in electromagnetic theory.

(iii) *If Poynting vector is zero then no electromagnetic energy can flow across a closed surface but if no net field energy is flowing across a closed surface the Poynting vector may or may not be zero.* For example in case of the field due to a point charge in the presence of a magnet at rest as shown in fig. or a calrged capacitor placed between the poles of a permanent magnet if E is not parallel to H, the Poynting vector is not zero as

$$|\vec{N}| = \vec{E} \times \vec{H} = EH \sin\theta \neq 0$$

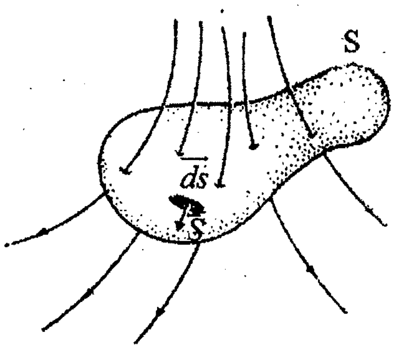
While the flow of energy across any closed surface is zero as

$$\oint_S \vec{N} \cdot ds = \int_\tau \nabla \cdot \vec{N}\, d\tau = \int_\tau \nabla \cdot \left(\vec{E} \times \vec{H}\right) d\tau$$



(a)
For steady fields $N \neq 0$
But there is no flow of energy

Fig.



(b)
$N \cdot ds \neq 0.$
$\oint N \cdot ds = 0.$

i.e. $\oint_S \vec{N} \cdot \vec{ds} = \int_\tau \left(\vec{H} \cdot \text{curl}\,\vec{E} - \vec{E} \cdot \text{curl}\,\vec{H}\right) d\tau = 0$

(as E and H are constant for steady fields).

This is because, N does not determine the rate of flow through small element ds at a point but implies that only the flux of S across a closed surface is significant.

(iv) In case of time varying fields $\vec{N} = \vec{E} \times \vec{H}$ gives the instantaneous value of the Poynting vector and the average value is defined as the average over one complete period i.e.

$$\langle \vec{N} \rangle = \frac{1}{T}\int_0^{2\tau} \left(\vec{E} \times \vec{H}\right) dt.$$

For example in the case of sinusoidally varying fields *i.e.*

$$\vec{E} = \vec{E}_0 \sin \omega t \text{ and } \vec{H} = \vec{H}_0 \sin \omega t$$

$$\langle \vec{N} \rangle = \frac{1}{T}\int_0^{T} \left(\vec{E}_0 \sin \omega t\right) \times \left(\vec{H}_0 \sin \omega t\right) dt$$

i.e. $$\langle \vec{N} \rangle = \frac{1}{T}\left(\vec{E}_0 \times \vec{H}_0\right)\int_0^{2\pi/\omega} \sin^2 \omega t \, dt$$

i.e. $$\langle \vec{N} \rangle = \frac{1}{2}\left(\vec{E}_0 \times \vec{H}_0\right) \frac{\vec{E}_0}{\sqrt{2}} \times \frac{\vec{H}_0}{\sqrt{2}} = \vec{E}_{av} \times \vec{H}_{av}.$$

The importance of Poynting vector lies in the fact that with its help we can interpret various optical phenomena.

❑ **Problem.**

(a) Calculate the value of Poynting vector at the surface of the sun if the power radiated by sun is 3.8×10^{26} watts while its radius is 7×10^8 m.

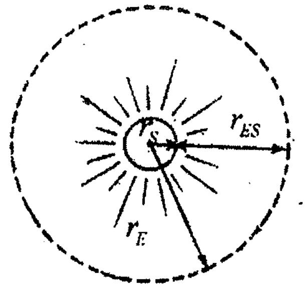
(b) If the average distance between the sun and earth is 1.5×10^{11} meters, show that the average solar energy incident on the earth is ~2 cal./cm². min (called solar constant). Calculate also the amplitudes of the electric and magnetic vectors at the surface of earth assuming that the radiation is a plane wave.

Solution.

(a) If $\langle N_S \rangle$ is the average Poynting vector at the surface of the sun then by its definition

$$\langle N_S \rangle \times 4\pi r^2 = P$$

i.e. $$\langle N_S \rangle = \frac{P}{4\pi r^2} = \frac{3.8 \times 10^{26}}{4\pi \times \left(7 \times 10^8\right)^2}$$



i.e. $\langle N_S \rangle = 6.175 \times 10^7$ watt/meter²

(b) If $\langle N_E \rangle$ is the average value of poynting vector at the surface of earth then

$$\langle N_E \rangle 4\pi r_E^2 = \langle N_S \rangle 4\pi r_S^2 = P$$

i.e. $\langle N_E \rangle = \frac{r_S^2}{r_E^2}\langle N_S \rangle = \frac{r_S^2}{(r_{ES}+r_S)^2}\langle N_S \rangle \approx \frac{r_S^2}{r_{ES}^2}\langle N_S \rangle$

i.e. $\langle N_E \rangle = \left[\frac{7\times10^8}{1.5\times10^{11}}\right]\times 6.175\times10^7 = 1.5\times10^3$ watts/m^2

i.e. $\langle N_E \rangle = \frac{1.5\times10^3\times60}{4.2\times10^4}$ cal / cm^2– min = 2.1 cal./cm^2.min.

Now as in a plane wave

$$H_{av} = \epsilon_0 cE_{av}$$

So $N_{av} = E_{av}H_{av} = \epsilon_0 cE_{av}^2 = \frac{1}{2}\epsilon_0 cE_0^2 \quad \left(\text{as } E_{av} = E_0/\sqrt{2}\right)$

i.e. $E_0\sqrt{\left(\frac{2S_{av}}{\epsilon_0 c}\right)} = \sqrt{\left(\frac{2\times1.5\times10^3}{9\times10^{-12}\times3\times10^8}\right)} \approx 1050$ volts/meter

and as $B_0 = \mu_0 H_r = \mu_0 \epsilon_0 cE_0 = \frac{E_0}{c}$

i.e. $B_0 = \frac{1050}{3\times10^8}$ web/m^2 $= 3.5\times10^{-6}$ web/m^2.

[Here $H_0 = \frac{B_0}{\mu_0} = \frac{3.5\times10^{-6}}{4\pi\times10^{-7}} = \frac{35}{4\pi}$ amp-turn/meter]

74.8 Electromagnetic Potentials and Gauge Transformation :

We know at every point of free space the field vectors satisfy the following equations :

$$\nabla\cdot\vec{D} = \rho$$

$$\nabla\cdot\vec{B} = 0$$

$$\nabla\cdot\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}.$$

Complete description of an electromagnetic field can be obtained by solving Maxwell's equations. The ocess becomes simple if the equations are written in a suitable form. It is often convenient to reduce the number equations by introducing new quantities called "electro-magnetic potentials". We have already adopted this technique in the treatment of static fields : The electrostatic field was expressed in terms of scalar potential $\left(\vec{E}=-\text{grad}\,\phi\right)$ and the magnetic fields in terms of a vector potential $\left(\vec{B}=\nabla\times\vec{A}\right)$. We shall now consider potentials in electromagnetic fields when electric and magnetic fields are time-varying.

In the the time-deponent case the eqution

$$\nabla\cdot\vec{B}=0 \qquad \text{.............. (1)}$$

still holds and hence we can express B in terms of a vector potential,

i.e. $$\boxed{\vec{B}=\nabla\times\vec{A}.} \qquad \text{.............. (2)}$$

We consider the equation which does not involve any currents or charges i.e.

$$\because \nabla\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0 \qquad \text{.............. (3)}$$

Since the curl of the gradient of a scalar function vanishes, the quantity within the brackets can be expressed as a gradient of a scalar function ϕ

$$\vec{E}+\frac{\partial\vec{A}}{\partial t}=-\nabla\phi$$

or, $$\boxed{\vec{E}=-\nabla\phi-\frac{\partial\vec{A}}{\partial t}} \qquad \text{..............(4)}$$

Once $\vec{A}$ and are determined $\vec{B}$ and $\vec{E}$ can be found from equation (2) and (4).

If we add the gradient of any arbitrary scalar function to the vector potential, i.e. say if $\vec{A}$ is changed to

$$\vec{A}=\vec{A}+\nabla\psi \qquad \text{.............. (5)}$$

the magnetic field remains unchanged. But will $\vec{E}$ remain unchanged? It will certainly change if some special precautions are not taken. In order that the addition of $\nabla\psi$ should not affect the electric field, the scalar potential ϕ must be simultaneoulsy transformed to ϕ ' where

$$\phi'=\phi-\frac{\partial\psi}{\partial t} \qquad \text{.............. (6)}$$

You can verify this by substituting $\vec{A}'$ and ϕ' in (4)

Any physical law that can be expressed in terms of the electromagnetic potentials $\vec{A}$ and ϕ remains unaffected by the transformations of the type (5) and (6). These transformations are called **gauge transformations.** Clearly, equations involving potentials must be **gauge invariant.**

In electrostatics we adopted the condition $\nabla \cdot \vec{A} = 0$ which together with $\vec{B} = \nabla \cdot \vec{A}$ specified A. In electromagnetism we have to make a different choice. In order to specify $\vec{A}$ we have to impose an additional condition on A in such a way that it does not change the physics. In other words, it must be consistent with the transformation (5) and (6) so that $\vec{E}$ and $\vec{B}$ remain unaffected.

Substituting (4) in equation $\nabla \cdot \vec{D} = \rho$ we have

$$\nabla \cdot \vec{D} = \nabla \cdot \epsilon_0 \vec{E} = \epsilon_0 \nabla \cdot \left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} \right) = \rho$$

i.e. $-\nabla^2\phi - \frac{\partial}{\partial t}\left(\nabla \cdot \vec{A}\right) = \rho/\epsilon_0$. .... (7)

Equation (*iv*) can be written as

$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$$

i.e. $\nabla \times \frac{\vec{B}}{\mu_0} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{J}$

i.e. $\frac{1}{\mu_0} \nabla \times \left(\nabla \times \vec{A}\right) - \epsilon_0 \frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} \right) = \vec{J}$

i.e. $\nabla \times \left(\nabla \times \vec{A}\right) - \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} \right) = \mu_0 \vec{J}$

i.e. $-\nabla^2 \vec{A} + \left(\nabla \cdot \vec{A}\right) + \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla\phi) + \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J}$ .... (8)

where we have made use of the indentity.

$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot \vec{A}\right) - V^2 \vec{A}.$$

Let us choose A and ϕ such that

$$\nabla \cdot \vec{A} = -\mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = \frac{1}{c^2}\frac{\partial \phi}{\partial t}. \quad \text{.................. (9)}$$

We see at once that with this substitution the two middle terms of (8) cancel and the equation reduces to

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \quad \text{................ (10)}$$

With the condition (9) the equation (7) becomes

$$-\nabla^2\phi - \frac{\partial}{\partial t}\left(\nabla \cdot \vec{A}\right) = -\nabla^2\phi + \mu_0 \in_0 \frac{\partial^2\phi}{\partial t^2} = \rho/\in_0,$$

or $$-\nabla^2\phi - \mu_0 \in_0 \frac{\partial^2\phi}{\partial t^2} = \rho/\in_0 . \qquad \text{................(11)}$$

The choice has yielded two independent equations : one for $\vec{A}$ (Equation 10) and the other for ϕ (Equation 11). $\vec{A}$ is connected with the vector $\vec{J}$ and ϕ with the scalar quantity ρ. Furthermore, both the equations have the same form i.e. both potentials satisfy the same eauations. The condition, thus, introduces complete symmetry between the scalar and vector potentials. For the steady-state, the time derivatives vanish and we have

$$\nabla^2\vec{A} = \mu\vec{J} \text{ and } \nabla^2\phi = -\rho/\in_0$$

The condition (9) is known as **Lorentz gauge condition.** The gauge used in manetostatics viz. $\nabla \cdot \vec{A} = 0$ is called **Coulomb gauge.**

We know that the electric field E and the magnetic field B are invariant under transformation (5) and (6). The potentials thus transformed will have to satisfy the Lorentz condition. Hence, the gauge function ψ which so far remained arbitrary will have to satisfy a certain condition.

Since the original and the transformed potentials have to satisfy Lorentz condition, we have

$$\nabla^2\vec{A} + \mu_0 \in_0 \frac{\partial\phi}{\partial t} = 0 \qquad \text{................(12)}$$

and $$\nabla^2 \cdot \vec{A}' + \mu_0 \in_0 \frac{\partial\phi'}{\partial t} = 0 \qquad \text{...............(13)}$$

i.e. $$\nabla \cdot \left(\vec{A} + \nabla\psi\right) + \mu_0 \in_0 \frac{\partial}{\partial t}\left(\phi - \frac{\partial\psi}{\partial t}\right) = 0$$

i.e. $$\nabla \cdot \vec{A} + \nabla^2\psi + \mu_0 \in_0 \frac{\partial\phi}{\partial t} - \mu_0 \in_0 \frac{\partial^2\psi}{\partial t^2} = 0$$

Hence,

$$\nabla^2\psi - \mu_0 \in_0 \frac{\partial^2\psi}{\partial t^2} = 0. \qquad \text{................(14)}$$

Thus, the restricted gauge transformation

$$\vec{A}' \rightarrow \vec{A} + \nabla\psi$$

$$\phi' \rightarrow \phi - \frac{\partial \psi}{\partial t} \qquad \text{.................(15)}$$

where ψ satisfies the equation.

$$\nabla^2 \psi - \mu_0 \in_0 \frac{\partial^2 \psi}{\partial t^2} = 0$$

preserve the Lorentz condition i.e. Lorentz condition is gauge invariant.

$$\nabla \cdot \vec{A} + \mu \in \frac{\partial \phi}{\partial t} = 0 \rightarrow \textbf{Lorentz Condition}$$

when the vector and scalar potential satisfy it, the gauge is known as **Lorentz gauge.**

Equations (1) and (11) can be written as

$$\Box^2 A = -\mu \vec{J} \qquad \text{.............(16)}$$

and $$\Box^2 \phi = -\rho / \in_0 \qquad \text{...........(17)}$$

Equation (16) and (17) are known as D'Alembertian equations; and can be solved in general by a method similar to that we used to solve Poisson's equation.

❍ **Problem :**

Show that an electromagnetic field possesses momentum. What is electromagnetic momentum density? Write the relation between electromagnetic momentum density vector and Poynting vector.

Solution :

We know the force on a region containing both charges and currents is

$$\vec{F} = \int_\tau \left(\rho \vec{E} + \vec{J} \times \vec{B}\right) d\tau.$$

If P_{mech} is the sum of momenta of all the particles

$$\frac{D\vec{P}_{mech}}{dt} = \int_\tau \left(\rho \vec{E} + \vec{J} \times \vec{B}\right) d\tau.$$

From Maxwell's equations

$$\rho = \vec{\nabla} \cdot \vec{D}; J = \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t}$$

$$\therefore \frac{d\vec{P}_{mech}}{dt} = \int_\tau \left\{ (\nabla\cdot\vec{D})\vec{E} + \left(\nabla\times\vec{H} - \frac{\partial\vec{D}}{\partial t}\right)\times\vec{B} \right\} d\tau$$

$$= \int_\tau \left\{ (\nabla\cdot\vec{D})\vec{E} + \vec{B}\times\frac{\partial\vec{D}}{\partial t} - \vec{B}\times(\nabla\times\vec{H}) \right\} d\tau$$

Since $\frac{\partial}{\partial t}(\vec{D}\times\vec{B}) = \vec{D}\times\frac{\partial\vec{B}}{\partial t} + \frac{\partial\vec{D}}{\partial t}\times\vec{B}$

$$\frac{d\vec{P}_{mech}}{dt} = \int_\tau \left[(\nabla\cdot\vec{D})\vec{E} + \left(\vec{D} - \frac{\partial\vec{B}}{\partial t}\right) - \frac{\partial}{\partial t}(\vec{D}\times\vec{B}) - \vec{B}\times(\nabla\times\vec{H}) \right] d\tau$$

Because $\nabla\cdot\vec{B} = 0$, addition of $(\nabla\cdot\vec{B})\vec{H}$ to the square bracket does not alter the result.

$$\therefore \frac{d\vec{P}_{mech}}{dt} + \frac{d}{dt}\int_\tau (\vec{D}\times\vec{B})\, d\tau$$

$$= \int_\tau \left[(\nabla\cdot\vec{D})\vec{E} + (\nabla\cdot\vec{B})\vec{H} - \{\vec{D}\times(\nabla\times\vec{E})\} \right.$$

$$= \int_\tau \left[(\nabla\cdot\vec{D})\vec{E} + (\nabla\cdot\vec{B})\vec{H} - \{\vec{D}\times(\nabla\times\vec{E})\} - \{\vec{B}\times(\nabla\times\vec{H})\} \right] d\tau \qquad \left(\because \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\right)$$

Clearly, the integral in the second term of the left-hand side represents momentum. Since it is not associated with the mass of particles and consists only of fields, we identify it as the electromagnetic momentum P_{field}.

The vector $\vec{g} = [\vec{D}\times\vec{B}]$ is called **electromagnetic momentum density.** The right-hand side can be converted into a surface integral and identified as momentum flow.

The momentum density vector g is related to the Poynting vector N and the relation is given below :

$$\vec{g} = [\vec{D}\times\vec{B}] = [\epsilon\vec{E}\times\mu\vec{H}] = \mu\epsilon[\vec{E}\times\vec{H}] = \mu\epsilon\vec{N}$$

❑ 74.8. Summary :

(i) Equation of continuity : $\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$

(ii) Equation for the decay of free-charge : $\rho = \rho_0 e^{-t/\tau}; \tau = \frac{\epsilon}{\sigma}$ = relaxation time.

(iii) Displacement current density $\vec{J}_d = \frac{\partial\vec{D}}{\partial t}$.

(iv) Differential form of Maxwell's four field equation :

$$\nabla \cdot \vec{D} = \rho \text{ (Gauss's law)}$$

$$\nabla \cdot \vec{B} = \rho \text{ (no name)}$$

$$\nabla \cdot \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \text{ (Faraday's law)}$$

$$\nabla \cdot \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \text{ (Ampere's law with Maxwell's corrections)}$$

(v) Integral form of Maxwell's field equations :

$$\int_S \vec{D} \cdot \overline{ds} = \int_\tau \rho \, d\tau$$

$$\int_S \vec{B} \cdot \overline{ds} = 0$$

$$\oint \vec{E} \cdot \overline{dl} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot \overline{ds}$$

$$\oint \vec{H} \cdot \overline{dl} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \overline{ds}$$

(vi) Poynting Theorem :

$$\int_\tau \left(\vec{J} \cdot \vec{E} \right) d\tau + \int_\tau \frac{1}{2} \frac{\partial}{\partial t} \left(\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B} \right) d\tau + \oint \left(\vec{E} \times \vec{H} \right) \cdot \overline{ds} = 0$$

Poynting vector : $\vec{N} = \vec{E} \times \vec{H}$.

(vii) Fields interms of Electromagnetic potentials :

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}.$$

(viii) Gauge Transformation :

$$\vec{A}' = \vec{A} + \nabla \psi$$

$$\phi' = \phi - \frac{\partial \psi}{\partial t}.$$

(ix) Lorentz gauge condition : $\nabla \cdot \vec{A} = -\mu_0 \in_0 \dfrac{\partial \phi}{\partial t}$.

Coulomb gauge condition : $\nabla \cdot \vec{A} = 0$.

(x) Electromagnetic momentum density : $\vec{g} = (\vec{D} \times \vec{B})$

Relation between momentum density and Poynting vector

$$\vec{g} = \mu \in \vec{N}.$$

❑ 74.9. Self-Assessment Questions :

1. Starting from Maxwell's equations prove (i) Coulomb's law; (ii) Equation of contunity.
2. Show that the total current flowing out of some volume must be equal to the rate of decrease of charge within the volume, assuming that no sources and sinks are present within that volume.
3. Show that the potentials at the position defined by the vector $\vec{r}$ in uniform electric and magnetic fields may be written as,

$$\phi = -\vec{E} \cdot \vec{r} \text{ and } \vec{A} = \frac{1}{2}(\vec{B} \times \vec{r})$$

4. In a source free region if

$$\vec{A} = \hat{i}\, x^4 + \hat{k}\, Z^2 t^2$$

Compute field vectors $\vec{E}$ and $\vec{B}$.

5. Proving that Ampere's circuital law is not applicable with conservation of charge in general, explain how Maxwell resolved this contradiction by introducing the idea of displacement current.
6. Give physical significance of (i) Displacemtn current and (ii) Poynting vector.
7. State and prove Maxwell equations. What form these equations will presume if the medium is non-conducting.
8. State the law of conservation of charge; hence deduce the equation of continuity.
9. Write down Maxwell's field equations and prove Poynting's theorem relating to the flow of energy at a point in space in an electromagnetic field.
10. State Ampere's circuital law and discuss why and how it was modified to include the displacement current.
11. Express electromagnetic fields interms of electromagnetic potentials A and ϕ. Explain Lorentz condition.
12. Show that the electromagnetic potentials define the field vectors uniquely but they themselves are non-unique.

74.10. References :

1. Introduction to Electrodynamics – David J. Giffiths. Prenctice Hall of India Pvt. Ltd.
2. Elementary Electromagnetic Theory (3 Vol.) – Chirgwin, Plumpton, Kilmister. Pergamon Press. Oxford.
3. Introduction to Electromagnetic field and waves – Corson and Lorrain. Freeman, Sau Francisco.
4. The Fundamentals of Electromagnetism – Cullwick. Cambridge University Press.
5. Applied Electromagnetism – Hammond, Pergamon Press.
6. Field and Wave Electrodynamics – Johnson. McGraw-Hill, New York.
7. Foundations of Electromagnetic Theory – Reitz and Milford. Addison-Wesley.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper-VII

Group-A

Module No. - 75
(Electromagnetic Theory)

Module Structure :

❑ 75.1 Introduction :

Maxwell's field equations predict the existance of electromagnetic waves. The character of Propagation of electromagnetic waves in free space, non-conducting and conducting media can be predicted from the mathematical analysis. The nature and velocity of electromagnetic waves can also be known. The interaction of electromagnetic waves with matter will give the laws of reflection and transmission.

In this connection it is mentioned that, a wave is a disturbance of a continuous medium that propagates with a fixed shape at constant velocity. Electromagnetic waves can propogate even in vacuum. In the presence of absorption, the wave generally diminishes in size as it moves; if the medium is dispersive different frequencies travel at different speeds; in two or three dimensions, as the wave spreads out its amplitude will decrease. The general wave equation in one dimension,

$$\frac{\partial^2 f}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2};$$

v = velocity of propagation

$f(x,t)=g(x-vt); g$ = any function.

Most general solution to the wave equation,

$$f(x,t)=g(x-vt)+h(x+vt)$$

A wave is said to be a plane wave, as long as the wavelength is much less than the radius of curvature of the wave-front.

It is found that electromagnetic energy can be radiated only if a charged particle is undergoing acceleration.

To get the quantitative idea regarding total radiation emitted and the angular distribution of radiation the velocity dependent electromagnetic potentials (retarded potentials) are incorporated.

❑ 75.2 Objectives :

After going through this module one will be able to know the following topics :

- Idea and nature of electromagnetic waves;
- Propagation of electromagnetic waves in vacuum;
- Propagation of electromagnetic waves in dielectrics;
- Propagation of electromagnetic waves in conducting media;
- Skin affect and skin depth;
- Reflection and Transmission of electromagnetic waves;
- Fresnel's formula;
- Retarded Potentials;
- Lienard-Wiechart Potentials;
- Field produced by moving charged particles;
- Radiation produced from an accelerated charged particle at low velocity and high velocity.

❑ 75.3 Propagation of Plane Electromagnetic Waves :

Maxwell's four field equations are :

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

We know also for homogeneous, isotropic and source free medium,

$\vec{D} = \in \vec{E}; \in =$ permittivity

$\vec{B} = \mu \vec{H}; \mu =$ permeability

$\vec{J} = \sigma\vec{E}; \sigma =$ conductivity

- **Homogeneous medium :** The medium for which $\in, \mu$ and σ are constant throughout is called ***homogeneous medium***. Here the properties of the medium do not vary from point to point.
- **Isotropic medium :** The medium for which $\in$ is a scalar constant so that $\vec{D}$ and $\vec{E}$ are same in all direction is called ***isotropic medium.***
- **Source free medium :** The medium which does not contain any impressed voltages or currents is called ***source free medium.***

The laws of field vectors $\vec{E}$ and $\vec{H}$ are derived in studying the propagation of plane electromagnetic waves. The solutions of the differential equations (Maxwell equations) will provide us the desired results.

75.3.1 Electromagnetic waves in Free Space :

Free Space means where there is no charge or current.

$$\vec{J} = 0$$
$$\rho = 0$$
$$\sigma = 0$$
$$\mu = 0$$
$$\in = 0$$
$$\in_r = \frac{\in}{\in_0} = 1$$
$$\vec{D} = \in_0 \vec{E}$$
$$\vec{B} = \mu_0 \vec{H}$$

$\in_0 =$ permittivity in free space $= 8.8542 \times 10^{-12}\ F/m.$

$\mu_0 =$ permeability in free space $= 4\pi \times 10^{-7}\ H/m.$

- **The Wave Equation for $\vec{E}$ and $\vec{B}$ in Free Space :**

We know $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$

$$\therefore \nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

or, $\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}=-\frac{\partial}{\partial t}(\nabla\times\vec{B})$

or, $\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}=-\frac{\partial}{\partial t}(\nabla\times\mu_0\vec{H})$

or, $\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}=-\mu_0\frac{\partial}{\partial t}(\nabla\times\vec{H})$

or, $\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}=-\mu_0\frac{\partial}{\partial t}\left(\frac{\partial\vec{D}}{\partial t}\right)$

or, $\nabla^2\vec{E}=\mu_0\in_0\frac{\partial^2\vec{E}}{\partial t^2}$ .... (1)

$[\because\nabla\cdot\vec{E}=0]$

Similarly, using $\nabla\times\vec{B}=\mu_0\in_0\frac{\partial\vec{E}}{\partial t}$, we get,

$$\nabla\times(\nabla\times\vec{B})=\nabla(\nabla\cdot\vec{B})-\nabla^2\vec{B}=\nabla\times\left(\mu_0\in_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\nabla(\nabla\cdot\vec{B})-\nabla^2\vec{B}=\mu_0\in_0\frac{\partial}{\partial t}(\nabla\times\vec{E})$$

or, $-\nabla^2\vec{B}=-\mu_0\in_0\frac{\partial^2\vec{B}}{\partial t^2}$ .... (2)

$\left[\because\nabla\cdot\vec{B}=0\ \&\ \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\right]$

Equations (1) and (2) are separate equations for $\vec{E}$ and $\vec{B}$ but they are of second order.

In vacuum, then each Cartesian components of $\vec{E}$ and $\vec{B}$ satisfies the three-dimensional wave equations.

$$\nabla^2 y=\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}.$$

So, Maxwell's equations imply that empty space supports the propagation of electromagnetic waves, travelling at a speed,

$$v=\frac{1}{\sqrt{\mu_0\in_0}}$$

$=3.00\times10^8$ m/s [Putting the values $\in_0,\mu_0$]

$= c$ (velocity of light in vacuum)

Hence, we can conclude that,

(a) *there exists electromagnetic waves in space;*

(b) *electromagnetic waves travel in free space with the velocity of light in vacuum.*

Indirectly it may be inferred that light is an electromagnetic wave.

The differential wave equation in vacuum for $\vec{E}$ and $\vec{H}$ are,

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

and $$\nabla^2 \vec{H} - \frac{1}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

Let the solution of the above two equation be,

$$\vec{E} = \vec{E}_0 e^{-i(wt - \vec{k}\cdot\vec{r})}$$

& $$\vec{H} = \vec{H}_0 e^{-i(wt - \vec{k}\cdot\vec{r})}$$

where $\vec{k}$ = propagation vector $= \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi f}{c}\hat{n} = \frac{w}{c}\hat{n}$;

here $\hat{n}$ = unit vector in the direction of wave propagation.

$$\therefore \nabla \rightarrow i\vec{k} \text{ and } \frac{\partial}{\partial t} \rightarrow (-iw)$$

So, Maxwell's equations in free space can be written as,

$$\vec{k}\cdot\vec{E} = 0 \text{ .... } (i)$$

$$\vec{k}\cdot\vec{H} = 0 \text{ .... } (ii)$$

$$-\vec{k}\times\vec{H} = w\epsilon_0 \vec{E} \text{ .... } (iii)$$

and $$\vec{k}\times\vec{E} = w\mu_0\vec{H} \text{ .... } (iv)$$

Hence from the above equations, regarding plane electromagnetic waves in free space, it may be inferred that,

(a) From equation (*i*) $\vec{E}$ is perpendicular to the direction of propagation, and from equation (*ii*) $\vec{H}$ is perpendicular to the direction of propagation. So, **electromagnetic waves are transverse in nature.**

(b) From equation (*iii*) $\vec{E}$ is perpendicular to both $\vec{H}$ and $\vec{k}$; and from equation (*iv*) $\vec{H}$ is perpendicular to both $\vec{k}$ and $\vec{E}$, field vectors $\vec{E}$ and $\vec{H}$ are also mutually perpendicular in a plane wave. This all in turn implies that **in a plane electromagnetic waves vectors $\vec{E}$, $\vec{H}$ and $\vec{k}$ are orthogonal.**

(c) From equation (*iv*),

$$\vec{k}\times\vec{E} = w\mu_0\vec{H}$$

$$\therefore \vec{H} = \frac{k}{w\mu_0}(\hat{n}\times\vec{E}) \quad [\because \vec{k} = \hat{n}k]$$

$$\vec{H} = \frac{\hat{n}\times\vec{E}}{c\mu_0} = c\in_0(\hat{n}\times\vec{E})$$

$$\left[\because k = \frac{w}{c} \, and \, \mu_0\in_0 = \frac{1}{c^2}\right]$$

or, $$\vec{B} = \frac{\vec{n}\times\vec{E}}{c}$$

and $$\left|\frac{E}{H}\right| = \frac{E_0}{H_0} = c\mu_0 = \sqrt{\left(\frac{\mu_0}{\in_0}\right)} = Z_0\,(\text{say}) = 377\Omega.$$

As the ratio $\left|\frac{E}{H}\right|$ is real and positive, the vectors $\vec{E}$ and $\vec{H}$ are in plase; i.e. when E has its maximum value H has also maximum value. It is also clear that in an electro-magnetic wave the amplitude of electric vector $\vec{E}$ is Z_0 times that of the magnetic vector $\vec{H}$.

(d) The Poynting vector for a plane electromagnetic wave will be given by,

$$\vec{N} = \vec{E}\times\vec{H} = \vec{E}\times\frac{\hat{n}\times\vec{E}}{c\mu_0}$$

or, $$\vec{N} = \frac{(\vec{E}\cdot\vec{E})\hat{n} - (\vec{E}\cdot\hat{n})\vec{E}}{c\mu_0}$$

$$= \frac{1}{c\mu_0}E^2\hat{n} \quad [\because \vec{E}\cdot\hat{n} = 0]$$

$$= \in_0 cE^2\hat{n}$$

$$= \frac{E^2\hat{n}}{Z_0} \quad \left[\because \frac{1}{c\mu_0} = c\in_0 = \frac{1}{Z_0}\right]$$

$$\therefore \langle N \rangle = \epsilon_0 c \langle E^2 \rangle \hat{n}$$

$$= \frac{\langle E^2 \rangle \hat{n}}{Z_0}$$

Conclusion ;

For electromagnetic waves in free space

(*i*) *e.m. waves travels with speed of light;*

(*ii*) *e.m. waves are transverse in nature;*

(*iii*) $\vec{E}$, $\vec{H}$ *and* $\vec{k}$ *constitute a right hand orthogonal set;*

(*iv*) *the field vectors* $\vec{E}$ *and* $\vec{H}$ *are in the same phase;*

(*v*) *the direction of energy flow is the direction of wave propagation;*

(*vi*) *the energy density associated with e.m. waves in free space propagates with the speed of light.*

75.3.2 Electromagnetic waves in anisotropic dielectric medium

A non-conducting medium whose properties depend on direction is called **anisotropic dielectric.**

In anistropic dielectric medium the relative permitivity is not longer a scalar and to deal with wave propagation we refer all fields to the principal axes so that

$$D_x = \epsilon_x \epsilon_0 E_x ; D_y = \epsilon_y \epsilon_0 E_y \text{ and } D_z = \epsilon_z \epsilon_0 E_z \quad \text{.................} (1)$$

Further since the medium is non conducting *i.e.*

$$J = 0, \ \rho = 0 \text{ and } \mu_r = 1$$

So Maxwell's equation in an anisotropic dielectric medium reduces to

$$\left.\begin{array}{ll} \text{div}\,\vec{D} = 0 & (a) \\ \text{div}\,\vec{H} = 0 & (b) \\ \text{curl}\ \vec{H} = \dfrac{\partial \vec{D}}{\partial t} & (c) \\ \text{curl}\ \vec{E} = -\mu_0 \dfrac{\partial \vec{H}}{\partial t} & (d) \end{array}\right\} \quad \text{.................} (2)$$

It is important to note that in this case though

$$\text{div}\,\vec{D} = 0 \quad \text{div}\,\vec{E} \neq 0$$

because $\vec{D}$ in general is not in the direction of $\vec{E}$.

Now let us consider a plane wave advancing with phase velocity v along the direction of wave normal $\vec{n}$ (*i.e.* wave vector $\vec{k}$). Let it be,

$$\begin{Bmatrix} \vec{E} \\ \vec{H} \end{Bmatrix} = \begin{Bmatrix} \vec{E}_0 \\ \vec{H}_0 \end{Bmatrix} e^{-i(\omega t - \vec{k}\cdot\vec{r})} \qquad \text{.................. (3)}$$

So the operators ∇ and $\frac{\partial}{\partial t}$ will be

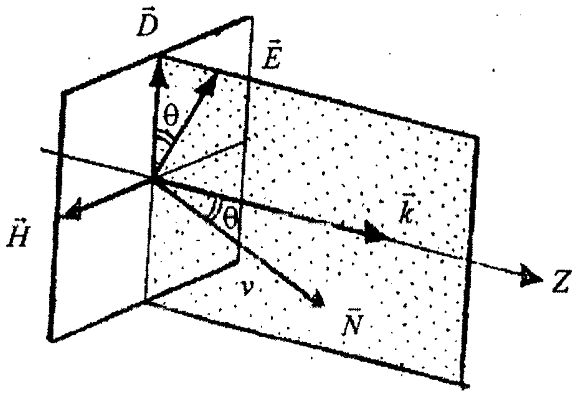
$$\rightarrow \vec{ik} \text{ and } \frac{\partial}{\partial t} \rightarrow (-i\omega).$$

And in terms of these operations equation (2) and (3) can be written as

$$\left.\begin{array}{ll} \vec{k}\cdot\vec{D} = 0 & (a) \\ \vec{k}\cdot\vec{H} = 0 & (b) \\ -\vec{k}\times\vec{H} = \omega\vec{D} & (c) \\ \vec{k}\times\vec{E} = \mu_0\omega\vec{H} & (d) \end{array}\right\} \qquad \text{.................. (4)}$$

From these form of Maxwell's equations it is clear that :

(*i*) *The E.M.W. are transverse in nature w.r.t.* $\vec{D}$ *and* $\vec{H}$ (and not w.r. to $\vec{E}$ and $\vec{H}$ as in a isotropic



media). It is because according to 4 (*a*) $\vec{k}$ is $\perp$ to $\vec{D}$ while according to 4 (*b*) $\vec{k}$ is $\perp$ to $\vec{H}$ *i.e.* $\vec{k}$ is $\perp$ to both $\vec{H}$ and $\vec{D}$ as shown below in fig.

(*ii*) *The vector $\vec{D}$ and $\vec{H}$ and $\vec{k}$ are orthogonal* it is because according to equation 4(*b*) $\vec{k}$ is $\perp$ to $\vec{H}$ while according to equation 4(*c*) $\vec{D}$ is $\perp$ to both $\vec{k}$ and $\vec{H}$.

(*iii*) *The vectors $\vec{D}$, $\vec{E}$ and $\vec{k}$ are co-planer.* This is because according to equation 4(*c*).

$$\vec{D} = -\frac{\vec{k}\times\vec{H}}{\omega} \qquad \text{...............(5)}$$

while according to 4 (*d*)

$$\vec{H} = \frac{\vec{k}\times\vec{E}}{\mu_0\omega}. \qquad \text{.................(6)}$$

So from equations (5) and (6)

$$\vec{D} = -\frac{1}{\mu_0\omega^2}\left[\vec{k}\times\left(\vec{k}\times\vec{E}\right)\right]$$

i.e. $$\vec{D} = -\frac{1}{\mu_0\omega^2}\left[\left(\vec{k}\cdot\vec{E}\right)\vec{k} - k^2\vec{E}\right] \qquad \text{...............(7)}$$

(*iv*) *In an anisotropic media energy is not propagated in general in the direction of wave propagation (i.e. the direction of $\vec{k}$ and $\vec{N}$ are not same) and the Poynting vector $\vec{N}$ is coplaner with $\vec{D}, \vec{E}$ and $\vec{k}$. This is because the Poynting vector is given by*

$$\vec{N} = \vec{E}\times\vec{H}$$

i.e. $\vec{N}$ is normal to the plane of $\vec{E}$ and $\vec{H}$ and not to the plane of $\vec{D}$ and $\vec{H}$ (which is the direction of $\vec{k}$).

Example. ***Show that in case of propagation of plane electro-magnetic waves through an anisotropic dielectric***

$$\frac{\cos^2\alpha}{v^2 - v_x^2} + \frac{\cos^2\beta}{v^2 - v_y^2} + \frac{\cos^2\gamma}{v^2 - v_z^2} = 0$$

where v is the phase velocity of the wave, α, β and γ are the angles which the wave vector makes with the principla axes and $v_x = c/\sqrt{(\in_z)}$, *and*

Solution. In case of propogation of plane E.M.W. in an anisotropic dielectric we know that

$$\vec{D} = -\frac{1}{\mu_0 \omega^2}\left[\left(\vec{k}\cdot\vec{E}\right)\vec{k} - k^2\vec{E}\right] \text{ (from equation. 7)}$$

i.e. $$\vec{D} = -\frac{k^2}{\mu_0 \omega^2}\left[\vec{E} - \left(\vec{n}\cdot\vec{E}\right)\vec{n}\right] \quad \left(\text{as } \vec{k} = \vec{n}\cdot\vec{E}\right)$$

i.e. $$\vec{D} = -\frac{k^2}{\mu_0 v^2}\left[\vec{E} - \left(\vec{n}\cdot\vec{E}\right)\vec{n}\right] \quad \left(\text{as } k = \omega/v\right) \qquad \text{.............(1)}$$

Equation (1) in terms of components can be written as

$$D_x = -\frac{k^2}{\mu_0 v^2}\left[\vec{E} - \left(\vec{n}\cdot\vec{E}\right)\vec{n}\right]_x$$

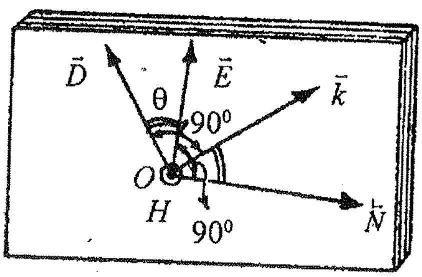
i.e. $$D_x = -\frac{1}{\mu_0 v^2}\left[E_x - \left(\vec{n}\cdot\vec{E}\right)\cos\alpha\right] \qquad \text{.............(2)}$$

Similarly

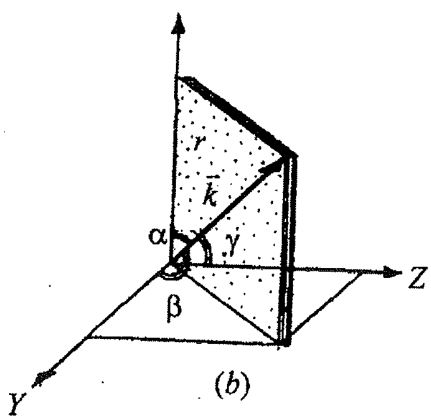
$$D_y = -\frac{1}{\mu_0 v^2}\left[E_y - \left(\vec{n}\cdot\vec{E}\right)\cos\beta\right] \qquad \text{.............(3)}$$

And $$D_z = -\frac{1}{\mu_0 v^2}\left[E_z - \left(\vec{n}\cdot\vec{E}\right)\cos\gamma\right] \qquad \text{.............(4)}$$

Now as in an anisotropic medium



(*a*)
Vector *H* is normal to the plane of the paper and outward



(*b*)

$$k_x = k\cos\alpha$$

$$k_y = k\cos\beta$$

$$k_z = k\cos\gamma$$

$$D_x = \epsilon_x \epsilon_0 E_x \qquad D_y = \epsilon_y \epsilon_0 E_Y \qquad \text{and} \qquad D_z = \epsilon_z \epsilon_0 E_z$$

i.e. $$E_x = \frac{D_x}{\epsilon_x \epsilon_0} \qquad E_z = \frac{D_z}{\epsilon_z \epsilon_0} \qquad \text{and} \qquad E_z = \frac{D_z}{\epsilon_z \epsilon_0}$$

or $$E_x = \frac{c^2 \mu_0}{\epsilon_x} D_x \qquad E_y = \frac{c^2 \mu_0}{\epsilon_y} D_y \qquad \text{and} \qquad E_z = \frac{c^2 \mu_0}{\epsilon_z} D_z$$

(as $1/\epsilon_0 = \mu_0 c^2$)

or $$E_x = \mu_0 v_x^2 D_x \qquad E_y = \mu_0 v_y^2 D_Y \qquad \text{and} \qquad E_z = \mu_0 v_z^2 D_z \qquad \text{.............(5)}$$

[as $c/\sqrt{(\epsilon_x)} = v_x, c/\sqrt{(\epsilon_y)} = v_y$ and $c/\sqrt{(\epsilon_z)} = v_z$]

So on substituting the values of E_x, E_y and E_z from equation (5) in 2, 3 and 4 respectively we get

$$D_x = \frac{1}{\mu_0 v^2}\left[\mu_0 v_x^2 D_x - \left(\vec{n} \cdot \vec{E}\right)\cos\alpha\right]$$

i.e. $$D_x = \frac{\cos\alpha}{\mu_0\left[v_x^2 - v^2\right]}\left(\vec{n} \cdot \vec{E}\right) \qquad \text{.............(6)}$$

Similarly

$$D_y = \frac{\cos\beta}{\mu_0\left[v_y^2 - v^2\right]}\left(\vec{n} \cdot \vec{E}\right) \qquad \text{.............(7)}$$

and $$D_z = \frac{\cos\gamma}{\mu_0\left[v_z^2 - v^2\right]}\left(\vec{n} \cdot \vec{E}\right) \qquad \text{.............(8)}$$

Now as vectors $\vec{D}$ and $\vec{k}$ are perpendicular to each other in case of a plane electromagnetic wave propagating through an anisotropic dielectric *i.e.*

$$\vec{k} \cdot \vec{D} = 0$$

i.e. $$k_x D_x + k_y D_y + k_z D_z = 0$$

i.e. $$k\left[D_x \cos\alpha - D_y \cos\beta + D_z \cos\gamma\right] = 0$$

(as $k_x = k \cos\alpha$ and so on)

i.e. $$D_x \cos\alpha + D_y \cos\beta + D_z \cos\gamma = 0 \qquad \text{.............(9)}$$

On substituting the values of D_x, D_y and D_z from equation (6), (7) and (8) in (9) we get

$$\frac{\cos^2\alpha}{\left(v_x^2-v^2\right)}\frac{\left(\vec{n}\cdot\vec{E}\right)}{\mu_0}+\frac{\cos^2\beta}{\left(v_y^2-v^2\right)}\frac{\left(\vec{n}\cdot\vec{E}\right)}{\mu_0}+\frac{\cos^2\gamma}{\left(v_z^2-v^2\right)}\frac{\left(\vec{n}\cdot\vec{E}\right)}{\mu_0}=0$$

i.e. $$\frac{\cos^2\alpha}{v^2-v_x^2}+\frac{\cos^2\beta}{v^2-v_y^2}+\frac{\cos^2\gamma}{v^2-v_z^2}=0.$$

This is the required result and is known as **Fresnel law of normal velocities** or **Fresnel law for the wave or phase velocity.**

❑ 75.3.3 Electromagnetic waves in conducting medium :

We know that Maxwell equations are

$$\left.\begin{aligned}&\text{div}\,\vec{D}=\rho\\&\text{div}\,\vec{B}=\rho\\&\text{curl}\,\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}\\&\text{curl}\,\vec{E}=-\frac{\partial\vec{B}}{\partial t}\end{aligned}\right\}\quad\text{with}\quad\left\{\begin{aligned}&\vec{J}=\sigma\vec{E}\\&\vec{D}=\in\vec{E}\\&\vec{B}=\mu\vec{H}\end{aligned}\right.$$

In case of a conducting media as $\rho=0$, so field equations reduces to

$$\left.\begin{aligned}&\text{div}\,\vec{E}=0 &&\text{.......}(a)\\&\text{div}\,\vec{H}=0 &&\text{.......}(b)\\&\text{curl}\,\vec{H}=\sigma\vec{E}+\in\frac{\partial\vec{E}}{\partial t} &&\text{.......}(c)\\&\text{curl}\,\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t} &&\text{.......}(d)\end{aligned}\right\}\qquad\text{.............(1)}$$

Now if

(*i*) we take the curl of equation (c) then

$$\nabla\times\left(\nabla\times\vec{H}\right)=\nabla\times\left[\sigma\vec{E}+\in\frac{\partial\vec{E}}{\partial t}\right]$$

i.e. grad div $\vec{H} - \nabla^2 \vec{H} = \sigma\left(\nabla \times \vec{E}\right) + \in \frac{\partial}{\partial t}\left(\nabla \times \vec{E}\right)$ .... (2)

But from equations 1 (*b*) and 2 (*d*)

$$\nabla \cdot \vec{H} = 0 \qquad \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}.$$

So equation (2) reduce to

$$\nabla^2 \vec{H} = \sigma\mu \frac{\partial \vec{H}}{\partial t} + \mu \in \frac{\partial^2 \vec{H}}{\partial t^2}$$

i.e. $\nabla^2 \vec{H} - \sigma\mu \frac{\partial \vec{H}}{\partial t} - \mu \in \frac{\partial^2 \vec{H}}{\partial t^2} = 0.$ .... (*A*)

(*ii*) we take the curl of equation (*d*) then

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\mu \nabla \times \left(\frac{\partial \vec{H}}{\partial t}\right)$$

i.e. grad div $\vec{E} - \nabla^2 \vec{E} = \mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right).$ .... (3)

But from equations 1 (*a*) and 2 (*c*)

$$\nabla \cdot \vec{E} = 0 \quad \text{and} \quad \nabla \times \vec{H} = \sigma \times \vec{E} = \sigma \vec{E} + \in \frac{\partial \vec{E}}{\partial t}.$$

So equation (3) reduces to

$$\nabla^2 \vec{E} = \mu \frac{\partial}{\partial t}\left(\sigma \vec{E} + \in \frac{\partial \vec{E}}{\partial t}\right)$$

i.e. $\nabla^2 \vec{E} - \sigma\mu \frac{\partial \vec{E}}{\partial t} - \mu \in \frac{\partial^2 \vec{E}}{\partial t^2} = 0.$ .... (*B*)

Equations (*A*) and (*B*) are known as '**equations of telegraphy**'. It is worthy to note here that second term in equation (*A*) arises due to conduction current *J*, while third arises due to displacement current $\left(\partial \vec{D}/\partial t\right)$. Equations (*A*) and (*B*) are of the from

$\nabla^2 \psi - \sigma\mu \frac{\partial \psi}{\partial t} - \mu \in \frac{\partial^2 \psi}{\partial t^2} = 0.$ .... (4)

Now as equation (4) represents an attenuated wave of the form

$$\psi = \psi_0 e^{-i(\omega t - \vec{k}^* \cdot \vec{r})}$$

where $\vec{k}^*$ is the complex wave vector, the solutions of equations (*A*) and (*B*) must be of the form

$$\begin{Bmatrix} \vec{E} \\ \vec{H} \end{Bmatrix} = \begin{Bmatrix} \vec{E}_0 \\ \vec{H}_0 \end{Bmatrix} e^{-i(\omega t - \vec{k}^* \cdot \vec{r})} \qquad \text{.............. (C)}$$

The above from of field vectors suggest operator ∇ is equivalent to $\overline{ik^*}$ while $\partial/\partial t$ to $(-i\omega)$. And so equations (*B*) and (*A*) yields

$$\left(-k^{2*} + i\sigma\mu\omega + \mu \in \omega^2\right)\begin{Bmatrix} \vec{E} \\ \vec{H} \end{Bmatrix} = 0.$$

Since the above result is valid for any arbitrary $\vec{E}$ or $\vec{H}$

$$k^{2*} - i\sigma\mu\omega - \mu \in \omega^2 = 0$$

i.e. $\quad k^{2*} = \mu \in \omega^2 \left[1 + \dfrac{i\sigma}{\in \omega}\right]$ ....: (5)

Equation (5) shows that propagation constant is complex and may be expressed as

$k^* = \alpha + i\beta$ .... (6)

i.e. $\quad k^{2*} = \alpha^2 - \beta^2 + i2\alpha\beta$ .... (7)

Comparing equation (5) and (7) we get

$\alpha^2 - \beta^2 = \mu \in \omega^2$ .... (8)

and $\quad 2\alpha\beta = \mu\sigma\omega$ .... (9)

and on substituting the value of β from equation (9) and (8)

$$\alpha^2 - \frac{\mu^2\sigma^2\omega^2}{4\alpha^2} = \mu \in \omega^2$$

i.e. $\quad \alpha^4 - \mu \in \omega^2\alpha^2 - \dfrac{\mu^2\sigma^2\omega^2}{4} = 0$

which gives

$$\alpha^2 = \frac{\mu \in \omega^2 \pm \sqrt{\left[\left(\mu \in \omega^2\right)^2 + \mu^2\sigma^2\omega^2\right]}}{4}$$

i.e. $\alpha = \pm\omega\sqrt{\left(\frac{\mu \in}{2}\right)}\left[1 \pm \sqrt{1+\left(\frac{\sigma}{\in \omega}\right)^2}\right]^{1/2}$ .... (10)

Now as in the limit $\sigma \to 0, \mu \to \mu_0, \in \to \in_0$ and so equation (5) reduces as

$$k^{2*} = \mu_0 \in_0 \omega^2$$

while equation (6) reduces to

$$k^* = \alpha.$$

So that

$$\alpha \to \omega\sqrt{(\mu_0 \in_0)}.$$

This in turn implies that correct value of α given by equation (10) is

$$\alpha = \omega\sqrt{\left(\frac{\mu \in}{2}\right)}\left[1+\sqrt{\left\{1+\left(\frac{\sigma}{\in \omega}\right)^2\right\}}\right]^{1/2}$$

i.e. $\alpha = \omega\sqrt{\left(\frac{\mu \in}{2}\right)}\left[\sqrt{\left\{1+\left(\frac{\sigma}{\in \omega}\right)^2\right\}}+1\right]^{1/2}$ .... (D)

Similarly putting the value of α in terms of β from equation (9) in (8) and solving we get

$\beta = \omega\sqrt{\left(\frac{\mu \in}{2}\right)}\left[\sqrt{\left\{1+\left(\frac{\sigma}{\in \omega}\right)^2\right\}}-1\right]^{1/2}$ .... (E)

and in terms of these values of α and β the field vectors given by equation (C) may expressed as

$$\begin{Bmatrix} \vec{E} \\ \vec{H} \end{Bmatrix} = \begin{Bmatrix} \vec{E}_0 \\ \vec{H}_0 \end{Bmatrix} e^{-i[\omega t-(\alpha+i\beta)\vec{n}\cdot\vec{r}]}$$

i.e. $\begin{Bmatrix} \vec{E} \\ \vec{H} \end{Bmatrix} = \begin{Bmatrix} \vec{E}_0 \\ \vec{H}_0 \end{Bmatrix} e^{-\beta\vec{n}\cdot\vec{r}}\, e^{-i(\omega t-\alpha\vec{n}\cdot\vec{r})}$ .... (11)

From equation (11) it is clear that the field vectors are spatially attenuated and are propated through conducting medium at a wave speed

$$v = \frac{\omega}{\alpha} = \sqrt{\left(\frac{2}{\mu \in}\right)}\left[\sqrt{1+\left(\frac{\sigma}{\in \omega}\right)^2}+1\right]^{-1/2}$$

The quantity β is a measure of attenuation and is called **absorption coefficient.** So if we get

$$\vec{n}\cdot\vec{r} = \delta = 1/\beta.$$

Equation (11) yields

$$\vec{E}_{(\delta)} = \frac{1}{2}\left[\vec{E}_0 e^{-i(\omega t - \alpha \vec{n}\cdot\vec{r})}\right]$$

i.e. $[E_0]_\delta = \frac{1}{e}[E_0]_{surface}$ .... (12)

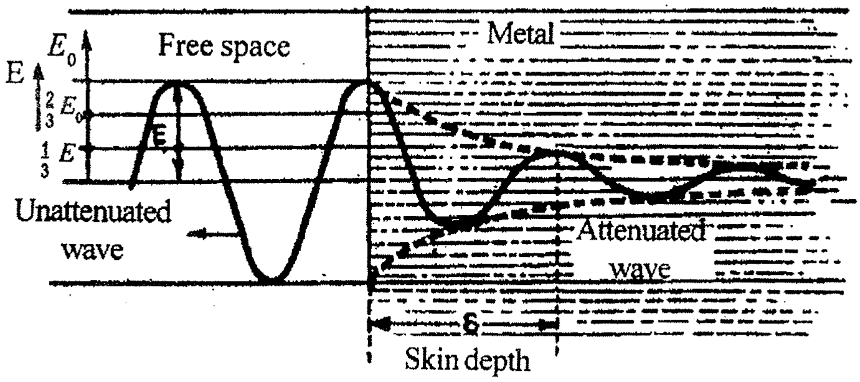
i.e. at a distance δ the amplitude of the wave becomes (1/*e*) *i.e.* 0.369 times of its value at the surface of the conducting medium. This distance is called **skin depth.** *i.e.*

$$\delta = \frac{1}{\beta} = \sqrt{\left(\frac{2}{\mu \in \omega^2}\right)}\left[\sqrt{\left\{1+\left(\frac{\sigma}{\in \omega}\right)^2\right\}}-1\right]^{-1/2}$$ .... (13)

From equation (13) it is clear that for good conductors

$$\delta \approx \sqrt{\left(\frac{2}{\mu \in \omega^2}\right)\times\left(\frac{\sigma}{\in \omega}\right)^{-1}} = \sqrt{\left(\frac{2}{\mu \in \omega}\right)}$$

at high frequencies the current will flow only on the surface of a conductor. For example in silver or copper



($\sigma \sim 10^7$ mho/m) at typical micro wave frequency of 100M Hertz $\delta = 10^{-4}$ cm. and in sea waver ($\sigma \sim 5$ mho/m) at frequency of 30K hertz $\delta \approx 10^2$ cm. This is why for micro wave frequency transmission it is necessary only to have a thin coating of silver (or copper) on even a poor conductor and radio communication with submerged submarines become increasingly difficult at depths of several meters. Further if $\sigma \to \infty, \delta \to 0$ *i.e.* $\vec{E}$ and $\vec{H}$ are both zero inside supper conductors.

Now as according to equation (C)

$$\nabla \to \overline{ik^*} \text{ and } \frac{\partial}{\partial t} \to (-i\omega)$$

So Maxwell solid equations (i) in terms of complex wave vector can be written as

$$\left.\begin{aligned} &\vec{k}^* \cdot \vec{E} = 0 && \text{............ }(a) \\ &\vec{k}^* \cdot \vec{H} = 0 && \text{............ }(b) \\ &-\vec{k}^* \times \vec{H} = (\in \omega + i\sigma)\vec{E} && \text{............ }(c) \\ \text{and } &-\vec{k}^* \times \vec{E} = \mu\omega\vec{H} && \text{............ }(d) \end{aligned}\right\} \qquad \text{.............(14)}$$

From equations 14 (a) it is obvious that $\vec{E}$ is $\perp$ to $\vec{k}^*$ while according to 14 (b) $\vec{H}$ is $\perp$ to $\vec{k}^*$ *i.e.* the vectors $\vec{E}$ and $\vec{H}$ are perdicur to the direction of propagation *i.e.* electromagnetic wave is transverse w.r.t. $\vec{E}$ and $\vec{H}$. And as according to equation 4(d) $\vec{H}$ is $\perp$ to $\vec{k}$ and $\vec{E}$ both, $\vec{E}$ and $\vec{H}$ are also mutually $\perp$.

$$\vec{H} = \frac{k^*}{\mu\omega}(\vec{n} \times \vec{E}) = \frac{\alpha + i\beta}{\mu\omega}(\vec{n} \times \vec{E})$$

i.e. complex quantity *i.e.* the vectors $\vec{E}$ and $\vec{H}$ in a conductor are not in phase. So writing

$$k^* = \alpha + i\beta = re^{i\phi} = r\cos\phi + ir\sin\phi$$

$$\left.\begin{aligned} &i.e.\ r = \sqrt{(\alpha^2 + \beta^2)} = \omega\sqrt{(\mu \in)}\left[1 + \left(\frac{\sigma}{\omega \in}\right)^2\right]^{1/4} \\ &\text{and } \phi = \tan^{-1}\frac{\beta}{\alpha} = \frac{1}{2}\tan^{-1}\left[\frac{\sigma}{\omega \in}\right] \end{aligned}\right\} \qquad \text{.............}(F)$$

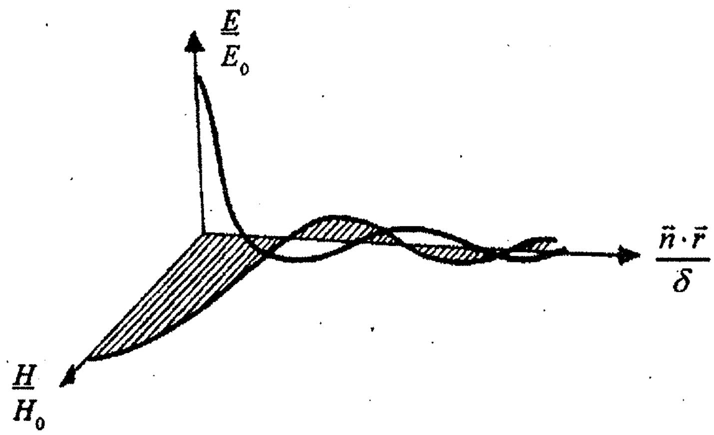
we get

$$\vec{H} = \frac{1}{\omega\mu}\omega\sqrt{(\mu \in)}\left[1 + \left(\frac{\sigma}{\in \omega}\right)^2\right]^{1/4} e^{i\phi}(\vec{n} \times \vec{E})$$

i.e. $\vec{H} = \sqrt{\left(\frac{\epsilon}{\mu}\right)}\left[1+\left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/4} e^{i\phi}\left(\vec{n}\times\vec{E}\right)$ ....(15)

Equation (15) clearly shows that there is a time lag of H, behind E by an amount of phase angle ϕ (given by equation F) and that the magnitude of electric and magnetic vectors are related by

$$|H| = \sqrt{\left(\frac{\epsilon}{\mu}\right)}\left[1+\left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/4} |E| \quad ....(16)$$



Attenuated electromagnetic wave in which E is leading H is phase

Further the Poynting vector in this case will be

$$\langle S\rangle = \frac{1}{2}\mathrm{Re}\left(\bar{E}\times\vec{H}^{*}\right)$$

i.e. $\langle S\rangle = \frac{1}{2}\mathrm{Re}\sqrt{\left(\frac{\epsilon}{\mu}\right)}\left[\bar{E}\times\left(\vec{n}\times\bar{E}^{*}\right)\right]\left[1+\left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/4} e^{i(-\phi)}$

i.e. $\langle S\rangle = \frac{1}{2}\sqrt{\left(\frac{\epsilon}{\mu}\right)}\left[E_0^2 n e^{-2\beta(\vec{n}\cdot\vec{r})}\right]\left[1+\left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/4} \cos\phi$

For good condensors $\phi = \frac{\pi}{4}$ and $\frac{\sigma}{\epsilon\omega} \gg 1$ so

$$\langle S \rangle = 1\sqrt{\left(\frac{\sigma}{2\mu\omega}\right)} E_{rms}^2 e^{-2\beta(\hat{n}\cdot\vec{r})} \vec{n} \qquad \text{...........(17)}$$

And the energy density u will be given by

$$u = u_e + u_m$$

$$\text{with } u_e = \frac{1}{2}\text{Re}\left(\frac{\in}{2}\vec{E}\cdot\vec{E}^*\right) = \frac{\in}{2}E_{rms}^2 e^{-2\beta(\hat{n}\cdot\vec{r})} \qquad \text{...........(18)}$$

$$\text{and } u_m = \frac{1}{2}\text{Re}\left(\frac{\mu}{2}\vec{H}\cdot\vec{H}^*\right)$$

$$= \frac{\mu}{2}\left[\frac{\in}{\mu}\left\{1+\left(\frac{\sigma}{\in\omega}\right)^2\right\}^{1/2}\right] E_{rms}^2 e^{-2\beta(\hat{n}\cdot\vec{r})}$$

$$i.e. \quad u_m \left[1+\left(\frac{\sigma}{\in\omega}\right)^2\right]^{1/2} u_e \qquad \text{...........(19)}$$

From equations (16) and (17) it is evident that in a conducting medium, the intensity of the wave (energy flow) and the energy density are damped or attenuated as the wave progress. This energy is lost because of the resistive heating of the medium. It is also obvious from equation (19) that the magnetic and electric energy densities are different with in a conducting medium. In fact as $\sigma \rightarrow \infty$ the energy density becomes directly magnetic (as in the case of super conductors) and as $\sigma \rightarrow 0, u_m = u_e$ (as in case of insulators). For $\omega \rightarrow 0$ the energy density is again entirely magnetic and is in agreement with the fact that a conductor can not support a static electric field.

