DISTANCE LEARNING MATERIAL

VIDYASAGAR UNIVERSITY

DIRECTORATE OF DISTANCE EDUCATION

MIDNAPORE- 721 102

M. SC. IN APPLIED MATHEMATICS
WITH OCEANOLOGY & COMPUTOR PROGRAMMING

PART - I

Paper : IV

Module No. : 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47 & 48

CONTENT

PART – I

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-IV Group-A

Module No. - 37
Principle of Mechanics
(PRELIMINARIES)

CONTENT :

Classical mechanics which was primarily supposed as the study of motions of physical objects such as motion of celestial bodies, is now considered as the part of mechanics deling with the objects neither too big so that there exists a close agreement between theory and experiment, nor too small interacting object so that systems are considered on an atomic scale.

Classical mechanics may be classified into three sub-sections:

(i) **Kinematics:** It deals with the all possible motions of material systems without reference to the agency which causes motion.

(ii) **Dynamics:** It deals with the motions to the forces associated with it along with the properties of moving bodies. As such this subsection introduces the concept of force and explains possibility of most favourable types of motion which take place under the action of given forces.

(iii) **Statics:** It deals with the system of forces which actually give no motion to the system. For equilibrium of the system the net effect of the forces should be zero.

Objectives

- Mechanics of a particle
- Kinetic and potential energies
- Conservative force
- Conservation laws
- Mechanics of a system of particles
- Kinetic and potential energies for a system of particles
- Conservation laws for a system of particles
- Virial theorem
- Generalised coordinates and constraints
- Principle of virtual works
- D'Alembert's principle
- Generalised forces
- Exercise.

1.1 Mechanics of a Particle

Let $\vec{r}$ be the position vector of a particle with respect to some given origin and $\vec{v}$ be its velocity at time t then

$$\vec{v} = \frac{d\vec{r}}{dt}. \tag{1.1}$$

The **linear momentum** $\vec{p}$ of the particle is defined as the product of the mass and its velocity, i.e.,

$$\vec{p} = m\vec{v}. \tag{1.2}$$

The Newton's second law states that there exist a frame of reference in which the motion of the particle is described by the following well known equation

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}). \tag{1.3}$$

It also known that the equation (1.3) does not valid in all reference frames. A reference frame in which equation (1.3) valid is called an **inertial** or **Galilean system**. This frame, even within classical mechanics, is an idealization. In practice, it is usually feasible to set up a coordinate system that comes as close to the desired properties as may be required. In the 'laboratory system' a reference frame is fixed in the Earth and it is sufficient consideration to an inertial system. But, in astronomical study it may be required to construct an inertial system by reference to the most distant galaxies.

Work

Suppose a particle is displaced at a distance $\vec{r}$ due to the application of the force $\vec{F}$. Then the work done dW by the force on the particle due to the displacement $d\vec{r}$ is

$$dW = \vec{F}.d\vec{r}. \tag{1.4}$$

If the particle is displaced from point P_1 to P_2 with position vectors $\vec{r}_1$ and $\vec{r}_2$ respectively along any path, then the work done by the force $\vec{F}$ acting on the particle is given by

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}.d\vec{r}. \tag{1.5}$$

Kinetic energy

The kinetic energy of a particle is defined as a scalar quantity equal to half the product of the mass of the particle and the square of its velocity. That is, if a particle of mass m is moving with velocity $\vec{v}$, then its kinetic energy is given by

$$T = \frac{1}{2}mv^2. \tag{1.6}$$

Let $\vec{v}$ be the velocity of the particle of mass m acted upon by a force $\vec{F}$. By Newton's second law of motion, we have

$$\vec{F} = m\frac{d\vec{v}}{dt}.$$

Multiplying both sides scalarly by $\vec{v}$, we get

$$\vec{F}.\vec{v} = m\vec{v}.\frac{d\vec{v}}{dt}$$

or $$\vec{F}.\vec{v} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right)$$

or $$\frac{dW}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right).$$

If v_1 and v_2 are the initial and final velocities at times t_1 and t_2 respectively, then the work done by the applied forces on the particle is given by

$$\begin{aligned} W &= \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{1}{2}mv^2\right)dt \\ &= \int_{v_1}^{v_2} d\left(\frac{1}{2}mv^2\right) \\ &= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \end{aligned}$$

= $T_2 - T_1$, where T_1 and T_2 are the initial and the final kinetic energies of the particle

= gain in K.E.

Thus the gain in kinetic energy of a moving system is equal to the work on the system by the applied forces.

Conservative forces

If the force filed is such that the work W is the same for any physical possible path between initial and final points, then the force (and the system) is said to be **conservative**.

An alternative description of a conservative system is obtained by imagining the particle being taken from point P_1 to the point P_2 by one poissible path and then being returned to point P_1 by another path. The independence of the work done W_{12} from the point P_1 to the point P_2 on the particular path implies that the work done around such a closed circuit is zero, that is,

$$\oint \vec{F}.d\vec{s} = 0. \tag{1.7}$$

Physically, it is clear that a system cannot be conservative if friction or other dissipation forces are present, because $\vec{F}.d\vec{s}$ due to friction is always positive and the integral cannot be vanish.

The necessary and sufficient condition that the work W_{12} be independent of the path of the physical path taken by the particle is that $\vec{F}$ be the gradient of some scalar function ϕ of position, i.e.,

$$\vec{F} = \nabla\phi. \tag{1.8}$$

The function ϕ is called the scalar potential.

Again, the necessary and sufficient condition for the force $\vec{F}$ to be conservative if $curl\,\vec{F} = 0$. Thus, if $\vec{F}$ is conservative then $\vec{F}$ can be expressed as $grad\,\phi = \vec{F}$ where ϕ is a scalar potential.

Potential energy

In a conservative field of force $\vec{F}$, the potential energy (P.E.) V of a particle situated at P is defined as the work done by $\vec{F}$ in moving the particle from P to some fixed position O. Thus the P.E. V_p is given by

$$V_P = \int_{PO} \vec{F}.d\vec{r}.$$

Since, $\vec{F}$ is conservative, $\vec{F} = grad\,\phi$

$$V_P = \int_{PO} \vec{F}.d\vec{r} = \int_{PO} grad\,\phi.d\vec{r} = \int_{PO} \frac{\partial\phi}{\partial r} dr = \phi_0 - \phi_P$$

Thus $V = -\phi + \phi_0$.

Now, O can be any fixed point the potential energy at any point V is unique to within an additive constant. It may be noted that the P.E. is a function of position only, i.e., $V = V(\vec{r})$.

Thus in conservative system

$$\vec{F} = grad\,\phi = grad(-V + \phi_0) = -grad\,V. \tag{1.9}$$

where V is potential energy.

Many important conclusion of mechanics can be expressed in the form of conservation theorems. Some of them are presented below.

Theorem 1.1 (First conservation law) *If total force $\vec{F}$ acting on the particle is zero then the linear momentum $\vec{p}$ is conserved.*

Proof. Conservation of linear momentum follows from Newton's second law of motion in the form

$$\frac{d}{dt}(m\vec{v}) = \frac{d\vec{p}}{dt} = \vec{F}, \text{ where } \vec{p} = m\vec{v}.$$

Thus if the total force $\vec{F}$ is zero, then $\frac{d\vec{p}}{dt} = 0$, i.e., $\vec{p} = $ constant.

Hence the linear momentum is conserved.

Corollary 1.1 *If the total forces acting in any given direction is zero, the linear momentum in that direction is conserved.*

Theorem 1.2 (Conservation theorem on angular momentum) *The total torque $\vec{N}$ acting on the particle is zero, then the angular momentum $\vec{L}$ is conserved.*

Proof. The angular momentum $\vec{L}$ with respect to some reference origin O is defined as $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{r}$ is the position vector of a particle with respect to O and $\vec{p}$ is linear momentum.

The torque, i.e., the moment of the applied force on the particle about O defined as $\vec{N} = \vec{r} \times \vec{F}$.

Now,

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{v} \times \vec{p} + \vec{r} \times \vec{F} = \vec{v} \times (m\vec{v}) + \vec{r} \times \vec{F}$$

$$= \vec{r} \times \vec{F} = \vec{N}$$

If the torque is zero i.e., $\vec{N} = 0$ then $\frac{d\vec{L}}{dt} = 0$, i.e. $\vec{L} =$ constant.

Hence angular momentum is conserved.

Theorem 1.3 (Conservation theorem of energy) *If the forces acting on a particle are conservative then the total energy of the particle is conserved, i.e. $T + V =$ constant.*

Proof. The K.E. $T = \frac{1}{2} m\vec{v}^2 = \frac{1}{2} m\dot{\vec{r}}.\dot{\vec{r}}$.

Therefore,

$$\frac{dT}{dt} = \frac{1}{2} m \frac{d}{dt}\left(\dot{\vec{r}}.\dot{\vec{r}}\right) = \frac{1}{2}\left(\ddot{\vec{r}}.\dot{\vec{r}} + \dot{\vec{r}}.\ddot{\vec{r}}\right) = m\ddot{\vec{r}}.\dot{\vec{r}}$$

$$= \vec{F}.\dot{\vec{r}} = grad\,\phi.\dot{\vec{r}}$$

$$= \left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial \phi}\vec{k}\right)\cdot\left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right)$$

$$= \frac{d\phi}{dt} = \frac{d}{dt}(-V + \phi_0) = -\frac{dV}{dt}$$

where V is P.E. Therefore, $\frac{d}{dt}(T+V)=0$ i.e., $T+V=$ constant. Hence total energy is conserved.

1.2 Mechanics of a System of Particles

Let us consider a system of particles with masses $m_1, m_2, \ldots, m_n$ and position vectors $\vec{r}_1, \vec{r}_2, \ldots \vec{r}_n$.. For the system of particles, we have to consider the external forces acting on the particles due to some outside sources and the internal forces on, say, some particle i due to all other particles in the system. Then by Newton's second law

$$\frac{d\vec{p}_i}{dt} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \vec{F}_{ji} + \vec{F}_i^{(e)}, \tag{1.10}$$

where $F_i^{(e)}$ is the external force applied on the ith particle and $\vec{F}_{ji}$ is the internal force on the ith particle due to the jth particle.

Now summation is taken over all particles,

$$\sum_{i=1}^{n} \frac{d\vec{p}_i}{dt} = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq 1}}^{n} \vec{F}_{ji} + \sum_{i=1}^{n} \vec{F}_i^{(e)}. \tag{1.11}$$

The term, $\sum_{i=1}^{n} \vec{F}_i^{(e)}$ represents the total external force $\vec{F}^{(e)}$ applied on the system, while the first term vanishes, since the law of action and reaction states that each pair $\vec{F}_{ji} + \vec{F}_{ij}$ is zero.

The left hand side is

$$\sum_{i=1}^{n} \frac{d\vec{p}_i}{dt} = \sum_{i=1}^{n} \frac{d}{dt}\left(m_i \dot{\vec{r}}_i\right) = \frac{d^2}{dt^2}\left(\sum_{i=1}^{n} m_i \vec{r}_i\right).$$

To further reduce this expression we define a vector $\vec{R}$ as the average of the radii vectors of the particles, weighted in proportional to their mass, i.e.,

$$\vec{R} = \frac{\sum m_i \vec{r}_i}{\sum m_i} = \frac{\sum m_i \vec{r}_i}{M}, \tag{1.12}$$

where M is the total mass of the system. The vector $\vec{R}$ defines a point known as the **center of mass**, or more loosely as the center of gravity, of the system. Then equation (1.11) reduces to

$$\frac{d^2}{dt^2}\left(M\vec{R}\right)=\vec{F}^{(e)} \quad \text{or} \quad M\frac{d^2\vec{R}}{dt^2}=\vec{F}^{(e)} \tag{1.13}$$

This states that the center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass. Purely internal force, if they obey Newton's third law, therefore have no effect on the motion of the center of mass.

Now, the total linear momentum is

$$\vec{P}=\sum_{i=1}^{n} m_i\vec{v}_i=\sum_{i=1}^{n} m_i\dot{\vec{r}}_i=\frac{d}{dt}\left(\sum_{i=1}^{n} m_i\vec{r}_i\right)=\frac{d}{dt}\left(M\vec{R}\right)=M\frac{d\vec{R}}{dt}=M\vec{V}, \tag{1.14}$$

where $\vec{V}$ is the velocity of the center of mass.

Hence the total linear momentum of the system is the total mass of the system times the velocity of the centre of mass. This states that the total linear momentum of the system is the same as if the entire mass where concentrated at the centre of mass and moving with it.

Consequently, the equation of motion for the center of mass (1.14) conclude the following result.

Theorem 1.4 (Conservation theorem for linear momentum). *If the total external force acting on a system of particles is zero, the total linear momentum is conserved.*

Angular momentum for a system of particles

Let $\vec{L}_i$ be the angular momentum of the *i*th particle. Then $\vec{L}_i=\vec{r}_i\times\vec{p}_i$.

Then the total angular momentul $\vec{L}$ of the system is

$$\vec{L}=\sum_{i=1}^{n}\vec{L}_i=\sum_{i=1}^{n}(\vec{r}_i\times\vec{p}_i)$$

Then

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \frac{d}{dt}\left\{\sum_{i=1}^{n}(\vec{r}_i\times\vec{p}_i)\right\}\\ &= \sum_{i=1}^{n}\left(\frac{d\vec{r}_i}{dt}\times\vec{p}_i\right)+\sum_{i=1}^{n}\left(\vec{r}_i\times\frac{d\vec{p}_i}{dt}\right)\\ &= 0+\sum_{i=1}^{n}\left(\vec{r}_i\times\frac{d\vec{p}_i}{dt}\right)\end{aligned}$$

$$= \sum_{i=1}^{n}\left[\vec{r}_i \times \left(\vec{F}_i^{(e)} + \sum_{j=i}^{n} \vec{F}_{ji}\right)\right]$$

$$= \sum_{i=1}^{n}\left(\vec{r}_i \times \vec{F}_i^{(e)}\right) + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n}\left(\vec{r}_i \times \vec{F}_{ji}\right)$$

The second term can be considered as a sum of the pairs, i.e., $\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij} = \vec{r}_i \times \vec{F}_{ji} - \vec{r}_j \times \vec{F}_{ji} = (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji}$.

But, $\vec{r}_i - \vec{r}_j$ is identical with the vector $\vec{F}_{ij}$ from j to i and therefore, $\vec{r}_{ij} \times \vec{F}_{ij} = 0$. Since $\vec{F}_{ij}$ is along the line between the two particles, i.e., $\vec{r}_{ij}$ and hence the sum vanish. That is,

$$\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} \vec{r}_i \times \vec{F}_{ji} = 0$$

Hence

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i^{(e)} = \sum_{i=1}^{n} \vec{N}_i^{(e)} = N^{(e)} \tag{1.15}$$

which is the total applied torque of the system of particles.

Theorem 1.5 (Conservation theorem of angular momentum for a system of particles). *If the total applied torque for a system of particles is zero then the total angular momentum for the system is conserved.*

Proof. From the above relation, we have $\frac{d\vec{L}}{dt} = \vec{N}^{(e)}$.

If $\vec{N}^{(e)}$ is zero then $\frac{d\vec{L}}{dt} = 0$. Hence $\vec{L}$ = constant.

Hence the total angular momentum is conserved.

K.E. and P.E. for a system of particles

Let W_i be the work done by the external force $\vec{F}_i$ acting to ith particle in displacing it form position P_1 to P_2. Then

$$W_i = \left[\frac{1}{2} m_i v_i^2\right]_{P_1}^{P_2} = \left[T_i\right]_{P_1}^{P_2}$$

where T_i is the kinetic energy of ith particle.

Now summing over all the particles of the system, we have

$$\sum W_i = \sum [T_i]_{P_1}^{P_2} \text{ or } W_{12} = T_2 - T_1 \tag{1.16}$$

where W_{12} is the total work done by the external force, and T_2 and T_2 are the final and initial values of total K.E. of the system.

For a conservative system, the force $\vec{F}_i$ acting on ith particle is expressed as the gradient of some scalar function, i.e.,

$$\vec{F}_i = -\nabla V_i$$

where V_i is the P.E. of ith particle.

Considering directional derivative, we get

$$\nabla V_i = \frac{\partial V_i}{\partial S_i}.$$

Therefore,

$$\vec{F}_i = -\frac{\partial V_i}{\partial S_i} \tag{1.17}$$

Work done by the external force $\vec{F}_i$ acting on the ith particle in displacing it form position P_1 to P_2 is given by

$$W_i = \int_{P_1}^{P_2} \vec{F}_i . dS_i = -\int_{P_1}^{P_2} \frac{\partial V_i}{\partial S_i} dS_i = -[V_i]_{P_1}^{P_2}.$$

Now, summing over all the particles of the system, we have

$$\sum W_i = \sum [-V_i]_{P_1}^{P_2} \quad \text{or} \quad W_{12} = V_1 - V_2 \tag{1.18}$$

where W_{12} is the total work done by the external forces, and V_1 and V_2 are the initial and final values of total potential energy of the system.

Comparing equations (1.16) and (1.18), we have

$$T_2 - T_1 = V_1 - V_2$$

or, $T_2 + V_2 = T_1 + V_1 = E$ = total energy of the system. This result leads to the following theorem.

Theorem 1.6 *If the forces acting on the system of particles are conservating, the total energy of the system of particles which is the sum of the total K.E. and the total P.E. of the system is conserved.*

1.3 Virial Theorem

Let us consider a system of particles with masses $m_1, m_2, \ldots, m_n$ and position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$. Let $\vec{F}_i$ be the force applied to the ith particle. Then the equation of motion of ith particle is given by

$$\vec{F}_i = \dot{\vec{p}}_i \tag{1.19}$$

where $\vec{p}_i$ is the momentum of ith particle.

Let us introduce a quantity ϕ such that

$$\phi = \sum \vec{p}_i \cdot \vec{r}_i \tag{1.20}$$

where the summation is taken over all the particles of the system.

The derivative of ϕ is given by

$$\frac{d\phi}{dt} = \sum \vec{p}_i \cdot \dot{\vec{r}}_i + \sum \vec{r}_i \cdot \dot{\vec{p}}_i = \sum m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i + \sum \vec{F}_i \cdot \vec{r}_i,$$

where m_i is the mass of the ith particle.

$$\frac{d\phi}{dt} = \sum m_i \dot{\vec{r}}_i^{\,2} + \sum \vec{F}_i \cdot \vec{r}_i\,,$$

or $$\frac{d\phi}{dt} = 2T + \sum \vec{F}_i \cdot \vec{r}_i \tag{1.21}$$

where T is the K.E. of the system.

The time average of equation (1.21) over a time interval τ is given by

$$\frac{1}{\tau}\int_0^\tau \frac{d\phi}{dt}\,dt = 2\overline{T} + \overline{\sum \vec{F}_i \cdot \vec{r}_i}, \tag{1.22}$$

where $\overline{T}$ is the average K.E. of the system over time interval τ and $\overline{\sum \vec{F}_i \cdot \vec{r}_i}$ is the average value of $\sum \vec{F}_i \cdot \vec{r}_i$ over the time interval τ.

The equation (1.22) can be written as

$$\frac{1}{\tau}\left[\phi(\tau) - \phi(0)\right] = 2\overline{T} + \overline{\sum \vec{F}_i \cdot \vec{r}_i}. \tag{1.23}$$

Now, if the motion is periodic then all the coordinates repeat after a certain time. If this is chosen to be the time period of motion, which is generally the case, then $\phi(\tau) = \phi(0)$ and hence the L.H.S. of equation (1.22) vanishes.

Hence

$$2\overline{T} + \overline{\sum \vec{F}_i . \vec{r}_i} = 0.$$

Again, if the motion is non-periodic then consider the motion in which the coordinates and velocities for all particles remains finite, so that there is an upper bound to ϕ. Then by choosing τ sufficiently long the L.H.S. of equation (1.21) can be made as small as we please. Hence for a proper choice of τ, L.H.S. of equation (1.21) is zero. Thus, in both the cases L.H.S. of equation (1.22) is zero and we have

$$2\overline{T} + \overline{\sum \vec{F}_i . \vec{r}_i} = 0 \quad \text{or} \quad \overline{T} = \overline{-\frac{1}{2}\sum \vec{F}_i . \vec{r}_i}, \tag{1.24}$$

which is the **Virial theorem.** The R.H.S. is called the **Virial of Clausius.**

Example 1.1 *If the particles attract each other according to inverse square law of force, prove that* $2T + V = 0$, *where T and V represent respectively the total K.E. and P.E. of the particles.*

SOLUTION: For a conservative system, the forces $\vec{F}_i$ are derivable from a scalar potential function by taking the gradient of the latter, so that $\vec{F}_i = -\nabla V$, then by Virial theorem

$$\overline{T} = \frac{1}{2}\overline{\sum \nabla V . \vec{r}_i}.$$

For a single particle moving under a central force, the above equation reduces to

$$\overline{T} = \frac{1}{2}\frac{\partial V}{\partial r} r.$$

If the force law varies as r^n, then

$$V = kr^{n+1},$$

so that

$$\frac{\partial V}{\partial r} r = (n+1)kr^n . r = (n+1)kr^{n+1} = (n+1)V.$$

Thus

$$\overline{T} = \frac{n+1}{2}\overline{V}.$$

In the case of inverse square law of force $n = -2$, then above equation becomes

$$\overline{T} = -\frac{\overline{V}}{2} \quad \text{or} \quad 2\overline{T} + \overline{V} = 0$$

1.4 Generalised Coordinates

The cartesian coordinates system is not only possible system of coordinates required to define the position of a mathematical system. In fact, the choice of the coordinates is perfectly arbitrary and is determined on the basis of the problem concerned.

Now we may think a wide variety of possible coordinate system, any set of parameters which give an unambiguous representation of the configuration of the system of coordinates in more general sense.

Any quantities $q_1, q_2, ..., q_n$ which define completely the position of a mechanical system will be called **generalised coordinates** of the system and $\dot{q}_1, \dot{q}_2, ..., \dot{q}_n$ are called **generalised velocities**.

The generalised coordinates may be any parameters. They may be angles, axes, moments or any set of suitable parameters. Following are the some examples of generalised coordinates:

1. A dynamical system be a simple pendulum of length l, the corresponding generalised coordinate is θ, the angular displacement from the vertical.
2. A particle on the surface of a sphere, generalised coordinates are θ, ϕ where θ, ϕ are the polar coordinates on the surface.
3. A lamina lying on a plane, generalised coordinates are x, y, θ, where (x, y) are the coordinates of the centroid and θ is the angle made by a line fixed in the plane.
4. A rod lying on a plane surface, generalised coordinates are x, y, θ, where (x, y) are the coordinates of one end of teh rod and θ is the angle between x-axis and the rod.

Degrees of freedom

The number of independent parameters which can be independently varied and are required to define uniquely. The number of independent variables of a system is called the number of **degrees of freedom or d.o.f.** of the system.

If the system is connected by k constraints for n generalised coordinates then d.o.f. will be $n - k$.

The d.o.f. of 3-dimensional cartesian coordinates system is 3. But, if a particle is moving along the surface $x^2 + y^2 + z^2 = a^2$ then the number of degrees of freedom is $3 - 1 = 2$.

1.5 Classification of Dynamical System

Any restriction on the motion of a system is known as **constraint** and the force responsible is called **force of constraint.**

In rigid body the constraints on the motion of the particle are the distance r_{ij} remain unchanged. This constraints reduce to the number of degrees of freedom of a body and therefore also reduce the number of its independent generalised coordinates. The constraints may be classified as follows:

If the constraints involved in a system be such that the condition of constraints can be expressed as equation in terms of coordinates and possibly in time t in the form

$$f_j(q_1, q_2 ... q_n) = 0, j = 1,2..., k(< n). \quad (1.25)$$

Then the system is called **connected holonomic system** and the constraints are called **holonomic constraints.**

Example 1.2 *The distance between any two points r_i and r_j of a rigid body remains unchanged, i.e., $(r_i - r_j)^2 = c_{ij}^2$. This is an example of a holonomic constraints.*

Constrains in which the time t does not appear explicitly are known as **scleronomic constraints.** If the time t appear explicitly in constraints they are called **rheonomic constraints.**

Example 1.3 *A pendulum with a fixed support is scleronomic, whereas, the pendulum for which the point of support is given an assigned motion is rheonomic.*

If the constraints cannot express as equation i.e., in the form of (1.25), then the constraints are called non-holonomic constraints and the system involved with non-holonomic constraints are called **non-holonomic system.**

Some of the constraints of a system may be expressed as

$$f_i(q_1, q_2, ..., q_n) \leq 0, j = 1,2,..., k(< n). \quad (1.26)$$

Example 1.4 *The walls of a gas container constitute a non-holonomic constraints. The constraints of a particle placed on the surface of a sphere is also non-holonomic. It can be expressed as $(r^2 - a^2) \geq 0$, where a is the redius of the sphere.*

There are other types of constraints for which the condition of the constraints cannot be expressed in the form of (1.25) or (1.26), but are expressible in terms of non-integrable relations of the form

$$a_i dt + (a_{i1} dq_1 + a_{i2} dq_2 + ... + a_{in} dq_n) = 0 \quad (1.27)$$

where a's are in general functions of q's and t. The system with constraints (1.26) and (1.27) are called non-holonomic system.

The constraints of the form (1.25) and (1.27) are called **bilateral constraints, i.e., if one imagine a small** allowable displacement from any configuration of the system. The negative of the displacement is also allowable, assuming any fixed value of time such bilateral constraints are always expressed as an equality.

As non-integrable nature of the form of (1.27), one cannot obtain of the form (1.25) and use these to eliminate some of the variable. Hence non-holonomic system always required more coordinates for their description then their d.o.f.

1.5.1 Principle of Virtual Work

A virtual displacement is an arbitrary, instantaneous, infinitesimal displacement of a dynamical system independent of time and consistent with the constraints of the system. The work due to virtual displacement is known as **virtual work.**

Example 1.5 *A particle is constrained to move on a surface, the force of constraint is perpendicular to the surface, while the virtual displacement must be tangent to it, and hence the virtual work vanishes.*

Suppose the system is in equilibrium, i.e., total force $\sum \vec{F}_i$ on every particle is zero, then the work done by this force in a small virtual displacement δr_i will also vanish i.e., $\sum \vec{F}_i \cdot \delta\vec{r}_i = 0$.

Let this total force be expressed as sum of applied force $\vec{F}_i^{(a)}$ and forces of constraints $\vec{f}_i$. Then the above equation takes the form

$$\sum_i \vec{F}_i^{(a)}.\delta\vec{r}_i + \sum_i \vec{f}_i.\delta\vec{r}_i = 0. \qquad (1.28)$$

If the virtual work done by the forces of constraints will be zero then

$$\sum_i \vec{F}_i^{(a)}.\delta\vec{r}_i = 0. \qquad (1.29)$$

This equation is called the **principle of virtual work.**

The following is an example where the work done by constraints force is zero.

Example 1.6 *A particle be constraints to move on a smooth surface so that the forces of constraints being perpendicular to the surface while virtual displacement tangential to it, then virtual work done by forces of constraints will be zero.*

1.5.2 D'Alembert's principle

We consider the system which will remains in equilibrium under the action of forces $\vec{F}_i$ plus a reversed effected force $\dot{\vec{p}}_i$.

Therefore, $\vec{F}_i + (-\dot{\vec{p}}_i) = 0$. Hence by the principle of virtual work,

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i).\delta\vec{r}_i = 0$$

or, $$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i).\delta\vec{r}_i + \sum_i \vec{f}_i.\delta\vec{r}_i = 0$$

Using $\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i$.

Dealing with the system for which the virtual work of the forces of constraints is zero.

We write

$$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta\vec{r}i = 0. \qquad (1.30)$$

This is called D'Alembert's principle.

This implies that the sum of the virtual work done by the external force and the reverse effective force is zero.

1.6 Generalised Force in Holonomic System

Consider a dynamical system with n particles of masses $m_i, i = 1,2,....n,$ with position vectors $\vec{r}_i$ with respect to the origin O. We suppose that the particles undergo a virtual displacement.

Let the ith particle m_i at position $\vec{r}_i$ at time t undergoes a virtual displacement to position $\vec{r}_i + \delta\vec{r}_i$. Suppose $\vec{F}_i$ and $\vec{F}_i'$ are the external and internal forces acting on m_i, then the vertual work done on m_i is $(\vec{F}_i + \vec{F}_i') \cdot \delta\vec{r}i.$

So, the total virtual work done on all the particles of the system is

$$\delta W = \sum_{i=1}^{N} (\vec{F}_i + \vec{F}_i') \cdot \delta\vec{r}_i = \sum_{i=1}^{N} \vec{F}_i \cdot \delta\vec{r}_i + \sum_{i=1}^{N} \vec{F}_i' \cdot \delta\vec{r}_i, \qquad (1.31)$$

where $\sum_{i=1}^{N} \vec{F}_i' \cdot \delta\vec{r}_i$ is the total work done by the internal forces of the system, when we consider to be zero. Then from (1.31)

$$\delta W = \sum_{i=1}^{N} \vec{F}_i.\delta\vec{r}_1. \qquad (1.32)$$

Let us suppose that the system is holonomic and the generalised coordinates are $q_1, q_2, \ldots q_n$ then

$$\vec{r}_i = r_i(q_1, q_2, \ldots q_n).$$

Thus,

$$\delta\vec{r}_i = \frac{\partial\vec{r}_i}{\partial q_1}\delta q_1 + \frac{\partial\vec{r}_i}{\partial q_2}\delta q_2 + \ldots + \frac{\partial\vec{r}_i}{\partial q_n}\delta q_n = \sum_{k=1}^{n}\frac{\partial\vec{r}_i}{\partial q_k}\delta q_k. \tag{1.33}$$

Using (1.33), equation (1.32) becomes

$$\delta W = \sum_{i=1}^{N}\left(\vec{F}_i \cdot \sum_{k=1}^{n}\frac{\partial\vec{r}_i}{\partial q_k}\delta q_k\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{N}\vec{F}_i \cdot \frac{\partial\vec{r}_i}{\partial q_k}\delta q_k\right)$$

$$= \sum_{k=1}^{N} Q_k \delta q_k \tag{1.34}$$

where

$$Q_k = \sum_{i=1}^{N}\vec{F}_i \cdot \frac{\partial\vec{r}_i}{\partial q_k}. \tag{1.35}$$

The coefficients $Q_1, Q_2, \ldots, Q_n$ are called **components of generalised force associated with the coordinates** $q_1, q_2, \ldots, q_n$ respectively. It is called **generalised force** because Q_k is the coefficient of δq_k in (1.35).

If the system is conservative and if V be the potential energy of the system in any configuration space, we have

$$\delta W = -\delta V. \tag{1.36}$$

In this case, $V = V(q_1, q_2, \ldots, q_n)$. Thus from (1.34) and (1.36), we have

$$\sum_{k=1}^{n} Q_k \delta q_k = -\sum_{k=1}^{n}\frac{\partial V}{\partial q_k}\delta q_k. \tag{1.37}$$

Since the dynamical system is holonomic, $dq_1, dq_2, \ldots dq_n$ are independent and arbitrary.

Thus from (1.37), we get

$$Q_k = -\frac{\partial V}{\partial q_k}, \quad k = 1, 2, \ldots, n. \tag{1.38}$$

1.7 Unit Summary

In this unit, the mechanics of a particle and a system of particles are studied. The basic terms such as work, kinetic energy, conservative forces, potential energy are defined. The conservation laws viz., conservation theorem of linear and angular momentum, conservation theorem of energy are stated and proved. Also, these conservation theorems are proved for a system of particles. The virial theorem is stated and proved in this unit. The most fundamental coordinates, i.e., generalised coordinates are defined. Different types of constraints and systems like hononomic, non-holonomic, seleronomic etc. are defined. The principle of virtual work and D'Alembert's principle are stated. The generalised forces are defined here.

1.8 Self Assessment Questions

1. Define the following terms :

   (i) work, (ii) kinetic and potential energies, (iii) conservative and non-conservative forces.
2. State and prove conservation law of linear momentum and angular momentum.
3. State and prove conservation law of energy.
4. Show that the rate of change of angular momentum is equal to the applied torque for a system ofparticles.
5. State and prove virial theorem.
6. Define generalised coordinated with example. Also, define degrees of freedom of a system.
7. Define constrains. Define the following terms with examples :

   (i) holonomic constrains, (ii) non-holonomic constrains.
8. State principle of virtual work and D'Alembert's principle.
9. What do you mean by generalised forces? Find the expression of it in terms of generalised coordinates.

1.9 Suggested Further Readings

1. H. Goldstein, *Classical Mechanics,* Addison-Wesley, Cambridge, 1950.
2. T.W.B. Kibble, *Classical Mechanics,* Orient Longman, London, 1985.
3. L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed., Pergamon Press, Oxford, 1976.
4. A. Sommerfeld, *Mechanics,* Academic Press, New York, 1964.
5. J. Synge and B. Griffith, *Principles of Mechanics,* 2nd ed., McGraw Hill, New York, 1949.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART - I

Paper-IV Group-A

Module No. - 38
Principle of Mechanics
(ROTATING FRAMES)

CONTENT :

A rigid body is defined as a system of points subject to the holonomic constaraints where the distance between all pairs of points remain constant throughout the motion.

A rigid body with n particles can at most have $3n$ degrees of freedom. These degrees of freedom are greatly reduced by the constraints, which can be expressed as,

$$r_{ij} = c_{ij} \tag{2.1}$$

where r_{ij} is the distance between the ith and jth particles and c_{ij} 's are constants. The actual number of degrees of freedom cannot be obtained simply by subtracting the number of constraint equations from $3n$, for there are $\frac{1}{2}n(n-1)$ possible equations of the form of (2.1), which exceeds $3n$ for large n. But, the equations (2.1) are not all independent. To fix a point in the rigid body it is not necessary to specify its distance to all other points in the body, one needs only the distances at any three other non-collinear points.

Thus once the positions of three particles of a rigid body are determined, the constraints fix the positions of all remaining particles. The number of degrees of freedom therefore cannot be more than nine. But, the three reference points are themselves not independent, there are in fact three equations of rigid constrains imposed on them

$$r_{12} = c_{12}, \qquad r_{23} = c_{23}, \qquad r_{13} = c_{13}$$

which reduce the number of degrees of freedom to six. Therefore, only six coordinates are needed. A rigid body in space needs six independent generalised coordinates to specify its configuration.