❍ **Spl. Note :** As

$$\left|\frac{J}{J_d}\right| = \left|\frac{\sigma E}{\in(\partial E/\partial t)}\right| = \left|\frac{\sigma E}{-i\omega\in E}\right| = \frac{\sigma}{\in\omega}$$

So there are two possibilities.

Case I. $\left(\frac{\sigma}{\in\omega}\right) \ll 1$. The conduction current is much less than the displacement current. Such a situation holds good for a poor conductor or for even moderately good conductor at high frequency. In such a situation

$$\alpha = \omega\sqrt{(\mu\in)}\left[1+\frac{1}{4}\left(\frac{\sigma}{2\in\omega}\right)^2\right]^{1/2} \text{ and } \beta = \frac{\sigma}{2}\sqrt{\left(\frac{\mu}{\in}\right)}$$

Therefore we see that, in so far as the frequency dependence of the conductivity may be neglected, the attennation factor β is independent of the frequency.

If σ is sufficiently small, we have

$$\alpha \to \omega\sqrt{(\mu \in)}\ \beta \to 0$$

so that

$$k \to \alpha \to \omega\sqrt{(\mu \in)} \to k, \phi \to 0 \text{ and } |H| \to \sqrt{(\in/\mu)}|E|$$

These results are exactly the same as in case of propagation of E.M.W. through isotrpic delectric. So we conclude that at very high frequencies a conductor can behave like dielectrics for the propagation of E.M.W.

Case II. $\left(\frac{\sigma}{\in \omega}\right) \gg 1$. The conduction current is much greater than the metals $(\sigma/\varepsilon) \approx 10^{17}$, so the conduction current always dominates for frequencies below about 10^{16} Hertz which includes all of the radio waves, micro waves and visible light as well as part of the X-ray region. In such a situation

$$\alpha \approx \beta = \frac{1}{\delta} = \sqrt{\left(\frac{\omega\mu\sigma}{2}\right)}.$$

so that

$$|k^*| \approx \sqrt{(\omega\mu\sigma)}, \phi \approx \frac{\pi}{4} \text{ and } |H| = \sqrt{\left(\frac{\sigma}{\mu\omega}\right)}|E|.$$

Thus *H* and *E* are approximately 45⁰ out phase and *H* dominates $|E|$ in magnitude.

○ Conclusion :

In case of propagation of E.M.W. in conducting medium :

(*i*) The wave gets attenuated with penetration.

(*ii*) The wave is transverse w.r.t. *E* and *H*.

(*iii*) The vectors *E* and *H* are mutually perpendicular. *H* is much greater than *E* in magnitude but lags in phase.

(*iv*) Energy flows in the direction of wave propagation but is damped off exponentially.

(*v*) Magnetic energy density is much greater then electric energy density and both are damped off exponentially.

75.4. Reflection & Refraction of electromagnetic waves :

Here we shall discuss about the phenomena of reflection and refraction of electromagnetic waves i.e. in general the interaction of electromagnetic waves with matter. At the boundary between two dielectrics, the electromagnetic waves obey the familiar laws of reflection and refraction. The derivations of these phenomena will be based on general electromagnetic equations. Here we shall discuss first the boundary conditions which the electric and magnetic fields must satisfy at the boundary.

75.4.1. Boundary conditions for the electromagnetic field vectors :

The boundary conditions are –

(*i*) The normal component of magnetic induction $\vec{B}$ is continuous across boundary, i.e.

$$Bn_1 = Bn_2$$

(*ii*) The tangential component of E is continuous across the interface, i.e.

$$Et_1 = Et_2$$

(*iii*) The normal component of electric displacement $\vec{D}$ is discontinuous across the interface, i.e.

$$Dn_1 - Dn_2 = \sigma; \quad \sigma = \text{surface-charge density at the interface.}$$

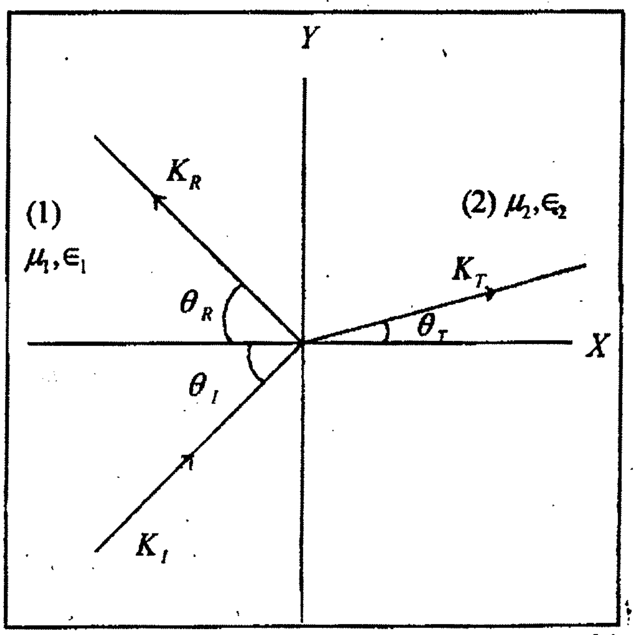
(*iv*) The tangential component of magnetic intensity $\vec{H}$ is continuous across the surface separating the two dielectrics.

$$Ht_1 = Ht_2$$

75.4.2. Reflection and Refraction at the plane boundary of two dielectric media :

Let us consider two non-conducting $(\sigma = 0)$ dielectric media designated as '1' and '2' characterized by constants μ_1, ϵ_1 and μ_2, ϵ_2 and separated by a plane $x = 0$.

Let a plane electromagnetic wave is incident obliquely on the plane boundary, as shown in fig.



We can express the fields for the incident, reflected and transmitted waves as :

For incident wave : $\vec{E}_I = \vec{E}_{0I} e^{i(\vec{k}_I \cdot \vec{r} - \omega_I t)}$; $\vec{H}_I = \dfrac{\vec{K}_I \times \vec{E}_I}{\mu_1 \omega_I}$

For reflected wave : $\vec{E}_R = \vec{E}_{OR} e^{i(\vec{k}_R \cdot \vec{r} - \omega_R t)}$; $\vec{H}_R = \dfrac{\vec{K}_R \times \vec{E}_R}{\mu_1 \omega_R}$

For transmitted wave : $\vec{E}_T = \vec{E}_{OT} e^{i(\vec{k}_T \cdot \vec{r} - \omega_T t)}$; $\vec{H}_T = \dfrac{\vec{K}_T \times \vec{E}_T}{\mu_2 \omega_T}$

E_{OI}, E_{OR}, E_{OT} are time indpendent scalar amplitudes which may be complex.

The tangential components of $\vec{E}$ and $\vec{H}$ can be continuous across the boundary at all points and at all times only if the exponentials are the same at the boundary for all three fields. This is possible if

$$w_I = w_R = w_T \text{ and } \vec{K}_I \cdot \vec{r} = \vec{K}_R \cdot \vec{r} = \vec{K}_T \cdot \vec{r}.$$

(i) **The frequency remains unchanged in the reflected and transmitted waves, i.e. the frequency of the wave remains unchanged by reflection and refraction :**

(ii) $\vec{K}_I \cdot \vec{r} = \vec{K}_R \cdot \vec{r} = \vec{K}_T \cdot \vec{r}$, shows that **all the propagation vectors are coplaner, i.e. the incident, reflected and refracted waves all lie in the same plane but normal to the boundary surface.**

Now if we choose $\vec{r}$ to lie in the boundary plane (i.e. $\hat{n} \cdot \vec{r} = 0$ where $\hat{n}$ is a unit vector normal to the plane) and in the plane of the propagation vector, it follows that,

$$K_I \sin\theta_I = K_R \sin\theta_R = K_T \sin\theta_T$$

Now $K_I = K_R$ as they are in the same medium

$$\therefore \theta_I = \theta_R$$

(iii) **In case of reflection, the angle of incidence is equal to the angle of reflection.**

Since $K_I \sin\theta_I = K_T \sin\theta_T$.

$$\therefore \frac{\sin\theta_I}{\sin\theta_T} = \frac{K_T}{K_I} = \sqrt{\frac{\mu_2 \in_2}{\mu_1 \in_1}} \qquad \left[\because K = \omega\sqrt{\mu \in}\right]$$

For non-magnetic materials $\mu_2 = \mu_1$

$$\therefore \frac{\sin\theta_I}{\sin\theta_T} = \sqrt{\frac{\in_2}{\in_1}} = \frac{n_2}{n_1}; \text{ **Snell's law.**}$$

where n_1, n_2 are the refractive indices of the medium '1' and '2' respectively.

(iv) In case of refraction or transmission, the ratio of the sine of the angle of incidence to the sine of angle of refraction is equal to the ratio of refractive indices of the two media. This is Snell's law

75.4.3. Fresnel's laws :

The formulae relating the amplitude of the reflected and transmitted waves with that of incident wave are known as **Fresnel's fumulae or Fresnel's laws.**

The boundary conditions of D_n and B_n are automatically satisfied provided the conditions on E_t and H_t are met. The conditions are :

$$\left(\vec{E}_I + \vec{E}_R\right)\times\hat{n} = \vec{E}_T \times \hat{n} \qquad \text{.................... (1)}$$

and $$\left(\vec{H}_I + \vec{H}_R\right)\times\hat{n} = \vec{H}_T \times \hat{n} \qquad \text{.................... (2)}$$

The equation (2) can be written as,

$$\left(\vec{K}_I \times \vec{E}_I + \vec{K}_I \times \vec{E}_R\right)\times\hat{n} = \left(\vec{K}_T \times \vec{E}_T\right)\times\hat{n} \qquad [\because \mu_1 = \mu_2] \qquad \text{.................... (3)}$$

$$[\text{and } K_I = K_R]$$

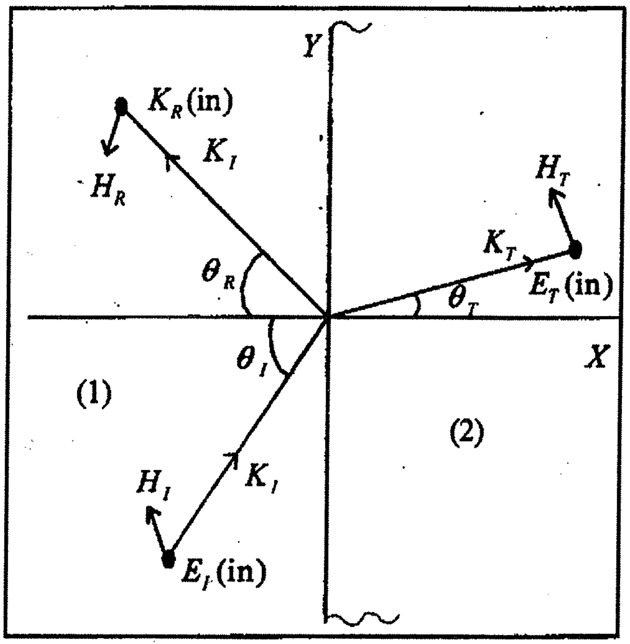
Let us consider two seperate situations :

(i) **$\vec{E}$ is polarised perpendicular to the plane of incidence;**

(ii) **$\vec{E}$ is polarized parallel to the plane of incidence.**

75.4.3.1 E polarized perpendicular to the plane of incidence.

The field vectors corresponding to this situation are shown in fig.



Here the electric field vectors are directed away from the observer.

The conditions (1) and (3) give,

$$E_{OI} + E_{OR} = E_{OT} \text{ .... (4)}$$

and $K_I E_{OI} \cos\theta_I - K_I E_{OR} \cos\theta_R = K_T E_{OT} \cos\theta_T$

or, $\left(E_{OI} - E_{OR}\right)\cos\theta_I = \frac{K_T}{K_I} E_{OT} \cos\theta_T$ .... (5) $\left[\because \theta_I = \theta_R\right]$

Solving these two equations, we get

$$\frac{E_{OR}}{E_{OI}} = \frac{\cos\theta_I - \frac{K_T}{K_I}\cos\theta_T}{\cos\theta_I + \frac{K_T}{K_I}\cos\theta_T}$$

$$= \frac{\cos\theta_I - \frac{n_2}{n_1}\cos\theta_T}{\cos\theta_I + \frac{n_2}{n_1}\cos\theta_T}$$

$$= \frac{\cos\theta_I - \frac{\sin\theta_I}{\sin\theta_T}\cos\theta_T}{\cos\theta_I + \frac{\sin\theta_I}{\sin\theta_T}\cos\theta_T} \qquad \left[\because \frac{n_2}{n_1} = \frac{\sin\theta_I}{\sin\theta_T}\right]$$

or, $$\frac{E_{OR}}{E_{OI}} = \frac{\sin\left(\theta_T - \theta_I\right)}{\sin\left(\theta_T + \theta_I\right)} \qquad \text{..................} (A)$$

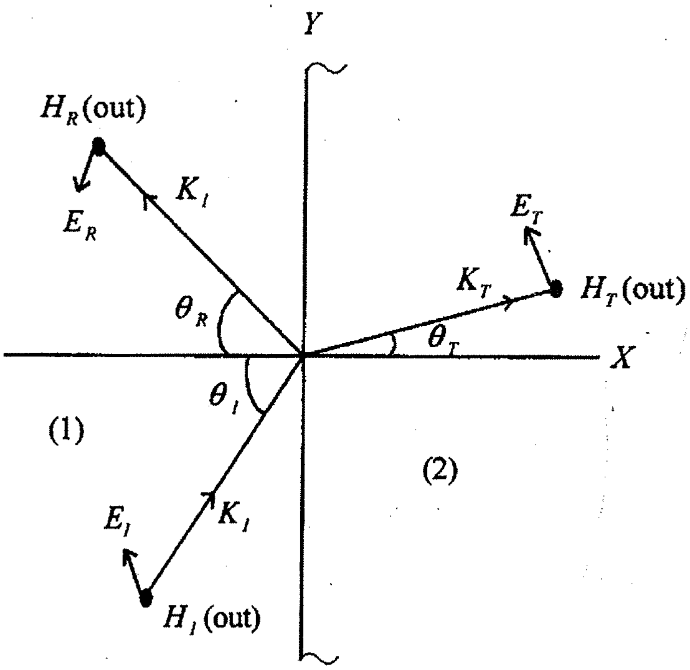
and $$\frac{E_{OT}}{E_{OI}} = \frac{2\cos\theta_I}{\cos\theta_I + \frac{K_T}{K_I}\cos\theta_T}$$

or, $$\frac{E_{OT}}{E_{OI}} = \frac{\sin\theta_T}{\sin\left(\theta_I + \theta_T\right)} \qquad \text{..................} (B)$$

Equation (A) gives the ratio of the amplitudes of the reflected and incident waves. If $n_2 > n_1$, the ratio is –ive, indicating that the reflection of the wave results in a phase change of π i.e. the electric vector of the reflected wave oscillates 180°out of phase with that in the incident wave. The ratio $\frac{E_{OT}}{E_{OI}}$ (equation-B) is always positive.

❑ 75.4.3.2. *E* in the plane of incidence

The field vectors corresponding to this situation are shown in fig.



Here the boundary conditions give,

$$E_{OI}\cos\theta_I - E_{OR}\cos\theta_R = E_{OT}\cos\theta_T$$

i.e. $(E_{OI} - E_{OR})\cos\theta_I = E_{OT}\cos\theta_T$ $\quad [\because \theta_I = \theta_R]$ .... (6)

and $K_I E_{OI} + K_I E_{OR} = K_T E_{OT}$ $\quad [\because K_I = K_R]$

i.e. $E_{OI} + E_{OR} = \frac{n_2}{n_1} E_{OT}$ $\quad \left[\because \frac{K_T}{K_I} = \frac{n_2}{n_1}\right]$ .... (7)

Solving we get,

$$\frac{E_{OR}}{E_{OI}} = \frac{\cos\theta_I - \frac{n_2}{n_1}\cos\theta_T}{\cos\theta_I + \frac{n_2}{n_1}\cos\theta_T}$$

or, $\frac{E_{OR}}{E_{OI}} = \frac{\tan(\theta_I - \theta_T)}{\tan(\theta_I + \theta_T)}$ .... (C)

and $\frac{E_{OT}}{E_{OI}} = \frac{2\sin\theta_T\cos\theta_I}{\sin(\theta_I + \theta_T)\cos(\theta_I - \theta_T)}$ .... (D)

The *A*, *B*, *C* and *D* equations are called **Fresnel's formulae or Fresnel's laws.**

❑ 75.5. Retarded Potentials :

Under Lorentz condition $\left[i.e.\ \nabla\cdot\vec{A}+\mu_0\in_0\frac{\partial\theta}{\partial t}=0\right]$ the Maxwell's equations interms of electromagnetic potentials are :

$$\left.\begin{aligned}\nabla^2\vec{A}-\mu_0\in_0\frac{\partial^2\vec{A}}{\partial t^2}&=-\mu_0\vec{J}\\ \text{and}\quad \nabla^2\phi-\mu_0\in_0\frac{\partial^2\phi}{\partial t^2}&=-\rho/\in_0\end{aligned}\right\}\qquad\text{.............(1)}$$

The solutions of the above equations yield the relationship between the retarded potentials $\vec{A}$ and ϕ say at field point (r, t) with the respective sources $\vec{J}$ and ρ, say at corresponding source point (r',t') such that the source and field point times are related to each other,

$$t'=\left(t-\frac{r}{c}\right)\qquad\text{.............(2)}$$

where r/c = time taken by electromagnetic signal to reach from the source to field point with free space signal velocity c. Equation (2) states that the signal perceived at field point at time t should have emanated from source (r/c) times earlier and hence potentials given by the following equation are '**retarded**'.

The **retarded potential** solutions of equation (1) are written as

$$\left.\begin{aligned}\vec{A}(r,t)&=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(r',t')}{r}d\tau'\\ \phi(r,t)&=\frac{1}{4\pi\in_0}\int\frac{\rho(r',t')}{r}d\tau'\end{aligned}\right\}\qquad\text{.............(3)}$$

In case of small charge (say electron),

$$\left.\begin{aligned}\int\rho d\tau'&=e\\ \int\vec{J}d\tau'&=e\vec{v}\end{aligned}\right\}\qquad\text{.............(4)}$$

e = electronic charge, $\vec{v}$ = velocity of the electron.

For electron or small charge equation (3) i.e. **retarded potentials** can be written as.

$$\left.\begin{aligned} \vec{A} &= \frac{\mu_0 \in_0}{4\pi \in_0} \frac{e\vec{v}}{[r]} = \frac{1}{4\pi \in_0 c} \frac{\vec{\beta} e}{[r]} \\ \phi &= \frac{1}{4\pi \in_0} \frac{e}{[r]} \end{aligned}\right\} \qquad \text{.............(5)}$$

Here $\mu_0 \in_0 = \frac{1}{c^2}; \beta = v/c$

when relativistic influence on retarded potential is considered, $[r]$ is replaced by $(r - \vec{\beta}\cdot\vec{r})$. The expression for the **relativistic retarded potentials** for a uniformly moving point charge are :

$$\left.\begin{aligned} \vec{A} &= \frac{e}{4\pi \in_0 c} \frac{\vec{\beta}}{(r - \vec{\beta}\cdot\vec{r})} \\ \phi &= \frac{e}{4\pi \in_0} \frac{1}{(r - \vec{\beta}\cdot\vec{r})} \end{aligned}\right\} \qquad \text{.............(6)}$$

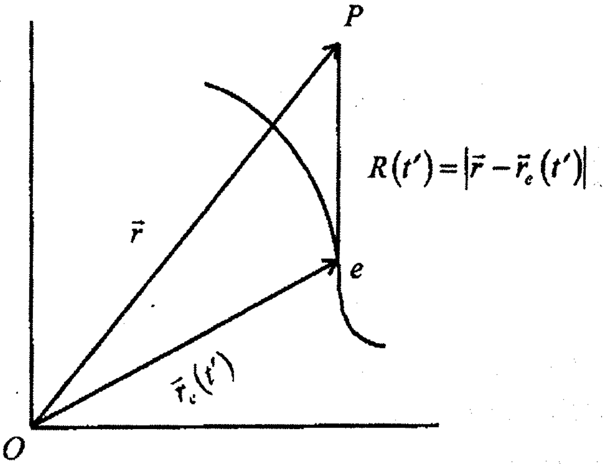
$\beta \to 0$ (non relativistic) implies $A \to 0$ i.e. no magnetic effect;

and $\phi = \frac{e}{4\pi \in_0 r}$ (electrostatic case).

Lienard and Wiechert transformed retarded potetials into relativistic ones for the first time. Hence relativistic retarded potentials are known as **Lienard-Wiechert potentials.**

❑ 75.6. Lienard-Wiechert Potentials :

Let us consider the application of the retarded potentials to compute the radiation from a single charged particle, say, an electron, in arbitrary motion. Since the calculation of the potnetials depends upon the position and velocity of the charge at the retarded time $t - \frac{|\vec{r} - \vec{r}'|}{c}$, we must know the details of the motion of the charge. In Fig. a trajectory of the electron described by the radius vector $\vec{r}_e(t')$ is shown.The calculation of the potentials as given in equation (3) involves a retarded time integration over the entire volume containing the charge. Now we do not know how the charge is distributed geometrically within the electron. The only thing that we know is that it has certain total charge. If we assume the electron to have zero physical extent we will land into difficulties. We, therefore, assume that the electron has a finite radius, but shall consider only those properties which are independent



of the magnitude of the radius. For an electron we may express the retarded potential in terms of Dirac δ-function. Thus,

$$\phi(r,t) = \frac{e}{4\pi \in_0} \int_{-\infty}^{\infty} \frac{\delta\left\{t' - \left(t - \frac{|\vec{r} - \vec{r}_e t|}{c}\right)\right\}}{|\vec{r} - \vec{r}_e|} dt'. \qquad \text{................ (7)}$$

If we put the integral in the form $\int_{-\infty}^{\infty} f(x)\delta(x - x')dx$, following a property of delta-function, its integral is readily found and is equal to $f(x')$.

We, therefore, introduce a now variable t'' such that

$$t'' = t' - t + \frac{|\vec{r} - \vec{r}_e(t')|}{c} \qquad \text{................ (8)}$$

$$dt'' = dt' + \frac{1}{c}\frac{d}{dt'}\left[\vec{r} - \vec{r}_e(t')\right]dt' \qquad \text{................ (9)}$$

Here we have takne $dt = 0$, because the observation is made at a fixed time t. Let the coordinates of the fixed point P be x_1, x_2, x_3 and those of the electron $x_{e_1}(t'), x_{e_2}(t'), x_{e_3}(t')$.

Now $|\vec{r} - \vec{r}_e(t')| = \sqrt{\sum_i \left\{x_i - x_{e_i}(t')\right\}^2}$. .... (10)

$$\therefore \frac{1}{c}\frac{d}{dt'}|\vec{r} - \vec{r}_e(t')| = \frac{1}{c}\sum_i \frac{\partial}{\partial x_{e_i}}|\vec{r} - \vec{r}_e(t')|\frac{\partial x_{e_i}}{dt'} \qquad \text{................ (11)}$$

Since $\frac{\partial}{\partial x_{e_i}}\left|\vec{r}-\vec{r}_e(t')\right|$ are the components of the gradient of $\left|\vec{r}-\vec{r}_e(t')\right|$ and $\frac{dx_e}{dt'}$ are the components of $\frac{d\vec{r}_e}{dt}$ we can write

$$\frac{1}{c}\frac{d}{dt'}\left|\vec{r}-\vec{r}_e(t')\right|=\frac{1}{c}\,grad_{re}\left|\vec{r}-\vec{r}_e(t')\right|\cdot\frac{d\vec{r}_e}{dt'}. \qquad \text{................ (12)}$$

The gradient can readily be determined

$$grad_{re}\left|\vec{r}-\vec{r}_e(t')\right|=-\frac{\vec{r}-\vec{r}_e(t')}{\left|\vec{r}-\vec{r}_e(t')\right|}=-\frac{\vec{R}}{\left|\vec{R}\right|} \qquad \text{................ (13)}$$

where $\vec{r}-\vec{r}_e(t)=\vec{R}.$

We also know that

$$\frac{d\vec{r}_e}{dt}=\vec{u} \text{ (the velocity of the electron)}$$

$$\therefore \frac{1}{c}\frac{d}{dt'}\left|\vec{r}-\vec{r}_e(t')\right|=-\frac{1}{c}\frac{\vec{R}}{R}\cdot u=-\frac{\vec{\beta}\cdot\vec{R}}{\left|\vec{R}\right|} \qquad \text{................ (14)}$$

where $\vec{\beta}=\frac{\vec{u}}{c}.$

Therefore,

$$dt''=dt'\left[1-\frac{\vec{\beta}\cdot\vec{R}}{\left|\vec{R}\right|}\right].$$

i.e. $$dt'=\frac{\left|\vec{R}\right|}{\left|\vec{R}\right|-\vec{\beta}\cdot\vec{R}}dt''. \qquad \text{................ (15)}$$

Hence, the potential (9.44) can be expressed as

$$\phi(\vec{r},t)=\frac{e}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\delta(t'')}{\left|\vec{R}(t')\right|}\frac{\left|\vec{R}(t')\right|}{\left|\vec{R}(t')\right|-\vec{\beta}(t')\cdot\vec{R}(t')}dt''. \qquad \text{................ (16)}$$

$$=\frac{e}{4\pi\epsilon_0}\left[\frac{1}{\left|\vec{R}(t')\right|-\vec{\beta}(t')\cdot\vec{R}(t')}\right]_{t''=0} \qquad \text{................ (17)}$$

Since $t'' = 0$ implies $t' = t - R(t')/c$

$$\phi(\vec{r},t) = \frac{e}{4\pi \in_0} \frac{1}{\left[R - \vec{\beta}\cdot\vec{R}\right]_{t'=t-\frac{R(t')}{c}}} \qquad \text{................ (18)}$$

By similar arguments we find that the vector potential is given by

$$\vec{A}(\vec{r},t) = \frac{\mu_0 e}{4\pi}\left[\frac{\vec{u}}{R - \vec{\beta}\cdot\vec{R}}\right]_{t'=t-\frac{R(t')}{c}} \qquad \text{................ (19)}$$

The potentials ϕ (equation 18) and A (equation 19) are called **Lienard-Wiechaert potentials.** They are dependent on the velocity of the electron but independent of the extent of the charge, i.e. of any detailed electronic model.

❑ 75.7. Fields produced by an arbitrary moving changed particle :

Let $x_\alpha = x_1, x_2, x_3$ be the coordinates of the point of observation P Fig. and $x'_\alpha(t') = x'_1(t'), x'_2(t'), x'_3(t')$ be the corrdinates of the charge at time t' at which a signal propagated with velocity c is emitted at x'_α so as to arrive at x_α at time t.

$$R^2 = \sum(x_\alpha - x'_\alpha)^2.$$

Let, further that $x'_\alpha(t')$ is given.

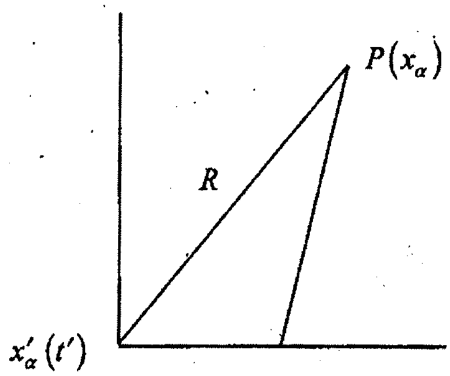


Fig.

The fields can be found from the relations

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

and $\vec{B} = \nabla \times \vec{A}$.

The potentials are given by

$$\phi = \frac{e}{4\pi \in_0} \frac{1}{\left[R - \vec{\beta}\cdot\vec{R}\right]} = \frac{e}{4\pi \in_0} \frac{1}{S} \qquad \text{.................(1)}$$

$$\vec{A} = \frac{\mu_0}{4\pi} e \left[\frac{\vec{u}}{R - \vec{\beta}\cdot\vec{R}}\right] = \frac{e}{4\pi \in_0} \frac{\vec{u}}{c^2 S}$$

where $S = \left(R - \vec{\beta}\cdot\vec{R}\right)$.

The components of ∇ are partial derivatives at constant time t and not at constant time t'. Since the time variation with respect to t' is given, in order to compute the fields, we have to transform $\left.\frac{\partial}{\partial t}\right|_{x_\alpha}$ and $\nabla|x_\alpha$ to expressions in terms of $\left.\frac{\partial}{\partial t'}\right|_{x_\alpha}$. This is necessary because in the case of an accelerated charge it is not possible, in general, to express the potentials in terms of the "present position" alone. Let us see how this can be done. From Fig. we have

$$R\left[x_\alpha, x'_\alpha(t')\right] = \left[\sum (x_\alpha - x'_\alpha)^2\right]^{1/2}$$

$$= c(t - t'). \qquad \text{.................(2)}$$

Since x'_α is given as a funtion of t', R is a function of x_α ant t'

$$R\left[x_\alpha, x'_\alpha(t')\right] = f(x_\alpha, t') = c(t - t'). \qquad \text{.................(3)}$$

Now

$$\frac{\partial R}{\partial t} = c\left(1 - \frac{\partial t'}{\partial t}\right). \qquad \text{.................(4)}$$

Also

$$\frac{\partial R}{\partial t} = \frac{\partial R}{\partial t'}\frac{\partial t'}{\partial t}. \qquad \text{.................(5)}$$

But

$$\frac{\partial R}{\partial t'} - \frac{\vec{R}\cdot\vec{u}}{R}$$

Hence

$$c\left(1-\frac{\partial t'}{\partial t}\right) = -\frac{\vec{R}\cdot\vec{u}}{R}\frac{\partial t'}{\partial t}$$

or $$\frac{\partial t'}{\partial t} = \frac{R}{R-\vec{\beta}\cdot\vec{R}} = \frac{R}{S}. \qquad \text{................ (6)}$$

Therefore,

$$\frac{\partial}{\partial t} = \frac{R}{S}\frac{\partial}{\partial t'}. \qquad \text{................ (7)}$$

Let us now transform the operator ∇. Because R is a function of x_α ant t', we can write

$$\nabla R = \nabla_1 R + \frac{\partial R}{\partial t'}\nabla t' = \frac{\vec{R}}{R} - \frac{\vec{R}\cdot\vec{u}}{R}\nabla t' \qquad \text{................ (8)}$$

where ∇_1 implies differentiation with respect to x_α at constant reatrded time t'.

We have also from (3)

$$\nabla R = -c\nabla t'$$

$$\therefore c\nabla t' = \frac{\vec{R}}{R} - \frac{\vec{R}\cdot\vec{u}}{R}\nabla t'$$

i.e. $$\nabla t' = -\frac{\vec{R}}{Sc} \qquad \text{................ (9)}$$

Substituting this in equation (8) we see that we can write, in general, for ∇

$$\nabla = \nabla_1 - \frac{R}{Sc}\frac{\partial}{\partial t'}. \qquad \text{................ (10)}$$

We have thus found the required transformation of the operator $\frac{\partial}{\partial t}$ and ∇

We now compute $\vec{E}$ and $\vec{B}$

$$\vec{E} = -\frac{e}{4\pi\epsilon_0}\nabla\left(\frac{1}{S}\right) - \frac{e}{4\pi\epsilon_0}\frac{\partial}{\partial t}\left(\frac{\vec{u}}{Sc^2}\right)$$

$$= \frac{e}{4\pi \epsilon_0} \left[\frac{1}{S^2} \nabla_1 S - \frac{\vec{R}}{S^3 c} \frac{\partial S}{\partial t'} - \frac{R}{S^2 c^2} \vec{u} \cdot \frac{R\vec{u}}{c^2 S^3} \frac{\partial S}{\partial t'} \right] \qquad \text{................(11)}$$

$$= \frac{e}{4\pi \epsilon_0} \left[\frac{\vec{R}}{s^2 R} - \frac{\vec{u}}{cS^2} + \frac{\vec{R}}{S^3 c} \left(\frac{\vec{R} \cdot \vec{u}}{R} \right) - \frac{R}{S^3 c} \frac{u^2}{c} + \frac{R}{S^3 c} \left(\frac{\vec{R} \cdot \vec{u}}{c} \right) \right]$$

$$- \frac{Ru}{S^2 c^2} - \frac{R}{S^3 c^2} \vec{u} \left(\frac{\vec{R} \cdot \vec{u}}{R} \right) + \frac{R}{S^3 c^3} \vec{u} u^2 - \frac{R}{c^2 S^3} \vec{u} \left(\frac{\vec{R} \cdot \vec{u}}{c} \right) \qquad \text{................(12)}$$

$$\left(\therefore \nabla_1 S = \frac{\vec{R}}{R} - \frac{u}{c} \right).$$

Rearranging and combining the terms, we have

$$\vec{E} = \frac{e}{4\pi \epsilon_0} \left[\frac{1}{S^3} \left(\vec{R} - \frac{R\vec{u}}{c} \right) \left(1 - \frac{u^2}{c^2} \right) + \frac{1}{c^2 S^3} \left\{ \vec{R} \times \left(\vec{R} - \frac{R\vec{u}}{c} \right) \times \vec{u} \right\} \right]. \qquad \text{................(13)}$$

Similarly

$$\vec{B} = \frac{e}{4\pi \epsilon_0 c^2} \left[\frac{\vec{u} \times \vec{R}}{S^3} \left(1 - \frac{u^2}{c^2} \right) + \frac{1}{c^2 S^3} \frac{\vec{R}}{R} \times \left\{ \vec{R} \times \left(\vec{R} - \frac{R\vec{u}}{c} \right) \times \vec{u} \right\} \right]. \qquad \text{................(14)}$$

It is seen that, $\vec{E}$ is composed of two components. The first component given by the first term is a function of velocity $\vec{u}$, while the second is a function of acceleration. We can, therefore, write

$$\vec{E} = \vec{E}_v + \vec{E}_a \qquad \text{................(15)}$$

where $\vec{E}_v$ is the velocity field and $\vec{E}_a$ the acceleration field. We further see that $\vec{E}_v \propto (1/R^2)$ while $\vec{E}_a \propto (1/R)$. If we compute the Poynting vector for the fields, we find that the contribution to this vector due to the two components is

$$\vec{N}_v \propto \frac{1}{R^4}$$

$$\vec{N}_a \propto \frac{1}{R^2}. \qquad \text{................(16)}$$

To find the energy radiated by the particle, we have to integrate the normal component of $\vec{N}$ over the surface of a sphere of radius R. Because the element of surface area involves R^2, the integral containing $\vec{N}_v$ varies as $1/R^2$ while that involing $\vec{N}_a$ remains finite. Therefore, for large R, the contribution due to N_v tends to zero while that due to $\vec{N}_a$ is finite. We conclude, therefore, that **a particle moving with a uniform velocity cannot radiate energy. Energy can be radiated only by accelerated charges.**

75.8. Radiation from an accelerated charged particle at low velocity :

If the velocity of the particle is so small that u/c can be neglected, then $S \approx R$ and fields as obtained from equation (13) and equation (14) are :

$$\vec{E}_a = \frac{e}{4\pi \in_0 c^2 R^3}\left\{\vec{R}\times\left(\vec{R}\times\dot{\vec{u}}\right)\right\} \qquad \text{................ (17)}$$

$$\vec{B}_a = \frac{e}{4\pi \in_0 c^3 R^2}\left(\dot{\vec{u}}\times\vec{R}\right). \qquad \text{................ (18)}$$

The Poynting vector that contributes to the radiation is

$$\vec{N}_a = \vec{E}_a \times \vec{H}_a = \vec{E}_a \times \frac{\vec{B}_a}{\mu_0} = \vec{E}_a \times \frac{1}{\mu_0 c}\left(\frac{\vec{R}\times\vec{E}_a}{R}\right).$$

Since $\vec{N}_a$ is perpendicular to $\vec{R}$

$$\vec{N}_a = \frac{1}{\mu_0 c}E_a^2\hat{n} = \sqrt{\frac{\in_0}{\mu_0}}E_a^2\hat{n} \qquad \text{................ (19)}$$

Now $\quad \vec{E}_a = \frac{e}{4\pi \in_0 c^2 R^3}\left\{\vec{R}\times\left(\vec{R}\times\dot{\vec{u}}\right)\right\}$

$$= \frac{e}{4\pi \in_0 c^2 R^5}\left\{\left(\vec{R}\cdot\dot{\vec{u}}\right)\vec{R}-\left(\vec{R}\cdot\vec{R}\right)\dot{\vec{u}}\right\}$$

$$= \frac{e}{4\pi \in_0 c^2 R^3}\left\{R\dot{\vec{u}}\cos\theta\,\vec{R}-R^2\dot{\vec{u}}\right\}$$

where θ is the angle between $\vec{R}$ and $\dot{\vec{u}}$.

Therefore,

$$\vec{N}_a = \frac{1}{\mu_0 c}\frac{e^2}{16\pi^2 \in_0^2 c^4 R^6}\left\{R\dot{\vec{u}}\cos\theta\,\vec{R}-R^2\dot{\vec{u}}\right\}^2\hat{n}$$

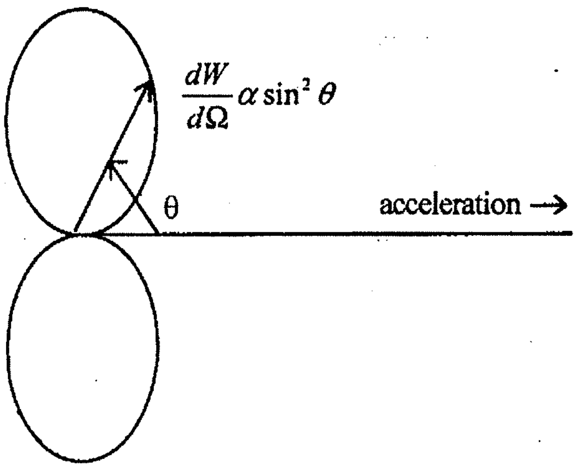
$$= \frac{e^2}{16\pi^2 \in_0 c^3 R^6} \left\{ R^4 (\dot{u})^2 \cos^2\theta + R^4 (\dot{u})^2 - 2R^4 (\dot{u})^2 \cos^2\theta \right\} \hat{n}$$

$$= \frac{e^2 (\dot{u})^2}{16\pi^2 \in_0 c^3 R^2} \left(1 - \cos^2\theta\right) \hat{n} = \frac{e^2 (\dot{u})^2}{16\pi^2 \in_0 c^3 R^2} \sin^2\theta \, \hat{n}. \qquad \text{................ (20)}$$

The Poynting vector gives us the energy flow per unit area per unit time. The power radiated per unit solid angle can be found by multiplying by R^2 which is the area per unit solid angle.

$$\therefore \frac{dW}{d\Omega} = \frac{e^2 (\dot{u})^2}{16\pi^2 \in_0 c^3} \sin^2\theta \qquad \text{................ (21)}$$

The angular distribution of energy, therefore, is just the $\sin^2\theta$ distribution (Fig.)



To obtain the totald radiated power, we have to integrate over the whole sphere

$$W = \frac{e^2 (\dot{u})^2}{16\pi^2 \in_0 c^3 R^2} \int_0^{2\pi} \int_0^{\pi} \left(1 - \cos^2\theta\right) R^2 \sin\theta \, d\theta \, d\phi$$

or, $$W = \frac{e^2 (\dot{u})^2}{16\pi^2 \in_0 c^3} 2\pi \frac{4}{3} = \frac{e^2 (\dot{u})^2}{6\pi^2 \in_0 c^3} = \frac{e^2 a^2}{6\pi \in_0 c^3}$$ where a = acceleration = $= \dot{u}$ .... (22)

This is known as **Larmor formula.**

It can be shown that for charged particle at **high velocity** the total power radiated will be

$$W = \frac{e^2}{6\pi \in_0 c} \gamma^6 \left\{ \dot{\beta}^2 - \left(\vec{\beta} \times \dot{\vec{\beta}}\right)^2 \right\} \qquad \text{................ (23)}$$

$$\left[\text{where}\ \gamma = \frac{1}{\sqrt{1-u^2/c^2}};\ \beta = \frac{u}{c};\ \dot{\beta} = \frac{a}{c};\ a = \dot{u}\right]$$

❑ 75.9. Summary

(i) Electromagnetic waves travel in free space with the velocity of light in vacuum.

(ii) Electromagnetic waves are transverse waves.

(iii) Skin depth for good conductors

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

(iv) Fresnel's laws:

(a) For E polarized perpendicular to the plane of incidence :

$$\frac{E_{OR}}{E_{OI}} = \frac{\sin(\theta_T - \theta_I)}{\sin(\theta_T + \theta_I)}$$

$$\frac{E_{OT}}{E_{OI}} = \frac{2\cos\theta_I \sin\theta_T}{\sin(\theta_I + \theta_T)}$$

(b) for E in the plane of incidence :

$$\frac{E_{OR}}{E_{OI}} = \frac{\tan(\theta_I - \theta_T)}{\tan(\theta_I + \theta_T)}$$

$$\frac{E_{OT}}{E_{OI}} = \frac{2\sin\theta_T \cos\theta_I}{\sin(\theta_I + \theta_T)\cos(\theta_I + \theta_T)}$$

(v) Retarded potentials

$$\bar{A}(r,t) = \frac{\mu_0}{4\pi}\int \frac{J(r',t')}{r} d\tau'$$

$$\phi(r,t) = \frac{1}{4\pi \in_0}\int \frac{J(r',t')}{r} d\tau'$$

$$t' = \left(t - r/c\right)$$

(vi) Lienard-Wiechaert Potentials

$$\vec{A} = \frac{e}{4\pi \in_0 c} \frac{\vec{\beta}}{\left(r - \vec{\beta}\cdot\vec{r}\right)}$$

$$\phi = \frac{e}{4\pi \in_0} \frac{1}{\left(r - \vec{\beta}\cdot\vec{r}\right)}$$

(vii) Energy can be radiated only by accelerated charges.

(viii) Larmor formula : $W = \dfrac{e^2 a^2}{6\pi \in_0 c^3}$

❑ 75.10. Self Assessment questions :

1. Discuss the propagation of plane electromagnetic waves in a conducting medium.
2. What is skin effect? Show that for a good conductor the skin depth,

   $\delta = \sqrt{\dfrac{2}{\mu\omega\sigma}}$ where notations have their usual meaning.
3. Show that the speed of electromagnetic waves in isotropic dielectrics is less than the speed of electromagnetic waves in free space.
4. Show that in a plane electromagnetic wave $\vec{E}, \vec{H}$ and $\vec{K}$ vectors are orthogonal.
5. Find the 'equations of telegraphy' using Maxwell's field equations.
6. Show that in a conductor energy flows in the direction of wave propagation but is damped off exponentially.
7. Write the boundary conditions for the electromagnetic field vectors. Using electromagnetic wave equations prove the kinematic laws of reflection and refraction.
8. Deduce Fresnel's laws.
9. What is retarded potential? Write down Lienard-Wiechert potentials.
10. Find the express in for the fields produced by an arbitrary moving charged particle.
11. Obtain the total radiated power from an accelerated charged particle at low velocity.

❑ 75.11. References :

1. Classical Electrodynamics – john David Jackson. John Wiley & Sons.
2. Electrodynamics and Classical Theory of Fields and particles – A.O. Barut. Macmillan, New York.
3. Electromagnetic Energy, Transmission and Radiation. R B. Adler *et al.*, Wiley, New York.
4. Electromagnetism and Relativity – E.G. Cullwick. Longman.
5. Electromagnetic Theory – J.A. Stratton. McGraw-Hill.
6. Electromagnetic Fields – J. Van Bladel. McGraw-Hill.
7. Introduction to Electrodynamics – David J. Griffiths. Prenctice Hall of India Pvt. Ltd.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper-VII

Module No. - 76

FUZZY SETS

(INTRODUCTION TO FUZZY SETS)

Content

76.1 Introduction

The classical sets divide the world into two distinct classes viz. white and black, true and false.

As for example in a collection "all integers" if we consider a subcollection of all "even integers" and ask the question whether an integer belongs to that subcollection or not, the answer will be either yes or no. No ambiguity remains in the answer. But in the same collection of all integers if we consider a subcollection of all "large integers" and ask whether an integer, say 2500790, belongs to the subcollection or not, then ambiguity comes in the answer. In some situation the answer will be yes (e.g. in the study of number of students in different classes in a school), in some situation the answer will be no (e.g. in the study of number of atoms taking part in a chemical reaction), again in some situation ambiguity occurs in the answer, the answer may be neither yes nor no. Let us consider another example. In the collection of "all boys" if we consider the subcollection "good boys" and ask the question whether a particular boy is a member of this subcollection or not, the answers will be different from different persons. His friend or his mother will answer 'yes', his enimy will answer 'no', whereas a common people will answer "I don't no". Someone may say that the boy is neither good nor bad.

Thus in our natural language, there is a great deal of imprecision, vagueness or fuzziness. The following are some more examples.

i) The classification of certain objects as "small".

ii) The description of a human characteristic such as "healthy" or as "tall".

iii) The classification of people by age such as "old".

iv) The classification of patients as "depressed".

v) The classification of flowers as "red".

vi) The classification of students as "intelligent".

In the above examples it may be impossible to decide whether an individual object belongs to the subset or not. There is no sharp boundary between members and non members and hence the concept of gradation of membership, or degree of membership becomes necessary. To discuss the situation of partial membership, let us consider the following example. Let us consider the universal set as "all students attending the inaugural ceremony held in a room of a school". Let the students be listening the function in that packed up room both in sitting and standing position and the standing students are standing both inside and outside of that specified room. Let us consider the subset "the students remaining inside of the room".

Here the student remaining completely inside of the room has full membership having membership grade one and the student remaining completely outside of the room has no membership having membership grade zero. Now question arises "what about the membership grade of a student standing at the door whose some part of the body is inside and some part outside of the room?" Naturally, this student will have a partial membership, and the grade of membership will be some number in between zero and one. The value of membership grade depends on the percentage of his body remaining inside of the room. If he has 50% of his body inside then the grade of membership is $\frac{1}{2}$, whereas if he has 75% of his body inside then the grade of membership is $\frac{3}{4}$. In general if he has x% of his body inside then the grade of membership is $x/100$.

Thes esituations where multigrade of membership is needed, "fuzzy set" is the tool. Fuzzy sets deals with objects that are "matter of degree" with all possible grades of truth between yes and no, and the shades of grey between white and black.

Let us consider one more example. If someone ask the question "Is Ram a student? "The answer is definite, Yes or No. This is a crisp situation. But if the question is "Is Ram honest?" The answer here is not definite. A variety of answers will come as "yes honest" or "extremely honest" or "extremely dishonest" or "honest at times" or "very honest" or "No" etc. This situation is fuzzy.

76.2 The Birth of Fuzzy Set Theory

In July, 1964, Zadeh was in New York city visiting his parents. He was then invited by Richard Belman to spend part of the summer at Rand Corp to work on problems in "pattern classification" and "system analysis". With this upcoming work on his mind, his thoughts often turned to the use of "imprecise categories for classification".

One night in New York, Zadeh had a dinner engagement with some friends. But it was cancelled, and he spent the evening alone in his parents apartment, and the idea of grade of membership, which is the backbone of fuzzy set theory, occured in his mind. This important event gave the birth of fuzzy logic technology and fuzzy set theory with the publication of his seminal paper on fuzzy sets in 1965.

The concept of fuzzy sets had to encounter sharp and strong criticism from academic community. Some rejected it because of the name, without knowing the content in detail. Others rejected it because of the theory's empahsis on imprecision. The funding agency of Zadeh "National Science Foundation" even was suggested by Congress as "Not to waste Government Funds".

76.3 Transition from traditional view to modern view

A paradigmatic change in science occured with the concpet of uncertainty. In science, this change occured as a gradual transition. The traditional view insisted that uncertainty is undesirable in science and should be avoided by all possible means. According to the traditional view, science should strive for certainty in all its manifestations and so science should deal with only precision, specificity, sharpness, consistency etc. Accordingly uncertain situations like imprecision, nonspecificity, vagueness, inconsistency etc. should be avoided as they are regarded unscientific.

The transition from the traditional view to the modern view of uncertainty began in the 19th century when study of molecular level became essential in physics. The need for fundamentally different approach to the study of physical processes at the molecular level motivated the development of relevant statistical methods viz statistical mechanics. The role played in Newtonian mechanics by the calculus, which involves no uncertainty, is replaced in statistical mechanics by probability theory. The analytic methods and statistical methods are highly complementary. The analytic methods based upon the calculus are applicable only to problems involving a very small number of variables that are related to one another in a predictable way. The statistical method on the other hand has exactly opposite characteristic as they require a very large number of variables which are related to one another in a very high unpredicable manner.

The purpose of probability theory is to capture uncertainty of a particular type known as random uncertainty. But there are many uncertainties which are not of random type. They are called non-random uncertainties and are associated with vagueness, with imprecision and with lack of information regarding a particular element of the problem at hand. Fuzzy set theory is a marvellous tool for handling these non-random uncertainties. The underlying power of fuzzy set theory is that it can use linguistic variables rather than quantitative variables, to represent imprecise concepts.

76.4 Concept of uncertainty

Uncertainty arise from the following

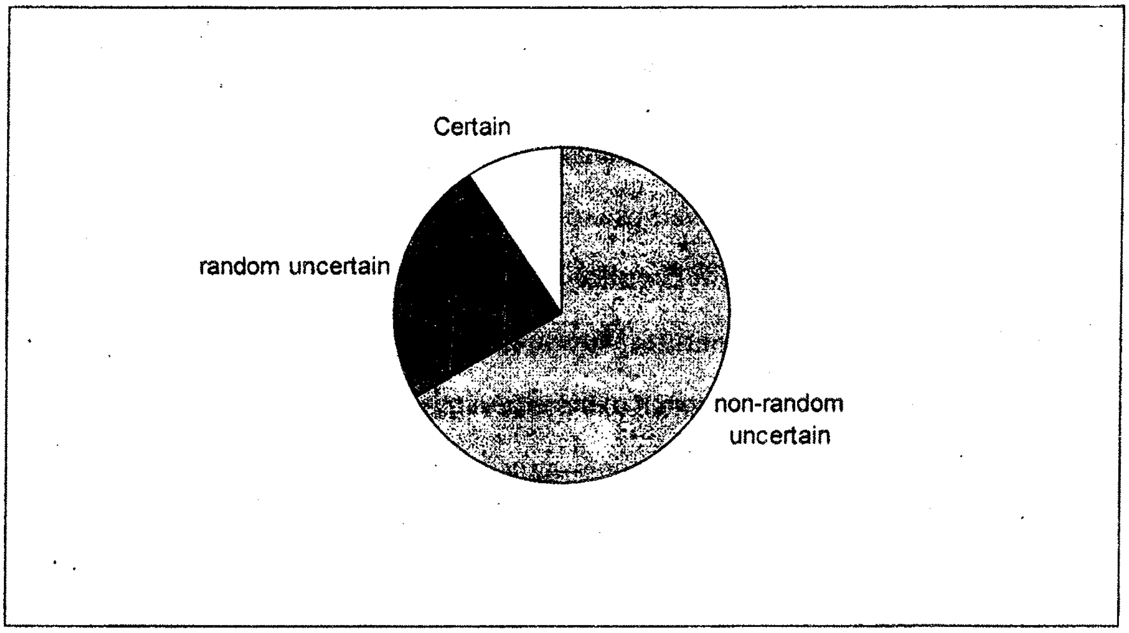
i) complexity ii) ignorance iii) chance iv) randomness v) imprecision vi) inability to perform adequate measurements vii) inconsistency viii) vagueness from natural language etc.

Uncertainties can be divided into following two types (i) Random uncertainty and (ii) Non-random uncertainty.

Random uncertainty occurs when due to lack of information, the future state of the system is not known completely. It describes uncertainty in the occurrance of the event. This type of uncertainty is handled by probability theory.

Non random uncertainty occurs due to vagueness concerning the description of the semantic meaning of the events, phenomena or statements. It describes the ambiguity of an event. This type of uncertainty is handled by fuzzy set theory.

Only a small portion of the "information world" is certain, a vast portion of the information world is actually uncertain. Again in the uncertain information world the portion of the non-random type uncertainty is much larger than the portion of the random type uncertainty. The following pi-diagram shows these proportions.



Information World (real situations)

76.5 Random uncertainty verses Fuzzy uncertainty

The classical concept of set holds for both the deterministic and the stochastic cases. The random uncertainty occurs if the future state of the system is not known. It is handled by probability theory. Stationary random processes are those that arise out of chance, where the chances represent frequencies of occurence that can be measured. Problems like drawing balls from an urn, tossing coin and dice, drawing cards from a pack are examples of stationary random processes.

Now we see how to recognise the random behavior of uncertainties? For example, are the following uncertainties random?

i) uncertainty in the weather prediction

ii) uncertainty in choosing clothes for the next day

iii) uncertainty in buying a car

iv) uncertainty in your preference in colors

v) uncertainty in your ability in parking a car

vi) uncertainty in causing cancer for consuming tabaco

Although it is possible to model all of these forms of uncertainty with various classes of random processes, the solution obtained may not be reliable. Treatment of these forms of uncertainty using fuzzy logic should also be done with caution. We should study the character of the uncertainty first, then we are to choose an appropriate approach to develop a model of the process. Again same problem may have many features. As for example, let the weather report suggests that "there is a 80% chance of rain tomorrow". It may mean that there has been rain on tomorrow's date for 80 of the last 100 years. It may mean that somewhere in your community 80% of the land area will receive rain. Again it may mean that 80% of the time of tomorrow it will be raining. Also humans often deal with these forms of uncertainty linguistically such as "it will likely rain tomorrow". And with this crude assessment of the possibility of rain, humans can still make appropriately accurate decisions about the weather.

Another important point is to be noted here. The statement "I think it will rain today" is not certain. This statement may be true with a degree of certainty. Let the level of certainty be 0.8. It is the truth value of the statement. The degree of certainty sounds like probability. But it is not quite the same. Probabilities for mutually exclusive events cannot add up to more than one, but their fuzzy values may. Suppose that probability of a cup of tea being hot is 0.8 and so probability of being cold is 0.2. The probabilities must add up to 1. On the other hand, the truth value of the proposition "a cup of tea is hot" may be 0.8 and the truth value of the proposition "a cup of tea is cold" may be 0.3. The sum of these two truth values here is 1.1 not 1.

The problems occuring in the real world are in general complex owing to an element of uncertainty either in the parameters which define the problem or in the situation in which the problem occurs. Probabitity theory can be applied only to a situation whose characteristics are based on random process i.e. process in which the occurance of events is strictly determined by chance. In reality there are a large class of problems whose uncertainty is characterized by a non-random process. Here the uncertainty may airse due to following reasons:

i) due to partial information about the problem

ii) due to information which is not fully reliable

iii) due to inherent imprecision in the language

iv) due to receipt of information from more than one source which are conflicting.

Fuzzy set theory has immense potential for effective solving of uncertainty of above types which are non-random in nature.

We should not be confused between probability value and membership value. If we ask the question "what is the probability of an individual x to be a member of a subset A?" The answer may be "the probability for *x* to be a member of A is 90%". Here the chance of the correct prediction for membership of *x* is 90% membership in the set A and 10% non-membership in the same set. We may note that in the classical set theory, it is not permissible for an individual to be a patial member of a set. Partial membership is only permissible in fuzzy set theory.

Let us consider an example and see how we can combine two types of uncertainties. Let a bag contains ten identical red balls with different gradation of red colour. Let the grades of 10 balls be 0.95, 0.93, 0.91, 0.9, 0.9, 0.7, 0.7, 0.0, 0.0, 0.0. Here the balls are identical and three balls have membership grade 0.0 i.e. three balls are completely non-red. The other seven balls are red having different gradation in red colour.

If a ball is drawn from the bag at random then the probability that the ball drawn is red is given by 7/10 i.e. 0.7 as in the bag there are 7 red balls and 3 non-red balls. Here the drawn ball may be any one of the 10 balls, even non-red one. If the experiment is performed a large number of times then we expect 70% of the drawn balls to be red.

On the other hand "Grade of membership of a ball is 0.7" means a particular ball whose grade of redness is 0.7, it can never be any other ball having grade different from 0.7, however it may be any one of the two balls having grade 0.7.

To be more precise in the experiment of drawing red balls at random from the bag we note that all balls are not red of same grade i.e. they have diffrent gradetion of being red.So the statement that there are 7 red balls out of 10 balls is not fully correct as someone may disagree to regard the balls with gradetion 0.7 as red.

We give the following argument to tackel the situation.First,we have to select a value of membership, above which we would be willing to regard the colour as red. For example, any ball with a membership value above 0.8 in the fuzzy set of "red balls" would be considered as red. Secondly we would have then to know the proportion of the balls in the bag that have membership values above 0.8. The number of such balls is 5 having membership grade 0.95,0.93,0.91,0.9 & 0.9.

Thus the probability of randomly selecting red balls from be bag is $\frac{5}{10}=\frac{1}{2}$. On the other hand if we regard the balls having grade more than 0.9 as red then the probability is $\frac{3}{10}$.

Hence first we have to access the ambiguity of redness and then we are to determine the probability. Thus we have been able to combine both types of uncertainties random and non-random.

Finally we again recollect that

i) Random uncertainty describes uncertainty in the occurance of the event i.e. uncertainty arising due to random occurance is handled by probability theory.

ii) Non-random uncertainity describes ambiguity of an event. It arises due to belongingness as in handled by fuzzy set theory.

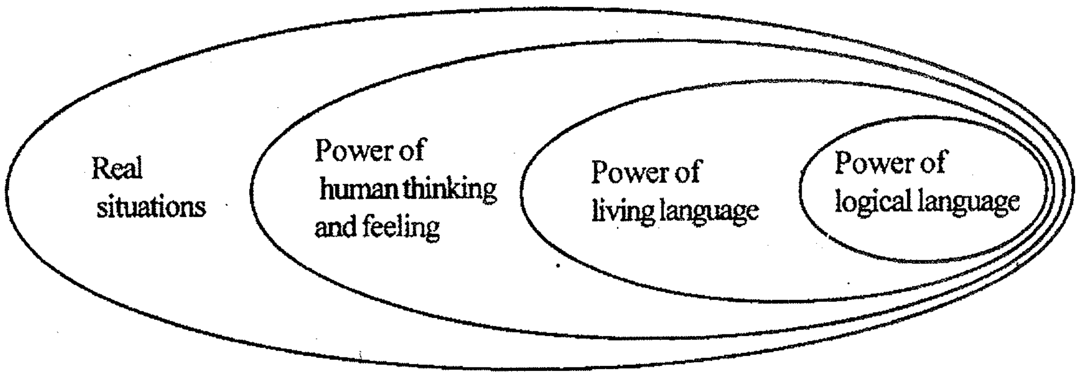
76.6 Power of humanity thinking

Our common way to convey information is living language. By the very nature of living language,it is vague and impresics. Yet it is most powerful form of communication and information exchange among humans.

Despite the vagueness, humans have very little trouble in understanding one another's concept and ideas. Our understaning is based largely on imprecise human reasoning. This imprecision is a form of information that can be quite useful to humans.

Human thinking and feeling, in which ideas, pictures, images and value systems are formed, has certainly more concepts or comprehensions than our daily language has words. Our thinking is unlimited but words in a dictionary is definitely limited.

The following diagram shows the real situations and our power of thinking.



76.7. Applications of fuzzy set theory

Fuzzy set occurs almost in all areas in which human judgement, evaluations and decisions are important. These are the areas of decision making, reasoning, learning and so on.

More specifically application area of fuzzy set theory covers

i) Engineering

ii) Phychology

iii) Medicine

iv) Ecology

v) Artificial Intelligence

vi) Decision theory

vii) Pattern recognition

viii) Sociology

ix) Meteorology

x) Computer science

xi) Manufacturing and so on.

Practical implementation on fuzzy set theory are as follows :

i) Fuzzy air conditioner that controls temparature changes according to human comfort.

ii) Fuzzy washing machine which detect the colour and the kind of cloth present in the machine and accordingly acts i.e. controls the revolution and select the type and amount of detergent.

iii) Fuzzy videography offering fuzzy focussing and image stabilization.

iv) Fuzzy computer which controls a number of stations in the subway system, the ride is so smooth that the riders do not need to hold anything.

v) Fuzzy anti-skid braking system to luxary cars.

vi) Fuzzy rice cookers.

vii), Fuzzy vacuum cleaners.

and so on.

Important Quotations

Relating to the notion of fuzzy set theory and fuzzy logic great thinkers and philosophers made remarkable statements. We state here some of them.

i) **Charles Sanders Pierce (1839-1914) :** He laughed at the 'sheep and goat separators' who split the world into true and false. "All that exists is continuous and such continuums govern knowledge".

ii) **Bertrand Russell (1872-1970) :** "Both vagueness and precision are featurs of language, not reality. Vagueness clearly is a matter of degree". All traditional logic assumes precise symbols. So traditional logic is not applicable to this terrestrial life.

iii) **Jan Lukasiewicz (1878-1956) :** He proposed a formal model of vagueness, a logic 'based on more values than TRUE or FALSE'. 1 stands for TRUE, 0 stands for FALSE and $^1/_2$ stands for possible.

(Actually the trhee-valued logic by Lukasiewicz stayed just one step away from the multivatued fuzzy logic by Zadeh and can be considered as its closest relative).

iv) **Max Black (1909-89) :** He proposed a degree as a measure of vagueness.

v) **Albert Einstein (1879-1955) :** "So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality".

vi) **Lotfi Zadeh (1923) :** He introduced fuzzy sets and logic theory. 'As the complexity of a system increases, our ability to make precise and significant statements about its behaviour diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics.....'. A corollary principle may be stated succinctly as "The closer one looks at a real-world problem, the fuzzier becomes its solution".

76.9 Mathematical modeling of fuzzy sets

The mathematical modeling of fuzzy concepts was presented by Zadeh in 1965. His contention is that meaning in natural language is a mater of degree. If we have a proposition such as "Ram is old", then it is not always possible to assert that it is either true or false. When we know that Ram's age is x, then the 'truth', or more correctly, the "compatibility" of x with "is old" is a matter of degree. It depends on our understanding of the concept "old". If the proposition is "Ram is under 50 years old" and we know Ram's age, then we can give a yes or no answer to whether the proposition is true or not. This can be formalized a fit by considering possible ages to be the interval $[0, \propto)$, letting A be the subset $\{x : x \in [0,\propto)\}$ and x<22}, and then determining whether or not Ram's age is in A. But "old" cannot be defined as an ordinary subset of $[0,\propto)$. This led Zadeh was led to the notion of fuzzy subset. Clearly, 60 and 70 years olds are old, but with different degree as 70 is older than 60. This suggests that membership in a fuzzy subset should not be on a 0 or 1 basis, but rather on a 0 to 1 scale. So the membership should be an element of the interval [0, 1].

An ordinary subset A of a set X is determined by its characteristic function χ_A defined by

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

The characteristic function of the subset A of the universal set X specifies whether or not an element is in A. If the value of the function is 1 then the element is in A and if the value is 0 then the element is not in A. There is only two possibility either the element is in A or is not in A. The characteristic function can take only two possible values 0 and 1. i.e. the range of the characteristic function is {0, 1}. This notion is generalized by allowing range of the function to be the closed interval [0, 1]. This generalized function of the characteristic function is called membership function and is denoted by $\mu_A(x)$ and the corresponding fuzzy subset will be denoted by $\underset{\sim}{A}$. Thus $\mu_A(x): X \to [0,1]$ whereas $\chi_A(x): X \to \{0,1\}$.

Hence the functions whose images are contained in {0, 1} correspond to ordinary or crisp subset of X and the functions whose images are contained in [0, 1] correspond to fuzzy subset of X. It is common to refer to a fuzzy subet simply as a fuzzy set. henceforth we also will do that.

Let us again consider the set "old persons". Here "old" is not well defined in the sense of classical mathematics and cannot be precisely measured.

If we know that age of Ram is 55 years, it is not clear if Ram is old, also it is not clear whether Ram is old if his age is 49 years or 61 years. In classical set theory, we may draw a line at the exact age of say 80. As a result, a person who is exactly 80 years old belongs to the set and is considered to be "old" but another person of one-day less than 80 years will not be considered "old". This distinction is mathematically correct, but practically unreasonable. So we need to quantify the concept "old". Instead of a sharp cut at the exact age of 80, we use common sense and say "absolutely old" persons are those who are 80 year old or older and say "absolutely young" persons are those who are 30 years old or younger. All the other persons are old as well as young at the same time, with different degrees of oldness and youngness depending on their actual ages. Thus the membership function of the fuzzy set $\underset{\sim}{A}$={old persons} may be defined as

$$\mu_{\underset{\sim}{A}}(x)=\begin{cases}0, x\leq 30\\ \dfrac{x-30}{50}, 30<x<80\\ 1, x\geq 80\end{cases}$$

The graph of $\mu_{\underset{\sim}{A}}$ is given in Fig. 1.

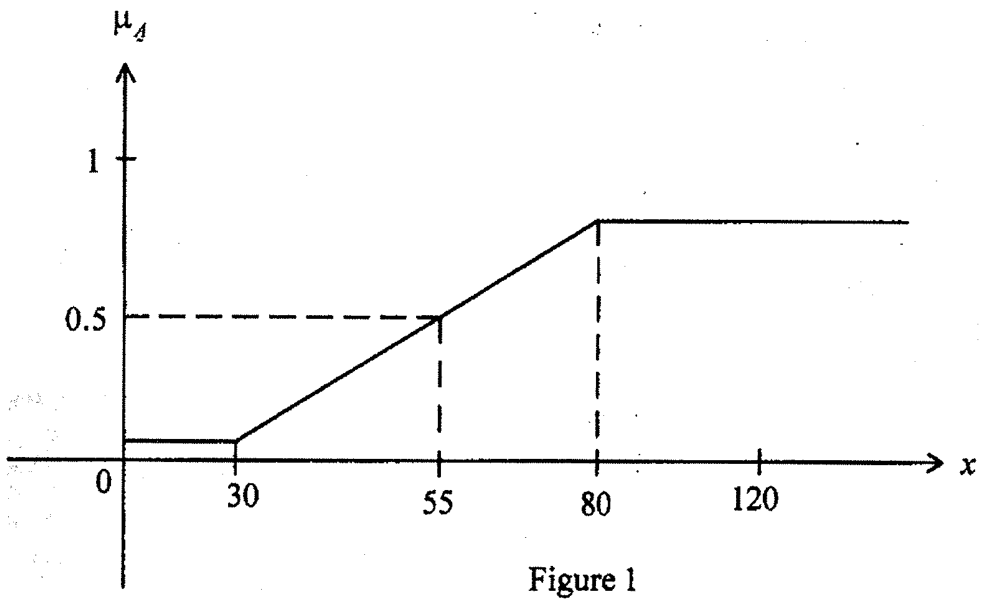


Figure 1

A person of 55 years old is considered to be "old" with degree 0.5 and at the same time 'young' with degree 0.5. A person of 40 years old is considered to be "old" with degree $1/5$ and 'young' with degree $4/5$. We note that a person of age 30 yrs to 80 yrs. is neither a member of $\underset{\sim}{A}$ fully nor he is non-member of $\underset{\sim}{A}$ fully. He has a partial membership to the fuzzy set $\underset{\sim}{A}$.

Depending on the concept of "old" the membership function $\mu_{\underset{\sim}{A}}(x)$ will change. It may be linear as well as non-linear. So the set $\underset{\sim}{A}$ can have infinite possible membership function $\mu_{\underset{\sim}{A}}(x)$. Figures 2 and 3 shows two other $\mu_{\underset{\sim}{A}}(x)$.

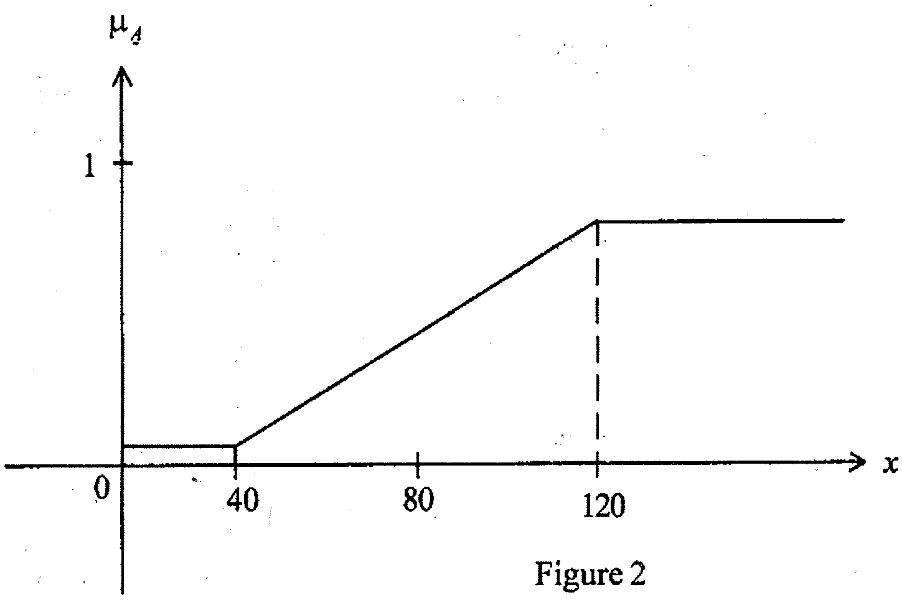


Figure 2

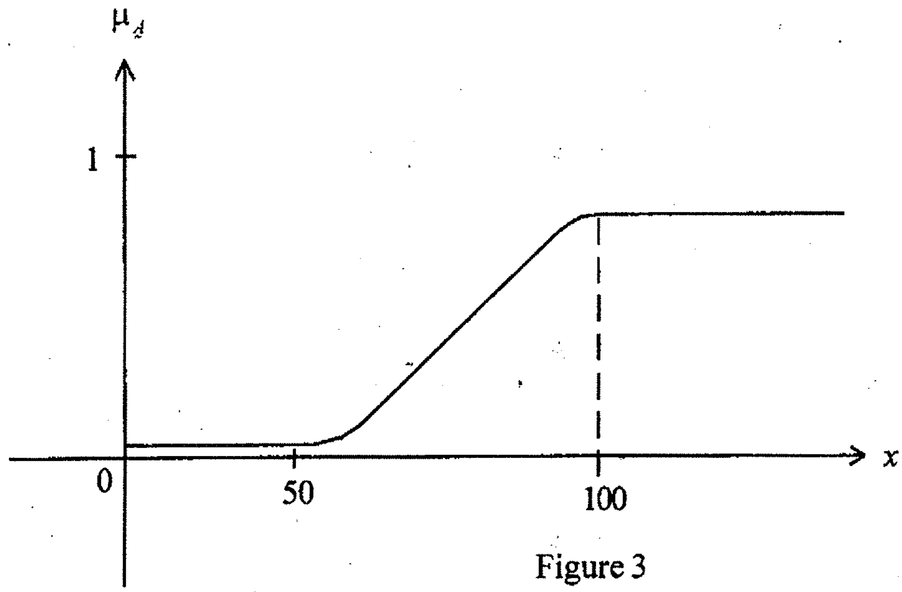


Figure 3

Let $\underset{\sim}{B}$ be the fuzzy set "young persons". We take the universal set as persons of all ages i.e. set of all positive real numbers. The membership function $\mu_{\underset{\sim}{B}}(x)$ may be defined as

$$\mu_{\underset{\sim}{B}}(x) = \begin{cases} 1, x \leq 30 \\ \dfrac{40-x}{10}, 30 < x < 40 \\ 1, x \geq 40 \end{cases}$$

Figure 4 shows the graph of this function.

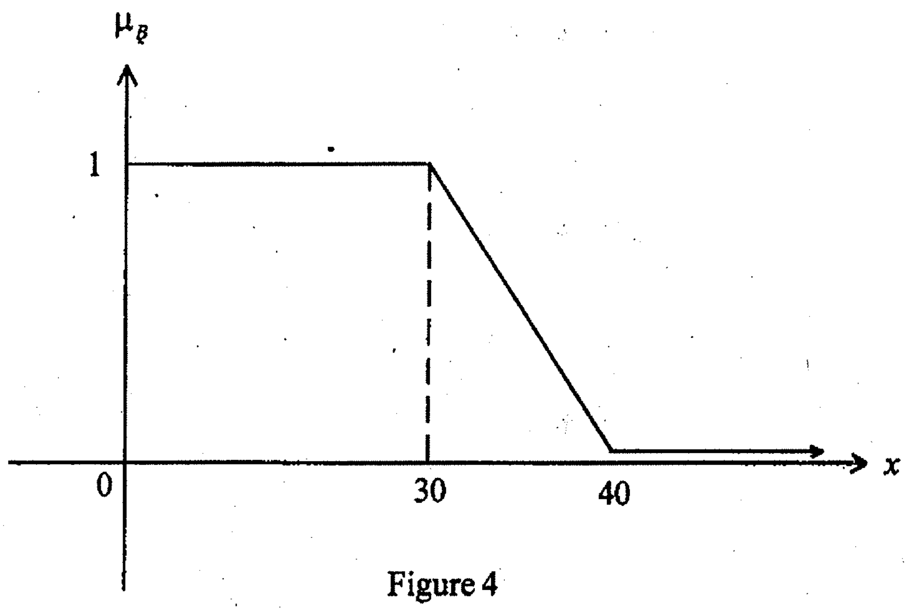


Figure 4

Another membership function of $\underset{\sim}{B}$ may be taken as

$$\mu_{\underset{\sim}{B}}(x) = \begin{cases} 1, if\ x \leq 30 \\ \dfrac{60-x}{30} \text{ if } 30 < x < 40 \\ \dfrac{50-x}{15} \text{ if } 40 \leq x \leq 50 \\ 0\ if\ x > 50 \end{cases}$$

Figure 5 shows the graph of this function. It is piecewise linear.

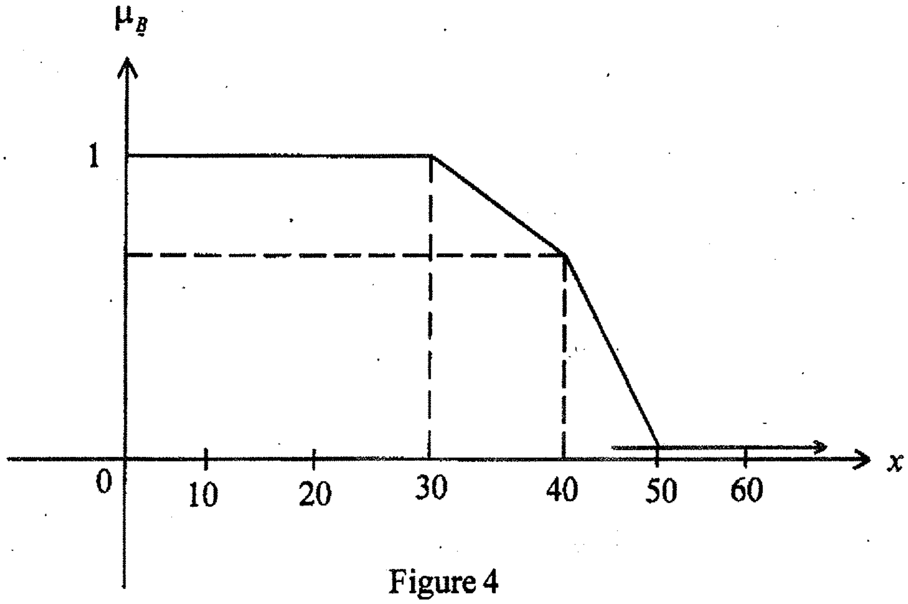


Figure 4

Let us consider another fuzzy set $\underset{\sim}{C}$ = "real numbers close to 5". One membership function of this fuzzy set $\underset{\sim}{C}$ is given by

$$\mu_{\underset{\sim}{C}}(x) = \begin{cases} 0 \text{ if } x \le 4.09 \\ \dfrac{x-4.09}{.01} \text{ if } 4.09 < x < 5 \\ \dfrac{5.01-x}{.01} \text{ if } 5 \le x \le 5.01 \\ 0 \text{ if } x \ge 5.01 \end{cases}$$

The graph of this function is shown in Figure 6.

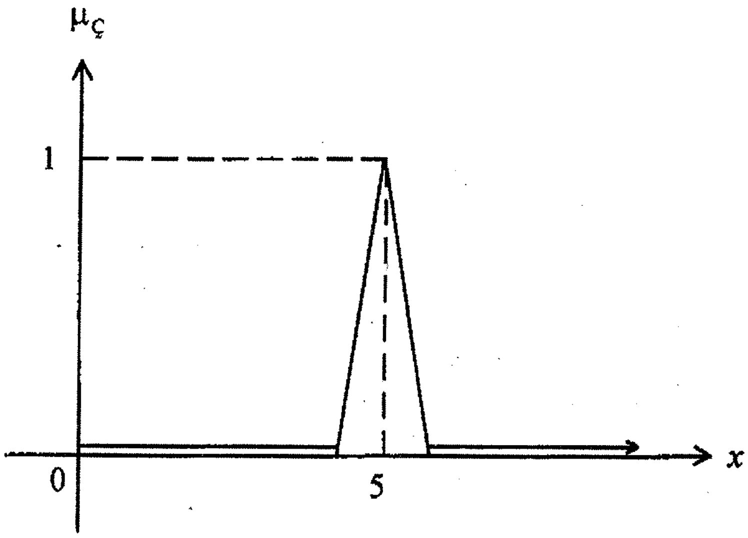


Figure 6

Another membership function (non-linear) of this fuzzy set $\underset{\sim}{C}$ is given in Fig. 7.

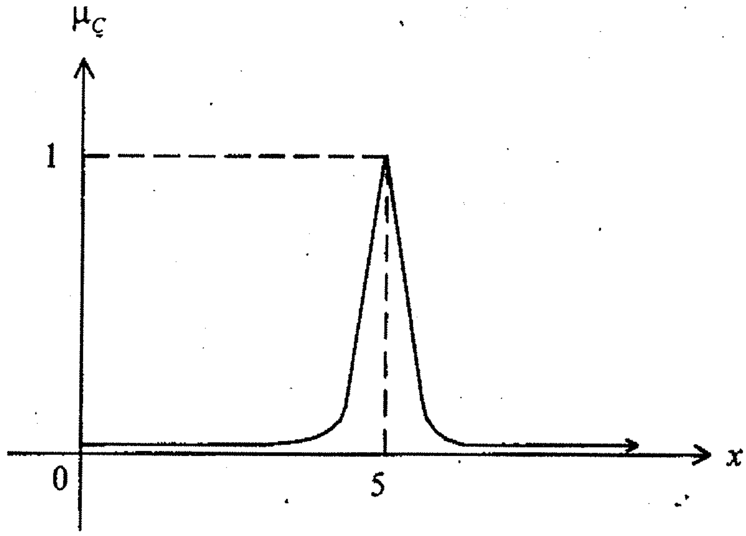


Figure 7

One more membership function of the fuzzy set may be taken as

$$\mu_{\underset{\sim}{C}}(x) = \frac{1}{1+(x-5)^2}$$

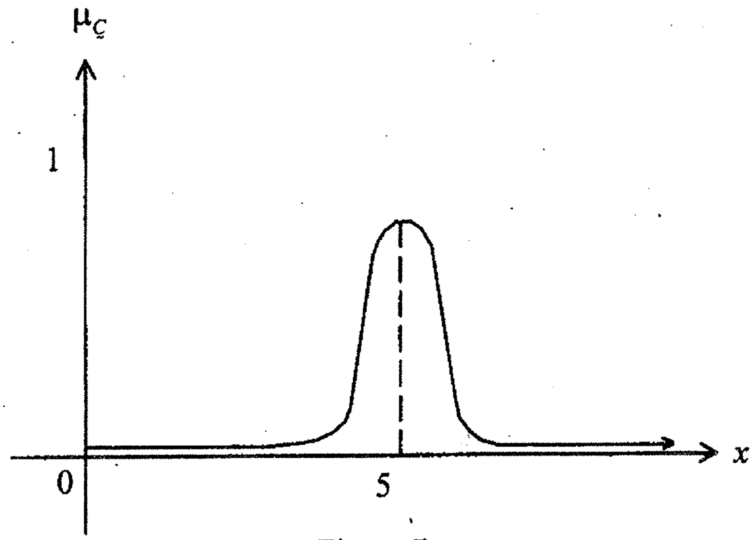


Figure 7

The graph of this function is given in Figure 8.

76.9.1 Fuzzy sets with a finite support

So far we have considered fuzzy sets on infinite support i.e. with the universal set as infinite set. Now we consider situations where the universal set is a finite set. Let the finite universal set be $X = \{x_1, x_2, ..., x_n\}$. Let $\underset{\sim}{A} \subset X$ and grade of membership of $x_i \in \underset{\sim}{A}$ be a_i. Then the fuzzy set $\underset{\sim}{A}$ is expressed by the notation

$$\{(x_i, a_i) \in \underset{\sim}{A} \times [0,1] \subset X \times [0,1]\}$$

Here $a_i = \mu_{\underset{\sim}{A}}(x_i)$,. So the notation becomes

$$\{(x_i, \mu_{\underset{\sim}{A}}(x_i)) \in \underset{\sim}{A} \times [0,1] \subset X \times [0,1]\}$$

Often in the literature the following notation is used

$$\underset{\sim}{A} = a_1/x_1 + a_2/x_2 + ... + x_n/x_n$$

i.e. $\underset{\sim}{A} = \mu_{\underset{\sim}{A}}(x_1)/x_1 + \mu_{\underset{\sim}{A}}(x_2)/x_2 + ... + \mu_{\underset{\sim}{A}}(x_n)/x_n$.

Here the slash is employed to link the elements of the support with their grades of membership in $\underset{\sim}{A}$, and the plus sign indicates, rather than any sort of algebraic addition, that the listed pairs of elements and membership grades collectivley form the definition of the set $\underset{\sim}{A}$. For the case in which a fuzzy set $\underset{\sim}{A}$ is defined on a universal set that is finite or countable, we may write, respectively,

$$\underset{\sim}{A} = \sum_{i=1}^{n} a_i / x_i \quad \text{or,} \quad \underset{\sim}{A} = \sum_{i=1}^{\infty} a_i / x_i$$

i.e. $$\underset{\sim}{A} = \sum_{i=1}^{n} \mu_{\underset{\sim}{A}}(x_i) / x_i \quad \text{or,} \quad \underset{\sim}{A} = \sum_{i=1}^{\infty} \mu_{\underset{\sim}{A}}(x_i) / x_i.$$

76.9.2 Example

Let us consider the fuzzy set $\underset{\sim}{A}$ consisting of six ordered pairs as

$$\underset{\sim}{A} = \{(x_1, 0.2), (x_2, 1), (x_3, 0.8), (x_4, 0.3), (x_5, 0.5), (x_6, 0.1)\}.$$

The elements x_i, i = 1, 2, ..., 6 are not necessary numbers. They belong to the classical set $\{x_1, x_2, \ldots\ldots, x_6\}$ which is a subset of a certain universal set X. Here the membership function $\mu_{\underset{\sim}{A}}(x_i)$ of $\underset{\sim}{A}$ takes the following values on [0, 1].

$$\mu_{\underset{\sim}{A}}(x_1) = 0.2, \mu_{\underset{\sim}{A}}(x_2) = 1, \mu_{\underset{\sim}{A}}(x_3) = 0.8$$

$$\mu_{\underset{\sim}{A}}(x_4) = 0.3, \ \mu_{\underset{\sim}{A}}(x_5) = 0.5, \ \mu_{\underset{\sim}{A}}(x_6) = 0.1.$$

The following interpretation could be given to $\mu_{\underset{\sim}{A}}(x_i), i =,1,2,\ldots\ldots 6$. The element x_2 is a full member of the fuzzy set $\underset{\sim}{A}$, while the element x_6 is a member of A a little, x_1 and x_4 are a little more members of $\underset{\sim}{A}$; the element x_3 is almost a full member of $\underset{\sim}{A}$, while x_5 is more or less a member of $\underset{\sim}{A}$.

Now we specify in two different way the element x_i in $\underset{\sim}{A}$.

i) First we assume that x_i are integers e.g. $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6$; they belong to the set $A = \{1, 2, 3, 4, 5, 6\}$, a subset of the universe $N = \{1, 2, 3, \ldots. \infty\}$. The fuzzy set $\underset{\sim}{A}$ then becomes

$$\underset{\sim}{A} = \{(1, 0.1), (2, 1), (3, 0.8), (4, 0.3), (5, 0.5), (6, 0.1)\}$$

The membership function $\mu_{\underset{\sim}{A}}(x)$ are shown in Fig. 9.

ii) Secondly, let us consider the universal set X as "All friends of Ram" & $\underset{\sim}{A}$ be the set "close friends of Ram".

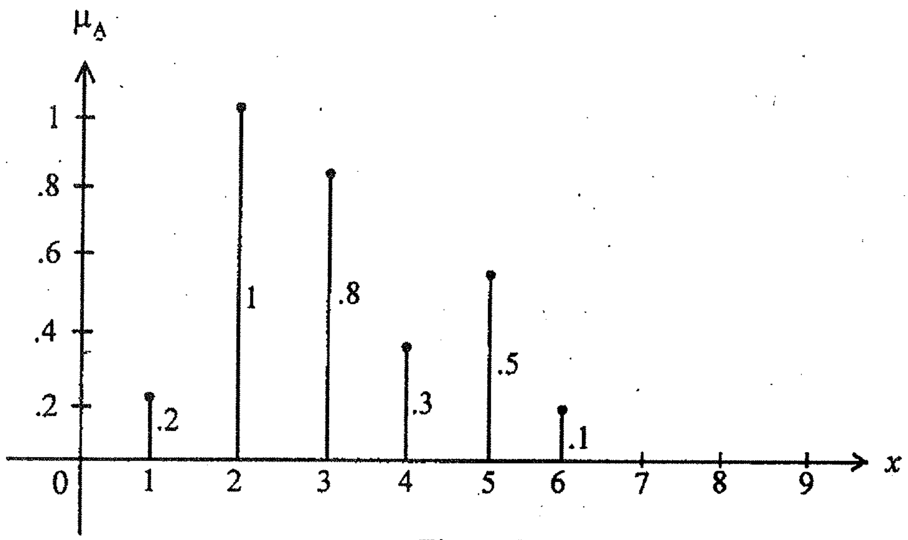


Figure 9

Let $\underset{\sim}{A}$ = {(Rahim, 0.2), (Jadu, 1), (Kamal, 0.8), (Bimal, 0.3), (Amal, 0.5), (Tapan, 0.1)}

Here x_1 = Rahim, x_2 = Jadu, x_3 = Kamal, x_4 = Bimal, x_5 = Amal and x_6 = Tapan.

We note that regarding closeness in friendship of Ram : Jadu is closest, Tapan is little close, Rahim and Bimal are a little more close, Kamal is almost close, while Amal is more or less close.

76.10 Summary

In this module we have inroduced the notion of fuzzy sets, its necessity, its application area. Also we have discussed the concept of uncertainty and its types. Finally, the mathematical modeling of fuzzy sets is done.

76.11 Suggested Further Readings

1. Kaufmann, A [1975] Introduction to the Theory of Fuzzy Subsets New York, London, San Francisco
2. Klir, G.J. and Folger, T.A. [1988], Fuzzy Sets, Uncertainty and Information, Englewood Cliffs

3. Klir, G.J. and Yuan, B. [1997], Fuzzy Sets and Fuzzy Logic, Theory and Applications, Prentice Hall of India

4. Zimmermann, H.J. [1996], Fuzzy Set Theory and Its Applications, Allied Publishers Ltd.

5. Ross, T.J. [1997], Fuzzy Logic with Engineering Applications, McGraw Hill, Inc.

6. Bart Kosko, Fuzzy Thinking [1993], Flamingo, Harper Collins Publishers

7. Bojadziev & Bojadziev, Fuzzy Sets, Fuzzy Logic & Applications; World Scientific.

8. Chen, G. and Pham, T.T.;[2006] A first course in Fuzzy Logic, Chapman & Hall/CRC

9. Nguyen, H.T. & Walker, E.A. [1999] A first course in Fuzzy Logic, Chapman & Hall/CRP

10. Bector, C.R. & Chandra, S. [2004], Fuzzy Mathematical Programming and Fuzzy Matrix Games, Springer.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper-VII Group – B

Module No. - 77

FUZZY SETS

(INTERVALS, FUZZY SETS, FUZZY NUMBERS AND THEIR ARITHMETIC)

Content

77.1 Introduction

In this module we first consider the definition and arithmetic of Intervals. The notionof Interval arithmetic is then used to develop the arithmetic of Fuzzy numbers. Fuzzy numbers are nothing but particular fuzzy sets. Hence operations on fuzzy sets are discussed first. Then the notion of Interval arithmetic and operations on fuzzy sets are used for the development of fuzzy numbers arithmetic.

77.2 Interval Numbers

77.2.1 Definition

An interval number is defined as an ordered pairs of finite real numbers $[a, b]$ where $a \leq b$. When $a = b$ the interval number $[a, b]$ degenerates to the scalar real number 'a'.

An interval number can be thought as

(i) an extension of the concept of a real number and also as a subset of the rea line [Moore 1979, Alefeld & Herzberger (1983)].

(ii) a simplest form tolerance-type uncertainty with no information about the probabilities within this tolerance range (Nauyen & Kreinovich, 2005).

(iii) a grey number whose exact value is unknown but a range within which the value lies is known [Liu & Lin, 1998].

Thus an interval number represents a set of possible values that a particular entity or variable may assume, without any prior assumption about exact value and probability measure. In other words, interval numbers should be used whenever decision variables can assume different values, but a probability measure on these values is not available or justifiable. In reality, inexactness of this kind occurs in countless numbers. An interval number may also be called as an interval.

77.2.2. Set Operations on Intervals

77.2.2.1 Definition

(i) Equality : Two intervals $[a, b]$ and $[c, d]$ are said to be equal if and only if $a = c$ and $b = d$.

(ii) Intersection : The intersection of two intervals $[a, b]$ and $[c, d]$ is defined as

$$[a,b] \cap [c,d] = [\max\{a,c\}, \min\{b,d\}]$$

Note : $[a,b]\cap[c,d]=\phi$ if and only if $a>d$ or $c>b$.

(iii) Union : The union of two intervals $[a, b]$ and $[c, d]$ is defined as

$$[a,b]\cup[c,d]=[\min\{a,c\},\max\{b,d\}]$$

provided that $[a,b]\cap[c,d]\neq\phi$

(iv) Inclusion : The interval $[a, b]$ is said to be included in $[c, d]$ if and only if both $c<a$ and $b<d$. It is written as $[a,b]\subset[c,d]$.

For given two intervals $I_1=[a, b]$ and $I_2=[c, d]$

the following six cases may arise :

(i) $a>d$ (ii) $c>b$ (iii) $a>c$ and $b<d$

(iv) $c>a$ and $d<b$ (v) $a<c<b<d$ and (vi) $c<a<d<b$.

Table 77.2.2.1 shows the various combinations of set-theoretic intersection and set-theoretic union for these six possible combinations of a, b, c and d.

Cases	Intersection ($\cap$)	Union ($\cup$)
i) $a>d$	ϕ	$[c,d]\cup[a,b]$
ii) $c>b$	ϕ	$[a,b]\cup[c,d]$
iii) $a>c$, $b<d$	$[a, b]$	$[c, d]$
iv) $c>a$, $d<b$	$[c, d]$	$[a, b]$
v) $a<c<b<d$	$[c, b]$	$[a, d]$
vii) $c<a<d<b$	$[a, d]$	$[c, b]$

Table : 77.2.2.1

77.2.3. Interval Arithmetic

Let $[a_1,b_1],[a_2,b_2]$ and $[a, b]$ be intervals. Then addition, subtraction, multiplication and division are defined as follows;

(i) **Addition :**

$$[a_1,b_1]+[a_2,b_2]=[a_1+a_2,b_1+b_2]$$

(ii) **Subtraction :**

$$[a_1,b_1]-[a_2,b_2]=[a_1-b_2,b_1-a_2]$$

(iii) **Multiplication :**

$$[a_1,b_1]\times[a_2,b_2]=[\min\{a_1a_2,a_1b_2,b_1a_2,b_1b_2\},\max\{a_1a_2,a_1b_2,b_1a_2,b_1b_2\}]$$

(iv) **Division :**

$$[a_1,b_1]/[a_2,b_2]=[\min\{a_1/a_2,a_1/b_2,b_1/a_2,b_1/b_2\};\max\{a_1/a_2,a_1/b_2,b_1/a_2,b_1/b_2\}]$$

provided that $0\notin[a_2,b_2]$.

(v) **Scalar Multiplication :**

$$k[a,b]=[ka,kb]\ for\ k\geq 0$$
$$=[kb,ka]\ for\ k<0.$$

(vi) **Reciprocal :**

If $0\notin[a,b]$ then $[a,b]^{-1}=\left[\min\left\{\frac{1}{a},\frac{1}{b}\right\},\max\left\{\frac{1}{a},\frac{1}{b}\right\}\right]$

If $0\in[a,b]$ then $[a,b]^{-1}$ is undefined.

For non-negative intervals multiplication, division and reciprocal reduces to the following.

Multiplication :

$$[a_1,b_1]\times[a_2,b_2]=[a_1a_2,b_1b_2]$$

Division :

$$[a_1,b_1]/[a_2,b_2]=[a_1/b_2,b_1/a_2]$$

Inverse :

$$[a,b]^{-1}=[1/b,1/a]$$

Remarks :

From [3, 14] + [5,20] = [8,34] we note that for any $x \in [3,14]$ and any $y \in [5,20]$, it is guaranteed that $x + y \in [8,34]$. Also from [2, 8] – [3, 10] = [–8, 5] we note that for any $x \in [2,8]$ and for any $y \in [3,10]$, it is guaranteed that $x - y \in [-8,5]$. So interval arithmetic intends to obtain on interval as the result of an operation such that the resulting interval contains all possible solutions.

Again interval arithmetic may produce some unusual results that could seem to be inconsistent with the ordinary numerical solutions. As for example ordinary results gives [2, 6] – [2, 6] = [0, 0], but from interval arithmetic we have [2, 6] – [2, 6] = [–4, 4] and not [0, 0]. Here we note that $[0,0] \in [-4,4]$ i.e. [–4, 4] contains 0 but not only 0, many others also i.e. 0 as well as all other possible solutions.

77.2.4 Algebraic Properties of Interval Arithmetic

We can easily prove the following properties of interval arithmetic.

Let X, Y, Z be intervals then we have

i) $X + Y = Y + X$

ii) $(X + Y) + Z = X + (Y + Z)$

iii) $(XY)Z = X(YZ)$

iv) $XY = YX$

v) $Z + 0 = 0 + Z = Z$

and $Z0 = 0Z = 0$ where $0 = [0, 0]$

vi) $ZI = IZ = Z$ where $I = [1, 1]$

vii) $Z(X + Y) \neq ZX + ZY$, except when :

(a) $Z = [z, z]$ is a point or

(b) $X = Y = 0$ or

(c) $xy \geq 0$ for all $x \in X$ and $y \in Y$.

In general, only the subdistributive law holds :

$$Z(X+Y)\subseteq ZX+ZY.$$

77.3 Operations on Fuzzy Sets

Now we proceed to define certain standard set theoretic operations for fuzzy sets.

77.3.1 Definition : Empty fuzzy set

A fuzzy set $\underset{\sim}{A}$ defined over the universe X is said to be empty if its membership function is identically zero, i.e. if $\mu_{\underset{\sim}{A}}(x)=0$ for all $x\in X$.

77.3.2 Definition : Subset

A fuzzy set $\underset{\sim}{A}$ is said to be a subset of a fuzzy set $\underset{\sim}{B}$ if $\mu_{\underset{\sim}{A}}(x)\leq\mu_{\underset{\sim}{B}}(x)$ for all $x\in X$. This is denoted by $\underset{\sim}{A}\subseteq\underset{\sim}{B}$.

77.3.3 Definition : Equality of fuzzy sets

Two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are said to be equal if $\underset{\sim}{A}\subseteq\underset{\sim}{B}$ and $\underset{\sim}{B}\subseteq\underset{\sim}{A}$ i.e. if $\mu_{\underset{\sim}{A}}(x)=\mu_{\underset{\sim}{B}}(x)$ for all $x\in X$.

77.3.4 Definition : Complement

The complement of a fuzzy set $\underset{\sim}{A}$ defined over the universal set X is another fuzzy set $\underset{\sim}{A}'$ defined by the membership function

$$\mu_{\underset{\sim}{A}'}(x)=1-\mu_{\underset{\sim}{A}}(x) \text{ for all } x\in X.$$

77.3.5 Definition : Union

The union of two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is another fuzzy set $\underset{\sim}{C}$ defined by the membership function

$$\mu_{\underset{\sim}{C}}(x)=\max\{\mu_{\underset{\sim}{A}}(x),\mu_{\underset{\sim}{B}}(x)\} \text{ for all } x\in X.$$

77.3.6 Definition : Intersection

The intersection of two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is another fuzzy set $\underset{\sim}{C}$ defined by the membership function

$\mu_{\underset{\sim}{C}}(x) = \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$ for all $x \in X$.

Before studing the properties of fuzzy sets we state the standard properties of crisp sets.

77.3.7 Properties of Crisp Sets

The following are the important properties of crips sets.

i) **Commutativity :** $A \cup B = B \cup A$

$A \cap B = B \cap A$

ii) **Associativity :** $(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

iii) **Distributive laws :** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

iv) **De Morgan's laws :** $(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

v) **Law of contradiction :** $A \cap A' = \phi$

vi) **Law of excluded middle :** $A \cup A' = X$.

77.3.8 Properties of Fuzzy Sets

Using the definitions of union, intersection and complement of fuzzy sets we now prove the properties of fuzzy sets. It is seen that all the properties stated above for crisp sets holds good also for fuzzy sets except the law of contadiction and the law of excluded middle. In the following theorem we prove this.

77.3.9 Theorem : Prove that for fuzzy sets commutative law, associative law, distributive law and De Morgan's law are true.

Proof. We prove distributive law and De Morgan's law. Commulative law and associative law can be proved easily.

Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ and $\underset{\sim}{C}$ be fuzzy sets with membership functions $\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)$ and $\mu_{\underset{\sim}{C}}(x)$ respectively.

We prove the distributive law

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cap (\underset{\sim}{A} \cup \underset{\sim}{C})$$

We have

$$\mu_{\underset{\sim}{A} \cup (\underset{\sim}{B} \cap \underset{\sim}{C})}(x)$$

$$= \max\left[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B} \cap \underset{\sim}{C}}(x)\right]$$

$$= \max\left[\mu_{\underset{\sim}{A}}(x), \min\left\{\mu_{\underset{\sim}{B}}(x), \mu_{\underset{\sim}{C}}(x)\right\}\right]$$

$$= \max\left[\alpha, \min\{\beta, \gamma\}\right] \text{ where } \alpha = \mu_{\underset{\sim}{A}}(x), \beta = \mu_{\underset{\sim}{B}}(x) \text{ and } \gamma = \mu_{\underset{\sim}{C}}(x)$$

$$= f(x) \text{ [say]}$$

Again

$$\mu_{(\underset{\sim}{A} \cup \underset{\sim}{B}) \cap (\underset{\sim}{A} \cup \underset{\sim}{C})}(x)$$

$$= \min\left[\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x), \mu_{\underset{\sim}{A} \cup \underset{\sim}{C}}(x)\right]$$

$$= \min\left[\max\left\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\right\}, \max\left\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{C}}(x)\right\}\right]$$

$$= \min\left[\max\{\alpha, \beta\}, \max\{\alpha, \gamma\}\right]$$

$$= g(x) \text{ [say]}$$

For any fixed $x \in X$, there arise following six cases

i) $\alpha \le \beta \le \gamma$

ii) $\alpha \le \gamma \le \beta$

iii) $\beta \le \gamma \le \alpha$

iv) $\beta \le \alpha \le \gamma$

v) $\gamma \le \alpha \le \beta$

vi) $\gamma \le \beta \le \alpha$

We consider all these six cases in the following table :

Case	$\min\{\beta,\gamma\}$	$f(x)$	$\max\{\alpha,\beta\}$	$\max\{\alpha,\gamma\}$	$g(x)$
i) $\alpha\le\beta\le\gamma$	β	β	β	γ	β
ii) $\alpha\le\gamma\le\beta$	γ	γ	β	γ	γ
iii) $\beta\le\gamma\le\alpha$	β	α	α	α	α
iv) $\beta\le\alpha\le\gamma$	β	α	α	γ	α
v) $\gamma\le\alpha\le\beta$	γ	α	β	α	α
vi) $\gamma\le\beta\le\alpha$	γ	α	α	α	α

In all these six cases we see that $f(x) = g(x)$. This is true for any $x \in X$. Hence we have

$$\mu_{\underset{\sim}{A}\cup(\underset{\sim}{B}\cap\underset{\sim}{C})}(x) = \mu_{(\underset{\sim}{A}\cup\underset{\sim}{B})\cap(\underset{\sim}{A}\cup\underset{\sim}{C})}(x) \text{ for all } x \in X.$$

This proves that

$$\underset{\sim}{A}\cup(\underset{\sim}{B}\cap\underset{\sim}{C}) = (\underset{\sim}{A}\cup\underset{\sim}{B})\cap(\underset{\sim}{A}\cup\underset{\sim}{C}).$$

We now prove the De Morgan's law

$$(\underset{\sim}{A}\cup\underset{\sim}{B})' = \underset{\sim}{A}'\cap\underset{\sim}{B}'$$

We have

$$\mu_{(\underset{\sim}{A}\cup\underset{\sim}{B})'}(x)$$

$$= 1 - \mu_{\underset{\sim}{A}\cup\underset{\sim}{B}}(x)$$

$$= 1 - \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$$

$$= 1 - \max\{\alpha,\beta\} \text{ where } \alpha = \mu_{\underset{\sim}{A}}(x) \text{ and } \beta = \mu_{\underset{\sim}{B}}(x)$$

$$= f(x) \text{ (say)}$$

and $\mu_{\underset{\sim}{A}'\cap\underset{\sim}{B}'}(x)$

$$= \min\{\mu_{\underset{\sim}{A}'}(x), \mu_{\underset{\sim}{B}'}(x)\}$$

$= \min\{1-\mu_{A}(x), 1-\mu_{B}(x)\}$

$= \min\{1-\alpha, 1-\beta\}$

$= g(x)$ (say)

For any fixed $x \in X$, two cases will arise

Case (i) $\alpha \le \beta$

Case (ii) $\alpha > \beta$

He consider these two cases in the following table

	Case	$\max\{\alpha,\beta\}$	$f(x)$	$\min\{1-\alpha,1-\beta\}$	$g(x)$
i)	$\alpha \le \beta$	β	$1-\beta$	$1-\beta$	$1-\beta$
ii)	$\alpha > \beta$	α	$1-\alpha$	$1-\alpha$	$1-\alpha$

In both the cases we see that $f(x) = g(x)$. This is true for any $x \in X$. Hence $(A \cup B)' = A' \cup B'$.

77.3.10 Theorem. Prove that the law of contraction and law of excluded middle do not hold for fuzzy sets.

Proof. Law of contradiction is $A \cap A' = \phi$

and law of excluded middle is $A \cup A' = X$

We have $\mu_{A \cap A'}(x)$

$= \min\{\mu_{A}(x), \mu_{A'}(x)\}$

$= \min\{\mu_{A}(x), 1-\mu_{A}(x)\}$

$= \min\{\alpha, 1-\alpha\}$ where $\alpha = \mu_{A}(x)$

$\mu_{\phi}(x) = 0$ for $x \in X$

For any $\alpha \in (0,1), \min\{\alpha, 1-\alpha\} \ne 0$

For $\alpha = 0$ for $\alpha = 1$ only min $\{\alpha, 1-\alpha\} = 0$

Thus min $\{\alpha, 1-\alpha\} = 0$ is not true in general for all $x \in X$.

i.e. $\mu_{A \cap A'}(x) = \mu_{\phi}(x)$ is not true in general for all $x \in X$.

i.e. $A \cap A' = \phi$ is not true.

i.e. Law of contradiction is not true for fuzzy sets.

Again $\mu_{A \cup A'}(x)$

$= \max\{\mu_A(x), 1 - \mu_A(x)\}$

$= \max\{\alpha, 1-\alpha\}$ where $\alpha = \mu_A(x)$

$$= \begin{cases} 1 \textit{ for } \alpha = 0 \textit{ or } 1 \\ 1-\alpha \textit{ for } 0 < \alpha < 1/2 \\ \alpha \textit{ for } 1/2 \le \alpha < 1 \end{cases}$$

$\therefore \mu_{A \cup A'}(x) = 1$ is not true in general for all α i.e. for all $x \in X$.

i.e. $A \cup A' = X$ is not true.

Thus law of excluded middle is not true for fuzzy sets.

77.4 Some Definitions

To develop the notion of fuzzy numbers and for the study of the arithmetic of fuzzy numbers we need certain crisp sets associated with fuzzy sets under consideration. These crips sets are called α-cut and is defined below.

77.4.1 Definition : α-cut of fuzzy set A.

The α-cut of the fuzzy set A defined over the universal set X is the crisp set A_α defined by

$A_\alpha = \{x \in X : \mu_A(x) \ge \alpha\}$. Here α is any number in $(0,1]$.

77.4.1 Support of a fuzzy set

Let A be a fuzzy set in X. Then the support of A, denoted by $S(A)$ is the crisp set defined by

$S(A) = \{x \in X : \mu_A(x) > 0\}$.

77.4.2 Height of a fuzzy set.

The height of a fuzzy set is $\underset{\sim}{A}$ is defined as

$$h(\underset{\sim}{A}) = \sup\{\mu_{\underset{\sim}{A}}(x) : x \in X\}$$

77.4.3 Normal fuzzy set

A fuzzy set $\underset{\sim}{A}$ is said to be normal if its height is one i.e. if $\sup\{\mu_{\underset{\sim}{A}}(x) : x \in X\} = 1$.

It fuzzy set $\underset{\sim}{A}$ is not normal we can normalize it by redefining the membership function as $\mu_{\underset{\sim}{A}}(x)/h(\underset{\sim}{A}), x \in X$.

77.4.4 Convex fuzzy set

A fuzzy set $\underset{\sim}{A}$ in R^n is said to be a convex set if and only if for all $x_1, x_2 \in R^n$ and $0 \le \lambda \le 1$,

$$\mu_{\underset{\sim}{A}}\{\lambda x_1 + (1-\lambda)x_2\} \ge \min\{\mu_{\underset{\sim}{A}}(x_1), \mu_{\underset{\sim}{A}}(x_2)\}$$

77.4.5 Fuzzy Number

A convex normal fuzzy set is called a fuzzy number.

The following theorem establishes a relation between the membership function and α-cuts of fuzzy set.

77.4.6 Theorem. Let $\underset{\sim}{A}$ be a fuzzy set in X with the membership function $\mu_{\underset{\sim}{A}}(x)$. Let A_α be the α-cuts of $\underset{\sim}{A}$ and $\chi_{A_\alpha}(x)$ be the characteristic function of the crisp set A_α for $\alpha \in (0,1]$. Then for each $x \in X$

$$\mu_{\underset{\sim}{A}}(x) = \sup\{\alpha \wedge \chi_{A_\alpha}(x) : 0 < \alpha \le 1\}.$$

Proof. We have $\chi_{A_\alpha} = \begin{cases} 1 \text{ if } x \in A_\alpha \\ 0 \text{ if } x \notin A_\alpha \end{cases}$

$\therefore$ For $x \in A_\alpha$ we have $\chi_{A_\alpha} = 1$ and $\mu_{\underset{\sim}{A}}(x) \ge \alpha$

and for $x \notin A_\alpha$ we have $\chi_{A_\alpha} = 0$ and $\mu_{\underset{\sim}{A}}(x) < \alpha$.

Now $\sup\{\alpha \wedge \chi_{A_\alpha}(x) : 0 < \alpha \le 1\}$

$$= \sup\{\alpha \wedge \chi_{A_\alpha}(x) : 0 < \alpha \le \mu_A(x)\} \vee \sup\{\alpha \wedge \chi_{A_\alpha}(x) : \mu_A(x) < \alpha \le 1\}$$

$$= \sup\{\alpha \wedge 1 : 0 < \alpha \le \mu_A(x)\} \vee \sup\{\alpha \wedge 0 : \mu_A(x) < \alpha \le 1\}$$

$$= \sup\{\alpha : 0 < \alpha \le \mu_A(x)\}$$

$$= \mu_A(x)$$

Remark : For given a fuzzy set $\underset{\sim}{A}$ in X we consider a special fuzzy set denoted by αA_α for $\alpha \in (0,1]$ whose membership function is defined as

$$\mu_{\alpha A_\alpha}(x) = \alpha \wedge \chi_{A_\alpha}(x) \text{ for all } x \in X.$$

Let the set S_A be defined as

$$S_A = \{\alpha : \mu_A(x) = \alpha\} \text{ for some } x \in X\}.$$

We call this set as level set of $\underset{\sim}{A}$.

Result : From above theorem we now get the following theorem

77.4.7 Theorem. The fuzzy set $\underset{\sim}{A}$ in X can be expressed in the form

$$\underset{\sim}{A} = \bigcup\{\alpha A_\alpha : \alpha \in S_A\}$$

where $\bigcup$ denotes the standard fuzzy union.

This theorem is called the representation theorem of fuzzy sets. This theorem essentially tell that a fuzzy set $\underset{\sim}{A}$ in X can always be expressed in terms of its α-cuts without explicitely resorting to its membership function $\mu_A(x)$.

Theorem 77.3.7 is explained in the following example.

77.4.8 Example

Let $\underset{\sim}{A}$ be the fuzzy set defined by the membership function

$$\mu_A(x) = \begin{cases} 0, x \le 1 \\ (x-1)/2, 1 < x \le 3 \\ (5-x)/2, 3 < x \le 5 \\ 0, x > 5 \end{cases}$$

Its graph is shown in Fig. 77.1.

Let us consider four α-cuts viz. $A_{.2}, A_{.4}, A_{.6}$ and $A_{.8}$.

In the following Figures 77.2 – 77.5 their corresponding membershhip functions $\mu_{\cdot 2A_2}(x), \mu_{\cdot 4A_4}(x), \mu_{\cdot 6A_6}(x)$ are shown.

Finally, in Fig. 77.6 the union of these four fuzzy sets i.e. $(\cdot 2A_2)\cup(\cdot 4A_4)\cup(\cdot 6A_6)\cup(\cdot 8A_8)$ is shown

i.e. $\cup\{\alpha A_\alpha : \alpha = \cdot 2, \cdot 4, \cdot 6, \cdot 8\}$ is shown.

It is seen that this is close to the graph of $\underset{\sim}{A}$ (Fig. 77.1).

This explains the fact that if we consider all $\alpha \in (0,1]$ then we get the graph of $\underset{\sim}{A}$ i.e.

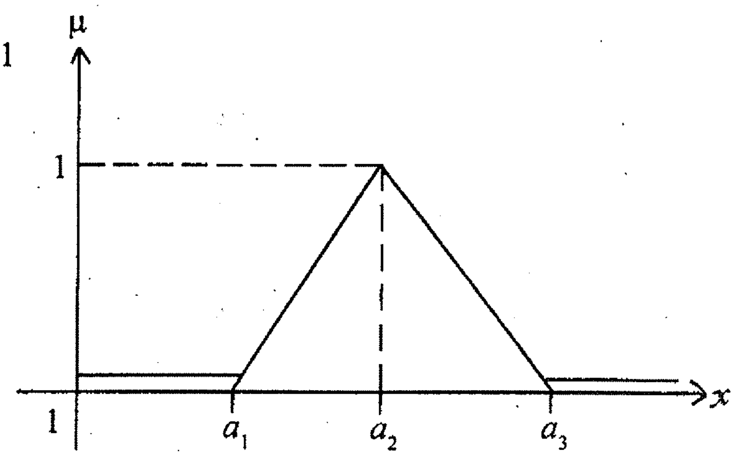
$$\cup\{\alpha A_\alpha : 0 < \alpha \le 1\} = \underset{\sim}{A}.$$

77.5 Some Useful and Important Fuzzy Numbers

Triangular Fuzzy Number : The graph of the membership function of trinagular fuzzy number is of triangular shape. It is described by a triplet i.e. $\underset{\sim}{A} = (a_1, a_2, a_3)$. The membership function is given by

$$\mu_{\underset{\sim}{A}}(x) = \begin{cases} 0 & for\ x \le a_1 \\ (x-a_1)/(a_2-a_1) & for\ a_1 < x < a_2 \\ (a_3-x)/(a_3-a_2) & for\ a_2 \le x < a_3 \\ 0 & for\ x \ge a_3. \end{cases}$$

The graph is shown in Fig. 77.5.1



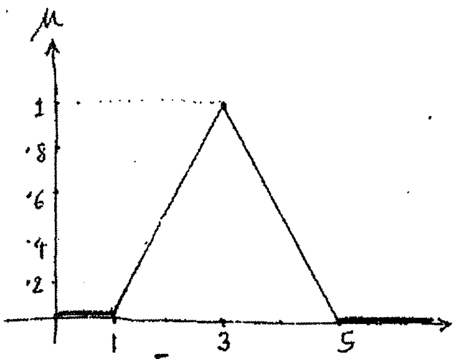


Fig 77.1

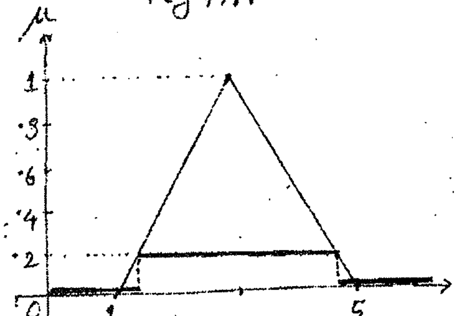


Fig 77·2

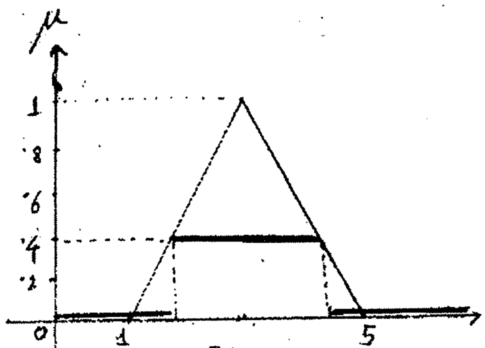


Fig 77·3

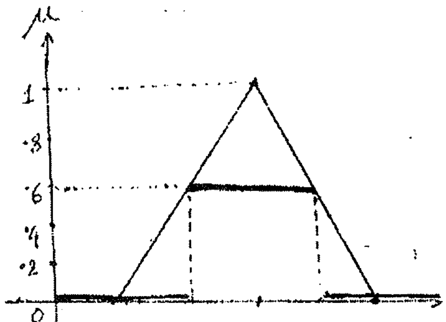


Fig 77·4

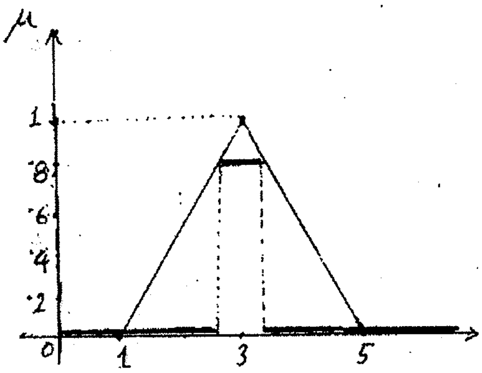


Fig 77·5

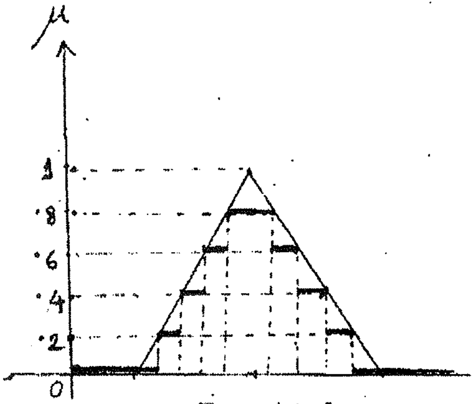


Fig 77·6

Trapezoidal Fuzzy Number : The graph of a trapezoidal fuzzy number is of the shape of a trapezium. It is described by a quadruplet i.e. $\underset{\sim}{A} = (a_1, a_2, a_3, a_4)$

The membership function of such a number is given by

$$\mu_{\underset{\sim}{A}}(x) = \begin{cases} 0 & \text{for } x \le a_1 \\ (x-a_1)/(a_2-a_1) & \text{for } a_1 < x \le a_2 \\ 1 & \text{for } a_2 < x \le a_3 \\ (a_4-x)/(a_4-a_3) & \text{for } a_3 < x < a_4 \\ 0 & \text{for } x \ge a_4. \end{cases}$$

The graph of trapezoidal fuzzy number is shown in Fig. 77.5.2.

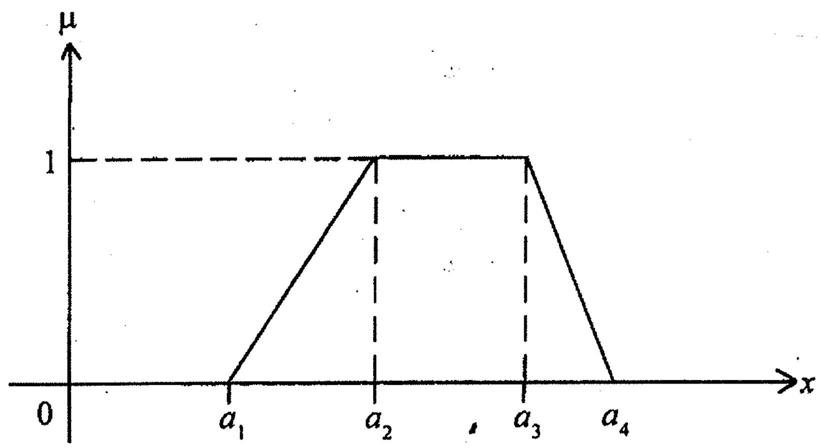


Fig. 77.5.2

Rectangular Fuzzy Number (Interval Number) : Its graph looks like a rectangle. It is a special case of trapezoidal fuzzy number. It is nothing but an interval number. It is also represented by a quadruplet i.e. $\underset{\sim}{A} = (a_1, a_2, a_3, a_4)$ whose first two and last two compondnets are alike i.e. $\underset{\sim}{A} = (a_1, a_1, a_2, a_2)$. The membership function is given by

$$\mu_{\underset{\sim}{A}}(x) = \begin{cases} 0 \text{ for } x < a_1 \\ 1 \text{ for } a_1 \le x \le a_2 \\ 0 \text{ for } x > a_2. \end{cases}$$

Its graph is shown in Fig. 77.5.3.

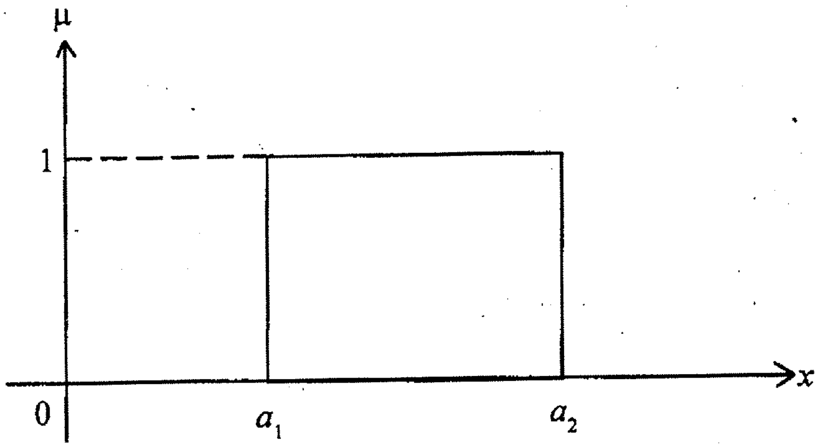


Fig. 77.5.3

Note : We note the following important facts that

A trapezoidal number becomes $\underset{\sim}{A} = (a_1, a_2, a_3, a_4)$ becomes

i) a triangular numbr if $a_2 = a_3$.

ii) an interval or a rectangular number if $a_1 = a_2$ and $a_3 = a_4$

iii) a real number if $a_1 = a_2 = a_3 = a_4$.