Generalised coordinates of a rigid body

Before setting up the motion of a rigid body which is free to rotate about a fixed point it will be necessary to identify three independent parameters specifying the orientation of a rigid body. The most common and useful set of parameters are Eulerian angles.

The Eulerian angles θ, ϕ, ψ form a set of generalised coordinates for a rigid body with a fixed point. They can also be used as part of a set of generalised coordinates for a rigid body free to move in space. Let $O(x, y, z)$

be the origin at the body set of axes with respect to space. Then the cartesian coordinates x, y, z together with the Eulerian angles θ, ϕ, ψ describe the configuration of the body. Since the number $x, y, z, \theta, \phi, \psi$ can be varied independently without violating the rigidity of the body. It is clear that a rigid body, free to move in space, has six degrees of freedom.

Objectives

- Generalised coordinates of a rigid body
- Euler's angles
- Velocity and angular momentum in rotating frames
- Coriolis force
- Focault pendulum
- Euler's equations
- Invariable line and plane
- Exercise.

2.1 Eulerian Angles

The Eulerian angles are defined as the three successive angles of rotation of a rigid body about a point fixed. The sequence will be started by rotating the initial system of axes *Oxyz* (fixed in space) by an angle ϕ counter-clockwise about the z-axis and the resultant coordinate system will be labelled as axes $\xi\eta\zeta$. In the second stage the intermediate axes $\xi\eta\zeta$ are rotated about the ξ-axis counter-clockwise by an angle θ to produce another set, the axes $\xi'\eta'\zeta'$. The ξ'-axis is at the intersection of the xy- and $\xi'\eta'$-planes and is known as the line of nodes. Finally the axes $\xi'\eta'\zeta'$ are rotated counter-clockwise by an angle ψ about ζ'-axis to produce the desired $x'y'z'$ system of axes (fixed in the body). The angles ϕ, θ, ψ are known as Eulerian angles. These angles will completely specify the motion of a rigid body about a point fixed in the body. Therefore, they act as the three needed generalised coordinates. We observe that all possible positions of the body can be obtained by assigning values to ϕ, θ, ψ, in the ranges $0 \le \phi \le 2\pi, 0 \le \theta \le \pi, 0 \le \psi \le 2\pi$.

2.1.1 Representation of coordinates x', y', z' in terms of x, y, z

Let (x, y, z) be the orthogonal space set of axes and (x', y', z') be that of body set of axes. In order to account for the rotatory motion, we shall carry out the transformation from space set of axes to body set of axes. The transformation is worked out through three successive rotations performed in a certain order.

First rotation

First of all space set of axes are rotated about z-axis so that xy-plane takes a new position ξ, η. The rotational angle is ϕ. The transformation of this new set of axes ξ, η, ζ from x, y, z can be represented by

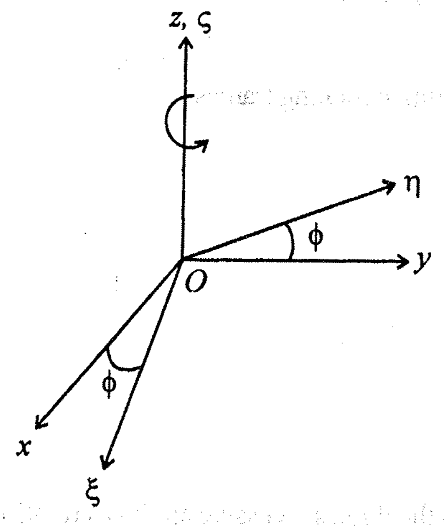

FIGURE 2.1: First rotation

$$\xi = x\cos\phi + y\sin\phi + 0.z$$
$$\eta = -x\sin\phi + y\cos\phi + 0.z$$
$$\zeta = 0.x + 0.y + 1.z$$

or,

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = D \begin{bmatrix} x \\ y \\ x \end{bmatrix}$$

where

$$D = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Second rotation

The second rotation is performed about new ξ axis so that ζ axis coincide with body z'-axis. The axes ζ' and η' obtained after rotation about ξ through an angle θ.

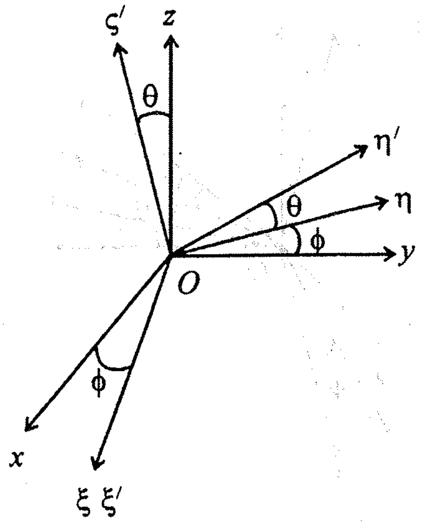

FIGURE 2.2: Second rotation

The transformation can be represented by

$$\begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} = C \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$$

where

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}.$$

Third rotation

The third rotation is performed about ζ' axis, i.e., about z'-axis so that new axes ξ'' coincides with the body x'-axis and the axis η'' coincides with the body y'-axis. This completes the transformation from space set of axes to body set of axes. The rotation angle is ψ.

The transformation can be written as

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = B \begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix}$$

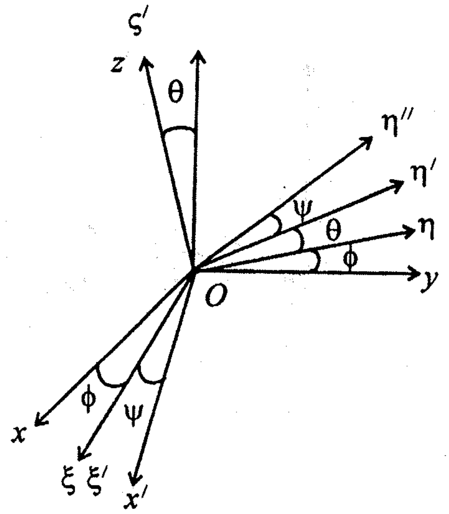

FIGURE 2.3: Third rotation

where

$$B = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Tehrefore, we have arrived at body set of axes after three successive and sequential rotation of space set of axes. The complete matrix of transformation will be

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = B \begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} = BC \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = BCD \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where

$$A = BCD = \begin{bmatrix} \cos\theta\cos\psi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\cos\phi\sin\psi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}.$$

This relation gives the relation between old and new set of coordinates.

2.2 Components of Angular Velocity in Terms of Euler Angles

If is often convenient to express the angular velocity vector in terms of the Euler angles and their time derivatives. Suppose a rigid body rotates about a point *O* fixed in both body and space. Let *Ox*, *Oy*, *Oz* be three mutually perpendicular axes through *O* and fixed in space. At any time *t*, suppose the body has an angular velocity ω referred to these axes. Then this angular velocity ω can be considered as consisting of three successive rotations with angular velocities

$$\omega_\phi = \dot{\phi}, \qquad \omega_\theta = \dot{\theta}, \qquad \omega_\psi = \dot{\psi},$$

where ω_ϕ is along the space *z*-axis, ω_θ is along the line of nodes (along ξ'-axis) and ω_ψ is along the body z'-axis. Since ω_ϕ is parallel to the space *z*-axis, its components along the body axes x', y', z' are given by

$$\begin{aligned}
(\omega_\phi)_{x'} &= \dot{\phi}\sin\theta\sin\psi \\
(\omega_\phi)_{y'} &= \dot{\phi}\sin\theta\cos\psi \\
(\omega_\phi)_{z'} &= \dot{\phi}\cos\theta
\end{aligned}$$

The line of nodes, which is the direction of ω_θ, coincides with ξ'-axis, so the components of ω_0 with respect to the body axes are

$$\begin{aligned}
(\omega_\theta)_{x'} &= \dot{\phi}\cos\psi \\
(\omega_\theta)_{y'} &= -\dot{\phi}\sin\psi \\
(\omega_\theta)_{z'} &= 0.
\end{aligned}$$

The components of ω_ψ lies along the z'-axis. Now adding the components of the separate angular velocities, we obtain the components of ω along the body set axes:

$$\begin{aligned}
\omega_{x'} &= \dot{\phi}\sin\theta\sin\psi + \dot{\phi}\cos\psi \\
\omega_{y'} &= \dot{\phi}\sin\theta\cos\psi - \dot{\phi}\sin\psi \\
\omega_{z'} &= \dot{\phi}\cos\theta + \dot{\psi}.
\end{aligned}$$

Rotating axes *xyz* by an angle ϕ counter-clockwise about the *z*-axis we get the resultant coordinate system as the axes $\xi\eta\zeta$. Again rotate $\xi\eta\zeta$ about ξ-axis counter-clockwise by an angle θ to produce $\xi'\eta'\zeta'$. The ξ'-axis is at the intersection of the *xy* and $\xi'\eta'$ planes and is known as the line of nodes. Finally $\xi'\eta'\zeta'$ axes are rotated counter-clockwise by an angle ψ about the ζ'-axis to produce the $x'y'z'$ system of axes, fixed in the body.

2.3 Frames of Reference Rotating with Constant Angular Velocity

Case I: When the origins of the two frames coincide

Consider two systems *S* and *S′* the latter being in uniform rotating to the former which is at rest in absolute space. Let the origins of both the systems coincide. Let $\vec{i}, \vec{j}, \vec{k}$ be the unit vectors associated with axes *X, Y, Z* of system *S* and $\vec{i}', \vec{j}', \vec{k}'$ the unit vectors associated with axes *X′, Y′, Z′* of system *S′*. Then the position vector $\vec{R}$ in terms of its components are

$$\vec{R} = x\vec{i} + y\vec{j} + z\vec{k} \tag{2.2}$$

and $$\vec{R} = x'\vec{i}' + y'\vec{j}' + z'\vec{k}'. \tag{2.3}$$

An observer in system *S′* situated at the origin *O* will observed the time derivative of $\vec{R}$ to be

$$\frac{d\vec{R}}{dt} = \frac{dx'}{dt}i' + \frac{dy'}{dt}j' + \frac{dz'}{dt}k'. \tag{2.4}$$

But the time derivative of $\vec{R}$ relative to an observer in the system *S* will be

$$\frac{d\vec{R}}{dt} = \frac{dx'}{dt}\vec{i}' + \frac{dy'}{dt}\vec{j}' + \frac{dz'}{dt}\vec{k}' + x'\frac{d\vec{i}'}{dt} + y'\frac{d\vec{j}'}{dt} + z'\frac{d\vec{k}'}{dt} \tag{2.5}$$

since the unit vectors $\vec{i}', \vec{j}', \vec{k}'$ actually change with velocity to the observer fixed in system *S*.

Since $\vec{i}'$ is a unit vector $\frac{d\vec{i}'}{dt}$ is perpendicular to i' and must therefore lie in the plane of $\vec{j}'$ and $\vec{k}'$. Then

$$\frac{d\vec{i}'}{dt} = A_1\vec{j}' + A_2\vec{k}'. \tag{2.6}$$

Similarly,

$$\frac{d\vec{j}'}{dt} = A_3\vec{k}' + A_4\vec{j}'. \tag{2.7}$$

$$\frac{d\vec{k}'}{dt} = A_5\vec{i}' + A_6\vec{j}'. \tag{2.8}$$

Differentiating the relation $\vec{i}'.\vec{j}' = 0$, we get

$$\vec{i}'.\frac{d\vec{j}'}{dt} + \frac{d\vec{i}'}{dt}.\vec{j}' = 0 \tag{2.9}$$

Using (2.6) $\times\vec{j}'$ and (2.7) $\times\vec{i}'$ we have

$$\vec{j}'.\frac{d\vec{i}'}{dt} = A_1 \text{ and } \vec{i}'.\frac{d\vec{j}'}{dt} = A_4.$$

Substituting in (2.9) we get

$$A_4 + A_1 = 0. \quad (2.10)$$

Similarly, from $\vec{i}'.\vec{k}' = 0$ we get $A_5 = -A_2$ and from $\vec{j}'.\vec{k}' = 0$ we get $A_6 = -A_3$.

Equations (2.6), (2.7) and (2.8) become

$$\frac{d\vec{i}'}{dt} = A_1\vec{j}' + A_2\vec{k}'$$
$$\frac{d\vec{j}'}{dt} = A_3\vec{k}' - A_1\vec{i}'$$
$$\frac{d\vec{k}'}{dt} = -A_2\vec{i}' - A_3\vec{j}'.$$

Then we have

$$x'.\frac{d\vec{i}'}{dt} + y'.\frac{d\vec{j}'}{dt} + z'.\frac{d\vec{k}'}{dt}$$
$$= x'\left(A_1\vec{j}' + A_2\vec{k}'\right) + y'\left(A_3\vec{k}' + A_1\vec{i}'\right) + z'\left(-A_2\vec{i}' - A_3\vec{j}'\right)$$
$$= \left(-A_1y' - A_2z'\right)i' + \left(A_1x' - A_3z'\right)j' + \left(A_2x' - A_3y'\right)\vec{k}'$$
$$= \begin{vmatrix} \vec{i}' & \vec{j}' & \vec{k}' \\ A_3 & -A_2 & A_1 \\ x' & y' & z' \end{vmatrix}. \quad (2.11)$$

Choosing $A_3 = w_x, -A_2 = w_y$ and $A_1 = w_z$ where w_x, w_y, w_z are the components of angular velocity vector $\vec{w}$ of system S', i.e., $\vec{w} = \vec{i}'w_x + j\vec{w}_y' + \vec{k}'w_z$.

Then (2.11) may be written as

$$x'\frac{d\vec{i}'}{dt} + y'\frac{d\vec{j}'}{dt} + z'\frac{d\vec{k}'}{dt} = \begin{vmatrix} \vec{i}' & \vec{j}' & \vec{k}' \\ w_x & w_y & w_z \\ x' & y' & z' \end{vmatrix} = \vec{w} \times \vec{R}. \quad (2.12)$$

From equation (2.11) and (2.12), we get

$$\frac{d\vec{R}}{dt}=\frac{dx'}{dt}\vec{i}'+\frac{dy'}{dt}\vec{j}'+\frac{dz'}{dt}\vec{k}'+\vec{w}\times\vec{R}.$$

That is,

$$\left(\frac{d\vec{R}}{dt}\right)_S=\left(\frac{d\vec{R}}{dt}\right)_{S'}+\vec{w}\times\vec{R}. \qquad (2.13)$$

Case II: When the origins of the two frames do not coincide

Let there be two systems S and S', the latter being in uniform rotation relation to former which is at rest in absolute space. Let $\vec{i},\vec{j},\vec{k}$ be the unit vectors associated with axes X, Y, Z of the system S and $\vec{i}',\vec{j}',\vec{k}'$, the unit vectors associated with axes X', Y', Z' of the system S'. Let O and O' be the origins of the system S and S' which do not coincide.

Let $\vec{r}_0$ be the position vector of origin O' of system S' relative to the origin O of the system S.

If $\vec{r}$ and $\vec{R}$ are the position vectors of the particle P relative to the origins O and O respectively, then the velocity of the particle P relative to the moving system S' is given by

$$\frac{d'\vec{r}}{dt}=\vec{i}'\frac{dx'}{dt}+\vec{j}'\frac{dy'}{dt}+k'\frac{dz'}{dt}. \qquad (2.14)$$

The position vector of P relative to system S is given by

$$\vec{R}=\vec{r}_0'+\vec{r}.$$

Then the velocity of the particle relative to system S is given by

$$\frac{d\vec{R}}{dt}=\frac{d}{dt}(\vec{r}_0+\vec{r})=\frac{d\vec{r}_0}{dt}+\frac{d\vec{r}}{dt}$$

or, $$\frac{d\vec{R}}{dt}=\frac{d\vec{r}_0}{dt}+\frac{d'\vec{r}}{dt}+w\times\vec{r}. \qquad (2.15)$$

Here $\dot{\vec{r}}_0$ is the velocity of origin O' from equation (2.13) relative to O.

2.4 Coriolis and Centripetal Forces

We recall the equation (2.13)

$$\left(\frac{d\vec{R}}{dt}\right)_{Space}=\left(\frac{d\vec{R}}{dt}\right)_{Body}+w\times\vec{R}. \qquad (2.13)$$

This equation is the basis of kinematical law upon which the dynamical equations of motion for a rigid body are found.

Let $\vec{r}$ be the radius vector from the origin of the terrestial (body set) system to the given particle.

Then we have

$$\left(\frac{d\vec{r}}{dt}\right)_{Space} = \left(\frac{d\vec{r}}{dt}\right)_{Body} + w\times\vec{r}.$$

We denote $\left(\frac{d\vec{r}}{dt}\right)_{Space}$ and $\left(\frac{d\vec{r}}{dt}\right)_{Body}$ by $\vec{V}_s$ and $\vec{V}_b$ respectively. Using these notation the above relation becomes

$$\vec{V}_s = \vec{V}_b + \vec{w}\times\vec{r}. \tag{2.16}$$

Thus,

$$\left\{\frac{d}{dt}(\vec{V}_s)\right\}_s = \left\{\frac{d}{dt}(\vec{V}_s)\right\}_b + \vec{w}\times\vec{V}_s.$$

This leads to

$$\begin{aligned}\vec{a}_s &= \frac{d}{dt}\left(\vec{V}_b + \vec{w}\times\vec{r}\right) + \vec{w}\times\left(\vec{V}_b + \vec{w}\times\vec{r}\right)\\ &= \frac{d}{dt}\left(\vec{V}_b\right) + \frac{d\vec{w}}{dt}\times\vec{r} + \vec{w}\times\frac{d\vec{r}}{dt} + \vec{w} + \vec{V}_b + \vec{w}\times\left(\vec{w}\times\vec{r}\right)\\ &= \vec{a}_b + \frac{d\vec{w}}{dt}\times\vec{r} + 2\vec{w}\times\vec{V}_b + \vec{w}\times\left(\vec{w}\times\vec{r}\right),\end{aligned} \tag{2.17}$$

where $\vec{a}_s$ and $\vec{a}_b$ are the accelerations of the particle in the two systems. But, the equation of motion in the inertial system (space set) is

$$\vec{F} = m\vec{a}_s = m\vec{a}_b + m\frac{d\vec{w}}{dt}\times\vec{r} + 2m\left(\vec{w}\times\vec{V}_b\right) + m\vec{w}\times\left(\vec{w}\times\vec{r}\right)$$

or,

$$\vec{F} - 2m\left(\vec{w}\times\vec{V}_b\right) - m\vec{w}\times\left(\vec{w}\times\vec{r}\right) - m\left(\frac{d\vec{w}}{dt}\times\vec{r}\right) = m\vec{a}_b. \tag{2.18}$$

Therefore, to an observer in the rotating system it appears as if the particle is moving under the influence of an effective force. That is,

$$\vec{F}_{eff} = m\vec{a}_b = \vec{F} - 2m(\vec{w}\times\vec{V}_b) - m\vec{w}\times(\vec{w}\times\vec{r}) - m\frac{d\vec{w}}{dt}\times\vec{r}.$$

The term $2m(\vec{w}\times\vec{V}_b)$ is called the **Coriolis force,** it is perpendicular to the plane containing $\vec{w}$ and $\vec{V}_b$. The order of the magnitude of these force may easily be calculated for a particle on the earth surface.

The term $m\vec{w}\times(\vec{w}\times\vec{r})$ is a vector normal to $\vec{w}$ and pointing outwards. Further its magnitudes $mw^2r\sin\theta$ and this term is known as the **centrifugal force.**

When the particle is fixed with respect to the body set of axes then $\vec{V}_b = 0$. Then the Coriolis force is zero.

Again, for a uniform rotating system, $\vec{w}$ is constant and hence the term $m\frac{d\vec{w}}{dt}\times\vec{r} = 0$. Therefore, the acceleration $\vec{a}_b$ referred to the uniformly rotating system is given by

$$\vec{a}_b = \frac{\vec{F}}{m} - 2(\vec{w}\times\vec{V}_b) - \vec{w}\times(\vec{w}\times\vec{r}).$$

That is

$$\vec{a}_b = \vec{g}_e - 2(\vec{w}\times\vec{V}_b),$$

where

$$\vec{g}_e = \frac{\vec{F}}{m} - w'\times(\vec{w}\times\vec{r})$$

and $\vec{F}$ is the force due to the earths attraction only.

Example 2.1 *Find the horizontal component of the Coriolis force acting on a body of mass 0.1 kg. moving northward with a horizontal velocity 100 m/sec. at 30⁰N latitude on the earth.*

SOUTHTION: Taking X', Y' and Z' axes along east, north and vertical directions respectively, the velocity of the body moving towards north is given by

$$\vec{v}' = 100\vec{j}'m/\sec = 10^4 j'cm/\sec.$$

The angular velocity vector $\vec{w}$ of the earth is given by

$$\vec{w} = w\cos\lambda\vec{j} + w\sin\lambda\vec{k}.$$

Here $\lambda = 30^0$. Therefore,

$$\vec{w} = w\cos 30^0 \vec{j}' + w\sin 30^0 \vec{k}' = \frac{\sqrt{3}}{2} w\vec{j}' + \frac{1}{2} w\vec{k}'.$$

The Coriolis force is given by

$$\vec{F} = -m(\vec{w}\times\vec{v}') = -m\left[\left(\frac{\sqrt{3}}{2} w\vec{j}' + \frac{1}{2} w\vec{k}'\right)\times 10^4 \vec{j}'\right]$$

$$= -mw\times\frac{1}{2}\times 10^4 i' = -\frac{1}{2}\times 10^4 mw\vec{i}'.$$

Here $m = 0.1$ kg. = 100 gm. and $w = \dfrac{2\pi}{24\times 60\times 60}$ rad./sec.

Therefore, $\vec{F} = \dfrac{1}{2}\times 10^4 \times 100 \times \dfrac{2\pi}{24\times 60\times 60}\vec{i}$ dynes $= 36\vec{i}'$ dynes.

That is, the horizontal component of Coriolis force of 36 dynes is acting along the east.

2.5 Deviation of Freely Falling Bodies from the Vertical

Let us consider the problem of free fall of a body on the earth surface. The earth is rotating with an angular velocity $\vec{F} = 0.7\times 10^{-4}$ rad./sec. with north-south as the axes of rotation. Let X', Y', Z' be the axes along the east, north and vertical directions respectively and $\vec{i}', \vec{j}', \vec{k}'$ be the unit vectors along X', Y' and Z' axes respectively. Let the body of mass m be falling freely on the surface of the earth. Let $\vec{v}'$ be the velocity of the body at any instant t.

Then

$$\vec{w} = w\cos\lambda\vec{j}' + w\sin\lambda\vec{k}' \text{ and } \vec{v}' = -v'\vec{k}',$$

where v' is the magnitude of velocity, $\vec{v}'$, negative sign represents that the body is moving vertically downwards, while positive z-axis is vertically upwards.

The Coriolis force is given by

$$\vec{F} = -2m(\vec{w}\times\vec{v}')$$

$$= -2m\left[\left(w\cos\lambda\vec{j}' + w\sin\lambda\vec{k}'\right)\times\left(-v'\vec{k}'\right)\right]$$

$$= -2m\left\{-wv'\cos\lambda\vec{j}'\right\} = -2mwv'\cos\lambda\vec{i}'. \qquad (2.19)$$

The equation represents that the Coriolis force, in this case, is acting along positive x-axis. Therefore, the direction of the body will be towards east in the northern hemisphere.

If $\frac{d^2x}{dt^2}$ is the acceleration along x-axis, then the equation of motin along x-axis is given by

$$m\frac{d^2x}{dt^2}\vec{i}' = 2mv'w\cos\lambda\vec{i}'.$$

That is,

$$\frac{d^2x}{dt^2} = 2v'w\cos\lambda. \tag{2.20}$$

If g is the acceleration due to gravity, then $\vec{v}' = gt$ (using the formula $v=u+ft$, here $u=0$ and $f=g$). Using this result equation (2.20) becomes

$$\frac{d^2x}{dt^2} = 2gw\cos\lambda t. \tag{2.21}$$

Integrating with respect to t, we get

$$\frac{dx}{dt} = 2gw\cos\lambda.\frac{t^2}{2} + A, A \text{ is a constant.} \tag{2.22}$$

From the initial conditions we know that $\frac{dx}{dt} = 0$ at $t=0$. With this substitution equation (2.22) yields $A=0$ and takes the form

$$\frac{dx}{dt} = gw\cos\lambda.t^2 \tag{2.23}$$

Again integrating we get,

$$x = gw\cos\lambda\frac{t^3}{3} + B, B \text{ is the constant.} \tag{2.24}$$

From the initial conditions it is obvious that initially the deviation is zero, i.e., $x=0$ when $t=0$. The above equation takes the form

$$x = \frac{t^3}{3}gw\cos\lambda. \tag{2.25}$$

This equation represents the deviation along x-axis at any time t at latitude λ. If h is the height traversed by the body in time t, then we have

$$h = \frac{1}{2}gt^2 \quad \text{(using the formula } s = ut + \frac{1}{2}ft^2\text{)}$$

or, $$t = \sqrt{\frac{2h}{g}}.$$

Substituting value of t from above equation in (2.25), we get

$$x = \frac{1}{3} gw\cos\lambda\left(\frac{2h}{g}\right)^{3/2}$$

$$= \frac{2}{3} hw\cos\lambda\left(\frac{2h}{g}\right)^{1/2}. \qquad (2.26)$$

This is the deviation of the falling body from the vertical due to the rotation of the earth towards east in the northern hemisphere and towards south in the southern hemisphere.

Example 2.2 *Calculate the deviation of freely falling body from a height of 100 meters at latitude 45^0N due to Coriolis acceleration.*

SOLUTION: The deviation of freely falling body from the vertical at latitude λ is given by

$$x = \frac{2}{3} wh\cos\lambda\left(\frac{2h}{g}\right)^{1/2}$$

Here $w = \dfrac{2\pi}{24 \times 60 \times 60}$ rad./sec.

$h = 100$ meter $= 10^4$ cm.

$\lambda = 45^0$ and $g = 980$ cm/sec^2.

Therefore, $x = \dfrac{2}{3} \times \dfrac{2\pi}{24 \times 60 \times 60} \times 10^4 \times \cos 45^0 \times \left(\dfrac{2 \times 10^4}{980}\right)^{1/2} = 1.55$ cm.

2.6 Focault Pendulum

In a frame rotating relative to an inertial frame, some additional forces such as Coriolis force and the centrifugal force appear. Thus unlike uniform motion in a straight line rotation may be said to have an absolute meaning. Mechanics make no distinction between two frames in uniform relative velocity. But in case there is a question of rotation, we may just go to investigate whether mechanical phenomena require some Coriolis type of force for their explanation. If they do, we say that the frame is rotating meaning thereby that it is rotating relative to an inertial frame and the magnitude -and direction of the Coriolis force would determine the magnitude and direction of angular velocity w.

Precisely this is the idea behind the Foucault pendulum experiment. Our frame is one that is bound to the earth. We shall develop the theory assuming that the earth is not an inertial frame and assume for the moment that the rotation is about the polar axis of the earth with angular velocity w.

Apparently all the heavenly bodies are revolving round us about once in a day. But it would be simpler to think that we ourselves are rotating rather than the millions of stars that we see. In any case, the earth-bound frame and the astronomical frame (i.e. the rest frame of the distant stars) cannot be both inertial. Foucault's pendulum experiment is designed to decide which one of these, if at all any one, is inertial and we make the calculations provisionally assuming that the earth frame is non-inertial.

We can consider a simple pendulum, its point of support is fixed to the earth. One could look at it as a problem with a time-dependent constraint-the point of support sharing the motion of the earth. But we shall rather treat it as a pendulum with a time-independent constraint but in a non-inertial frame.

We adopt a cartesian coordinate system with the x-axis towards the east, the y-axis towards the north and the z-axis vertically upwards. Of course by vertical we mean the earth's radial direction. The latitude of the place is ϕ. We have $w_z = w\sin\phi$. The equation of motion of the pendulum bob will be of the form

$$m\ddot{\vec{r}} - m\vec{g} - \vec{F}_{const} = 2m(\vec{v}\times\vec{w}).$$

Centrifugal term is omitting. As it is a constant force during the motion of the bob, we are absorbing it within mg. Of course due to the centrifugal force as well as the peculiarity in the shape of the earth, $\vec{g}$ will not be exactly towards the centre of the earth.

We may neglect this slight departure. The force of constraint $\vec{F}_{const}$ has already been evaluated by Lagrange's equation of the first kind in connection with the spherical pendulum and assuming as before that the amplitude is small (so that $\dot{z} = \ddot{z} = 0, z = -l$), we get

$$\ddot{x} + \frac{g}{l}x = 2\dot{y}w\sin\phi \tag{2.27}$$

$$\ddot{y} + \frac{g}{l}x = -2\dot{x}w\sin\phi. \tag{2.28}$$

In the Coriolis force expression $\dot{z}w_x$ and $\dot{z}w_y$ have been neglected, thus only $w_z = w\sin\phi$ is effective. Multiplying equation (2.28) by i, adding it to equation (2.27) and writing $\eta = x + iy$, we get

$$\ddot{\eta} + \frac{g}{l}\eta + 2iw\sin\phi\dot{\eta} = 0.$$

Let $\eta = Ae^{i\alpha t}$ be a trial solution, then

$$\alpha^2 + 2\alpha w \sin\phi - \frac{g}{l} = 0.$$

or,

$$\alpha^2 = -w\sin\phi \pm \left(w^2 \sin^2\phi + \frac{g}{l}\right)^{1/2}. \qquad (2.29)$$

Therefore, the solution is

$$\eta = e^{-iw\sin\phi t}\left[A_1 e^{ipt} + A_2 e^{-ipt}\right] \qquad (2.30)$$

where

$$p = \left(w^2 \sin^2\phi + \frac{g}{l}\right)^{1/2}.$$

In (2.29) A_1 and A_2 are arbitrary constants. As they are in general complex, there are in reality four arbitrary constants. As is the usual method, they are determined by some initial (or boundary) conditions about, say, $x, y, \dot{x}, \dot{y}$. However as there may be a wide variety of these conditions, this procedure could lead to tedious (but by no means difficult) calculations and would not be quite illuminating for our general purpose.

Consider the term $e^{-iw\sin\phi t}$. If we take w to be due to the diurnal rotation of the earth $w \simeq 2\pi/24hrs \simeq 10^{-4}$ *rad./sec*. Thus within a time of the order of a few minutes (more precisely a time which is small compared to a day) the term $e^{-iw\sin\phi t}$ would not appreciably vary and the dominant variation will be due to the term within the brackets. The latter term represents a simple harmonic motion with period $2\pi/p \simeq 2\pi\sqrt{l/g}$, as the w term is comparatively small. Thus observation over small intervals of time would show only a simple harmonic motion for both x and y (which are combined in η), as in the ordinary spherical pendulum in an inertial system. If however one makes an observation after a fairly long time (i.e., a time comparable with a day, say 3 or 4 hours), the modulating factor $e^{-iw\sin\phi t}$ would have undergone a significant change. To understand the influence of this change, assume for simplicity that the term within the brackets in (2.30) is purely real, say, $A\cos(pt+\alpha)$. Then at time $t = 0$

$$x = A\cos(pt+\alpha)$$
$$y = 0.$$

i.e., it is simple harmonic motion in x-direction. But at time $t = T$ (say).

$$x = A\cos(w\sin\phi T)\cos(pt+\alpha).$$

$$y = -A\sin(w\sin\phi T)\cos(pt+\alpha).$$

Now the oscillation will appear to be simple harmonic motion in a direction making an angle $(-w\sin\phi T)$ with the x-axis, i.e., the plane of oscillation apparently rotates with an angular velocity $w\sin\phi$ about the z-axis.

The conclusion is by no means dependent on the assumption that the term in brackets is purely real. Consider instead that it is purely imaginary, say, $iB\cos(pt+\beta)$. Then at time $t \simeq 0$,

$$x = 0$$
$$y = B\cos(pt+\beta)$$

i.e., it is a motion purely in the y-direction. Again after time T,

$$x = B\sin(w\sin\phi T)\cos(pt+\beta)$$
$$y = B\cos(w\sin\phi T)\cos(pt+\beta).$$

Thus the new plane of oscillation makes an angle $(\pi/2 - w\sin\phi T)$ with the x-axis instead of $\pi/2$ previously and we have the same conclusion as before.

Thus the observation will consist in noting whether the plane of oscillation changes and if it does to calculate the value of w from the rate of rotation of the plane of oscillation. Actual determination of w in this way shows that it agrees with the value calculated from astronomical observations, i.e. $w \simeq 2\pi/24hrs.$

This apparently shows that the astronomical frame is an inertial one while the earth-bound frame is rotating relative to it. This fact that the astronomical frame is an inertial frame has been confirmed to a very high degree of accuracy by later observations and analysis.

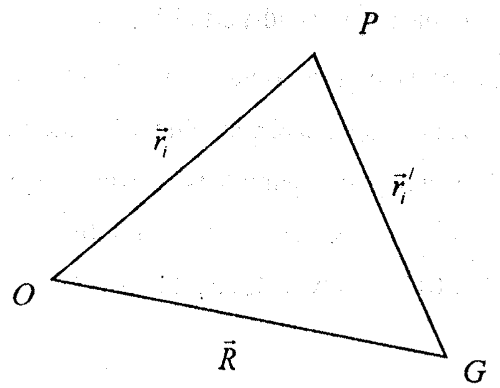

FIGURE 2.4 :

2.7 Dynamics of Rigid Body when Rotating About a Fixed Point

2.7.1 Components of velocity

Let O be a fixed point about which the rigid body is rotating and G be the centre of mass.

Let $\vec{OG} = \vec{R}, \vec{OP} = \vec{r}_i, \vec{GP} = \vec{r}_i'$. Let $\vec{w}$ be the angular velocity of rigid body whose axis along a direction with components (w_x, w_y, w_z). Let $\vec{v}_i$ be the relative velocity of the *i*th point P.

Therefore,

$$\vec{w} = w_x\vec{i} + w_y\vec{j} + w_z\vec{k}.$$

Therefore,

$$\vec{v}_i = \vec{w} \times \vec{r}_i = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ w_x & w_y & w_z \\ x_i & y_i & z_i \end{vmatrix}$$

$$= \vec{i}\left(w_y z_i - w_z y_i\right) + \vec{j}\left(w_z x_i - w_x z_i\right) + \vec{k}\left(w_x y_i - w_y x_i\right).$$

Now, $(v_i)_x = w_y z_i - w_z y_i, (v_i)_y = w_z x_i - w_x z_i, (v_i)_z = w_x y_i - w_y x_i.$

These are the components of the velocity at the point P about a fixed point O.