Gaussian Fuzzy Number : It is described by a triplet $\underset{\sim}{A} = (m, \sigma_1, \sigma_2)$. The membership function is given by

$$\mu_{\underset{\sim}{A}}(x) = \begin{cases} e^{-\frac{x-m}{\sigma_1^2}} & \text{for } x \le m \\ e^{-\frac{x-m}{\sigma_2^2}} & \text{for } x > m \end{cases}$$

Its graph is shown in Fig. 77.5.4. It is a symmetric curve about the line $x = \text{m}$ if $\sigma_1 = \sigma_2$. In general it is not symmetric.

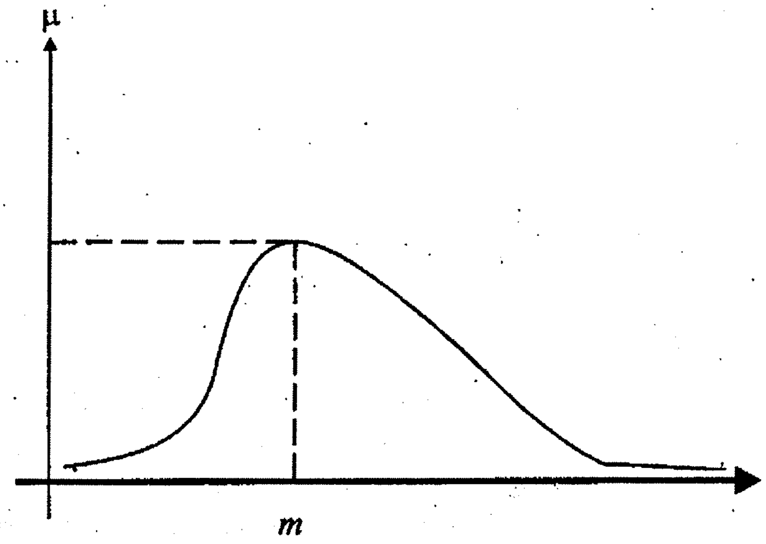


Fig. 77.5.4

77.6 Zadeh's Extension Principle

Zadeh's extension principle is a very important tool in fuzzy mathematics. This principle provides a procedure to fuzzify a crisp function. This type of fuzzification helps us to study mathematical relationships between fuzzy entities. Fuzzy arithmatic with fuzzy numbers is based on this principle.

Let $f : X \to Y$ be a crisp function. Let $P(X)$ and $P(Y)$ be the sets of all fuzzy sets of X and Y respectively. The function $f : X \to Y$ induces the function $f : P(X) \to P(Y)$ and the extension principle of Zadeh gives formulas to compute the membership function of the fuzzy set $f(\underset{\sim}{A})$ in Y in terms of the membership function of fuzzy set $\underset{\sim}{A}$ in X.

77.6.1 Definition : Zadeh's Extension Principle

Let $f : X \to Y$ be a mapping of the form $y = f(x)$ and $\underset{\sim}{A}$ be any fuzzy set of the fuzzy power set $P(X)$ of X. If $\underset{\sim}{A}$ is mapped to $\underset{\sim}{B}$ by f i.e. if $f(\underset{\sim}{A}) = \underset{\sim}{B}$ then the membership function of $\underset{\sim}{B}$ is given by

$$\mu_{f(\underset{\sim}{A})}(y) = \mu_{\underset{\sim}{B}}(y) = \sup\{\mu_{\underset{\sim}{A}}(x) : x \in X, y = f(x)\}$$

More generally, let fuzzy sets $\underset{\sim}{A}_1, \underset{\sim}{A}_2, ..., \underset{\sim}{A}_n$ be defined on the universe $X_1, X_2, ..., X_n$ respectively. The mapping $f : X_1 \times X_2 \times ... \times X_n \to Y$ of the form $f(x_1, x_2, ..., x_n) = y$ allows us to determine the membership function of the fuzzy set $f(\underset{\sim}{A}_1, \underset{\sim}{A}_2, ..., \underset{\sim}{A}_n) = \underset{\sim}{B}$ as follows by the extension principle.

$$\mu_{\underset{\sim}{B}}(y) = \sup\left[\min\left\{\mu_{\underset{\sim}{A}_1}(x_1), \mu_{\underset{\sim}{A}_2}(x_2), \ldots, \mu_{\underset{\sim}{A}_n}(x_n)\right\} : y = f(x_1, x_2, \ldots, x_n)\right]$$

Zadeh's extension pirnciple is a very powerful idea and is one of the fundamentals of fuzzy set theory. It gives us the rule of calculation of an output of a fuzzy system when we know the structure of the input fuzzy system.

77.7 Arithmetic of Fuzzy Numbers

Using Zadeh's extension principle, the arithmetic operations on fuzzy numbers are defined as follows.

Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be two fuzzy numbers then addition subtraction, multiplication and division are defined as follows.

Addition : $\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(z) = \sup\limits_{z=x=y}\left[\min\left\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)\right\}\right].$

Subtraction : $\mu_{\underset{\sim}{A}-\underset{\sim}{B}}(z) = \sup\limits_{z=x-y}\left[\min\left\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)\right\}\right]$

Multiplication : $\mu_{\underset{\sim}{A}\cdot\underset{\sim}{B}}(z) = \sup\limits_{z=x\cdot y}\left[\min\left\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)\right\}\right]$

Division : $\mu_{\underset{\sim}{A}/\underset{\sim}{B}}(z) = \sup\limits_{z=x/y}\left[\min\left\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)\right\}\right]$

77.7.1 Example. Using the addition rule for fuzzy numbers show that 3+7=10 for real numbers.

Solution. We know that every real number is a particular fuzzy number. Let $3 = \underset{\sim}{A}$ & $7 = \underset{\sim}{B}$.

Then their membership functions are

$$\mu_{\underset{\sim}{A}}(x) = \begin{cases} 1 \text{ for } x = 3 \\ 0 \text{ for } x \neq 3 \end{cases}$$

and $\mu_{\underset{\sim}{B}}(y) = \begin{cases} 1 \text{ for } y = 7 \\ 0 \text{ for } y \neq 7. \end{cases}$

From definition the membership function of $\underset{\sim}{A} + \underset{\sim}{B}$ is given by

$$\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(z) = \sup\limits_{z=x+y}\left[\min\left\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)\right\}\right]$$

For $z = 10$ we have $\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(z) = \sup_{x+y=10}\left[\min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)\}\right]$

$$= \sup_x\left[\min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(10-x)\}\right] = \sup_x[\beta(x)](\text{say})$$

x	$\mu_{\underset{\sim}{A}}(x)$	$\mu_{\underset{\sim}{B}}(10-x)$	$\beta(x)$
$x=3$	1	1	1
$x \neq 3$	0	0	0

$\therefore \mu_{\underset{\sim}{A}+\underset{\sim}{B}}(z) = \sup\{1,0\} = 1.$

For $z \neq 10$ we have $\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(z) = \sup_x\left[\min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(z-x)\}\right] = \sup[\beta(x)], (\text{say})$

x	$\mu_{\underset{\sim}{A}}(x)$	$\mu_{\underset{\sim}{B}}(z-x)$	$\beta(x)$
$x=3$	1	0	0
$x \neq 3$	0	0 or 1	0

$\therefore \mu_{\underset{\sim}{A}+\underset{\sim}{B}}(z) = \sup\{0,0\} = 0.$

Hence we have $\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(z) = 1$ for $z = 10$

$= 0$ for $z \neq 10.$

This proves that $\underset{\sim}{A} + \underset{\sim}{B} = 10$

77.7.2 Example. Using addition rule for fuzzy numbers, prove that [3, 5] + [4, 8] = [7, 13]

Solution. Let $\underset{\sim}{A} = [3,5], \underset{\sim}{B} = [4,8]$ and $\underset{\sim}{C} = [7,13]$

Then $\mu_{\underset{\sim}{A}}(x) = \begin{cases} 0 \text{ for } x < 3 \\ 1 \text{ for } 3 \leq x \leq 5 \\ 0 \text{ for } x > 5 \end{cases}$

$$\mu_{\underset{\sim}{B}}(y) = \begin{cases} 0 \text{ for } y < 4 \\ 1 \text{ for } 4 \leq y \leq 8 \\ 0 \text{ for } y > 8 \end{cases}$$

and $\mu_{\underset{\sim}{C}}(z)=\begin{cases}0 \text{ for } z<7\\ 1 \text{ for } 7\le z\le 13\\ 0 \text{ for } y>13.\end{cases}$

We have to prove that $\underset{\sim}{A}+\underset{\sim}{B}=\underset{\sim}{C}$

From the addition rule for fuzzy numbers we have

$$\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(z)=\sup_{x+y=10}\left[\min\{\mu_{\underset{\sim}{A}}(x),\mu_{\underset{\sim}{B}}(y)\}\right]$$

$$=\sup_{x+y=z}\left[\min\{\mu_{\underset{\sim}{A}}(x),\mu_{\underset{\sim}{B}}(z-x)\}\right]$$

$$=\sup_{x+y=z} g(x) \text{ where } g(x)=\min\{\mu_{\underset{\sim}{A}}(x),\mu_{\underset{\sim}{B}}(z-x)\}.$$

For any $z<7$ we have

x	$\mu_{\underset{\sim}{A}}(x)$	$\mu_{\underset{\sim}{B}}(z-x)$	$g(x)$
$x<3$	0	0 or 1	0
$3\le x\le 5$	1	0 as $-\infty<z-x<4$	0
$x>5$	0	0 as $-\infty<z-x<2$	0

$\therefore$ For $z<7$ we have $\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(z)=\min\{0,0,0\}=0$

For any z with $7\le z\le 13$ we have

x	$\mu_{\underset{\sim}{A}}(x)$	$\mu_{\underset{\sim}{B}}(z-x)$	$g(x)$
$x<3$	0	0 or 1 as $4\le z-x<\infty$	0
$3\le x\le 5$	1	0 or 1 as $2<z-x<10$	0 or 1
$x>5$	0	0 or 1 as $-\infty<z-x<\infty$	0

$\therefore$ When $7\le z\le 13$ then $\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(z)=\sup\{0,1\}=1.$

For any z with $z > 13$ we have

x	$\mu_{\underset{\sim}{A}}(x)$	$\mu_{\underset{\sim}{B}}(z-x)$	g(x)
$x < 3$	0	0 as $10 < z-x < \infty$	0
$3 \le x \le 5$	1	0 as $8 < z-x < \infty$	0
$x>5$	0	0 or 1 as $-\infty < z-x < \infty$	0

$\therefore$ When $z > 13$ then $\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(z) = \sup\{0,0,0\} = 0$

Thus $\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(z) \begin{cases} 0 \text{ for } z < 7 \\ 1 \text{ for } 7 \le z \le 13 \\ 0 \text{ for } z > 13. \end{cases}$

Note : In general using addition, subtraction, multiplicaton and division rule for fuzzy numbers we can prove the laws of addition, subtraction, multiplication and division for intervals. This is shown below.

77.8. Arithmetic Operations on Fuzzy Numbers using α-cuts.

In Theorem 77.4.7 we have proved that any fuzzy number can be expressed in the form

$$\underset{\sim}{A} = \bigcup\{\alpha A_\alpha : \alpha \in S_{\underset{\sim}{A}}\}$$

where αA_α is a special fuzzy set with membership function

$$\mu_{\alpha A_\alpha}(x) = \alpha \wedge \chi_{A_\alpha}(x) \text{ for all } x \in X.$$

Since fuzzy number is normal convex set, it follows that αA_α is nothing but an interval number. We denote it by $(\underset{\sim}{A})_\alpha$.

Thus above result becomes $\underset{\sim}{A} = \bigcup\{(\underset{\sim}{A})_\alpha : 0 < \alpha \le 1\}$

To perform arithmetic operations on fuzzy numbers using this result we proceed as follows.

Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ denote fuzzy numbers and * denote any of the four basic arithmetic operations $\{+, -, \cdot, \div\}$.

Thus for $\underset{\sim}{A} = \bigcup\{(\underset{\sim}{A})_\alpha : 0 < \alpha \le 1\}$ and $\underset{\sim}{B} = \bigcup\{(\underset{\sim}{B})_\alpha : 0 < \alpha \le 1\}$.

Using above result we get

$$\underset{\sim}{A} * \underset{\sim}{B} = \cup\{(\underset{\sim}{A})_\alpha * (\underset{\sim}{B})_\alpha : 0 < \alpha \le 1\}.$$

77.8.1 Example : Let us consider two traingular numbers $\underset{\sim}{A}$ and $\underset{\sim}{B}$ with membership functions given by

$$\mu_{\underset{\sim}{A}}(x) = \begin{cases} 0 & \text{for } x \le -1 \\ (x+1)/2 & \text{for } -1 < x \le 1 \\ (3-x)/2 & \text{for } 1 < x \le 3 \\ 0 & \text{for } x > 3 \end{cases}$$

$$\mu_{\underset{\sim}{B}}(x) = \begin{cases} 0 & \text{for } x \le 1 \\ (x-1)/2 & \text{for } 1 < x \le 3 \\ (5-x)/2 & \text{for } 3 < x \le 5 \\ 0 & \text{for } x > 5. \end{cases}$$

To get α-cut of $\underset{\sim}{A}$ we have from $\mu_{\underset{\sim}{A}}(x)$

$$(x+1)/2 = \alpha \ \& \ (3-x)/2 = \alpha$$

i.e. $x = 2\alpha - 1 \ \& \ x = 3 - 2\alpha$

$\therefore \alpha - \text{cut } of \ \underset{\sim}{A} \text{ is } (\underset{\sim}{A})_\alpha = [2\alpha - 1, 3 - 2\alpha].$

Similarly, to get α-cut of $\underset{\sim}{B}$ we have from $\mu_{\underset{\sim}{B}}(x)$

$$(x-1)/2 = \alpha \ \& \ (5-x)/2 = \alpha$$

i.e. $x = 2\alpha + 1 \ \& \ x = 5 - 2\alpha$

$\therefore \alpha - \text{cut of } \underset{\sim}{B} \text{ is } (\underset{\sim}{B})_\alpha = [2\alpha + 1, 5 - 2\alpha].$

Now $(\underset{\sim}{A})_\alpha + (\underset{\sim}{B})_\alpha = [2\alpha - 1, 3 - 2\alpha] + [2\alpha + 1, 5 - 2\alpha]$

$$= [4\alpha, 8 - 4\alpha]$$

$$(\underset{\sim}{A})_\alpha - (\underset{\sim}{B})_\alpha = [2\alpha - 1, 3 - 2\alpha] - [2\alpha + 1, 5 - 2\alpha]$$

$$= [4\alpha - 6, 2 - 4\alpha]$$

$$\therefore \underset{\sim}{A} + \underset{\sim}{B} = \cup\{[4\alpha, 8 - 4\alpha] : 0 < \alpha \le 1\}$$

and $\underset{\sim}{A}-\underset{\sim}{B}=\bigcup\{[4\alpha-6,2-4\alpha]:0<\alpha\leq 1\}$

Now $4\alpha=x$ gives $\alpha=x/4$

and $8-4\alpha=x$ gives $\alpha=(8-x)/4$

$$\therefore \underset{\sim}{A}+\underset{\sim}{B}=\begin{cases}0 & \text{for } x\leq 1\\ x/4 & \text{for } 0<x\leq 4\\ (8-x)/4 & \text{for } 4<x<8\\ 0 & \text{for } x\geq 8.\end{cases}$$

Again $4\alpha-6=x$ gives $\alpha=(x+6)/4$

$2-4\alpha=x$ gives $\alpha=(2-x)/4$

$$\therefore \underset{\sim}{A}-\underset{\sim}{B}=\begin{cases}0 & \text{for } x\leq -6\\ (x+6)/4 & \text{for } -6<x\leq -2\\ (2-x)/4 & \text{for } -2<x<2\\ 0 & \text{for } x\geq 2.\end{cases}$$

In terms of triplet notation we see that

$\underset{\sim}{A}=[-1,1,3],\underset{\sim}{B}=[1,3,5]$

and $\underset{\sim}{A}+\underset{\sim}{B}=[0,4,8],\underset{\sim}{A}-\underset{\sim}{B}=[-6,-2,2]$

Note : We see that $\underset{\sim}{A}+\underset{\sim}{B}=[-1+1,1+3,3+5]$

and $\underset{\sim}{A}-\underset{\sim}{B}=[-1-5,1-3,3-1]$

77.8.2 Rules for addition subtraction and scalar multiplication of triangular fuzzy numbers

Theorem : If $\underset{\sim}{A}=[a_1,b_1,c_1]$ and $\underset{\sim}{B}=[a_2,b_2,c_2]$ then prove that

$\underset{\sim}{A}+\underset{\sim}{B}=[a_1+a_2,b_1+b_2,c_1+c_2],\underset{\sim}{A}-\underset{\sim}{B}=[a_1-c_2,b_1-b_2,c_1-a_2]$ and

$$k\underset{\sim}{A}=\begin{cases}[ka_1,kb_1,kc_1] & \text{for } k\geq 0\\ [kc_1,kb_1,ka_1] & \text{for } k<0.\end{cases}$$

Proof. Here $\underset{\sim}{A}=[a_1,b_1,c_1]$ and $\underset{\sim}{B}=[a_2,b_2,c_2]$.

The membership functions of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are given respectively by

$$\mu_{\underset{\sim}{A}}(x)=\begin{cases}0 & \text{for } x\le a_1\\ (x-a_1)/(b_1-a_1) & \text{for } a_1<x\le b_1\\ (c_1-x)/(c_1-b_1) & \text{for } b_1<x<c_1\\ 0 & \text{for } x\ge c_1.\end{cases}$$

$$\mu_{\underset{\sim}{B}}(x)=\begin{cases}0 & \text{for } x\le a_2\\ (x-a_2)/(b_2-a_2) & \text{for } a_2<x\le b_2\\ (c_2-x)/(c_2-b_2) & \text{for } b_2<x<c_2.\\ 0 & \text{for } x\ge c_2.\end{cases}$$

To get α-cut of $\underset{\sim}{A}$ we have from $\mu_{\underset{\sim}{A}}(x)$

$(x-a_1)/(b_1-a_1)=\alpha$ and $(c_1-x)/(c_1-b_1)=\alpha$

From these $x=\alpha(b_1-a_1)+a_1$ and $x=c_1-\alpha(c_1-b_1)$.

$\therefore (\underset{\sim}{A})_\alpha=\left[\alpha(b_1-a_1)+a_1, c_1-\alpha(c_1-b_1)\right]$

To get α-cut of $\underset{\sim}{B}$ we have from $\mu_{\underset{\sim}{B}}(x)$

$(x-a_2)/(b_2-a_2)=\alpha$ and $(c_2-x)/(c_2-b_2)=\alpha$

From these $x=\alpha(b_2-a_2)+a_2$ and $x=c_2-\alpha(c_2-b_2)$

$\therefore (\underset{\sim}{B})_\alpha=\left[\alpha(b_2-a_2)+a_2, c_2-\alpha(c_2-b_2)\right]$

Using addition rule for Interval numbers we get

$(\underset{\sim}{A})_\alpha+(\underset{\sim}{B})_\alpha=\left[\alpha(b_1+b_2-a_1-a_2)+a_1+a_2, c_1+c_2-\alpha(c_1+c_2-b_1-b_2)\right]$

$\therefore \underset{\sim}{A}+\underset{\sim}{B}=\bigcup\left\{\left[\alpha(b_1+b_2-a_1-a_2)+a_1+a_2, c_1+c_2-\alpha(c_1+c_2-b_1-b_2)\right]:0<\alpha\le 1\right\}$

From $\alpha(b_1+b_2-a_1-a_2)+a_1+a_2=x$ we have

$\alpha=(x-a_1-a_2)/(b_1+b_2-a_1-a_2)$

From $x=c_1+c_2-\alpha(c_1+c_2-b_1-b_2)$ we have

$\alpha=(c_1+c_2-x)/(c_1+c_2-b_1-b_2)$

Hence we have

$$\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(x)=\begin{cases}0 & \text{for } x\le a_1+a_2\\ (x-a_1-a_2)/(b_1+b_2-a_1-a_2) & \text{for } a_1+a_2<x\le b_1+b_2\\ (c_1+c_2-x)/(c_1+c_2-b_1-b_2) & \text{for } b_1+b_2<x<c_1+c_2\\ 0 & \text{for } x\ge c_1+c_2.\end{cases}$$

i.e. $\underset{\sim}{A}+\underset{\sim}{B}=[a_1+a_2, b_1+b_2, c_1+c_2]$.

To get subtraction rule we have

$$(\underset{\sim}{A})_\alpha-(\underset{\sim}{B})_\alpha=[\alpha(b_1-a_1)+a_1-c_2+\alpha(c_2-b_2), c_1-\alpha(c_1-b_1)-\alpha(b_2-a_2)-a_2]$$
$$=[\alpha(b_1-a_1+c_2-b_2)+a_1-c_2, c_1-a_2-\alpha(c_1-b_1+b_2-a_2)]$$

$$\therefore \underset{\sim}{A}-\underset{\sim}{B}=\bigcup\{[\alpha(b_1-a_1+c_2-b_2)+a_1-c_2, c_1-a_2-\alpha(c_1-b_1+b_2-a_2)]:0<\alpha\le 1\}$$

From $\alpha(b_1-a_1+c_2-b_2)+a_1-c_2=x$ we get

$$\alpha=(x-a_1+c_2)/(b_1+c_2-a_1-b_2)$$

From $c_1-a_2-\alpha(c_1+b_2-b_1-a_2)=x$ we get

$$\alpha=(x-c_1+a_2)/(c_1+b_2-b_1-a_2)$$

$$\therefore \mu_{\underset{\sim}{A}-\underset{\sim}{B}}(x)=\begin{cases}0 & \text{for } x\le a_1-c_2\\ (x-a_1+c_2)/(b_1+c_2-a_1-b_2) & \text{for } a_1-c_2<x\le b_1-b_2\\ (x-c_1+a_2)/(c_1+b_2-b_1-a_2) & \text{for } b_1-b_2<x<c_1-a_2\\ 0 & \text{for } x\ge c_1-a_2\end{cases}$$

i.e. $\underset{\sim}{A}-\underset{\sim}{B}=[a_1-c_2, b_1-b_2, c_1-a_2]$

To prove the scalar multiplication rule we note that the α-cut of $\underset{\sim}{A}$ is

$$(\underset{\sim}{A})_\alpha=[\alpha(b_1-a_1)+a_1, c_1-\alpha(c_1-b_1)]$$

From the rule of scalar multiplication of intervals we have

$$k(\underset{\sim}{A})_\alpha=\begin{cases}[k\alpha(b_1-a_1)+ka_1, kc_1-k\alpha(c_1-b_1)] & \text{for } k\ge 0\\ [kc_1-k\alpha(c_1-b_1), k\alpha(b_1-a_1)+ka_1] & \text{for } k<0\end{cases}$$

$\therefore$ For $k \geq 0$ we have

$k\underset{\sim}{A} = \bigcup\{[k\alpha(b_1 - a_1) + ka_1, kc_1 - k\alpha(c_1 - b_1)] : 0 < \alpha \leq 1\}$

Now $x = k\alpha(b_1 - a_1) + ka_1$ and $x = kc_1 - k\alpha(c_1 - b_1)$ gives

$\alpha = (x - ka_1)/k(b_1 - a_1)$ and $\alpha = (kc_1 - x)/k(c_1 - b_1)$.

$\therefore$ For $k \geq 0$ we have

$$\mu_{k\underset{\sim}{A}}(x) = \begin{cases} 0 & \text{for } x \leq ka_1 \\ (x - ka_1)/k(b_1 - a_1) & \text{for } ka_1 < x < kb_1 \\ (kc_1 - x)/k(c_1 - b_1) & \text{for } kb_1 \leq x < kc_1 \\ 0 & \text{for } x \geq kc_1 \end{cases}$$

This gives $k\underset{\sim}{A} = [ka_1, kb_1, kc_1]$.

Similarly for $k < 0$ we have

$k\underset{\sim}{A} = \bigcup\{[kc_1 - k\alpha(c_1 - b_1), k\alpha(b_1 - a_1) + ka_1]$

Now $x = kc_1 - k\alpha(c_1 - b_1)$ and $x = k\alpha(b_1 - a_1) + ka_1$ gives

$\alpha = (kc_1 - x)/k(c_1 - b_1)$ and $\alpha = (x - ka_1)/k(b_1 - a_1)$.

$\therefore$ For $k < 0$ we have

$$\mu_{k\underset{\sim}{A}}(x) = \begin{cases} 0 & \text{for } x \leq kc_1 \\ (kc_1 - x)/k(c_1 - b_1) & \text{for } kc_1 < x < kb_1 \\ (x - ka_1)/k(b_1 - a_1) & \text{for } kb_1 \leq x < ka_1 \\ 0 & \text{for } x \geq ka_1 \end{cases}$$

This gives $k\underset{\sim}{A} = [kc_1, kb_1, ka_1]$

Hence we get finally $k\underset{\sim}{A} = \begin{cases} [ka_1, kb_1, kc_1] \text{ for } k \geq 0 \\ [kc_1, kb_1, ka_1] \text{ for } k < 0. \end{cases}$

77.8.3 Rules for addition subtraction and scalar multiplication of trapezoidal fuzzy numbers.

Proceeding exactly in the same way as triangular fuzzy numbers we can easily prove the following rules.

Addition : If $\underset{\sim}{A} = (a_1, a_2, a_3, a_4)$ and $\underset{\sim}{B} = (b_1, b_2, b_3, b_4)$ then

$$\underset{\sim}{A} + \underset{\sim}{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$$

Subtractioin : If $\underset{\sim}{A} = (a_1, a_2, a_3, a_4)$ and $\underset{\sim}{B} = (b_1, b_2, b_3, b_4)$ then

$$\underset{\sim}{A} - \underset{\sim}{B} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$$

Scalar Multiplication : If $\underset{\sim}{A} = (a_1, a_2, a_3, a_4)$ and k is a scalar then

$$k\underset{\sim}{A} = \begin{cases} [ka_1, ka_2, ka_3, ka_4] \text{for } k \geq 0 \\ [ka_4, ka_3, ka_2, ka_1] \text{for } k < 0. \end{cases}$$

77.8.4 Rules for addition and subtraction for Gaussian fuzzy numbers

Addition : If $\underset{\sim}{A} = (m_a, \sigma_{1a}, \sigma_{2a})$ and $\underset{\sim}{B} = (m_b, \sigma_{1b}, \sigma_{2b})$ then

$$\underset{\sim}{A} + \underset{\sim}{B} = (m_a + m_b, \sigma_{1a} + \sigma_{1b}, \sigma_{2a} + \sigma_{2b})$$

Subtraction : If $\underset{\sim}{A} = (m_a, \sigma_{1a}, \sigma_{2a})$ and $\underset{\sim}{B} = (m_b, \sigma_{1b}, \sigma_{2b})$ then

$$\underset{\sim}{A} - \underset{\sim}{B} = (m_a - m_b, \sigma_{1a} + \sigma_{1b}, \sigma_{2a} + \sigma_{2b}).$$

77.9 Illustrative Examples

77.9.1 Example. Show that for interval numbers distributive law does not hold in general.

Solution. Let $X = [1, 4]$, $Y = [2, 5]$ and $Z = [3, 8]$.

$\therefore X + Y = [1+2, 4+5] = [3, 9]$

Now $(X + Y)Z$

$= [3, 9]\ [3, 8]$

$= [9, 72]$

and $XZ + YZ$

$= [3, 32] + [6, 40]$

$= [9, 72]$

$\therefore (X + Y)Z = XZ + YZ$

Again let $X = [-1, 3]$, $Y = [2, 4]$, $Z = [-3, -1]$

$\therefore (X+Y)Z$

$= [1, 7][-3, -1]$

$= [\min\{-3, -1, -21, -7\}, \max\{-3, -1, -21, -7\}]$

$= [-21, -1]$

$XZ + YZ$

$= [\min\{3, 1, -9, -3\}, \max\{3, 1, -9, -3\}] + [\min\{-6, -2, -12, -4\}, \max\{-6, -2, -12, -4\}]$

$= [-9, 3] + [-12, -2]$

$= [-21, 1]$

$\therefore (X+Y)Z \subset XZ + YZ.$

Hence distributive law does not hold in general.

77.9.2 Example. Show that the fuzzy set with following membership function is neither normal nor convex.

$$\mu_A(x) = \begin{cases} 0 & \textit{for } x \in 1 \\ 3(x-1)/8 & \textit{for } 1 < x \leq 3 \\ (6-x)/4 & \textit{for } 3 < x \leq 4 \\ (3x-2)/20 & \textit{for } 4 < x \leq 6 \\ 3(7-x)/5 & \textit{for } 6 < x < 7 \\ 0 & \textit{for } x \geq 7. \end{cases}$$

Solution. We first draw the graph of this fuzzy set.

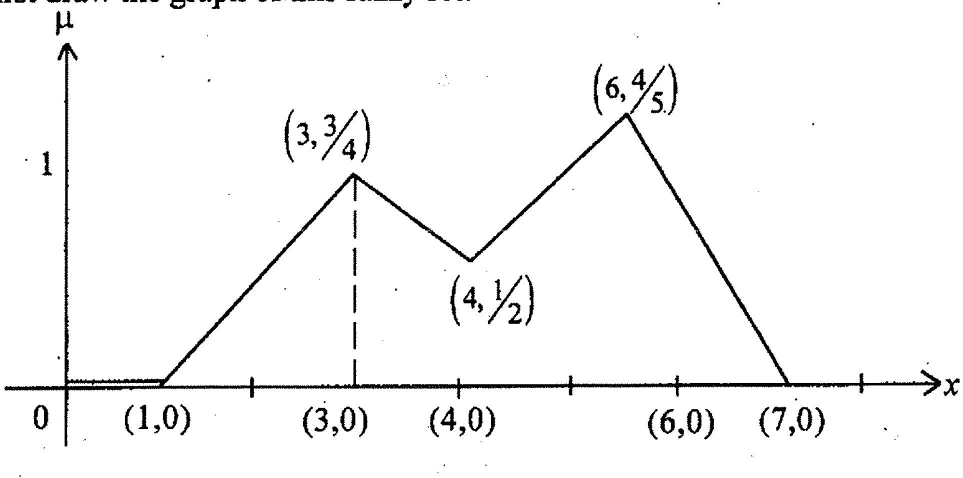


Fig. 77.9.2

From the figure we see that the height of this fuzzy set is $4/5$ which is less than one. Hence the fuzzy set is not normal.

To show that the fuzzy set is non-convex we consider two points $x_1=3$ and $x_2=6$

Now $\mu_{\underset{\sim}{A}}(x_1) = 3/4$ and $\mu_{\underset{\sim}{A}}(x_2) = 4/5$

$$\therefore \min\{\mu_{\underset{\sim}{A}}(x_1), \mu_{\underset{\sim}{A}}(x_2)\} = \min\{3/4, 4/5\} - 3/4$$

Again for $\lambda = 2/3$

$$\lambda x_1 + (1-\lambda)x_2 = \frac{2}{3}\times 3 + \frac{1}{3}\times 6 = 4$$

$$\therefore \mu_{\underset{\sim}{A}}\{\lambda x_1 + (1-\lambda)x_2\} = \mu_{\underset{\sim}{A}}(4) = 1/2$$

But $\frac{1}{2} < \frac{3}{4} \therefore \mu_{\underset{\sim}{A}}\{\lambda x_1 + (1-\lambda)x_2\} < \min\{\mu_{\underset{\sim}{A}}(x_1), \mu_{\underset{\sim}{A}}(x_2)\}$.

This shows that $\underset{\sim}{A}$ is not convex set.

77.9.3 Example. Evaluate the following

2 (5, 6, 8, 12) + 3 (–1, 3, 4) –5 [–3, 2] + 8.

Solution. 2 (5, 6, 8, 12) + 3 (–1, 3, 4) – 5 [–3, 2] + 8

= 2 (5, 6, 8, 12) + 3 (–1, 3, 3, 4) – 5 (–3, –3, 2, 2) + 8 (1, 1, 1, 1)

= (10, 12, 16, 24) + (–3, 9, 9, 12) – (–15, –15, 10, 10) + (8, 8, 8, 8)

= (15, 29, 33, 44) – (–15, –15, 10, 10)

= (15 – 10, 29 – 10, 30 + 15, 44 + 15)

= (5, 19, 45, 59).

77.10 Summary

In this module the notion of interval numbers is introduced first. Then operations on fuzzy sets are introduced. Fuzzy numbers are defined and arithmetic operations on them are discussed. The famous extension principle of Zadeh is taken into account for this purpose. The arithmetic operations on fuzzy numbers are considered as an extension of arithmetic operations on interval numbers by representing fuzzy numbers as union of α-cuts. All these are illustrated with the help of examples.

77.11. Suggested Further Readings

1. Kaufmann, A [1975] Introduction to the Theory of Fuzzy Subsets New York, London, San Francisco
2. Klir, G.J. and Folger, T.A. [1988], Fuzzy Sets, Uncertainty and Information, Englewood Cliffs
3. Kir, G.J. and Yuan, B [1997], Fuzzy Sets and Fuzzy Logic, Theory and Applications, Prenctice Hall of India.
4. Zimmermann, H.J. [1996], Fuzzy Set Theory and Its Applications, Allied Publishers Ltd.
5. Ross, T.J. [1997], Fuzzy Logic with Engineering Applications, McGraw Hill, Inc.
6. Bart Kosko, Fuzzy Thinking [1993], Flamingo, Harper Collins Publishers.
7. Bojadziev & Bojadziev, Fuzzy Sets, Fuzzy Logic & Applications; World Scientific.
8. Chen, G. and Pham, T.T; [2006] Introduction to Fuzzy Systems, Chapman & Hall/CRC.
9. Nguyen, H.T. & Walker, E.A. [1999] A first coruse in Fuzzy Logic, Chapman & Hall/CRC.
10. Bector, C.R. & Chandra, S [2004], Fuzzy Mathematical Programming and Fuzzy Matrix Games, Springer.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper-VII Group – B

Module No. - 78
FUZZY SETS
(APPLICATION OF FUZZY SETS)

Content

77.1 Introduction

In the crisp linear programming problem, the aim is to maximize or minimize a linear objective function subject to some linear constraints. But in many real life practical situations the LPP can not be specified precisely. The objective function and/or the constraint functions appears in the problem in the fuzzy sense having a vague meaning. To handle such problems fuzzy linear programming problem is introduced. In such problems the decision maker has more flexibility. Fuzziness may occur in a linear programming problem in many ways. The objective function may be fuzzy, the inequalities may be fuzzy or the problem parameters c, A, b may be in terms of fuzzy numbers. Different methods are there to solve fuzzy LPP depending on the character of fuzziness. Some of them will be discussed in detail in this module.

78.2 Classification of fuzzy LPP

The crisp linear programming problem may be stated as

Optimize $z = cx$

subject to the constraints $Ax \leq = \geq b$

and $x \geq 0$

where $c \in R^n, b^T \in R^m, x^T \in R^n$ and A is mxn real matrix.

We shall use the following notations to represents fuzzy quantitites.

$\tilde{z}$ for fuzzy objective

$\tilde{b}$ for fuzzy resource

$\tilde{c}$ for fuzzy costs

$\tilde{A}$ for fuzzy coefficients matrix

$\lesssim$ for fuzzy inequality.

In a fuzzy LPP, the fuzzy environment may occur in the following possible ways

(i) Instead of maximizing or minimizing the objective function the decision maker needs to achieve some aspiration level which itself may not even be definable crisply. As for example the decision maker may have a target to "improve the present sales situation considerably".

(ii) The constraints appeared in the LPP might be vague. The inequalities "$\leq$ or $=$ or $\geq$" may not mean in the strict mathematical sense. Some violations may be acceptable within some tolerence limit. As for example the decision maker might say "try to contact about 1800 customers per week and it must not be less than 1600 customers per week in any situation".

(iii) The components of the cost vector c, the requirement vector b and the coefficient matrix A may not be crisp numbers instead some or all of them may be fuzzy numbers. The inequalities in such situation may be interpreted in terms of ranking of fuzzy numbers.

The class of fuzzy LPP can be broadly classified as follows.

i) LPP with fuzzy inequalities and crisp objective function.

ii) LPP with crips inequalities and fuzzy objective function.

iii) LPP with fuzzy inequalities and fuzzy objective function.

iv) LPP with fuzzy resources and fuzzy coefficient i.e. LPP with fuzzy parameters i.e. elements of c, b and A are fuzzy numbers.

We have noted that there are different types of fuzzy LPP. Depending on the types of the fuzzy LPP the methods of solving them are also different. The following table shows the types of the fuzzy LPP and the standard available method for solving them.

	Types	**Methods**
1.	Crisp LPP	Simplex method
2.	$\tilde{b}$	i) Parametric Programming
		ii) Verdegay's method
		iii) Chana's method
3.	$\tilde{z}$ and $\tilde{b}$	i) Werner's method
		ii) Zimmermann's method
		iii) Lai and Hwang's method
4.	$\tilde{c}$	Parametric Programming

5.	$\tilde{A}$	Carlsson and Korhonen's method
6.	$\tilde{b}$ and $\tilde{c}$	
7.	$\tilde{A}$ and $\tilde{b}$	
8.	$\tilde{A}$ and $\tilde{c}$	
9.	$\tilde{A}$ and $\tilde{b}$ and $\tilde{c}$	
10.	$\tilde{z}$ and $\tilde{A}$	Lai and Hwang's method
11.	$\tilde{z}$ and $\tilde{A}$ and $\tilde{b}$	

78.3 Bellman and Zadeh's Principle

Let the fuzzy environment has a set of p goals $G_1, G_2, \ldots, G_p$ along with a set of n constraints $C_1, C_2, \ldots, C_n$ and each of them is expressed by fuzzy sets on the universal set X. For such a model of decision making, Bellman and Zadeh proposed that a fuzzy decision is determined by an appropriate aggregation of the fuzzy sets $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_p$ and $\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_n$. In this approach the symmetry between goals and constraints is the main feature. Bellmann and Zadeh suggested the aggregation operator to be the fuzzy intersection. The fuzzy decision $\tilde{D}$ is defined as the intersection of all $\tilde{G}_i$ and $\tilde{C}_j$ i.e. $\tilde{D} = \left(\tilde{G}_1 \cap \tilde{G}_2 \cap \ldots \cap \tilde{G}_p\right) \cap \left(\tilde{C}_1 \cap \tilde{C}_2 \cap \ldots \cap \tilde{C}_n\right)$. The membership function of $\tilde{D}$ is given by

$$\mu_{\tilde{D}}(x) = \min\left\{\mu_{\tilde{G}_1(x)}, \mu_{\tilde{G}_2(x)}, \ldots, \mu_{\tilde{G}_p(x)}, \mu_{\tilde{C}_1(x)}, \mu_{\tilde{C}_2(x)}, \mu_{\tilde{C}_n(x)}\right\}$$

Once the fuzzy decision $\tilde{D}$ is found, the optimal decision x^* is determined as $x^* \in X$ satisfying

$$\mu_{\tilde{D}}(x^*) = \max_x \mu_{\tilde{D}}(x)$$

78.3.1 Illustration of Bellman and Zadeh's Principle

Zimmermann considered the fuzzy decision problem in which we are to find a real number x which is in the vicinity of 15 and is substantially larger than 10. The constaint of the point lying in the vicinity of 15 may be regarded as a fuzzy constraint $\tilde{C}$ and the goal of having its value larger than 10 is regarded as a fuzzy goal $\tilde{G}$.

Let us take the membership function of $\tilde{C}$ and $\tilde{G}$ as follows.

$$\mu_{\tilde{C}}(x) = \left\{1+(x-15)^4\right\}^{-1}$$

$$\mu_{\tilde{G}}(x) = \begin{cases} 0 & \text{for } x \le 10 \\ \left\{1+(x-10)^{-2}\right\}^{-1} & \text{for } x > 10. \end{cases}$$

By the principle of Bellman and Zadeh, the fuzzy decision $\tilde{D}$ is given by $\tilde{C} \cap \tilde{G}$.

$\therefore$ The membership function of $\tilde{D}$ is given by

$$\mu_{\tilde{D}}(x) = \min\left\{\mu_{\tilde{C}}(x), \mu_{\tilde{G}}(x)\right\}$$

The following graph (Fig. 78.3.1) represents the fuzzy decision and optimal solution x^* where

$$\mu_{\tilde{D}}(x^*) = \max_x \mu_{\tilde{D}}(x).$$

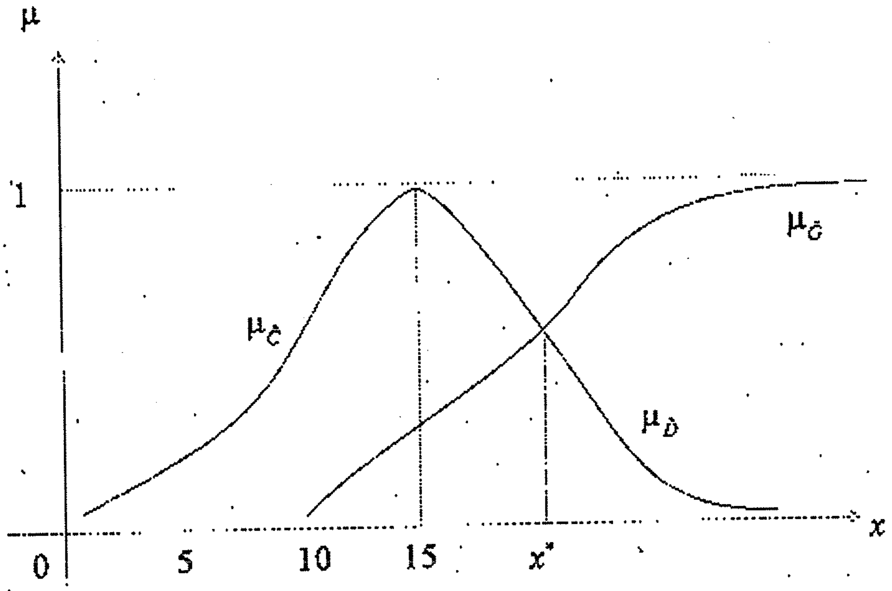


Fig. 78.3.1

To get the optimal decision x^* we proceed as follows.

We have $\mu_{\tilde{D}}(x^*) = \max_x \mu_{\tilde{D}}(x)$.

For $\alpha \in (0,1)$ we first determine all points for which $\mu_{\tilde{D}}(x) \geq \alpha$. These decisions x satisfying $\mu_{\tilde{D}}(x) \geq \alpha$ will have at least α degree of membership value. So particular x^* for which α becomes maximum will be the required optimal decision (as for α maximum $\mu_{\tilde{D}}(x)$ will also become maximum).

Hence the optimal decision x^* is the solution of the problem :

Maximize α

subject to $\mu_{\tilde{G}_i}(x) \geq \alpha, i = 1, 2, ..., p$

$\mu_{\tilde{C}_j}(x) \geq \alpha, j = 1, 2, ..., n$

$0 \leq \alpha \leq 1$

and $x \geq 0$.

78.3.2 Another classification of fuzzy LPP

The class of fuzzy LPP can be classified also as

(i) Symmetric fuzzy LPP and

(ii) Non symmetric fuzzy LPP.

Symmeric fuzzy LPP : The symmetric models are based on the definition of fuzzy decision as proposed by Bellman and Zadeh. The basic feature here is the symmetry of objectives and constraints. The decision set here is obtained as the intersection of the fuzzy sets corresponding to the objectives and constraints.

Non Symmetric fuzzy LPP : In the non-symmetric fuzzy LPP the constraints and the objectives are regarded as distinct entity. There are two approaches for non-symmetric model. In the first approach a fuzzy set of decisions is determined first and then the crisp objective function is optimized over this fuzzy set of decisions. This approach leads to a parametric LPP. In the second approach also a fuzzy set of decisions is determined first and then a suitable membership function is determined for the objective function. The problem is then solved as the symmetric case.

78.4 Verdegay's approach to solve fuzzy LPP

Verdegay considered the fuzzy LPP where the inequality is fuzzy or the resource is fuzzy.

The general model of fuzzy LPP with fuzzy inequality is

Maximum $z = cx$

subject to $(Ax)_i \lesssim b_i, i = 1, 2, \ldots, m$

$x \geq 0$

Here the fuzzy constraint $(Ax)_i \lesssim b_i$ has the meaning that the constraint $(Ax)_i \leq b_i$ is absolutely satisfied, whereas the constraint $(Ax)_i > b_i + p_i$ is absolutely violated. Here p_i is the maximum tolerance from p_i as determined by the decision maker.

The general model of fuzzy LPP with fuzzy resources is

Maximize $z = cx$

subject to $(Ax)_i \leq \tilde{b}_i, i = 1, 2, \ldots, m$

$x \geq 0$

where b_i for all i are in $[b_i, b_i + p_i]$ with given p_i.

If in both the LPP with fuzzy constraints and in fuzzy resources, the tolerance limit p_i is same and both the LPP has same membership function then Vardegay proved that both the problems are equivalent.

Verdagay showed that this fuzzy LPP is equivalent to a crisp parametric LPP. The fuzzy constraint or the fuzzy resources are transformed into crisp constraint by choosing appropriate membership function for each constraint. Here $(Ax)_i \in [b_i, b_i + p_i]$ and the membership function is taken as a monotonically decreasing function and the decrease is taken along a linear function.

Thus the membership function corresponding to the ith constraint is taken as

$$\mu_i(x)=\begin{cases} 1 & \text{for } (Ax)_i < b_i \\ \{b_i + p_i - (Ax)_i\}/p_i & \text{for } b_i \le (Ax)_i \le b_i + p_i \\ 0 & \text{for } (Ax)_i > b_i + p_i \end{cases}$$

The crisp LPP equivalent to this fuzzy LPP is taken as

Maximize $z = cx$

subject to $\mu_i(x) \ge \alpha, i = 1,2,\ldots,m$

$x \ge 0$

$0 \le \alpha \le 1.$

i.e., Maximize $z = cx$

subject to $(Ax)_i \le b_i + (1-\alpha)p_i$

$x \ge 0$

$0 \le \alpha \le 1.$

This LPP is a standard parametric LPP with $\theta = 1-\alpha$ as parameter. So the solution of the given fuzzy LPP is obtained by solving this equivalent crisp parametric LPP.

Here, we note that we have an optimal solution for each $\alpha \in [0,1]$. So the solution with α grade of membership is actually fuzzy. Also we note that this is a non-symmetric model.

To develop the idea of fuzzy LPP we consider the following problem. Also the notion of determining the membership function of fuzzy constraint will be clear from this example.

78.4.1 Example. Three metals namely iron, copper and zinc are required to produce two alloys A and B. To produce 1 metre rod of A, 1 kg iron, 1 kg copper and 0.5 kg zinc and to produce 1 metre rod of B, 1 kg copper and 1 kg zinc are needed. Total available quantitites of metals ranges as follows

iron : 3 kg to 9 kg, copper : 4 kg to 8 kg and

zinc : 3 kg to 5 kg. The profits of selling one unit of A and B are respectively Rs. 2 and Re 1.

Find the maximum profit.

Solution. All informations of the problem can be put in the following table.

Alloy	A	B	Available quantity
Iron	1 kg	0 kg	3 kg to 4 kg
Copper	1 kg	1 kg	4 kg to 6 kg
Zinc	0.5 kg	1 kg	3 kg to 5 kg
Profit	Rs. 2	Re 1	

Here the available quantities of the metals are not a fixed amount, they are given in a range. So the problem is not a crisp problem, it becomes a fuzzy problem. To formulate this problem as a LPP, let x_1 metre of alloy A and x_2 metre of alloy B be produced.

Then the fuzzy LPP becomes

Maximize $z = 2x_1 + x_2$

subject to $x_1 + ox_2 \leq 3 \text{ to } 4$

$x_1 + x_2 \leq 4 \text{ to } 6$

$0.5x_1 + x_2 \leq 3 \text{ to } 5$

$x_1, x_2 \geq 0$

78.4.2 Membership function of the *i*th constraint

The graph of the LPP with lower limits of the available quantities of iron, copper and zinc i.e. 3 kg iron, 4 kg copper and 3 kg zinc is given in the Fig. 78.4.2. Also lines are drawn with quantities as upper limits i.e. 4 kg iron, 6 kg copper and 5 kg zinc. The thick lines *AB*, *BC*, *CD* represents respectively the lower limits i.e. 3 kg iron, 4 kg copper and 3 kg zinc whereas the dotted lines $A'B', B'C'$ and $C'D'$ represents respectively the upper limits i.e. 4 kg iron, 6 kg copper and 5 kg zinc.

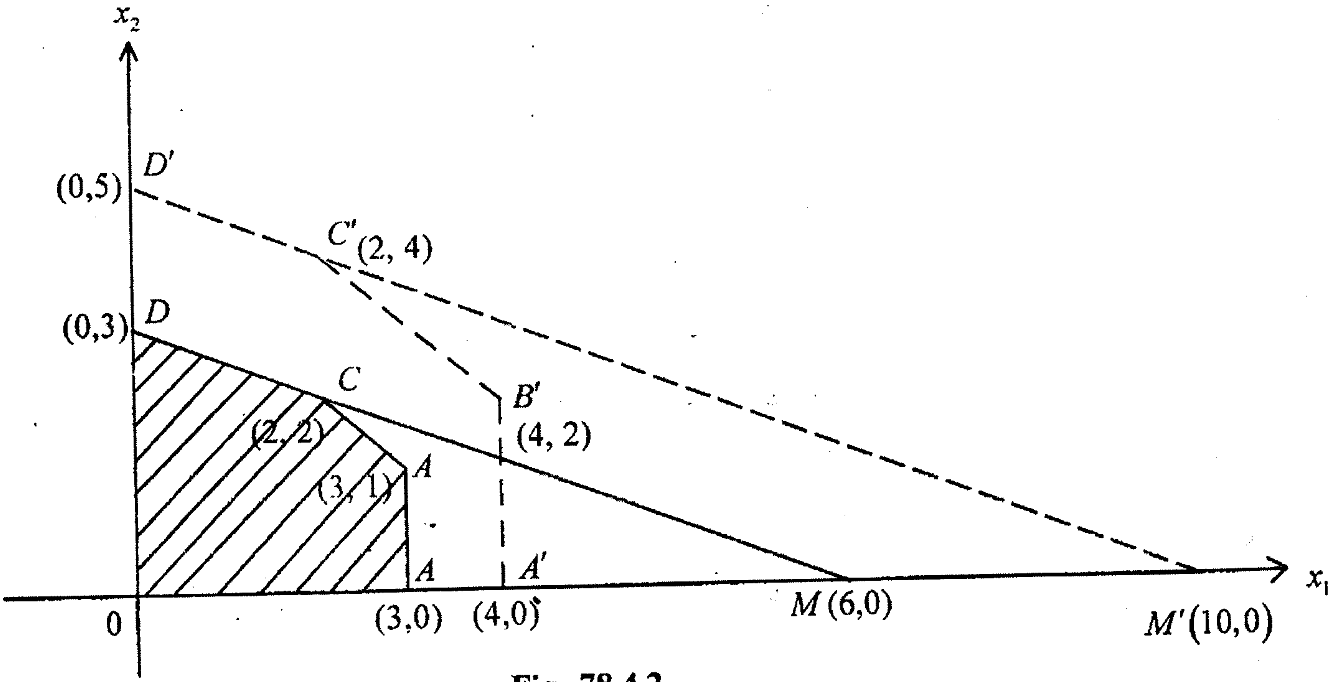


Fig. 78.4.2

In the Fig. 78.4.2 the line DM represents zinc = 3 kg and the line $D'M'$ represents zinc = 5 kg. So in the region ODM zinc $\leq$ 3 kg which is always available and hence in this region the membership function $\mu_3(x)$ should have a value 1. Again in the region beyond the line $D'M'$, amount of zinc is more than 5 kg which is not available, hence in this region the membership function should have a value zero. In the region between the lines DM and $D'M'$, the value of the membership function should lie in the interval (0, 1), as the availability of zinc there is in between 3 kg to 5 kg which is a doubtful situation. The membership function $\mu_3(x)$ should change its value there linearly from 1 on DM to 0 on $D'M'$. Hence the membership function $\mu_3(x)$ is defined as

$$\mu_3(x) = \begin{cases} 1 & \text{for } x \in \text{region ODM} \\ (5 - x)/2 & \text{for } x \in \text{region DM M'D'} \\ 0 & \text{for } x \in \text{beyond D'M'} \end{cases}$$

$$\text{i.e. } \mu_3(x_1,x_2)=\begin{cases}1 & \text{for } 0.5x_1+x_2\le 3\\ (5-0.5x_1-x_2)/2 & \text{for } 3<0.5x_1+x_2<5\\ 0 & \text{for } 0.5x_1+x_2\ge 5.\end{cases}$$

Similarly, the membership function $\mu_1(x)$ corresponding to the metal iron and $\mu_2(x)$ corresponding to the metal copper are defined as follows

$$\mu_1(x_1,x_2)=\begin{cases}1 & \text{for } x_1\le 3\\ (4-x_1) & \text{for } 3<x_1<4\\ 0 & \text{for } x_1\ge 4\end{cases}$$

$$\mu_2(x_1,x_2)=\begin{cases}1 & \text{for } x_1+x_2\le 4\\ (6-x_1-x_2)/2 & \text{for } 4<x_1+x_2<6\\ 0 & \text{for } x_1+x_2\ge 6.\end{cases}$$

To discuss Verdegay's approach we consider the following example.

78.4.3 Example. A company produces four items *A*, *B*, *C* and *D*. The inputs for the production are man-weeks, material *X* and material *Y*. The availability of the resources and profits corresponding to the items *A*, *B* and *C* are shown in the table below. Using Verdegay's method find its solution.

Item	Man Weeks	Material *X*	Material *Y*	Unit Profit
A	1	7	3	4
B	1	5	5	5
C	1	3	10	9
D	1	2	15	11
Availability	15 to 18	120	100 to 120	Maximize

Solution. Here the availability of the material *X* is 120 unit which is a precise quantity. But the available total man-weeks and material *Y* are imprecise and their maximum tolerances are respectively

3 and 20 units respectively as 18–15=3 and 120– 100=20. Let x_1, x_2, x_3 and x_4 be the amount produced for the items *A*, *B*, *C* and *D* respectively. Then the problem can be formulated as the following fuzzy LPP.

Maximize $z = 4x_1 + 5x_2 + 9x_3 + 11x_4$

subject to $x_1 + x_2 + x_3 + x_4 \le 15 \text{ to } 18$

$7x_1 + 5x_2 + 3x_3 + 2x_4 \le 120$

$3x_1 + 5x_2 + 10x_3 + 15x_4 \le 100 \text{ to } 120$

$x_1, x_2, x_3, x_4 \ge 0.$

Let $g_1(x) = x_1 + x_2 + x_3 + x_4$

$g_2(x) = 7x_1 + 5x_2 + 3x_3 + 2x_4$

$g_3(x) = 3x_1 + 5x_2 + 10x_3 + 15x_4$

$cx = 4x_1 + 5x_2 + 9x_3 + 11x_4$

Hence the problem becomes

Maximize $z = cx$

subject to $g_1(x) \le 15 \text{ to } 18$

$g_2(x) \le 120$

$g_3(x) \le 100 \text{ to } 120$

$x \ge 0.$

The membership functions of the first and third constraints are given by

$$\mu_1(x) = \begin{cases} 1 & \text{for } g_1(x) \le 15 \\ \{18 - g_1(x)\}/3 & \text{for } 15 < g_1(x) < 18 \\ 0 & \text{for } g_1(x) \ge 18 \end{cases}$$

$$\mu_3(x) = \begin{cases} 1 & \text{for } g_1(x) \le 100 \\ \{120 - g_3(x)\}/20 & \text{for } 100 < g_3(x) < 120 \\ 0 & \text{for } g_3(x) \ge 120. \end{cases}$$

The graphs of μ_1 and μ_3 are shown below.

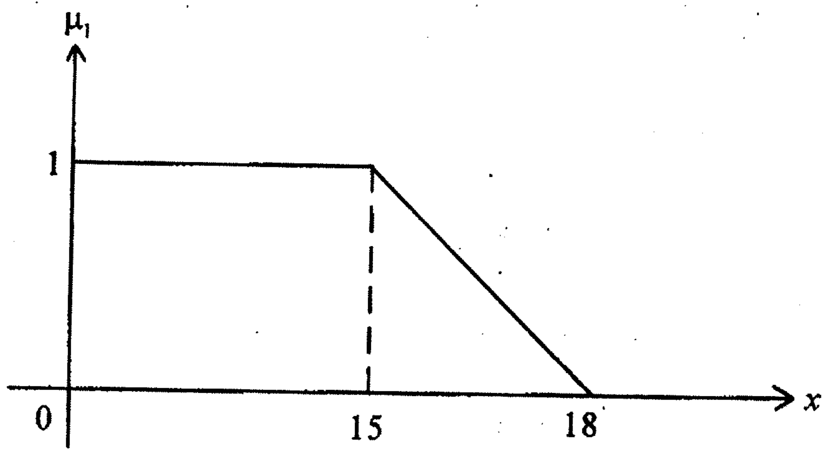


Fig. 78.4.3.1

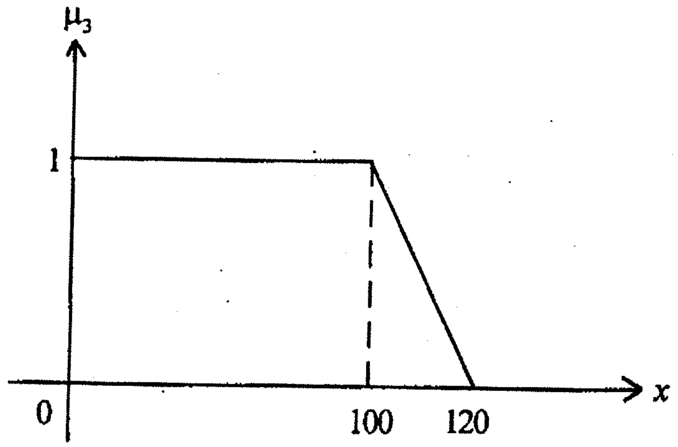


Fig. 78.4.3.2

Using Verdegay's method the crips parametric programming problem equivalent to the given fuzzy LPP is given by

Maximize $= z = cx$

subject to $\mu_1(x) \geq \alpha$

$g_2(x) \leq 120$

$\mu_3(x) \geq \alpha$

$x \geq 0$

$0 \leq \alpha \leq 1$

i.e. Maximize $z = cx$

subject to $g_1(x) \leq 15 + (1-\alpha)3$

$g_2(x) \leq 120$

$g_3(x) \leq 100 + (1-\alpha)20$

$x \geq 0$

$0 \leq \alpha \leq 1$.

i.e. Maxmize $z = 4x_1 + 5x_2 + 9x_3 + 11x_4$

subject to $x_1 + x_2 + x_3 + x_4 \leq 15 + 3\theta$

$7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 120 + 0\theta$

$3x_1 + 5x_2 + 10x_3 + 15x_4 \leq 100 + 20\theta$

$x_1, x_2, x_3, x_4 \geq 0$

where $\theta = 1 - \alpha$ is the parameter and $0 \leq \theta \leq 1$.

To solve this parametric programming problem we first solve the corresponding LPP obtained by taking θ=0 using simplex method. The tables are shown below where the variables x_5, x_6 and x_7 are slack variables.

		c	4	5	9	11	0	0	0	
c_B	x_B	b	y_1	y_2	y_3	y_4	y_5	y_6	y_7	min ratio
0	x_5	15	1	1	1	1	1	0	0	15
0	x_6	120	7	5	3	2	0	1	0	60
0	x_7	100	3	5	10	$\boxed{15}$	0	0	1	$100/15$
$z=0$		z_j-c_j	-4	-5	-9	-11	0	0	0	
0	x_5	$25/3$	$\boxed{4/5}$	$2/3$	$1/3$	0	1	0	$-1/15$	$125/12$
0	x_6	$320/3$	$33/5$	$13/3$	$5/3$	0	0	1	$-2/15$	$1600/99$
11	x_4	$20/3$	$1/5$	$1/3$	$2/3$	1	0	0	$1/15$	$100/3$
$z=220/3$		z_j-c_j	$-9/5$	$-4/3$	$-5/3$	0	0	0	$11/15$	
4	x_1	$125/12$	1	$5/6$	$5/12$	0	$5/4$	0	$-1/12$	25
0	x_6	$455/12$	0	$-7/6$	$-13/12$	0	$-33/4$	1	$5/12$	
11	x_4	$55/12$	0	$1/6$	$\boxed{7/12}$	1	$-1/4$	0	$1/12$	$55/7$
$z=1105/12$		z_j-c_j	0	$1/6$	$-11/12$	0	$9/4$	0	$7/12$	
4	x_1	$50/7$	1	$5/7$	0	$-5/7$	$10/7$	0	$-1/7$	
0	x_6	$325/7$	0	$-6/7$	0	$13/7$	$-61/7$	1	$4/7$	
9	x_3	$55/7$	0	$2/7$	1	$12/7$	$-3/7$	0	$1/7$	
$z=695/7$		z_j-c_j	0	$3/7$	0	$11/7$	$13/7$	0	$5/7$	

Using parametric programming technique the final table of this simplex method can be used to get the optimal values of the basic variables and the corresponding value of the objective function for the parametric LPP as follows.

The optimal values of the basic variables for the parametric LPP are given by

$$x_B = b + 3\theta y_5 + 0\theta y_6 + 20\theta y_7$$

or, $$\begin{bmatrix} x_1 \\ x_6 \\ x_3 \end{bmatrix} = \begin{bmatrix} 50/7 \\ 325/7 \\ 55/7 \end{bmatrix} + 3\theta \begin{bmatrix} 10/7 \\ -61/7 \\ -3/7 \end{bmatrix} + 0\theta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 20\theta \begin{bmatrix} -1/7 \\ 4/7 \\ 1/7 \end{bmatrix}$$

or, $$\begin{bmatrix} x_1 \\ x_6 \\ x_3 \end{bmatrix} = \begin{bmatrix} 50/7 + 10\theta/7 \\ 325/7 - 103\theta/7 \\ 55/7 + 11\theta/7 \end{bmatrix}$$

The optimal value of the objective function is given by

$$Z^* = Z + 3\theta(Z_5 - C_5) + 0\theta(Z_6 - C_6) + 20\theta(Z_7 - C_7)$$

$$= 695/7 + 3\theta(13/7) + 0 + 20\theta(5/7)$$

$$= (695 + 139\theta)/7$$

Hence the optimal solution of the parametric LPP i.e. of the given fuzzy LPP is

$$x_1 = (50 + 10\theta)/7$$

$$x_2 = 0$$

$$x_3 = (55 + 11\theta)/7$$

$$x_4 = 0$$

and $Z_{max} = (695 + 139\theta)/7$ where $0 \le \theta \le 1$.

We note that the answer depends on the choice of the value of θ by the decision maker.

78.5 Werners' method for solving fuzzy LPP

The general fuzzy LPP with fuzzy inequality is

Maximize $z = cx$

subject to $(Ax)_i \lesssim b_i, i = 1,2,...,m$

$x \geq 0.$

Werners proposed that because of fuzzy inequality constraint its effect will fall on the objective function and as a result the objective function should also be fuzzy.

Let the tolerances for the m constraints because of fuzzy inequalities be $p_1, p_2, ..., p_m$. So the lower and upper limits of the resources will be b_i and $b_i + p_i$ for each $i = 1, 2, ..., m$. Here we note that the given fuzzy LPP may be given equivalently also as fuzzy resource lying in $(b_i, l_i + p_i)$.

$\therefore$ The constraints $(Ax)_i \leq b_i, i = 1,2,...,m$ are satisfied completely and the constraints $(Ax)_i > b_i + p_i, i = 1,2,...,m$ are never satisfied. The constraints $(Ax)_i \leq b_i'$ where $b_i' \in (b_i, b_i + p_i)$ are satisfied partly. Thus the value of the membership function for $(Ax)_i \leq b_i$ should be 1 for $(Ax)_i \leq b_i', b_i < b_i' < b_i + p_i$ should lie in (0, 1) and for $(Ax)_i > b_i + p_i$ it should be 0.

Hence the membership function for ith constraint ($i = 1,2, ...m$) is given by

$$\mu_i(x) = \begin{cases} 1 & \text{for } (Ax)_i \leq b_i \\ \{b_i + p_i - (Ax)_i\}/p_i & \text{for } b_i < (Ax)_i < b_i + p_i \\ 0 & \text{for } (Ax)_i \geq b_i + p_i \end{cases}$$

To construct the membership function for the objective function Werners suggested to solve two LPP one with lower limit of resources and other with upper limit of resources. These two LPP's are thus

Maximize $z = cx$

subject to $(Ax)_i \leq b_i, i = 1,2,...,m$ .... (1)

$x \geq 0$

and Maximize $z = cx$

subject to $(Ax)_i \leq b_i + p_i, i = 1,2,...,m$ .... (2)

$x \geq 0.$

The optimal values of the basic variables for the parametric LPP are given by

$$x_B = b + 3\theta y_5 + 0\theta y_6 + 20\theta y_7$$

or, $$\begin{bmatrix} x_1 \\ x_6 \\ x_3 \end{bmatrix} = \begin{bmatrix} 50/7 \\ 325/7 \\ 55/7 \end{bmatrix} + 3\theta \begin{bmatrix} 10/7 \\ -61/7 \\ -3/7 \end{bmatrix} + 0\theta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 20\theta \begin{bmatrix} -1/7 \\ 4/7 \\ 1/7 \end{bmatrix}$$

or, $$\begin{bmatrix} x_1 \\ x_6 \\ x_3 \end{bmatrix} = \begin{bmatrix} 50/7 + 10\theta/7 \\ 325/7 - 103\theta/7 \\ 55/7 + 11\theta/7 \end{bmatrix}$$

The optimal value of the objective function is given by

$$Z^* = Z + 3\theta(Z_5 - C_5) + 0\theta(Z_6 - C_6) + 20\theta(Z_7 - C_7)$$

$$= 695/7 + 3\theta(13/7) + 0 + 20\theta(5/7)$$

$$= (695 + 139\theta)/7$$

Hence the optimal solution of the parametric LPP i.e. of the given fuzzy LPP is

$$x_1 = (50 + 10\theta)/7$$

$$x_2 = 0$$

$$x_3 = (55 + 11\theta)/7$$

$$x_4 = 0$$

and $Z_{max} = (695 + 139\theta)/7$ where $0 \le \theta \le 1$.

We note that the answer depends on the choice of the value of θ by the decision maker.

78.5 Werners' method for solving fuzzy LPP

The general fuzzy LPP with fuzzy inequality is

Maximize $z = cx$

5.1 Example to explain Werners' method

Let the LPP with fuzzy resources be

Maximize $Z = 4x_1 + 5x_2 + 9x_3 + 11x_4$

subject to $g_1(x) = x_1 + x_2 + x_3 + x_4 \le \widetilde{15}$

$$g_2(x) = 7x_1 + 5x_2 + 3x_3 + 2x_4 \le \widetilde{80}$$

$$g_3(x) = 3x_1 + 5x_2 + 10x_3 + 15x_4 \le \widetilde{100}$$

$$x_1, x_2, x_3, x_4 \ge 0$$

and the tolerances as $p_1 = 5, p_2 = 40, p_3 = 30$.

To get membership function for the objective function we have to solve two LPPs one with the er limits of fuzzy resources and other with the upper limits of fuzzy resources. These two LPP are ollows.

Maximize $Z = 4x_1 + 5x_2 + 9x_3 + 11x_4$

subject to $x_1 + x_2 + x_3 + x_4 \le 15$

$$7x_1 + 5x_2 + 3x_3 + 2x_4 \le 80$$

$$3x_1 + 5x_2 + 10x_3 + 15x_4 \le 100$$

$$x_1, x_2, x_3, x_4 \ge 0$$

and $Z = 4x_1 + 5x_2 + 9x_3 + 11x_4$

subject to $x_1 + x_2 + x_3 + x_4 \le 20$

$$7x_1 + 5x_2 + 3x_3 + 2x_4 \le 120$$

$$3x_1 + 5x_2 + 10x_3 + 15x_4 \le 130$$

$$x_1, x_2, x_3, x_4 \ge 0.$$

The optimum value of the objective function of these LPPs are respectively $z_0 = 99.29$ and $z_1 =$ 0. The membership functions of the objective function and the costraints are as follows.

$$\mu_0(x) = \begin{cases} 1 & \text{for } cx \geq 130 \\ (cx - 99.29)/30.71 & \text{for } 99.29 < cx < 130 \\ 0 & \text{for } cx \leq 99.29 \end{cases}$$

$$\mu_1(x) = \begin{cases} 1 & \text{for } g_1 \leq 15 \\ (20 - g_1)/5 & \text{for } 15 < g_1 < 20 \\ 0 & \text{for } g_1 \geq 20 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & \text{for } g_2 \leq 80 \\ (120 - g_2)/40 & \text{for } 80 < g_2 < 120 \\ 0 & \text{for } g_2 \geq 120 \end{cases}$$

$$\mu_3(x) = \begin{cases} 1 & \text{for } g_3 \leq 100 \\ (130 - g_3)/30 & \text{for } 100 < g_3 < 130 \\ 0 & \text{for } g_3 \geq 130. \end{cases}$$

Using Werners' method the crisp LPP equivalent to the given fuzzy LPP is

Maximize $z = \alpha$

subject to $\mu_0(x) \geq \alpha$

$\mu_1(x) \geq \alpha$

$\mu_2(x) \geq \alpha$

$\mu_3(x) \geq \alpha$

$x \geq 0$

$0 \leq \alpha \leq 1$

i.e. Maximize $z = \alpha$

subject to $4x_1 + 5x_2 + 9x_3 + 11x_4 - 30.71\alpha \leq 99.29$

$x_1 + x_2 + x_3 + x_4 + 5\alpha \leq 20$

$7x_1 + 5x_2 + 3x_3 + 2x_4 + 40\alpha \leq 120$

$$3x_1 + 5x_2 + 10x_3 + 15x_4 + 30\alpha \le 130$$

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + \alpha \le 1$$

and $\quad x_1, x_2, x_3, x, \alpha \ge 0.$

The optimum solution is obtained as

$$x_1 = 8.57, x_2 = 0, x_3 = 8.93, x_4 = 0$$

$$z_{max} = 114.64, \alpha = 0.5.$$

Actual used resources are found as

17.5, 86.78 and 115.01 respectively.

78.6 Zimmermann's method to solve fuzzy LPP

The general model of a LPP with fuzzy objective and fuzzy constraints is given by

$$\widetilde{\max}\ z = cx$$

$$\text{subject to } (Ax)_i \lesssim b_i, i = 1, 2, \ldots, m$$

$$x \ge 0.$$

The fuzzy constraint $(Ax)_i \lesssim b_i$ for each $i = 1, 2, \ldots, m$ has the meaning that if $(Ax)_i \le b_i$ then the ith constraint is absolutely satisfied, if $(Ax)_i \ge b_i + p_i$ then the ith constraint is absolutely violated, where p_i is the maximum tolerance from b_i. If $b_i < (Ax)_i < b_i + p_i$ then the ith constraint is satisfied partially. For $(Ax)_i \in (b_i, b_i + p_i)$, the membership function is monotonically decreasing as a linear function. The membership function is defined for each $i = 1, 2, \ldots, n$, as

$$\mu_i(x) = \begin{cases} 1 & \text{for } (Ax)_i \le b_i \\ \{b_i + p_i - (Ax)_i\}/p_i & \text{for } b_i < (Ax)_i < b_i + p_i \\ 0 & \text{for } (Ax)_i \ge b_i + p_i \end{cases}$$

The fuzzifier $\widetilde{\max}$ is understood in the sense of the satisfaction of an aspiration level z_0 as best as possible. Let p_0 be the permissible tolerance for the objective function. The membership function $\mu_0(x)$ for the objective function is taken to be nondecreasing and continuous and is defined as

$$\mu_0(x)=\begin{cases}1 & \text{for } cx \ge z_0\\ (cx+p_0-z_0)/p_i & \text{for } z_0-p_0<cx<z_0\\ 0 & \text{for } cx\le z_0-p_0\end{cases}$$

To identify the fuzzy decision Zimmermann employed Bellman and Zadeh principle. This leads to the following crips LPP

$$\begin{aligned}&\text{Maximize } && z=\alpha\\ &\text{subject to } && \mu_0(x)\ge\alpha\\ &&& \mu_i(x)\ge\alpha,\ i=1,2,\ldots,m\\ &&& x\ge 0\\ &&& 0\le\alpha\le 1\end{aligned}$$

$$\begin{aligned}&\text{or, Maximize } && z=\alpha\\ &\text{subject to } && cx\ge z_0-(1-\alpha)p_0\\ &&& (Ax)_i\le b_i+(1-\alpha)p_i,\quad i=1,2,\ldots,m\\ &&& x\ge 0\\ &&& 0\le\alpha\le 1\end{aligned}$$

We note here that if (x^*,α^*) is the optimal solution of this equivalent crisp LPP then α^* is the degree upto which the aspiration level z_0 of the decision maker is met.

To expain Zimmermann's method for solving fuzzy LPP Zimmermann considered the following example.

78.6.1 Example. Zimmermann considered the example

$$\begin{aligned}&\text{Max } z=x_1+x_2\\ &\text{subject to } && -x_1+3x_2\lesssim 21\\ &&& x_1+3x_2\lesssim 27\\ &&& 4x_1+3x_2\lesssim 45\\ &&& 3x_1+x_2\lesssim 30\\ &&& x_1,x_2\ge 0\end{aligned}$$

The aspiration level z_0 and tolerance levels p_i are taken as $z_0 = 14.5, p_0 = 2, p_1 = 3, p_2 = 6$ and p_3=6.

Using Zimmermann's method the crisp LPP equivalent to this fuzzy LPP is given by

Maximize $z = \alpha$

$$\begin{aligned}
\text{subject to}\quad & x_1 + x_2 \geq 14.5 - 2(1-\alpha) \\
& -x_1 + 3x_2 \leq 21 + 3(1-\alpha) \\
& x_1 + 3x_2 \leq 27 + 6(1-\alpha) \\
& 4x_1 + 3x_2 \leq 45 + 6(1-\alpha) \\
& 3x_1 + x_2 \leq 30 \\
& \alpha \leq 1 \\
& x_1, x_2, \alpha \geq 0
\end{aligned}$$

or, Maximize $z = \alpha$

$$\begin{aligned}
\text{subject to}\quad & 2\alpha - x_1 - x_2 \leq -12.5 \\
& 3\alpha - x_1 + 3x_2 \leq 24 \\
& 6\alpha + x_1 + 3x_2 \leq 33 \\
& 6\alpha + 4x_1 + 3x_2 \leq 51 \\
& 3x_1 + x_2 \leq 30 \\
& \alpha \leq 1 \\
& x_1, x_2, \alpha \geq 0
\end{aligned}$$

Using simplex method the optimal solution is obtained as $x_1^* = 6, x_2^* = 7.75, z_{max} = 13.75$ and $\alpha^* = 0.625$.

78.7 Illustrative Examples

78.7.1 Example. Using Verdegay's method solve the fuzzy LPP considered in example 78.4.1

Solution. The fuzzy LPP is

Maximize $z = 2x_1 + x_2$

subject to $x_1 + 0x_2 \leq 3$ to 4

$x_1 + x_2 \leq 4$ to 6

$0.5x_1 + x_2 \leq 3$ to 5

$x_1, x_2 \geq 0.$

The membership functions of the constraints are given by

$$\mu_1(x_1,x_2) = \begin{cases} 1 & \text{for } x_1 \leq 3 \\ (4-x_1) & \text{for } 3 < x_1 < 4 \\ 0 & \text{for } x_1 \geq 4 \end{cases}$$

$$\mu_2(x_1,x_2) = \begin{cases} 1 & \text{for } x_1 + x_2 \leq 4 \\ (6-x_1-x_2)/2 & \text{for } 4 < x_1 + x_2 < 6 \\ 0 & \text{for } x_1 + x_2 \geq 6 \end{cases}$$

$$\mu_3(x_1,x_2) = \begin{cases} 1 & \text{for } 0.5x_1 + x_2 \leq 3 \\ (5-0.5x_1-x_2)/2 & \text{for } 3 < 0.5x_1 + x_2 < 5 \\ 0 & \text{for } 0.5x_1 + x_2 \geq 5 \end{cases}$$

Using Verdegay's method the crisp parametric programming problem equivalent to the given fuzzy LPP is given by

Maximize $z = 2x_1 + x_2$

subject to $\mu_1(x) \geq \alpha$

$\mu_2(x) \geq \alpha$

$\mu_3(x) \geq \alpha$

$\alpha \geq 0$

$\alpha, x \geq 0$ where $x = (x_1, x_2)$

or, Maximize $z = 2x_1 + x_2$

subject to $x_1 \leq 3 + (1-\alpha)$

$x_1 + x_2 \leq 4 + (1-\alpha)2$

$$0.5x_1 + x_2 \leq 3 + (1-\alpha)2$$

$$\alpha \leq 1$$

$$\alpha, x_1, x_2 \geq 0.$$

Let $\theta = 1 - \alpha$. Since $0 \leq \alpha \leq 1$ we have $0 \leq \theta \leq 1$

$\therefore$ The LPP becomes

Maximize $z = 2x_1 + x_2$

subject to $x_1 \leq 3 + \theta$

$$x_1 + x_2 \leq 4 + 2\theta$$

$$0.5x_1 + x_2 \leq 3 + 2\theta$$

$$\theta \geq 0$$

$$\theta \leq 1$$

$$x_1, x_2 \geq 0$$

or, Maximize $z = 2x_1 + x_2$

subject to $x_1 \leq 3 + \theta$

$$x_1 + x_2 \leq 4 + 2\theta$$

$$x_1 + 2x_2 \leq 6 + 4\theta$$

$$x_1, x_2 \geq 0$$

$$0 \leq \theta \leq 1.$$

To solve this parametric LPP we first solve the LPP taking $\theta = 0$ i.e. we solve the following LPP by simplex method.

Maximize $z = 2x_1 + x_2$

subject to $x_1 \quad \leq 3$

$$x_1 + x_2 \leq 4$$

$$x_1 + 2x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

Introducing slack variables x_3, x_4, x_5 we get

Maximize $z = 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5$

subject to $x_1 + 0x_2 + x_3 = 3$

$x_1 + x_2 + x_4 = 4$

$x_1 + 2x_2 + x_5 = 6$

$x_1, x_2, x_3, x_4, x_5 \geq 0.$

		c_j	2	1	0	0	0	
c_B	x_B	b	y_1	y_2	y_3	y_4	y_5	min ratio
0	y_3	3	[1]	0	1	0	0	3
0	y_4	4	1	1	0	1	0	4
0	y_5	6	1	2	0	0	1	6
$z = 0$		$z_j - c_j$	-2	-1	0	0	0	
2	y_1	3	1	0	1	0	0	–
0	y_4	1	0	[1]	-1	1	0	1
0	y_5	3	0	2	-1	0	1	$3/2$
$z = 6$		$z_j - c_j$	0	-1	2	0	0	
2	y_1	3	1	0	1	0	0	
1	y_2	1	0	1	-1	1	0	
0	y_5	1	0	0	1	-2	1	
$z = 7$		$z_j - c_j$	0	0	1	1	0	

From this final table we get the optimal valus of the basic variables for the parametric LPP as

$$x_B = b + \theta \cdot y_3 + 2\theta \cdot y_4 + 4\theta \cdot y_5$$

or, $\begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + \theta \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 2\theta \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + 4\theta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

or, $\begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3+\theta \\ 1-\theta+2\theta \\ 1+\theta-4\theta+4\theta \end{bmatrix}$

or, $\begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3+\theta \\ 1+\theta \\ 1+\theta \end{bmatrix}$

The optimal value of the objective function is given by

$$z^* = z + \theta(z_3 - c_3) + 2\theta(z_4 - c_4) + 4\theta(z_5 - c_5)$$
$$= 7 + \theta \cdot 1 + 2\theta \cdot 1 + 4\theta \cdot 0$$
$$= 7 + 3\theta.$$

Hence the optimal solution of the parametric LPP is

$x_1 = 3 + \theta$

$x_2 = 1 + \theta$

and $z_{max} = 7 + 3\theta$ where $0 \le \theta \le 1$.

78.7.2 Example. Using Werners' method solve the fuzzy LPP considered in Example 78.4.1.

Solution. The fuzzy LPP is

Maximize $z = 2x_1 + x_2$

subject to $x_1 + 0x_2 \le 3$ to 4

$x_1 + x_2 \le 4$ to 6

$0.5x_1 + x_2 \le 3$ to 5

$x_1, x_2 \ge 0.$

In the Werners' method the membership function of the objective function is found with the help of optimal values of the objective function of the following two LPP.

$$\text{Maximize } z = 2x_1 + x_2$$

$$\text{subject to } \quad x_1 + 0x_2 \le 3$$
$$x_1 + x_2 \le 4$$
$$0.5x_1 + x_2 \le 3$$
$$x_1, x_2 \ge 0.$$

$$\text{and Maximize } z = 2x_1 + x_2$$

$$\text{subject to } \quad x_1 + 0x_2 \le 4$$
$$x_1 + x_2 \le 6$$
$$0.5x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0.$$

The optimal solution of the first LPP is $x_1=3, x_2=1$ and maximum value of z is 7 $\therefore z_0 = 7$.

The optimal solution of the second LPP is $x_1 = 4, x_2 = 2$ and the maximum value of z is 10 $\therefore z_1 = 10$.

The membership function of the objective function is given by

$$\mu_0(x) = \begin{cases} 1 & \text{for } 2x_1 + x_2 \ge 10 \\ (2x_1 + x_2 - 7)/3 & \text{for } 7 < 2x_1 + x_2 < 10 \\ 0 & \text{for } 2x_1 + x_2 \le 7 \end{cases}$$

The membership function of the constraints are given by

$$\mu_1(x) = \begin{cases} 1 & \text{for } x_1 \le 3 \\ (4 - x_1) & \text{for } 3 < x_1 < 4 \\ 0 & \text{for } x_1 \ge 4 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & \text{for } x_1 + x_2 \le 4 \\ (6 - x_1 - x_2)/2 & \text{for } 4 < x_1 + x_2 < 6 \\ 0 & \text{for } x_1 + x_2 \ge 6 \end{cases}$$

$$\mu_3(x) = \begin{cases} 1 & \text{for } 0.5x_1 + x_2 \le 3 \\ (5 - 0.5x_1 - x_2)/2 & \text{for } 3 < 0.5x_1 + x_2 < 5 \\ 0 & \text{for } 0.5x_1 + x_2 \ge 5. \end{cases}$$

Using Werners' method the crisp LPP equivalent to the given fuzzy LPP is given by

Maximize $z = \alpha$

subject to $cx \ge z_1 - (1-\alpha)(z_1 - z_0)$

$(Ax)_i \ge b_i + (1-\alpha)p_i$

$x \ge 0$

$0 \le \alpha \le 1.$

or, Maximize $z = \alpha$

subject to $2x_1 + x_2 \ge 10 - (1-\alpha)\cdot 3$

$x_1 \le 3 + (1-\alpha)\cdot 1$

$x_1 + x_2 \le 4 + (1-\alpha)\cdot 2$

$0.5x_1 + x_2 \le 3 + (1-\alpha)2$

$x_1, x_2 \ge 0$

$0 \le \alpha \le 1.$

or, Maximize $z = \alpha$

subject to $2x_1 + x_2 - 3\alpha \ge 7$

$x_1 + 0x_2 + \alpha \ge 4$

$x_1 + x_2 + 2\alpha \le 6$

$x_1 + 2x_2 + 4\alpha \le 10$

$\alpha \le 1$

$\alpha, x_1, x_2 \ge 0.$

Solution of this crisp LPP gives the optimal solution of the given fuzzy LPP.

78.7.3 Example. Using Zimmermann's method solve the fuzzy LPP considered in the example 78.4.1

Solution. The fuzzy LPP is

$\widetilde{\text{Maximize}}\ z = 2x_1 + x_2$

subject to $x_1 + 0x_2 \le 3$ to 4

$x_1 + x_2 \le 4$ to 6

$0.5x_1 + x_2 \le 3$ to 5

$x_1, x_2 \ge 0.$

Let us take here the aspiration level of the objective function value as 12 and the permissible tolerance of it as 3.

According to Zimmermann's method the membership function of the objective function is given by

$$\mu_0(x_1,x_2) = \begin{cases} 1 & \text{for } 2x_1 + x_2 \ge 12 \\ (2x_1 + x_2 - 9)/3 & \text{for } 9 < 2x_1 + x_2 < 12 \\ 0 & \text{for } 2x_1 + x_2 \le 9 \end{cases}$$

The membership functions of the constraints are

$$\mu_1(x_1,x_2) = \begin{cases} 1 & \text{for } x_1 \le 3 \\ (4 - x_1) & \text{for } 3 < x_1 < 4 \\ 0 & \text{for } x_1 \ge 4 \end{cases}$$

$$\mu_2(x_1,x_2) = \begin{cases} 1 & \text{for } x_1 + x_2 \le 4 \\ (6 - x_1 - x_2)/2 & \text{for } 4 < x_1 + x_2 < 6 \\ 0 & \text{for } x_1 + x_2 \ge 6 \end{cases}$$

$$\mu_3(x_1,x_2) = \begin{cases} 1 & \text{for } 0.5x_1 + x_2 \le 3 \\ (5 - 0.5x_1 - x_2)/2 & \text{for } 3 < 0.5x_1 + x_2 < 5 \\ 0 & \text{for } 0.5x_1 + x_2 \ge 5 \end{cases}$$

Using Zimmermann's method the crisp LPP equivalent to the given fuzzy LPP is given by

Maximize $z = \alpha$

subject to $\mu_0(x_1, x_2) \geq \alpha$

$\mu_i(x_1, x_2) \geq \alpha, \quad i = 1,2,3$

$\alpha, x_1, x_2 \geq 0$

$\alpha \leq 1.$

or, Maximize $z = \alpha$

subject to $2x_1 + x_2 \geq 12 - (1-\alpha)3$

$x_1 \leq 3 + (1-\alpha)\cdot 1$

$x_1 + x_2 \leq 4 + (1-\alpha)\cdot 2$

$0.5x_1 + x_2 \leq 3 + (1-\alpha)2$

$\alpha, x_1, x_2 \geq 0$

$\alpha \leq 1$

or, Maximize $z = \alpha$

subject to $2x_1 + x_2 - 3\alpha \geq 9$

$x_1 + 0x_2 + \alpha \leq 4$

$x_1 + x_2 + 2\alpha \leq 6$

$0.5x_1 + x_2 + 2\alpha \leq 5$

$\alpha \leq 1$

$\alpha, x_1, x_2 \geq 0.$

Using simplex method we get the optimal solution of this crips LPP and that is the optimal solution of the given fuzzy LPP.

78.8 Summary

In this module we have discussed applications of the fuzzy set theory developed in the earlier modules. Applicatons area are confined here mainly to fuzzy linear programming. Classification of fuzzy LPP is discussed. The pioneering work of Bellman and Zadeh for getting decision of fuzzy environment is considered for solving fuzzy LPP. Different methods developed by Verdegay, Werners and Zimmermann are discussed in details with examples to explain the methods.

78.9 Suggested Further Readings

1. Kaufmann, A [1975] Introduction to the Theory of Fuzzy Subsets New York, London, San Francisco
2. Klir, G.J. and Folger, T.A. [1988], Fuzzy Sets, Uncertainty and Information, Englewood Cliffs
3. Kir, G.J. and Yuan, B [1997], Fuzzy Sets and Fuzzy Logic, Theory and Applications, Prenctice Hall of India.
4. Zimmermann, H.J. [1996], Fuzzy Set Theory and Its Applications, Allied Publishers Ltd.
5. Ross, T.J. [1997], Fuzzy Logic with Engineering Applications, McGraw Hill, Inc.
6. Bart Kosko, Fuzzy Thinking [1993], Flamingo, Harper Collins Publishers.
7. Bojadziev & Bojadziev, Fuzzy Sets, Fuzzy Logic & Applications; World Scientific.
8. Chen, G. and Pham, T.T; [2006] Introduction to Fuzzy Systems, Chapman & Hall/CRC.
9. Nguyen, H.T. & Walker, E.A. [1999] A first coruse in Fuzzy Logic, Chapman & Hall/CRC.
10. Bector, C.R. & Chandra, S [2004], Fuzzy Mathematical Programming and Fuzzy Matrix Games, Springer.

M.Sc. Course

in

Applied Mathematics with Oceanology

and

Computer Programming

PART-II

Paper-VII

Group-C

Module No. - 79

IRROTATIONAL MOTION

Objectives

The main objective of this module is to find the complex potential for the motion of circular cylinder in a uniform stream or liquid streaming past a fixed cylinder. the complex velocity potential for a circulation round a cylinder is also obtained and discussed.

Structures

1.1 Introduction

We shall consider an irrotational motion of a liquid in two dimensions. Let u, v be the velocity components and are functions of x, y only. The component w is zero. The motion takes place in a series of planes parallel to xy and is the same in each of these planes. This type of flow is said to be two-dimensional. All physical quantities say, velocity, pressure, density, etc. are independent of z-coordinate. In this module, we consider special methods for the solution this class of problem and confine attention to inviscid incompressible fluids throughout. The solutions of several problems are obtained analytically and are of great interest. First, we shall discuss irrotational motion in two-dimensions and the motion of a cylinder in two-dimensions. In this module we mainly discuss the general motion of a cylinder in two-dimensions. The ideas of velocity potential, stream-function, and complex potential are presented first.

The necessary and sufficient condition that the right hadn side of (3) is an exact differential, is that

$$rot\,\vec{q}=0, \text{ or, } \nabla\times\vec{q}=0 \tag{5}$$

The function $\phi(x,y,z,t)$ given by (4) is known as the velocity poential for the flow field $\vec{q}$ and the surfaces

$$\phi(x,y,z,t) = \text{constant} \tag{6}$$

are called equipotentials. Equations (1) and (2) show that at all points of the field of flow the equipotentials are cut orthogonally by the streamlines.

The negative sign in equation

$$\vec{q}=-\vec{\nabla}\phi \tag{7}$$

is a convention. It ensures that flow takes place from the higher to lower potentials. When (7) holds, we get

$$rot\ \vec{q}^{1}=-\nabla\times\left(\vec{\nabla}^{1}\phi\right)\equiv 0 \tag{8}$$

i.e., the vorticity vector vanishes. Such types of motion in which the vorticity vector vanishes through out the flow field is known as Irrotational Motion. This type of flow is called potential kind.

If is a region of flow $rot\,\vec{q}$ does not vanish, the flow is called rotational flow field.

1.2 Velocity Potential : Irrotational Motion

Let the velocity at time t is $\vec{q}=(u,v,w)$ in Cartesian co-ordinate system.

Then the equation of the stream lines at that instant are given by

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w} \tag{1}$$

These curves cut the surfaces

$$udx+vdy+wdz=0 \tag{2}$$

orthogonally, provided they exist.

Suppose that at the instant t, we can find a scalar function $\phi(x,y,z,t)$ uniform throughout the entire field of flow, such that

$$-d\phi = udx+vdy+wdz \tag{3}$$

i.e., $udx+vdy+wdz=\left[\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz\right]$

Therefore,

$$\left.\begin{aligned} u=-\frac{\partial\phi}{\partial x}, v=-\frac{\partial\phi}{\partial y}, w=-\frac{\partial\phi}{\partial z} \\ \text{i.e.,} \qquad \vec{q}=-grad\phi \end{aligned}\right\} \tag{4}$$

In an incompressible flow, the equation of continuity is

$$div\vec{q}=0 \qquad (9)$$

i.e., $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$

If the motion be irrotational, then putting $\vec{q}=-\vec{\nabla}\phi$ in equation (9),

we get

$$div\, grad\, \phi=0$$

i.e., $\nabla^2\phi=0$ (10)

Thus, the velocity potential ϕ satisfies the Laplace equation (10) in an incompressible fluid.

Note : In spherical polar co-ordinates, if q_r, q_θ, q_w be the velocity components in the r, θ, w directions, respectively, then in terms of velocity potential, we have

$$q_r=-\frac{\partial\phi}{\partial r}, q_\theta=-\frac{\partial\phi}{r\partial\theta}, q_w=-\frac{\partial\phi}{r\sin\theta dw}.$$

Problem - 1 :

At a point in an incompressible fluid having spherical polar co-ordinates (r,θ,w), the velocity components are $\left(2\Pi\cos\theta/r^3, \Pi\sin\theta/r^3, 0\right)$. Show that the motion is irrotational and the velocity potential $\phi(r,\theta,w)$ is given by

$$\phi(r,\theta,w)=\frac{\pi\cos\theta}{r^2}.$$

Find also the equations of the stream lines.

1.3 General two-dimensional Motion

The fluid motion is said to be two-dimensional when the flow pattern in a certain plane, say *XOY* plane at any given instant is same as that in all other parallel planes within the fluid. In this case, all physical quantities, say velocity, pressure, density, etc. are independent of z. So, the velocity components u, v are functions of xy and t and $w = 0$ (there is no velocity perpendicular to the X-Y plane) for two-dimensional motion.

1.4 Motion in two-dimension

If (u, v, w) denote the velocity components then

$$u = v\ (x, y, z),\ v = v\ (x, y, z),\ w = 0$$

and all flow variables are independent of the z-co-ordinate.

The stream lines in the x-y plane are given by

$$\frac{dx}{u}=\frac{dy}{v}\Rightarrow vdx-udy=0 \qquad (1)$$

For incompressible flow, the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \text{ i.e., } \frac{\partial v}{\partial y}=\frac{\partial}{\partial x}(-u) \qquad (2)$$

The condition (2) shows that the left hand side of (1) is an exact differential, $d\psi$, say

i.e., $vdx-udy=d\psi$ (3)

Then (1) gives us

$$d\psi=vdx-udy=0$$

By integration we get

$$\psi(x,y,z)=c(t) \qquad (4)$$

where $c(t)$ is an arbitrary function of time t.

The function $\psi(x,y,z)$, defined by the equation (3) is called the stream function and the stream lines are given by equation (4). Also from (3), the velocity components are given by

$$u=\frac{\partial\psi}{\partial y}, v=\frac{\partial\psi}{\partial x} \qquad (5)$$

Thus whether the motion be steady or unsteady, rotational or irrotational, a stream function always exist for a two-dimensional incompressible flow.

1.5 Physical significance of stream function

Volume flux of liquid across any curve joining the points A and B from right to left as an observer moves from A to B is given by

$$\int_A^B (v\,dx - u\,dy) = \int_A^B d\psi = \psi_B - \psi_A$$

= (Value of ψ at B) – (Value of ψ at A)

Thus the flux depends on the positions of the points A and B and is independent of the curve joining there.

If q_n be the velocity normal to an arc ds from right to left,

$$q_n = \frac{\partial \psi}{\partial s}$$

Equation of continuity in polar co-ordinates is

$$\frac{\partial}{\partial r}(rq_r) + \frac{\partial}{\partial \theta}(q_\theta) = 0.$$

If ψ be the stream function, then

$$rq_r = -\frac{\partial \psi}{\partial \theta}, q_\theta = \frac{\partial \psi}{\partial r} r$$

i.e., $q_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}, q_\theta = \frac{\partial \psi}{\partial r}$

This gives us the velocity components in polar co-ordinates.

Along a stream line ψ = constant and hence $\frac{\partial \psi}{\partial s} = 0$. But $q_n = \frac{\partial \psi}{\partial s}$.

Hence there is no flow across a stream line.

1.6 Complex Potential

Let us consider a two-dimensional irrotational motion of a liquid. If ϕ be the velocity potential, the velocity components are given by

$$u = \frac{\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y} \quad (1)$$

Equation of continuity gives us

$$div\, \vec{q} = 0$$

i.e., $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial t} = 0 \quad (2)$$

Putting (1) in (2) we get

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \qquad (3)$$

i.e. ϕ satisfies Laplaces equation. Let ψ be the stream function.

Then

$$u=-\frac{\partial\psi}{\partial y}, v=\frac{\partial\psi}{\partial x} \qquad (4)$$

The condition of irrotationality gives us

$$rot\ \vec{q}=0$$

i.e., $\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}=0$ (5)

putting (4) in (5) we obtain

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \qquad (6)$$

Thus both ϕ and ψ satisfy Laplaces equation.

Also from (1) and (4) we get

$$\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}=(-u)v+(-v)(-u)=0.$$

This shows that the families of curves ϕ = constant and ψ = constant cut one another orthogonally.

The equations (1) and (4) shows that

$$\frac{\partial \phi}{\partial x}=\frac{\partial \psi}{\partial y},\frac{\partial \phi}{\partial y}=-\frac{\partial \psi}{\partial x} \tag{7}$$

Hence ϕ,ψ satisfy Cauchy-Riemann's conditions. This shows the existence of a complex function $\phi+i\psi$ of the complex variable $z=x+iy, i=\sqrt{-1}$

i.e. $\phi+i\psi=f(z)$ (8)

If we write $w=\phi+i\psi$, then the complex function $w(z)$ is known as the complex potential. The function ϕ,ψ are complex conjugates w is analytic at all points where the motion is continuous.

We have

$$\frac{dw}{dz}=\frac{d}{dz}(\phi+i\psi)=\frac{\partial \phi}{\partial x}+i\frac{\partial \psi}{\partial x}$$

$$= -u + iv \tag{9}$$

$$\therefore \left|\frac{dw}{dz}\right|=\sqrt{u^2+v^2}=q,$$

where q is the magnitude of the velocity.

If θ be the angle made by the velocity with the x-axis, then

$$u=q\cos\theta, v=q\sin\theta$$

$$\therefore \frac{dw}{dz} = -q\cos\theta + iq\sin\theta$$

$$= -qe^{-i\theta} = qe^{i(\pi-\theta)}$$

$\frac{dw}{dz}$ is known as the complex velocity. Thus any relation $w = f(z)$ i.e., $\phi + i\psi = f(x+iy)$ represents a two-dimensional irrotational motion in which the complex velocity is $\frac{dw}{dz} = qe^{i(\pi-\theta)}$

Example – 1 :

Consider the complex potential

$$W = UZ \qquad (1)$$

where U is real

Now, $\phi + i\psi = U.(x+iy)$

i.e., $\phi = Ux, \psi = Uy$

This gives us

$$u = \frac{\partial\phi}{\partial x} = U, v = 0.$$

Thus the complex potential given by (1) represents a uniform flow in the negative direction of the x-axis.

Example-2 :

Considered the uniform flow U making an angle α with the x-axis. Find ϕ and ψ.

Here the complex velocity is given by

$$\frac{dw}{dz} = Ue^{i(\pi-\alpha)}$$

$$\therefore w = Ue^{i(\pi-\alpha)}z$$

and $$\phi + i\psi = -U(\cos\alpha - i\sin\alpha)(x+iy)$$

$$\therefore \phi = -U(x\cos\alpha - y\sin\alpha)$$

$$\psi = U(x\cos\alpha - y\sin\alpha).$$

1.7 Motion of a circular cylinder in a uniform stream

To obtain the motion of a circular cylinder moving in a infinite mass of liquid at rest at infinity, with velocity U in the direction of x-axis.

The velocity potential ϕ satisfies the Laplace's equation

$$\nabla^2\phi = 0$$

at every point of the liquid. In polar co-ordinates in two dimensions $\nabla^2\phi = 0$ takes the following form

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0. \qquad (1)$$

Hence the sum of any number f terms of the form

$$A_n r^n \cos n\theta, B_n r^n \sin n\theta$$

is also a solution of (1). Here n is any integer, positive or negative.

Prescription of boundary conditios :

(*i*) Normal velocity at any point of the cylinder = Velocity of the liquid at that point in that direction, i.e., we have

$$-\frac{\partial\phi}{\partial r}=U\cos\theta \text{ at } r=a, \qquad (2)$$

the radius of the circular cylinder.

(*ii*) Since the liquid is at rest at infinity, velocity must be zero there. Thus, we get

$$-\frac{\partial\phi}{\partial r}=0 \text{ and } -\frac{1}{r}\frac{\partial\phi}{\partial r}=0 \text{ at } r=\infty. \qquad (3)$$

Solution :

The above considerations suggest that we must assume the following suitable form of ϕ. Since the equation is linear, a more general type of solution is as follows.

$$\phi = Ar\cos\theta + \frac{B}{r}\cos\theta. \qquad (4)$$

From equation (4), we get

$$-\frac{\partial\phi}{\partial r} = -\left(A - \frac{B}{r^2}\right)\cos\theta. \qquad (5)$$

Putting $r = a$ in equation (5) and using equation (2), we get

$$U\cos\theta = -\left(A - \frac{B}{a^2}\right)\cos\theta.$$

or, $$-U = \left(A - \frac{B}{a^2}\right)$$

Putting $r = \infty$ in equation (5) and using the condition (3), we get

$$A = 0.$$

Then from equation (6), we get the constant B as

$$B = Ua^2.$$

Hence from (4), the velocity poential ϕ is given by

$$\phi = \frac{Ua^2}{r}\cos\theta \tag{7}$$

It may be noted that (7) also satisfies the second condition given by (3). Hence (7) gives the required velocity potential. But we have the relation

$$\frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}. \tag{8}$$

where ψ be the stream function.

Therefore,

$$\frac{\partial\psi}{\partial r} = \frac{Ua^2}{r^2}\sin\theta.$$

After integrating, we obtain

$$\psi = -\frac{Ua^2}{r}\sin\theta \tag{9}$$

which gives the stream function of the motion. The complex potential w is given by

$$w = \frac{Ua^2}{r}(\cos\theta - i\sin\theta) = \frac{Ua^2}{z}, \tag{10}$$

where $z = re^{i\theta} = r(\cos\theta + i\sin\theta), i = \sqrt{-1}.$

1.8 Complex potential for a fixed circular cylinder in a stream

Let a circular cylinder having radius 'a' be fixed at the origin and x-axis be chosen in the opposite direction of the stream $\vec{U}$. We shallf ind the velocity potential stream function and the corresponding potential

Let R be the region $r \geq a$. Now the velocity potnetial ϕ satisfies the equation

$$\nabla^2\phi = 0 \text{ in } R. \qquad (1)$$

The boundary conditions are given by

$$\phi \sim US \text{ at infinity,}$$

and $-\frac{\partial\phi}{\partial r} = 0$ on the boundary of cylinder.

The flow is irrotational kind and two-dimensional. The velocity potential due to the uniform stream is ($Ur\cos\theta$). When the cylinder is placed, it will produce a perturbation of the flow. This perturbation must satisfy Laplace equation and become vanishingly small for large r.

Let us take the velocity poential ϕ as

$$\phi = Ur\cos\theta + \phi_1, \qquad (2)$$

where ϕ_1 is the contribution due to the presence of the cylinder.

The boundary conditions give

$$\phi_1 \to 0 \text{ at infinity} \tag{3}$$

and $$-\frac{\partial \phi_1}{\partial r} = U\cos\theta \text{ on } c: r = a. \tag{4}$$

Now since ϕ is harmonic so ϕ_1 is harmonic and its normal derivative prescribed on the boundary.

Now, let us assume ϕ_1 to be of the following form

$$\phi_1 = \left(Ar + \frac{B}{r}\right)\cos\theta.$$

From equation (3) (boundary condition) we get,

$$A = 0,$$

and from equation (4) (another boundary condition) we obtain

$$B = a^2 U.$$

Hence the velocity potential is given by

$$\phi(r,\theta) = Ur\cos\theta + \frac{Ua^2}{r}\cos\theta.$$

Hence the velocity components at any point $P(r,\theta,z)$ are

$$q_r = -\frac{\partial \phi}{\partial r} = -U\cos\theta\left(1 - \frac{a^2}{r^2}\right)$$

$$q_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\cos\theta\left(1 + \frac{a^2}{r^2}\right)$$

$$q_z = -\frac{\partial \phi}{\partial z} = 0.$$

As $r \to \infty, q_r \to -U\cos\theta, q_\theta \to U\sin\theta,$ approximately.

Again, we have the relation

$$\frac{\partial \psi}{\partial r} = -\frac{1}{r}\frac{\partial \phi}{\partial \theta},$$

This gives the stream function ψ as

$$\psi = Ur\sin\theta - \frac{Ua^2}{r}\sin\theta.$$

Hence, the complex potential

$$w(z) = Uz + \frac{Ua^2}{z} \text{ in } R'\left(\text{since } z = re^{i\theta}, z^{-1} = re^{-i\theta}, i = \sqrt{-1}\right)$$

Therefore, the equation of stram line is

$$\psi = \text{constant}$$

So, $\left(r - \frac{a^2}{r}\right) = \text{constant}$

or, $\frac{r^2 - a^2}{r}$ = constant

or, $\left(y - \frac{a^2 y}{x^2 + y^2}\right)$ = constant.

The complex velocity is given by

$$-\frac{dw}{dz} = -U_0\left(1 - \frac{a^2}{z^2}\right).$$

So, $\frac{dw}{dz} = 0$ gives $z = \pm a.$

Therefore, $z = \pm a$ are stagnation points of the flow.

1.9 Circulation about a circular cylinder

The circulation Γ round any closed curve c surrounding the origin and in the plane of flow is given by $\Gamma = \oint_c \bar{q}.ds$. Let k be the constant circulation about the cylinder. Then the suitable form of velocity potential ϕ may be obtained by equating to k the circulation round a circle of radius r. Thus, we have the relation

$$\left(-\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)(2\pi r) = \kappa.$$

Integrating this, we get

$$\phi = -\frac{k\theta}{2\pi}.$$

Again, we have

$$\frac{\partial \psi}{\partial r} = -\frac{1}{r}\frac{\partial \phi}{\partial \theta},$$

This gives

$$\psi = \frac{k}{2\pi}\ln r.$$

Thus, the complex potential due to the circulation about a circular cylinder is given by

$$w = \frac{ik}{2\pi}(\ln r + i\theta), i = \sqrt{-1}$$

or, $$w = \frac{ik}{2\pi}\ln z, \text{ since } z = re^{i\theta} = r(\cos\theta + i\sin\theta).$$

This gives the complex potential for the circulation of strength k round the circular cylinder. It implies that we may introduce a solid boundary on the circle $r = a$. Thus the complex velocity potential $w = \frac{ik}{2\prod \log z}, |z| \geq a$ will give irrotational flow outside the cylinder $|z| = a$ of infinite length. The fluid is at rest at infinity and having a circulation of amount k about the cylinder.

1.10 Circulation about a fixed circular cylinder in a uniform stream

The complex potential (w_1) due to the circulaton of strength k about the cylinder is given by

$$w_1 = \frac{ik}{2\pi} \ln z. \tag{1}$$

$i = \sqrt{-1}$, k being the strength of circulation round the cylinder.

Again, the complex potential (w_2) for streaming past a fixed circular cylinder of radius a with velocity U in the negative direction of x-axis is given by

$$w_2 = \left(Uz + \frac{Ua^2}{z} \right). \tag{2}$$

Hence the complex potential w due to the combined effects at any point z is given by

$$w = w_1 + w_2$$

$$= U\left(z + \frac{a^2}{z} \right) + \frac{ik}{2\pi} \ln z. \tag{3}$$

Also, we know that the complex potential w can be expressed as

$$w = \phi + i\psi, i = \sqrt{-1}. \tag{4}$$

where ϕ and ψ be the velocity potential and stream function respectively.

Comparing the equations (3) and (4), we obtain

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{k\theta}{2\pi}$$

and $$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta - \frac{k}{2\pi}\ln r.$$

Since the velocity will be only tangential at the boundary of the cylinder, $-\left(\frac{\partial\phi}{\partial r}\right) = 0$ and hence the magnitude of the velocity q is given by

$$\left|\left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\right| = \left|2U\sin\theta + \frac{k}{2\pi a}\right|.$$

If there are no cylinder ($k = 0$), there would be points of zero velocity on the cylinder at $\theta = 0$ and $\theta = \pi$, the former being the point at which the incoming stream divides. However, in the presence of circulation, the stagnation points are given by $q = 0$.

i.e.,

$$\sin\theta = -\frac{k}{4\pi Ua} \tag{5}$$

and such points exist when the following inequality holds.

$$|k| < 4\pi Ua. \tag{6}$$

We now determine the pressure at points of the cylinder. The pressure is given by Bernoulli's equation

$$\frac{p}{\rho} = C(t) - \frac{1}{2}q^2. \tag{7}$$

Let π be the pressure at infinity. Then $q = U$, so that from (7)

$$\frac{\pi}{\rho} = C(t) - \frac{1}{2}U^2 \tag{8}$$

Then, from (7) and (8) we get

$$\frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{1}{2}\left(U^2 - q^2\right).$$

or, $$p = \frac{\Pi}{\rho} + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(2U\sin\theta + \frac{k}{2\pi a}\right)^2 \tag{9}$$

If X, Y be the components of the thrust on the cylinder, we have

$$X = -\int_0^{2\pi} p\cos\theta a d\theta, \tag{10}$$

$$Y = -\int_0^{2\pi} p\sin\theta a d\theta. \tag{11}$$

Using (9), the components of the thrust are given by

$$X = 0$$

and $$Y = \rho k U,$$

Thus the thrust on,the cylinder acts perpendicularly to the uniform stream at infinity. We can show that no couple acts on the cylinder. So, the cylinder experinces an upward lift.

1.11 Equations of motion of a circular cylinder

A circular cylinder is moving in a liquid at rest at infinity. The forces acting on the cylinder due to the pressure of the fluid are calculated in the following way.

Let U, V be the components of the velocity of the cylinder when the center of the cross-section O is (x_0, y_0). Then, we have

$$U = \dot{x}_0 \text{ and } V = \dot{y}_0. \tag{1}$$

Let $z_0 = x_0 + iy_0$ and $(z - z_0) = re^{i\theta}, i = \sqrt{-1}$ (2)

Here r denotes the distance from the axis of the cylinder.

On the surface of the cylinder $r = a$, we must have, velocity of the liquid normal to the cylinder = normal velocity of the cylinder, i.e.

$$-\frac{\partial \phi}{\partial r} = U\cos\theta + V\sin\theta \tag{3}$$

Since the liquid is at rest at infinity,

$$-\frac{\partial \phi}{\partial r}=0 \text{ as } r \to \infty. \tag{4}$$

The conditions (3) and (4) suggest the velocity potential ϕ may be taken as follows.

$$\phi=\left(Ar+\frac{B}{r}\right)\cos\theta+\left(Cr+\frac{D}{r}\right)\sin\theta. \tag{5}$$

We have to find the constants A, B, C, D in the following way.

Now,

$$\frac{\partial \phi}{\partial r}=\left(A-\frac{B}{r^2}\right)\cos\theta+\left(C-\frac{D}{r^2}\right)\sin\theta. \tag{6}$$

Using the boundary condition (3) in (6), we get

$$U=\frac{B}{a^2}-A$$

and $$V=\frac{D}{a^2}-C.$$

Again, from (4) and (6) we obtain

$$A=0 \text{ and } C=0.$$

Thus, we get

$$B=a^2U \text{ and } D=a^2V.$$

Hence, the velocity potential ϕ reduces to the following form as

$$\phi=\frac{a^2}{r}(U\cos\theta+V\sin\theta). \tag{7}$$

Again, we have the following relation

$$\frac{\partial\psi}{\partial r}=-\frac{1}{r}\frac{\partial\phi}{\partial\theta},$$

Using (7) and integrating this equation, we obtain

$$\psi=\frac{a^2}{r}(-U\sin\theta+V\cos\theta) \tag{8}$$

Hence, the complex potential is given by

$$w=\frac{a^2e^{-i\theta}}{r}(U+iV),i=\sqrt{-1} \tag{9}$$

then using (2), we get

$$w=\frac{a^2(U+iV)}{(z-z_0)}. \tag{10}$$

Now, $$\frac{\partial w}{\partial t}=\frac{a^2(U+iV)}{(z-z_0)}+\frac{a^2(U+iV)^2}{(z-z_0)^2}. \tag{11}$$

Also, $$\frac{\partial w}{\partial t}=\frac{\partial\phi}{\partial t}+i\frac{\partial\psi}{\partial t}. \tag{12}$$

Comparing the equations (11) and (12) we obtain

$$\frac{\partial\phi}{\partial t}=\frac{a^2}{r}(U\cos\theta+V\sin\theta)+\frac{a^2}{r^2}\left[\left(U^2-V^2\right)\cos 2\theta+2UV\sin 2\theta\right]. \qquad (13)$$

The velocity q is given by

$$q^2=\left|\frac{dw}{dz}\right|^2=\left|-a^2\frac{U+iV}{(z-z_0)}\right|^2=\frac{a^4\left(U^2+V^2\right)}{r^4}. \qquad (14)$$

Omitting the external forces, the pressure at any point is given by Bernoulli's equation as

$$\frac{p}{\rho}=C(t)+\frac{\partial\phi}{\partial t}-\frac{1}{2}q^2 \qquad (15)$$

Using (13) and (14), (15) reduces to

$$\frac{p}{\rho}=C(t)+\frac{a^2}{r}(U\cos\theta+V\sin\theta)+\frac{a^2}{r^2}\left[\left(U^2-V^2\right)\cos 2\theta+2UV\sin 2\theta\right] \qquad (16)$$

$$-\frac{1}{2}\frac{a^4}{r^4}\left(U^2+V^2\right)$$

Let p_1 be the pressure at (a,θ) on the boundary of the cylinder. The p_1 is given by putting $r=a$ in (16), thus we get

$$p_1=\rho C(t)+\rho a\left(\dot{U}\cos\theta+\dot{V}\sin\theta\right)+\rho\left[\left(U^2-V^2\right)\cos 2\theta+2UV\sin 2\theta\right] \qquad (17)$$

$$-\frac{1}{2}\left(U^2+V^2\right)$$

Let X and Y be the components of the force on the cylinder due to fluid thrusts. Then, we have

$$X = -\int_0^{2\pi} ap_1 \cos\theta d\theta, \tag{18}$$

$$Y = -\int_0^{2\pi} ap_1 \sin\theta d\theta. \tag{19}$$

Using (17), (18) gives

$$X = -\rho a^2 \int_0^{2\pi} \dot{U} \cos^2\theta d\theta$$

$$= -\pi r^2 \rho \dot{U}$$

$$= -M\dot{U}, \tag{20}$$

where $M' = \pi a^2 \rho =$ the mass of the liquid displaced by the cylinder of unit length.

Similarly,

$$Y = -\pi a^2 \rho \dot{V} = -M'\dot{V}. \tag{21}$$

1.12 Keywords

Irrotational motion, velocity potential, equipotential stream function, complex velocity potential, motion of a cylinder in a uniform stream, circulation round the cylinder.

1.13 Exercises

1. Show that for an incompressible two dimensional irrotational flow the stream function and the velocity potential satisfy Laplace equation.

2. If $u=\frac{ax-by}{x^2+y^2}$, $v=\frac{ay+bx}{x^2+y^2}$, $w=0$ investigate the nature of the motion.

3. Prove that the stream function for a two dimensional flow is constant along a steam line.

4. Prove that if the Co-ordinates (x, y) of an element at any time in a two-dimensional motion be expressed in terms of initial co-ordinates (x_0, y_0) and the time, the motions is irrotational if $\frac{\partial(\dot{x},x)}{\partial(x_0,y_0)}+\frac{\partial(\dot{y},y)}{\partial(x_0,y_0)}=0$, where $\dot{x}=\frac{dx}{dt}$ and $\dot{y}=\frac{dy}{dt}$.

5. Show that in two-dimensional motion htere exists a stream function whether the motion is irrotational or rotational.

6. Show that a stream line cuts itself at a point of zero velocity in a twodimensional motion and the two branches are at right angles when the motion is irrotational.

7. Show that when a cylinder moves uniformly in a given straight line in an infinite liquid, the path of any point is given by the equations

$$\frac{dz}{dt}=\frac{Va^2}{(z'-Vt)^2};\frac{dz'}{dt}=\frac{Va^2}{(z-Vt)^2},$$

where V = velocity of cylinder, a its radius, and z, z' are $x + iy$, $x - iy$ where x, y are the coordinates measured from the starting point of the axis, along and perpendicular to its direction of motion.

8. If a long circular cylinder of radius a moves in a straight line at right angles to its length in liquid at rest at infinity, show that when a particle of liquid in the plane of symmetry, initially at distance b in advance of the axis of the cylinder has moved through a distance c, then the cylinder has moved through a distance

$$c+\frac{b^2+a^2}{b+a\coth(c/a)}.$$

9. A circular cylinder of radius a and infinite length lies on a plane in an infinite depth of liquid. The velocity of liquid at a great distance from the cylinder is U perpendicular to the generators, and the motion is irrotational and two-dimensional. Verify that the stream function is the imaginary part of $w=\pi aU\coth(\pi a/z)$, where z is a complex variable zero on the line of contact and real on the plane. Prove that hte pressure at the two ends of the diameter of the cylinder normal to the plane differs by

$$(1/32)\pi^4\rho U^2.$$

10. The space between two concentric spherical shells of radii a and b $(a>b)$ is filled with an incompressible fluid of density ρ and the shells suddenly begin to move with velocities U, V in the same direction; prove that resultant impulsive pressure on the inner shell is

$$\frac{2\pi\rho b^3}{3\left(b^3-a^3\right)\left[3a^2U-\left(a^2+2b^2\right)V\right]}.$$

11. Find the equations of the stream lines due to uniform line sources of strength m through the points $A\,(-C, 0)$, $B\,(C, 0)$ and a uniform line sink of strength m through the origin.

12. Describe the irrotational motion of an incompressible liquid for which the complex potential is $w = ik \log z$.

1.14 Further Readings

1. Milne-Thomson, L. M., Theoretical Hydrodynamics, Macmilan & Co. Ltd., London, 1955.

2. Ramsey, A.S., A Treatise on Hydromechanics, CBS Publishers & Distributors, New Delhi, 2000.

3. Chorlton, F., Textbook of Fluid Dynamics, CBS Publishers & Distributors, New Delhi, 2003.

4. Panton, L. Ronald, Incompressible Flow, A Wiley-Interscience Publication, New York, 1984.

M.Sc. Course
in
Applied Mathematics with Oceanology
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Computer Programming

PART-II

Paper-VII

Group-C

Module No. - 80

MOTION OF AN ELLIPTIC CYLINDER

Objectives

The main objective of this module is to find the complex potential due to the motion of an elliptic cylinder in an infinite mass of liquid at rest at infinity or when a elliptic cylinder is inserted in a uniform stream.

Structures

2.1 Introduction

The present unit is devoted to study two-dimensional irrotational motion produced by an elliptic cylinder in an infinite mass of liquid at rest at infinity. We discuss mainly motion of an elliptic cylinder in an infinite liquid. The equations of motion of an elliptic cylinder rotating in an infinite mass of liquid at rest at infinity are obtained and discussed.

2.2 Elliptic Co-ordinates

We use elliptic co-ordinates for analyzing the two-dimensional irrotational flow produced by an elliptic cylinder. Let $z = c\cosh\zeta$ where $\zeta = \xi + i\eta, i = \sqrt{-1}.$

So,

$$x + iy = c\cosh(\xi + i\eta)$$

$$= c(\cosh\xi\cos\eta + i\sinh\xi\sin\eta)$$

$$\therefore x = c\cosh\xi\cos\eta, y = \sinh\xi\sin\eta \quad (1)$$

Eliminating η from (1), we get

$$\frac{x^2}{c^2\cosh^2\xi} + \frac{y^2}{c^2\sin^2\xi} = 1 \quad (2)$$

Again, eliminating ξ from (1), we get

$$\frac{x^2}{c^2\cos^2\eta}-\frac{y^2}{c^2\sin^2\eta}=1 \qquad (3)$$

If a and b are the semi-axes of the ellipse (2) when $\xi=\alpha$ and $2c$ be the distance between the foci. Then we get the following relations as

$$a=c\cosh\alpha, b=c\sinh\alpha, a^2-b^2=c^2$$

$$(a+b)=ce^{\alpha},(a-b)=ce^{-\alpha},e^{2\alpha}=\frac{a+b}{a-b}$$

The parameters ξ, η are called elliptic co-ordinates.

2.3 Motion of an Elliptic cylinder

(1) Find the velocity potential and stream function when an elliptic cylinder moves in an infinite liquid with velocity U parallel to the axis plane through the major axis of a cross-section.

For any cylinder moving with velocities U and V parallel to axes and rotating with an angular velocity ω, the stream function ψ is given by

$$\psi=Vx-Uy+\frac{1}{2}\omega\left(x^2+y^2\right)+A,$$

A being a constant.