2.7.2 Components of angular momentum

Let $\vec{h}$ be the angular momentum of the rigid body. From the definition

$$\vec{h} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \vec{r}_i \times (m_i\vec{v}_i) = \sum_i m_i(\vec{r}_i \times \vec{v}_i)$$

$$= \sum_i m_i \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_i & y_i & z_i \\ (v_i)_x & (v_i)_y & (v_i)_z \end{vmatrix}$$

$$= \sum_i m_i\left[\{y_i(v_i)_z - (v_i)_y z_i\}\vec{i} + \{z_i(v_i)_x - (v_i)_z x_i\}\vec{j} + \{x_i(v_i)_y - (v_i)_x y_i\}\vec{k}.\right.$$

Since $\vec{h} = h_x\vec{i} + h_y\vec{j} + h_z\vec{k}$. Therefore,

$$h_x = \sum m_i\{y_i(v_i)_z - z_i(v_i)_y\}$$

$$= \sum m_i\{y_i(y_i w_x - x_i w_y) - z_i(x_i w_z - z_i w_x)\}$$

$$= w_x \sum m_i(y_i^2 + z_i^2) - w_y \sum m_i x_i y_i - w_z \sum m_i x_i z_i.$$

Let us introduce the notations

$$A = \sum m_i(y_i^2 + z_i^2), B = \sum m_i(z_i^2 + x_i^2), C = \sum m_i(x_i^2 + y_i^2),$$

$$D = \sum m_i y_i z_i,\ \ E = \sum m_i z_i x_i,\ \ F = \sum m_i x_i y_i,$$

where A, B, C are called moment of inertia while D, E, F are called product of inertia about Ox, Oy, Oz respectively.

Then

$$h_x = Aw_x - Ew_z - Fw_y. \tag{2.31}$$

Similarly,

$$h_y = Bw_y - Dw_z - Fw_x, \tag{2.32}$$

$$h_z = Cw_z - Dw_y - Ew_x. \tag{2.33}$$

Note: If the axes are principal axes then

$D = E = F = 0$ and hence $h_x = Aw_x, h_y = Bw_y, h_z = Cw_z.$ (2.34)

2.7.3 Kinetic energy of a rigid body

The K.E. (T) is given by

$$T = \frac{1}{2}\sum m_i v_i^2$$

$$= \frac{1}{2}\sum m_i\{(v_i)_x^2 + (v_i)_y^2 + (v_i)_z^2\}$$

$$= \frac{1}{2}\sum m_i\{(z_i w_y - y_i w_z)^2 + (x_i w_z - z_i w_x)^2 + (y_i w_x - x_i w_y)^2\}$$

$$= \frac{1}{2}w_x^2\{\sum m_i(y_i^2 + z_i^2)\} + \frac{1}{2}w_y^2\{\sum m_i(z_i^2 + x_i^2)\} + \frac{1}{2}w_z^2\{\sum m_i(x_i^2 + y_i^2)\}$$

$$-w_y w_z \sum m_i y_i z_i - w_x w_z \sum m_i z_i x_i - w_x w_y \sum m_i x_i y_i$$

$$= \frac{1}{2}(Aw_x^2 + Bw_y^2 + Cw_z^2) - Dw_y w_z - Ew_z w_x - Fw_x w_y.$$

Note: If the axes are principal axes then

$$T = \frac{1}{2}\left(Aw_x^2 + Bw_y^2 + Cw_z^2\right).$$

2.8 Euler's Equations of Motion for a Rigid Body Rotation

Let a rigid body be rotating about a point *O* fixed in both body and space. Let us take the coordinate axes *Ox, Oy, Oz* along the three principal axes of inertia of the body at *O*. These axes are arbitrary orthogonal and fixed in the body. Let $\vec{i}, \vec{j}, \vec{k}$ be the unit vectors along *Ox, Oy, Oz*. At time *t*, the body has an angular velocity *w* and the angular momentum $\vec{h}$ as

$$\vec{w} = w_x\vec{i} + w_y\vec{j} + w_z\vec{k} \text{ and } \vec{h} = h_x\vec{i} + h_y\vec{j} + h_z\vec{k},$$

where w_x, w_y, w_z are the components of $\vec{w}$ along $\vec{i}, \vec{j}, \vec{k}$ and h_x, h_y, h_z are the components of $\vec{h}$ along *Ox, Oy, Oz*.

Consider a particle of mass *m* of the body at *P(x, y, z)* so that $\vec{r} = \overrightarrow{OP}$. If $\vec{v}$ be the velocity of *P* then $\vec{v} = \vec{w} \times \vec{r}$. Now

$$\vec{h} = \sum \vec{r} \times m\vec{v} = \sum \vec{r} \times m(\vec{w} \times \vec{r})$$
$$= \sum m\{(\vec{r}\cdot\vec{r})\vec{w} - (\vec{r}\cdot\vec{w})\vec{r}\}$$
$$= \sum m\left[(x^2 + y^2 + z^2)\vec{w} - (xw_x + yw_y + zw_z)\vec{r}\right].$$

Then

$$h_x = \sum m\left[(x^2 + y^2 + z^2)w_x - (xw_x + yw_y + zw_z)x\right]$$

or

$$h_x = w_x\sum m(y^2 + z^2) - w_y\sum mxy - w_z\sum mzx$$

or

$$h_x = Aw_x - Fw_y - Ew_z. \tag{2.36}$$

Similarly

$$h_y = -Fw_x + Bw_y - Dw_z \tag{2.37}$$
$$h_z = -Ew_x - Dw_y + Cw_z \tag{2.38}$$

where *A, B, C, D, E, F* are moments and products of inertia with respect to the axes *Ox, Oy, Oz*. Since *Ox, Oy,*

Oz are the principal axes of inertia of the body at O, so $D = E = F = 0$. Then equation (2.36) - (2.38) becomes

$$h_x = Aw_x \tag{2.39}$$

$$h_y = Bw_y \tag{2.40}$$

$$h_z = Cw_z \tag{2.41}$$

where A, B, C are now the principle moments of inertia at O. Now let

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

be the external applied force on the particle of mass m at P. We have from Newton's second law of motion

$$\sum \vec{r} \times m\frac{d^2\vec{r}}{dt^2} = \sum \vec{r} \times \vec{F}. \tag{2.42}$$

But,

$$\vec{h} = \sum \vec{r} \times m\vec{v} = \sum \vec{r} \times m\frac{d\vec{r}}{dt}.$$

Therefore,

$$\frac{d\vec{h}}{dt} = \sum\left(\frac{d\vec{r}}{dt} \times m\frac{d\vec{r}}{dt} + \vec{r} \times m\frac{d^2\vec{r}}{dt^2}\right) = \sum \vec{r} \times m\frac{d^2\vec{r}}{dt^2}.$$

Then equation (2.42) becomes

$$\frac{d\vec{h}}{dt} = \sum \vec{r} \times \vec{F} = \vec{N} \text{ (say)}$$

where $\vec{N}$ is the moment of the force $\vec{F}$ about O. Therefore,

$$\frac{d\vec{h}}{dt} + \vec{w} \times \vec{h} = N_x\vec{i} + N_y\vec{j} + N_z\vec{k} \tag{2.43}$$

where

$$N_x = \sum\left(yF_z - zF_y\right)$$

$$N_y = \sum\left(zF_x - xF_z\right)$$

$$N_z = \sum\left(xF_y - yF_x\right)$$

Here N_x, N_y, N_z are the moments of external forces about Ox, Oy, Oz respectively. Now, from (2.43) we have

$$\dot{h}_x\vec{i} + \dot{h}_y\vec{j} + \dot{h}_z\vec{k} + \vec{i}\left(w_yh_z - w_zh_y\right) + \vec{j}\left(w_zh_x - w_xh_z\right) + \vec{k}\left(w_xh_y - w_yh_x\right) = N_x\vec{i} + N_y\vec{j} + N_z\vec{k}$$

Comparing the coefficients, we get

$$\dot{h}_x + w_x h_z - w_y h_y = N_x \quad (2.44)$$

$$\dot{h}_y + w_z h_x - w_x h_z = N_y \quad (2.45)$$

$$\dot{h}_z + w_x h_y - w_y h_x = N_z. \quad (2.46)$$

When the axes are taken as principal axes then the above equations become

$$A\dot{w}_x - w_y w_z (B - C) = N_x \quad (2.47)$$

$$B\dot{w}_y - w_z w_x (C - A) = N_y \quad (2.48)$$

$$C\dot{w}_z - w_x w_y (A - B) = N_z. \quad (2.49)$$

These equations are known as Euler's dynamical equation for motion of a rigid body about a fixed point.

Note: If there is no force, then $\vec{F} = 0$. In this case $N_x = N_y = N_z = 0$. Then above equations become

$$A\dot{w}_x - (B - C) w_y w_z = 0 \quad (2.50)$$

$$B\dot{w}_y - (C - A) w_z w_x = 0 \quad (2.51)$$

$$C\dot{w}_z - (A - B) w_x w_y = 0. \quad (2.52)$$

2.8.1 Conservation of K.E. and angular momentum

Multiplying equations (2.50), (2.51) and (2.52) by w_x, w_y and w_z respectively and adding we get

$$A w_x \dot{w}_x + B w_y \dot{w}_y + C w_z \dot{w}_z = 0.$$

Integrating, we get

$$A w_x^2 + B w_y^2 + C w_z^2 = \text{constant}$$

or,

$$2T = \text{constant}.$$

Therefore, K.E. of a rigid body is conserved.

Again, multiplying equations (2.50), (2.51) and (2.52) by Aw_x, Bw_y and Cw_z respectively and adding we get

$$A^2 w_x \dot{w}_x + B^2 w_y \dot{w}_y + C^2 w_z \dot{w}_z = 0.$$

Integrating, we get

$$A^2 w_x^2 + B^2 w_y^2 + C^2 w_z^2 = \text{constant} = H_2.$$

$\left[\text{as } h_x = Aw_x, h_y = Bw_y, h_z = Cw_z, H^2 = h_x^2 + hh_y^2 + h_z^2 = A^2w_x^2 + B^2w_y^2 + C^2h_z^2\right]$

Therefore, the magnitude of resultant angular momentum is conserved.

2.8.2 Invariable line and invariable plane

A line through the point O in the fixed direction of $\vec{h}$ is called the **invariable line**. Let P be a point such that $\vec{OP} = \vec{w}$ at any instant. We draw a line PN perpendicular from P on OQ (see Figure 2.5).

We have

$$\vec{h} = Aw_x\vec{i} + Bw_y\vec{j} + Cw_z\vec{k}$$

and

$$\vec{w} = w_x\vec{i} + w_y\vec{j} + w_z\vec{k}.$$

Therefore,

$$\vec{h}.\vec{w} = Aw_x^2 + Bw_y^2 + Cw_z^2 = 2T.$$

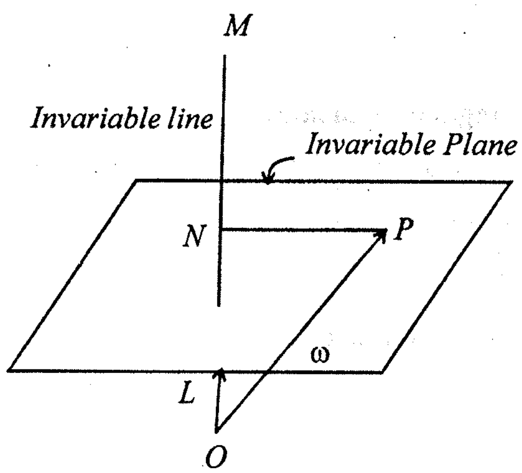

FIGURE 2.5: Invariable plane.

Thus, during the motion

$$\vec{h}.\vec{w} = \text{constant}. \qquad (2.53)$$

Then from equation (2.53) we have

$$ON = \text{Projection of } w \text{ in the direction of } \vec{h} = \text{constant}.$$

Thus N is a fixed point during the motion. The plane through N perpendicular to the invariable line OQ is a fixed plane. This fixed plane is called the **invariable plane**.

The motion of a rigid body about a fixed point (under no forces) may be represented by the rolling of an ellipsoid fixed in the body upon an invariable plane. The invariant plane is fixed in space. This ellipsoid is known as **Poinsot ellipsoid**. The point of contact of the ellipsoid and the invariable plane lies on the instantaneous axis of rotation and so there can be no slipping at this point. Hence the ellipsoid rolls on the invariable plane. The curve traces out by the point of contact on the ellipsoid is known as the **polhode** while similar curve on the invariable plane is called the **herpolhode.**

2.8.3 Instantaneous axis of rotation

Let the rigid body be turn about the fixed point *O* and at time *t*, let w_x, w_y, w_z be the components of angular velocity about the principle axes *OA, OB, OC* then the direction cosine of the resultant axis of rotation are $\left\{\frac{w_x}{w}, \frac{w_y}{w}, \frac{w_z}{w}\right\}$ where $w_x^2 + w_y^2 + w_z^2$. But, w_x, w_y, w_z have been expressed in terms of Eulerian angles θ, ϕ, ψ which change from time to time. Hence w_x, w_y, w_z as changes as time changes. Consequently, the resultant axis of rotation also changes implying that at different instant there are different axis of rotation. In view of this, the resultant axis of rotation is known as instantaneous axis of rotation or simply the instantaneous axis.

2.9 Solution of Euler's Dynamical Equations

We have the following three equations

$$Aw_x^2 + Bw_y^2 + Cw_z^2 - 2T = 0$$

$$A^2w_x^2 + B^2w_y^2 + C^2w_z^2 - H^2 = 0$$

$$w_x^2 + w_y^2 + w_z^2 - w^2 = 0.$$

Solving these equation by Cramers rule, we get

$$\frac{w_x^2}{\begin{vmatrix} -2T & B & C \\ -H^2 & B^2 & C^2 \\ -w^2 & 1 & 1 \end{vmatrix}} = \frac{w_y^2}{\begin{vmatrix} A & -2T & C \\ A^2 & -H^2 & C^2 \\ 1 & -w^2 & 1 \end{vmatrix}} = \frac{w_z^2}{\begin{vmatrix} A & B & -2T \\ A^2 & B^2 & -H^2 \\ 1 & 1 & -w^2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} A & B & C \\ A^2 & B^2 & C^2 \\ 1 & 1 & 1 \end{vmatrix}}.$$

Therefore

$$w_x^2 = \frac{-BC(B-C)}{(A-B)(B-C)(C-A)}\left[w^2 \frac{2T(B+C)-H^2}{BC}\right]$$

$$=\frac{BC}{(C-A)(A-B)}\left[\lambda_1^2-w^2\right], \text{ where } \lambda_1^2=\frac{2T(B+C)-H^2}{BC}.$$

Similarly,

$$w_y^2=\frac{CA}{(A-B)(B-C)}\left[\lambda_2^2-w^2\right], \text{ where } \lambda_2^2=\frac{2T(A+C)-H^2}{AC},$$

$$w_z^2=\frac{AB}{(B-C)(C-A)}\left[\lambda_3^2-w^2\right], \text{ where } \lambda_3^2=\frac{2T(A+B)-H^2}{AB}.$$

Now, $w^2=w_x^2+w_y^2+w_z^2$.

Differentiating w.r.t. time t, we get

$$w\dot{w}=w_x\dot{w}_x+w_y\dot{w}_y+w_z\dot{w}_z$$

or,

$$w\dot{w}=w_xw_yw_z\left[\frac{B-C}{A}+\frac{C-A}{B}+\frac{A-B}{C}\right]$$

or,

$$w\frac{dw}{dt}=\pm\frac{ABC}{(C-A)(A-B)(B-C)}\sqrt{\left(\lambda_1^2-w^2\right)\left(\lambda_2^2-w^2\right)\left(\lambda_3^2-w^2\right)}$$

$$\times\frac{(B-C)(C-A)(A-B)}{ABC}$$

$$=\pm\sqrt{\left(\lambda_1^2-w^2\right)\left(\lambda_2^2-w^2\right)\left(\lambda_3^2-w^2\right)}$$

[by substituting the values of w_x, w_y, w_z.]

or,

$$\frac{wdw}{\sqrt{\left(\lambda_1^2-w^2\right)\left(\lambda_2^2-w^2\right)\left(\lambda_3^2-w^2\right)}}=\pm dt.$$

Integrating we get

$$\int\frac{wdw}{\sqrt{\left(\lambda_1^2-w^2\right)\left(\lambda_2^2-w^2\right)\left(\lambda_3^2-w^2\right)}}=\pm(t+c).$$

Substituting $w^2 = u$, therefore, $2wdw = du$. Then

$$\int \frac{du}{\sqrt{(\lambda_1^2 - u)(\lambda_2^2 - u)(\lambda_3^2 - u)}} = \pm 2(t + c),$$

which is an elliptic integral, this gives the time t in terms of w.

2.9.1 Integrable cases of Euler dynamical equations

Case I. $A = B$.

In this case the Euler's dynamical equations reduce to

$$A\dot{w}_x - (A - C)w_y w_z = 0 \tag{2.54}$$

$$B\dot{w}_y - (C - A)w_x w_z = 0 \tag{2.55}$$

$$C\dot{w}_z = 0 \tag{2.56}$$

From (2.56) w_z = constant = η (say)

Multiplying (2.54) and (2.55) by w_x and w_y respectively and adding we get

$$A\left(w_x \dot{w}_x + w_y \dot{w}_y\right) = 0$$

or,

$$w_x^2 + w_y^2 = \text{constant}$$

or,

$$w^2 = w_x^2 + w_y^2 + w_z^2 = \text{constant}.$$

Therefore, w, the resultant angular velocity is constant.

Now,

$$\cos^{-1}\left(\frac{w_z}{w}\right) = p \text{ (say)}$$

is constant.

Therefore,

$$w_z = w\cos p. \tag{2.57}$$

or,

$$w_x^2 + w_y^2 = w^2 - w_z^2 = w^2 \sin^2 p \text{ (by (2.57))}.$$

Let

$$w_x = w \sin p \cos \chi \text{ and } w_y = w \sin p \sin \chi \qquad (2.58)$$

Substituting the values of w_x, w_y, w_z in (2.54), we get

$$-Aw \sin p \sin \chi \frac{d\chi}{dt} = (A-C) w \sin p \sin \chi . \eta$$

or,

$$\frac{d\chi}{dt} = \frac{A-C}{A} . \eta$$

or,

$$\chi = -\frac{A-C}{A} \eta (t - t_0),$$

where t_0 is the constant of integration.

Therefore,

$$w_x = w \sin p \cos \left\{ -\frac{A-C}{A} \eta (t - t_0) \right\}$$

$$w_y = w \sin p \sin \left\{ -\frac{A-C}{A} \eta (t - t_0) \right\}$$

$$w_z = w \cos p.$$

Therefore, the components of angular velocity are known.

Case II. *Let B be the mean of moment of inertia so that either A > B > C or A < B < C and also $H^2 = 2BT$ where T is the K.E. and H is the resultant angular momentum.*

In this case, the Euler's dynamical equations of motion reduce to

$$A\dot{w}_x - (B-C) w_y w_z = 0 \qquad (2.59)$$

$$B\dot{w}_y - (C-A) w_x w_z = 0 \qquad (2.60)$$

$$C\dot{w}_y - (A-B) w_x w_y = 0 . \qquad (2.61)$$

Multiplying (2.59), (2.60), (2.61) by w_x, w_y and w_z respectively and adding we get

$$A w_x \dot{w}_x + B w_y \dot{w}_y + C w_z \dot{w}_z = 0.$$

Integrating

$$A w_x^2 + B w_y^2 + C w_z^2 = \text{constant} = 2T = H^2 / B$$

Again, multiplying (2.59), (2.60), (2.61) by Aw_x, Bw_y, Cw_z we get

$$A^2 w_x \dot{w}_x + B^2 w_y \dot{w}_y + C^2 w_z \dot{w}_z = 0$$

or,

$$A^2 w_x^2 + B^2 w_y^2, C^2 w_z^2 = \text{constant} = H^2.$$

The above two equations can be written as

$$ABw_x^2 + BCw_z^2 - \left(H^2 - Bw_y^2\right) = 0$$

$$A^2 w_x^2 + C^2 w_z^2 - \left(H^2 - B^2 w_y^2\right) = 0.$$

Solving these equations we get

$$\frac{w_x^2}{C^2 - BC} = \frac{w_z^2}{AB - A^2} = \frac{H^2 - B^2 w_y^2}{ABC^2 - A^2 BC}.$$

Therefore,

$$w_x^2 = \frac{B-C}{A-C}\frac{1}{AB}\left(H^2 - B^2 w_y^2\right)$$

$$w_z^2 = \frac{A-B}{A-C}\frac{1}{BC}\left(H^2 - B^2 w_y^2\right).$$

Putting w_x and w_z in (2.60) we get

$$B\frac{dw_y}{dt} = (C - A) w_x w_z$$

$$= \pm(C - A)\sqrt{\frac{B-C}{A-C}\frac{1}{AB}\left(H^2 - B^2 w_y^2\right)\frac{A-B}{A-C}\frac{1}{BC}\left(H^2 - B^2 w_y^2\right)}$$

$$= \pm\sqrt{\frac{(B-C)(A-P)}{AC}}\frac{1}{B}\left(H^2 - B^2 w_y^2\right)$$

$$\frac{Bdw_y}{H^2 - B^2 w_y^2} = \pm\sqrt{\frac{(B-C)(A-B)}{AC}}\frac{1}{B}dt.$$

Integrating we have

$$\frac{1}{H}\tanh^{-1}\left(\frac{Bw_y}{H}\right) = \pm\sqrt{\frac{(B-C)(A-B)}{AC}}\frac{1}{B}(t - t_0)$$

$$= \pm\frac{\lambda}{H}(t - t_0)$$

$$\text{where } \lambda = \frac{H}{B}\sqrt{\frac{(B-C)(A-B)}{AC}}$$

or,

$$w_y = \pm\frac{H}{B}\tanh\{\lambda(t-t_0)\}.$$

Therefore,

$$w_x^2 = \frac{B-C}{A-C}\frac{1}{AB}\{H^2 - H^2\tanh^2(\lambda(t-t_0))\}$$

$$= \frac{B-C}{A-C}\frac{1}{AB}H^2\sec h^2(\lambda(t-t_0))$$

$$w_z^2 = \frac{A-B}{A-C}\frac{1}{BC}H^2\sec h^2(\lambda(t-t_0)).$$

Corollary 2.1 *As* $t\to\infty$, $\tanh\{\lambda(t-t_0)\}\to 1$ *and* $\sec h\{\lambda(t-t_0)\}\to 0$. *In this case,* $w_x\to 0, w_z\to 0$, $w_y\to \pm H/B \text{ as } t\to\infty$.

So finally the rigid body rotates about the axis of mean moment of inertia.

2.10 Worked out Examples

Example 2.3 *(a) A rigid body is rotating about a fixed point at which A, A, C are principal moments of inertia under a couple* $-\lambda w_1, -\lambda w_2, -\lambda w_3$, *where* w_1, w_2, w_3, *are the angular velocity components under the principal axes. Prove that at time t,*

$$w_1 = ae^{-\lambda t/A}\sin\left(\frac{\sigma C}{\lambda}e^{-\lambda t/C} + \varepsilon\right)$$

$$w_2 = ae^{-\lambda t/A}\cos\left(\frac{\sigma C}{\lambda}e^{-\lambda t/C} + \varepsilon\right)$$

$$w_3 = ne^{-\lambda t/C} \quad \textit{where } \sigma = \frac{n(C-A)}{A}$$

a, n are constants.

(b) Also, show that instantaneous axes would approach ultimately to coincides with the least axis of the rigid body.

SOULUTION: (a) Here the Euler's equations are

$$A\dot{w}_1 - (A-C)w_2w_3 = -\lambda w_1 \tag{2.62}$$

$$A\dot{w}_2 - (C-A)w_1w_3 = -\lambda w_2 \tag{2.63}$$

$$C\dot{w}_3 - (A-A)w_1w_2 = -\lambda w_3 \tag{2.64}$$

From (2.64),

$$\frac{\dot{w}_3}{w_3} = -\frac{\lambda}{C}$$

or, $\log w_3 = -\frac{\lambda}{C}t + \log n, n$ constant

or,

$$w_3 = ne^{-\lambda t/C}. \tag{2.65}$$

Multiplying equation (2.62) by w_1 and (2.63) by w_2 and adding we obtain

$$w_1\dot{w}_1 + w_2\dot{w}_2 = -\frac{\lambda}{A}\left(w_1^2 + w_2^2\right)$$

or,

$$\frac{1}{2}\frac{d}{dt}\left(w_1^2 + w_2^2\right) = -\frac{\lambda}{A}\left(w_1^2 + w_2^2\right)$$

or,

$$\frac{d\left(w_1^2 + w_2^2\right)}{\left(w_1^2 + w_2^2\right)} = -\frac{2\lambda}{A}dt$$

Integrating we get

$$w_1^2 + w_2^2 = a^2e^{-2\lambda t/A},\ a \text{ is a constant.}$$

Let $w_1 = ae^{-\lambda t/A}\sin\chi$ and $w_2 = ae^{-\lambda t/A}\cos\chi$.

Putting these values in (2.62) we have

$$Aa\left[-\frac{\lambda}{A}e^{-\lambda t/A}\sin\chi + e^{-\lambda t/A}\cos\chi\frac{d\chi}{dt}\right] - (A-C)ae^{-\lambda t/A}\cos\chi . ne^{-\lambda t/C} = -\lambda ae^{-\lambda t/A}\sin\chi$$

or, $A\frac{d\chi}{dt} - n(A-C)e^{-\lambda t/C} = 0$

or,

$$\frac{d\chi}{dt} = \frac{n(A-C)}{A} e^{-\lambda t/C}$$

$$= -\sigma e^{-\lambda t/C} \quad \text{where } \sigma = \frac{n(A-C)}{A}$$

Therefore,

$$\chi = -\sigma(-C/\lambda)e^{-\lambda t/C} + \varepsilon = \frac{\sigma C}{\lambda} e^{-\lambda t/C} + \varepsilon.$$

Hence

$$w_1 = ae^{-\lambda t/A} \sin\left\{\frac{C\sigma}{\lambda} e^{-\lambda t/C} + \varepsilon\right\}$$

$$w_2 = ae^{-\lambda t/A} \cos\left\{\frac{C\sigma}{\lambda} e^{-\lambda t/C} + \varepsilon\right\}$$

$$w_3 = ne^{-\lambda t/C}.$$

These are the required components of the angular velocity.

(b) We have

$$w_1 = ae^{-\lambda t/A} \sin\left\{\frac{C\sigma}{\lambda} e^{-\lambda t/C} + \varepsilon\right\}$$

$$w_2 = ae^{-\lambda t/A} \cos\left\{\frac{C\sigma}{\lambda} e^{-\lambda t/C} + \varepsilon\right\}$$

$$w_3 = ne^{-\lambda t/C}.$$

Direction cosine of the instantaneous axes is given by

$$\frac{w_1}{\sqrt{w_1^2 + w_2^2 + w_3^2}}, \frac{w_2}{\sqrt{w_1^2 + w_2^2 + w_3^2}}, \frac{w_3}{\sqrt{w_1^2 + w_2^2 + w_3^2}}.$$

But

$$w_1^2 + w_2^2 + w_3^2 = a^2 e^{-2\lambda t/A} + n^2 e^{-2\lambda t/C}$$

$$= n^2 e^{-2\lambda t/C}\left\{1 + \frac{a^2}{n^2} e^{-2\lambda t(1/A - 1/C)}\right\}$$

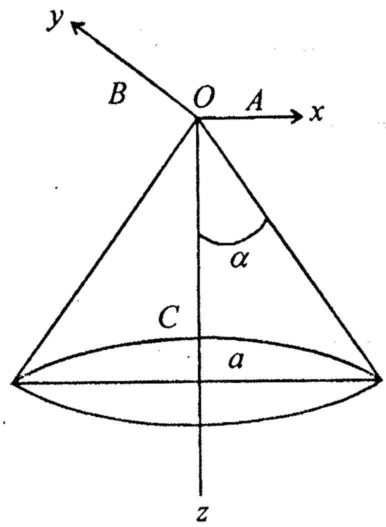

FIGURE 2.6:

Therefore, the direction cosine are 0, 0, 1 as $t \to \infty$ for $C > A$.

Hence the instantaneous axes ultimately coincides with the least axis i.e., with the z-axis.

Example 2.4 *A uniform right circular cone vertical angle 2α moves under no forces except at its vertex which is fixed. It is set rotating about a generator. Show that its axis describes in space a right cone of angle 2β where*

$$\tan\beta = \frac{1}{2}\tan\alpha + 2\cot\alpha.$$

SOLUTION: Let O be the vertex of the cone about which it rotates. The principal axes at the vertex O are the axes of the cone and two lines through O at right angles to the axis of the cone. Thus Ox, Oy, Oz are the principal axes at O.

Let w be the initial angular velocity with which the cone is set rotating about the generator. Thus initially $w_1 = w\sin\alpha,\ w_2 = 0,\ w_3 = w\cos\alpha$.

Here $A = B$ = moment of inertia about the vertex is

$$\frac{2M}{20}\left(a^2 + 4h^2\right),$$

where $h = a\cot\alpha$, h = height, a = radius of the base and $C = \frac{3Ma^2}{10}$.

In this problem the Euler's equations are

$$A\dot{w}_1 - (B-C)w_2w_3 = 0$$
$$B\dot{w}_2 - (C-A)w_3w_1 = 0$$
$$C\dot{w}_3 - (A-B)w_1w_2 = 0$$

Here $B = A$, the above equation become

$$A\dot{w}_1 - (A-C)w_2w_3 = 0$$

or, $$\dot{w}_1 = \frac{A-C}{A}w_2w_3 \qquad (2.66)$$

$$A\dot{w}_2 - (C-A)w_3w_1 = 0$$

or, $$\dot{w}_2 = -\frac{A-C}{A}w_1w_3 \qquad (2.67)$$

$$C\dot{w}_3 = 0$$

or, $$\dot{w}_3 = 0 \qquad (2.68)$$

Integrating (2.68), we get $w_3 = a, a$ is a constant.

But, initially, $w_3 = w\cos\alpha$. Thus $a = w\cos\alpha$.

Hence

$$w_3 = w\cos\alpha. \qquad (2.69)$$

Dividing (2.66) by (2.67) we obtain

$$\frac{\dot{w}_1}{\dot{w}_2} = -\frac{w_2}{w_1} \quad \text{and} \quad w_1\dot{w}_1 + w_2\dot{w}_2 = 0$$

Integrating we get, $w_1^2 + w_2^2 = b$, b is a constant.

Initially, $w_1 = w\sin\alpha$, $w_2 = 0$. Therefore,

$$b = w^2\sin^2\alpha \cdot \qquad (2.70)$$

Hence

$$w_1^2 + w_2^2 = w^2\sin^2\alpha.$$

The invariable line is fixed in space and its direction cosines are proportional to Aw_1, Bw_2, Cw_3.

If θ be the angle the invariable line makes with OZ, then

$$\cos\theta = \frac{Cw_3}{\sqrt{A^2w_1^2 + B^2w_2^2 + C^2w_3^2}}$$

$$= \frac{Cw_3}{\sqrt{A^2(w_1^2 + w_2^2) + C^2w_3^2}} \text{ [since } B = A\text{]}$$

$$= \frac{C}{\sqrt{A^2w^2 \sin^2\alpha + C^2w^2\cos^2\alpha}}$$

$$= \frac{C}{\sqrt{A^2\tan^2\alpha + C^2}}$$

That is,

$$\tan\theta = \frac{A\tan\alpha}{C} = \frac{1}{2}\frac{a^2 + 4h^2}{a^2} \times \tan\alpha$$

$$= \frac{1}{2}(1 + 4\cot^2\alpha)\tan\alpha \text{ as } \frac{h}{a} = \cot\alpha$$

$$= \frac{1}{2}\tan\alpha + 2\cot\alpha$$

which is constant as α is constant.

Therefore,

$$\tan\beta = \frac{1}{2}\tan\alpha + 2\cot\alpha \text{ as } \theta = \beta$$

Hence axis of z i.e., axis of the cone describes about the invariable line a right cone of vertical angle 2β.

Example 2.5 *A body moves about a point O under no forces. The principal moments of inertia at O being 3A, 5A and 6A. Initially the angular velocity has components* $w_1 = n, w_2 = 0, w_3 = n$ *about the corresponding principal axes. Show that at any time t.*

$$w_2 = \frac{3n}{\sqrt{5}}\tan\left(\frac{nt}{\sqrt{5}}\right)$$

and that the body ultimately rotates the mean axis.

Solution: Since there are no forces, the Euler's equations of motion under usual notations are

$$A\dot{w}_1 - (B - C)w_2w_3 = 0 \tag{2.71}$$

$$B\dot{w}_2 - (C - A)w_3 w_1 = 0 \tag{2.72}$$

$$C\dot{w}_3 - (A - B)w_1 w_2 = 0. \tag{2.73}$$

Substituting the values of *A, B, C* we have

$$3\dot{w}_1 = -w_2 w_3 \tag{2.74}$$

$$5\dot{w}_2 = 3w_3 w_1 \tag{2.75}$$

$$3\dot{w}_3 = -w_1 w_2. \tag{2.76}$$

From equations (2.74) and (2.75), we obtain

$$9w_1\dot{w}_1 + 5w_2\dot{w}_2 = 0.$$

Integrating, we get

$$9w_1^2 + 5w_2^2 = \text{constant} = a \text{ (say)}$$

Initially, $w_1 = n,\ w_2 = 0$, therefore,

$$a = 9n^2. \tag{2.77}$$

From (2.74) and (2.76), we have

$$w_1\dot{w}_1 - w_3\dot{w}_3 = 0.$$

Integrating we get

$$w_1^2 - w_3^2 = b \text{ (say)}$$

Initially when $w_1 = n,\ w_3 = n,$ we get $b = 0$. Thus,

$$w_1^2 = w_3^2. \tag{2.78}$$

Then (2.75) becomes $5\dot{w}_2 = 3w_3 w_1 = 3w_1^2$.