Since the cylinder moves with velocity U parallel to the axial plane through major cross-section, so we get

$$V = 0, w = 0.$$

Hence, the stream function is given by

$$\psi = -Uy + A. \qquad (1)$$

Let the cross section be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is the same as $\varepsilon = \alpha$, if $a = c\cosh \alpha a, b = c\sinh \alpha$ and $c^2 = a^2 - b^2$, where

$$x = c\cosh \varepsilon \cos \eta, \qquad (2)$$

and

$$y = c\sinh \varepsilon \sin \eta. \qquad (3)$$

Using (2) and (3), (1) becmes

$$\psi = -Uc\sinh \alpha \sin \eta + a. \qquad (4)$$

Since ψ contains sin η and the liquid is at rest at infinity, ψ must be of the form $e^{-\varepsilon} \sin \eta$. We therefore, assume that

$$\phi + i\psi = Be^{-(\varepsilon + i\eta)} \tag{5}$$

so that $\psi = -Be^{-\eta}\sin\eta.$ (6)

Then at boundary $\varepsilon = \alpha$, we must have

$$-Be^{-\alpha}\sin\eta = -Uc\sinh\alpha\sin\eta + A.$$

This gives the values of A and B as

$$A = 0,$$

$$B = Uce^{\alpha}\sinh\alpha.$$

So, $\psi = -Uce^{\alpha-\varepsilon}\sinh\alpha\sin\eta$ (7)

is a stream function which will make the boundary of the ellipse a stream line, when the cylinder moves with velocity U.

But we have

$$c\sinh\alpha = b$$

and $e^{\alpha} = \left(\dfrac{a+b}{a-b}\right)^{\frac{1}{2}}$ (8)

Using (8), (7) can be written in the form

$$\psi = -Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{\varepsilon}\sin\eta. \tag{9}$$

Also from (5),

$$\phi = Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{\varepsilon} \cos\eta. \tag{10}$$

Hence we obtain

$$w = \phi + i\psi = Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\varepsilon+i\eta)} \tag{11}$$

(b) Find the velocity potential and the stream function when an elliptic cylinder moves in an infinite with velocity V parallel to the axial plane through the minor axis of a cross-section.

Solution :

Proceeding as case (a), we can obtain the velocity potential and stream function as

$$\phi = Va\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\varepsilon} \cos\eta, \tag{12}$$

$$\psi = Va\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\varepsilon} \cos\eta, \tag{13}$$

and $$w = \phi + i\psi = iVa\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\varepsilon+i\eta)}. \tag{14}$$

(c) Find the complex potential when an elliptic cylinder moves in an infinite liquid with a velocity v in a direction making an angle θ with the major axis of the cross section of the cylinder.

Solution :

The components of v along coordinate axes are given by

$$U = v\cos\theta$$

and $$V = v\sin\theta.$$

Let w_1 and w_2 be the complex potentials corresponding to the motion of the cylinder with velorities U and V respectively. Then from the above problem we obtain w_1 and w_2 as follows.

$$w_1 = Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\varepsilon+i\eta)}.$$

$$= bv\cos\theta\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\varepsilon+i\eta)}$$

and $$w_2 = iVa\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\varepsilon+i\eta)}$$

$$= iav\sin\theta\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\varepsilon+i\eta)}$$

Hence, the complex potential due to velocity v is given by where Thus, we get

$$w = w_1 + w_2.$$

$$= v\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\zeta}(b\cos\theta + ia\sin\theta),$$

$$= v\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}. e^{-\zeta}(c\sinh\alpha\cos\theta + ic\cosh\alpha\sin\theta)$$

where $\zeta = \xi + i\eta; b = c\sinh\alpha, a = c\cosh\alpha$. Thus, we get

$$w = cv\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\zeta}\sinh(\alpha + i\theta)$$

2.4 Liquid streaming past a fixed elliptic cylinder

To obtain ϕ and ψ for a liquid streaming past a fixed elliptic cylinder with velocity U parallel to majro axis of the section.

Let us consider a velocity U on the cylinder and on liquid both in the sense opposite to the velocity of the liquid. This brings the liquid at rest and the cylinder in motion with velocity U. Hence, some suitable term must be added to each of the expressions for ϕ and ψ obtained in the case (a) of the previous section. When the stream flows from positive x-axis to negative x-axis, we get the following equations as

$$-\frac{\partial\phi}{\partial x}=-\frac{\partial\psi}{\partial y}=-U. \qquad (1)$$

Accordingly, we must add a term Ux to ϕ and Uy to ψ. Thus, we have

$$\phi=Ux+Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}e^{-\xi}\cos\eta$$

$$=U\left(a^2-b^2\right)^{\frac{1}{2}}\cosh\xi\cos\eta+Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}e^{-\xi}\sin\eta. \qquad (2)$$

and the stream function as

$$\psi=Uy-Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}e^{-\xi}\sin\eta$$

$$=U\left(a^2-a^2\right)^{\frac{1}{2}}\sinh\xi\sin\eta-Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}e^{-\xi}\sin\eta. \qquad (3)$$

The complex potential is given by

$$w=\phi+i\psi=Uz+Ube^{\alpha-\zeta} \qquad (4)$$

Another form of ϕ,ψ and complex potential w, we can be obtained as follows.

$$\phi=Uce^{\alpha}\cos\eta\cosh(\xi-\alpha), \qquad (5)$$

$$e^{\alpha}=\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}$$

and, $$\psi = Uce^{\alpha}\sin\eta\sinh(\xi-\alpha), \quad (6)$$

and

$$w=\phi+i\psi=U(a+b)\left[\cos\eta\cosh(\xi-\alpha)+i\sin\eta\sinh(\xi-\alpha)\right]$$

$$=U(a+b)\cosh\left[(\xi-\alpha)+i\eta\right]$$

$$=U(a+b)\cosh(\xi-\alpha). \quad (7)$$

2.5 Rotating elliptic cylinder

Find the velocity potential ϕ and the stream function ψ when an elliptic cylinder is rotating with angular velocity ω in an infinite mass of the liquid at infinity.

Solution :

For any cylinder moving with velocity U and V parallel to axes and rotating with an angular velocity ω, we know that the stream function ψ is

$$\psi = Vx - Uy + \frac{1}{2}w\left(x^2+y^2\right)+A, \quad (1)$$

A be the constant.

Let the cross-section be the ellipse, as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

This is the same as $\xi=\alpha$, if $a=c\cosh\alpha, b=c\sinh\alpha$ and $c^2 = a^2 - b^2$. The elliptic coordinates (ξ,η) are given as follows.

Let $z=c\cosh\zeta$, where $\zeta=\xi+i\eta$. Then

$$x+iy=c\cosh(\xi+i\eta)$$

$$=c\cosh\zeta\cos\eta+ic\sinh\xi\sin\eta.$$

Then, we get

$$x=c\cosh\xi\cos\eta, \qquad (2)$$

$$y=c\sinh\xi\sin\eta. \qquad (3)$$

Here

$$U=V=0.$$

So using (2) and (3), the stream function ψ of (1) reduces to

$$\psi=\frac{1}{4}\omega c^2(\cosh 2\xi+\cos 2\eta)+A. \qquad (4)$$

Since, ψ contains $\cos 2\eta$ and the liquid is at rest at infinity, ψ must be taken in the form

$$\psi = Be^{2-\xi}\cos 2\eta \tag{5}$$

and hence the velocity potential

$$\phi = Be^{-2\xi}\sinh 2\eta. \tag{6}$$

So at the boundary $\xi = \alpha$, we get the following relation

$$Be^{-2\alpha}\cos 2\eta = \frac{1}{4}\omega c^2(\cosh 2\alpha + \cos 2\eta) + A$$

This gives

$$B = \frac{1}{4}wc^2 e^{2\alpha}$$

and $A = -\frac{1}{4}wc^2\cosh 2\alpha.$

Then ϕ and ψ reduce to

$$\phi = \frac{1}{4}w(a+b)^2 e^{-2\xi}\sin 2\eta, \tag{7}$$

$$\psi = \frac{1}{4}w(a+b)^2 c^{-2\xi}\sin 2\eta. \tag{8}$$

Thus the complex potential function w is

$$w = \frac{1}{4}i\omega(a+b)^2 e,\ \text{since } \zeta = (\xi + i\eta), i = \sqrt{-1}. \tag{9}$$

2.6 Motion of a liquid in rotating elliptic cylinder

Let the elliptic cylinder containing liquid rotate with angular velocity ω.

The stream function ψ must satisfy the Laplace's equation given by

$$\nabla^2\psi = 0$$

and on the boundary it satisfies the condition

$$\psi = \frac{1}{2}w\left(x^2 + y^2\right) + A. \tag{1}$$

We assume that

$$\psi = B\left(x^2 - y^2\right). \tag{2}$$

On the boundary of the cylinder, we must have

$$\left(B - \frac{1}{2}\omega\right)x^2 - \left(B + \frac{1}{2}\omega\right)y^2 = A$$

or,
$$\frac{x^2}{A/\left(B - \frac{1}{2}w\right)} + \frac{y^2}{A/\left(-B - \frac{1}{2}w\right)} = 1. \tag{3}$$

We also know that the boundary of the cylinder is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{4}$$

Comparing the equatons (3) with (4) we get

$$B = \frac{1}{2}\omega\frac{a^2 - b^2}{a^2 + b^2}.$$

So that

$$\psi = \frac{1}{2}\omega\frac{a^2 - b^2}{a^2 + b^2}(x^2 - y^2). \qquad (5)$$

The expression of ψ suggests that we must take velocity potential ϕ as

$$\phi = -\omega\frac{a^2 - b^2}{a^2 + b^2}xy. \qquad (6)$$

The velocity q is given by

$$q^2 = \left(-\frac{\partial\phi}{\partial x}\right)^2 + \left(-\frac{\partial\phi}{\partial y}\right)^2$$

$$= \omega^2\left(\frac{a^2 - b^2}{a^2 + b^2}\right)(x^2 + y^2). \qquad (7)$$

The kinetic energy of the liquid contained in rotating cylinder is given by

$$T = \frac{1}{2}\rho\iint q^2 dxdy$$

$$=\frac{1}{2}\rho\omega^2\left(\frac{a^2-b^2}{a^2+b^2}\right)^2\iint\left(x^2+y^2\right)dxdy$$

$$=\frac{1}{8}\pi\eta abw^2\frac{\left(a^2-b^2\right)^2}{a^2+b^2},$$

where ρ being the density of the liquid.

$$\left[\because\iint x^2dxdy=\frac{1}{4}\Pi ab^3,\iint y^2dxdy=\frac{1}{4}\Pi ba^3\right]$$

Example – 1:

If an elliptic cylinder having semi-axes a, b filled with a liquid rotates with a uniform velocity about its axes. Prove that the kinetic energy of the liquid. Contained is less than if it were moving as solid in the ratio $\left(a^2-b^2\right)^2:\left(a^2+b^2\right)^2$.

Solution :

Let T_1 be the kinetic energy of the liquid contained in the cylinder and T_2 be the kinetic energy. When the liquid rotates with the boundary as rigid mass with angular velocity ω.

So,

$$T_1=\frac{1}{8}\Pi ab\omega^2\frac{\left(a^2-b^2\right)^2}{a^2+b^2}.$$

and $T_2 = \frac{1}{2}k^2\omega^2$.

$$= \frac{1}{2}(\Pi ab\rho)\frac{a^2+b^2}{4}\omega^2$$

$$= \frac{1}{8}\Pi ab\rho\omega^2(a^2+b^2)$$

$$\left[\because M = \Pi ab\rho, k^2 = \frac{a^2+b^2}{4}\right]$$

So, $T_1 : T_2 = (a^2-b^2)^2 : (a^2+b^2)^2$.

Hence proved.

2.7 Keywords

Motion of an elliptic cylinder, Motion of a liquid in a rotating cylinder, Velocity potential, Kinetic Energy.

2.8 Exercises

1. Prove that if $2a$, $2b$ are axes of the cross-section of an elliptic cylinder placed across a stream in which the velocity at infinity is U parallel to the major axis of the cross-section, the velocity at a point $(a\cos\eta, b\sin\eta)$ on the surface is

$$\frac{U(a+b)\sin\eta}{\left(b^2\cos^2\eta+a^2\sin^2\eta\right)^{1/2}}$$

and that, in consequence of the motion, the resultant thrust per unit length on that half cylinder on which the stream impinges is diminished by

$$\frac{2b^2\rho U^2}{a-b}\left[1-\left(\frac{a+b}{a-b}\right)^{1/2}\tan^{-1}A\left(\frac{a-b}{a+b}\right)^{1/2}\right],$$

where ρ is the density of the liquid.

2. An elliptic cylinder, the semi-axes of whose cross-sections are a and b, is moving with velocity U parallel to the major axis of the cross-section, through an infinite liquid of density ρ which is at rest at infinity, the pressure there being Π. Prove that in order that the pressure may everywhere be positive

$$\rho U^2 < \frac{2a^2\Pi}{2ab+b^2}.$$

3. An elliptic cylinder, semi-axes a and b, is held with its length perpendicular to, and its major axis making an angle θ with the direction 0of a stream of velocity V. Prove that the magnitude of the couple per unit length on the cylinder due to the fluid pressure is

$$\Pi\rho\left(a^2-b^2\right)V^2\sin\theta\cos\theta$$

and determine its sense.

4. If an elliptic cylinder of semi-axes a, b filled with a liquid, rotates with a uniform velocity about its axes, show that the kinetic energy of liquid contained is less than if it were moving as solid in the ratio $\left(a^2-b^2\right)^2:\left(a^2+b^2\right)^2$.

5. If the ellipse $a\left(x^2-y^2\right)+2bxy-\frac{1}{2}w\left(x^2+y^2\right)+c=1$ is full of liquid and is rotated round the origin with angular velocity w. Show that the stream function ψ is given by $\psi=a\left(x^2-y^2\right)^2+2bxy$.

6. A thin shell in the form of an infinite long elliptic cylinder with semi-axes a and b is rotating about its axes in an infinite liquid otherwise at rest. It is filled with the same liquid. Prove that the ratio of the kinetic

energy of the liquuid inside to that of that of the liquid outside is

$2ab : (a^2 + b^2)$

2.9 Further readings

1. Mlne-Thomson, L.M., Theoretical Hydrodynamics, Macmilan & Co. Ltd., London, 1955.

2. Ramsey, A.S., A. Treatise on Hydromechanics, CBS Publishers & Distributors, New Delhi, 2000.

3. Chorlton, F., Textbook of Fluid Dynamics, CBS Publishers & Distributors, New Delhi, 2003.

4. Panton, L. Ronald, Incompressible Flow, A Wiley-Interscience Publication, New York, 1984.

5. Raisinghania, M.D. Fluid Dynamics, S. Chand, Sixth Edition, 2005.

M.Sc. Course
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PART-II

Paper-VII Group-C

Module No. - 81
BLASIUS, KUTTA-JOUKOWSH THEOREMS AND CONFORMAL MAPPING

Objectives

The main objective of this module is to prove Blasius, Kutta and Jukowski theorems. Applications of conformal mapping are given.

Structures

3.1 Introduction

Blasius (1910) developed an elegant method of computing the force and torque exerted on a body that is held stationary in an ambient steady flow. A long cylinder is placed with its generators perpendicular to the incident stream of a moving incompressible fluid containing hydrodynamical singularities. The cylinder experiences forces tending to produce translation as well as rotation. The forces are calculated using a theorem due to Blasius. Conformal mapping allows us to calculate potential flows in two-dimensional domains with complex geometrics from a knowledge of elementary flows in domains with simpler geometrics. The conformal mapping is discussed in brief and has many applications in fluid flow problems. Some problems are worked out.

3.2 Milne-Thomson's circle theorem

Statement : Let us consider a two-dimensional irrotational motion of an inviscid liquid in the $x-y$ plane. Let there be no rigid boundary and the complex potential of the flow be $f(z)$. Further we assume that all the similarities of $f(z)$ be at a distance greater than 'a' from the origin.

If a circular cylinder whose cross-section is $|z|=a$ be introduced in the flow field the complex potential of the modified flow is given by

$$w=f(z)+\bar{f}\left(\frac{a^2}{z}\right)$$

where $\bar{f}$ is the complex conjugate of f.

Proof.

The complex potential w of the modified flow has to satisfy the following conditions :

(i) As there can be no flow across the circle $C:|z|=a$, therefore the circle is a stream line, say $\psi=0$ and hence w is real on the circle C.

(ii) w and $f(z)$ have the same singularities outside C.

The function $f(z)+\bar{f}(\bar{z})$ is real every where.

Also on, C, $z\bar{z}=a^2$ (1)

If we take

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) \qquad (2)$$

then w is real on C. If the point z lie outside C, the point $\frac{a^2}{z}$ is within C and vice-versa. Since, by hypothesis, all the singularities of $f(z)$ are exterior to C, all the singularities of $f\left(\frac{a^2}{z}\right)$ are within C and hence $\bar{f}\left(\frac{a^2}{z}\right)$ has no singularity outside C.

Thus all the conditions are satisfied by w given by (2), so that (2) represents the required complex potential of the modified flow.

3.3 Uniform flow past a circle

Let there be a uniform two-dimensional flow parallel to the x-axis. The complex potential due to the uniform flow is

$$f(z) = Uz$$

when. the circle $|z|=a$ is introduced, by the circle theorem, the complex potential is given by

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$$

$$= Uz + U\left(\frac{a^2}{z}\right) = U\left(z + \frac{a^2}{z}\right) \quad (1)$$

Introducing polar co-ordinates (r, θ) by we get

$$w = U\left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right)$$

$$= Ur(\cos\theta + i\sin\theta) + \frac{Ua^2}{r}(\cos\theta + i\sin\theta)$$

$$= U\left\{\left(r + \frac{a^2}{r}\right)\cos\theta + i\left(r - \frac{a^2}{r}\right)\sin\theta\right\}.$$

Hence the velocity and the stream function are given by

$$\phi(r,\theta) = U\left(r + \frac{a^2}{r}\right)\cos\theta \quad (2)$$

$$\psi(r,\theta) = U\left(r - \frac{a^2}{r}\right)\cos\theta \quad (3)$$

$$= U\left(1-\frac{a^2}{r^2}\right) r\sin\theta$$

The stram line $\psi = 0$ given by $r \sin\theta = 0$, i.e., $y = 0$ and $r = a$.

Hence the line $\psi = 0$ consists of the circle $r = a$ and the part of the x-axis outside the circle, it advances from $-\infty$ along the x-axis towards the circle, until it meets the circle at $A(-a, 0)$, say, where it divides and proceeds in the opposite directions round the cylinder, joins up again at B and moves off along the x-axis to $+\infty$. This stream line which divides on the contour is called the dividing stream line.

From (1), the complex velocity is given by

$$\frac{dw}{dz} = -u + iv = U\left(1-\frac{a^2}{z^2}\right)$$

$$\therefore \frac{dw}{dz} = 0 \text{ where } z = \pm a, \text{ i.e., at } B(a, 0) \text{ and } A(-a, 0).$$

The points A, B where the velocity vanish and the stream line divides are called the stagnation points.

$$\frac{dw}{dz} = U\left\{1-\frac{a^2}{z^2}(\cos 2\theta - i\sin 2\theta)\right\}$$

$$\therefore q^2 = U^2\left[\left(1-\frac{a^2}{r^2}\cos 2\theta\right)+\frac{a^4}{r^4}\sin^2 2\theta\right]$$

$$= U^2\left[1-\frac{a^4}{r^4}-\frac{2a^2}{r^2}\cos 2\theta\right].$$

3.4 Blasius theorem

Statement : In a stready, two-dimensional motion of an inviscid liquid under no external forces past a fixed infinite cylinder, if $w = f(z), i.e., \phi + i\psi = f(x+iy), i = \sqrt{-1}$ represents the complex potential of the flow in a plane perpendicular to the axis of the cylinder, thus the component ρ, X, Y of the thrust on unit length of the cylinder and he couple M on it about the origin are given by

$$X - iY = \frac{1}{2} i\rho \oint_C \left(\frac{dw}{dz}\right)^2 dz$$

$$M = \text{Real part of } -\frac{1}{2}\rho \oint_C z\left(\frac{dw}{dZ}\right)^2 dz.$$

where the contour c represents the section of the cylinder, ρ the density of fluid.

Proof :

The components of the thrust on an element *ds* of the conform *c* of the cylinder (per unit length) are given by

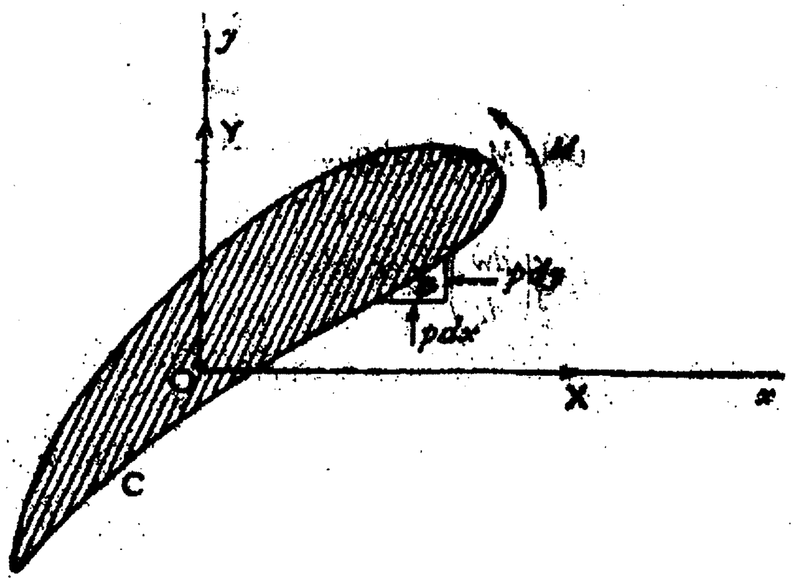
$$dX = -pdy,\ dY = pdx \qquad (1)$$

and the couple about the origin is

$$dM = x.pdx + y.pdy$$

$$= p(xdx + ydy) \qquad (2)$$

where p represents the fluid pressure.



Since the motion is steady, by Beroulli's equation, the pressure is given by

$$p = A - \frac{1}{2}\rho q^2$$

$$= A - \frac{1}{2}\rho\left|\frac{dw}{dz}\right|^2 \quad \left[\because |\bar{q}| = \left|\frac{dw}{dz}\right|\right]$$

$$= A - \frac{1}{2}\rho\frac{dw}{dz}\left(\frac{d\bar{w}}{dz}\right), \text{ A being a constant}$$

$$\therefore X - iY = \oint_c(-pdy - ipdx) = -i\int_c p(dx - idy)$$

$$= -i\oint_c\left\{A - \frac{1}{2}\rho\left(\frac{dw}{dz}\right)\left(\frac{d\bar{w}}{dz}\right)\right\}(dx - idy)$$

$$= -i\oint_c\left\{A - \frac{1}{2}\rho\frac{dw}{dz}\left(\frac{d\bar{w}}{d\bar{z}}\right)\right\}d\bar{z}\,(\because d\bar{z} = dx - idy)$$

$$= -i\oint_c A d\bar{z} + \frac{1}{2}i\rho\oint_c\frac{dw}{dz}\frac{d\bar{w}}{dz}d\bar{z}$$

$$= -\frac{1}{2}i\rho\oint_c\frac{dw}{dz}d\bar{w}\left[\because \oint_c A d\bar{z} = 0\right]$$

Since the contour c is a stress line, on c,

ψ =const w hence $d\psi$ =0 mc.

On c, $\quad dw = d\phi + id\psi = d\phi$

$$= d\phi + id\psi = d\bar{w}$$

$$= X - iy = \frac{1}{2} d\rho \oint_c \frac{dw}{dz} dw$$

$$= \frac{1}{2} i\rho \oint_c \left(\frac{dw}{dz} \right)^2 dz \qquad (3)$$

From equation (2),

$$dM = p\ (xdx + ydy)$$

$$= \text{Real part of } pzd\bar{z}.$$

The result and couple about the origin is given by

$$M = \text{Real part of } \oint_c \left(A - \frac{1}{2} \rho \frac{dw}{dz} \frac{d\bar{w}}{dz} \right) zd\bar{z}$$

$$= \text{Re} - \frac{1}{2} \rho \oint_c z \frac{dw}{dz} \frac{d\bar{w}}{dz} d\bar{z}$$

$$= \text{Re} - \frac{1}{2} \rho \oint_c z \frac{dw}{dz} d\bar{w}$$

$$= \text{Re} - \frac{1}{2} \rho \oint_c z \frac{dw}{dz} dw \qquad [\because cuc\ d\psi = 0, dw = d\bar{w}]$$

$$= \text{Re} - \frac{1}{2} \rho \oint_c z \left(\frac{dw}{dz} \right)^2 dz \qquad (4)$$

Note : The contour integrations in (3) and (4) may be takes about any other contour c' which is reconcilable with c, provided there is no singularity of the integrands between c and c'.

Such singularities can only occur in hydrodynamics when the fluid contains sources or vortices.

Example -1 :

A source and a sink of equal strength are placed at the points $\left(\pm\frac{1}{2}a, 0\right)$ within a fixed circular boundary $x^2+y^2=a^2$. Show that the stream lines are given by, $\left(r^2-\frac{a^2}{4}\right)\left(r^2-4a^2\right)-4a^2y^2=ky\left(r^2-a^2\right)$, k being an arbitrary constant.

Solution :

The complex potential due to a source of strength m at $\left(\frac{1}{2}a, 0\right)$ and a sink m at $\left(-\frac{1}{2}a, 0\right)$ in the absence of boundary is

$$f(z)=-m\log\left(z-\frac{1}{2}a\right)+m\log\left(z+\frac{1}{2}a\right)$$

$$\therefore \bar{f}\left(\frac{a^2}{z}\right)=-m\log\left(\frac{a^2}{z}-\frac{a}{2}\right)+m\log\left(\frac{a^2}{z}+\frac{a}{2}\right)$$

when the boundary is inserted, the complex potential is given by

$$w(z)=f(z)+\bar{f}\left(\frac{a^2}{z}\right)$$

$$=-m\log\left(z-\frac{1}{2}a\right)+m\log\left(z+\frac{1}{2}a\right)$$

$$-m\log\left(\frac{a^2}{z}-\frac{1}{2}a\right)+m\log\left(\frac{a^2}{z}+\frac{1}{2}\right)$$

$$=-m\log\left(z-\frac{1}{2}\right)+m\log\left(z+\frac{1}{2}\right)$$

$$-m\log(z-2a)+m\log z+m\log(z+2a)$$

$$-m\log z+\text{ constant}$$

$$=m\left[\log\left(x+\frac{a}{2}+iy\right)-\log\left(x-\frac{a}{2}+iy\right)\right]$$

$$+\log(x+2a+iy)-\log(x-2a+iy)+\text{ constant}$$

$$\therefore \psi=m\left[\tan^{-1}\frac{y}{x+a/2}-\tan^{-1}\frac{y}{x-a/2}\right.$$

$$\left.+\tan^{-1}\frac{y}{x+2a}-\tan^{-1}\frac{y}{x-2a}\right]$$

$$= m\left[\tan^{-1}\left(\frac{\frac{y}{x+a/2}-\frac{y}{x-a/2}}{1-\frac{y^2}{x^2+a^2/4}}\right)+\tan^{-1}\frac{\frac{y}{x+2a}-\frac{y}{x-2a}}{1-\frac{y^2}{x^2-4a^2}}\right]$$

Example-2 :

Within a circular boundary of radius a there is a two dimensional liquid motion due to a source producing liquid at the rate m at a ditance f from the centre and an equal sink at the centre. Find the velocity potential and show that the resultant of the pressure in the boundary is

$$\frac{\rho m^2 f^3}{2\pi a^2\left(a^2-f^2\right)}$$

where ρ is the density of the liquid. Deduce, as a limit the velocity potential due to the double at the centre.

Solution :

The complex potential is given by

$$w(z) = -\frac{m}{2\pi}\log(z-f)+\frac{m}{2\pi}\log(z)$$

$$-\frac{m}{2\pi}\log\left(z-a^2/f\right) \qquad (1)$$

$$\phi(x,y) = \frac{m}{2\pi}\left[\log(OP)-\log(PA)-\log(PA')\right]$$

$$=\frac{m}{2\pi}\log\left(\frac{OP}{PA.PA'}\right)$$

$$\frac{dw}{dz}=-\frac{m}{2\pi}\left[\frac{1}{z-f}-\frac{1}{z}+\frac{1}{z-a^2/f}\right] \tag{2}$$

If X, Y be the components of the thrust exerted on the boundary, then by Blasius theorem, we get

$$X-iY=\frac{1}{2}i\rho\oint_c\left(\frac{dw}{dz}\right)^2 dz$$

$=\frac{1}{2}i\rho\times 2\pi i$ [sum of the residues of $=\left(\frac{dw}{dz}\right)^2$ at the poles within the circle]

(3)

Now,

$$\left(\frac{dw}{dz}\right)^2=\frac{m^2}{4\pi^2}\left[\frac{1}{(z-f)^2}+\frac{1}{z^2}+\frac{1}{(z-a^2/f)^2}-\frac{z}{z(z-f)}\right.$$

$$\left.-\frac{2}{z(z-a^2/f)^2}+\frac{2}{(z-f)(z-a^2/f)}\right]$$

$$=\frac{m^2}{4\pi^2}\left[\frac{1}{(z-f)^2}+\frac{1}{z^2}+\frac{1}{(z-a^2/f)^2}+\frac{2}{fz}-\frac{2}{(z-f)f}+\frac{2f}{za^2}\right.$$

$$+\frac{2}{(z-f)\left(f-a^2/f\right)}+\frac{2}{\left(\frac{a^2}{f}-f\right)\left(z-\frac{a^2}{f}\right)}\Bigg]$$

The singularities of $\left(\frac{dw}{dz}\right)^2$ within the circle $|z|=a$ are at $z=0$ and $z=f(<a)$.

Residue at $z=0$ is (coefficient of $1/z$)

$$=\frac{m^2}{4\pi^2}\left\{\frac{2}{f}+\frac{2f}{a^2}\right\}=m^2\frac{a^2+f^2}{2\pi^2 fa^2}$$

Residue at $z=f$ is $=\frac{m^2}{4\pi^2}\left[-\frac{2}{f}+\frac{2}{f-a^2/f}\right]$

$$=\frac{m^2}{4\pi^2}\left[-\frac{2}{f}+\frac{2f}{f^2-a^2}\right]$$

$\therefore$ Sum of the residues

$$=\frac{m^2}{4\pi^2}\left[\frac{2}{f}+\frac{2f}{a^2}-\frac{2}{f}+\frac{2f}{f^2-a^2}\right]$$

$$=\frac{2fm^2}{4\pi^2}\frac{f^2-a^2+a^2}{a^2\left(f^2-a^2\right)}=-\frac{m^2 f^3}{2\pi^2 a^2\left(a^2-f^2\right)}$$

Hence, we get

$$X-iY=-\rho\pi=\frac{\left(-m^2f^3\right)}{2\pi^2a^2\left(a^2-f^2\right)}=\frac{\rho m^2f^3}{2\pi a^2\left(a^2f^2\right)}$$

Hence the force components are

$$X=\frac{\rho m^2f^3}{2\pi a^2\left(a^2-f^2\right)}, Y=0.$$

Second part :

The combination (source and sink) forms a doublet if $f\to 0$ and $m\to\alpha$ such that

$$\lim\frac{m}{2\pi}f=\mu,\ \text{strength of foublet} \qquad (a)$$

Expanding (1) in powers of f, we get

$$w(z)=-\frac{m}{2\pi}\left[\log z+\log\left(1-\frac{f}{2}\right)-\log z\right.$$

$$\left.+\log\left(1-\frac{fz}{a^2}\right)\right], \text{(neglectig other terms)}$$

$$=-\frac{m}{2\pi}\left[\left(-\frac{f}{z}-\frac{1}{2}\frac{f^2}{z^2}-...\right)-\left(\frac{fz}{a^2}+\frac{1}{2}\frac{f^2z^2}{a^2}+...\right)\right]$$

$$=\frac{1}{z}\left(\frac{mf}{2\pi}\right)+\frac{mf}{2\pi}\frac{z}{a^2}+m\sigma\left(f^2\right).$$

$\frac{\mu}{z}+\frac{\mu z}{a^2}$ in the limit,

Example- 3:

A source of fluid situated in space of two dimension is of such strength that $2\pi\rho\mu$ represents mass of the fluid of density ρ emitted per unit of time. So that the force necessary to hold a circular disc at rest in the plane of source is

$$\frac{2\pi\rho\mu^2 a^2}{r\left(r^2-a^2\right)}$$

where a is the radius of the disc and r be the distance of the source from its centre. In what direction the disc and r be the distance of the source from its centre. In what direction the disc is urged by the pressure?

Solution :

Since $2\pi\rho\mu$ is the mass of the fluid emitted per unit time, then the strength of source is u. Let 0 be the entire of the disc whose radius is a. The source of strength μ is situated at the point S whose distance from the centre is r i.e., $OS = r$.

Let S' be the inverse point of S with respect to the circular disc.

$$\therefore OS.OS' = a^2 \text{ or, } OS' = \frac{a^2}{OS} = \frac{a^2}{r} \quad [\because OS = r]$$

The system can formulate as follows.

(i) a source of strength μ at $z = r$,

(ii) a source of strength μ at $z = 0$.

Taking OS as real axis, the complex potential at any point is given by

$$W = -\mu \log(z-r) - \mu \log\left(z - \frac{a^2}{r}\right) + \mu \log z \qquad (1)$$

Let X and Y represent the components of the resultant force on the circular disc along the coordinate axes.

Then, by Blasius theorem, we have

$$X - iY = \frac{i\rho}{2} \oint_c \left(\frac{dW}{dz}\right)^2 dz \qquad (2)$$

Now, from (1), we have

$$W = -\mu \log(z-r) - \mu \log\left(z - \frac{a^2}{r}\right) + \mu \log z$$

$$\therefore \frac{dW}{dz} = -\mu\frac{1}{z-r} - \mu\frac{1}{z-\frac{a^2}{r}} + \frac{\mu}{z}$$

or, $$\therefore \left(\frac{dW}{dz}\right)^2 = \mu^2\left[\frac{1}{z-r} + \frac{1}{z-\frac{a^2}{r}} + \frac{1}{z}\right]^2$$

or, $$\frac{1}{\mu^2}\left(\frac{dW}{dz}\right)^2 = \frac{1}{(z-r)^2} + \frac{1}{\left(z-\frac{a^2}{r}\right)} + \frac{1}{z^2}$$

$$+\frac{2}{(z-r)\left(z-\frac{a^2}{r}\right)} - \frac{2}{(z-r)} - \frac{2}{z\left(z-\frac{a^2}{r}\right)}$$

Clearly, the function $\frac{1}{\mu^2}\left(\frac{dW}{dz}\right)^2$ has poles at $z = 0$, $z = \frac{a^2}{r}$ inside the circular disc.

Now the residue at $z = 0$ is the coefficient of $\frac{1}{z}$ when $z = 0$

i.e.,

$$\text{Res}(at\ z=0) = \left[-\frac{2}{z-r} - \frac{2}{z-\frac{a^2}{r}}\right]_{z=0} = 0$$

$$=\frac{2}{r}+\frac{2}{\frac{a^2}{r}}=\frac{2}{r}+\frac{2r}{a^2}$$

Similarly, the residue at $z=\frac{a^2}{r}$ is the coefficient of $\frac{1}{z-\frac{a^2}{r}}$ when $z=\frac{a^2}{r}$

$$\therefore \text{Res}\left(at\ z=\frac{a^2}{r}\right)=\left[\frac{2}{z-r}-\frac{2}{z}\right]_{z=\frac{a^2}{r}}$$

$$=\frac{2}{\frac{a^2}{r}-r}-\frac{2}{\frac{a^2}{r}}$$

$$=\frac{2r}{a^2-r^r}-\frac{2r}{a^2}$$

Hence the sum of the residues

$$=\frac{2}{r}+\frac{2r}{a^2}+\frac{2r}{a^2-r^r}-\frac{2r}{a^2}=\frac{2a^2-2r^2+2r^2}{r\left(a^2-r^2\right)}=\frac{2a^2}{r\left(a^2-r^2\right)}$$

By Cauchy's residue theorem, we have

$$\oint_c \frac{1}{\mu^2}\left(\frac{dW}{dz}\right)^2 dz=2\pi i.\frac{2a^2}{r\left(a^2-r^2\right)}$$

or, $$\oint_c \left(\frac{dW}{dz}\right)^2 dz = \frac{4\pi i a^2 \mu^2}{r\left(a^2 - r^2\right)}$$

now putting the above value $\oint_c \left(\frac{dW}{dz}\right)^2 dz$ in (2) we have

$$X - iY = \frac{i\rho}{2} \cdot \frac{4\pi i a^2 \mu^2}{r\left(a^2 - r^2\right)}$$

or, $$X - iY = \frac{i\rho}{2} \cdot \frac{4\pi i a^2 \mu^2}{r\left(a^2 - r^2\right)}$$

or, $$X - iY = \frac{4\pi a^2 \mu^2 \rho}{r\left(r^2 - a^2\right)}$$

Now, equating real and imaginary parts of both sides we get

$$X = \frac{2\pi a^2 \mu^2 \rho}{r\left(r^2 - a^2\right)} \text{ and } Y = 0$$

This implies that hte force is purely along $O\vec{S}$, so that the disc will move along $O\vec{S}$.

3.5 d' Alembert's paradox

If we place an obstacle in the middle of the tube in which an inviscid liquid is following with constant speed U. The flow in the immediate neighbourhood of obstacle will be deranged, but at a great distance either upstream or downstream the flow will be undisturbed. A force and a couple required to hold the obstacle at rest. Let F be the component of the force in the direction parallel to the flow, we can prove that $F = 0$. This is known as d' Alembert's paradox. In case of flow of an ideal fluid past a cylinder, we get $F = (M + M')\frac{dU}{dt}$. Here F is the force acting per unit length of the cylinder moving with velocity U, M' the mass of fluid displaced by the unit lengthf of the cylinder and M the mass of the cylinder per unit length. This equation has the form of Newton's second law of motion with $(M + M')$ appearing instead of mass of the cylinder. One surprising conclusion from the above equation is that there is no force exerted on a cylinder moving through a fluid with uniform speed U. This result is not followed our everyday experience that a fluid exerts a drag force o any object moving through it. This paradox arises because we have neglected fluid viscosity.

3.6 Conformal Mapping

A transformation or mapping is mapped the point (x_0, y_0) of the xy plane into the point (u_0, v_0) of of the uv-plane through the set of equations repreented by

$$\left.\begin{aligned} u &= u(x,y) \\ v &= v(x,y) \end{aligned}\right\} \qquad (1)$$

The curves c_1 and c_2, say intersecting at (x_0, y_0) are mapped into curves c_1' and c_2'. If the transformation is such that hte angle at (x_0, y_0) between c_1 and c_2 is equal to the angle at (u_0, v_0) between both c_1' and c_2' both in magnitude and sense, then the transformation or mapping is said to be conformal at (x_0, y_0).

Important results of conformal transformations

(a) In a conformal transfomation, a source is transformed into an equal source, a sink into an equal sink and a doublet into an equal doublet;

(b) The complex potential $w = \phi + i\psi$ is invariant under a conformal transformation;

(c) Let $\xi = f(z)$ be the conformal transformation. The total k. E. of fluid in z-plane (per unit depth) = Total k.E of the liquid in ξ-plane (per unit depth);

(d) A stream line in z-plane is transformed into, a stream line in ξ-plane.

We begin developing the method by introducing the complex variable $\zeta = \xi + i\eta$, where ξ, η are two real variables, and the complex function $F(z)$

that is analytic in a certain region of the complex z plane, and maps a point in the z-plane to another point in the ξ-plane, so that $\zeta = F(z)$.

Where the function $F(z)$ is multivalued, we introduce an appropriate branch cut in the z-plane so as to render the mapping unique. Furthermore, we introduce the inverse mapping function that maps a point in the ζ plane back to a point in the z-plane $z = f(\zeta)$.

3.7 The Schwarz-Christoffel Transformations

Consider a polygon in the w-plane having vertices at $w_1, w_2, ..., w_n$ with corresponding interior angle $\alpha_1, \alpha_2, ..., \alpha_n$, respectively. Let the points $w_1, w_2, ..., w_n$ map into points $x_1, x_2, ..., x_n$ as the real axis of the z-plane. A transformation which maps the interior R of the polygon of the w-plane on the upper half R' of the z-plane and the boundary of the polygon nonto the real axis is represented by

$$\frac{dw}{dz} = A(z-x_1)^{(\alpha_1/\pi-1)}(z-x_2)^{(\alpha_2/\pi-1)}...(z-x_n)^{(\alpha_n/\pi-1)}$$

or, $$w = A\int(z-x_1)^{(\alpha_1/\pi-1)}\int(z-x_2)^{(\alpha_2/\pi-1)}...\int(z-x_n)^{(\alpha_n/\pi-1)}dz + B$$

where, A, B are complex constants.

3.8 The Joukoswski Transformations

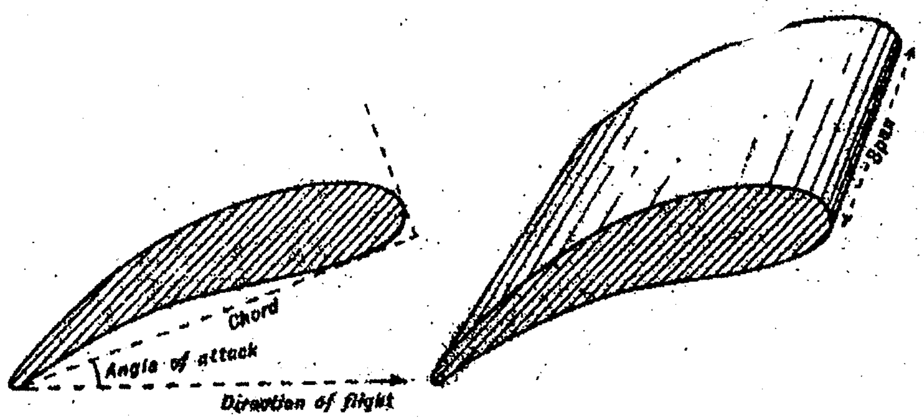
The transformation

$$z = Z + \frac{c^2}{4Z} \qquad (1)$$

is one of the simplest and most important transformations of two-dimensional motion. By means of this transformation we can map the Z-plane on the z-plane, and vice verse. It can be shown that when $|z|$ is large, we have $Z = z$ nearly, so that the distant parts of the two-planes unaltered. Thus, a uniform stream at infinity in the z-plane will correspond to a uniform stream of the same strength and direction in the Z-plane.

3.9 The aerofoil

The aerofoil used in modern aeroplanes has a profile of "fish" type as depicted in the following figure. Such an aerofoil has a blunt leading edge and a sharp trailing edge. The projection of the profile on the double tangent, as shown in the diagram, is the chord. The ratio of the span to the chord is the aspect ratio.



The following assumptions are made for the theory of flow round the aerofoil.

(i) That the air behaves as an incompressible inviscid fluid;

(ii) That the aerofoil is a cylinder whose cross-section is a curve of the above type;

(iii) That the flow is two-dimensional, irrotational cyclic motion.

The above assumptions are of course only approximations to the actual state of affairs but by making these simplifications it is possible to arrive at a general understanding of the principles involved. There is a considerable and interesting literature on this subject which is not possible to discuss in the present.

It has been found that profiles obtained by conformal transformation of circle by the simple Joukowski transformation make good wing shapes, and the lift can be calculated from the known flow with respect to a circular cylinder.

3.10 The theorem of Kutta and Joukowski

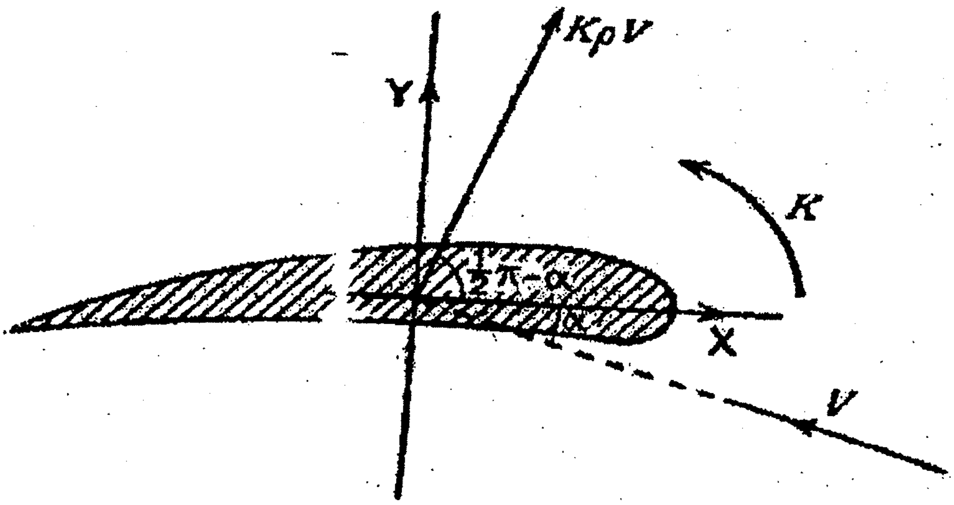
An aerofoil at rest in a uniform wind of speed V, with circulation K round the aerofoil, undergoes a lift $K\rho V$ perpendicular to the wind. The direction of the lift vector is got by rotating the wind velocity vector through a right angle in the sense opposite to that of the circulation.

Proof :

Since there is a uniform wind, the velocity at a great distance from the aerofoil must tend to the wind velocity, and therefore if $|z|$ is sufficiently large, we may write

$$\frac{dw}{dz} = -Ve^{i\alpha} + \frac{A}{z} + \frac{B}{z^2} + \ldots \qquad (1)$$

where α is the angle of incidence or angle of attack (see figure below).



Thus, we have

$$w = Vze^{i\alpha} - A\ln z + \frac{B}{z} + \ldots$$

and since there is circulation K, we must have

$$-A = \frac{iK}{2\pi} \qquad (2)$$

for $\ln z$ increases by $2\pi i$ when we go once round the aerofoil in the positive sense.

From (1) and (2) we get,

$$\left(\frac{dw}{dz}\right)^2 = V^2 e^{2i\alpha} + \frac{iKV}{\pi z} e^{i\alpha} - \frac{K^2 + 8\pi^2 BVe^{i\alpha}}{4\pi^2 z^2} - \ldots \qquad (3)$$

If we now integrate round a circle whose radius is sufficiently large for the expression (3) to be valid, the theorem of Blasius gives

so that, changing the sign of i we obtain

$$X-iY=\left(\frac{1}{2}i\rho\right)2\pi i\left(\frac{iKVe^{i\alpha}}{\pi}\right)$$

$$=-iK\rho Ve^{i\alpha}$$

so that, changing the sign of i we obtain

$$X+iY=iK\rho Ve^{-i\alpha}=K\rho Ve^{i\left(\frac{\pi}{2}-\alpha\right)}.$$

Comparison with above figure shows that this force has all the properties stated in the enunciation. The moment about the origin is obtained from the theorem of Blasius as M = Real part of

$$2\pi i\rho BVe^{i\alpha}.$$

3.11 Applications of conformal mapping to flow past two-dimensional bodies

To compute the velocity field corresponding to uniform flow past a two-dimensional body with an arbitrary cross-section, we map the exterior of the body ifn the z plane to the exterior of a disk of radius c centered at the origin in the ζ plane, and then recover the flow in the physical plane from the flow in the image plane using the exact solution, which in this case becames

$$W(\zeta)=V^{*}\zeta+V\frac{c^{2}}{\zeta}+\frac{k}{2\pi i}\ln\frac{\zeta}{c} \qquad (1)$$

To ensure that the far flows in the two planes behave in a similar manner, so that uniform flow in the ζ plane is also uniform flow in the z plane far from the body, we require that the mapping function $\zeta=F(z)$ and its inverse $z=f(\zeta)$ behave in a linear manner far from the body as $|z|$ tends to infinity, so that their first derivatives tend to a constant.

3.12 Flow past an Elliptical Cylinder

As a first application, let us consider uniform flow past a cylinder with an elliptical cross-section. One may readily verify that the inverse mapping function

$$z=f(\zeta)=\zeta+\frac{e^{2}}{4}\frac{a^{2}}{\zeta} \qquad (2)$$

where $e=\left[1-\left(\frac{b}{a}\right)^{2}\right]^{1/2}$ is the eccentircity, maps the exterior of a disk with radius $c=\frac{1}{2}(a+b)$ centered at the origin to the exterior of an ellipse with major and minor semi-axes equal to a and b also centerd at the origin.

Furthermore, f exhibits the required linear behaviour at infinity and is thus acceptable for the study of uniform flow.

Decomposing Eq. (2) into its real and imaginary parts, we obtain the explicit coordinate transformations

$$z=\xi\left(1+\frac{e^2}{4}\frac{a^2}{|\zeta|^2}\right),\ y=\eta\left(1-\frac{e^2}{4}\frac{a^2}{|\zeta|^2}\right) \tag{3}$$

The inverse transformations are found by solving the quadratic equation (2) for ζ. Since the root with the negative sign corresponds to a point inside the ellipse, we maintain the root with positive sign and obtain

$$\zeta=F(z)=\frac{1}{2}\left[z+\left(z^2-a^2+b^2\right)^{1/2}\right] \tag{4}$$

The value of the square root on the right-hand side of Eq. (4) becomes unique by introducing a branch cut along the x axis extending from $-ae$ to ae.

The complex potential of the flow in the z plane is found readily by substitutiong Eq. (4) into Eq. (1) and setting yielding

$$w(z)=V^*\frac{1}{2}\left(z+\sqrt{z^2-a^2+b^2}\right)+V\frac{1}{2}\frac{(a+b)^2}{z+\sqrt{z^2-a^2+b^2}}$$

$$+\frac{k}{2\pi i}\ln\frac{z+\sqrt{z^2-a^2+b^2}}{a+b} \tag{5}$$

Setting $a = b$ produces the solution for flow past a circular cylinder.

3.13 Flow past a Flat Plane

Letting b/a tend to zero, in which case e tends to unity, reduces the ellipse to a flat plate of length equal to $2a$. The transformation (2) becomes

$$z = f(\zeta) = \zeta + \frac{1}{4}\frac{a^2}{\zeta} \tag{6}$$

which maps the exterior of a disk of radius $c = a/2$ centered at the origin to the whole complex plane; the contour of the disk is mapped to the flat plate. A different method of arriving at Eq. (6). The inverse transformation (4) becomes

$$\zeta = F(z) = \frac{1}{2}\left[z + \left(z^2 - a^2\right)^{1/2}\right] \tag{7}$$

The branch of the square root coincides with the length of the plate.

Substituting Eq. (7) into Eq. (1) and setting $c = a/2$, or applying Eq. (5) with $b = 0$, yields the complex potential of the flow in the z plane

$$w(z) = V^* \frac{1}{2}\left[z + \left(z^2 - a^2\right)^{1/2}\right] + V\frac{1}{2}\frac{a^2}{z + \left(z^2 - a^2\right)^{1/2}}$$

$$+ \frac{k}{2\pi i}\ln\left(\frac{z + \left(z^2 - a^2\right)^{1/2}}{a}\right) \tag{8}$$

which may be simplified to

$$w(z) = V_x z - iV_y\left(z^2 - a^2\right)^{1/2} + \frac{k}{2\pi i}\ln\left(\frac{z + \left(z^2 - a^2\right)^{1/2}}{a}\right) \tag{9}$$

The velocity field is given by

$$u - iv = \frac{dw}{dz} = V_x - i\left(V_y z + \frac{k}{2\pi}\right)\frac{1}{\left(z^2 - a^2\right)^{1/2}} \tag{10}$$

The tangential velocities on the upper and lower surface of the plate, designated, respectively, by the plus and minus superscripts, are given by

$$u^{\pm} = V_x \mp \left(V_y x + \frac{k}{2\pi}\right)\frac{1}{\left(a^2 - x^2\right)^{1/2}} \tag{11}$$

where $-a < x < a$. Note that the velocity diverges at both ends at $x = \pm a$.

3.14 Keywords

Blasius and Kutta-Jukowski theorems, D. Alemberts paradox, Conformal mapping and its applications.

3.15 Exercise

1. State and prove Blasius theorem.
2. State and prove Kutta-Joukowski's theorem.
3. A circular cylinder of radius 'a' is placed across a uniform stream of velocity U with circulation k round the cylinder. Find the 'lift' on the cylinder. Also find the maximum velocity of the liquid on the surface of the cylinder assuming $|k| < 2aU$.
4. For a uniform flow about a fixed circular cylinder, about which there is a circulation, find the complex potential and hence obtain the stagnation points in different cases. Also, find the thrust on the cylinder.

3.16 Further Readings

1. Milne-Thomson, L. M., Theoretical Hydrodynamics, Macmilan & Co. Ltd, London, 1955.

2. Ramsey, A.S., A Treatise on Hydromechanics, CBS Publishers & Distributors, New Delhi, 2000.

3. Chorlton, F., Taxtbook of Fluid Dynamics, CBS Publishers & Distributors, New Delhi, 2003.

4. Panton, L. Ronald, Incompressible Flow, A Wiley-Interscience Publication, New York, 1984.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper-VII Group-C

Module No. - 82
VORTEX MOTION

Objectives

The objectives of this module are to discuss the different types of vortex motion and Helmholtz's theorems on vortex dynamics, to find the velocity vector from known vorticity vector. The Karman vortex street is also discussed.

Structures

4.1 Introduction

The vorticity in fluid motion is very important and the corresponding dynamics even in case of irrotation fluid motion is very interesting and helpful to understand the flow dynamics physically. In this unit, we shall discuss the fluid motion in terms of vorticity vector. The first a vorticity dynamics was given by Helmholtz and later by Kelvin, Kirchhoff and others. Kelvin's theorem indicates that the vortices move with the fluid. The evidence for this is found by observing that voracities in rivers are carried with the general flows of the rivers.

4.2 Vortex Motion

If $\vec{q}\,(x, y, z)$ be the velocity vector at a point $P(x, y, z)$ is a moving fluid, then the vector rot $\vec{q}$ is called the vorticity vector at that point.

A fluid motion is said to be rotational if the vorticity vector does not vanish in a region R.

A vortex line is a curve drawn in a moving fluid such that the tangent to it at each point is in the direction of the vorticity vector at that point.

If (ξ,η,ζ) be the components of the vorticity vector $(\vec{q}) = \nabla \times \vec{q}$ then the vortex lines are given by

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}$$

4.3 Vortex tube and Vortex filament

If through each point of *a* closed curve, we draw the vortex lines we obtain a tubular region which is called a vortex tube. A vortex tube of very small cross section is known as a vortex filament or simply a vortex. It is to be noted that vortex lines and tubes cannot originate or terminate at internal points in a fluid. They can only form closed curves terminate on boundaries. Further vorticity vector $\vec{\Omega} = 2\vec{\omega}, \vec{\omega}$ being angular velocity.

If through each point of an open curve we draw vortex lines, we get a vortex surface.

4.4 Helmholtz's Theorems on vorticity

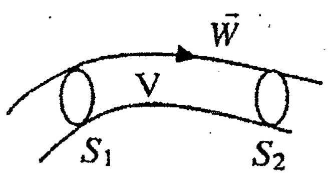
We shall now give some remarkable theorem on vorticity.

Theorem - 1 :

The product of the cross-sectional area and the magnitudes of the vorticity at each point of a vortex filament has the same value all along the filament and for all times.

Proof :

Let us consider a portion V of the vortex filament between two cross-sections of areas σ_1 and σ_2. Applying Gauss theorem to the vector rot $(\vec{q})(=\vec{w}, say)$ in this region, we get



$$\int_v div\,\vec{w}\,d\tau = \iint_{\sigma_1+\sigma_2+s} \vec{w}\cdot\vec{n}ds \quad (1)$$

or, $$0=\vec{w}_1\vec{n}_1\sigma_1+\vec{w}_2\vec{n}_2\sigma_1+0 \quad (2)$$

Since $\text{div}(\vec{w})=div(\text{rot}\,\vec{q})=\vec{\nabla}\cdot(\vec{\nabla}\times\vec{q})=0$ and on $S_1, \vec{w}$ is perpendicular to $\vec{n}$ and σ_1, σ_2 are very small.

i.e., $-w_1\sigma_1 + w_2\sigma_2 = 0 \qquad w_2\sigma_2 = w_1\sigma_1 =$ constant

along the tube, since σ_1, σ_2 are taken arbitrarily.

If the external forces are conservative and the pressure is a function density only then from Kelvin's theorem, the circulation in any closed circuit moving with the fluid is constant for all time. Let G be the curve bounding the section σ_1. The circulation about G is given by

$$\Gamma = \int_G \bar{q}d\vec{s} = \iint_{s+\sigma_2} rot\, \bar{q}\bar{n}ds$$

$$\Gamma = \int_G \bar{q}d\ddot{s} = \iint_{s+\sigma_2} rot\, \bar{q}\bar{n}ds$$

$w_2\sigma_2$, [since σ_2 is very small all the filament by equation (1).]

= constant.

By Kelvin's theorem Γ, i.e. (σw) remains constant for all times. Hence the theorem. The product vorticity X cross-sectional area of a vortex filament is known is the strength of the vortex.

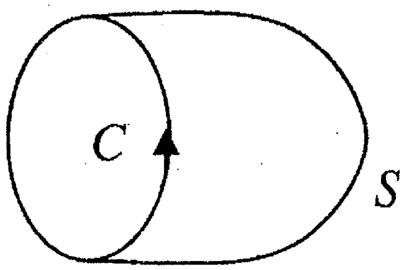
Theorem - 2 :

"Vortex lines move with the fluid i.e., the vortex lines are composed of the same fluid particles."

Proof :

Let s be a surface with c as its rim in a moving fluid. Circulation along C is given by

$$\Gamma = \int_c \vec{q} \cdot \vec{ds} = \iint_s \text{rot}\,\vec{q} \cdot \vec{n} ds = \iint_s \vec{w} \cdot \vec{n} ds$$



If s be a vortex surface, then $\vec{w} \cdot \vec{n}$ on s, i.e. $\Gamma = 0$. Conversely, if $\Gamma = 0$ on every circuit c drawn in certain surface s, then $\vec{w} \cdot \vec{n}$ =0, i.e., ξ is a vortex surface.

At Time t, let Γ be the circulation along a closed circuits c and a vortex surface s, then $\Gamma = 0$.

At time $(t + \delta t)$ the particles that formed the surface s now be on another surface s' and the circuits c on s becomes c' on s'. By Kelvin's theorem, the circulation along c' remains zero. This is true for every circuit c' on , s', so that s' is a vortex surface. Hence vortex surfaces move with the fluid. The interaction of two vortex surfaces is a vortex line and hence vortex lines move with the fluid.

Theorem - 3 :

"Vortex lines must either form closed curves or have their extremities on the boundary on the liquid, i.e., they cannot begin or end at any point within the liquid."

Proof.

Suppose a vortex tube terminates abruptly at a point P within a liquid. We consider a circuit C embracing the tube and let s be a surace with C as since, which partly lies out side the tube and the rest forms the part of the tube. If Γ be the circulation along C, then

$$\Gamma = \oint_{c} \vec{q} \cdot \vec{ds} = \iint_{s} (\text{rot}\, \vec{q}) \cdot \vec{n} ds$$

But $(\nabla \times \vec{q}) \cdot \vec{n} = 0$ on the tube and rot $\vec{q} = 0$ outside the tube, so that the surface integral vanishes. This shows that $\Gamma = 0$ which contradicts. The Theorem 1 one vortex motion. Hnce the theorem.

4.5 To determine the velocity vector when the vorticity at every point of a fluid is known

For an incompressible flow, the equation of continuity is

$$\text{div } \vec{q} = 0, \text{ i.e., } \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \qquad (1)$$

Equation (1) suggests the existence of a vector function such $\vec{A}$ that

$$\vec{q} = \text{rot } \vec{A} \qquad (2)$$

If ξ, η, ζ be the vorticity components then we get

$$\vec{w} = (\xi, \eta, \zeta) = rot\vec{q}$$

$$= \vec{\nabla} \times (\vec{\nabla} \times A)$$

$$= \vec{\nabla}(\vec{\nabla} \cdot A) - \nabla^2 A$$

i.e., $\xi = \frac{\partial}{\partial x}(div\,\vec{A}) - \nabla^2 A_x$

$$\eta = \frac{\partial}{\partial y}(div\vec{A}) - \nabla^2 A_y$$

$$\zeta = \frac{\partial}{\partial z}(div\vec{A}) - \nabla^2 A_z \text{ where } \vec{A} = (A_x, A_y, A_z)$$

Eqn. (2) can be satisfied if we can determine A_x, A_y, A_z such that

$$div\vec{A}=0 \tag{3}$$

$$\nabla^2 A_x = -\xi, \nabla^2 A_y = -\eta, \nabla^2 A_z = -\zeta \tag{4}$$

The solution of (4) is of the form

$$A_x(x,y,z)=\frac{1}{4\pi}\int_V \frac{\xi'}{r} dx'dy'dz'$$

i.e., $$\vec{A}(x,y,z)=\frac{1}{4\pi}\int_V \frac{\vec{w}'}{r} dx'dy'dz' \tag{5}$$

where $\vec{w}'$ denotes the value of $\vec{w}$ at the point $Q(x',y',z'),V$ the volume of the fluid and

$$r^2=(x-x')^2+(x-y')^2+(z-z')^2$$

$$\vec{r}=(x-x',y-y',z-z')$$

We now show that solution given by (5) satisfies the condition (3). We have

$$div\ \vec{A}=\frac{1}{4\pi}\vec{\nabla}\cdot\int_V \frac{\vec{w}'}{r} d\tau' [d\tau'=dx'dy'dz']$$

$$=\frac{1}{4\pi}\int_V \vec{w}'\cdot\vec{\nabla}\left(\frac{1}{r}\right)d\tau' [\because \vec{\nabla} \text{ apply to } x,y,z]$$

$$=\frac{1}{4\pi}\int_V \vec{w}'\cdot\vec{\nabla}\left(\frac{1}{r}\right)d\tau'\left[\because\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=\frac{\partial}{\partial x'}\left(\frac{1}{r}\right)\right]$$

$$=-\frac{1}{4\pi}\int_V\left[\vec{\nabla}\left(\frac{\vec{w}'}{r}\right)-\frac{1}{r}\vec{\nabla}\cdot\vec{w}'\right]d\tau'$$

$$=-\frac{1}{4\pi}\int_V\vec{\nabla}\cdot\left(\frac{\vec{w}'}{r}\right)d\tau' \qquad \left[\because\vec{\nabla}\cdot\vec{w}'\equiv 0\right]$$

$$=-\frac{1}{4\pi}\iint_s\frac{\vec{n}\cdot\vec{w}'}{r}ds,\vec{n}$$

$\vec{n}$ being the outward normal to ds, s = boundary of V.

If $\vec{w}'$ vanishes on s and if s be such that $\vec{w}'\vec{n}=0$ on s, then the surface inegral vanishes and the condition $\bar{A}=0$ is satisfied. Finally, the velocity at P is given by

$$\vec{q}=rot\bar{A}=\vec{\nabla}\times\frac{1}{4\pi}\int_V\frac{\vec{w}'}{r}d\tau'$$

$$=\frac{1}{4\pi}\int_V\vec{\nabla}\times\left(\frac{\vec{w}'}{r}\right)d\tau=-\frac{1}{4\pi}\int_V\vec{w}'\times\vec{\nabla}\left(\frac{1}{r}\right)d\tau$$

$$=\frac{1}{4\pi}\int_V\frac{\vec{w}'\times\vec{r}}{r^3}d\tau' \quad \left[\because rot(\phi\vec{A})=\phi rot\vec{A}+grad\phi\times\vec{A}\right]$$

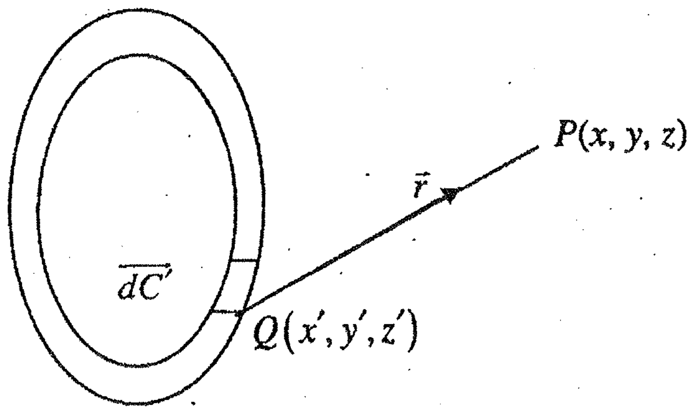
where $\vec{r}=(x-x',y-y',z-z')$

The above result means that the velocity $\vec{q}$ at $P(x,y,z)$ can be regarded as the vector sum of elementary velocities $d\vec{q}$, that corresponding to the vorticity $\vec{w}'$ in the volume element $d\tau'$ at a (x',y',z'), the elementary velocity is given by

$$d\vec{q} = \frac{\vec{w}' \times \vec{r}}{4\pi r^3} d\tau'.$$

4.6 Velocity potential due to a single closed vortex filament

Let then be a single closed vortex filament of strength K in a given mass of liquid. If $d\vec{s}'$ be an arc, σ' the cross-section of the filament at a point $Q(x',y',z')$, the elementary velocity $d\vec{q}$ at a point $P(x,y,z)$ in the fluid is given by



$$d\vec{q} = \frac{\vec{w}' \times \vec{r}}{4\pi r^3} d\tau'$$

$$= \frac{\vec{w}' \times \vec{r}\sigma' ds'}{4\pi r^3} = \frac{w'\sigma' d\vec{s}' \times \vec{r}}{4\pi r^3} \qquad [\because \vec{w}' // d\vec{s}']$$

$$= \frac{k}{4\pi} = \frac{d\vec{s}' \times \vec{r}}{r^3} \text{ where } k = w'\sigma' \text{ strength}$$

$$\therefore K = \oint_c \frac{K}{4\pi r^3}\{(z-z')dy' - (y-y')dz'\}$$

$$= \oint_c \frac{K}{4\pi}\left\{\frac{\partial}{\partial z'}\left(\frac{1}{r}\right)dy' - \frac{\partial}{\partial y'}\left(\frac{1}{r}\right)dz'\right\}$$

$$= \oint_c \frac{K}{4\pi}\{Xdx' + Ydy' + Zdz'\}$$

where $X = 0, Y = \frac{\partial}{\partial z'}\left(\frac{1}{r}\right), Z = -\frac{\partial}{\partial y'}\left(\frac{1}{r}\right)$

Applying Stoke's Theorem, we get

$$u = \frac{K}{4\pi}\iint_S \left[l\left(\frac{\partial Z}{\partial y'} - \frac{\partial Y}{\partial z'}\right) + m\left(\frac{\partial X}{\partial z'} - \frac{\partial Z}{\partial x'}\right) + n\left(\frac{\partial Y}{\partial x'} - \frac{\partial X}{\partial y'}\right)\right] ds'$$

where S is any open surface, within the fluid with c, the filament as its rims and (l, m, n) are the of the *d.c.'s* of the normal to ds'.

Putting for X, Y, Z we get

$$u=\frac{K}{4\pi}\iint_S\left[-l\left(\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}\right)\left(\frac{1}{r}\right)+m\frac{\partial^2}{\partial x'\partial y'}\left(\frac{1}{r}\right)+n\frac{\partial^2}{\partial x'\partial z'}\left(\frac{1}{r}\right)\right]ds'$$

$$=\frac{K}{4\pi}\iint_S l\left(\frac{\partial}{\partial x'}+m\frac{\partial}{\partial y'}+n\frac{\partial}{\partial z'}\right)\frac{\partial}{\partial x'}\left(\frac{1}{r}\right)ds' \qquad \left[\because \nabla^2\left(\frac{1}{r}\right)=0\right]$$

$$=-\frac{K}{4\pi}\frac{\partial}{\partial x}\iint_S\left(l\frac{\partial}{\partial x'}+m\frac{\partial}{\partial y'}+n\frac{\partial}{\partial z'}\right)\frac{1}{r}ds'$$

$$=-\frac{\partial\phi}{\partial x}$$

where

$$\phi=\frac{K}{4\pi}\iint_S\left(l\frac{\partial}{\partial x'}+m\frac{\partial}{\partial y'}+n\frac{\partial}{\partial z'}\right)\frac{1}{r}ds'$$

Similarly,

$$v=-\frac{\partial\phi}{\partial y},\; w=-\frac{\partial\phi}{\partial z}$$

Thus the velocity corresponents can be obtained from a velocity potential ϕ

given by

$$\Phi=\frac{K}{4\pi}\iint_S\left(l\frac{\partial}{\partial x'}+m\frac{\partial}{\partial y'}+n\frac{\partial}{\partial z'}\right)\left(\frac{1}{r}\right)ds'$$

$$=\frac{K}{4\pi}\iint_S\frac{\cos\theta}{r^2}ds'$$

where θ is the angle $P(x, y, z)$ between the normal $\vec{n}$ at ds' and $\vec{r}$.

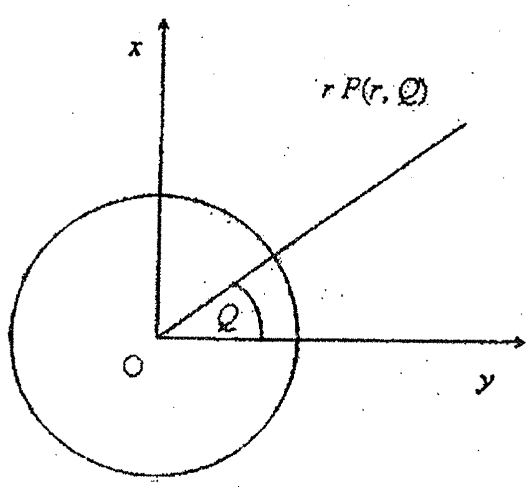
$$\therefore \Phi = \frac{K}{4\pi}\Omega$$

where Ω is the solid angle substituted at the point $P(x, y, z)$ by a surface s whose rim is the vortex filament.

The velocity potential is a multi-valued function.

4.7 Circular Vortex

Let there be a cylindrical vortex column whose cross-section is a circule of radius 'a' in an infinite mass of inviscid liquid, otherwise at rest. We assume that the vorticity over a circulr section of the column has a constant value, outside the circle, the vorticity is zero.



Taking the centre 0 of the circle as origin, the $x - y$ plane along a section of the cylinder, z-axis along the axis of the cylinder, the components of the vorticity are given by

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \equiv 0$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \equiv 0$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Introducing the stream function ψ as

$$u = -\frac{\partial \psi}{\partial y}, \; v = \frac{\partial \psi}{\partial x}$$

We get $\zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$

If (r, θ) be the polar co-ordinates of a point in the $x - y$ plane with 0 as pole the n from the given conditions

$$\nabla_1^2 \psi = \zeta \text{ for } r < a \tag{1}$$

$$= 0 \text{ for } r > a \tag{2}$$

Since the motion is symmetric about 0 in the $x - y$ plane, in polar co-ordinates we get from (1) and (2)

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}=\xi \quad \text{for } r<a \tag{3}$$

$$=0 \text{ for } r>a \tag{4}$$

since $\frac{\partial\psi}{\partial\theta}+=0.$

We can write (1) as

$$\frac{d}{dr}\left(r\frac{\partial\psi}{dr}\right)=\zeta r,\ r<a \tag{5}$$

Integrating (5) we get

$$\frac{\partial\psi}{dr}=\frac{1}{2}\zeta r^2+A \tag{6}$$

and $\psi=\frac{1}{4}\zeta r^2+A\log e+B$ for $r<a$ (7)

where A, B are arbitrary constants.

Since for $r>a$, $\zeta=0$, we get from (5)

$$\psi=c\log r+D,\ r>a \tag{8}$$

for $r>a$, C, D being arbitrary constants.

Velocity components are given by

$$q_r=-\frac{\partial\psi}{r\partial\theta}=0;\ q_\theta=\frac{\partial\psi}{\partial r} \tag{9}$$

From (6), for $r < a$,

$$\frac{\partial\psi}{dr} = \frac{1}{2}\zeta r + \frac{A}{r} \tag{10}$$

Since the velocity is finite at the centre $r = 0$, from (10) we get $A = 0$.

Also $\psi, \dfrac{\partial\psi}{dr}$ are continuous across $r = a$. We take $\psi = 0$ on $r = a$.

Then $B = -\dfrac{1}{4}\zeta a^2, D = -c\log a$.

From (10) $\dfrac{1}{2}\zeta a = \dfrac{c}{a}$

Thus, $\psi = \dfrac{1}{4}\xi\left(r^2 - a^2\right)$ for $r \leq a$ (11)

and $\psi = \dfrac{1}{2}\xi a^2 \log r/a$ for $r \leq a$ (12)

The velocity components are given by

$$q_r = -\frac{\partial y}{r\partial\theta} \equiv 0$$

$$q_\theta = \frac{\partial\psi}{\partial r} = \frac{1}{2}\xi r, \quad r \leq a$$

$$= \frac{\xi a^2}{2r}, r > a$$

Also at $r = 0$, Hence the center of the circle remains at rert.

We shall now calculate the pressure. The Euler's equation of motion

viz.

$$\frac{\partial \vec{q}}{\partial t} - \vec{q} \times rot\vec{q} = -\vec{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}q^2\right)$$

given as

$$-q_\theta \xi = -\frac{\partial}{\partial r}\left(\frac{p}{\rho} + \frac{1}{2}q_\theta^2\right) \qquad [\because q_r = 0]$$

By inegrating, inside the vortext, we get

$$\frac{p}{\rho} + \frac{1}{2}q_\theta^2 = \int \zeta \frac{1}{2}\zeta r = \frac{\zeta^2}{2}\frac{r^2}{2} + cont$$

i.e., $$\frac{p}{\rho} = \frac{\zeta^2 r^2}{4} - \frac{\zeta^2 r^2}{8} + \frac{p_0}{\rho}$$

$$= \frac{\zeta^2 r^2}{8} + \frac{p_0}{\rho}$$

where p_0 is the pressure at the center.

Outside the circle, $r > a$ we have

$$\frac{p}{\rho} = \frac{1}{2}\frac{\zeta^2 a^4}{4r^2} - = const = \frac{\pi}{\rho}$$

i.e., $$\frac{p}{\rho} = \frac{\pi}{\rho} - \frac{\zeta^2 a^4}{8r^2}$$

where Π is the pressure at infinity.

Since the pressure is continuous at $r = a$, we must have

$$\frac{\zeta^2 a^2}{8} + \frac{p_0}{\rho} = \frac{\Pi}{\rho} - \frac{\zeta^2 a^2}{8}$$

i.e., $p_0 = \Pi - \frac{\zeta^2 a^2 \rho}{4}$.

Since $p_0 > 0$, we must have

$$\Pi > \frac{\zeta^2 a^2 \rho}{4}$$

The circulation of circular vortex is obtained as follows.

If K be the circulation about a circle of radius $r > a$, we have

$$K = \int_c \vec{q} \cdot d\vec{s} = \int_0^{\pi} q_\theta r d\theta, \quad r > a$$

$$= \int_0^{2\pi} \frac{\zeta a^2}{2r} r d\theta = \frac{\zeta a^2}{2} \times 2\pi = \pi \zeta a^2$$

Outside the circle $r = a$, the motion is irrotational, with a velocity potential ϕ such that

$$q_r = -\frac{\partial \phi}{\partial r} = 0$$

$$q_\theta = -\frac{\partial\phi}{r\partial r} = \frac{\zeta a^2}{2r}$$

$$\therefore \phi = -\frac{1}{2}\zeta a^2\theta.$$

The complex potential is given by

$$w = \phi + i\psi = -\frac{1}{2}\zeta a^2\theta + i\frac{1}{2}\zeta a^2 \log r + \text{constant}$$

$$= \frac{i}{2}\zeta a^2 [\log r + i\theta], \text{ neglecting the constant}$$

$$= \frac{i}{2}\zeta a^2 \log z.$$

If we make $\zeta \to \infty$ and $a \to 0$ such that

$\pi\zeta a^2$ = finite = K, then

$$w = \frac{ik}{2\pi}\log z.$$

This gives the complex potential due to a straight vortex filament with circulation k.

Example - 1 :

Find the necessary and sufficient conditions that vortex lines may be at right angles to the streamlines.

Solution :

In Cartesian co-ordinate system, the streamlines and the vortex lines are given by

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w} \qquad (1)$$

and $$\frac{dx}{\Omega x}=\frac{dy}{\Omega y}=\frac{dz}{\Omega z} \qquad (2)$$

These will be at right angles if

$$u\Omega_x+v\Omega_y+w\Omega_z=0 \qquad (3)$$

But

$$\Omega_x=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),\Omega_y=\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right),\Omega_z=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \qquad (4)$$

using (4), equation (3) may be written as

$$u\left(\frac{\partial w}{\partial y}-\frac{\partial u}{\partial z}\right)+v\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+w\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=0$$

which is necessary and sufficient condition that $udx+vdy+wdz$ may be a perfect differential. So we write

$$udx+vdy+wdz=\mu d\phi=\mu\left(\frac{\partial \phi}{\partial x}dx+\frac{\partial \phi}{\partial y}dy+\frac{\partial \phi}{\partial z}dz\right).$$

Thus the necessary and sufficient condition that vortex lines may be at right angles to the stream lines are

$$u = \mu\frac{\partial\phi}{\partial x}, v = \mu\frac{\partial\phi}{\partial y}, w = \mu\frac{\partial\phi}{\partial z}.$$

4.8 Vortex pair

Consider the case of two vortices of strengths k_1 and k_2 at a ditance r_0 apart. Let A, B be their centers and 0 be the center of the system. The point 0 divides AB in the ratio $k_2 : k_1$. The motion of each vortex as a whole is entirely due to the other, and is therefore always perpendicular to AB. Hence the two vortices remain always at hte same distance from one another, and rotate with constant angular velocity about the point 0, which is fixed. The velocities at the two vortices at A and B are respectively $\frac{k_1}{2\pi r_0}$ and $\frac{k_2}{2\pi r_0}$. To obtain the angular velocity w of the system, we divide the velocity of the vortex A by the distance AO where

$$AO = \frac{k_2}{k_1+k_2} \cdot AB = \frac{k_2 r_0}{k_1+k_2}$$

$\therefore$ The angular velocity is given by

$$w = \frac{velocity\ of\ the\ vortex\ at\ A}{AO} = \frac{k_1 + k_2}{2\pi r_0^2}.$$

If k_1, k_2 be f the same sign, i.e., if the direction of rotation in the two vortices be the same then O lies between A and B, otherwise O lies in AB or BA, produced.

If $k_1 = -k_2$, O is at infinity. However, A, B move with equal velocities $\frac{k_1}{2\pi r_0}$ at right angles to AB, which remains fixed in direction. Such a combinatin of two equal and opposite vortices may be called a *vortex pair.*

4.9 Vortex doublet

Consider a vortex pair, k at $ae^{i\alpha}$ in the complex z-plane where $z = x + iy$. If we let $a \to 0$ and $k \to \infty$ so that $2ak = \mu$ is a finite constant, we get a vortex doublet of strength μ inclined at an angle α to the x-axis.

The direction of the doublet is determined from the vortex of negative rotation to that of positive rotation. The complex potential is given by

$$w = \lim_{a \to 0} \frac{ik}{2\pi} \left\{ \log\left(z - ae^{i\alpha}\right) - \log\left(z + ae^{i\alpha}\right) \right\}$$

$$= \lim_{a \to 0} \frac{ik}{2\pi}\left\{\frac{ae^{i\alpha}}{z} + \frac{a^2e^{2i\alpha}}{2z^2} - \ldots - \frac{ae^{i\alpha}}{z} - \frac{a^2e^{i\alpha}}{2z^2} - \ldots\right\} = \frac{i\mu}{2\pi z}e^{2i\alpha}$$

The stream function is $\psi = -\frac{\mu}{2\pi r}\cos(\alpha - \theta)$.

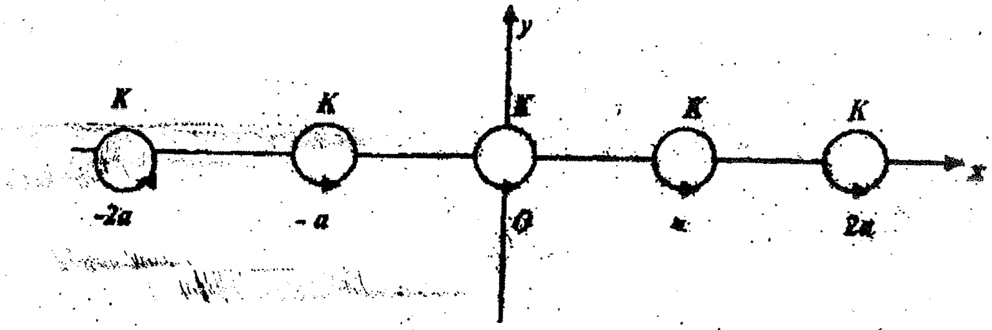
If, in particular, we take the vortex doublet to be the origin and along the axis of y, we have $\psi = -\frac{\mu \sin\theta}{2\pi r}$. If we put $\frac{\mu}{2\pi} = Ub^2$, we obtain $\psi = -\frac{Ub^2 \sin\theta}{r}$ which is the stream function for a circular cylinder of radius b moving with velocity U along the x-axis.

Thus the motion due to a circular cylinder is the same as that due to a suitable vortex doublet placed at the center, and with its axis perpendicular to the direction of motion.

4.10 Infinite row of parallel rectilinear vortices

We shall find the complex potential for single infinite row of vortices.

Let us consider an infinite row of vortices each of strength k at the points $0, \pm a, \pm 2a, \ldots, \pm na, \ldots$ (as shown in the figure below).



The complex potential of the $(2n + 1)$ vortices nearest to the origin is

$$w_n = \frac{ik}{2\pi}\log z + \frac{ik}{2\pi}\log(z-a) + \ldots + \frac{ik}{2\pi}\log(z-na)$$

$$+\frac{ik}{2\pi}\log(z+a) + \ldots + \frac{ik}{2\pi}\log(z+na)$$

$$= \frac{ik}{2\pi}\log z\left\{z\left(z^2-a^2\right)\left(z^2-2^2a^2\right)\left(z^2-n^2a^2\right)\right\}$$

$$= \frac{ik}{2\pi}\log\left\{\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\ldots\left(1-\frac{z^2}{n^2a^2}\right)\right\}$$

The constant term may be omitted, so that we write

$$w_n = \frac{ik}{2\pi}\log\left\{\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\ldots\left(1-\frac{z^2}{n^2a^2}\right)\right\} \quad (1)$$

Now, sin x can be expressed as an infinite product in the following form

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\ldots\left(1-\frac{x^2}{n^2\pi^2}\right). \qquad (2)$$

Thus, letting $n \to \infty$ in (1), we get by using (2),

$$w = \frac{ik}{2\pi}\log\sin\left(\frac{\pi z}{a}\right) \qquad (3)$$

Now, consider the vortex at $z = 0$. Since its motion is due to the other vortices, the complex velocity of the vortex at the oritin is given by

$$-\frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}-\frac{ik}{2\pi}\log z\right\}_{z=0} = -\frac{ik}{2\pi}\left(\frac{\pi}{a}\cdot\cot\frac{z\pi}{a}-\frac{1}{z}\right)_{z=0}$$

$$= -\frac{ik}{2\pi}\lim_{z\to 0}\left[\frac{\pi\cos(\pi z/a)}{a\sin(\pi z/a)}-\frac{1}{z}\right]$$

$$= -\frac{ik}{2\pi}\lim_{z\to 0}\frac{\pi z\cos(\pi z/a)-a\sin(\pi z/a)}{z\sin(\pi z/a)}$$

$$= -\frac{ik}{2\pi}\times 0 \quad \text{[by using l'Hospital's rule]}$$

$$= 0$$

Hence the vortex at the origin is at rest. Similarly it can be shown that the remaining vortices are also at rest. Thus the vortex row induces no velocity on itself.

To determine the stream function we note that

$$w(z)=\phi+i\psi,\ \bar{w}(\bar{z})=\phi-i\psi$$

so that we get

$$2i\psi,\ w(z)-\bar{w}(\bar{z})=\frac{ik}{2\pi}\log\left\{\sin\frac{\pi z}{a}\sin\frac{\pi\bar{z}}{a}\right\},$$

or, $$\psi=\frac{k}{4\pi}\log\left[\frac{1}{2}\left(\cosh\frac{2\pi y}{a}-\cos\frac{2\pi x}{a}\right)\right]$$

or, $$\psi=\frac{k}{4\pi}\log\frac{1}{2}\left(\cosh\frac{2\pi y}{a}-\cos\frac{2\pi x}{a}\right).$$

The required stream lines are given by

$$\cosh\left(\frac{2\pi y}{a}\right)-\cos\left(\frac{2\pi x}{a}\right)=\text{constant.}$$

For large values of $\frac{y}{a}$, we neglect the term $\cos\frac{2\pi x}{a}$, for its modulus never exceeds unity, and therefore along the streamlines y = constant. Thus at a great distance from the row the stream lines are parallel to the row.

Again, if v_1, v_2 are the complex velocitieis at the points $z, \bar{z}$ respectively, we have

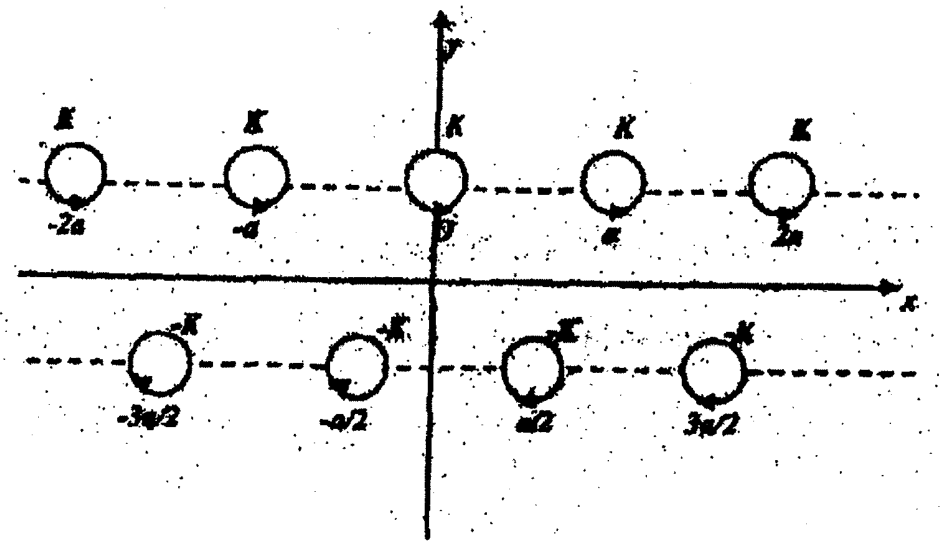
$$v_1 + v_2 = -\frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}\right\}_{z=z} - \frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}\right\}_{z=\bar{z}}$$

$$= -\frac{ik}{2a}\cot\frac{\pi z}{a} - \frac{ik}{2a}\cot\frac{\pi\bar{z}}{a} = -\frac{ik}{2a}\frac{2\sin\frac{2\pi x}{a}}{\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}}$$

which is purely imaginary and tends to zero when y tends to infinity. Thus the velocities along the distanct streamlines are parallel to the row but in opposite directions.

4.11 Karman Vortex Street

Let there be two parallel rows of vortices of equal but opposite strengths. This consists of two parallel infinite rows of the same spacing, say '*a*', but of opposite vortex strengths k and $-k$, arranged that each vortex of the upper row is directly above the mid point of the line joining two vortieces of the lower row and vice-versa. Taking the configuration at time $t = 0$, we take the axes as shwon in the following figure, the x-axis being midway between and parallel to the rows which are at the distance b apart. At this instant the vortices in the upper row are at the points $ma + \frac{1}{2}ib$, and those in the lower row at the points $\left(m+\frac{1}{2}\right)a-\frac{1}{2}ib$, where $m=0,\pm1,\pm2,....$



Now, the complex potential at the instant $t=0$, is given by

$$w=\frac{ik}{2\pi}\log\sin\frac{\pi}{a}\left(z-\frac{ib}{2}\right)-\frac{ik}{2\pi}\log\sin\frac{\pi}{a}\left(z-\frac{1}{2}+\frac{ib}{2}\right).$$

[since lower vortices are of opposite strength]

Since neither row induces any velocity in itself, the velocity of vortex at $z=\left(\frac{a}{2}-\frac{ib}{2}\right)$ will be given by

$$-u_1+iv_1=\left[\frac{d}{dz}\frac{ik}{2\pi}\log\sin\frac{\pi}{a}\left(z-\frac{ib}{2}\right)\right]_{z=\frac{a}{2}-\frac{ib}{2}}$$

$$=\frac{ik}{2a}\cot\left(\frac{\pi}{2}-\frac{i\pi b}{a}\right)=-\frac{k}{2a}\tanh\frac{\pi b}{a}.$$

Thus the lower row advanced with velocity

$$V=\frac{k}{2a}\tanh\frac{\pi b}{a},$$

and similarly the upper row advances with the same velocity. The rows will advance the distance a in time $T=\frac{a}{V}$ and the configuration will be the same after this interval as at the initial instant.

Example -1 :

Three parallel rectilinear vortices of the same strength K and in the same sense meet any plane perpendicular to them in an equilateral triangle of side a. Prove that the vortices all move round the same cylinder with uniform speed in time $\frac{2\pi a^2}{3K}$.

Solution :

Let r be the radius of the circumcircle of the equilateral triangle ABC. Lt 0 be the circumcentre. so, $r=OB=\left(\frac{a}{2}\right)\sec 30^0=\frac{a}{\sqrt{3}}$. There are three vortices of strength $K\left(=\frac{k}{2\pi}\right)$ at A, B, C which are situated at the points

$z_m re^{2m\pi i/3}, m=1,2,3$. Then the complex potential of the vortices at B, C, A, is given by

$$w=\frac{ik}{2\pi}\left[\log\left(z-re^{\frac{2i\pi}{3}}\right)+\log\left(z-re^{\frac{4\pi i}{3}}\right)+\log\left(z-re^{\frac{6\pi i}{3}}\right)\right]$$

$$=\left(\frac{ik}{2\pi}\right)\log\left(z^3-r^3\right)$$

Now the velocity induced at $z=re^{6\pi i/3}=r$, by other vortices is given by

$$u_1-iv_1=-\frac{d}{dz}\left[\frac{ik}{2\pi}\log\left(z^3-r^3\right)-\frac{ik}{2\pi}(z-r)\right]_{at\ r}$$

$$=-\frac{ik}{2\pi}\frac{2z+r}{z^2+zr+r^2}$$

Thus, $\quad q=|u_1-iv_1|=\frac{k}{2\pi}\left[\frac{2z+r}{z^2+zr+r^2}\right]_{z=r}$

$$=\frac{k}{2\pi}\left[\frac{3r}{3r^2}\right]=\frac{k}{2\pi r}=\frac{k}{r}$$

So, the required time $=\dfrac{\text{circumference of the circumcircle}}{\text{velocity at } z=r}$

$$=\frac{2\pi\frac{a}{\sqrt{3}}}{\frac{k}{r}}=\frac{2\pi a^2}{3k} \qquad \left[\because r=\frac{a}{\sqrt{3}}\right].$$

Example - 2:

An infinite row of equidistant rectilinear vortices are at a distance a apart. The vortices are of the same strength k but they are alternately of opposite signs. Find complex potential and the stream function. Show also that, if α be the radius of a vortex, the amount of flow between two vortex and the next is $\left(\frac{k}{\pi}\right)\log\cot\left(\frac{\pi\alpha}{2a}\right)$.

Solution :

Let the row of vortices be taken along the x-axis. Let there be vortices of strength k each situated at the points

$$(0,0),(\pm 2a,0),(\pm 4a,0),...$$

According tot he problem the strengths is $-k$ eah at the point

$$(\pm a,0),(\pm 3a,0),(\pm 5a,0),...$$

The complex potential of the entire system is given by

$$w=\frac{ik}{2\pi}\Big[\{\log z+\log(z-2a)+\log(z+2a)+\log(z-4a)+\log(z+4a)$$

$$+...\}-\{\log(z-a)+\log(z+a)+\log(z-3a)+\log(z+3a)+...\}\Big]$$

$$=\frac{ik}{2\pi}\log\frac{z(z^2-2^2a^2)(z^2-4^2a^2)...}{(z^2-a^2)(z^2-3^2a^2)...}$$

$$= \frac{ik}{2\pi} \log \frac{\frac{z}{2a}\left[1-\left(\frac{z}{2a}\right)^2\right]\left[1-\left(\frac{z}{4a}\right)^2\right]\ldots}{\left[1-\left(\frac{z}{a}\right)^2\right]\left[1-\left(\frac{z}{3a}\right)^2\right]\ldots} + \text{constant}$$

$$= \frac{ik}{2\pi} \log \frac{\sin\left(\frac{\pi z}{2a}\right)}{\cos\left(\frac{\pi z}{2a}\right)} = \frac{ik}{2\pi} \log \tan\left(\frac{\pi z}{2a}\right), \qquad (1)$$

This is the complex potential of the problem. From (1), we get

$$\phi + i\psi = \frac{ik}{2\pi} \log \tan \frac{\pi}{2a}(x+iy)$$

and $\phi - i\psi = -\frac{ik}{2\pi} \log \tan \frac{\pi}{2a}(x-iy)$

$$\therefore 2i\psi = \frac{ik}{2\pi}\left[\log \tan \frac{\pi}{2a}(x+iy) + \log \tan \frac{\pi}{2a}(x-iy)\right]$$

$$\therefore \psi = \frac{k}{4\pi} \log \frac{\sin \frac{\pi}{2a}(x+iy) \sin \frac{\pi}{2a}(x-iy)}{\cos \frac{\pi}{2a}(x+iy) \cos \frac{\pi}{2a}(x-iy)}$$

$$= \frac{k}{4\pi} \log \frac{\cosh\left(\frac{\pi y}{a}\right) - \cos\left(\frac{\pi x}{a}\right)}{\cosh\left(\frac{\pi y}{a}\right) + \cos\left(\frac{\pi x}{a}\right)} \qquad (2)$$

Since the motion of the vortex at the origin is due to other vortices only, the velocity, q_0 of vortex at the origin is given by

$$q_0 = -\left\{\frac{d}{dz}\left[\frac{ik}{2\pi}\log\tan\frac{\pi z}{2a} - \frac{ik}{2a}\log z\right]\right\}_{z=0}$$

$$= -\frac{ik}{2\pi}\left[\frac{\sec^2\left(\frac{\pi z}{2a}\right)}{\tan\left(\frac{\pi z}{2a}\right)} \times \frac{\pi}{2a} - \frac{1}{z}\right]_{z=0} = 0.$$

Hence the vortex at origin is at rest. Similarly, it can be shown that the remaining vortices are also at rest. Thus we find that the vortex row induces no velocity on itself.

We now determine the amount flow between two vortex. For any point on the x-axis, $y=0$ and hence ψ' at any point on the x-axis is given by [putting $y = 0$ in (2)], we get

$$\psi' = \frac{k}{4\pi}\log\frac{1-\cos\left(\frac{\pi x}{a}\right)}{1+\cos\left(\frac{\pi x}{a}\right)} = \frac{k}{4\pi}\log\frac{2\sin^2\left(\frac{\pi x}{2a}\right)}{2\cos^2\left(\frac{\pi x}{2a}\right)}$$

or, $\psi' = 2\frac{k}{4\pi}\log\tan\frac{\pi x}{2a}$

The amount flow between two consecutive vortices is given by

$$= (\psi')_{a-\alpha} - (\psi')_{a} = \frac{k}{2\pi}\left[\log\tan\frac{\pi(a-\alpha)}{2a} - \log\tan\frac{\pi\alpha}{2a}\right]$$

$$= \frac{k}{2\pi}\log\frac{\tan\left(\frac{\pi}{2}-\frac{\pi\alpha}{2a}\right)}{\tan\frac{\pi\alpha}{2a}} = \frac{k}{2\pi}\log\cot^2\frac{\pi\alpha}{2a}$$

$$= \frac{k}{\pi}\log\cot\frac{\pi\alpha}{2a}.$$

Hence proved.

Example - 3 :

An infinite street of linear parallel vortices is given as $x = ra, y = b$, strength k; $x = ra$, $y = -b$ of strength $= -k$, where r is any positive or negative integer or zero. Prove that if the liquid at infinity is at rest, the vortex street moves as a whole, in the direction of its length with the speed $\left(\frac{k}{2a}\right)\coth\left(\frac{2\pi b}{a}\right)$.

Solution :

The vortices of strength k each are situated in the first row at points

$(0,b),(\pm a,b),(\pm 2a,b),\ldots$

and the vortices of strength $-k$ each are situated in the second row at points

$$(0,b),(\pm a,-b),(\pm 2a,-b),...$$

Hence the complex potential w of the above system of vortices is given by

$$w=\frac{ik}{2\pi}\log\sin\frac{\pi(z-ib)}{a}-\frac{ik}{2\pi}\log\sin\frac{\pi(z+ib)}{a} \qquad (1)$$

Let w' be the complex potential of the vortex at $z = ib$ due to vortices situated at the remaining points. To find the motion of vortex at $z = ib$, we must omit the part due to it. From (1), we get w' as

$$w'=\frac{ik}{2\pi}\log\sin\frac{\pi(z-ib)}{a}-\frac{ik}{2\pi}\log\sin\frac{\pi(z+ib)}{a}-\frac{ik}{2\pi}\log(z-ib)$$

Let u, v be the velocity components of the vortex at $z = ib$. So we have

$$u-iv=-\left(\frac{dw'}{dz}\right)_{z=ib}$$

$$=-\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi(z-ib)}{a}-\frac{\pi}{a}\cot\frac{\pi(z+ib)}{a}-\frac{1}{z-ib}\right]_{z=ib}$$

$$=\frac{k}{2a}\coth\frac{2\pi b}{a}$$

$$\therefore u = \frac{k}{2a}\coth\frac{2\pi b}{a} \text{ and } v = 0.$$

Hence the required velocity of the vortex street is

$$q = \sqrt{(u^2 + v^2)} = u = \frac{k}{2a}\coth\frac{2\pi b}{a}.$$

Hence proved.

4.12 Keywords

Vortex motion, Helmholtz's theorem, circular vortex, Infinite row of parallel rectilinear vortices.

4.13 Exercise

1. If $udx + vdy + wdz = d\theta + \lambda d\chi$, where θ, λ, χ are function of x, y, z, t, prove that the vortex lines at any time ar the lines of intersection of the surfaces λ = constant and χ = constant.

2. Define a 'vortex tube' and a 'vortex filament'. Find the velocity field $\overline{q}$ due to a closed vortex filament in an infinite liquid at rest at infinity in the form.

$$\bar{q} = \frac{k}{4\pi}\oint \frac{d\bar{s} \times \bar{r}}{r^3}$$

where the integration is taken along the filament and $\bar{r}$ is the vector to te field point from the elements ds' of the filament.

3. State and prove Helmoholtz first and second theorems connecting the fundamental properties of a vortex motion.

4. The velocity vector in the flow field is given by $\vec{q} = [(Az - By), (Bx - Cz), (Cy - Ax)]$ where A, B, and C are non-zero constants. Find the equations of the vortex lines.

5. In an incompressible fluid, the vorticity at every point is constant in magnitude and direction. Show that the components of vorticity are the solutions of Laplace's equation.

6. A velocity field is given by

$$\vec{q} = (-\hat{i}y + \hat{j}x)/(x^2 + y^2)$$

Find the circulation round a unit circle with center at origin.

7. Prove that for an inviscid fluid moving under a system of conservative forces, the vorticity vector $\vec{w}$ satisfies the equation

$$\frac{D}{Dt}\left(\frac{\vec{w}}{\rho}\right)=\left(\frac{\vec{w}}{\rho}.\vec{\nabla}\right)\vec{q},$$

provided the pressure p is a function of the density ρ alone.

8. If the vortices are of the same strength and the spin is in same sense both, show that the relative streamlines are given by

$$\log\left(r^4+b^4-2b^2r^2\cos 2\theta\right)-\left(r^2/2b\right)^2 = \text{constant}.$$

9. Three parallel rectilinear vortices of the same strength K and in the same sense meet any plane perpendiculat to them in an equilateral triangle of side a. Show that thte vortices all move round the same cylinder with uniform speed in time $\frac{2\pi a^2}{3K}$.

10. If $(r_1,\theta_1),(r_2,\theta_2),....,$ be polar coordinates at time t of a system of rectilinear vortices of strength k_1, k_2, ..., prove that

$$\sum kr^2 = \text{constant and } \sum kr^2\dot{\theta}=\left(\frac{1}{2\pi}\right)\sum k_1k_2.$$

11. An infinite street of linear parallel vortices is given as : $x = ra, y = b$, strength $=-k$, where r is any positive or negative integer or zero. Prove that if the liquid at infinity is at rest, the street moves as a whole, in the direction of its length with the speed $\left(\frac{k}{2k}\right)\coth\left(\frac{2\pi b}{a}\right)$.

4.14Further Readings

1. Milne-Thomson, L. M., Theoretical Hydrodynamics, Macmilan & Co. Ltd., London, 1955.

2. Ramsey, A.S., A Treatise on Hydromechanics, CBS Publishers & Distributors, New Delhi, 2000.

3. Chorlton, F., Textbook of Fluid Dynamics, CBS Publishers & Distributors, New Delhi, 2003.

4. Raisinghania, M.D., Fluid Dynamics, S. Chand, 2005.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

Paper - VII

Group - D **Marks - 20**

MAGNETOHYDRODYNAMICE

Module No. 83 and 84

STRUCTURE :

Paper - VII, Group - D, Marks - 20

MAGNETOHYDRODYNAMICS

1. INTRODUCTION

Magnetofluiddynamics (MFD) is the study concerning with the motion of electrically conducting fluid subjected to a magnetic field. If the fluid be incompressible, such as liquid mercury, and its other properties like viscosity, thermal conductivity, electrical conductivity etc. be regarded as constants, then we use the word *Magnetohydrodynamics (MHD)* of *Hydromagnetics*. On the other hand, if the fluid be compressible such as ionized gas, and if its other physical properties specially temperature, be variable, then the title *magnetofluiddynamics* is selected. In the study of MFD we generally consider the continuum approach regarding the fluid to be a continuous medium.

Faraday (1832) observed that if an electrically conducting fluid moves in a magnetic field then electric currents are induced in the fluid producing their own magnetic field, called induced magnetic field, thereby modifying the original magnetic field. In addition to this, the induced currents interact with the magnetic field and produces electric magnetic force that perturb the original motion. Thus the two important basic effects of magnetofluid dynamics are (i) *the motion of the fluid affects the magnetic field, and* (ii) *the magnetic field affects the motion of the fluid.* In fact the motion of the fluid slows down due to these electromagnetic forces unless we apply sufficiently large electrical field opposite to the direction or the induced magnetic field to overcome its effect as a result of which the net electromagnetic force accelerates the fluid motion.

Although some interesting results of MFD can be achieved in laboratory, its importance lies in *cosmic* problems in geophysics and astrophysics. These are the problems of earth's interior, of the sun, stars or interstellar space. Some of the application of MFD are :

(i) MHD power generator

In turbogenerators electricity is generated by the motion of a conductor through a magnetic field (Faraday's law). The conductor is moved by a compressible fluid which expands through a nozzle which transfers the internal energy into mechanical energy of the conductor and this in turn is transformed into electrical energy.

(ii) MFD flowmeter.

MFD flowmeter is used to measure ship's speed and is based on the principle that the induced voltage is proportional to the flow rate. This technique is widely applied in oceanography.

(iii) MFD submarines.

MFD submarines obtain their thrust from the Lorentz force which is produced by tranverse electric and magnetic fields. These pump the electrically conducting sea water through or past the submarine.

Some more applications of the subject are : radio wave propagation in ionosphere, space communication system, diagnostic techniques, solar flares etc.

2. OBJECTIVES

Our main objectives are to apply Faraday's laws of electromagnetism to the motion of conducting fluids, e.g. mercury, liquid sodium, human blood etc. In the case when the conductor is either a solid or gas, electromagnetic forces are generated which may be of the same order of magnitude as the hydrodynamical or inertial forces. Thus the equations of motion for the fluid must take these electromagnetic forces into account in addition to other forces.

MHD effects in conducting liquids have been studied in the laboratory by Hartmann and Williams. It is seen that the viscosity of mercury or molten sodium is enhanced when the flow takes place in a strong magnetic field.

3. KEY WORDS

Alfven waves, Ferraro's law of isorotation, Hartmann number, Lorentz force, magnetic diffusivity, magnetic energy, magnetic Reynolds number; Maxwell's equations.

4. MAIN DISCUSSIONS

4.1 MAXWELL'S ELECTROMAGNETIC FIELD EQUATIONS OF MOVING MEDIA :

Before we proceed with the mathematical theory of magnetohydrodynamics, we summarize Maxwell's electromagnetic field equations whose derivations may be found in any standard book on electromagnetic theory (e.g. V.C.A. Ferraro-Electromagnetic Theory, Athlone Press). Throughout our discussions we use rationalized (m.k.s.a.) quantities, based on length, mass, time and current.

In an inertial frame of reference the magnetic and electric fields $\mathbf{B}$ $(=\mu\mathbf{H})$ and $\mathbf{E}$ satisfy the equations

$$\nabla.\mathbf{B} = \mathbf{0} \text{ (magnetic field continuity equation)} \qquad (1)$$

$$\nabla.\mathbf{E} = \frac{q}{\varepsilon} \text{ (differential form of Gauss' law or charge continuity equation)} \qquad (2)$$

$$\nabla\times\mathbf{E} = -\frac{\delta B}{\delta t} \text{ (Faraday's law)} \qquad (3)$$

$$\nabla \times \mathbf{H} = \mathbf{j} \text{ (differential form of Amper'e's law).} \quad (4)$$

In the above equations, **E** is the electric field intensity, ε the permittivity, **B** the magnetic induction vector, μ the permeability, **q** the charge density of the medium per unit volume, **t** the time and **j** signifies the electric current density vector. The permeability μ and the permittivity ε are taken to be those pertaining to free space so that

$$\mu = 4\pi \times 10^{-7} \text{ henry/meter,}$$

$$\varepsilon = (30\pi \times 10^{9})^{-1} \text{ farad/meter} \quad (5)$$

and

$$\mu\varepsilon = \frac{1}{c^2}, \quad (6)$$

where **c** (≈ 3×10^8 **meter/second)** is the velocity of light. In all these equations all vectors and scalars are taken to be collective, i.e., they are all averaged in large-scale quantities over regions in comparison with the scale of random fluctuations.

Let us now transform the different quantities to a frame of reference moving with a fluid element. Since the velocity **v** relative to the inertial frame is supposed to be small compared with c, we may neglect v^2/c^2 in relativistic transformation formulae given approximately by

$$\left.\begin{aligned} E' &= E + v \times B, \\ B' &= B - \mu\varepsilon v \times E, \end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned} q' &= q - \frac{v.j}{c^2}, \\ j' &= j - qv, \end{aligned}\right\} \quad (8)$$

where the accented quantities are measured in the moving system. We also need one more equation, namely the law of induction. For stationary conductor, Ohm's law is

$$j = \sigma E, \quad (9)$$

σ being the electrical conductivity. Now since the conductivity depends on the local state of the conducting fluid, so it must be evaluated in the moving frame of reference. Thus using the first of equations (7), the modified Ohm's law is

$$j' = \sigma E' = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (10)$$

Simplification of electromagnetic field equations

In magnetohydrodynamics, we deal with conducting fluids of very large conductivity but finite electric currents. It, therefore, follows from equation (10) that for large σ, the electric field $\mathbf{E}'$ in a frame moving with the bulk velocity is very nearly zero and we may set $\mathbf{E}'=0$, where by the first equation of (7) we get

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}. \tag{11}$$

Substituting this in the second of equations (7), we see that the second term is of order $\mu\varepsilon v^2 B = \frac{v^2}{c^2} B$ which is negligible compared to $\mathbf{B}$. The equation (7) can now approximated as

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \ \ \mathbf{B}' = \mathbf{B} \tag{12}$$

Thus we may consider a magnetic field without specifying the frame of reference in which it is measured. However, the electric field is not invariant but transforms according to the law given in (12).

To estimate the order of magnitude of the derivative of any quantity $\mathbf{Q}$ with respect to a length $\mathbf{x}$, we suppose that $\mathbf{L}$ is a characteristic length so that $\frac{dQ}{dx} \approx \frac{Q}{L}$. Then by (2) and (1) we get

$$q \approx -\mu\varepsilon \nabla.(v \times H) \approx \frac{vH}{c^2 L} \tag{13}$$

The convention of this space charge by the fluid motion yields a contribution to the total current-density

$$|qv| \approx \frac{v^2 H}{c^2 L} \approx \frac{v^2}{c^2} |\nabla \times H|. \tag{14}$$

But the total current density is $\mathbf{j} = \nabla \times \mathbf{H}$ by (4). Thus the equation (14) implies that the convection of the net charge makes a negligible contribution to the total current. This is, therefore, mainly due to the conduction current, i.e., the drift of electric charges in the conducting fluid.

Again we have.

$$\frac{q}{ne} \approx \frac{vH}{c^2 L.ne} \approx \frac{vj}{c^2 ne} = \frac{v}{c}.\frac{V}{c}, \tag{15}$$

where $\mathbf{e}$ is the electronic charge, $\mathbf{n}$ the electron density and $\mathbf{V}$ is the drift velocity of the electrons. In most cases $\mathbf{V} \ll \mathbf{v}$ so that the charge separation is small in non-relativistic theory.

Let us now proceed to justify the neglect of the displacement current in equation (4). Let $\mathbf{T}$ be the time characteristic of the temporal variations of the field quantities. Then the displacement current $\varepsilon \frac{\delta E}{\delta t}$, which has been omitted in the right hand side of the equation (4), is of order

$$\varepsilon\left|\frac{\delta E}{\delta t}\right| \approx \frac{\varepsilon|E|}{T}$$

and this is negligible in comparison with $|\nabla \times H|$ if

$$\frac{\varepsilon|E|}{T} << |\nabla \times H| \approx \frac{H}{L}.$$

Using (11), i.e., $|E| \approx \mu v H$ the above condition becomes

$$\frac{\varepsilon\mu v H}{T} << \frac{H}{L}, \text{ i.e. } \frac{v}{c}.\frac{L}{c} \tag{16}$$

on using the relation $\varepsilon\mu = \frac{1}{c^2}$. For quasi-stationary states $T >> \frac{L}{c}$, i.e. the time that an electromagnetic wave takes to traverse a distance L must be short compared with the time variations of the field quantities; the relation (16) only introduces an extra small factor numerically less than unity.

With these approximations, Maxwell's equations are

$$\left.\begin{array}{l} \text{(i) } \nabla.E = \frac{q}{\varepsilon}, \\ \text{(ii) } \nabla.B = 0, \\ \text{(iii) } \nabla \times H = j, \\ \text{(iv) } \nabla \times E = -\frac{\delta B}{\delta t}. \end{array}\right\} \tag{17}$$

These equations are invariant under the Galilean transfromation

$$\left.\begin{array}{l} r' = r - vt, \\ t' = t, \end{array}\right\} \tag{18}$$

provided that **E** and **B** transform according to (12) and **q** and **j** according to

$$\left.\begin{array}{l} q' = q - \frac{v.j}{c^2}, \\ j' = j, \end{array}\right\} \tag{19}$$

(since the convection current qv is negligible). Since j/=j, we have by using (10)

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{20}$$

4.2 THE ELECTROMAGNETIC EFFECTS AND THE MAGNETIC REYNOLDS NUMBER

Combing equations (4) and (20) we get

$$\nabla \times H = \sigma(E + v \times B). \tag{21}$$

Multiplying both sides by μ and then operating with curl we obtain

$$\nabla \times \nabla \times \mu H = \mu\sigma\{\nabla \times E + \nabla \times (v \times B)\},$$

$$\text{or,} \quad \nabla \times \nabla \times B = -\mu\sigma \frac{dB}{\nabla t} + \mu\sigma\nabla \times (v \times B)\Big\}, \quad [\text{using (3)}]$$

$$\text{or,} \quad \nabla(\nabla.B) - \nabla^2 B = -\mu\sigma \frac{\delta B}{\delta t} + \mu\sigma\nabla \times (v \times B)\Big\},$$

so that

$$\frac{\delta B}{\delta t} = \nabla \times (v \times B) + \eta\nabla^2 B, \quad [\text{since } \nabla.B = 0 \text{ by } (1)] \tag{22}$$

where

$$\eta = \frac{1}{\mu\sigma} \tag{23}$$

is called the *magnetic diffusivity* or *magneticviscosity*.

Now, let **L** be a characteristic length and **V** be a characteristic velocity. Then the order of the magnitude of the first and second terms on the right-hand side of the equation (22) are $\frac{VB}{L}$ and $\frac{\eta B}{L^2}$ respectively. Their ratio

$$R_m = \frac{VL}{\eta} \tag{24}$$

is called the *magnetic Reynolds number*. If $R_m << 1$, we can neglect the first term on the right hand side of the equation (22) leading to

$$\frac{\delta B}{\delta t} = \eta\nabla^2 B. \tag{25}$$

On the other hand, if $R_m >> 1$, the first term on the right hand side of (22) is predominant and the equation reduces to

$$\frac{\delta B}{\delta t} = \nabla \times (v \times B). \tag{26}$$

4.3 LORENTZ FORCE

It is known that the electric field strength **E** is defined by the ratio $E = \frac{F}{q}$, where **F** is the force experienced by the charge density **q** initially at rest in the field. We now generalize the expression for electrostatic force **F** = **qE** to include the effect of moving charges, i.e. **F** = **qE** + **F**′.

Now the interaction of currents or charges in motion is described in terms of the magnetic field **B**. It is known from very precise measurements that a test particle moving in this field experiences a force **F**′ proportional to the strength of the magnetic field **B** and perpendicular to the velocity **v** of the particle. Thus, we may define

$$\mathbf{F}' = q(v \times B). \qquad (27)$$

Hence the total acting on the charge q moving with velocity **v** is

$$\mathbf{F} = \mathbf{q}[\mathbf{E} + \mathbf{v} \times \mathbf{B}]. \qquad (28)$$

The vorce **F** is known as the *Lorentz force*.

4.4 THE EQUATIONS OF MOTION OF A CONDUCTING FLUID

So far we have considered the electromagnetic field equations of a moving conducting fluid. Now we derive the concerned hydrodynamical equations of motion.

We have that the Navier-Stokes equation of motion of a veicous fluid is given in vector form by

$$\rho\frac{Dv}{Dt} = F - \nabla p + \frac{1}{3}\rho\nu\nabla(\nabla.v) + \rho\nu\nabla^2 v, \qquad (29)$$

where $\frac{D}{Dt} \equiv \frac{\delta}{\delta t} + (v.\nabla)$, ρ is the fluid density, **v** the fluid velocity, **F** the body force per unit volume, **p** the fluid pressure, ν the kinematic coefficient of viscosity.

Now if the fluid moves in an electric and magnetic field, then the body force **F** per unit volume consists of three parts : (i) gravitational force ρ **g**, **g** being the acceleration due to gravity, (ii) electrical force and (iii) magnetic force. An elemental volume $\delta\tau$ of the fluid contains a charge of amount $q\delta\tau$ so that the force on it due to an electric field of intensity **E** is $(q\delta\tau)E$ and hence the eletrical body force per unit volume is **qE**.

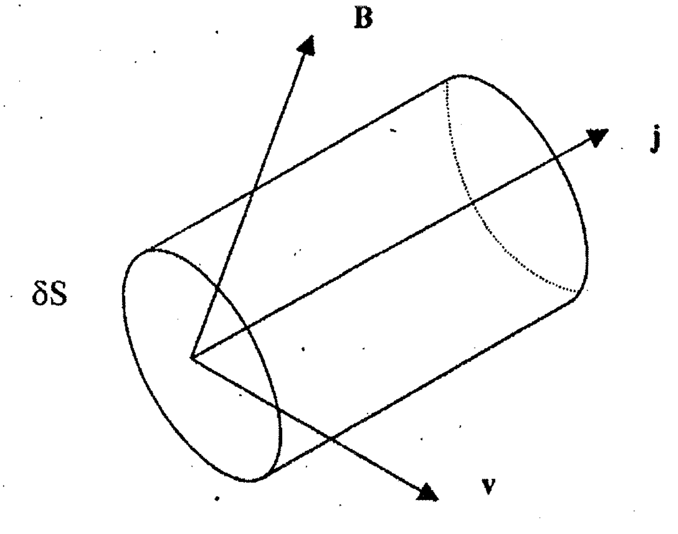


Figure 1

For the magnetic body force per unit volume, we note that the total current density vector is $\mathbf{j} + q\mathbf{v}$ in which the fluid element chosen moves along with local velocity $\mathbf{v}$. Thus the conductive component $\mathbf{j}$ makes an effective contribution to the magnetic body force but not the convective part $q\mathbf{v}$. Let δS be the normal cross-section of a fluid element whose length δs lies along the direction of $\mathbf{j}$. This element moves along with the local fluid velocity $\mathbf{v}$ in a magnetic field of intensity $\mathbf{H}$. Thus the current flowing through the element is $i = |j|\delta S$. The Biot-Savart law then gives the magnetic force in the element as

$$\delta F_1 = i\delta s \times B = |j|\delta s\delta S \times B = (j \times B)\delta s\delta S,$$

so that the magnetic body force per unit volume is

$$\mathbf{j} \times \mathbf{B} = \mu\, \mathbf{j} \times \mathbf{H}.$$

Hence the total body force per unit volume is

$$\mathbf{F} = \rho\, \mathbf{g} + q\, \mathbf{E} + \mu\, \mathbf{j} \times \mathbf{H}.$$

We can now rewrite the equation (29) as

$$\rho \frac{Dv}{Dt} = \rho g + qE + \mu j \times H - \nabla p + \frac{1}{3}\rho v \nabla(\nabla . v) + \rho v \nabla^2 v. \qquad (30)$$

In addition the equation of continuity is

$$\frac{\delta \rho}{\delta t} + \nabla.(\rho v) = 0. \qquad (31)$$

4.5 ALFVE$'$N'S THEOREM

Statement : *In a perfectly conducting fluid moving in a magnetic field, the vlux of magnetic field intensity through any closed circuit moving along with the fluid is constant.*

Proof : Consider an open surface Σ of fluid particles which at time **t** is bounded by a closed curve C (Figure-2). Then the flux of the magnetic field **B(r, t)** through the surface Σ at time t is given by

$$F = \int_{\Sigma} B(r, t).dS, \qquad (32)$$

where r is the position vector of a point lying on the surface Σ and dS is an element of area of Σ oriented along the normal to the surface associated in the sense of description of C. Now the surface Σ and the curve C will move with the fluid to new position Σ' and C'

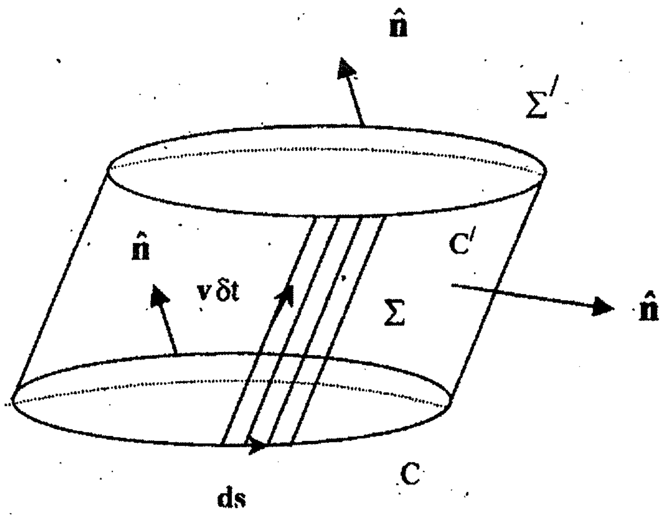


Figure 2

respectively at time t + δt (Fig.-2). Let r/ be ue the position vector of a point of $\Sigma^/$. Then the change of the flux F as Σ moves with the fluid is

$$\delta F = \int_{\Sigma^/} B(r^/, t+\delta t).dS - \int_{\Sigma} B(r,t).dS$$

$$= \left[\int_{\Sigma^/} B(r^/, t+\delta t).dS - \int_{\Sigma} B(r,t).dS\right]$$

$$+ \left[\int_{\Sigma} B(r, t+\delta t).dS - \int_{\Sigma} B(r,t).dS\right]$$

$$= I_1 \_ 1_2, \text{ say.} \qquad (33)$$

Here I_1 represents the change in the flux F at time t + δt due to the displacement of Σ while 1_2 is the change in the flux through Σ during the time interval δt, i.e. it represents the local rate of change of F. Thus

$$I_2 = \int_{\Sigma} \{B(r, t+\delta t) - B(r,t)\}.dS = \delta t \int_{\Sigma} \frac{\delta B}{\delta t}.dS \qquad (34)$$

to the first order in δt.