Therefore,

$$5\frac{dw_2}{dt} = \frac{1}{3}\left(9n^2 - 5w_2^2\right)$$

or,

$$t = 15\int_{t=0}^{t} \frac{dw_2}{9n^2 - 5w_2^2} = 3\int \frac{dw_2}{9n^2/5 - w_2^2} = \frac{\sqrt{5}}{n}\tanh^{-1}\left(\frac{\sqrt{5}}{2n}w_2\right).$$

Therefore,

$$w_2 = \frac{3n}{\sqrt{5}}\tanh\left(\frac{nt}{\sqrt{5}}\right).$$

Further when $t \to \infty$

$$\tanh\left(\frac{nt}{\sqrt{5}}\right) \to 1.$$

Therefore, $w_2 = 3n/\sqrt{5}$. Now putting $w_2 = 3n/\sqrt{5}$ in (2.77), we get $w_1 = 0$. From (2.78), $w_3 = 0$. Thus, we see that the rotation is about the mean axis.

2.11 Unit Summary

The generalised coordinates of rigid body are defined. The Euler's angles are introduced. Components of angular velocity in terms of Euler's angles are deduced. Expression for Coriolis force is deduced and its effect in freely falling bodies is shown in this unit. A note on Focault pendulum is given. Euler's equation of motion for a rigid body when rotating about a fixed point is deduced. Some integrable cases of this equation are discussed. An exercise is given.

2.12 Self Assessment Questions

1. Prove that (i) the K.E., (ii) the angular momentum $(\vec{h})$, and (iii) the magnitude of the angular momentum (H^2) are constant throughout the motion.
2. A solid cube is in motion about an angular point which is fixed. If there are no external forces and w_1, w_2, w_3 are the angular velocities about the edges through the fixed point, prove that
   $$w_1 + w_2 + w_3 = \text{constant and } w_1^2 + w_2^2 + w_3^2 = \text{constant.}$$
3. If a rectangular parallelepiped with its edges $2a,2a,2b$ rotates about its centre of gravity under no forces. Prove, its angular velocity about one principal axis is constant and about the other axis it is periodic.
4. The principal moments of inertia of a body at the centre of mass are $A,3A,6A$. The body is so rotated that its angular velocities about the axes are $3n,2n,n$ respectively. If in the subsequent motion under no force w_1, w_2, w_3 denote the angular velocities about the principal axes at that time t, show that
   $$w_1 = 3w_3 = \frac{9n}{\sqrt{5}} \text{ sech} u \text{ and } w_2 = 3n \tanh u.$$
   where
   $$u = 3nt + \frac{1}{2}\log_e 5.$$

5. Calculate the eastward deviation of a particle of mass 10 gm. falling freely from a height of 100 metre above the surface of the earth at (i) equator and (ii) latitude 60^0N.
6. Prove that acceleration is same in fixed and in rotating axes.
7. Write a short note on 'moving frames of reference'.

2.13 Suggested Further Readings

1. H. Goldstein, *Classical Mechanics*, Addison-Wesley, Cambridge, 1950.
2. T.W.B. Kibble, *Classical Mechanics*, Orient Longman, London, 1985.
3. L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed., Pergamon Press, Oxford, 1976.
4. A. Sommerfeld, *Mechanics*, Academic Press, New York, 1964.
5. J. Synge and B. Griffith, *Principles of Mechanics*, 2nd ed., McGraw Hill, New York, 1949.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-IV Group-A

Module No. - 39

Principle of Mechanics

(The Langrangian and Hamiltonian Formulations)

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CONTENT :

The Lagrangian and Hamiltonian functions have may applicatins in modelling mechanical and physical problems. Due to their wide applications these functions are studied extensively. In this unit, we introduced Lagrangian and Hamiltonain functions and their applications to solve mechanical problems.

Objectives

- Lagrangian and Hamiltonail functions.
- Lagrangian equations of motion in different cases.
- K.E. of a dynamical system.
- Hamilton's equations of motion.
- Advantage of Hamiltonian.
- Cyclic or Ignorable coordinates.
- Routhian and routhian equations of motion.
- Worked out examples.
- Exercise.

First of all we present two important results which are essential to deduce Lagrange formulation.

Lemma 3.1 : If $\vec{r} = \vec{r}(q_1, q_2, ..., q_n, t)$ then

$$\frac{\partial \dot{\vec{r}}}{\partial \dot{q}_j} = \frac{\partial \vec{r}}{\partial q_j} \text{ for all } j.$$

Proof : Here $\vec{r} = \vec{r}(q_1, q_2, ..., q_n, t)$. Let q_j be independent of t. Then

$$\frac{d\vec{r}}{dt} = \frac{\partial \vec{r}}{\partial q_1}\frac{dq_1}{dt} + ... + \frac{\partial \vec{r}}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial \vec{r}}{\partial t}$$

$$= \dot{q}_1 \frac{\partial \vec{r}}{\partial q_1} + \dot{q}_2 \frac{\partial \vec{r}}{\partial q_2} + ... + \dot{q}_n \frac{\partial \vec{r}}{\partial q_n} + \frac{\partial \vec{r}}{\partial t}.$$

Again, differentiating w.r.t. $\dot{q}_1, \dot{q}_2, ..., \dot{q}_n$ we have

$$\frac{\partial \dot{\vec{r}}}{\partial \dot{q}_1} = \frac{\partial \vec{r}}{\partial q_1}, \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_2} = \frac{\partial \vec{r}}{\partial q_2} + ... + \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_n} = \frac{\partial \vec{r}}{\partial q_n}.$$

In general, $\dfrac{\partial \dot{\vec{r}}}{\partial \dot{q}_j} = \dfrac{\partial \vec{r}_j}{\partial q_j}$, $j = 1, 2, ..., n.$

Lemmma 3.2 : If $\vec{r} = \vec{r}(q_1, q_2, ..., q_n . t)$ then

$$\frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial q_j}\right)=\frac{\partial \dot{\vec{r}}}{\partial q_j} \text{ for all } j.$$

Proof : From Lemma 3.1,

$$\frac{d}{dt}(\vec{r})=\frac{\partial \vec{r}}{\partial t}+\sum_i \frac{\partial \vec{r}}{\partial q_i}\dot{q}_i.$$

Replacing $\vec{r}$ by $\frac{\partial \vec{r}}{\partial q_j}$ we get

$$\frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial q_j}\right)=\frac{\partial}{\partial t}\left(\frac{\partial \vec{r}}{\partial q_j}\right)+\sum_i \frac{\partial \vec{r}}{\partial q_i}\left(\frac{\partial r}{\partial q_j}\right)\dot{q}_i$$

$$=\frac{\partial^2 \vec{r}}{\partial t \partial q_j}+\sum_i \frac{\partial^2 \vec{r}}{\partial q_i \partial q_j}\dot{q}_i$$

$$=\frac{\partial}{\partial q_j}\left\{\frac{\partial \vec{r}}{\partial t}+\sum_i \frac{\partial \vec{r}}{\partial q_i}\dot{q}_i\right\}$$

$$=\frac{\partial}{\partial q_j}\left\{\frac{d\vec{r}}{dt}\right\}=\frac{\partial \dot{\vec{r}}}{\partial q_j}, \text{ for all } j.$$

That is, the order of the operaior $\frac{\partial}{\partial q_i}$ and $\frac{d}{dt}$ have been interchanged.

3.1 Lagrange's Equations of Motion

Let $q_1, q_2, \ldots, q_n$ be n generalised coordinates of a dynamical system with n degrees of freedom. Suppose, the number of particles in the system be N and m_i be the mass of a typical particle and $\vec{r}_i$ be its position vector at time t with respect to the origin, so that

$$\vec{r}_i=\vec{r}_i(q_1,q_2,\ldots,q_n,t),\ i=1,2,\ldots,N. \tag{3.1}$$

Differentiating w.r.t. t we get

$$\frac{d\vec{r}_i}{dt}=\frac{\partial \vec{r}_i}{\partial q_1}\frac{dq_1}{dt}+\ldots+\frac{\partial \vec{r}_i}{\partial q_n}\frac{dq_n}{dt}+\frac{\partial \vec{r}_i}{\partial t}$$

$$=\sum_{j=1}^{n}\frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j+\frac{\partial \vec{r}_i}{\partial t}.$$

Thus the infinitesimal displacement $\delta\vec{r}_i$ can be connected with δq_j as

$$\delta\vec{r}_i = \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j + \frac{\partial \vec{r}_i}{\partial t}\delta t, \qquad (3.2)$$

but the last term is zero since in virtual displacement only coordinate displacement is considered and not that of time.

$$\therefore \delta\vec{r}_i = \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j. \qquad (3.3)$$

The virtual work done

$$\delta w = \sum_{i=1}^{N} \vec{F}_i.\delta\vec{r}_i = \sum_{i=1}^{n} \vec{F}_i.\left[\sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j\right]$$

$$= \sum_j \sum_i \vec{F}_i.\frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = \sum_j Q_j \delta q_j, \qquad (3.4)$$

where $Q_j = \sum_i \vec{F}_i.\frac{\partial \vec{r}_i}{\partial q_j}$ (3.5)

Q_j is known as generalised component of force.

It is observed that q's are not necessarily have the dimension of length and similarly Q's not necessarily have the dimension of force and hence from this analogy, Q's are generalised components of force.

Now, $\sum_{i=1}^{N} \dot{\vec{p}}_i.\delta\vec{r}_i = \sum_i m_i \ddot{\vec{r}}.\delta\vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i.\left\{\sum_j \frac{\partial \vec{r}_i}{\partial r_j}\delta q_j\right\}$ [From (3.3)]

$$= \sum_i \sum_j m_i \ddot{\vec{r}}_i.\frac{\partial \vec{r}_i}{\partial q_j}\delta q_j. \qquad (3.6)$$

Again, $\sum_i m_i \ddot{\vec{r}}_i.\frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \frac{d}{dt}\left(m_i \dot{\vec{r}}_i.\frac{\partial \vec{r}_i}{\partial q_j}\right) - \sum_i m_i \dot{\vec{r}}_i.\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$

$= \sum_i \frac{d}{dt}\left(m_i \dot{\vec{r}}_i.\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right) - \sum_i m_i \dot{\vec{r}}_i.\frac{\partial}{\partial q_j}\left(\frac{d\vec{r}_i}{dt}\right)$ [by Lemmas 3.1 and 3.2]

$\therefore \sum_i \dot{\vec{p}}_i.\delta\vec{r}_i = \sum_j \left\{\sum_i \frac{d}{dt}\left(m_i \dot{\vec{r}}_i.\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right) - \sum_i m_i \dot{\vec{r}}_i.\frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right\}\delta q_j$ [From (3.6)]

$$= \sum_j \left\{ \frac{d}{dt} \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^2 \right) - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^2 \right) \right\} \delta q_j$$

$$= \sum_j \left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right\} \delta q_j, \tag{3.7}$$

where $T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^2$.

Now, from D'Alembert's principle.

$$\sum_i \vec{F}_i . \delta\vec{r}_i - \sum_i \dot{\vec{p}}_i . \delta\vec{r}_i = 0$$

or, $\sum_j Q_j \delta q_j - \sum_j \left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right\} \delta q_j = 0$ [using (3.5) and (3.7)]

or, $$\sum_j \left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right\} \delta q_j = 0. \tag{3.8}$$

Case 1. Unconnected holonomic system.

In this case, the coordinates $q_1, q_2, \ldots, q_n$ are independent. Therefore, the virtual displacements δq_j are independent. Thus, the coefficients of (3.8) are separately vanish.

Therefore, we have

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = 0, j = 1, 2, \ldots, n. \tag{3.9}$$

These n equations are called the Lagrange's equation of motion of the first kind for an unconnected holonomic system.

Case 1.1 Conservative system

Let V be the P.E. function.

$\therefore \vec{F}_i = -(\Delta V)_i = -\frac{\partial V}{\partial r_i}$. Then

$$Q_j = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_{i=1}^{N} \frac{\partial V}{\partial r_i} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}.$$

Since V is a function of coordinates only,

$$\frac{\partial V}{\partial \dot{q}_j} = 0.$$

Using these results (3.9) reduces to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0$$

or, $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j}\right) - \left(\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j}\right) = 0$$

or, $$\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_j}(T-V)\right\} - \frac{\partial}{\partial q_j}(T-V) = 0$$

or, $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, j = 1,2,\ldots,n, \tag{3.10}$$

where $L = T - V$ is known as **Lagrange's function** of the system. It is also called as the connective potential of the system.

Equations (3.10) are called Lagrange's equation of second kind for conservative forces.

Case 1.2 Force is not fully conservative.

In this case Q_j can be expressed as

$$Q_j = -\frac{\partial V}{\partial q_j} + Q_j'.$$

Then from (3.5),

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} + Q_j'$$

or, $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j' \text{ where } L = T - V. \tag{3.11}$$

Case 2 Connected holonomic system

Let $q_1, q_2, \ldots, q_n$ be the the n-coordinates of the system connected by k independent equations.

$$f_i(q_1, q_2, \ldots, q_n, t) = 0, i = 1,2,\ldots,k(<n). \quad (3.12)$$

Then $\delta f_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j}\delta q_j = 0, i = 1,2,\ldots,k.$ (3.13)

Let each generalised force is obtained from a potential function V as $Q_j = -\frac{\partial V}{\partial q_j}$.

Let the constraints are workless. Therefore, the generalised constraint forces C_j must need the condition

$$\sum_j C_j \delta q_j = 0. \quad (3.14)$$

Using the method of Lagrange's multipliers with λ_i, we have $\lambda_i \delta f_i = 0$.

or, $\lambda_i \sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j}\delta q_j = 0,\ i = 1,2,\ldots,k.$ [from (3.13)]

Summing over i, we get

$$\sum_{i=1}^{k}\left\{\lambda_i \sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j}\delta q_i\right\} = 0. \quad (3.15)$$

Therefore, from (3.14) and (3.15)

$$\sum_{i=1}^{k}\lambda_i \sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j}\delta q_j - \sum_{j=1}^{n} C_j \delta q_j = 0$$

or, $$\sum_j \left\{\sum_i \lambda_i \frac{\partial f_i}{\partial q_j} - C_j\right\}\delta q_j = 0. \quad (3.16)$$

We choose λ 's such that

$$C_j = \sum_{i=1}^{k}\lambda_i \frac{\partial f_i}{\partial q_j}, j = 1,2,\ldots,n.$$

Therefore, the total generalised force

$$= Q_j + C_j = -\frac{\partial V}{\partial q_j} + \sum_{i=1}^{k}\lambda_i \frac{\partial f_i}{\partial q_j}.$$

Now, from case 1,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}= \text{total generalised force}$$

or, $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial V}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=-\frac{\partial V}{\partial q_j}+\sum_i \lambda_i \frac{\partial f_i}{\partial q_j}$$

or, $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=\sum_i \lambda_i \frac{\partial f_i}{\partial q_j}, j=1,2,\ldots,n.$$

The above n equations are the system of $(n+k)$ unknown of which n are q's and k are λ 's.

To solve these equations, required k more equations which are supplied by k constraints.

Case 3. Non-Holonomic system

For a non-holonomic system, there must be more generalised coordinates than the number of degrees of freedom. Therefore, δq 's are no longer independent if we assume a virtual displacement consistents with the constraints. Let $q_1, q_2, \ldots, q_n$ be n generalized coordinates of the system. Let there be k non-holonomic constraint equations of the form of k non-integrable relations as

$$\sum_{j=1}^{n} a_{ij} dq_j + a_i dt = 0, i = 1,2,\ldots,k, \tag{3.17}$$

where a's the functions of coordinates.

Now for a virtual displacement at time t, we have

$$\sum_{j=1}^{n} a_{ij} \delta_{qj} = 0, i = 1,2,\ldots,k(<n). \tag{3.18}$$

Let us assume again that generalized applied force Q_j is obtained from a potential function as

$$-\frac{\partial V}{\partial q_j}, j=1,2,\ldots,n.$$

Let the constraints be workless. So, the generalized constraints forces C_j must be satisfied by the condition

$$\sum_{j=1}^{n} C_j dq_j = 0 \tag{3.19}$$

for any constraints.

Now, we multiply equation (3.18) by a factor λ_i, the Lagrange multiplier and obtain the k equations as

$$\lambda_i \sum_{j=1}^{n} a_{ij}\delta q_j = 0, \quad i = 1,2,\ldots,k(<n)$$

or, $$\sum_{i=1}^{k}\lambda_i \sum_{j=1}^{n} a_{ij}\delta q_j = 0.$$

Substracting these equations from (3.19), we have

$$\sum_{j=1}^{n} C_j\delta q_j - \sum_{i=1}^{k}\lambda_i \sum_{j=1}^{n} a_{ij}\delta q_j = 0.$$

Interchanging the order of summation we have

$$\sum_{j=1}^{n}\left[C_i - \sum_{i=1}^{k}\lambda_i a_{ij}\right]\delta q_j = 0, \tag{3.20}$$

Upto this point, the λ 's has been considered to be arbitrary and if we choose λ 's such that

$$C_j = \sum_{i=1}^{k}\lambda_i a_{ij}, j = 1,2,\ldots,n,$$

then the coefficient of δq 's are zero and the equation (3.20) will apply for any set of δq 's. In other words, the δq 's can be choosen independently.

Now, the total generalised force is

$$Q_j^{total} = Q_j + C_j = -\frac{\partial V}{\partial q_j} + \sum_{i=1}^{k}\lambda_i a_{ij}.$$

Substitution these in the equation for non-conservative system where,

$Q_j = \sum_{i=1}^{k}\lambda_i a_{ij}$, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \sum_{i=1}^{k} a_{ij},\lambda_i, j = 1,2,\ldots,n. \tag{3.21}$$

Then $(n+k)$ unknowns i.e., n numbers of q's and k numbers of λ 's are evaluated with the help of the equation (3.21) and the equation of constraints which can be written as

$$\sum_{j=1}^{n} a_{ij}dq_j + a_i dt = 0, i = 1,2,\ldots,k(<n)$$

or, $\sum_{j=1}^{n} a_{ij}\dot{q}_j + a_i = 0, i = 1,2,\ldots,k.$

3.1.1 Expression for K.E. of the system

Let us consider a system of N particles with n generalized coordinates $q_1, q_2, \ldots, q_n$ and having position vector $\vec{r}_i$ for the ith particle with mass m_i.

Then K.E. T is

$$T = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^2 = \frac{1}{2}\sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial t}\dot{q}_j\right)^2$$

$$= \frac{1}{2}\sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2 + \sum_i m_i \frac{\partial \vec{r}_i}{\partial t}\sum_{j=1}^{n}\frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j$$

$$+ \frac{1}{2}\sum_i m_i \left(\sum_{j=1}^{n}\frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j \sum_{k=1}^{n}\frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k\right)$$

$$= \frac{1}{2}\sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2 + \sum_j \left\{\sum_i m_i \frac{\partial \vec{r}_i}{\partial t}\frac{\partial \vec{r}_i}{\partial q_j}\right\}\dot{q}_j$$

$$+ \sum_j \sum_k \left(\sum_i \frac{1}{2} m_i \frac{\partial \vec{r}_i}{\partial q_j}\frac{\partial \vec{r}_i}{\partial q_k}\right)\dot{q}_j \dot{q}_k$$

$$= T_0 + T_1 + T_2$$

where $T_0 = \frac{1}{2}\sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2$

$$T_1 = \sum_j \left\{\sum_i m_i \frac{\partial \vec{r}_i}{\partial t}\frac{\partial \vec{r}_i}{\partial q_j}\right\}\dot{q}_j$$

and $T_2 = \sum_j \sum_k \left(\sum_i \frac{1}{2} m_i \frac{\partial \vec{r}_i}{\partial q_j}\frac{\partial \vec{r}_i}{\partial q_k}\right)\dot{q}_j \dot{q}_k.$

The function T_2 is a quadratic homogeneous function of $\dot{q}'s$, T_1 is a homogeneous function of $\dot{q}'s$ and T_0 is a function of $q's$ and t.

If the transformation equation does not contain the time explicitly, i.e. $\frac{\partial \vec{r}_i}{\partial t} = 0$, then only the last term of the above expression, i.e., T_2 is non-vanishing, i.e., in that case K.E. is always a homogeneous quadratic function in generalized velocities.

Example 3.1.1 A particle of mass *m* moves in a plane. Write down the Lagrange's equations of motion for this particle using plane polar coordinates.

Solution. We have $x = r\cos\theta,\ y = r\sin\theta$.

Therefore, $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta,\ \ \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$.

Then $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\dot{r}^2 + \left(r\dot{\theta}\right)^2$.

Also, $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$. Thus $Q_r = \vec{F}\cdot\frac{\partial \vec{r}}{\partial r} = F_r$ and $Q_\theta = \vec{F}\cdot\frac{\partial \vec{r}}{\partial \theta} = \vec{F}\cdot r\vec{n} = rF_\theta, \vec{n}$ is the unit vector perpendicular to the direction of $\vec{r}$. We have

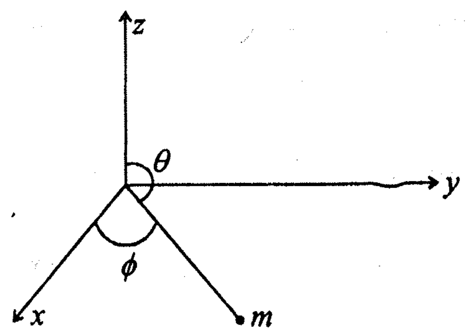
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r \text{ or, } \frac{d}{dt}(m\dot{r}) - m\dot{\theta}^2 r = F_r \quad \text{(i)}$$

Also, $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta \text{ or, } \frac{d}{dt}\left(mr^2\dot{\theta}\right) = rF_\theta,$$

or, $$2m\dot{r}\dot{\theta} + rm\ddot{\theta} = F_\theta. \quad \text{(ii)}$$

Equations (i) and (ii) are the required equations.

Example 3.1.2. Find the Lagrange's equation of motion for a pendulum in spherical polar coordinates, of length *l*.



Solution. Here $x = \ell\sin\theta\cos\phi$

$y = \ell\sin\theta\sin\phi$

$z = \ell\cos\theta.$

Then $\dot{x} = l\dot{\theta}\cos\theta\cos\phi - l\dot{\phi}\sin\theta\sin\phi$

$\dot{y} = l\dot{\theta}\cos\theta\sin\phi + l\dot{\phi}\sin\theta\cos\phi$

$\dot{z} = l\sin\theta.\dot{\theta}.$

$\therefore$ K.E. $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$= \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$

and P.E. $V = -mgl\cos(\pi - \theta) = mgl\cos\theta$.

The Lagrangian $L = T - V$

$= \frac{1}{2}ml^2[\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta] - mgl\cos\theta.$

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0,\ q_1 = \theta, q_2 = \phi.$$

<u>For the coordinate</u> θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

or, $ml^2\ddot{\theta} - ml^2\dot{\phi}^2\sin\theta\cos\theta - mgl\sin\theta = 0.$

<u>For the coordinate</u> ϕ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

or, $ml^2\ddot{\phi}\sin^2\theta + 2ml^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta = 0.$

These equations are the required equations of motion.

Example 3.1.3 A particle is constrained to move on the plane curve $xy = c$ under gravity. Obtain Lagrange's equation.

Solution. Here the K.E. $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$, $V = mgy$.

But, $y = c/x$, or, $\dot{y} = -\frac{c}{x^2}\cdot\dot{x}.$

Hence $L \quad = T-V=\frac{1}{2}m\left[\dot{x}^2+\frac{c^2}{x^4}\dot{x}^2\right]-\frac{mgc}{x}$

$$=\frac{1}{2}m\left(1+\frac{c^2}{x^4}\right)\dot{x}^2-\frac{mgc}{x}.$$

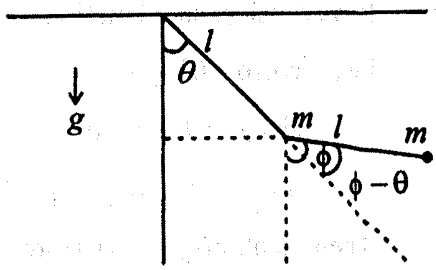
Now, the Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$$

or, $$\frac{d}{dt}\left\{m\left(1+\frac{c^2}{x^4}\right)\dot{x}\right\}-\left\{\frac{1}{2}m\dot{x}^2\left(-\frac{4c^2}{x^5}\right)+\frac{mgc}{x^2}\right\}=0$$

or, $$m\left(x^5+c^2x\right)\ddot{x}-2mc^2\dot{x}^2-mgcx^3=0.$$

Example 3.1.4 A double pendulum consists of two particles suspended by mass-less rods as shown in the following figure. Assume that all motion take place in a vertical plane. Find the differential equation of motion, linearizing the equations assuming small motion.



Solution. Let $v_1=\ell\dot{\theta}$ be the velocity of the first particle and the velocity of the second be $\ell\dot{\phi}$ and there difference of direction is $(\phi-\theta)$.

By cosine formula, we find, v_2, the velocity of the lower particle with respect to the origin O as

$$v_2^2=\left(\ell\dot{\theta}\right)^2+\left(\ell\dot{\phi}\right)^2+2\left(\ell\dot{\theta}\right)\left(\ell\dot{\phi}\right)\cos(\phi-\theta)$$

The total K.E. of the system is

$$T=\frac{1}{2}mv_1^2+\frac{1}{2}mv_2^2$$

$$=\frac{1}{2}m\ell^2\left[2\dot{\theta}^2+\dot{\phi}^2+2\dot{\theta}\dot{\phi}\cos(\phi-\theta)\right].$$

The total P.E. is

$$V=-mg\ell\cos\theta-\left(mg\ell\cos\theta+mg\ell\cos\phi\right).$$

$\therefore$ the Lagrangian $L=T-V$

$$= \frac{1}{2} m\ell^2 \left[2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\phi - \theta)\right] + mg\ell(2\cos\theta + \cos\phi).$$

Lagrange's equation for θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

or, $m\ell^2\left[2\ddot{\theta} + \ddot{\phi}\cos(\phi - \theta) - \dot{\phi}^2 \sin(\phi - \theta)\right] + 2mg\ell\sin\theta = 0.$ (i)

Lagrange's equation for ϕ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

or, $ml^2\left[\ddot{\phi} + \ddot{\theta}\cos(\phi - \theta) - \dot{\theta}^2 \sin(\phi - \theta)\right] + mg\ell\sin\phi = 0.$ (ii)

Equations (i) and (ii) are the equations of motion of the double pendulum.

If we consider small motion, i.e., ϕ, θ and their time derivative are small

i.e., $\cos(\phi - \theta) \simeq 1$

$\sin(\phi - \theta) \simeq \phi - \theta$

$\sin\theta \simeq \theta, \sin\phi \simeq \phi.$

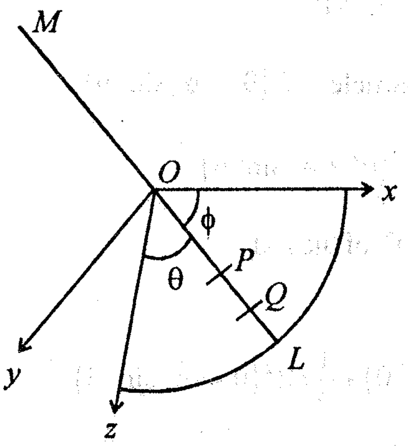
Then equations (i) and (ii) are reduce to

$$m\ell^2\left[2\ddot{\theta} + \ddot{\phi}\right] + 2mg\ell\theta = 0$$

and $m\ell^2\left[\ddot{\phi} + \ddot{\theta}\right] + mg\ell\phi = 0.$

Example 3.1.5 A uniform rod of mass $3m$ and length $2l$, has its middle point fixed and a mass m attached at one extremity. The rod when in horizontal position is set rotating about a vertical axis through its centre with an angular velocity equal to $\sqrt{2xg/\ell}$. Show that the heavy end of the rod will fall till the inclination of the rod to the vertical is $\cos^{-1}\left[\sqrt{(n^2+1)} - n\right]$ and will then rise again.

Solution. Let LM be the rod whose middle point is O. The mass m is attached at L. Initially, let the rod lies along OX in the plane of the paper. On the rod ML, take a point P such that $OP = \xi$, the element $PQ = d\xi$. Further, at any time t, let the plane through ML and the vertical have turned through an angle ϕ from its initial position and let the rod be inclined at an angle θ to the vertical. Then the generalised coordinates are θ and ϕ.



Taking O, the mid-point of the rod as the origin and OX and OY (a line perpendicular to the plane of the paper) and OZ (a line perpendicular to OX) as axes of reference, the coordinates of the point P on the rod are given by

$$x = \xi \sin\theta\cos\phi, y = \xi\sin\theta\sin\phi, z = \xi\cos\theta.$$

Therefore, $\dot{x} = \xi\dot{\theta}\cos\theta\cos\phi - \xi\dot{\phi}\sin\theta\sin\phi$

$$\dot{y} = \xi\dot{\theta}\cos\theta\sin\phi + \xi\dot{\phi}\sin\theta\cos\phi$$

$$\dot{z} = -\xi\dot{\theta}\sin\theta.$$

Thus, square of velocity of

$$P = V_p^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$= \xi^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right).$$

Square of velocity of mass $m = V_L^2$

$$= \ell^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right).$$

The mass of the element $PQ = \frac{3m}{2\ell}d\xi = dm$ (say)

The K.E. $= \frac{1}{2}dm \cdot V_P^2 = \frac{1}{2}\frac{3m}{2\ell}d\xi\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)\xi^2$

$$= \frac{3m}{4\ell}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)\xi^2 d\xi.$$

K.E. of the rod $= \frac{3m}{4\ell}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)\int_{-\ell}^{l} \xi^2 d\xi$

$$= \frac{1}{2}m\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)\ell^2.$$

Again, square of velocity of the particle $= \ell^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$.

K.E. of the particle $= \frac{1}{2}m\ell^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$.

Thus, the total K.E. T = K.E. of the rod

+ K.E. of the particle

$$= \frac{1}{2}m\ell^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{1}{2}m\ell^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right).$$

$$= m\ell^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right).$$

Also, the work function is given by

$$W = mg\ell\cos\theta + c$$

where c is a constant.

<u>The Lagrange's equation for ϕ</u>

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$$

or, $\frac{d}{dt}\left(2m\ell^2\dot{\phi}\sin^2\theta\right) = 0.$

Integrating, we have

$\dot{\phi}\sin^2\theta = K$, where K is a constant.

Initially, when $\theta = \pi/2$, $\dot{\phi} = \sqrt{\frac{2ng}{\ell}}$.

$$\therefore K = \sqrt{\frac{2ng}{\ell}}$$

Thus, $\dot{\phi}\sin^2\theta = \sqrt{\dfrac{2ng}{\ell}}$. (i)

The Lagrange's equation for θ

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$$

or, $\dfrac{d}{dt}\left(2m\ell^2\dot{\theta}\right) - 2m\ell^2\dot{\phi}^2\sin\theta\cos\theta = -mg\ell\sin\theta$

or, $2\ell\ddot{\theta} - 2\ell\dot{\phi}^2\sin\theta\cos\theta = -g\sin\theta.$ (ii)

Substituting the value of $\dot{\phi}$ from (i) to (ii), we have

$$2\ell\ddot{\theta} - 4ng\cot\theta\,\mathrm{cosec}^2\theta = -g\sin\theta. \quad \text{(iii)}$$

Integrating, we get

$$2\ell\dot{\theta}^2 + 4ng\cot^2\theta = 2g\cos\theta + D.$$

Initially, when $\theta = \pi/2, \dot{\theta} = 0$, then $D = 0$.

Thus $2\ell\dot{\theta}^2 + 4ng\cot^2\theta = 2g\cos\theta.$ (iv)

Hence, the rod will fall till $\dot{\theta} = 0$. That is,

$$2n\cos^2\theta - \cos\theta\sin^2\theta = 0.$$

If $\cos\theta = 0$, then $\theta = \pi/2$; which gives the initial position. Also, if

$$2n\cos\theta - \sin^2\theta = 0 \text{ then}$$

$$\cos^2\theta + 2n\cos\theta - 1 = 0.$$

That is, $\cos\theta = -n + \sqrt{2n^2+1}.$

This proves the required result. If we substitute this value of θ in (iii), we find that $\ddot{\theta}$ comes out to be positive. Hence from this position, the rod will rise again.

Example 3.1.6 For a dynamical system

$$T = \frac{1}{2}\left\{(1+2k)\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2\right\},\ V = \frac{n^2}{2}\left\{(1+k)\theta^2 + \phi^2\right\},$$

θ, ϕ are coordinates, *n*, *k* are positive constants. Write down the Lagrange's equations of motion and deduce that

$$\left(\ddot{\theta}-\ddot{\phi}\right)+n^2\left(\frac{1+k}{k}\right)(\theta-\phi)=0$$

and if $\theta=\phi$, $\dot{\theta}=\dot{\phi}$ at $t=0$ then $\theta=\phi$ for all t.

Solution. The Lagrangian for this problem is

$$L=T-V=\frac{1}{2}\left\{(1+2k)\dot{\theta}^2+2\dot{\theta}\dot{\phi}+\dot{\phi}^2\right\}-\frac{n^2}{n}\left\{(1+k)\theta^2+\phi^2\right\}.$$

The Lagrangian equation for θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta}=0$$

or, $\frac{d}{dt}\left\{\frac{1}{2}\left[(1+2k)\left(2\dot{\theta}\right)+2\dot{\phi}\right]\right\}+\frac{n^2}{2}(1+k)(2\theta)=0$

or, $(1+2k)\ddot{\theta}+\ddot{\phi}+n^2(1+k)\phi=0$ (i)

The Lagrangian equation for ϕ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right)-\frac{\partial L}{\partial\phi}=0$$

or, $\frac{d}{dt}\left\{\frac{1}{2}\left(2\dot{\theta}+2\dot{\phi}\right)\right\}+n^2\phi=0$

or, $\ddot{\theta}+\ddot{\phi}+n^2\phi=0$ (ii)

Multiplying (ii) by $(1+k)$ and subtracting from (i), we have

$$\dot{\theta}\left\{(1+k)-(1+2k)\right\}+\ddot{\theta}\left\{(1+k)-1\right\}+n^2(1+k)\phi-n^2(1+k)\theta=0$$

or, $k\left(\ddot{\phi}-\ddot{\theta}\right)+n^2(1+k)(\phi-\theta)=0.$ (iii)

Let $\phi-\theta=x$. Then (iii) becomes

$$\ddot{x}+\frac{n^2(1+k)}{k}x=0,\ \text{or,}\ \ddot{x}+A^2x=0$$

where $A^2=\frac{n^2(1+k)}{k}$.

The general solution is

$x=B\cos At+C\sin At.$ (iv)

When $t = 0, \phi = \theta$. Then $x = 0$ (since $\theta = \phi$).

$\therefore B = 0$ and so $x = C \sin At$.

Also, $\dot{x} = AC \cos At$.

Again, when $\dot{\theta} = \dot{\phi}$ at $t = 0, \dot{x} = 0$.

$\therefore AC = 0$, or, $C = 0$.

Thus, $x = 0$ or, $\phi = \theta$ for all t.

This is the required solution.

3.1.2 Electrical Circuit

The Lagrangian for an electrical circuit consists of a finite numbers of capacitance, inductance and resistance, is

$$L_E = T_M - V_E$$

where L_E is the Lagrangian for electrical circuit.

L_M is the magnetic energy,

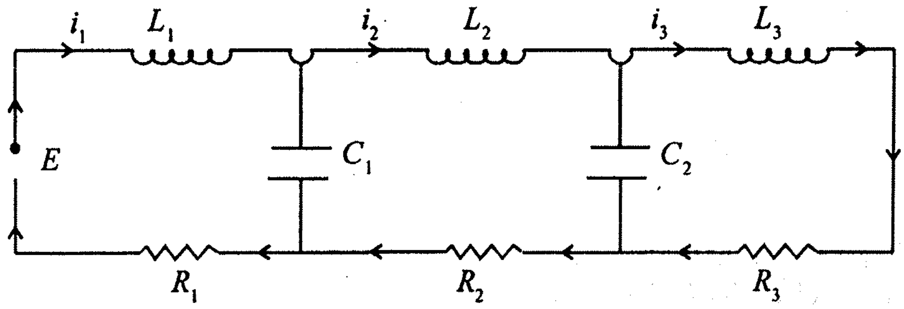
V_E is the electrical energy and the corresponding Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L_E}{\partial \dot{q}_j}\right) - \frac{\partial L_E}{\partial q_j} = Q_j,$$

where Q_j is frictional force of the system. If the system is free from friction then $Q_j = 0$ and consequently

$$\frac{d}{dt}\left(\frac{\partial L_E}{\partial \dot{q}_j}\right) - \frac{\partial L_E}{\partial q_j} = 0.$$

Example 3.1.7 Find the Lagrangian equation of motion for the following system



Solution. For the system, the magnetic energy $T_M = \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2\dot{q}_2^2 + \frac{1}{2}L_3\dot{q}_3^2$

and the electrical energy $V_E = \frac{1}{2}(q_1 - q_2)^2 \Big/ C_1 + \frac{1}{2}(q_2 - q_3)^2 \Big/ C_2 - q_1 E$

[Where q_1 is charge corresponding to the current i_1]

$$\therefore L_E = T_M - V_E = \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2\dot{q}_2^2 + \frac{1}{2}L_2\dot{q}_3^2$$

$$-\frac{1}{2}(q_1 - q_2)^2 \Big/ C_1 - \frac{1}{2}(q_2 - q_3)^2 \Big/ C_2 + q_1 E$$

and $Q_j = -R_j i_j$.

Lagrangian equation for the change q_1

$$\frac{d}{dt}\left(\frac{\partial L_E}{\partial \dot{q}_1}\right) - \frac{\partial L_E}{\partial q_1} = Q_1$$

or, $$\frac{d}{dt}(L_1\dot{q}_1) + \frac{q_1 - q_2}{C_1} - E = Q_1$$

or, $$L_1\ddot{q}_1 + \frac{q_1 - q_2}{C_1} - E = Q_1$$

For the charge q_2

$$\frac{d}{dt}(L_2\dot{q}_2) - \frac{q_1 - q_2}{C_1} + \frac{q_2 - q_3}{C_2} = Q_2$$

or, $$L_2\ddot{q}_2 - \frac{q_1}{C_1} + q_2\left(\frac{1}{C_1} + \frac{1}{C_2}\right) - \frac{q_3}{C_2} = Q_2$$

For the charge q_3

$$L_3\ddot{q}_3 - \frac{q_2}{C_2} + \frac{q_3}{C_2} = Q_3.$$

Hence the Lagrangian equations are

$$L_1\ddot{q}_1 + \frac{q_1}{C_1} - \frac{q_2}{C_2} = E - R_1\dot{q}_1$$

$$L_2\ddot{q}_2 - \frac{q_1}{C_1} + q_2\left(\frac{1}{C_1} + \frac{1}{C_2}\right) - \frac{q_3}{C_2} = -R_2\dot{q}_2$$

$$L_3\ddot{q}_3 - \frac{q_1}{C_1} + \frac{q_3}{C_2} = -R_3\dot{q}_3.$$

3.2 Hamilton's Equations of Motion

Let $\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots q_n)$ be the position of a particle. Then the K.E. of the system of N particles is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2}\sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \vec{r}_i}{\partial q_2}\frac{dq_2}{dt} + \ldots + \frac{\partial \vec{r}_i}{\partial q_n}\frac{dq_n}{dt}\right)^2$$

$$= \frac{1}{2}\sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \vec{r}_i}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial \vec{r}_i}{\partial q_n}\dot{q}_n\right)^2.$$

$$= \frac{1}{2}\sum_i m_i \left\{\left(\frac{\partial \vec{r}_i}{\partial q_1}\right)^2 \dot{q}_1^2 + \ldots + \left(\frac{\partial \vec{r}_i}{\partial q_n}\right)^2 \dot{q}_n^2 + 2\frac{\partial \vec{r}_i}{\partial q_1}\cdot\frac{\partial \vec{r}_i}{\partial q_2}\dot{q}_1\dot{q}_2 + \ldots\right\}$$

$$= p_{11}\dot{q}_1^2 + p_{22}\dot{q}_2^2 + \ldots + p_{nn}\dot{q}_n^2 + p_{12}\dot{q}_1\dot{q}_2 + \ldots$$

Let $p_j = \dfrac{\partial L}{\partial \dot{q}_j}$

Then $p_1 = \dfrac{\partial L}{\partial \dot{q}_1} = \dfrac{\partial T}{\partial \dot{q}_1} = 2p_{11}\dot{q}_1 + p_{12}\dot{q}_2 + p_{13}\dot{q}_3 + \ldots + p_{1n}\dot{q}_n$

Similarly, $p_2 = p_{21}\dot{q}_1 + 2p_{22}\dot{q}_2 + \ldots + p_{2n}\dot{q}_n$

- -

- -

In general,

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \mathrm{P}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}, \text{ where } \mathrm{P} = \begin{bmatrix} 2p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & 2p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & 2p_{nn} \end{bmatrix}.$$

From the above relation we observed that p_j (called the generalised momentum) can be used as a generalized coordinates.

We define the Hamiltonian H as

$$H(q_1,q_2,\ldots,q_n,p_1,p_2,\ldots,p_n,t)$$
$$=\sum p_j\dot{q}_j - L(q_1,q_2,\ldots,q_n,\dot{q}_1,\dot{q}_2,\ldots,\dot{q}_n,t) \qquad (3.22)$$

3.2.1 Deduction of Hamilton's equations of motion

Let $H(q_1,q_2,\ldots,q_n,p_1,p_2,\ldots,p_n,t)$ be the Hamiltonian of the dynamical system of d.o.f. n and $L(q_1,q_2,\ldots,q_n,\dot{q}_1,\dot{q}_2,\ldots,\dot{q}_n,t)$ be the Lagrangian of the system and q_j and p_j are the generalised coordinates and momentum respectively.

Now, $$dH=\sum\frac{\partial H}{\partial q_j}dq_j+\sum\frac{\partial H}{\partial p_j}dp_j+\frac{\partial H}{\partial t}dt. \qquad (3.23)$$

Since $L=L(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n,t)$,

$$dL=\sum\frac{\partial L}{\partial q_j}dq_j+\sum\frac{\partial L}{\partial \dot{q}_j}d\dot{q}_j+\frac{\partial L}{\partial t}dt$$
$$=\sum\dot{p}_jdq_j+\sum p_jd\dot{q}_j+\frac{\partial L}{\partial t}dt \qquad (3.24)$$

From Lagrange equation

$$\frac{\partial L}{\partial q_j}=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)=\frac{d}{dt}(p_j)=\dot{p}_j.$$

From (3.22), $H=\sum p_j\dot{q}_j-L$

or, $dH=\sum p_jd\dot{q}_j+\sum\dot{q}_jdp_j-dL$

$$=\sum p_jd\dot{q}_j+\sum\dot{q}_jdp_j-\sum\dot{p}_jdq_j-\sum p_jd\dot{q}_j-\frac{\partial L}{\partial t}dt \quad \text{[using (3.24)]}$$

$$=\sum\dot{q}_jdp_j-\sum\dot{p}_jdq_j-\frac{\partial L}{\partial t}dt. \qquad (3.25)$$

Equation (3.23) and (3.25) are identical. Comparing the coefficients of dp_j, dq_j and dt, we get

$$\dot{q}_j=\frac{\partial H}{\partial p_j},\dot{p}_j=-\frac{\partial H}{\partial q_j} \text{ and } \frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t}, j=1,2,\ldots,n. \qquad (3.26)$$

Equations (3.26) are known as Hamiltonian equations of motion.

Note: From (3.26) we have

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \text{ or, } \frac{\partial}{\partial t}(H+L)=0$$

or, $H+L$ is explicitly independent of time.

Also, $\frac{\partial}{\partial t}\left(\sum p_j\dot{q}_j\right)=\frac{\partial}{\partial t}(H+L)=0$ [From definition of Hamiltonian]

or, $\sum p_j\dot{q}_j$ is explicitly independent of time.

3.2.2 Advatange of Hamiltonian over Lagrangian

In Hamiltonian formulation there are two sets of first order differential equations combining to $2n$ degrees of freedom while in Lagrangian formulation, there are n second order differential equations corresponding to n degrees of freedom.

One reason of the importance of the Hamiltonian form of equation of motion is that it facilitates, the use of transformation is obtained in solutions. Later, we shall discuss the application of Hamiltonian in canonical transformation involving the pair of quantities (p_j, q_j) in the solution of Hamiltonian equation of motion.

Comparing the Lagrangian and Hamiltonian formulation, we see that either L or H can be regarded as descriptive function for the system from which a complete set of equations of motion can be derived.

Lemma 3.3 If H is not an explicit function of t, then H is a constant of the motion.

Proof.: Since H is independent of t, we have

$$H = H(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n).$$

Then
$$\frac{dH}{dt} = \sum_{j=1}^{n}\frac{\partial H}{\partial q_j}\dot{q}_j + \sum_{j=1}^{n}\frac{\partial H}{\partial p_j}\dot{p}_j$$

$$= \sum_{j=1}^{n}\frac{\partial H}{\partial q_j}\frac{\partial H}{\partial p_j} - \sum_{j=1}^{n}\frac{\partial H}{\partial p_j}\frac{\partial H}{\partial q_j}.$$

[using Hamilton's equations]

That is H = Constant.

Lemma 3.4 If the equations of transformation do not depend explicitly on time and if the potential energy is velocity independent, then H is the total energy of the system.

Proof. We consider a dynamical system with N particles. The position vector of the i-th particle is given by the assumption as

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots, q_n), i = 1,2,\ldots,n. \quad \text{(i)}$$

By the second assumption, we have

Potential energy = $V = V(q_1, q_2, \ldots, q_n)$

and kinetic energy = $T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\vec{r}}_i^2$.

Now, $\dot{\vec{r}}_i = \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j, i = 1,2,\ldots, N.$

Then $T = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j \right)^2.$

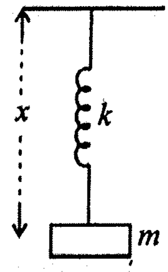
Thus T reduces to a homogeneous quadratic function $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$.

Therefore, by Euler's theorem on homogeneous function, $\sum_{j=1}^{n} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T.$ (ii)

$$\begin{aligned} \text{Now, } H &= \sum_{j=1}^{n} p_j \dot{q}_j - L \\ &= \sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L, \text{ since } p_j = \frac{\partial L}{\partial \dot{q}_j} \\ &= \sum_{j=1}^{n} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} - (T - V), \text{ since } \frac{\partial V}{\partial \dot{q}_j} = 0 \\ &= 2T - (T - V) \text{ [using (ii)]} \end{aligned}$$

Thus, $H = T + V$, which is the total energy of the system.

Example 3.2.1 Given a mass-spring system consisting of a mass *m* and a linear spring of stiffness *k* as follows. Find the equation of motion using the Hamiltonian procedure, assume that the displacement *x* is measured from the unstretched position of the spring.



Solution For this system $H = \sum p_j \dot{q}_j - L = p\dot{x} - L.$

The K.E. $T = \frac{1}{2}\dot{x}^2 m.$

Here or $F \propto x$ or, $F = -kx.$

In conservating system, $F = -\nabla V = -\frac{\partial V}{\partial x}.$

or, $V = -\int F dx = -\int(-kx)dx = \frac{kx^2}{2}.$

[when $x = 0, V = 0$ i.e., constant of integration is zero]

$$\therefore L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{kx^2}{2}.$$

and $H = p\dot{x} - \left(\frac{1}{2}m\dot{x}^2 - \frac{kx^2}{2}\right).$

Since $p = m\dot{x}$, then above equation becomes

$$H = \frac{p^2}{m} - \frac{1}{2}m\frac{p^2}{m^2} + \frac{kx^2}{2} = \frac{p^2}{2m} + \frac{kx^2}{2}. \qquad \text{(i)}$$

We know, $\dot{p}_j = -\frac{\partial H}{\partial q_j}$ and $\dot{q}_j = \frac{\partial H}{\partial p_j}.$

$$\left.\begin{aligned} &\text{Thus } \dot{p} = -\frac{\partial H}{\partial x} = -kx \\ &\text{and } \dot{x} = \frac{\partial H}{\partial p} = p/m. \end{aligned}\right\} \text{[Using (i)]}$$

Therefore, $\ddot{x} = \dot{p}/m = -\frac{kx}{m}$

or, $m\ddot{x} + kx = 0$, which represents a S.H.M.

3.3 Routhian of a Dynamical System

Cyclic or ignorable coordinaets:

We know, Lagrangian L is a function of generalised coordinates q_j, generalised velocities $\dot{q}_j$ and time t. Now, if coordinates q_j (say) is not in L then $\frac{\partial L}{\partial q_j} = 0$. Then this coordinate is known as cyclic or ignorable coordinate of the system.

Routhian:

Let $q_1, q_2, \ldots, q_n$ be the n generalised coordinates of an unconnected holonomic system. Let the coordinates $q_1, q_2, \ldots, q_k$ are cyclic.

Let us define a function

$$R = R(q_{k+1}, q_{k+2}, \ldots, q_n, \dot{q}_{k+1}, \dot{q}_{k+2}, \ldots, \dot{q}_n, \beta_1, \beta_2, \ldots, \beta_k, t)$$

$$= L(q_{k+1}, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) - \sum_{i=1}^{k} \beta_i \dot{q}_i. \tag{3.27}$$

or, $R = L - \sum_{i=1}^{k} \beta_i \dot{q}_i$, where $\beta_i = \frac{\partial L}{\partial \dot{q}_i}$.

The function R is called the Routhian of the system.

3.3.1 Routhian equations of motion

We have from (3.27),

$$R = L(q_{k+1}, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) - \sum_{i=1}^{k} \beta_i \dot{q}_i.$$

Taking an arbitrary variation of R, we get

$$\delta R = \sum_{i=k+1}^{n} \frac{\partial R}{\partial q_i} \delta q_i + \sum_{i=k+1}^{n} \frac{\partial R}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_{i=1}^{k} \frac{\partial R}{\partial \beta_i} \delta \beta_i + \frac{\partial R}{\partial t} \delta t \tag{3.28}$$

Variation on L gives

$$\delta L = \sum_{i=k+1}^{n} \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L}{\partial t} \delta t \tag{3.29}$$

Since $R = L - \sum_{i=1}^{k} \beta_i \dot{q}_i$ where $\beta_i = \frac{\partial L}{\partial \dot{q}_i}$.

$$\therefore \delta R = \delta L - \sum_{i=1}^{k} (\beta_i \delta \dot{q}_i + \dot{q}_i \delta \beta_i)$$

$$= \sum_{i=k+1}^{n} \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta \dot{q}_i + \frac{\partial L}{\partial t} \delta t - \sum_{i=1}^{k} (\beta_i \delta \dot{q}_i + \dot{q}_i \delta \beta_i) \qquad (3.30)$$

Now, $\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i = \sum_{i=1}^{k} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_{i=k+1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i$

$$= \sum_{i=1}^{k} \beta_i \delta \dot{q}_i + \sum_{i=k+1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i$$

Then (3.30) becomes

$$\delta R = \sum_{i=k+1}^{k} \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=1}^{k} \beta_i \delta \dot{q}_i + \sum_{i=k+1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L}{\partial t} \delta t - \sum_{i=1}^{k} (\beta_i \delta \dot{q}_i + \dot{q}_i \delta \beta_i)$$

$$= \sum_{i=k+1}^{n} \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=k+1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i - \sum_{i=1}^{k} \dot{q}_i \delta \beta_i + \frac{\partial L}{\partial q_i} \delta t. \qquad (3.31)$$

Then equation (3.28) and (3.31) must be identical. Therefore, comparing (3.28) and (3.31), we get

$$\frac{\partial R}{\partial q_i} = \frac{\partial L}{\partial q_i}, \frac{\partial R}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i}, i = k+1, \ldots, n$$

and $\frac{\partial R}{\partial t} = \frac{\partial L}{\partial t}$ and $\dot{q}_i = -\frac{\partial R}{\partial \beta_i}, i = 1, 2, \ldots, k$.

Thus the Routhian equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial \dot{q}_i} = 0$$

or, $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0, i = k+1, \ldots, n, \qquad (3.32)$

and $q_i = -\int \frac{\partial R}{\partial \beta_i} dt, i = 1, 2, \ldots, k. \qquad (3.33)$

Hence equation (3.32) gives the equation of motion for non-cyclic coordinates, i.e., for $q_{k+1}, \ldots, q_n$ and from (3.33) we get the cyclic coordinates $q_1, q_2, \ldots, q_n$.

Example 3.3.1 In a dynamical system of two degrees of freedom, the K.E.

$$T = \frac{1}{2}\frac{\dot{q}_1^2}{a+bq_2^2} + \frac{1}{2}\dot{q}_2^2$$

and P.E. $V = c + dq_2^2$

Find q_1, q_2 where a, b, c, d are constants.

Solution. The Lagrangian

$$L = T - V = \frac{1}{2}\frac{\dot{q}_1^2}{a+bq_2^2} + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2.$$

Here q_1 is cyclic coordinates, so we apply Routhian method.

In this problem q_1 is cyclic, $\frac{\partial L}{\partial q_1} = 0.$

$$\therefore \beta_1 = \frac{\partial L}{\partial \dot{q}_1} = \frac{\dot{q}_1}{a+bq_2^2}. \qquad \text{(i)}$$

The Routhian equation of motion for q_2 be

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_2}\right) - \frac{\partial R}{\partial q_2} = 0 \qquad \text{(ii)}$$

Where $R = L - \beta_1 \dot{q}_1$

$$= \frac{1}{2}\frac{\dot{q}_1^2}{a+bq_2^2} + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2 - \beta_1\dot{q}_1$$

$$= \frac{1}{2}\beta_1^2(a+bq_2^2) + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2 - \beta_1^2(a+bq_2^2)$$

$$= -\frac{1}{2}\beta_1^2(a+bq_2^2) + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2.$$

$\therefore$ the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_2}\right) - \frac{\partial R}{\partial q_2} = 0$$

or, $\frac{d}{dt}(\dot{q}_2)+(\beta_1^2 bq_2+2dq_2)=0$

or, $\ddot{q}_2+(\beta_1^2 b+2d)q_2=0$

or, $\ddot{q}_2+A^2q_2=0$ where $A^2=\beta_1^2 b+2d$.

or, $q_2=B\sin(At+\varepsilon)_1$, where ε and B are arbitrary constants.

Now, $\beta_1=\frac{\dot{q}_1}{a+bq_2^2}$.

or, $\dot{q}_1=\beta_1(a+bq_2^2)=\beta_1\{a+bB^2\sin^2(At+\varepsilon)\}$

or, $q_1=\int\beta_1\{a+bB^2\sin^2(At+\varepsilon)\}dt$

$$=\int\beta_1 a\,dt+\frac{1}{2}\int\beta_1 bB^2\{1-\cos 2(At+\varepsilon)\}dt$$

$$=\beta_1\left(a+\frac{1}{2}bB^2\right)\int dt-\frac{1}{2}\beta_1 bB^2\int\cos 2(At+\varepsilon)dt$$

$$=\beta_1\left(a+\frac{bB^2}{2}\right)t-\frac{bB^2\beta_1}{4A}\sin[2(At+\varepsilon)]+C,$$

C is constant.

Hence the required coordinates are

$$q_1=\beta_1\left(a+\frac{1}{2}b\beta^2\right)t-\frac{bB^2}{4A}\beta_1\sin 2(At+\varepsilon)+C$$

and $q_2=B\sin(At+\varepsilon)$.

3.4 Worked Out Examples

Example 3.4.1 Calculate the Lagrangian function and then solve the following problem.

A particle of mass m_2 is suspended by a light inextensible string of length l and another particle of mass m_1 at the point of support of m_2 and it can be moved on a horizontal line lying in the plane in which m_2 moves.

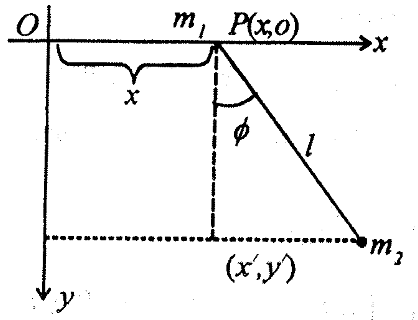
Solution. Let at time t the position of mass m_1 and m_2 are respectively $(x,0)$ and (x',y'). Also, m_2 inclined at a small angle ϕ at time t.

Then $x'=x+\ell\sin\phi$

$y' = \ell\cos\phi.$

$\therefore \quad \dot{x}' = \dot{x} + \dot{\phi}\ell\cos\phi$

$\dot{y}' = -\dot{\phi}\ell\sin\phi.$



$\therefore$ the K.E. of m_1 is $T_1 = \frac{1}{2}m_1\dot{x}^2$ and

P.E. is $V_1 = 0$.

The K.E. of m_2 is

$$T_2 = \frac{1}{2}m_2\left(\dot{x}'^2 + \dot{y}'^2\right)$$

$$= \frac{1}{2}m_2\left\{\dot{x}^2 + 2\ell\dot{x}\dot{\phi}\cos\phi + \ell^2\dot{\phi}^2\right\}$$

and P.E. of m_2 be

$$V_2 = -m_2gy' = -m_2g\ell\cos\phi.$$

Let T be the total K.E. of the system and V be the P.E.

Then $T = T_1 + T_2 = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left\{\dot{x}^2 + 2\left(\dot{x}\dot{\phi}\cos\phi + \ell^2\dot{\phi}^2\right)\right\}$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2\left(2\ell\dot{x}\dot{\phi}\cos\phi + \ell^2\dot{\phi}^2\right).$$

$$V = V_1 + V_2 = 0 - m_2g\ell\cos\phi = -m_2g\ell\cos\phi.$$

Therefore, the Lagrangian $L = T - V$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2\left(2\ell\dot{x}\dot{\phi}\cos\phi + \ell^2\dot{\phi}^2\right) + m_2g\ell\cos\phi$$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \left(\dot{x}\dot{\phi} + g\right)m_2\ell\cos\phi + \frac{1}{2}m_2\ell^2\dot{\phi}^2.$$

This is the required Lagrangian.

The Lagrangian equation for x is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

or, $\quad \frac{d}{dt}\left[(m_1 + m_2)\dot{x} + m_2\ell\cos\phi.\dot{\phi}\right] - 0 = 0$

or, $(m_1+m_2)\ddot{x}+m_2\ell\cos\phi.\ddot{\phi}-m_2\ell\sin\phi.\dot{\phi}^2=0.$ (i)

The Lagrangian equation for ϕ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right)-\frac{\partial L}{\partial\phi}=0$$

or, $\frac{d}{dt}\left[m_2\dot{x}\ell\cos\phi+m_2\ell^2\dot{\phi}\right]+(\dot{x}\dot{\phi}+g)m_2\ell\sin\phi=0$

or, $m_2(\ell^2\ddot{\phi}+\ell\ddot{x}\cos\phi)+m_2\ell(\dot{x}\dot{\phi}\sin\phi+g\sin\phi)$

$$-m_2\ell\dot{x}\dot{\phi}\sin\phi=0. \quad \text{(ii)}$$

If ϕ is small, then neglecting $\dot{\phi}^2$ and $\sin\phi\simeq\phi$, $\cos\phi\simeq 1$.

Then (i) and (ii) reduce to

$$(m_1+m_2)\ddot{x}+m_2\ell\ddot{\phi}=0. \quad \text{(iii)}$$

and $m_2\ell^2\ddot{\phi}+m_2\ell\ddot{x}+m_2\ell g\phi=0.$ (iv)

Integrating both sides of equation (iii) w.r.t. t, we get

$(m_1+m_2)\dot{x}+m_2\ell\dot{\phi}=c_1, c_1$ is a constant.

Again, integrating we get

$$(m_1+m_2)x=-m_2\ell\phi+c_1t+c_2$$

or, $x=-\frac{m_2\ell}{m_1+m_2}\phi+\frac{c_1t+c_2}{m_1+m_2}.$ (v)

Putting the value of $\ddot{x}$ from (iii) in (iv), we get

$$m_2\ell^2\ddot{\phi}-\frac{m_2^2\ell^2}{m_1+m_2}\ddot{\phi}+m_2\ell g\phi=0$$

or, $\left[\frac{m_1\ell}{m_1+m_2}\right]\ddot{\phi}+g\phi=0$

or, $\ddot{\phi}+w^2\phi=0$ where $w^2=\frac{g(m_1+m_2)}{m_1\ell}$

or, $\phi=A\sin(wt+\varepsilon),$ (vi)

where A and ε are arbitrary constants.

Putting this value of ϕ in equation (v), we get

$$x = -\frac{m_2 \ell}{m_1 + m_2} A \sin(wt + \varepsilon) + \frac{c_1 t + c_2}{m_1 + m_2}. \qquad \text{(vii)}$$

The equations (vi) and (vii) are the requierd solution of the given problem.

Example 3.4.2 A particle of mass m is attached to a fixed point O by an inverse square force $F_r = -\mu/r^2$, where μ is gravitational coefficient. Find the equation of motion in plane polar coordinates (r, θ).

Solution. Here $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$.

$$F = -\frac{\mu}{r^2} = -\text{ grad } V = -\frac{\partial V}{\partial r} \quad \text{or, } V = -\mu/r.$$

$\therefore$ Hamiltonian $H = H(r, \theta, p_r, p_\theta) = T + V$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu}{r}.$$

Now, $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ or, $\dot{r} = p_r/m$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\theta \quad \text{or, } \dot{\theta} = \frac{p_\theta}{mr^2}.$$

$$\therefore\ H = \frac{1}{2}m\left[\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2 r^4}\right] - \frac{\mu}{r}$$

The equation of motion for r is

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\left[-m\frac{p_\theta^2}{m^2 r^3} + \frac{\mu}{r^2}\right]$$

or, $\quad m\ddot{r} = \frac{p_\theta^2}{mr^3} - \frac{\mu}{r^2}$ $\quad$ or, $\quad m\ddot{r} - \frac{p_\theta^2}{mr^3} + \frac{\mu}{r^2} = 0.$

The equation of motion for θ is

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \quad \text{or, } mr^2\ddot{\theta} = 0 \quad \text{or, } \dot{\theta} = \text{constant} = \lambda \text{ (say)}$$

or, $\quad \theta = \lambda t + A.$

Example 3.4.3 Use Hamilton's equations to find the equations of motion of a projectile in space.

Solution. Let (x, y, z) be the coordinates of the projectile in space at time t. The K.E. is

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, where m is the mass of the projectile and the P.E. is $V = mgz$.

Therefore, $L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz.$ (i)

Since L does not involve t explicitly, so Hamiltonial is given by

$$H = T + V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz. \qquad \text{(ii)}$$

Now, $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad p_y = m\dot{y}, \quad p_z = m\dot{z}.$ (iii)

Substituting the values of $\dot{x}, \dot{y}, \dot{z}$ from (iii) to (ii), we get

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz. \qquad \text{(iv)}$$

The Hamilton's equations are

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0, \qquad \dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m},$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = 0, \qquad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m},$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -mg, \qquad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m},$$

The above equations can be written as

$$\ddot{x} = \frac{\dot{p}_x}{m} = 0$$

$$\ddot{y} = \frac{\dot{p}_y}{m} = 0$$

$$\ddot{z} = \frac{\dot{p}_z}{m} = -g.$$

These are the equations of motion of a projectile in space.

Example 3.4.4 The Hamiltonian of a dynamical system is given by $H = qp^2 - qp + bp$, where b is a constant. Solve the problem.

Solution. We have $H = qp^2 - qp + bp$ (i)

Hamilton equations of motion are

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

or, $\dot{q} = 2qp - q + b = (2p-1)q + b,$ (ii)

and $-\dot{p} = p^2 - p.$ (iii)

From (iii), we get

$$\frac{dp}{p^2 - p} = -dt \quad \text{or,} \quad \left(\frac{1}{p} - \frac{1}{p-1}\right)dp = dt$$

Integrating, we have $\log\frac{p}{p-1} = t + k$, where k is a constant of integration.

$$\therefore \frac{p}{p-1} = ce^{t+k}$$

or, $$p = \frac{e^{t+k}}{e^{t+k} - 1} = \frac{1}{2}\left[1 + \coth\left(\frac{t+k}{2}\right)\right].$$

Substituting the value of p in (ii), we get

$$\dot{q} = q\coth\left(\frac{t+k}{2}\right) + b$$

or, $$\frac{dq}{dt} - q\coth\left(\frac{t+k}{2}\right) = b.$$ (iv)

Equation (iv) is a linear ordinary differential equation.

Therefore, integrating factor

$$= e^{-\int \coth\left(\frac{t+k}{2}\right)dt} = e^{-2\log\left\{\sinh\left(\frac{t+k}{2}\right)\right\}}$$

$$= \frac{1}{\sinh^2\left(\frac{t+k}{2}\right)}.$$

Multiplying equation (iv) by the integrating factor, we get

$$\frac{d}{dt}\left[\frac{q}{\sinh^2\left(\frac{t+k}{2}\right)}\right]=\frac{b}{\sinh^2\left(\frac{t+k}{2}\right)}.$$

Integrating, we have

$$\frac{q}{\sinh^2\left(\frac{t+k}{2}\right)} = b\int \operatorname{cosech}^2\left(\frac{t+k}{2}\right)dt$$

$$= 2b\coth\left(\frac{t+k}{2}\right)+A,$$

where A is the constant of integration. Hence, the required result is

$$q = A\sinh^2\left(\frac{t+k}{2}\right)-2b\cosh\left(\frac{t+k}{2}\right)\sinh\left(\frac{t+k}{2}\right).$$

Example 3.4.5 Use Hamilton's equations, to find the equation of motion of a simple pendulum.

Solution. Let l be the length of the string, and m the mass of the bob, forming a simple pendulum.

At time t, let the string be inclined at an angle θ to the downward drawn vertical.

Velocity of the bob = $\ell\dot{\theta}$.

$$T=\frac{1}{2}m\ell^2\dot{\theta}^2, \qquad V=-mgl\cos\theta.$$

$$\therefore L=T-V=\frac{1}{2}ml^2\dot{\theta}^2+mgl\cos\theta$$

for which $p_\theta=\dfrac{\partial L}{\partial\dot{\theta}}=ml^2\dot{\theta}.$

Since L does not involve t explicitly, therefore Hamiltonian H is given by

$$H=T+V=\frac{1}{2}ml^2\dot{\theta}^2-mgl\cos\theta$$

$$=\frac{1}{2}m\frac{p_\theta^2}{m^2l^2}-mgl\cos\theta=\frac{1}{2ml^2}p_\theta^2-mgl\cos\theta.$$

Hence Hamilton's equations are

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta \qquad \text{(i)}$$

and $$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}. \qquad \text{(ii)}$$

Differentiating (ii) we have $\ddot{\theta} = \frac{1}{ml^2}\dot{p}_\theta$

$$= -\frac{1}{ml^2} mgl\sin\theta \text{ [by (i)]}$$

$$= -\frac{g}{l}\sin\theta.$$

Thus $\ddot{\theta} = -\frac{g}{l}\sin\theta$, which is the usual equation of a simple pendulum.

Example 3.4.6 Using cylindrical coordinates, write the Hamiltonian and Hamilton's equations for a particle of m ass m moving on the inside of a frictionless cone $x^2 + y^2 = z^2\tan^2\alpha$.

Solution. Let (ρ, ϕ, z) be the coordinates of any point in cylindrical coordinates.

Then $x = \rho\cos\phi,\ y = \rho\sin\phi$ and here $z = \rho\cot\alpha$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{\rho}\cot^2\alpha)$$

$$= \frac{1}{2}m(\dot{\rho}^2 \operatorname{cosec}^2\alpha + \rho^2\dot{\phi}^2)$$

$V = mgz = mg\rho\cot\alpha$, since the particle is above the vertex (origin).

$$\therefore L = T - V = \frac{1}{2}m(\dot{\rho}^2 \operatorname{cosec}^2\alpha + \rho^2\dot{\phi}^2) - mg\rho\cot\alpha$$

from which $p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}\operatorname{cosec}^2\alpha$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi}.$$

Since L does not involve t explicitly, therefore Hamiltonian H is

$$H = T + V = \frac{1}{2}m(\dot{\rho}^2 \operatorname{cosec}^2\alpha + \rho^2\dot{\phi}^2) + mg\rho\cot\alpha$$

$$= \frac{1}{2}m\left[\frac{p_\rho^2}{m^2 \operatorname{cosec}^2\alpha} + \frac{p_\phi^2}{m^2\rho^2}\right] + mg\rho\cot\alpha$$

$$= \frac{1}{2m}\left[\frac{p_\rho^2}{\operatorname{cosec}^2\alpha} + \frac{p_\phi^2}{\rho^2}\right] + mg\rho\cot\alpha.$$

Hamilton's equations are

$$\dot{p}_\rho = -\frac{\partial H}{\partial \rho} = \frac{p_\phi^2}{m\rho^2} - mg\cot\alpha, \quad \dot{\rho} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m\operatorname{cosec}^2\alpha}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \rho} = 0, \quad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\rho^2}$$

These are the Hamilton's equations.