Now consider the volume τ enclosed by surface Σ, $\Sigma^/$ and the cylindrical surface S traced out by C as it moves to C/. Then we have at time t + δt,

$$\int_{\Sigma'} B.dS - \int_{\Sigma} B.dS + \int_{s} B.dS = \int_{\tau} \nabla.Bd\tau = 0 \qquad \text{[by (1)]} \qquad (35)$$

in which **dS** in cach case is oriented in the oriented in the sense of the voctor $\hat{n}$ as shown in Fig.-2. Suppose an element **ds** of the curve C undergoes, to the first order, a displacement **v** δ**t** in time δ**t** and this displacement traces out a vectorial area **ds** × **v** δ**t**. Thus, to the first order in δ**t**, we have

$$\int_{s} B.dS = \oint_{c} B.(ds \times v)\delta t = \delta t \oint_{c} (v \times B).ds$$

$$= \delta t \int_{\Sigma} \nabla \times (v \times B).dS. \qquad \textbf{(by Stokes' theorem)}$$

Hence from (35) we get, to the first order of δ**t**,

$$I_1 = \int_{\Sigma'} B.dS - \int_{\Sigma} B.dS = -\delta t \int_{\Sigma} \nabla \times (v \times B).dS$$

at time t + δ**t**. Thus, we have finally

$$\delta F = I_1 + I_2 = \delta t \int_{\Sigma} \left\{ \frac{\delta B}{\delta t} - \nabla \times (v \times B) \right\}.dS$$

which, on proceeding to the limit ad $\delta t \to 0$, gives

$$\frac{dF}{dt} = \int_{\Sigma} \left\{ \frac{\delta B}{\delta t} - \nabla \times (v \times B) \right\}.dS = 0 \qquad \textbf{[by (26)].}$$

Hence the flux F is constant; in other words, the *magnetic lines of force are 'frozen' in the fluid* provided the equation (26) holds. The fluid, therefore, flows freely along the lines of magnetic force but any motion of the fluid perpendicular to the lines of force carries them with the fluid.

Ferraro's law of isorotation

As a consequence of the Alfvén theorem, we consider the motion of a rotating conducting fluid permeated by a magnetic field. This is of great interest mainly in astrophysics. Consider a star of high electrical conductivity possessing a magnetic field and suppose that the star rotates non-uniformly about the axis OZ with angular velocity ω. By Alfvèn's theorem, the lines of force are frozen in the material and so are carried round by rotation. Hence the magnetic field of the star can only be steady if it is symmetrical about the axis of rotation and each line of force lies wholly in a surface which is symmetrical about the axis and rotates with uniform angular velocity. *This is Ferraro's law of isorotation of the magnetic field.*

A simple analytical derivation of the law is given as following. Assuming axial symmetry about the z-axis (Fig.-3) and taking cylindrical polar coordinates (r, θ, z) with origin at the centre of the star, all variables are independent of θ and t, i.e., $\frac{\delta}{\delta\theta} \equiv 0, \ \frac{\delta}{\delta t} \equiv 0$. If the star has a poloidal magnetic field, the equation $\nabla . B = 0$ implies the existence of a scalar function $\psi(r, z)$ such that

$$B = (B_r, 0, B_z) = \left(\frac{1}{r}\frac{dY}{dz}, 0, -\frac{1}{r}\frac{dY}{dr} \right) \tag{36}$$

We call ψ the *magnetic stream function.*

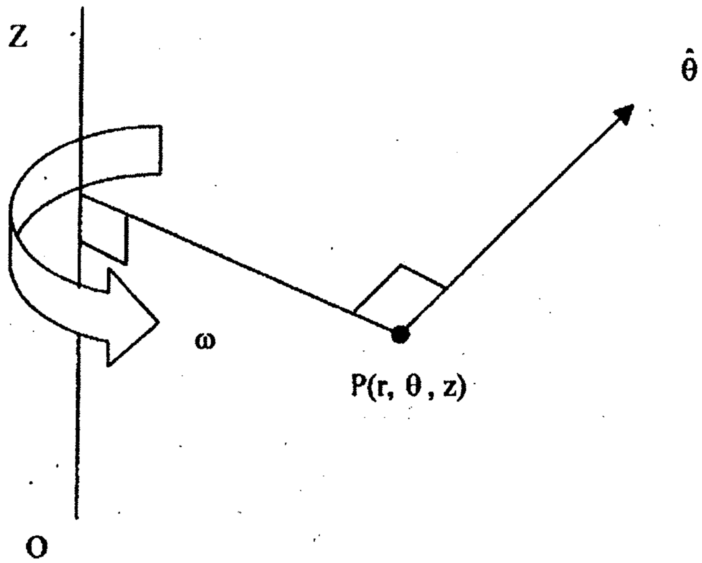


Figure -3

We assume that the fluid velocity **v** at a point P(r, θ, z) is

$$v = r\,\omega\,\hat{\theta}, \tag{37}$$

where $\omega = \omega(\mathbf{r}, \mathbf{z})$. Since **B** is independent of time, the equation (26), viz., $\frac{\delta B}{\delta t} = \nabla \times (v \times B)$ gives

$$\nabla \times (v \times B) = 0 \tag{38}$$

Now $$v \times B = r\omega\hat{\theta} \times \left(\frac{1}{r}\frac{\delta\psi}{\delta z}\hat{r} - \frac{1}{r}\frac{\delta\psi}{\delta r}\hat{z} \right) = -\omega\frac{\delta\psi}{\delta r}\hat{r} - \omega\frac{\delta\psi}{\delta z}\hat{z}$$

so that

$$\nabla \times (v \times B) = \frac{1}{1.r.1} \begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\delta}{\delta r} & \frac{\delta}{\delta \theta} & \frac{\delta}{\delta z} \\ -\omega \frac{\delta \psi}{\delta r} & 0 & -\omega \frac{\delta \psi}{\delta z} \end{vmatrix}$$

$$= \left[\frac{\delta}{\delta r}\left(\omega \frac{\delta \psi}{\delta z}\right) - \frac{\delta}{\delta z}\left(\omega \frac{\delta \psi}{\delta r}\right) \right] \hat{\theta}$$

$$= \left[\frac{\delta \omega}{\delta r}\frac{\delta \psi}{\delta z} - \frac{\delta \omega}{\delta z}\frac{\delta \psi}{\delta r} \right] \hat{\theta}$$

$$= \frac{\delta(\omega, \psi)}{\delta(r, z)} \hat{\theta}$$

giving with the help of (38)

$$\frac{\delta(\omega, \psi)}{\delta(r, z)} = 0$$

so that

$$\omega = f(\psi) \tag{39}$$

or that ω is a constant on the surface ψ = constant, that is, the angular velocity is constant over a surface generated by rotation of a line of magnetic force about the axis. Such surfaces are termed as *isorotational* or *isotachial* or *magnetic stream* surfaces. Any violation of the law will cause the lines of force to be drawn out in the direction of motion as a result or which there arises an azimuthal component of the field.

In a frame of reference rotationg with an isorotational surface, the electric field **E**′ vanishes and hence in an inertial frame of reference $\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}$. Thus the lines of force of the electrostatic field **E** are perpendicular to those of the magnetic field and, therefore, the *electrostatic potential over an isorotational surface is constant.*

4.6 MAGNETIC ENERGY

The magnetic energy is defined by

$$W_M = \frac{1}{2\mu} \int_\tau B^2 d\tau$$

where integration is taken over the volume occupied by the field. The rate of change of this expression with time is

$$\frac{\delta W_M}{\delta t} = \frac{1}{\mu}\int_\tau B.\frac{\delta B}{\delta t}d\tau$$

$$= -\int_\tau H.(\nabla \times E)d\tau \quad \left[\because \mathbf{B} = \mu\mathbf{H},\ \frac{\delta B}{\delta t} = -\delta \times E\right]$$

$$= -\int_\tau \nabla.(E \times H)d\tau - \int_\tau E.(\nabla \times H)d\tau$$

$$= -\oint_s (E \times H).dS - \int_\tau \left(\frac{j}{\sigma} - v \times B\right).jd\tau \quad [\because \mathbf{j} = \sigma(E + \mathbf{v} \times B)]$$

$$= -\oint_s (E \times H).dS - \int_\tau \frac{j^2}{\sigma}d\tau - \int_\tau v.(j \times B)d\tau \tag{40}$$

The first term on the right hand side of equation (40) represents the flow of Poynting vector over the surface Σ bounding the field. If the field extends to infinity, then this surface integral vanishes because $|E|$ and $|H|$ are at most of order (distance)$^{-2}$ in hydromagnetics. Thus in this case

$$\frac{\delta W_M}{\delta t} = -\int_\tau \frac{j^2}{\sigma}d\tau - \int_\tau v.(j \times B)\,d\tau. \tag{41}$$

The first term on the right of (41) represents the loss of magnetic energy as Joule heat at the rate of $\frac{j^2}{\sigma}$ per unit volume while the second term represents the work done by the material against the force exerted by the magnetic field on the currents during the motion.

4.7 MAGNETIC BODY FORCE

Using the equation (4), viz., $j = \nabla \times H$, the equation (30) can be written as

$$\rho\frac{Dv}{Dt} = \rho g + qE + \mu(\nabla \times H) \times H - \nabla p + \frac{1}{3}\rho\nu\nabla(\nabla.v) + \rho\nu\nabla^2 v.$$

Now

$$(\nabla \times H) \times H = -H \times \sum\left\{\hat{i} \times \frac{\delta H}{\delta x}\right\}$$

$$= -\sum\left\{\left(H.\frac{\delta H}{\delta x}\right)\hat{i}\right\} + \sum\left\{\left(H.\hat{i}\right)\frac{\delta H}{\delta x}\right\}$$

$$= -\sum\left\{\hat{i}\frac{\delta}{\delta x}\left(\frac{1}{2}H^2\right)\right\} + H.\left(\sum\hat{i}\frac{\delta}{\delta x}\right)H$$

$$= -\nabla\left(\frac{1}{2}H^2\right) + (H.\nabla)H.$$

To interpret the magnetic body force, we integrate $\mu(\nabla \times H) \times H$ throughout a volume $\nabla\tau$ bounded by a closed surface ∇S. Then from the above identity, we get

$$\int_{\Delta\tau}(\nabla \times H) \times H\, d\tau = \int_{\Delta\tau}\nabla\left(-\frac{1}{2}H^2\right)d\tau + \int_{\Delta\tau}(H.\nabla)H\, d\tau$$

$$= \int_{\Delta S}\nabla\left(-\frac{1}{2}H^2\right)\hat{n}\, d\tau + \int_{\Delta\tau}(H.\nabla)H\, d\tau. \qquad (42)$$

Now let ***a*** be an arbitrary non-zero constant vector. Then

$$a.\int_{\Delta\tau}(H.\nabla)H\, d\tau = \int_{\Delta\tau}a.\{(H.\nabla)H\}d\tau$$

$$= \int_{\Delta\tau}a.\sum\left\{\left(H.\hat{i}\right)\frac{\delta H}{\delta x}\right\}d\tau$$

$$= \int_{\Delta\tau}\sum\left\{\left(H.\hat{i}\right)\frac{\delta}{\delta x}(H.a)\right\}d\tau$$

$$= \int_{\Delta\tau}H.\left\{\sum\hat{i}\frac{d}{dx}(H.a)\right\}d\tau$$

$$= \int_{\Delta\tau}H.\nabla(H.a)\, d\tau$$

$$= \int_{\Delta\tau}\left[H.\nabla(H.a) + (\nabla.\mathbf{H})(H.a)\right]d\tau \quad [\because \nabla.\mathbf{H} = \mathbf{0}]$$

$$= \int_{\Delta\tau}\nabla.\left[H(H.a)\right]d\tau$$

$$= \int_{\Delta S} \hat{n}.[H(H.a)]\,dS$$

$$= \int_{\Delta S} (a.H)(\hat{n}.H)\,dS$$

$$= a.\int_{\Delta S} H(\hat{n}.H)\,dS.$$

Since ***a*** is arbitrary and non-zero, we must have

$$\int_{\Delta\tau} (H.\nabla)H\,d\tau = \int_{\Delta S} (\hat{n}.H)H dS. \quad (43)$$

The magnetic body force on $\Delta\tau$ is then given from (42) and (43) as

$$\mu \int_{\Delta\tau} (\nabla \times H) \times H\,d\tau = \int_{\Delta S} \left(-\frac{1}{2}\mu H^2\right)\hat{n} dS + \int_{\Delta S} \mu H(n.H) dS. \quad (44)$$

Thus the magnetic body force is equivalent to two of surface force on each surface element δS given by

$$-\frac{1}{2}\mu H^2 \hat{n}\delta S \qquad \text{and} \qquad \mu H(n.H)\delta S.$$

The surface force $\left(-\frac{1}{2}\mu H^2\right)\hat{n}\delta S$ represents of force $-\frac{1}{2}\mu H^2$ per unit area to the direction $-\hat{n}$. This is a hydrostatic pressure $\frac{1}{2}\mu H^2$.

To interpret the surface force $\mu H(\hat{n}.H)\delta S$, we note that $\hat{n}.H = H\,\cos\theta$, θ being the angle betwen $\hat{n}$ and **H** and we suppose that $\delta S'$ is the projection of δS normal to $\hat{n}$ so that $\delta S' = \delta S \cos\theta$. Then, if $H = H\hat{H}$, we have

$$\mu H(\hat{n}.H)\delta S = \mu H^2 \delta S' \hat{H}$$

representing the force μH^2 per unit area in the direction of **H**. This is a tensile force per unit area of amount μH^2 in the direction of the magnetic field.

Thus we conclude that *for a conducting fluid in a magnetic field, the magnetic body force per unit volume, viz.* $\mu(\nabla \times H) \times H$ *is equivalent to a tension* μH^2 *per unit area along the lines of force, together* *with a hydrostatic pressure* $\frac{1}{2}\mu H^2$.

4.8 SOME EXAMPLES OF MAGNETOHYDRODYNAMIC FLOWS

Example-1: *Steady laminar flow of a viscous conducting liquid between parallel walls in a transverse magnetic field (Hartmann, flow).*

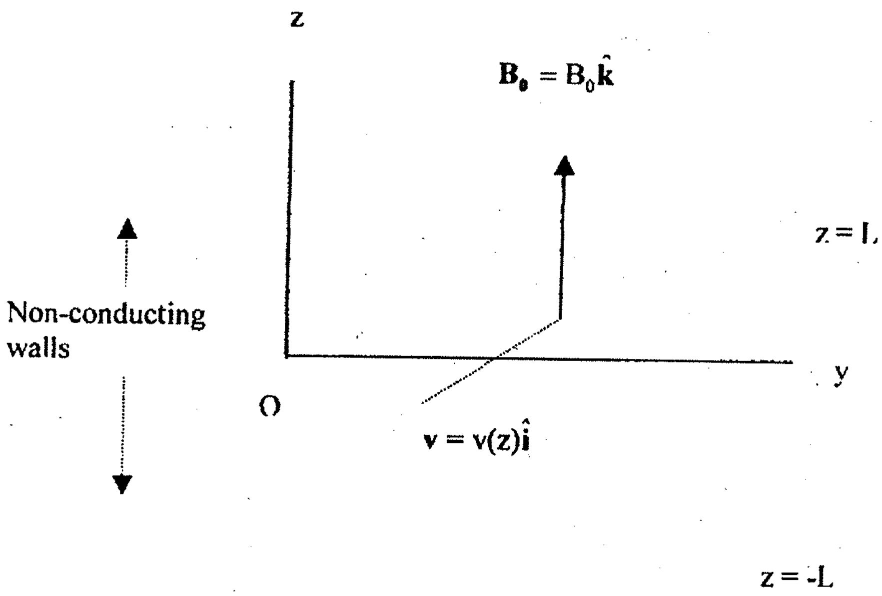


Figure -4

Let us suppose that a highly conducting viscous liquid, such as mercury, flows between two parallel non-conducting planes in a uniform transverse magnetic field perpendicular to the planes. Since the fluid particles tend to bind themselves to the magnetic field, so the field will in some way inhibit the motion of the liquid. The motion will produce tension to the lines of force which can revert to their initial positions because of the finite conductivity. Let the planes be $z = \pm L$ and the magnetic field across them is $B_0 = B_0\hat{k}$ (Fig.-4). The motion of the liquid across the field will induce electric current at right angles to the velocity $v = v(z)\hat{i}$ of the liquid and also to the applied magnetic field $B_0 = B_0\hat{k}$. The Lorentz force on the moving stream opposes the motion together with the viscous forces. The equation of continuity satisfied by the liquid velocity is $\nabla . v = 0$.

Now the motion of the liquid will produce a perturbation field intensity $b = b(z)\hat{i}$ because the liquid has the tendency to drag the lines of force in its direction of motion. Thus the total magnetic field is

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$$

which satisfies the magnetic field continuity equation (1), viz. $\nabla . B = 0$.

Let us assume the pressure p(x, z) in the liquid to be of the form

$$P(x, Z) = P_o(x) + P_1(z)$$

The first term po(x) gives rise to the pressure gradient $-\frac{dp_0}{dx}$ in the direction of motion, while the second term p_1 (z) is ascribable to hydrostatic stress.

Since we are considering steady flow, we must have $\frac{\delta B}{\delta t} = 0$ and then the magnetic induction equation (22) becomes

$$\nabla \times (v \times B) + \eta \nabla^2 B = 0, \quad \left[\eta = \frac{1}{\mu\sigma}\right] \tag{45}$$

The general equation of motion (30) for steady conditions gives

$$\rho(v.\nabla)v = -\rho g\hat{k} - \nabla(p_0 + p_1) + \frac{1}{\mu}(\nabla \times b) \times B + \nu\rho\nabla^2 v. \tag{46}$$

Noting that $\mathbf{v} \times \mathbf{B} = -\mathbf{v(z)B_o}\hat{j}$ and $\nabla \times (v \times B) = B_o v'(z)\hat{j}$, we have from (45)

$$\frac{d^2b}{dz^2} + \mu\sigma B_0 \frac{dv}{dz} = 0. \tag{47}$$

The equation (46) gives

$$0 = -\rho g\hat{k} - \frac{dp_0}{dx}\hat{i} - \frac{dp_1}{dz}\hat{k} + \frac{1}{\mu} b'(z)\hat{j} \times \left(b\hat{i} + B_0\hat{k}\right) + \nu\rho\frac{d^2v}{dz^2}\hat{i}$$

leading to

$$-\frac{dp_0}{dx} + \frac{B_0}{\mu}\frac{db}{dz} + \nu\rho\frac{d^2v}{dz^2} = 0 \tag{48}$$

and

$$\rho g + \frac{dp_1}{dz} + \frac{1}{\mu} b(z)\frac{db}{dz} = 0 \tag{49}$$

Now from (48), we get

$$\frac{B_0}{\mu}\frac{db}{dz} + \nu\rho\frac{d^2v}{dz^2} = \frac{dp_0}{dx} = -P, \quad \textbf{a constant.} \tag{50}$$

Thus for steady laminar flow, *the pressure gradient in the direction of motion remains constant throughout the liquid.* Again, integrating (49) with respect to z, we obtain

$$p_1(z) = c_1 - \rho g z - \frac{1}{2\mu}b^2. \tag{51}$$

Also, integration of (47) with respect to z leads to

$$\frac{db}{dz} + \sigma\mu B_0 v = c_2. \tag{52}$$

Now the equations $\nabla \times H = j$, i.e. $\nabla \times B = \frac{1}{\mu} j$, $j = \sigma(E + v \times B)$ and taking $\mathbf{j} = [j_1, j_2, j_3]$,

we get

$$\left.\begin{aligned} j_1 &= \sigma E_1 = 0, \\ j_2 &= \sigma(E_2 - B_0 v) = \frac{1}{\mu}\frac{db}{dz}, \\ j_3 &= \sigma E_3 = 0. \end{aligned}\right\} \tag{53 a, b, c}$$

Thus

$$\mathbf{E} = [0, E_2, 0] \tag{54}$$

where

$$E_2 = B_0 v + \frac{1}{\sigma\mu}\frac{db}{dz} = \frac{c_2}{\sigma\mu} \qquad \text{[using (52)]}.$$

$$\text{i.e.} \quad \frac{db}{dz} + \sigma\mu B_0 v = \sigma\mu E_2. \tag{55}$$

The equation (48) now gives

$$-\frac{dp_0}{dx} + \sigma B_0(E_2 - B_0 v) + \nu\rho\frac{d^2 v}{dz^2} = 0$$

$$\text{or,} \quad \nu\rho\frac{d^2 v}{dz^2} - \sigma B_0^2 v = -(P + \sigma B_0 E_2)$$

$$\text{or,} \quad \frac{d^2 v}{dz^2} - \frac{M^2}{L^2} v = -\frac{P + sB_0E_2}{\nu\rho} = \text{constant} = -\alpha(\text{say}), \tag{56}$$

where $M = B_0 L\sqrt{\frac{\sigma}{\nu\rho}}$ =*Hartmann number*, a dimensionless quantity. As there is no slipping on the boundaries, $v = 0$ on $z = \pm L$. Alos, $j_3 = 0$ at $z = \pm L$, since the walls are assumed to be non-conducting. It is readily verified that the later conditions are identically satisfied by (53c). Thesolution of (56) is given by

$$v(z)=\frac{\alpha L^2}{M^2\cosh M}\left[\cosh M-\cosh\left(\frac{M}{L}z\right)\right] \tag{57}$$

It, therefore, follows from (53b) and (57)

$$j_2=\sigma(E_2-B_0v)=\sigma\left[E_2-\frac{\alpha L^2B_0}{M^2\cosh M}\left\{\cosh M-\cosh\left(\frac{M}{L}z\right)\right\}\right].$$

If there is no externally applied current, then

$$\int_{-L}^{L} j_2dz=0$$

$$\text{or,}\quad \int_{-L}^{L}\left[\left\{E_2-\frac{\alpha L^2B_0}{M^2}\right\}+\frac{\alpha L^2B_0}{M^2\cos M}\cosh\left(\frac{M}{L}z\right)\right]dz=0$$

$$\text{or,}\quad 2L\left\{E_2-\frac{\alpha L^2B_0}{M^2}\right\}+2\frac{\alpha L^3B_0}{M^3\cosh M}\sinh M=0$$

$$\text{or,}\quad E_2-\frac{L^2B_0}{M^2}\frac{(P+\sigma B_0E_2)}{\nu\rho}\left(1-\frac{1}{M}\tanh M\right)=0$$

leading to

$$E_2=\frac{P}{\sigma B_0}(M\coth M-1), \tag{58}$$

whence (56) gives

$$v(z)=\frac{PM}{\sigma B_0^2\sinh M}\left\{\cosh M-\cosh\left(\frac{M}{L}z\right)\right\}. \tag{59}$$

Equation (55) integrates to give

$$b(z)=\frac{\mu PL}{B_0}\left[\frac{\sinh\left(\frac{M}{L}z\right)}{\sinh M}-\frac{z}{L}\right], \tag{60}$$

where we have used the condition $b=0$ at $z=\pm L$.

The mean velocity over the section is obtained from (60) as

$$\bar{v} = \frac{1}{2L}\int_{-L}^{L} v(z)dz = \frac{P(M\cosh M - \sinh M)}{\sigma B_0^2 \sinh M}$$

and, therefore,

$$v(z) = \frac{\bar{v}M\left\{\cosh M - \cosh\left(\frac{M}{L}z\right)\right\}}{M\cosh M - \sinh M}. \tag{61}$$

For a weak magnetic field, $M \approx 0$ and we have

$$v(z) = \lim_{M\to 0}\frac{\bar{v}M\left\{\cosh M - \cosh\left(\frac{M}{L}z\right)\right\}}{M\cosh M - \sinh M}$$

$$= \lim_{M\to 0}\frac{\bar{v}M\left\{\left(1-\frac{M^2}{2!}+\frac{M^4}{4!}-\ldots\ldots\right)-\left(1-\frac{M^2z^2}{2!L^2}+\frac{M^4z^4}{4!L^4}-\ldots\ldots\right)\right\}}{M\left(1-\frac{M^2}{2!}+\frac{M^4}{4!}-\ldots\ldots\right)-\left(M-\frac{M^3}{3!}+\frac{M^5}{5!}-\ldots\ldots\right)}$$

$$= \lim_{M\to 0}\frac{\bar{v}M^3\left\{\left(\frac{z^2}{2L^2}-\frac{1}{2}\right)+\frac{M^4}{4!}\left(1-\frac{z^4}{L^4}\right)+\ldots\ldots\right\}}{M^3\left\{\left(\frac{1}{3!}-\frac{1}{2}\right)+M^2\left(\frac{1}{4!}-\frac{1}{5!}\right)+\ldots\ldots\right\}}$$

$$= \frac{3}{2}\bar{v}\left\{1-\left(\frac{z}{L}\right)^2\right\}, \tag{62}$$

which is the parabolic velocity profile for viscous flow in the absence of a magnetic field. Figure 5 gives a sketch of the velocity profiles for various values of the Hartmann number.

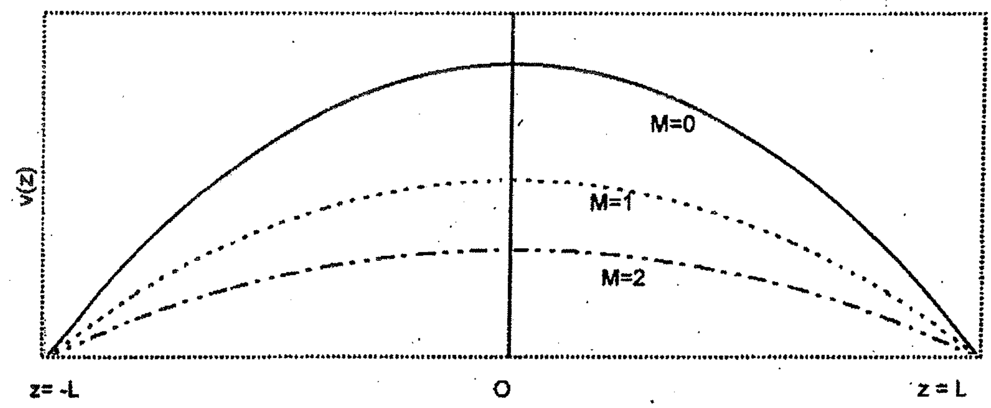


Figure 5 : Velocity distribution for different values of M

Example - 2: *Magnetohydrodynamic Couette flow*

Suppose a viscous incompressible conducting liquid of uniform density ρ is confined between the horizontal conducting plane $z = 0$ (lower) and non-conducting plane $z = L$ (upper). A uniform magnetic field $B_0\hat{k}$ acts vertically upwards. Let the lower plane $z = 0$ be held at rest while the upper one is moved horizontally with uniform velocity $V\hat{j}$, there being no pressure gradient in the liquid. We consider the motion of the liquid to be steady, the velocity of the liquid at (x, y, z) is $V(z)\hat{j}$ and the new magnetic field is $B = B_0\hat{k} + b(z)\hat{j}$. This satisfies $\nabla.B = 0$.

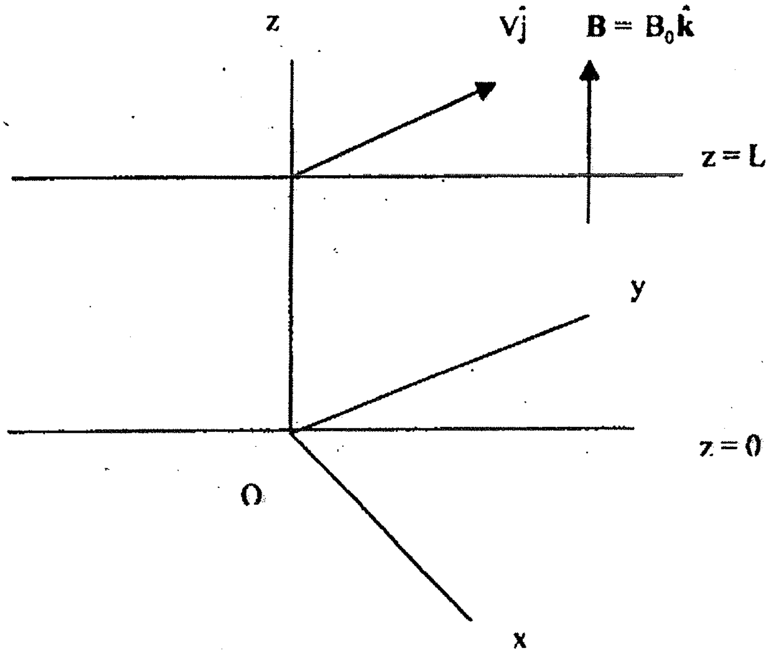


Figure - 6

Since the motion is steady $\frac{\delta B}{\delta t}=0,\ \frac{\delta v}{\delta t}=0$. The magnetic induction equation (22) then gives

$$\nabla\times(v\times B)+\eta\nabla^2 B=0 \qquad \left[\eta=\frac{1}{\mu\sigma}\right]$$

and noting that $v=v(z)\hat{j},\ B=B_0\hat{k}+b(z)\hat{j}$, the above equation becomes

$$B_0\frac{dv}{dz}+\eta\frac{d^2b}{dz^2}=0. \tag{63}$$

The general equation of motion (30) for steady condition gives

$$\rho(v.\nabla)v=-\rho g\hat{k}-\nabla p+\frac{1}{\mu}(\nabla\times b)\times B+\nu\rho\nabla^2 v$$

yielding

$$0=\frac{B_0}{\mu}\frac{db}{dz}+\nu\rho\frac{d^2v}{dz^2} \tag{64}$$

$$0=-\rho g-\frac{dp}{dz}-\frac{b}{\mu}\frac{db}{dz} \tag{65}$$

Integrating (65) with respect to z we get

$$p(z)+rgz+\frac{1}{2m}b^2(z)=\textbf{constant.} \tag{66}$$

Integration of (63) gives

$$B_0v+\eta\frac{db}{dz}=\textbf{constant}\ =\mathbf{c_1},\ \text{say.}$$

Noting that $v=0$ and $\frac{db}{dz}=0$ (since the planes are conducting) at $z=0$, we have $c_1=0$. Thus

$$\frac{db}{dz}=-\frac{B_0v}{\eta}. \tag{67}$$

Substituting this in (64) we botain

$$\nu\rho\frac{d^2v}{dz^2}-\frac{B_0^2v}{\mu\eta}=0$$

$$\text{or,}\quad \frac{d^2v}{dz^2}-\frac{M^2}{L^2}v=0,$$

where $M = B_0 L\sqrt{\frac{\sigma}{\nu\rho}}$ Hartmann number. Solution of this equation subject to the condition v = V at z = L and v = 0 at z = 0 is given by

$$v(z) = \frac{V\,\mathbf{sinh}\left(\frac{Mz}{L}\right)}{\mathbf{sinh}\,M}, \qquad (68)$$

which gives the velocity of the liquid.

Again, substituting (68) into (67) we derive

$$\frac{db}{dz} = -\frac{B_0 V\,\mathbf{sinh}\left(\frac{Mz}{L}\right)}{\eta\,\mathbf{sinh}\,M}.$$

Integrating we get

$$b(z) = \frac{B_0 VL}{M\eta\,\mathbf{sinh}\,M}\left[\mathbf{cosh}\,M - \mathbf{cosh}\left(\frac{ML}{z}\right)\right]$$

i.e., $$b(z) = \frac{\mu\sigma B_0 VL}{M\,\mathbf{sinh}\,M}\left[\mathbf{cosh}\,M - \mathbf{cosh}\left(\frac{ML}{z}\right)\right], \qquad (69)$$

where we have used the condition b = 0 at z = L. The relation (69) gives the required magnetic field.

Example - 3: *Magnetohydrodynamic* flow past a porous plate

Let us consider now the unsteady two-dimensional flow of an incompressible conducting viscous fluid past an infinite porous plate. We take the x and y axes along the surface in upward direction (i.e. opposite to the direction of gravity) and along the normal to the surface respectively. The fluid is absorbed by the surface with a periodic velocity and is in the direction parallel to the x-axis. The velocity of the fluid far away from the surface vibrates about a mean value. We also suppose that a uniform magnetic field $B_0\hat{j}$ is applied perpendicular to the direction of fluid flow and to simplify the problem it is assumed that the magnetic Reynolds number is very small so that we can neglect the induced electric and magnetic fields in the liquid (e.g. liquid sodium, blood etc.). For unsteady flow of a viscous conducting liquid bounded by an infinite vertical porous surface with periodic suction velocity and oscillatory free stream velocity U(t), we take **v** = [u, v, 0], **B** = [0, Bo, 0]. Then the equation of continuity of magnetic field $\nabla.B = 0$ is satisfied identically. The equation of continuity for liquid, viz. $\nabla.v = 0$ gives

$$\frac{\delta v}{\delta y} = 0, \qquad \text{i.e., } v = -v_0\left(1 + \varepsilon e^{i\omega t}\right), \text{ say} \qquad (70)$$

where v_0 is the cross-flow-velocity ($v_0 > 0$ represents suction and $v_0 < 0$ represents injection), ω the frequency of vibration and ε ($0 < \varepsilon < 1$) is a small parameter.

The equation of motion (30) yields

$$\rho\left(\frac{\delta u}{\delta t}+v\frac{\delta u}{\delta y}\right)=-\frac{\delta p}{\delta x}+v\rho\frac{\delta^2 u}{\delta y^2}-\sigma B_0^2 u, \tag{71}$$

$$0=-\frac{\delta p}{\delta y}. \tag{72}$$

The equation (72) shows that the pressure increase across the surface can be neglected. Hence the pressure is taken in a direction normal to the surface and may be assumed equal to that at the free stream. We may, therefore, write

$$-\frac{1}{\rho}\frac{\delta p}{\delta x}=\frac{dU}{dt}+\frac{\sigma B_0^2}{\rho}U. \tag{73}$$

Thus the equation (71) becomes

$$\rho\left[\frac{\delta u}{\delta t}-v_0\left(1+\varepsilon e^{i\omega t}\right)\frac{\delta u}{\delta y}\right]=\rho\frac{dU}{dt}+v\rho\frac{\delta^2 u}{\delta y^2}+\sigma B_0^2(U-u). \tag{74}$$

The boundary conditions are

$$\left.\begin{array}{lll} u=0 & \text{at} & y=0 \\ \text{and}\ \ u\rightarrow U(t) & \text{as } y\rightarrow\infty. \end{array}\right\} \tag{75}$$

For mathematical convenience, we introduce the following quantities:

$$\left.\begin{array}{l} y^*=\dfrac{yv_0}{v},\ t^*=\dfrac{v_0^2 t}{4v},\ u^*=\dfrac{u}{U_0},\omega^*=\dfrac{4v\omega}{v_0^2} \\ U^*(t)=\dfrac{U(t)}{U_0},\ m=\dfrac{\sigma B_0^2 v}{\rho v_0^2}(\text{magnetic field parameter}) \end{array}\right\} \tag{76}$$

where U_0 is a characteristic velocity.

Using (76), the equation (74) and boundary conditions (75) reduce to (omitting asterisks)

$$\frac{1}{4}\frac{\delta u}{\delta t}-\left(1+\varepsilon e^{i\omega t}\right)\frac{\delta u}{\delta y}=\frac{dU}{dt}+\frac{\delta^2 u}{\delta y^2}+m(U-u). \tag{77}$$

subject to

$$\left.\begin{array}{lll} u=0 & \text{at} & y=0 \\ \text{and}\ \ u\rightarrow U(t) & \text{as } y\rightarrow\infty. \end{array}\right\} \tag{78}$$

Assume that in the neighbourhood of the surface

$$\left.\begin{aligned} u(y,t) &= u_0(y)+\varepsilon u_1(y)e^{i\omega t} \\ \text{and in free stream} \qquad U(t) &= 1+\varepsilon e^{i\omega t}. \end{aligned}\right\} \tag{79}$$

Using (79) in (77) and (78) we get the following equations and boundary conditions on equating the coefficients of like powers of ε :

Zeroth order

$$u_0'' + u_0' - mu_0 = -m, \tag{80a}$$

subject to

$$\left.\begin{aligned} u = 0 \quad &\text{at} \quad y = 0 \\ \text{and} \quad u_0 \to U(t) \quad &\text{as } y \to \infty. \end{aligned}\right\} \tag{80b}$$

First order

$$u_1'' + u_1' - \left(m + i\frac{\omega}{4}\right)u_1 = -\left(m + i\frac{\omega}{4}\right) - u_0 \tag{81a}$$

$$\left.\begin{aligned} u = 0 \quad &\text{at} \quad y = 0 \\ \text{and} \quad u_0 \to U(t) \quad &\text{as } y \to \infty. \end{aligned}\right\} \tag{81b}$$

In the above, prime denotes differentiation with respect to y.

Solution of (80a) by using the conditions (80b) is

$$u_0 = 1 - e^{-m_1 y}, \tag{82a}$$

while the solution of (81a) on using (81b) is

$$u_0 = \left(1 - e^{-m_1 y}\right) + \frac{4im}{\omega}\left(e^{-m_1 y} - e\right. \tag{82b}$$

where

$$\left.\begin{aligned} m_1 &= \frac{1}{2}\left(1+\sqrt{1+4m}\right), \\ m_2 &= \frac{1}{2}\left(1+\sqrt{1+4m+i\omega}\right). \end{aligned}\right\} \tag{82c}$$

Thus the required solution for the velocity is

$$u(y,t) = u_0(y) + \varepsilon u_1(y)e^{i\omega t}, \tag{83}$$

where $u_0(y)$ and $u_1(y)$ are given by (82a) and (82b) respectively. It is to be noted that the velocity distribution in (83) is complex. However, we consider only the real part of u from the physical point of view.

4.9 MAGNETOHYDRODYNAMIC WAVES

We know according to Alfven's theorem that in a fluid of infinite electrical conductivity, the fluid particles are tied to the magnetic lines of force. Let $\mathbf{B}_0$ is the undisturbed field intensity in a tube of magnetic force of section δA so that the mass per unit length of such a tube is $m = \rho\delta A$, ρ being the density of the fluid. The magnetic forces acting on the tube are equivalent to a tension per unit area $\frac{B_0^2}{\mu}$ along the lines of force and a hydrostatic pressure $\frac{B_0^2}{2\mu}$. The latter can be balanced by a decrease in fluid pressure leaving the tubes in tension T along the lines of force, where $T = \frac{B_0^2 \delta A}{\mu}$. For incompressible fluid, the lines of force are like stretched strings in tension T and mass m per unit length. Thus if the liquid is slightly disturbed from rest, the lines of force will execute transverse vibrations, the phase velocity of the waves generated being

$$\left(\frac{\text{tension}}{\text{density}}\right)^{\frac{1}{2}} = \left(\frac{T}{m}\right)^{\frac{1}{2}} = \left(\frac{B_0^2}{\mu\rho}\right)^{\frac{1}{2}} = V_A, \text{ say.}$$

This is called *Alfve'n velocity* and the waves are known as *Alfve'n waves.*

In the case of perfectly conducting compressible fluid, longitudinal wave propagation is also feasible. The nature of the wave propagation depends on the direction of the magnetic field **B** relative to the particle velocity **v**.

When the particle velocity **v** and the direction of wave propagation are both parallel to $\mathbf{B}_0$ then since the fluid moves along the lines of force, no magnetic effects are called into play. Here the waves in the fluid will be ordinary acoustic waves, propagated with the velocity of sound c, since the motion of the particles parallel to the magnetic field will not give rise to any magnetic perturbation.

Now we suppose that the particle velocity is parallel to the direction of propagation and each is perpendicular to $\mathbf{B}_0$, the undisturbed magnetic field intensity. In such case we show that a new type of wave is excited. For a perfectly conducting liquid, we suppose that **B** is the field intensity at time **t** and **v** is the particle velocity. From (22), we have

$$\frac{\delta B}{\delta t} = \nabla \times (v \times B). \qquad (84)$$

Let $v = v(x)\hat{i}$, $B = B(x)\hat{j}$ and the motion is steady, i.e. $\frac{\delta B}{\delta t} = 0$. Then the equation (82) gives

$$-\frac{d}{dx}(vB)\hat{j} = 0$$

leading to vB = constant. Also the equation of continuity $\nabla.(\rho v) = 0$ gives ρv = constant. Hence from these two equations, we get

$$\frac{B}{\rho} = \text{constant} = \frac{B_0}{\rho_0},$$

where the suffix 0 refers to undisturbed conditions. Since the magnetic pressure is $\frac{B^2}{2\mu}$, the effective pressure is p^* where

$$p^* = p + \frac{B^2}{2\mu},$$

so that

$$\frac{dp^*}{d\rho} = \frac{dp}{d\rho} + \frac{B}{\mu}\frac{dB}{d\rho}$$

Since $\frac{dB}{d\rho} = \frac{B_0}{\rho_0} = \frac{B}{\rho}$ and $c^2 = \frac{dp}{d\rho}$, we have

$$\frac{dp^*}{d\rho} = c^2 + V_A^2,$$

where V_A is the Alfve'n wave velocity. Thus the speed of propagation is $\sqrt{(c^2 + V_A^2)}$. Such a wave is called a *magnetohydrodynamic wave.* Alfve'n waves are transverse and are propagated in conducting incompressible fluids, but magnetohydrodynamic waves are longitudinal and their propagation requires a compressible fluid of infinite electrical conductivity.

A more detailed discussion of Alfven waves

The velocity of propagation of Alfve'n wave lies along the lines of magnetic force and the particle velocity is at right angles to them. Consider an undisturbed uniform magnetic field $\mathbf{B_0} = \mathbf{B_0}\hat{\mathbf{k}}$ along the direction of the z-axis and **b** is the perturbation produced in the field due to a small disturbance so that the resultant field is

$$\mathbf{B_0} = \mathbf{B_0} + \mathbf{b}. \tag{85}$$

For conducting incompressible fluid, there is no charge accumulation at internal points. Thus, neglecting viscous effects, the equation of motion (30) gives

$$\rho\frac{Dv}{Dt}=-\nabla p+\frac{1}{\mu}(\nabla\times b)\times B. \tag{86}$$

The magnetic field continuity equation (1), viz. $\nabla.B=0$ gives

$$\nabla.b=0 \tag{87}$$

and equation of continuity for the fluid is

$$\nabla.v=0. \tag{88}$$

We suppose that the disturbances are so small that we can neglect the magnitude of the squares and product of **b** and **v**. Then

$$\frac{Dv}{Dt}=\frac{\delta v}{\delta t}+(v.\nabla)v\approx\frac{\delta v}{\delta t}$$

and

$$\begin{aligned}(\nabla\times\mathbf{b})\times B&=(\nabla\times\mathbf{b})\times(B_0+\mathbf{b})\\&\approx(\nabla\times\mathbf{b})\times B_0\\&=-B_0\times\sum\left(\hat{i}\times\frac{\delta\mathbf{b}}{\delta\mathbf{x}}\right)\\&=-\sum\left\{\hat{i}\left(B_0.\frac{\delta\mathbf{b}}{\delta\mathbf{x}}\right)\right\}+\sum\left\{(\hat{\mathbf{i}}.\mathbf{B}_0)\frac{\delta\mathbf{b}}{\delta\mathbf{x}}\right\}\\&=-\sum\left\{\hat{i}\frac{\delta}{\delta\mathbf{x}}(B_0.\mathbf{b})\right\}+B_0\frac{\delta\mathbf{b}}{\delta\mathbf{z}}\\&=-\nabla(B_0.\mathbf{b})+B_0\frac{\delta\mathbf{b}}{\delta\mathbf{z}}.\end{aligned}$$

Thus (84) approximates to

$$\nabla\left(\mathbf{p}+\frac{1}{\mu}B_0.\mathbf{b}\right)=\frac{B_0}{\mu}\frac{\delta\mathbf{b}}{\delta\mathbf{z}}-\rho\frac{\delta\mathbf{v}}{\delta\mathbf{t}}. \tag{89}$$

Taking divergence on both sides of this equation we get

$$\nabla.\nabla\left(\mathbf{p}+\frac{1}{\mu}B_0.\mathbf{b}\right)=\frac{B_0}{\mu}\frac{\delta}{\delta\mathbf{z}}(\nabla.\mathbf{v})-\rho\frac{\delta}{\delta\mathbf{t}}(\nabla.\mathbf{v})$$

or, $$\nabla^2\left(\mathbf{p}+\frac{1}{m}B_0.\mathbf{b}\right)=0\quad\textbf{[by using (87) and (88)]} \tag{90}$$

which shows that $p+\frac{1}{\mu}B_0.\mathbf{b}$ is a harmonic function. Here two cases arise: (i) the liquid is of infinite extent and (ii) the liquid is of finite eaten.

Case I : *Liquid is of infinite extent*

The solution of the equation (90) regular at all points including at infinity is

$$\mathbf{p}+\frac{1}{\mu}B_0.\mathbf{b} = \textbf{constant.}$$

Then the equation (87) gives

$$\rho\frac{\delta\mathbf{v}}{\delta\mathbf{t}}=\frac{B_0}{\mu}\frac{\delta\mathbf{b}}{\delta\mathbf{z}}. \qquad (91)$$

Now the equation (84) governing magnetic field variations reduces to the first order

$$\frac{\delta\mathbf{b}}{\delta\mathbf{t}}=\nabla\times(\mathbf{v}\times B_0)$$

$$=B_0\nabla\times(\mathbf{v}\times\hat{\mathbf{k}})$$

$$=B_0\left[(\nabla.\hat{\mathbf{k}})\mathbf{v}-(\nabla.\mathbf{v})\hat{\mathbf{k}}\right]$$

$$\text{i.e.,}\quad \frac{\delta\mathbf{b}}{\delta\mathbf{t}}=B_0\frac{\delta\mathbf{v}}{\delta\mathbf{z}}. \quad [\textbf{using (88)}] \qquad (92)$$

The equations (91) and (92) then lead to the wave equations

$$\left(\frac{\delta^2}{\delta t^2}-V_A^2\frac{\delta^2}{\delta z^2}\right)(\mathbf{b},\ \mathbf{v})=0 \qquad (93)$$

for **b** and **v**, where

$$V_A^2=\frac{B_0^2}{\mu\rho}$$

Thus the magnetic field and the fluid particles propagate as transverse waves along the lines of force with Alfve'n velocity V_A.

If a wave travels along the positive direction of the z-axis, then the solutions of the equations (93) are

$$\left.\begin{aligned}\mathbf{b}&=\mathbf{b}(\mathbf{z}-V_A\mathbf{t}),\\ \mathbf{v}&=\mathbf{v}(\mathbf{z}-V_A\mathbf{t}).\end{aligned}\right\} \qquad (94)$$

It follows from these equations that

$$\frac{\delta \mathbf{b}}{\delta \mathbf{t}} = -V_A \frac{\delta \mathbf{b}}{\delta \mathbf{z}}$$

so that from (92) we get

$$\frac{\delta \mathbf{b}}{\delta \mathbf{z}} = -\frac{B_0}{V_A}\frac{\delta \mathbf{v}}{\delta \mathbf{z}} = -(\mu\rho)^{\frac{1}{2}}\frac{\delta \mathbf{v}}{\delta \mathbf{z}}$$

which is satisfied by taking

$$\mathbf{b} = -(\mu\rho)^{\frac{1}{2}}\mathbf{v}. \tag{95}$$

Similarly, for a wave travelling along the negative direction of the z-axis we have

$$\mathbf{b} = (\mu\rho)^{\frac{1}{2}}\mathbf{v}. \tag{96}$$

The relations (95) and (96) are due to Wale'n. It follows from these equations that $\frac{\mathbf{b}^2}{2\mu} = \frac{1}{2}\rho v^2$. Thus the magnetic energy of the perturbed field is equal to the kinetic energy of the motion.

Case II : *Liquid of finite extent*

In this case $p + \frac{1}{\mu}B_0 . \mathbf{b}$ may not be constant and boundary reflections and transmissions would occur. Hence the solution of (90) would be complicated. However, taking curl on both sides of (89) we get

$$\nabla \times \nabla\left(\mathbf{P} + \frac{1}{\mu}B_0.\mathbf{b}\right) = \frac{B_0}{\mu}\frac{\delta}{\delta \mathbf{z}}(\nabla \times \mathbf{b}) - \rho\frac{\delta}{\delta \mathbf{t}}(\nabla \times \mathbf{v})$$

or,
$$0 = \frac{B_0}{\mu}\frac{\delta \mathbf{j}}{\delta \mathbf{z}} - r\frac{\delta \zeta}{\delta \mathbf{t}}. \tag{97}$$

where $\zeta = \nabla \times \mathbf{v}$ is the vorticity vector. Also from the equation (92) we get on taking curl

$$\frac{\delta}{\delta \mathbf{t}}(\nabla \times \mathbf{b}) = B_0 \frac{\delta}{\delta \mathbf{z}}(\nabla \times \mathbf{v})$$

or,
$$\frac{\delta j}{\delta t} = B_0 \frac{\delta \zeta}{\delta z}. \tag{98}$$

Equation (97) and (98) then lead to the wave equations

$$\left(\frac{\delta^2}{\delta t^2} - V_A^2 \frac{\delta^2}{\delta z^2}\right)(j, \zeta) = 0. \tag{99}$$

Thus the *vorticity vector ζ and the current density* **j** *propagate with speed* V_A *along the lines of force.*

Reflection and transmission of Alfven waves

Suppose a train of Alfve'n waves of finite amplitude be normally incident on the horizontal plane of two liquids permeated by a uniform vertically magnetic field. We take the z-axis vertically upwards so that the acceleration due to gravity, assumed to be uniform, has resolutes (0, 0, -g). Let the uniform magnetic field has resolutes $(0, 0, B_0)$ and all variables depend on z and t only. Also the fluid velocity is v = (0, v, 0), i.e. the motion of the fluid particles is purely horizontal. The equation of continuity $\frac{\delta\rho}{\delta t} = 0$ shows that the density ρ depends on z only and is unaffected by the passage of the waves. Further the disturbed magnetic field b = (0, b, 0) and the equations satisfied by v, b and the pressure p are

$$\left.\begin{aligned} \frac{\delta b}{\delta t} &= B_0 \frac{\delta v}{\delta z}, \\ \rho\frac{\delta v}{\delta t} &= \frac{B_0}{\mu}\frac{\delta b}{\delta z}, \end{aligned}\right\} \tag{100}$$

$$\frac{\delta b}{\delta t} + \frac{b}{\mu}\frac{\delta b}{\delta z} + \rho g = 0. \tag{101}$$

Consider first the propagation of waves in a liquid and assume harmonic vibrations of period $\frac{2\pi}{\lambda}$. Then the solutions of (100) are

$$\left.\begin{aligned} b &= a\exp\left\{i\lambda\left(t \pm \frac{z}{V_A}\right)\right\}, \\ v &= \pm(\mu\rho)^{\frac{1}{2}} a\exp\left\{i\lambda\left(t \pm \frac{z}{V_A}\right)\right\}, \end{aligned}\right\} \tag{102}$$

where *a* is the constant amplitude of the waves and V_A is the Alfven velocity. The equation (101) integrates to give

$$p + \frac{b^2}{2\mu} + \rho g z = F(t), \tag{103}$$

where F(t) is a function of t only.

Let us now consider the reflection and refraction of such a train of waves travelling upwards in a liquid of density ρ_1, occupying the region z < 0 at the plane interface z = 0 with another liquid of density ρ_2 occupying the region z > 0. We use the suffixes i, r and t to denote the incident, reflected and transmitted waves respectively. We then have

$$\left.\begin{aligned} b_i &= a_i \exp\left\{i\lambda\left(t-\frac{z}{V_1}\right)\right\}, v_i = -(\mu\rho_1)^{\frac{1}{2}} a_i \exp\left\{i\lambda\left(t-\frac{z}{V_1}\right)\right\}, \\ b_r &= a_r \exp\left\{i\lambda\left(t-\frac{z}{V_1}\right)\right\}, v_r = -(\mu\rho_1)^{\frac{1}{2}} a_r \exp\left\{i\lambda\left(t-\frac{z}{V_1}\right)\right\}, \\ b_t &= a_t \exp\left\{i\lambda\left(t-\frac{z}{V_1}\right)\right\}, v_t = -(\mu\rho_1)^{\frac{1}{2}} a_t \exp\left\{i\lambda\left(t-\frac{z}{V_1}\right)\right\}, \end{aligned}\right\} \quad (104)$$

where a_i, a_r and a_t are constant amplitudes of the magnetic fields of the waves and V_1, and V_2 are Alfve'n velocities in the two fluids given by

$$\left.\begin{aligned} V_1^2 &= \frac{B_0^2}{\mu\rho_1}, \\ V_2^2 &= \frac{B_0^2}{\mu\rho_2}. \end{aligned}\right\} \quad (105)$$

The linearity of b and v in the equations (100) shows that

$$\left.\begin{aligned} b &= b_1 = b_i + b_r, \quad v = v_1 = v_i + v_r \quad \text{for } z<0, \\ b &= b_2 = b_t, \quad v = v_2 = v_t \quad \text{for } z>0. \end{aligned}\right\} \quad (106)$$

We use the suffix 1 to denote the quantities for lower medium while the suffix 2 is used for the upper medium.

The boundary conditions at the plane of separation provide us the equations connecting the amplitude coefficients a_i, a_r and a_t and these boundary conditions are the continuity of (i) pressure, (ii) normal resolute of the fluid velocity, (iii) magnetic field and (iv) tangential resolute of the electric field. The first three conditions give

$$p_1 = p_2, \quad (107)$$

$$\hat{\mathbf{k}}.\mathbf{v}_1 = \hat{\mathbf{k}}.\mathbf{v}_2, \quad (108)$$

$$\mathbf{B}_1 = \mathbf{B}_2 \quad (109)$$

at $z=0$, $\hat{\mathbf{k}}$ being the unit vector along OZ. The condition (iv) gives

$$\mathbf{E}_1 - (\hat{\mathbf{k}}.\mathbf{E}_1)\hat{\mathbf{k}} = \mathbf{E}_2 - (\hat{\mathbf{k}}.\mathbf{E}_2)\hat{\mathbf{k}},$$

$$\text{i.e.,} \quad \hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{E}_1) = \hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{E}_2) \quad (110)$$

at $z=0$. Using the relation $\mathbf{E} = -\mathbf{v}\times\mathbf{B}$ and conditions (108) and (109) we have from (1 10)

$$(\hat{\mathbf{k}}.\mathbf{B}_1)\hat{\mathbf{k}}\times(\mathbf{v}_1 - \mathbf{v}_2) = 0 \quad (111)$$

at $z=0$. Since $\hat{\mathbf{k}}.\mathbf{B}_1 \neq 0$, so $\hat{\mathbf{k}}\times\mathbf{v}_1 = \hat{\mathbf{k}}\times\mathbf{v}_2$ at the plane $z=0$. Coupled with the condition (108), this implies

$v_1 = v_2$ at z = 0. Thus the boundary conditions at z = 0 reduce to

$$\mathbf{b}_1 = \mathbf{b}_2, \mathbf{v}_1 - \mathbf{v}_2 \qquad (112)$$

provided that arbitrary functions $\mathbf{F}_1(\mathbf{t}) \equiv \mathbf{F}_2(\mathbf{t})$ which arise in the pressure integrals.

Using these boundary conditions and equations (104) and (106) we have

$$a_i + a_r = a_t,$$

$$\rho_1^{\frac{1}{2}}(-\mathbf{a}_i + \mathbf{a}_r) = -\rho_2^{\frac{1}{2}}\mathbf{a}_t$$

so that

$$\mathbf{a}_r = \frac{\left(\sqrt{\rho_2} - \sqrt{\rho_1}\right)\mathbf{a}_i}{\left(\sqrt{\rho_2} + \sqrt{\rho_1}\right)}, \quad \mathbf{a}_t = \frac{\left(2\sqrt{\rho_2}\right)\mathbf{a}_i}{\left(\sqrt{\rho_2} + \sqrt{\rho_1}\right)} \qquad (113)$$

which give the amplitudes of the reflected and transmitted waves in terms of the amplitude of the incident wave.

The conditions at a free surface may be obtained from (113) by putting $\rho_2 = 0$ so that $a_r = -a_i$, $a_t = 0$. It, therefore, follows from (104) that a wave is reflected without change of phase in the velocity but there is a reversal of phase in the magnetic field. For a rigid surface, i.e. when $\rho_2 \to \infty$, these conditions are reversed because then $a_r = -a_i$, $a_t = 2a_i$.

5. UNIT SUMMARY

In our discussions in Section-4, we have derived the equations of motion of a conducting fluid and the equations satiesfied by the magnetic field with the help of Maxwell's electromagnetic field equations. The equations are then applied to consider some specific problems. The wave (Alfve'n wave) transmitted in conducting fluid has also been considered.

6. EXERCISES

1. Starting from the induction equation

$$\frac{\delta \mathrm{B}}{\delta t} = \nabla \times (\mathbf{v} \times \mathrm{B})$$

for an infinitely conducting fluid, show that the magnetic flux across any closed contour moving with the fluid remains constant. Interpret this result in terms of the motion of the lines of force.

2. An infinitely conducting fluid moves with velocity **v** in two dimensions relative to rectangular Cartesian axes x, y, z so that $\mathbf{v} = [\mathbf{v}_x, \mathbf{v}_y, \mathbf{0}]$. If the vector potential **A** of the magnetic field $\mathrm{B}(\mathrm{B} = \nabla \times \mathbf{A})$ is such that A = [0, 0, A] and all variables are independent of **z**, then show that

(i) $\frac{\delta A}{\delta t} = v \times B,$

(ii) $\frac{\delta A}{\delta t} = -(v.\nabla)A,$

and (iii) $\frac{DA}{Dt} = 0.$

Show that the last result is a special case of that in Exercise – 1.

3. Show that the vector potential **A** of the magnetic field **B** in fluid moving with velocity **v** may be defined so that

$$\frac{\delta A}{\delta t}.(\nabla \times A) = 0.$$

Deduce that if the vector potential is maintained constant $\left(\frac{\delta A}{\delta t} = 0\right)$ over a fixed surface **S** enclosed in a volume of infinitely conducting liquid τ,

$$I = \int_\tau A.(\nabla \times A)\, d\tau = \text{constant}.$$

Show also that the stationary values of the magnetic energy subject to the constraint **I** = constant with $\delta A = 0$ on **S** correspond to the force-free fields $\nabla \times B = \alpha B$, where α is constant.

4. Show that if a conducting liquid of uniform density ρ moves in a magnetic field of intensity **B** and in a field of conservative body force of potential Ω per unit volume (so that $F = -\nabla\Omega$) then when displacement currents are negligible and there is no volume distribution of electric charge, the equation of steady motion is

$$\rho(v.\nabla)v = -\nabla(p + \Omega) + \frac{1}{\mu}(\nabla \times B) \times B + \rho\nu\nabla^2 v.$$

Flow for which **B** and **v** are always parallel so that $B = \lambda v$ is called parallel flow. Show that for such flow the above equation becomes

$$\left(\rho - \frac{\lambda^2}{\mu}\right)(v.\nabla)v = -\nabla\left(p + \Omega + \frac{\lambda^2}{\mu}v^2\right) + \rho\nu\nabla^2 v.$$

5. If ψ represents the velocity stream function in an incompressible and Ψ represents the magnetic stream function and the flow is two-dimensional so that the Cartesian components of **v** and **B** are given by

$$v = \left[-\frac{\delta\psi}{\delta y}, \frac{\delta\psi}{\delta x}, 0\right], \qquad B = \left[-\frac{\delta\Psi}{\delta y}, \frac{\delta\Psi}{\delta x}, 0\right],$$

Show that the induction equation reduces to

$$\frac{\delta\Psi}{\delta t} = \frac{\delta(\Psi, \psi)}{\delta(x, y)}.$$

Show further that $\frac{\delta\Psi}{\delta t} = 0$ and deduce that the lines of force move with the fluid. Show also that when conditions are steady, $\Psi = f(\psi)$ so that the streamlines and the magnetic lines of force coincide.

7. SUGGESTED FURTHER READINGS

1. V.C.A. Ferraro and C. Plumpton : *An Introduction to Magneto-Fluid Mechanics,* Clarendon Press, Oxford.
2. F. Chorlton: *Textbook of Fluid Dynamics,* ELBS & Van Nostrand Reinhold Company, London.
3. A. Jeffrey : *Magnetohydrodynamics,* Oliver and Boyd, London.
4. T. G. Cowling: *Magnetohydrodynamics,* Interscience.
5. J.A. Shercliff: *A Textbook of Magnetohydrodynamics,* Pergamon Press.

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"Learner's Feed-back"

After going through the Modules/ Units please answer the following questionnaire.
Cut the portion and send the same to the Directorate.

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