3.5 Unit Summary

In this unit, two most fundamental functions of classical mechanics, Lagrangian and Hamiltonian are introduced. The Lagrange's equations of motion are deduced for holonomic and non-holonomic as well as conservative and non-conservative systems. The Hamilton's equations of motion are deduced for holonomic and conservative system. The cyclic coordinates of a dynamical system are defined. The Routhian function is defined and Routhian equations of motion are deduced from Lagrange's equations of motion. Many problems are solved using Lagrangian and Hamiltonian functions. An exercise is supplied with this unit.

3.6 Self Assessment Questions

3.1 A particle of mass m moves under the influence of gravity on the inner surface of paraboloid of revolution $x^2 + y^2 = az$ which is frictionless. Obtain the equations of motion.

3.2 The Hamiltonian of a dynamical system is given by $H = q_1 p_1 - q_2 p_2 - aq_1^2 + bq_2^2$, where a, b are constants. Solve the problem.

3.3 If all the coordinates of a dynamical system of n degrees of freedom are ignorable, prove that the problem can be solved completely by integration.

3.4 A particle of mass m moves in a force field of potential V. Write the Hamilton's equations of motion in spherical polar coordinates.

3.5 If the equations of transformation do not depend explicitly on time and if the potential energy is velocity independent, then H is the total energy of the system.

3.6 Prove that $\frac{dH}{dt} = \frac{\partial H}{\partial t}$, where H is the Hamiltonian function.

3.7 Construct the Routhian for the two-body problem, for which $L = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}\right) - V(r)$.

3.8 The Lagrangian for a system of one degree of freedom can be written as

$$L = \frac{m}{2}\left(\dot{q}^2 \sin wt + \dot{q}qw \sin 2wt + q^2w^2\right).$$

Determine the corresponding Hamiltonian.

3.9 If $2T = \dot{\theta}^2 + \theta^2\dot{\phi}^2$ and $V = \frac{1}{2}n^2\theta^2$ prove that the Hamilton's equations give

$$\theta^2 = a^2\cos^2(nt+\alpha) + b^2\sin^2(nt+\alpha)$$

and $\tan(\phi + \beta) = \frac{b}{a}\tan(nt + \alpha)$,

where a, b, α, β are constants.

3.7 Suggested Further Readings

1. H. Goldstein. *Classical Mechanics,* Addison-Wesley, Cambridge, 1950.
2. T.W.B. Kibble, *Classical Mechanics*, Orient Longman, London, 1985.
3. L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed., Pergamon Press, Oxford, 1976.
4. A. Sommerfeld, *Mechanics*, Academic Press, New York, 1964.
5. J. Synge and B. Griffith, *Principles of Mechanics*, 2nd ed., McGraw Hill, New York, 1949.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-IV | Group-A

Module No. - 40

PRINCIPLE OF MECHANICS

(VARIATIONAL PRINCIPLES)

Content :

Variational principle is used to solve a large number of optimization problems containing one or more independent variables. This principle is used to optimize a quantity which appears as an integral. In other words, it gives the necessary condition that the quantity appearing as an integral has either a minimum or a maximum, i.e., stationary value.

Objectives

- Variational principle.
- Euler-Lagrange equations of single and multiple dependent variables.
- Hamilton's principle.
- Lagrange's and Hamilton's equations of motion from Hamilton's principle.
- Modified Hamilton's principle.
- Principle of least action and its deduction.
- Exercise.

4.1 Calculus of Variations : Deduction of Euler - Lagrange Equation

Consider the simplest integral

$$J = \int_{x_0}^{x_1} F(y, y', x)\,dx, \tag{4.1}$$

where $y' = \frac{dy}{dx}$. Here F is a known function of y, y^1 and x, but the function $y(x)$ is unknown. The problem is to find a path $y = y(x)$, $x_0 \le x \le x_1$, which optimize the functional J.

The value of J is different along different paths connecting the point P and Q having coordinates (x_0, y_0) and (x_1, y_1) respectively. We have to choose the path of integration, $y(x)$, such that J has stationary value.

Consider two paths out of infinite number of possibilities, such that the difference between these two for the given value of x is the variation of y, i.e., δy and may described by introducing a new function $\eta(x)$ and ε to describe the arbitrary deformation of the path and the magnitude of variation respectively.

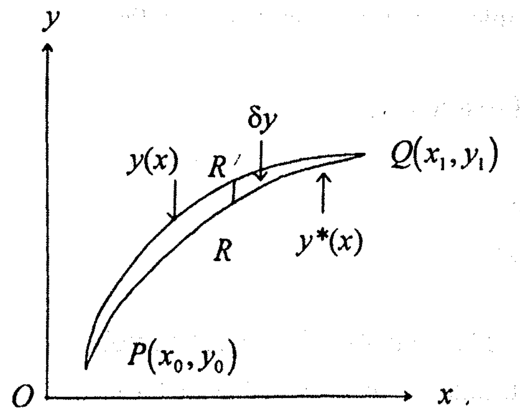


Fig. 4.1 Variation of path.

The function $\eta(x)$ must satisfy the following two conditions :

(i) All varied paths must pass through the fixed points P, Q, i.e., $\eta(x_0) = \eta(x_1) = 0$.

(ii) $\eta(x)$ must be differentiable.

Let PRQ be optimum path i.e., J is optimum along PRQ and let the equation of PRQ be $y=y^*(x)$.

Also, let $PR'Q$ be another path.

Therefore, $y = y^*(x) + \delta y = y^*(x) + \eta(x)\varepsilon$ and $y' = y^{*\prime}(x) + \eta'(x)\varepsilon$.

Thus the value of J in $PR'Q$ is

$$J = \int_{x_0}^{x_1} F\left(y^*(x) + \eta(x)\varepsilon, y^{*\prime}(x) + \eta'(x)\varepsilon, x\right)dx \quad [by (4.1)]$$

Hence for a given $\eta(x)$, J is a function of ε only.

Therefore,

$$J(\varepsilon) = \int_{x_0}^{x_1} F\left(y^*(x) + \eta(x)\varepsilon, y^{*\prime}(x) + \eta'(x)\varepsilon, x\right)dx$$

The condition for extremum of $J(\varepsilon)$ is

$$\frac{dJ(\varepsilon)}{d\varepsilon} = 0.$$

Again, from Figure 4.1 we see that $J(\varepsilon)$ will be optimum if $\varepsilon = 0$.

Thus, the necessary condition of optimization of J is

$$\frac{dJ(\varepsilon)}{d\varepsilon} = 0 \quad \text{for } \varepsilon = 0.$$

Now, $$\frac{dJ}{d\varepsilon} = \int_{x_0}^{x_1}\left(\frac{\partial F}{\partial y}\frac{dy}{d\varepsilon} + \frac{\partial F}{\partial y'}\frac{dy'}{d\varepsilon} + \frac{\partial F}{\partial y}\frac{dx}{d\varepsilon}\right)dx,$$

[where $y = y^*(x) + \eta(x)\varepsilon$ and $y' = y^{*\prime}(x) + \eta'(x)\varepsilon$]

$$= \int_{x_0}^{x_1}\left(\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x) + \frac{\partial F}{\partial x}.0\right)dx$$

$$= \int_{x_0}^{x_1}\left\{F_y\left(y^*(x) + \eta(x)\varepsilon, y^{*\prime}(x) + \eta'(x)\varepsilon, x\right)\cdot\eta(x)\right.$$

$$\left. + F_{y'}\left(y^*(x) + \eta(x)\varepsilon, y^{*\prime}(x) + \eta'(x)\varepsilon, x\right)\eta'(x)\right\}dx$$

Since $\frac{dJ}{d\varepsilon} = 0$ for $\varepsilon = 0$, therefore,

$$\int_{x_0}^{x_1} \left\{F_y(y^*, y^{*\prime}, x)\eta(x) + F_{y'}(y^*, y^{*\prime}(x), x)\eta'(x)\right\}dx = 0. \tag{4.2}$$

Now, we consider the term

$$\int_{x_0}^{x_1} F_{y'}(y^*, y^{*\prime}, x)\eta'(x)dx = \int_{x_0}^{x_1} F_{y'}\,\eta'(x)dx$$

$$= \left[F_{y'}\int \eta'(x)dx\right]_{x_0}^{x_1} - \int_{x_0}^{x_1}\left[\left\{\frac{d}{dx}(F_{y'})\right\}\int \eta'(x)dx\right]dx$$

$$= -\int_{x_0}^{x_1}\frac{d}{dx}(F_{y'})\eta(x)dx. \quad [\because \eta(x_0) = \eta(x_1) = 0]$$

Thus (4.2) becomes

$$\int_{x_0}^{x_1}\left\{\eta(x)F_y\,dx - \frac{d}{dx}(F_{y'})\eta(x)\right\}dx = 0$$

or, $$\int_{x_0}^{x_1}\left\{F_y - \frac{d}{dx}(F_{y'})\right\}\eta(x)\,dx = 0.$$

Since $\eta(x)$ is arbitrary deformation of the path, therefore,

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0. \tag{4.3}$$

This equation is known as Euter-Lagrange's equation or simply Euler's equation.

Example 4.1.1 : Prove that the shortest distance between two points in a plane is a straight line.

Solution : An element of distance between two points in xy-plane is given by

$$(ds)^2 = (dx)^2 + (dy)^2$$

or, $$ds = \left[(dx)^2 + (dy)^2\right]^{1/2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx.$$

The total distance between two points having coordinates (x_1, y_1) and (x_2, y_2) is given by

$$J = \int_{(x_1,y_1)}^{(x_2,y_2)} ds = \int_{x_1}^{x_2}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx \tag{i}$$

$$= \int_{x_1}^{x_2} F(y, y', x)dx,$$

where $F(y,y',x)=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}=(1+y'^2)^{1/2}.$

If J is to be minimum, then

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0 \text{ [Euler's equation]}$$

or $\quad \frac{d}{dx}\left[\frac{y'}{(1+y'^2)^{1/2}}\right]=0 \quad$ or, $\quad \frac{y'}{(1+y'^2)^{1/2}}=C,$ C is a constant.

or, $\quad y'=C(1+y'^2)^{1/2} \quad$ or, $\quad y'^2=\frac{C^2}{1-C^2}$

or, $\quad y'=\frac{C}{\sqrt{1-C^2}}=a$ (say)

or, $\quad \frac{dy}{dx}=a.$

Integrating, we get $y = ax + b$, where b is constant of integration, which is the required equation of the straight line.

Example 4.1.2 Prove that if F does not depend on x explicitly then $F-y'\frac{\partial F}{\partial y'}$ is constant.

Proof. The Euler's differential equation is

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0.$$

Multiplying above equation by y' and adding, and subtracting $y''\frac{\partial F}{\partial y'}$ (where $y''=\frac{\partial y'}{\partial x}$ and $y'=\frac{\partial y}{\partial x}$)

we get

$$y'\frac{d}{dx}\left(\frac{\partial F}{\partial y^1}\right)-y'\frac{\partial F}{\partial y}+y''\frac{\partial F}{\partial y^1}-y''\frac{\partial F}{\partial y^1}=0$$

or, $$y'\frac{d}{dx}\left(\frac{\partial F}{\partial y^1}\right)+y''\frac{\partial F}{\partial y'}-y''\frac{\partial F}{\partial y^1}-y'\frac{\partial F}{\partial y}=0$$

or, $\frac{d}{dx}\left(y'\frac{\partial F}{\partial y^1}\right)-y''\frac{\partial F}{\partial y'}-y'\frac{\partial F}{\partial y}-\frac{\partial F}{\partial x}+\frac{\partial F}{\partial x}=0$

or, $\frac{d}{dx}\left(y'\frac{\partial F}{\partial y^1}\right)-\left[y''\frac{\partial F}{\partial y'}+y'\frac{\partial F}{\partial y}+\frac{\partial F}{\partial x}\right]+\frac{\partial F}{\partial x}=0$

or, $\frac{d}{dx}\left(y'\frac{\partial F}{\partial y^1}\right)-\frac{dF}{dx}+\frac{\partial F}{\partial x}=0$

or, $\frac{d}{dx}\left[y'\frac{\partial F}{\partial y^1}-F\right]+\frac{\partial F}{\partial x}=0.$ (*i*)

If F does not depend upon x explicitly, then $\frac{\partial F}{\partial x}=0$ and hence we must have

$$\frac{d}{dx}\left[y'\frac{\partial F}{\partial y^1}-F\right]=0$$

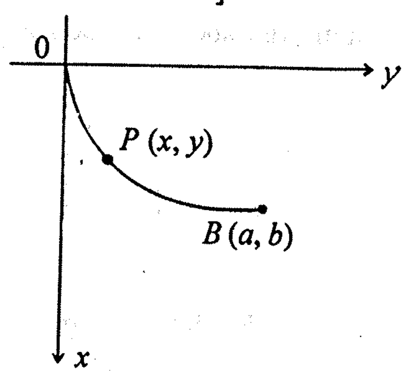
or, $y'\frac{\partial F}{\partial y'}-F=\text{constant}$

or, $F-y'\frac{\partial F}{\partial y^1}=\text{constant.}$ (*ii*)

Example 4.1.3 (Brachistochrone Problem).

Show that the path followed by a particle in sliding from one point to another in the absence of friction in the shortest time is a cycloid.

[The problem of finding path followed by a particle in sliding from one point to another in the absence of friction in the shortest (brachistos) time (chronos) is called the problem of Brachistochrone.]



Solution : Let the particle of mass m slide freely along a curve from the origin 0 (0, 0) to the point $B(a, b)$ under the influence of gravity. Let y axis be vertically upward and v the velocity of the particle at any point $P(x, y)$ of its path. Then from the principle of conservation of energy T + V = constant.

i.e. $\frac{1}{2}mv^2 - mgx = 0$ [At 0 K.E+P.E=0]

or, $v^2 = 2gx$

or, $\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] = 2gx$

or, $\left[\left(\frac{dy}{dx}\right)^2 + 1\right]\left(\frac{dx}{dt}\right)^2 = 2gx$

or, $\frac{dx}{dt} = \frac{\sqrt{2gx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$ or, $dt = \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{2gx}}dx$

Integrating between the limits $t = 0$ to $t = t$ and $x = 0$ to $x = a$, we have

$$t = \int_0^a \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{2gx}}dx = \int_0^a F(y, y', x)dx,$$

where $F(y, y', x) = \sqrt{\frac{1+y'^2}{2gx}}, y' = \frac{dy}{dx}$.

Now, t is a functional, according to the problem t is minimum. Then the problem is to find a path $y=y(x)$ such that t is minimum. Therefore, by Euler's theorem

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

or, $\frac{d}{dx}\left[\frac{1}{\sqrt{2gx}}\frac{y'}{\sqrt{1+y'^2}}\right] = 0 \left[\because \frac{\partial F}{\partial y} = 0\right]$

or, $\frac{y'}{\sqrt{2gx(1+y'^2)}} = c$, ($c$ is a constant) or, $y' = \frac{dy}{dx} = \sqrt{\frac{2gxc^2}{1-2gxc^2}}$

or, $y = \int\sqrt{\frac{2gxc^2}{1-2gxc^2}}dx = \int\sqrt{\frac{4gc^2x}{(1+1-4gc^2x)}}dx$

Substituting $x = \frac{1}{4gc^2}(1-\cos\theta) = a'(1-\cos\theta)$, where $a' = \frac{1}{4gc^2}$. (i)

$\therefore y = \int\frac{\sin\theta/2}{\cos\theta/2}dx = \int\frac{\sin\theta/2}{\cos\theta/2}\cdot 2a'\sin\theta/2\cos\theta/2\,d\theta$

$= \int a'(1-\cos\theta)d\theta = a'(\theta-\sin\theta)$. (ii)

The constant of integration is zero at (0, 0).

Therefore, the prarametric equation of the path is $x = a'(1-\cos\theta), y = a'(\theta-\sin\theta)$, which represents a cycloid.

Example 4.1.4 Show that a sphere is a solid figure of revolution which has maximum volume for a given surface area.

Solution. The volume of sphere may be supposed to be fixed of a large number of dics while the surface area may be supposed to be form a large number of rings.

The surface area of sphere

$A = 2\pi\int y ds = 2\pi\int y\left[(dx)^2+(dy)^2\right]^{1/2}$

$= 2\pi\int y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx$

$= 2\pi\int y\left[1+y'^2\right]^{1/2}dx,\ y' = \frac{dy}{dx}$. (i)

The volume of a sphere

$V = \pi\int y^2 dx$. (ii)

Combining (*i*) and (*ii*) we may write

$$\pi\int y^2 dx + \frac{\lambda}{2}\cdot 2\pi\int y\left(1+y'^2\right)^{1/2} dx = \text{Constant.}$$

[For fixed area and volume]

or, $$\int\left[y^2 + \lambda y\left(1+y'^2\right)^{1/2}\right]dx = \text{Constant}$$

or, $$\int\left[y^2 + \lambda y\left(1+y'^2\right)^{1/2}\right]dx = \text{extremum (say)} \qquad (iii)$$

Let $F = y^2 + \lambda y\left(1+y'^2\right)^{1/2}$

Since, F does not depend on x explicity, we have

$$y'\frac{\partial F}{\partial y'} - F = \text{Constant.} \qquad (iv)$$

Now, $$\frac{\partial F}{\partial y'} = 0 + \lambda y - \frac{1}{2}\left(1+y'^2\right)^{-1/2}\cdot 2y' = \frac{2yy'}{\sqrt{1+y'^2}}.$$

Therefore, (*iv*) gives

$$\left[y'\frac{\lambda yy'}{\left(1+y'^2\right)^{1/2}} - y^2 - \lambda y\left(1+y'^2\right)^{1/2}\right] = \text{Constant} = c. \qquad (v)$$

By y=0, at x =0 (at the origin), gives c=0.

$$\therefore \frac{\lambda yy'^2}{\left(1+y'^2\right)^{1/2}} - y^2 - \lambda y\sqrt{1+y'^2} = 0$$

or, $$\frac{\lambda yy'^2 - y^2\left(1+y'^2\right)^{1/2} - \lambda y\left(1+y'^2\right)}{\left(1+y'^2\right)^{1/2}} = 0$$

or, $$\lambda yy'^2 - y^2\left(1+y'^2\right)^{1/2} - \lambda y - \lambda yy'^2 = 0$$

or, $$-y^2\left(1+y'^2\right)^{1/2} - \lambda y = 0$$

or, $-y^2\left(1+y'^2\right)^{1/2} = \lambda y.$

Squaring, we have

$$y^2\left(1+y'^2\right) = \lambda^2.$$

Solving for y', we get

$$y' = \frac{dy}{dx} = \frac{\sqrt{\lambda^2 - y^2}}{y}.$$

Integrating, we obtain

$$-\sqrt{\lambda^2 - y^2} = x - x_0, \text{ [} x_0 \text{ the constant of integration]}$$

or, $(x - x_0)^2 + y^2 = \lambda^2.$

This equation represents a sphere with centre at x_0 on x-axis and radius λ. Hence we say that for the values of area A and volume V of sphere F is extremum.

Thus the sphere is the solid figure of revolution which has maximum volume for given surface area.

4.2 Derivation of Euler-Lagrange Differential Equations for Multiple Dependent Variables

Here we consider that the integrand F occur in the integral to be minimize or maximize is a function of one independent variable x and multiple dependent variables $y_1(x), y_2(x),.....$

Then the probelm is to find the functions $y_1(x), y_2(x),.....$ such that the integral

$$J = \int_{x_0}^{x_1} F\left(y_1, y_2, ..., y_1', y_2', ..., x\right) dx \tag{4.4}$$

may be stationary.

Consider two paths out of infinite number of possibilities, such that the difference between them may be described by introducing new functions $\eta_k(x)$ and ε.

The functions $\eta_k(x)$ must satisfy the following two conditions :

(i) All the neighbouring paths must pass through the fixed points P and Q, i.e., $\eta_k(x_0) = \eta_k(x_1) = 0$.

(ii) $\eta_k(x)$ must be differentiable.

Let PRQ (Figure 4.1) be the path along which J has stationary value and $PR'Q$ be the neighbouring path. If y_k^* and $y_k^{*\prime}$ are the values of y_k and y_k' along the varied path, then introducing $\eta_k(x)$ and ε, we have

$$y_1^{*} = y_1 + \delta y_1 = y_1 + \varepsilon\eta_1(x)$$
$$y_2^{*} = y_2 + \delta y_2 = y_2 + \varepsilon\eta_2(x)$$

....

....

$$y_k^{*} = y_k + \delta y_k = y_2 + \varepsilon\eta_k(x).$$

If the integral has stationary value along PRQ, then

$$J(\varepsilon) = \int_{x_0}^{x_1} F\left(y_1 + \varepsilon\eta_1, y_2 + \varepsilon\eta_2, \ldots, y_k + \varepsilon\eta_k, \ldots, y_1' + \varepsilon\eta_1', y_2' + \varepsilon\eta_2', \ldots, y_k' + \varepsilon\eta_k', \ldots, x\right)dx. \quad (4.5)$$

Now, $J(\varepsilon)$ is stationary for $\varepsilon=0$, as in case of single dependent variable.

Expanding (4.5) by Taylor's theorem and differentiating w.r.t. ε and then setting $\varepsilon=0$, we get

$$\left[\frac{dJ(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=0} = \int_{x_0}^{x_1}\sum_k\left[\eta_k\frac{\partial F}{\partial y_k} + \eta_k'\frac{\partial F}{\partial y_k'}\right]dx. \quad (4.6)$$

The condition for the path along which J has stationary value is

$$\left[\frac{dJ(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=0} = 0.$$

With this condition for J to be extremum, we have from (4.6)

$$\int_{x_0}^{x_1}\sum_k\left[\eta_k\frac{\partial F}{\partial y_k} + \eta_k'\frac{\partial F}{\partial y_k'}\right]dx = 0. \quad (4.7)$$

Integrating the second term by parts, we get

$$\int_{x_0}^{x_1}\eta_k'\frac{\partial F}{\partial y_k'}dx = \left[\eta_k\frac{\partial F}{\partial y_k'}\right]_{x_0}^{x_1} - \int_{x_0}^{x_1}\eta_k\frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)dx$$

$$= -\int_{x_0}^{x_1}\eta_k'\frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)dx$$

[since $\eta_k(x_0) = \eta_k(x_1) = 0$.]

Using this result equation (4.7) becomes

$$\int_{x_0}^{x_1}\sum_k\left[\eta_k\frac{\partial F}{\partial y_k} - \eta_k\frac{d}{dx}\left(\frac{\partial F}{dy_k'}\right)\right]dx = 0$$

or, $$\int_{x_0}^{x_1} \sum_k \eta_k \left[\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{dy_k'} \right) \right] dx = 0. \tag{4.8}$$

Since η_k are perfectly arbitrary and independent of one another, each of the terms in equation (4.8) must vanish independently, we have

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'} \right) = 0, k = 1,2,3,......$$

or, $$\frac{d}{dx}\left(\frac{\partial F}{\partial y_k'} \right) - \frac{\partial F}{\partial y_k} = 0. \tag{4.9}$$

Equation (4.9) represents a whole set of Euler-Lagrange equations each of which must be satisfied for an extreme value.

4.3. Hamilton's Principle

The principle :

The path actually traversed by a conservative, holonomic dynamical system from time t_0 and t_1 is one over which the inegral of the Lagrangian between limits t_0 and t_1 is stationary, i.e., the time integral of the Lagrangian is extremum.

Mathematically, $\int_{t_0}^{t_1} L\,dt = J =$ extremum, (4.10)

where J is the extremum value of the time integral of the Lagrangian and is known as Hamilton's principle function for the path.

Equation (4.10) can be represented as

$$\delta \int_{t_0}^{t_1} L\,dt = 0$$

where δ is the variation symbol.

This principle helps to distinguish the actual path from the neighbouring paths.

4.4 Deduction of Hamilton's Principle from D'Alembert's Principle

Let us consider that the conservative holonomic dynamical system moves from P to Q, where P and Q are initial and final configurations of the system at times t_0 and t_1 respectively. Let PRQ be the actual path and

$PR'Q, PR''Q$ the two neighbouring paths out of infinite number of possibilities.

For the deduction of Hamilton's principle the following two conditions must be satisfied :

(*i*) δt must be equal to zero at end points, i.e. at t_0 the particle must be at P and at t_1 the particle must be at Q.

(*ii*) δr must be equal to zero at end points, i.e., the points P and Q are fixed in space.

Let the system be acted upon by a number of forces represented by $\vec{F}$. Let ith particle of the system acted upon by force $\vec{F}_i$ acquire acceleration $\ddot{\vec{r}}_i$, so that we have

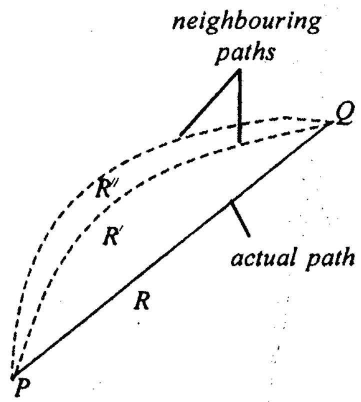


Fig. 4.2

$$\vec{F}_i = m_i\ddot{\vec{r}}_i.$$

From D'Alembert's pinciple, we have

$$\sum_i \left(\vec{F}_i - m_i\ddot{\vec{r}}_i\right)\cdot\delta\vec{r}_i = 0$$

or, $$\sum_i \vec{F}_i\cdot\delta\vec{r}_i - \sum_i m_i\ddot{\vec{r}}_i\cdot\delta\vec{r}_i = 0. \qquad (4.11)$$

The K.E. of the system is

$$T = \frac{1}{2}\sum_i m_i\dot{\vec{r}}_i^{\,2}. \qquad (4.12)$$

Taking small variation, we get

$$\delta T = \sum_i m_i\dot{\vec{r}}_i\cdot\delta\dot{\vec{r}}_i.$$

Now, $\frac{d}{dt}\left\{\sum_i m_i\dot{\vec{r}}_i\cdot\delta\vec{r}_i\right\}=\sum_i m_i\ddot{\vec{r}}_i\cdot\delta\vec{r}_i+\sum_i m_i\dot{\vec{r}}_i\cdot\frac{d}{dt}(\delta\vec{r}_i)$ (4.13)

$$=\sum_i m_i\ddot{\vec{r}}_i\cdot\delta\vec{r}_i+\sum_i m_i\dot{\vec{r}}_i\cdot\delta\dot{\vec{r}}.$$

Here we have interchanged the operation of variation and of differentiation w.r.t. time t, i.e.

$$\frac{d}{dt}(\delta\vec{r}_i)=\delta\left(\frac{d\vec{r}_i}{dt}\right).$$

It happens in virtual displacement.

Combining (4.11), (4.12) and (4.13), we have

$$\delta W=\sum_i\vec{F}_i\cdot\delta\vec{r}_i=\sum_i m_i\ddot{\vec{r}}_i\cdot\delta\vec{r}_i$$

$$=\sum_i\frac{d}{dt}\left\{\sum_i m_i\dot{\vec{r}}_i\cdot\delta\vec{r}_i\right\}-\sum_i m_i\dot{\vec{r}}_i\cdot\delta\dot{\vec{r}}_i$$

$$=\frac{d}{dt}\left\{\sum_i m_i\dot{\vec{r}}_i\cdot\delta\vec{r}_i\right\}-\delta T$$

or, $\delta T+\delta W=\frac{d}{dt}\left\{\sum_i m_i\dot{\vec{r}}\cdot\delta\vec{r}_i\right\},$

where δW is the virtual work of applied forces. Now, integrating this equation w.r.t. t between the limits t_0 and t_1.

$$\therefore\int_{t_0}^{t_1}(\delta T+\delta W)dt=\left[\sum_i m_i\dot{\vec{r}}_i\cdot\delta\vec{r}_i\right]_{t_0}^{t_1}=0. \qquad (4.14)$$

Since the configuration of the system specified at time t_0 and t_1, the variation $\delta\vec{r}_i$ are zero at t_0 and t_1.

In terms of generalised coordinates $q_1,q_2,\ldots,q_n$ the equation (4.14) can be written as

$$\int_{t_0}^{t_1}\left(\delta T+\sum_{j=1}^{n}Q_j\delta q_j\right)dt=0, \qquad (4.15)$$

where Q_j's are applied generalised forces.

The results (4.14) or (4.15) are often consider as a generalised version of Hamilton's principle.

For a conservative system

$$\delta W=-\delta V,$$

where V is the P.E. Then from (4.14)

$$\int_{t_0}^{t_1}(\delta T - \delta V)\,dt = 0$$

or, $$\int_{t_0}^{t_1}\delta(T - V)\,dt = 0$$

or, $$\int_{t_0}^{t_1}\delta L\; dt = 0$$

or, $$\delta\int_{t_0}^{t_1} L\; dt = 0 \qquad (4.16)$$

$\therefore \int_{t_0}^{t_1} L\,dt$ is extremum.

Example 4.4.1 Use Hamilton's principle to find the equation of motion of one dimensional harmonic oscillator.

Solution : The K.E. of harmonic oscillator is

$$T = \frac{1}{2}m\dot{x}^2.$$

The P.E. of harmonic oscillator,

$$V = -\int F\,dx = \int kx\,dx = \frac{1}{2}kx^2.$$

$\therefore$ the Lagramgian of the system

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

According to the Hamilton's principle

$$\delta\int_{t_0}^{t_1} L\,dt = 0$$

or, $$\delta\int_{t_0}^{t_1}\frac{1}{2}\left(m\dot{x}^2 - kx^2\right)dt = 0.$$

or, $$\delta\int_{t_0}^{t_1}\delta\left(m\dot{x}^2 - kx^2\right)dt = 0$$

or, $$\delta\int_{t_0}^{t_1}\left(m\dot{x}\delta\dot{x} - kx\,\delta x\right)dt = 0$$

But, $\delta\dot{x} = \frac{d}{dt}(\delta x).$

or, $\int_{t_0}^{t_1} m\dot{x}\frac{d}{dt}(\delta x)dt - \int_{t_0}^{t_1} kx\,\delta x\,dt = 0$

or, $[m\dot{x}\,\delta x]_{t_0}^{t_1} - \int_{t_0}^{t_1} m\frac{d}{dt}(\dot{x})\delta x\,dt - \int_{t_0}^{t_1} kx\,\delta x\,dt = 0.$ (*i*)

But, δx=0 at the fixed points, i.e. at instants t_0 and t_1.

$\therefore [m\dot{x}\,\delta x]_{t_0}^{t_1} = 0.$

The equatin (*i*) gives

$$-\int_{t_0}^{t_1} m\frac{d}{dt}(\dot{x})\,\delta x\,dt - \int_{t_0}^{t_1} kx\,\delta x\,dt = 0$$

or, $\int_{t_0}^{t_1}(m\ddot{x} + kx)\,\delta x\,dt = 0.$

Since δx is arbitrary, the above equation is satisfied only if $m\ddot{x} + kx = 0$, which is the equation of motion for one dimensional harmonic oscillator.

Example 4.4.2 Use Hamilton's principle to find the equations of motion of a particle of unit mass moving on a plane in a conservative field.

Solution. Let $P(x, y)$ be the position of a particle moving on the *xy*-plane under the action of the forces *X*, *Y* where

$$X = -\frac{\partial V}{\partial x},\ Y = -\frac{\partial V}{\partial y}.$$

Now, $L = T - V = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - V.$ (*i*)

By Hamilton's principle, we have

$$\delta\int_{t_0}^{t_1} L\,dt = 0 \text{ or, } \int_{t_0}^{t_1}\delta L\,dt = 0.$$

Therefore from (*i*), we get

$$\int_{t_0}^{t_1}(\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} - \delta V)\,dt = 0$$

or, $\int_{t_0}^{t_1}\left(\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} - \frac{\partial V}{\partial x}\delta x - \frac{\partial V}{\partial y}\delta y\right)dt = 0.$ (*ii*)

Now, $\int_{t_0}^{t_1} \dot{x}\delta\dot{x}\,dt = \int_{t_0}^{t_1} \dot{x}\frac{d}{dt}(\delta x)dt$

$$= [\dot{x}\,\delta x]_{t_0}^{t_1} - \int_{t_0}^{t_1} \ddot{x}\,\delta x\,dt = -\int_{t_0}^{t_1} \ddot{x}\,\delta x\,dt \qquad (iii)$$

[since $\delta x = 0$ at $t = t_0$ and $t = t_1$].

Similarly,

$$\int_{t_0}^{t_1} \dot{y}\,\delta\dot{y}\,dt = -\int_{t_0}^{t_1} \ddot{y}\,\delta y\,dt. \qquad (iv)$$

Using (*iii*) and (*iv*) equation (*ii*) becomes

$$\int_{t_0}^{t_1}\left[\left(\ddot{x}+\frac{\partial V}{\partial x}\right)\delta x+\left(\ddot{y}+\frac{\partial V}{\partial y}\right)\delta y\right]dt = 0.$$

Now, δx and δy are independent and arbitrary, therefore

$$\ddot{x} = -\frac{\partial V}{\partial x} = X, \ddot{y} = -\frac{\partial V}{\partial y} = Y.$$

These give the equations of motion.

4.5 Deduction of Lagrange's Equations of Motion from Hamilton's Principle

The Lagrangian L for the generalised coordinates $q_1, q_2, \ldots. q_n$ is

$$L = L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t).$$

Taking δ variation, we have

$$\delta L = \sum_{j=1}^{n}\frac{\partial L}{\partial q_j}\delta q_j + \sum_{j=1}^{n}\frac{\partial L}{\partial \dot{q}_j}\delta\dot{q}_j.$$

($\delta t = 0$, since in δ variation there is no time variation along any path and also at the end points).

Now, integrating between the limits t_0 and t_1 we obtain

$$\int_{t_0}^{t_1}\delta L\,dt = \int_{t_0}^{t_1}\sum_{j=1}^{n}\frac{\partial L}{\partial q_j}\delta q_j\,dt + \int_{t_0}^{t_1}\sum_{j=1}^{n}\frac{\partial L}{\partial \dot{q}_j}\delta\dot{q}_j\,dt.$$

According to Hamilton's principle,

$$\int_{t_0}^{t_1}\delta L\,dt = 0.$$

$$\therefore \int_{t_0}^{t_1}\sum_{j=1}^{n}\frac{\partial L}{\partial q_j}\delta q_j\,dt+\int_{t_0}^{t_1}\sum_{j=1}^{n}\frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j\,dt=0. \quad (4.17)$$

Now, $\int_{t_0}^{t_1}\sum_{j=1}^{n}\frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j\,dt=\int_{t_0}^{t_1}\sum_{j=1}^{n}\frac{\partial L}{\partial \dot{q}_j}\frac{d}{dt}(\delta q_j)\,dt$

$$=\sum_{j=1}^{n}\left\{\left[\frac{\partial L}{\partial \dot{q}_j}\delta q_j\right]_{t_0}^{t_1}-\int_{t_0}^{t_1}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j\,dt\right\}$$

$$=\int_{t_0}^{t_1}\sum_{j=1}^{n}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j\,dt$$

[Since $\delta q_j = 0$ at the ends points.]

Substituting this value in (4.17), we have

$$\int_{t_0}^{t_1}\sum_{j=1}^{n}\frac{\partial L}{\partial q_j}\delta q_j\,dt-\int_{t_0}^{t_1}\sum_{j=1}^{n}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j\,dt=0$$

or, $\int_{t_0}^{t_1}\sum_{j=1}^{n}\left[\frac{\partial L}{\partial q_j}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]\delta q_j\,dt=0.$

The variations of generalised coordinates are independent iff

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0,\ j=1,2....,n \quad (4.18)$$

which are Lagrange's equations of motion.

4.5.1 Deduction of Lagrange's equations of motion for non-conservative system

From generalised Hamilton's principle [equation (4.15)]

We have

$$\int_{t_0}^{t_1}\left(\delta T+\sum_{j}Q_j\delta q_j\right)dt=0. \quad (4.19)$$

Now, $\delta T = \sum_j \frac{\partial T}{\partial q_j}\delta q_j + \sum_j \frac{\partial T}{\partial \dot{q}_j}\delta \dot{q}_j$. Integrating, we get

or, $\int_{t_0}^{t_1} \delta T\, dt = \int_{t_0}^{t_1} \sum_j \frac{\partial T}{\partial \dot{q}_j}\delta \dot{q}_j\, dt + \int_{t_0}^{t_1} \sum_j \frac{\partial T}{\partial q_j}\delta q_j\, dt.$ (4.20)

Now, consider the first term of (4.20),

$$\int_{t_0}^{t_1} \sum_j \frac{\partial T}{\partial \dot{q}_j}\frac{d}{dt}(\delta q_j)\, dt = \left[\sum_j \frac{\partial T}{\partial \dot{q}_j}\delta q_j\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \sum_j \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\delta q_j\, dt$$

$$= -\int_{t_0}^{t_1} \sum_j \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\delta q_j\, dt. \text{ [at ends } \delta q_j = 0]. \quad (4.21)$$

$\therefore$ from (4.19), (4.20) and (4.21), we have

$$\int_{t_0}^{t_1} \sum_j \frac{\partial T}{\partial q_j}\delta q_j\, dt - \int_{t_0}^{t_1} \sum_j \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\delta q_j\, dt + \int_{t_0}^{t_1} \sum_j Q_j\, \delta q_j\, dt = 0$$

or, $\int_{t_0}^{t_1} \sum_j \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + Q_j\right]\delta q_j\, dt = 0.$

Since δq_j are arbitrary, the above relation will be valid iff

$$\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + Q_j = 0, j = 1,2,\ldots\ldots,n,$$

or, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j, j = 1,2,\ldots.,n,$

which are the Lagrange's equations for holonomic and non-conservative system.

4.6 Deduction of Hamilton's Equations of Motion from Hamilton's Principle

The Hamiltonian H for the generalised coordinates $q_1, q_2, \ldots, q_n$ and generalised momentum $p_1, p_2, \ldots., p_n$ is

$$H(p_j, q_j, t) = \sum p_j \dot{q}_j - L(q_j, \dot{q}_j, t).$$

or, $L(q_j, \dot{q}_j, t) = \sum p_j \dot{q}_j - H(p_j, q_j, t).$ (4.22)

Taking δ variation,

$$\delta L = \sum\left(p_j \delta \dot{q}_j + \dot{q}_j \delta p_j\right) - \left\{\sum \frac{\partial H}{\partial p_j}\delta p_j + \sum \frac{\partial H}{\partial q_j}\delta q_j\right\}.$$

Integrating between the limits t_0 and t_1, we obtain

$$\int_{t_0}^{t_1} \delta L\, dt = \int_{t_0}^{t_1}\left[\sum p_j\, \delta \dot{q}_j + \dot{q}_j\, \delta p_j\right]dt - \int_{t_0}^{t_1}\left\{\sum \frac{\partial H}{\partial p_j}\delta p_j + \sum \frac{\partial H}{\partial q_j}\delta q_j\right\}dt. \tag{4.23}$$

By Hamilton's principle,

$$\int_{t_0}^{t_1} \delta L\, dt = 0.$$

Therefore, (4.23) becomes

$$\int_{t_0}^{t_1} \delta L\, dt = \left[\sum p_j\, \delta \dot{q}_j + \dot{q}_j\, \delta p_j\right]dt - \int_{t_0}^{t_1}\left\{\sum \frac{\partial H}{\partial p_j}\delta p_j + \sum \frac{\partial H}{\partial q_j}\delta q_j\right\}dt = 0. \tag{4.24}$$

Now, consider the term

$$\int_{t_0}^{t_1} \sum p_j\, \dot{q}_j\, dt = \sum \int_{t_0}^{t_1} p_j \frac{d}{dt}\left(\delta q_j\right)dt$$

$$= \sum\left\{\left[p_j\, \delta q_j\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\left(p_j\right)\delta q_j\, dt\right\}$$

$$= -\int_{t_0}^{t_1} \sum \dot{p}_j\, \delta q_j\, dt. \quad [\because \delta q_j = 0 \text{ at the ends}]$$

Substituting this value in (4.24), we have

$$\int_{t_0}^{t_1}\left[\sum\left\{-\dot{p}_j\, \delta q_j + \dot{q}_j\, \delta p_j - \frac{\partial H}{\partial p_j}\delta p_j - \frac{\partial H}{\partial q_j}\delta q_j\right\}dt\right] = 0$$

or, $$\int_{t_0}^{t_1}\left\{\sum\left(\dot{q}_j - \frac{\partial H}{\partial p_j}\right)\delta p_j - \left(\dot{p}_j + \frac{\partial H}{\partial q_j}\right)\delta p_j\right\}dt = 0. \tag{4.25}$$

If δq_j 's and δp_j 's are independent of each other, the integrand (4.25) is satisfied only when the coefficient of δq_j and δp_j vanish separately i.e., when

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \text{ and } \dot{p}_j = -\frac{\partial H}{\partial q_j}, j = 1,2,....,n,$$

which are the required Hamilton's equation of motion.

4.7 Modified Hamilton's Principle

According to Hamilton's principle, we have

$$\delta J = \delta \int_{t_0}^{t_1} L\, dt = 0. \tag{4.26}$$

The relation between Lagrangian and Hamiltonian is

$$H = \sum_j p_j \dot{q}_j - L$$

or, $$L = \sum_j p_j \dot{q}_j - H.$$

Then (4.26) becomes

$$\delta J = \delta \int_{t_0}^{t_1} \left[\sum_j p_j \dot{q}_j - H \right] dt = 0 \tag{4.27}$$

or, $$\int_{t_0}^{t_1} \left[\sum_j p_j \, \delta \dot{q}_j + \delta p_j \, \dot{q}_j - \delta H \right] dt = 0. \tag{4.28}$$

This is called modified Hamilton's principle.

4.8 Derivation of Hamilton's Equations from Variational Principle

Consider two paths PRQ and $PR'Q$ out of infinite number of possibilities between P and Q and shown in Figure 4.3. In this case δ variation comprises independent variations of both q_j and p_j at constant time t. The difference between the two paths for the given value of t, may be described by introducing a parameter α common to all points of the path of integrating in phase space.

If q_j^* and p_j^* are the values of q_j and p_j for the varied paths $PR'Q$, we have

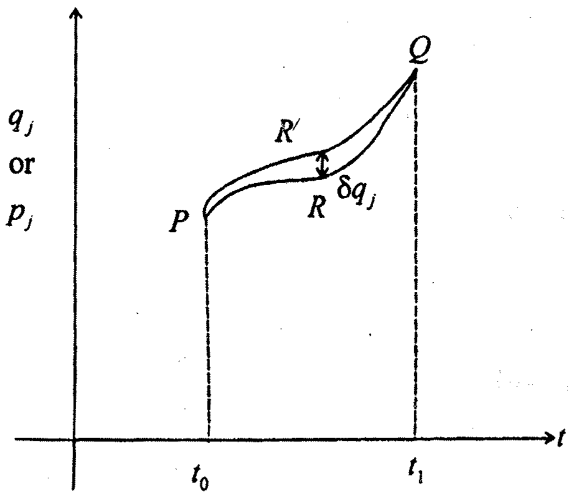


Figure 4.3

$$q_j^* = q_j + \delta q_j = q_j + \frac{\partial q_j}{\partial \alpha}\delta\alpha = q_j + \eta_j\,\delta\alpha \tag{4.29}$$

and $$p_j^* = p_j + \delta p_j = p_j + \frac{\partial p_j}{\partial \alpha}\delta\alpha = p_j + \xi_j\,\delta\alpha, \tag{4.30}$$

where $\frac{\partial q_j}{\partial \alpha} = \eta_j$ and $\frac{\partial p_j}{\partial \alpha} = \xi_j$. (4.31)

The variations in q_j and p_j are given by

$$\left.\begin{aligned} \delta q_j &= \eta_j \delta\alpha \\ \delta p_j &= \xi_j\,\delta\alpha. \end{aligned}\right\} \tag{4.32}$$

[From equations (4.29) and (4.30)]

If $\delta\alpha = 0$, then $\delta q_j = \delta p_j = 0$, so that $q_j^* = q_j$ and $p_j^* = p_j$, i.e., the varied path $PR'Q$ coincides with the actual path PQR.

As the times at end points are not varied so that

$$\left.\begin{aligned} \eta_j(t_0) &= \eta_j(t_1) = 0 \\ \xi_j(t_0) &= \xi_j(t_1) = 0. \end{aligned}\right\} \tag{4.33}$$

Interms of parameter α, Hamilton's principle becomes

$$\delta J = \frac{\partial J}{\partial \alpha}\delta\alpha = \delta\alpha \frac{\partial}{\partial \alpha}\int_{t_0}^{t_1}\left[\sum_j p_j \dot{q}_j - H\right]dt. \tag{4.34}$$

As the times at the end points are not varied and hence are the functions of α, so that the differentiation and integration may be interchanged.

Then equation (4.34) may be written as

$$\delta\alpha\int_{t_0}^{t_1}\frac{\partial}{\partial\alpha}\left[\sum p_j\dot{q}_j - H\right]dt = 0$$

or, $$\delta\alpha\int_{t_0}^{t_1}\sum_j\left[\frac{\partial p_j}{\partial\alpha}\dot{q}_j + \frac{\partial\dot{q}_j}{\partial\alpha}p_j - \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial\alpha} - \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial\alpha}\right]dt = 0 \quad (4.35)$$

But, $$\int_{t_0}^{t_1}\frac{\partial\dot{q}_j}{\partial\alpha}p_j\,dt = \int_{t_0}^{t_1}p_j\frac{d}{dt}\left(\frac{\partial q_j}{\partial\alpha}\right)dt$$

$$= \left[p_j\frac{\partial q_j}{\partial\alpha}\right]_{t_0}^{t_1} - \int_{t_0}^{t_1}\dot{p}_j\frac{\partial q_j}{\partial\alpha}dt$$

$$= \left[p_j\eta_j\right]_{t_0}^{t_1} - \int_{t_0}^{t_1}\eta_j\dot{p}_j\,dt \quad \text{[Using (4.31)]}$$

$$= \int_{t_0}^{t_1}\eta_j\dot{p}_j\,dt \quad (4.36)$$

since $\left[p_j\eta_j\right]_{t_0}^{t_1} = p_j\eta_j(t_1) - p_j\eta_j(t_0) = 0.$ [Using (4.33)]

Then using (4.31) and (4.29), equation (4.35) becomes

$$\delta\alpha\int_{t_0}^{t_1}\sum\left[\xi_j\dot{q}_j - \eta_j\dot{p}_j - \frac{\partial H}{\partial q_j}\eta_j - \frac{\partial H}{\partial p_j}\xi_j\right]dt = 0$$

or, $$\int_{t_0}^{t_1}\sum_j\left[\dot{q}_j\xi_j\delta\alpha - \eta_j\dot{p}_j\delta\alpha - \frac{\partial H}{\partial q_j}\eta_j\delta\alpha - \frac{\partial H}{\partial p_j}\xi_j\delta\alpha\right]dt = 0$$

or, $$\int_{t_0}^{t_1}\sum_j\left[\left(\dot{q}_j - \frac{\partial H}{\partial p_j}\right)\xi_j\delta\alpha - \left(\dot{p}_j + \frac{\partial H}{\partial q_j}\right)\eta_j\delta\alpha\right]dt = 0$$

or, $$\int_{t_0}^{t_1}\sum_j\left[\left(\dot{q}_j - \frac{\partial H}{\partial p_j}\right)\delta p_j - \left(\dot{p}_j + \frac{\partial H}{\partial q_j}\right)\delta q_j\right]dt = 0. \quad (4.37)$$

[Using 14.32)]

But the variations δq_j and δp_j are independent of each other; the integral (4.37) is satisfied only when the coefficients of δp_j and δq_j vanish separately, i.e., when

$$\dot{q}_j - \frac{\partial H}{\partial p_j} = 0 \quad \text{and} \quad \dot{p}_j + \frac{\partial H}{\partial q_j} = 0$$

i.e., $$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, j = 1,2,....n, \tag{4.38}$$

which are required Hamilton's equations.

4.9 Principle of Least Action

The time integral of twice the K.E. is called the **action**. The principle of least action states that

$$\Delta\int_{t_0}^{t_1} 2T\, dt = 0. \tag{4.39}$$

But, in system for which Hamiltonian H remains constant,

$$2T = \sum_j p_j \dot{q}_j .$$

Therefore, for such systems, the principle of least action may be written as

$$\Delta\int_{t_0}^{t_1} \sum_j p_j\, \dot{q}_j\, dt = 0, \tag{4.40}$$

where Δ represents a new type of variation of the path which allows time as well as position coordinates to vary.

4.10 Deduction of Principle of Least Action

To deduce the principle of least action we use a variation termed as Δ variation in which

(i) time as well as the position coordinates are followed to vary,

(ii) time t varies even at the end points of the path,

(iii) the position coordinates are held fixed at the end points of the path, i.e., $\Delta q_j = 0$ at the end points.

Let PRQ be the actual path and $P'R'Q'$, the varied path. The end point P and Q after time Δt take the positions of P' and Q' such that the position coordinates of P and Q are fixed while the time t is not fixed. A point R on the actual path now goes over R' on the varied path with the correspondence

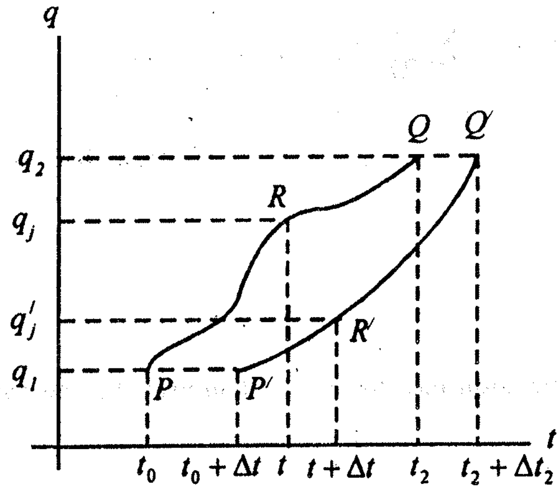


Fig. 4.4

$$q_j \to q_j' = q_j + \Delta q_j.$$

It α is variational parameter, then in δ process t is independent of α; but in Δ process t is function of α even at the end points, i.e. $t = t(\alpha)$. Thus the q_j depends on t and α throughout.

Analytically Δ variation is defined as

$$\Delta q_j = \left[\frac{d}{d\alpha} q_j(\alpha, t)\right] d\alpha$$

$$= \left[\frac{\partial q_j}{\partial \alpha} + \frac{dq_j}{dt}\frac{dt}{d\alpha}\right] d\alpha$$

$$= \frac{\partial q_j}{\partial \alpha} d\alpha + \dot{q}_j \frac{dt}{d\alpha} d\alpha$$

But, $\delta q_j = \frac{\partial q_j}{\partial \alpha} d\alpha$ [same as (4.32)] and $\delta \dot{q}_j \frac{\partial t}{\partial \alpha} = \dot{q}_j \,\Delta t$ (4.41)

$$\Delta q_j = \delta q_j + \dot{q}_j \Delta t. \qquad (4.42)$$

Analytically, Δ-variation of any function $f(q_j, \dot{q}_j, t)$ is given by

$$\Delta f = \sum_j \left(\frac{\partial f}{\partial q_j} \Delta q_j + \frac{\partial f}{\partial \dot{q}_j} \Delta \dot{q}_j + \frac{\partial f}{\partial t} \Delta t \right)$$

$$= \sum_j \frac{\partial f}{\partial q_j}\left(\delta q_j + \dot{q}_j \, \Delta t\right) + \sum_j \frac{\partial f}{\partial \dot{q}_j}\left(\delta \dot{q}_j + \ddot{q}_j \, \Delta t\right) + \frac{\partial f}{\partial t}\Delta t$$

$$= \delta f + \frac{df}{dt}\Delta t.$$

Thus $\Delta = \delta + \Delta t \frac{d}{dt}$. (4.43)

It may be noted that Δ operation and time differentiation are not interchangeble,

$J = \int_{t_0}^{t_1} L dt$ = extremum.

Taking Δ variation, we have

$$\Delta J = \Delta \int_{t_0}^{t_1} L \, dt = \left(\delta + \Delta t + \frac{d}{dt}\right) \int_{t_0}^{t_1} L \, dt$$

$$= \delta \int_{t_0}^{t_1} L \, dt + \Delta t \int_{t_0}^{t_1} dL$$

$$= \delta \int_{t_0}^{t_1} L \, dt + \left[L \, \Delta t\right]_{t_0}^{t_1}$$

$$= \int_{t_0}^{t_1} \delta L \, dt + \left[L \, \Delta t\right]_{t_0}^{t_1}$$

$$= \int_{t_0}^{t_1} \sum_j \left[\frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j\right] dt + \left[L \, \Delta t\right]_{t_0}^{t_1} \qquad (4.44)$$

From Lagrange's equation, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

or, $$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right). \qquad (4.45)$$

Also, we know, $\delta \dot{q}_j = \frac{d}{dt}\left(\delta q_j\right)$. (4.46)

Using (4.45) and (4.46), equation (4.44) becomes

$$\int_{t_0}^{t_1} L\,dt = \int_{t_0}^{t_1}\sum_j\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\frac{d}{dt}(\delta q_j)\right]dt + [L\,\Delta t]_{t_0}^{t_1}$$

$$= \int_{t_0}^{t_1}\sum_j\left[\dot{p}_j\,\delta q_j + p_j\frac{d}{dt}(\delta q_j)\right]dt + [L\,\Delta t]_{t_0}^{t_1}$$

$$\left[\because p_j = \frac{\partial L}{\partial q_j}\right]$$

$$= \int_{t_0}^{t_1}\sum_j\frac{d}{dt}(p_j\,\delta q_j)\,dt + [L\,\Delta t]_{t_0}^{t_1}$$

$$= \int_{t_0}^{t_1}\sum_j\frac{d}{dt}\{p_j(\Delta q_j - \dot{q}_j\,\Delta t)\}dt + [L\,\Delta t]_{t_0}^{t_1} \quad \text{[by (4.42)]}$$

$$= \int_{t_0}^{t_1}\sum_j d(p_j\,\Delta q_j) - \int_{t_0}^{t_1}\sum_j d(p_j\dot{q}_j\,\Delta t) + [L\,\Delta t]_{t_0}^{t_1}$$

$$= \sum_j[p_j\Delta q_j]_{t_0}^{t_1} - \sum_j[p_j\dot{q}_j\Delta t]_{t_0}^{t_1} + [L\,\Delta t]_{t_0}^{t_1} \tag{4.47}$$

But, $[p_j\Delta q_j]_{t_0}^{t_1} = 0$ since $\Delta q_j = 0$ at end points.

$\therefore$ equation (4.47) reduces to

$$\Delta\int_{t_0}^{t_1} L\,dt = \left[\left(L - \sum p_j\dot{q}_j\right)\Delta t\right]_{t_0}^{t_1}$$

$$= -[H\,\Delta t]_{t_0}^{t_1}. \tag{4.48}$$

If we consider the system for which $\frac{\partial H}{\partial t} = 0$, i.e., for which H remains constant, then

$$[H\,\Delta t]_{t_0}^{t_1} = \Delta\int_{t_0}^{t_1} H\,dt.$$

Substituting this in (4.48), we get

$$\Delta\int_{t_0}^{t_1} L\,dt = -\Delta\int_{t_0}^{t_1} H\,dt$$

or, $\Delta\int_{t_0}^{t_1}(L+H)dt=0.$ (4.49)

or, $\Delta\int_{t_0}^{t_1}\left(L+\sum_j p_j\dot{q}_j-L\right)dt=0$

or, $\Delta\int_{t_0}^{t_1}\sum_j p_j\dot{q}_j\,dt=0.$ (4.50)

This is the principle of least action. The quantity $\int_{t_0}^{t_1}\sum_j p_j\dot{q}_j\,dt$ is generally called Hamilton's characteristic function.

But $\sum_j p_j\dot{q}_j=2T.$

$\therefore\ \Delta\int_{t_0}^{t_1}2T\,dt=0,$ (4.51)

which is another form of principle of least action.

Example 4.10.1 Apply principle of least action to prove that out of all possible paths between two points, the system for which K.E. is conserved moves along the path for which the transit time is extremum.

Solution. According to the principle of least action,

$$\therefore\ \Delta\int_{t_0}^{t_1}2T\,dt=0.$$

If K.E. of the system tis conserved, then above equation yields

$$\Delta\int_{t_0}^{t_1}dt=0$$

or, $\Delta(t_1-t_0)=0,$

which states that the system moves along the path for which the transit time is extremum.

4.11 Unit Summary

In this unit, Euler-Lagrange's equation is deduced and some problems have been solved using this equation. The Hamilton's principle and its modification are stated. Using Hamilton's principle, Lagrange's and Hamilton's equations of motion are deduced. The principle of least action is stated and proved. An exercise is included with this unit.

4.12 Self Assessment Questions

4.1 Prove that $J=\int_{x_0}^{x_1} F(y,y',x)\,dx$ will be minumum only when $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0.$

4.2. Prove that $J=\int_{x_0}^{x_1} F(y_1,y_2,....,y_k,...,y_1',y_2',....,y_k',...,x)\,dx$

will be stationary only if

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)-\frac{\partial F}{\partial y_k}=0, k=1,2....$$

where $y_k'=\frac{\partial y_k}{\partial x}$.

4.3. State and explain Hamilton's principle and derived Lagrange's equation of motion from it. Discuss how the result will be modified if the forces are non-conservative.

4.4 Prove that the equation of curve for which surface area is minimum is a caterary

$$x=a\cosh\frac{y-b}{a},$$

where a and b are constants.

4.5 Derive Hamilton's equations of motion from the variational principle.

4.6 Prove that $J=\iiint F(u,u_x,u_y,u_z,x,y,z)\,dx\,dy\,dz$

will be minimum only if

$$\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)=0.$$

4.7 State Hamilton's principle and derive if from D'Alembert's principle.

4.8 State and prove the principle of least action.

4.9 Find x and y as functions of t, so that

$$J = \int_{t_0}^{t_1}\left[\frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mgy\right]dt$$

may have stationary value. It may be assumed that x and y are given at t_0 and t_1.

4.10 State Hamilton's principle. How can the principle be used to find the equation of one dimensional harmonic oscillator?

4.11 A particle of mass m moves under the influence of gravity on the inner surface of the paraboloid of revolution $x^2 + y^2 = az$ which is frictionless. Obtain the equations of motion.

4.13 Suggested Further Readings

1. H. Goldstein, *Classical Mechanics,* Addison-Wesley, Cambridge, 1950.
2. T.W.B. Kibble, *Classical Mechanics,* Orient Longman, London, 1985.
3. L.D. Landau and E.M. Lifshitz, *Mechanics,* 3rd ed., Pergamon Press, Oxford, 1976.
4. A. Sommerfeld, *Mechanics,* Academic Press, New York, 1964.
5. J. Synge and B. Griffith, *Principles of Mechanics,* 2nd ed., McGraw Hill, New York, 1949.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-IV Group-A

Module No. - 41

Principle of Mechanics

(Canonical Transformations, Bracket and Hamilton-Jacobi Method)

Content :

In this unit, canonical transformations, Poisson bracket and Hamilton-Jacobi method are introduced.

Objectives

- Canonical transformations.
- Legendre transformation.
- Generating functions.
- Necessary and sufficient condition for canonical transformation.
- Poincaré theorem.
- Poisson bracket and its properties.
- Hamilton's equations using Poisson bracket.
- Constant of motion.
- Jacobi's identity.
- Hamilton-Jacobi equation.
- Solution of one dimensional simple harmonic oscillator.
- Liouville's theorem.
- Exercise.

5.1 Canonical Transformations

There are a number of problems in mechanics for the solution of which, it is often desired to change one set of position and momentum coordinates into another set of position and momentum coordinates which may be rather suitable. For instance we assume that q_j and p_j are the old position and momentum coordinates and Q_j and P_j are the new position and momentum coordinates related by the transformations

$$\left.\begin{aligned} P_j &= P_j(p_1,p_2,\ldots,p_n,q_1,q_2,\ldots q_n,t) \\ Q_j &= Q_j(p_1,p_2,\ldots,p_n,q_1,q_2,\ldots q_n,t) \end{aligned}\right\} \tag{5.1}$$

or, $$P_j = P_j(p_j,q_j,t), Q_j = Q_j(p_j,q_j,t) \tag{5.2}$$

then if there exists a Hamiltonian $\overline{H}$ in the new coordinates, such the

$$\dot{P}_j = -\frac{\partial \overline{H}}{\partial Q_j} \quad \text{and} \quad \dot{Q}_j = \frac{\partial \overline{H}}{\partial P_j}. \tag{5.3}$$

The transformations (5.1) or (5.2) for which equations (5.3) valid are called as <u>canonical</u> or <u>contact transformations.</u>

Through in the Hamiltonian formulation the momenta are independent variables similar to generalised coordinates but the canonical transformations include the simultaneous transformation of the independent position and momentum coordinates q_j, p_j to the new set Q_j, P_j.

Here Q_j, P_j are referred as canonical coordinates.

Let H be the Hamiltonian in the old coordinates and $L, \bar{L}$ the Lagrangian in the old and new set of coordinates respectively, then by the definition of Hamiltonian, we have

$$H = \sum_{j=1}^{n} p_j \dot{q}_j - L \quad \text{and} \quad \bar{H} = \sum_{j=1}^{n} P_j \dot{Q}_j - \bar{L}. \tag{5.4}$$

5.2 Legendre Transformation

The Legendre transformation is a mathematical procedure used to changed the basis from the $(q_j, \dot{q}_j, t)$ set to the (q_j, p_j, t) set.

Let there be a function of only two variables $f(x, y)$, so that the differential of f may be expressed as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$= u\,dx + v\,dy, \tag{5.5}$$

where $u = \dfrac{\partial f}{\partial x}$ and $v = \dfrac{\partial f}{\partial y}$. (5.6)

Let us now change the basis from x, y to the independent variables u, y so that the differential quantities may be expressed in terms of du and dy.

Let there be another function g of variables u and y, such that

$$g = f - ux. \tag{5.7}$$

Therefore the differential of g is given by

$$dg = df - u\,dx - x\,du. \tag{5.8}$$

Substituting value of df from (5.5) in (5.8), we get

$$dg = udx + vdy - udx - xdu$$

or, $$dg = vdy - xdu, \tag{5.9}$$

which is the required form.

As $g = g(u, y)$, therefore

$$dg = \frac{\partial g}{\partial y} du + \frac{\partial g}{\partial y} dy. \tag{5.10}$$

Comparing (5.9) and (5.10), we get

$$x = -\frac{\partial g}{\partial u} \text{ and } v = \frac{\partial g}{\partial y}. \tag{5.11}$$

Thus by the help of (5.7) we can transform the basis from x, y to the independent variables u, y provided x and u satisfy equation (5.6).

5.3 Generating Functions and the Canonical Transformations

From Hamilton's principle, we have

$$\delta\int_{t_0}^{t_1} L dt = 0 \text{ and } \delta\int_{t_0}^{t_1} \overline{L} dt = 0, \tag{5.12}$$

where $L = \sum_{j=1}^{n} p_j . \dot{q}_j - H$ and $\overline{L} = \sum_{j=1}^{n} P_j \dot{Q}_j - \overline{H}$.

$$\left.\begin{aligned} &\text{Then } \delta\int_{t_0}^{t_1}\left(\sum p_j \dot{q}_j - H\right)dt = 0 \\ &\text{and } \delta\int_{t_0}^{t_1}\left(\sum P_j \dot{Q}_j - H\right)dt = 0. \end{aligned}\right\} \tag{5.13}$$

The two expressions of (5.13) are simultaneously valid if the integrands differ by a total time derivative of an arbitrary function say G, i.e. if

$$\left(\Sigma p_j \dot{q}_j - H\right) - \left(\Sigma P_j \dot{Q}_j - \overline{H}\right) = \frac{dG}{dt}. \tag{5.14}$$

The integrals (5.13), when combined become

$$\delta\int_{t_0}^{t_1}\left[\left(\sum_{j=1}^{n} p_j \dot{q}_j - H\right) - \left(\sum_{j=1}^{n} P_j \dot{Q}_j - \overline{H}\right)\right]dt = 0$$

or, $\delta\int_{t_0}^{t_1} \frac{dG}{dt} dt = 0$

or, $$\delta\int_{t_0}^{t_1} \frac{dG}{dt} dt = \delta\left[G(t_1) - G(t_0)\right] = 0. \tag{5.15}$$

[since the end points are fixed.]

The function G is called the generating function of the transformation. The left hand side of (5.14), the first expression is a function of q's, p's and t and the second one is a function of Q's, P's and t. Hence the generating function G in general is a function of $4n+1$ variables q's, p's, Q's, P's and t.

But, with the help of transformation (5.2), it is possible to reduce G to be a function of $2n+1$ independent variables one of which is t and the other $2n$ variables are from p's, q's, P's, Q's.

These are four possibilities of the generating function G:

(i) $G_1(q_j, Q_j, t)$, provided that only q_j, Q_j are treated independent.

(ii) $G_2(q_j, P_j, t)$, provided that only q_j, P_j are treated independent.

(iii) $G_3(p_j, Q_j, t)$, provided that only q_j, Q_j are treated independent.

(iv) $G_4(p_j, P_j, t)$, provided that only p_j, P_j are treated independent.

Case I The generating function G is a function of q_j, Q_j, t.

Let $G = G_1(q_j, Q_j, t)$

$$= G_1(q_1, q_2, \ldots, q_n, Q_1, Q_2, \ldots, Q_n, t).$$

Then from (5.14), we have

$$\left(\Sigma p_j \dot{q}_j - H\right) - \left(\Sigma P_j \dot{Q}_j - \overline{H}\right) = \frac{dG_1}{dt}$$

$$= \sum \frac{\partial G_1}{\partial q_j} \dot{q}_j + \sum \frac{\partial G_1}{\partial Q_j} \dot{Q}_j + \frac{\partial G_1}{\partial t}$$

or, $$\sum\left(p_j - \frac{\partial G_1}{\partial q_j}\right)\dot{q}_j - \sum\left(P_j + \frac{\partial G_1}{\partial Q_j}\right)\dot{Q}_j + \left(\overline{H} - H - \frac{\partial G_1}{\partial t}\right) = 0$$

or, $$\sum\left(p_j - \frac{\partial G_1}{\partial q_j}\right)dq_j - \sum\left(P_j + \frac{\partial G_1}{\partial Q_j}\right)dQ_j + \left(\overline{H} - H - \frac{\partial G_1}{\partial t}\right)dt = 0.$$

As Q's, q's and t are independent variables, we have

$$p_j = \frac{\partial G_1}{\partial q_j} \tag{5.16a}$$

$$P_j = -\frac{\partial G_1}{\partial Q_j} \tag{5.16b}$$

and $$\overline{H} = H + \frac{\partial G_1}{\partial t}. \tag{5.16c}$$

Equation (5.16a) states that Q_j can be determined in terms of q_j, p_j, t. It means that (5.16a) can give transformation $Q_j = Q_j(q_j, p_j, t)$. Knowing $Q(q_j, p_j, t)$ from (5.16b), we can determine $P_j = P_j(q_j, p_j, t)$. Finally, (5.16c) provides the connection between the new Hamiltonian and the old one.

Case II. The generating function G is a function of q_j, P_j, t.

Let $G_2(q_j, P_j, t) = G_1(q_j, Q_j, t) + \sum P_j Q_j$ (5.17)

Substituting the value of G_1 in (5.14), we have

$$\left(\sum p_j \dot{q}_j - H\right) - \left(\sum P_j \dot{Q}_j - \overline{H}\right)$$

$$= \frac{d}{dt}\left\{G_2(q_j, P_j, t) - \sum P_j Q_j\right\}$$

$$= \sum \frac{\partial G_2}{\partial q_j}\dot{q}_j + \sum \frac{\partial G_2}{\partial P_j}\dot{P}_j + \frac{\partial G_2}{\partial t} - \left(\sum P_j \dot{Q}_j + Q_j \dot{P}_j\right)$$

or, $$\sum\left(p_j - \frac{\partial G_2}{\partial q_j}\right)dq_j + \sum\left(Q_j - \frac{\partial G_2}{\partial P_j}\right)dP_j + \left(\overline{H} - H - \frac{\partial G_2}{\partial t}\right)dt = 0$$

as q_j, P_j and t are independent variables, we get

$$p_j = \frac{\partial G_2}{\partial q_j} \quad (5.18a)$$

$$Q_j = \frac{\partial G_2}{\partial P_j} \quad (5.19b)$$

and $$\overline{H} = H + \frac{\partial G_2}{\partial t}. \quad (5.19c)$$

Equations (5.18a), (5.19b) and (5.19c) represent the canonical transformation for the given generating function G_2.

Equation (5.19a) states that P_j can be determined in terms of q_j, p_j, t. Knowing $P(q_j, p_j, t)$, we can find $Q_j(q_j, p_j, t)$ from (5.19b), while equation (5.19c) provides the connection between the new Hamiltonian $\overline{H}$ and old Hamiltonian H.

Case III. The generating function G is a function of p_j, Q_j, t.

As before the relation between G_1 and G_3 can be obtain as

$$G_1(q_j, Q_j, t) = G_3(p_j, Q_j, t) + \sum p_j, q_j.$$

Substituting the value of G_1 in (5.14) and then proceeding as before we get

$$q_j = -\frac{\partial G_3}{\partial p_j} \quad (5.20a)$$

$$P_j = -\frac{\partial G_3}{\partial Q_j} \quad (5.20b)$$

and $$\overline{H} = H + \frac{\partial G_3}{\partial t}. \quad (5.20c)$$

Case IV. The generating function G is a function of p_j, P_j, t.

In this case, the generating function G_1 and G_4 are connected by

$$G_4(p_j, P_j, t) = G_1(q_j, Q_j, t) + \sum P_j Q_j - \sum p_j q_j.$$

Substituting the value of G_1 in (5.14) we get

$$\sum p_j \dot{q}_j - H = \sum P_j \dot{Q}_j - \overline{H} + \frac{\partial G_1}{\partial t}$$

$$= \sum P_j \dot{Q}_j - \overline{H} + \frac{\partial G_4}{\partial t} - \sum \dot{P}_j Q_j + \sum \dot{p}_j q_j - \sum P_j \dot{Q}_j + \sum p_j \dot{q}_j$$

or, $$-H = -\overline{H} + \frac{\partial G_4}{\partial t} - \sum \dot{P}_j Q_j + \sum \dot{p}_j q_j$$

or, $$-H = -\overline{H} + \sum \frac{\partial G_4}{\partial p_j}\dot{p}_j + \sum \frac{\partial G_4}{\partial P_j}\dot{P}_j + \frac{\partial G_4}{\partial t} - \sum \dot{P}_j Q_j + \sum \dot{p}_j q_j.$$

Comparing the coefficients of $\dot{p}_j$ and $\dot{P}_j$ we get

$$q_j = -\frac{\partial G_4}{\partial p_j} \quad (5.21a)$$

$$Q_j = \frac{\partial G_4}{\partial P_j} \quad (5.21b)$$

and $$\overline{H} = H + \frac{\partial G_4}{\partial t}. \quad (5.21c)$$

Equation (5.21a) states that P_j can be determined in terms of q_j, p_j, t. Knowing P_j, we can determine Q_j from (5.21b) in terms of q_j, p_j, t.

5.3.1 The necessary and sufficient condition for a transformation to be canonical.

Theorem 5.1 If $\sum(P_j dQ_j - p_j dq_j)$ or $\sum(p_j dq_j - P_j dQ_j)$ be an exact differential then the transformation $P_j = P_j(q_j, p_j, t)$ and $Q_j = Q_j(q_j, p_j, t)$ is canonical.

Proof. We know that for a canonical transformation, the relation (5.14) should be satisfied, i.e.,

$$\left(\sum p_j \dot{q}_j - H\right) - \left(\sum P_j \dot{Q}_j - \overline{H}\right) = \frac{dG}{dt}. \qquad (5.22)$$

If the generating function G does not include t explicitly then

$$\overline{H} = H + \frac{\partial G}{\partial t} = H \quad \left(\because \frac{\partial G}{\partial t} = 0\right).$$

Then (5.22) reduces to

$$\sum p_j \dot{q}_j - \sum P_j \dot{Q}_j = \frac{dG}{dt}$$

or, $\sum(p_j dq_j - P_j dQ_j) = dG$, where dG is the exact differential of G and G is the corresponding generating function.

5.3.2 Examples on canonical transformation

Example 5.3.1 If the transformation equations between two sets of coordinates are $P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p$, $Q = \log(1+\sqrt{q}\cos p)$,

(i) show that the transformation is canonical

(ii) the generating function of this transformation is $f_3 = -(e^Q - 1)^2 \tan p$.

Solution: Here $pdq - PdQ$

$$= pdq - 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p \times \frac{\cos p\, dq - 2q \sin p\, dp}{2\sqrt{q}(1+\sqrt{q}\cos p)}$$

$$= \left(p - \frac{1}{2}\sin 2p\right)dq + q(1 - \cos 2p)dp$$

$$= d\left\{q\left(p - \frac{1}{2}\sin 2p\right)\right\}.$$

This shows that $p\,dq - P\,dQ$ is exact. Hence the transformation is canonical.

(ii) We have $e^Q = 1 + \sqrt{q}\cos p$ or, $q = \left(\frac{e^Q - 1}{\cos p}\right)^2$. (i)

$$P = 2\left(1 + \sqrt{q}\cos p\right)\sqrt{q}\sin p \text{ or, } P = 2e^Q\left(e^Q - 1\right)\tan p. \quad \text{(ii)}$$

Since, $q = -\frac{\partial G_3}{\partial p}$, $P = -\frac{\partial G_3}{\partial Q}$, therefore

$$\frac{\partial G_3}{\partial p} = -\left(\frac{e^Q - 1}{\cos p}\right)^2 = -\left(e^Q - 1\right)^2 \sec^2 p \quad \text{(iii)}$$

and $\frac{\partial G_3}{\partial Q} = -2e^Q\left(e^Q - 1\right)\tan p.$ (iv)

Integrating,

$$G_3 = -\int \left(e^Q - 1\right)^2 \sec^2 p\,dp = -\left(e^Q - 1\right)^2 \tan p, \quad \text{(v)}$$

and $G_3 = -\int e^Q\left(e^Q - 1\right)\tan p\,dQ = -\left(e^Q - 1\right)^2 \tan p.$ (vi)

Thus $G_3 = -\left(e^Q - 1\right)^2 \tan p$, is the generating function.

Example 5.3.2 Consider the generating function

$$G_1 = \frac{m}{2}wq^2 \cot Q$$

where m and w are constants.

Solution. Then $P = \frac{\partial G_1}{\partial q} = mwq\cot Q$ (i)

and $P = -\frac{\partial G_1}{\partial Q} = \frac{mwq^2}{2\sin^2 Q}.$ (ii)

From (ii), we have

$$q = \sqrt{\frac{2P}{mw}} \sin Q. \qquad \text{(iii)}$$

Then from (ii) and (iii) we have

$$p = \sqrt{zmw\,P} \cos Q. \qquad \text{(iv)}$$

The generating function does not involve the time t explicitly, thus the value of the Hamiltonian is not affected by the transformation and it is only necessary to express H in terms of the new Q and P.

Suppose the Hamiltonian has the form

$$H = \frac{p^2}{2m} + \frac{mw^2}{2} q^2. \qquad \text{(v)}$$

Thus the Hamiltonian in new coordinates is

$$H = wP. \qquad \text{(vi)}$$

The equation of motion for Q is

$$\dot{Q} = \frac{\partial H}{\partial P} = w.$$

That is $Q = wt + \alpha$. Then from (iii),

$$q = \sqrt{\frac{2P}{mw}} \sin(wt + \alpha). \qquad \text{(vii)}$$

Now from (vi) and (vii), we have

$$q = \sqrt{\frac{2H}{mw^2}} \sin(wt + \alpha) = \sqrt{\frac{2E}{mw^2}} \sin(wt + \alpha), \qquad \text{(viii)}$$

where $H = E$ (the total energy) = constant, since H is independent of time.

The equation (viii) is the solution of a harmonic oscillator.

5.3.3 A property of canonical transformations

Theorem 5.2 (Poincaré theorem) Under canonical transformation, the integral

$$J = \int\int_S \sum_i dq_i dp_i$$

remain invariant, where S is a two-dimensional surface in phase space (phase space is a $2n$-dimensional space having coordinates $q_1, q_2, \ldots, q_n$ and $p_1, p_2, \ldots, p_n$).

Proof. Since the position of any point on the two-dimensional surface is specified by two parameters. Let u, v be the parameters. Then

$$q_i = q_i(u,v) \text{ and } p_i = p_i(u,v) \tag{5.23}$$

Therefore,

$$\sum_i dq_i dp_i = \sum_i \frac{\partial(q_i, p_i)}{\partial(u,v)} du\, dv, \tag{5.24}$$

where $\frac{\partial(q_i, p_i)}{\partial(u,v)}$ is the jacobian of q_i, p_i w.r.t. u, v.

Let the canonical transformation be

$$Q_k = Q_k(q_j, p_j, t) \text{ and } P_k = P_k(q_j, p_j, t). \tag{5.25}$$

So, we can write

$$\sum_k dQ_k dP_k = \sum_k \frac{\partial(Q_k, P_k)}{\partial(u,v)} du\, dv. \tag{5.26}$$

If J is invariant under canonical transformations as in (5.25), then we can write

$$\int\int_S \sum_i dq_i dp_i = \int\int_S \sum_k dQ_k dP_k$$

or, $$\int\int_S \sum_i \frac{\partial(q_i, p_i)}{\partial(u,v)} du\ dv = \int\int_S \sum_k \frac{\partial(Q_k, P_k)}{\partial(u,v)} du\ dv$$

or, $$\sum_i \frac{\partial(q_i, p_i)}{\partial(u,v)} = \sum_k \frac{\partial(Q_k, P_k)}{\partial(u,v)}. \tag{5.27}$$

Thus the proof of invariant of J under canonical transformation is equivalent to the proof of identity (5.27).

Let the generating function be $G_2(q_i, P_i, t)$ for canonical transformation from q_i, p_i to Q_i, P_i. Therefore,

$$p_i = \frac{\partial G_2}{\partial q_i}. \tag{5.28}$$

Now, $$\frac{\partial p_i}{\partial u} = \sum_k \left(\frac{\partial^2 G_2}{\partial q_i \partial q_k} \frac{\partial q_k}{\partial u} + \frac{\partial^2 G_2}{\partial q_i \partial P_k} \frac{\partial P_k}{\partial u} \right)$$

$$\text{and } \frac{\partial p_i}{\partial v} = \sum_k \left(\frac{\partial^2 G_2}{\partial q_i \partial q_k} \frac{\partial q_k}{\partial v} + \frac{\partial^2 G_2}{\partial q_i \partial P_k} \frac{\partial P_k}{\partial v} \right).$$

Here

$$\sum_i \frac{\partial(q_i, p_i)}{\partial(u,v)} = \sum_i \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_i}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_i}{\partial v} \end{vmatrix} \tag{5.29}$$

$$= \sum_i \sum_k \begin{vmatrix} \frac{\partial q_i}{\partial u} & \left(\frac{\partial^2 G_2}{\partial q_i \partial q_k} \frac{\partial q_k}{\partial u} + \frac{\partial^2 G_2}{\partial q_i \partial P_k} \frac{\partial P_k}{\partial u} \right) \\ \frac{\partial q_i}{\partial v} & \left(\frac{\partial^2 G_2}{\partial q_i \partial q_k} \frac{\partial q_k}{\partial v} + \frac{\partial^2 G_2}{\partial q_i \partial P_k} \frac{\partial P_k}{\partial v} \right) \end{vmatrix}$$

$$= \sum_i \sum_k \left[\frac{\partial^2 G_2}{\partial q_i \partial q_k} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial q_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial q_k}{\partial v} \end{vmatrix} + \frac{\partial^2 G_2}{\partial q_i \partial P_k} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial P_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial P_k}{\partial v} \end{vmatrix} \right]. \tag{5.30}$$

We have

$$\sum_i \sum_k \frac{\partial^2 G_2}{\partial q_i \partial q_k} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial q_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial q_k}{\partial v} \end{vmatrix} = \sum_i \sum_k \frac{\partial^2 G_2}{\partial q_k \partial q_i} \begin{vmatrix} \frac{\partial q_k}{\partial u} & \frac{\partial q_i}{\partial u} \\ \frac{\partial q_k}{\partial v} & \frac{\partial q_i}{\partial v} \end{vmatrix}$$

$$= -\sum_i \sum_k \frac{\partial^2 G_2}{\partial q_i \partial q_k} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial q_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial q_k}{\partial v} \end{vmatrix}$$

$$\text{or, } \sum_i \sum_k \frac{\partial^2 G_2}{\partial q_i \partial q_k} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial q_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial q_k}{\partial v} \end{vmatrix} = 0. \tag{5.31}$$

Replacing q by P, we have from (5.31)

$$\sum_i \sum_k \frac{\partial^2 G_2}{\partial P_i \partial P_k} \begin{vmatrix} \frac{\partial P_i}{\partial u} & \frac{\partial P_k}{\partial u} \\ \frac{\partial P_i}{\partial v} & \frac{\partial P_k}{\partial v} \end{vmatrix} = 0.$$

Now, equation (5.30) can be written as

$$\sum_i \frac{\partial(q_i, p_i)}{\partial(u,v)} = \sum_i \sum_k \left[\frac{\partial^2 G_2}{\partial P_i \partial P_k} \begin{vmatrix} \frac{\partial P_i}{\partial u} & \frac{\partial P_k}{\partial u} \\ \frac{\partial P_i}{\partial v} & \frac{\partial P_k}{\partial v} \end{vmatrix} + \frac{\partial^2 G_2}{\partial q_i \partial P_k} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial P_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial P_k}{\partial v} \end{vmatrix} \right]$$

$$= \sum_i \sum_k \begin{vmatrix} \left(\frac{\partial^2 G_2}{\partial P_k \partial P_i} \frac{\partial P_i}{\partial u} + \frac{\partial^2 G_2}{\partial P_k \partial q_i} \frac{\partial q_i}{\partial u} \right) & \frac{\partial P_k}{\partial u} \\ \left(\frac{\partial^2 G_2}{\partial P_k \partial P_i} \frac{\partial P_i}{\partial v} + \frac{\partial^2 G_2}{\partial P_k \partial q_i} \frac{\partial q_i}{\partial v} \right) & \frac{\partial P_k}{\partial v} \end{vmatrix}$$

$$= \sum_k \begin{vmatrix} \frac{\partial}{\partial u}\left(\frac{\partial^2 G_2}{\partial P_k} \right) & \frac{\partial P_k}{\partial u} \\ \frac{\partial}{\partial v}\left(\frac{\partial G_2}{\partial P_k} \right) & \frac{\partial P_k}{\partial v} \end{vmatrix} = \sum_k \begin{vmatrix} \frac{\partial Q_k}{\partial u} & \frac{\partial P_k}{\partial u} \\ \frac{\partial Q_k}{\partial v} & \frac{\partial P_k}{\partial v} \end{vmatrix}$$

$$= \sum_k \frac{\partial(Q_k, P_k)}{(u,v)}$$

where $\frac{\partial G_2}{\partial P_k} = Q_k$.

Therefore, $\sum_i \frac{\partial(q_i, p_i)}{\partial(u,v)} = \sum_k \frac{\partial(Q_k, P_k)}{\partial(u,v)}$. This completes the proof.

5.4 Poisson Bracket

The Poisson Bracket of two dynamical variables $X(q,p,t)$ and $Y(q,p,t)$ is defined as

$$[X,Y]_{q,p} = \sum_j \left(\frac{\partial X}{\partial q_i} \frac{\partial Y}{\partial p_i} - \frac{\partial X}{\partial p_j} \frac{\partial Y}{\partial q_j} \right). \qquad (5.32)$$

5.4.1 Properties of Poisson bracket

Property 5.4.1 Poisson bracket does not obey the commutative law, i.e.

$$[X,Y] = -[Y,X]. \tag{5.33}$$

Property 5.4.2 Poisson bracket obey's the distributive law of algebra i.e.,

$$[X+Y,Z] = [X,Z]+[Y,Z]. \tag{5.34}$$

Proof. $$[X+Y,Z] = \sum_j \left(\frac{\partial(X+Y)}{\partial q_j}\frac{\partial Z}{\partial p_j} - \frac{\partial(X+Y)}{\partial p_j}\frac{\partial Z}{\partial q_j} \right)$$

$$= \sum_j \left[\left(\frac{\partial X}{\partial q_j} + \frac{\partial Y}{\partial q_j} \right)\frac{\partial Z}{\partial p_j} - \left(\frac{\partial X}{\partial p_j} + \frac{\partial Y}{\partial p_j} \right)\frac{\partial Z}{\partial q_j} \right]$$

$$= \sum_j \left[\left(\frac{\partial X}{\partial q_j}\frac{\partial Z}{\partial p_j} - \frac{\partial X}{\partial p_j}\frac{\partial Z}{\partial q_j} \right) \right] + \sum_j \left[\left(\frac{\partial Y}{\partial q_j}\frac{\partial Z}{\partial p_j} - \frac{\partial Y}{\partial p_j}\frac{\partial Z}{\partial q_j} \right) \right]$$

$$= [X,Z]_{q,p} + [Y,Z]_{q,p}$$

$$= [X,Z]+[Y,Z].$$

Similarly, $[X,Y+Z] = [X,Y]+[X,Z]$.

Property 5.4.3 $[X,YZ] = Y[X,Z]+[X,Y]Z.$ (5.35)

Prof. $$[X,YZ] = \sum_j \left[\frac{\partial X}{\partial q_j}\frac{\partial (YZ)}{\partial p_j} - \frac{\partial X}{\partial p_j}\frac{\partial (YZ)}{\partial q_j} \right]$$

$$= \sum_j \left\{ \frac{\partial X}{\partial q_j}\left(Y\frac{\partial Z}{\partial p_j} + Z\frac{\partial Y}{\partial p_j} \right) - \frac{\partial X}{\partial p_j}\left(Y\frac{\partial Z}{\partial q_j} + Z\frac{\partial Y}{\partial q_j} \right) \right\}$$

$$= Y\left\{ \sum_j \left(\frac{\partial X}{\partial q_j}\frac{\partial Z}{\partial p_j} - \frac{\partial X}{\partial p_j}\frac{\partial Z}{\partial q_j} \right) \right\} + Z\left\{ \sum_j \left(\frac{\partial X}{\partial q_j}\frac{\partial Y}{\partial p_j} - \frac{\partial X}{\partial p_j}\frac{\partial Y}{\partial q_j} \right) \right\}$$

$$= Y[X,Z]+Z[X,Y].$$

Property 5.4.4 $[p_i, p_j]_{q,p} = 0 = [q_i, q_j]_{q,p}$ for $i \neq j$. (5.36)

Proof. $[p_i, p_j] = \sum_k \left(\frac{\partial p_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} \right)$

$= \sum_k [0.\delta_{jk} - \delta_{ik}.0] = 0$

[where δ_{jk} is the Kroneker delta].

Similarly, $[q_i, q_j] = 0$.

Property 5.4.5 $[q_i, p_j] = \delta_{ij}$. (5.37)

Proof. $[q_i, p_j] = \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial p_i}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_i}{\partial q_k} \right)$

$= \sum_k \delta_{ik}.\delta_{kj} - 0.0 = \delta_{ij}$.

Property 5.4.6 Poisson bracket of two dynamical variables is invariant under infinitesimal canonical transformation, i.e. $[X, Y]_{q,p} = [X, Y]_{Q,P}$.

Proof. $[X, Y]_{Q,P} = \sum_j \left(\frac{\partial X}{\partial Q_j}\frac{\partial Y}{\partial P_j} - \frac{\partial X}{\partial P_j}\frac{\partial Y}{\partial Q_j} \right)$

$$= \sum_j \left\{ \frac{\partial X}{\partial Q_j}\left[\sum_i \left(\frac{\partial Y}{\partial q_i}\frac{\partial q_i}{\partial P_j} + \frac{\partial Y}{\partial p_i}\frac{\partial p_i}{\partial P_j} \right) \right] - \frac{\partial X}{\partial P_j}\left[\sum_i \left(\frac{\partial Y}{\partial q_i}\frac{\partial q_i}{\partial Q_j} + \frac{\partial Y}{\partial p_i}\frac{\partial p_i}{\partial Q_j} \right) \right] \right\}$$

$$= \sum_i \left\{ \sum_j \left[\frac{\partial Y}{\partial q_i}\left(\frac{\partial X}{\partial Q_i}\frac{\partial q_i}{\partial P_j} - \frac{\partial X}{\partial P_i}\frac{\partial q_i}{\partial Q_j} \right) + \frac{\partial Y}{\partial p_i}\left(\frac{\partial X}{\partial Q_j}\frac{\partial p_i}{\partial P_j} - \frac{\partial X}{\partial P_j}\frac{\partial p_i}{\partial Q_j} \right) \right] \right\}$$

$$= \sum_i \left\{ \frac{\partial Y}{\partial q_i}[X, q_i]_{Q,P} + \frac{\partial Y}{\partial p_i}[X, p_i]_{Q,P} \right\} \quad (5.38)$$

Now, $[X, q_i]_{Q,P} = -[q_i, X]_{Q,P}$

$$= -\sum_m \left\{ \frac{\partial q_i}{\partial Q_m}\frac{\partial X}{\partial P_m} - \frac{\partial q_i}{\partial P_m}\frac{\partial X}{\partial Q_m} \right\}$$

$$= -\sum_m \left\{ \frac{\partial q_i}{\partial Q_m} \left[\sum_m \frac{\partial X}{\partial q_k} \frac{\partial q_k}{\partial P_m} + \frac{\partial X}{\partial p_k} \frac{\partial p_k}{\partial P_m} \right] - \frac{\partial q_i}{\partial P_m} \left[\sum_k \frac{\partial X}{\partial q_k} \frac{\partial q_k}{\partial Q_m} + \frac{\partial X}{\partial p_k} \frac{\partial p_k}{\partial Q_m} \right] \right\}$$

$$= -\sum_k \left\{ \frac{\partial X}{\partial q_k} \left[\sum_m \left(\frac{\partial q_i}{\partial Q_m} \frac{\partial q_k}{\partial P_m} - \frac{\partial q_i}{\partial P_m} \frac{\partial q_k}{\partial Q_m} \right) \right] \right.$$

$$\left. + \frac{\partial X}{\partial p_k} \left[\sum_m \left(\frac{\partial q_i}{\partial Q_m} \frac{\partial q_k}{\partial P_m} - \frac{\partial q_i}{\partial P_m} \frac{\partial q_k}{\partial Q_m} \right) \right] \right\}$$

$$= -\sum_k \left\{ \frac{\partial X}{\partial q_k} [q_i, q_k]_{Q,P} + \frac{\partial X}{\partial p_k} [q_i, p_k]_{Q,P} \right\}$$

$$= -\sum_k \left(\frac{\partial X}{\partial q_k} .0 + \frac{\partial X}{\partial p_k} .\delta_{ik} \right) = -\frac{\partial X}{\partial p_i}. \tag{5.39}$$

Similarly, $[X, p_i] = \dfrac{\partial X}{\partial q_i}.$ (5.40)

Using (5.39), (5.40), equation (5.38) becomes

$$[X,Y]_{Q,P} = \sum_i \left\{ \frac{\partial Y}{\partial q_i} \left(-\frac{\partial X}{\partial p_i} \right) + \frac{\partial Y}{\partial p_i} \left(\frac{\partial X}{\partial q_i} \right) \right\} = [X,Y]_{q,p}.$$

The value of the Poisson bracket remains the same in two different system.

Property 5.4.7 Condition for a transformation to be canonical, in terms of Poisson bracket.

The transformation $Q_j = Q_j(q_j, p_j), P_j = P_j(q_j, p_j)$ will be canonical if $[Q_j, Q_j]_{q,p} = 0 = [P_j, P_j]_{q,p}$ and $[Q_j, P_j]_{q,p} = 1$. (5.41)

Proof. $[Q_j, Q_j]_{q,p} = \sum_k \left(\dfrac{\partial Q_j}{\partial q_k} \dfrac{\partial Q_j}{\partial p_k} - \dfrac{\partial Q_j}{\partial p_k} \dfrac{\partial Q_j}{\partial q_k} \right) = 0.$

For the generating function $G_1(q_j, Q_j, t)$ we have

$$P_j = -\frac{\partial G_1}{\partial Q_j}, \; p_i = \frac{\partial G_1}{\partial q_i}.$$

From these relations

$$\frac{\partial p_i}{\partial Q_j} = \frac{\partial^2 G_1}{\partial Q_j \partial q_j} = \frac{\partial}{\partial q_j}\left(\frac{\partial G_1}{\partial Q_j}\right) = -\frac{\partial P_j}{\partial q_j} \tag{5.42}$$

Similarly, for the generating function $G_2(q_j, P_j, t)$, we have $p_i = \frac{\partial G_2}{\partial q_i}$, $Q_j = \frac{\partial G_2}{\partial P_j}$ from these relations

$$\frac{\partial p_i}{\partial P_j} = \frac{\partial^2 G_2}{\partial P_j \partial q_i} = \frac{\partial}{\partial q_i}\left(\frac{\partial G_2}{\partial P_j}\right) = \frac{\partial Q_j}{\partial q_i}$$

$$\therefore \frac{\partial p_i}{\partial P_j} = \frac{\partial Q_j}{\partial q_i} \tag{5.43}$$

Similarly, for $G_3(p_j, Q_j, t)$ and $G_4(p_j, P_j, t)$, we can get

$$\frac{\partial q_i}{\partial Q_j} = \frac{\partial P_j}{\partial p_i} \tag{5.44}$$

and $$\frac{\partial q_i}{\partial P_j} = -\frac{\partial Q_j}{\partial P_i} \tag{5.45}$$

Now, $[Q_i, P_j]_{q,p} = \sum_k \left(\frac{\partial Q_i}{\partial q_k}\frac{\partial P_j}{\partial p_k} - \frac{\partial Q_i}{\partial p_k}\frac{\partial P_j}{\partial q_k}\right)$

$= \sum_k \left(\frac{\partial Q_i}{\partial q_k}\frac{\partial q_k}{\partial Q_j} + \frac{\partial Q_i}{\partial p_k}\frac{\partial p_k}{\partial Q_j}\right)$ [using (5.42) and (5.44)]

$= \frac{\partial Q_i}{\partial Q_j} = \delta_{ij}$.

$\therefore [Q_j, P_j]_{q,p} = 1.$

Similarly, we can prove that

$$[Q_i, Q_j]_{q,p} = 0, [Q_i, Q_j]_{Q,P} = 0, [P_i, P_j]_{q,p} = 0 \text{ and } [P_i, P_j]_{Q,P} = 0 \tag{5.45}$$

Example 5.4.1 Show that the transformation

$q = \sqrt{2P}\sin Q, p = \sqrt{2P}\cos Q$ is canonical.

Solution. From the given equation we have

$\tan Q = q/p$ and $P = (p^2 + q^2)^{1/2}$.

Now, $[Q,Q]=0,[P,P]=0,$

and $[Q,P]=\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p}-\frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right)$

$$=\left(\frac{1}{p}\cos^2 Q\right)p+\frac{q}{p^2}(\cos^2 Q)q=(1+q^2/p^2)\cos^2 Q$$

$$=\cos^2 Q(1+\tan^2 Q)=1.$$

Hence the transformation is canonical.

Example 5.4.2 Using the fundamental Poisson bracket, find the values of α and β for which the equation

$Q=q^\alpha\cos\beta p,\ P=q^\alpha\sin\beta p$ represents a canonical transformation.

Solution. We know that the transformation is canonical if

$$[Q,P]=1,[Q,Q]=0,[P,P]=0.$$

Now, $[Q,P]=\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p}-\frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right)$

$$=\alpha q^{\alpha-1}\cos\beta p.q^\alpha\cos\beta p.\beta+q^\alpha\sin\beta p.\beta\alpha q^{\alpha-1}\sin\beta p$$

$$=\alpha\beta q^{2\alpha-1}(\cos^2\beta p+\sin^2\beta p)=\alpha\beta q^{2\alpha-1}.$$

If the transformation is canonical then $\alpha\beta q^{2\alpha-1}=1$.

Equating the coefficient of q on both sides which is possible only if we take

$$2\alpha-1=0 \text{ or, } \alpha=1/2.$$

Also, $\alpha\beta=1\ \therefore \beta=2$.

Hence the required values of α and β is $1/2$ and 2.

Property 5.4.8 $\frac{\partial}{\partial x}[X,Y]=\left[\frac{\partial X}{\partial x},Y\right]+\left[X,\frac{\partial Y}{\partial x}\right].$ (5.46)

Proof. $[X,Y]=\sum_j\left(\frac{\partial X}{\partial q_j}\frac{\partial Y}{\partial p_j}-\frac{\partial X}{\partial p_j}\frac{\partial Y}{\partial q_j}\right).$

$$\therefore \frac{\partial}{\partial x}[X,Y] = \frac{\partial}{\partial x}\left\{\sum_j\left(\frac{\partial X}{\partial q_j}\frac{\partial Y}{\partial p_j} - \frac{\partial X}{\partial p_j}\frac{\partial Y}{\partial q_j}\right)\right\}$$

$$= \sum_j\left\{\frac{\partial}{\partial q_j}\left(\frac{\partial X}{\partial x}\right)\frac{\partial Y}{\partial p_j} + \frac{\partial X}{\partial q_j}\frac{\partial}{\partial p_j}\left(\frac{\partial Y}{\partial x}\right) - \frac{\partial}{\partial p_j}\left(\frac{\partial X}{\partial x}\right)\frac{\partial Y}{\partial q_j} - \frac{\partial X}{\partial p_j}\frac{\partial}{\partial q_j}\left(\frac{\partial Y}{\partial x}\right)\right\}$$

$$= \left[\frac{\partial X}{\partial x}, Y\right] + \left[X, \frac{\partial Y}{\partial x}\right].$$

Property 5.4.9 Jacobi's Identity

If X, Y, Z are three dynamical variables then

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \qquad (5.47)$$

Proof. $[X,[Y,Z]] + [Y,[Z,X]]$

$$= [X,[Y,Z]] - [Y,[X,Z]].$$

$$= \left[X, \sum_i\left(\frac{\partial Y}{\partial q_i}\frac{\partial Z}{\partial p_i} - \frac{\partial Y}{\partial p_i}\frac{\partial Z}{\partial q_i}\right)\right] - \left[Y, \sum_i\left(\frac{\partial X}{\partial q_i}\frac{\partial Z}{\partial p_i} - \frac{\partial X}{\partial p_i}\frac{\partial Z}{\partial q_i}\right)\right]$$

$$= \left[X, \sum_i \frac{\partial Y}{\partial q_i}\frac{\partial Z}{\partial p_i}\right] - \left[X, \sum_i \frac{\partial Y}{\partial p_i}\frac{\partial Z}{\partial q_i}\right] - \left[Y, \sum_i \frac{\partial X}{\partial q_i}\frac{\partial Z}{\partial p_i}\right]$$

$$+ \left[Y, \sum_i \frac{\partial X}{\partial p_i}\frac{\partial Z}{\partial q_i}\right]$$

$$= \sum_i\left[X, \frac{\partial Y}{\partial q_i}\frac{\partial Z}{\partial p_i}\right] - \sum_i\left[X, \frac{\partial Y}{\partial p_i}\frac{\partial Z}{\partial q_i}\right] - \sum_i\left[Y, \frac{\partial X}{\partial q_i}\frac{\partial Z}{\partial p_i}\right] + \sum_i\left[Y, \frac{\partial X}{\partial p_i}\frac{\partial Z}{\partial q_i}\right]$$

$$= \sum_i\left\{\left[X, \frac{\partial Y}{\partial q_i}\right]\frac{\partial Z}{\partial p_i} + \left[X, \frac{\partial Z}{\partial q_i}\right]\frac{\partial Y}{\partial q_i} - \left[X, \frac{\partial Y}{\partial p_i}\right]\frac{\partial Z}{\partial q_i} - \left[X, \frac{\partial Z}{\partial q_i}\right]\frac{\partial Y}{\partial p_i}\right.$$

$$\left. - \left[Y, \frac{\partial X}{\partial q_i}\right]\frac{\partial Z}{\partial p_i} - \left[Y, \frac{\partial Z}{\partial p_i}\right]\frac{\partial X}{\partial q_i} + \left[Y, \frac{\partial X}{\partial p_i}\right]\frac{\partial Z}{\partial q_i} + \left[Y, \frac{\partial Z}{\partial q_i}\right]\frac{\partial X}{\partial q_i}\right\}$$

$$= \sum_i\left(\frac{\partial Z}{\partial p_i}\left\{\left[X, \frac{\partial Y}{\partial q_i}\right] - \left[Y, \frac{\partial X}{\partial q_i}\right]\right\} + \frac{\partial Z}{\partial q_i}\left\{\left[Y, \frac{\partial X}{\partial p_i}\right] - \left[X, \frac{\partial Y}{\partial p_i}\right]\right\}\right) + S$$

where

$$S=\sum_i\left\{\left[X,\frac{\partial Z}{\partial p_i}\right]\frac{\partial Y}{\partial q_i}-\left[X,\frac{\partial Z}{\partial q_i}\right]\frac{\partial Y}{\partial p_i}-\left[Y,\frac{\partial Z}{\partial p_i}\right]\frac{\partial X}{\partial q_i}\right.$$

$$\left.+\left[Y,\frac{\partial Z}{\partial q_i}\right]\frac{\partial X}{\partial p_i}\right\}.$$

It can be shown that by expanding the Poisson bracket $S = 0$. Then above expression will reduces to

$$[X,[Y,Z]]+[Y,[Z,X]]$$

$$=\sum_i\left(-\frac{\partial Z}{\partial p_i}\right)\left\{\left[Y,\frac{\partial X}{\partial q_i}\right]+\left[\frac{\partial Y}{\partial q_i},X\right]\right\}$$

$$-\frac{\partial Z}{\partial p_i}\left\{\left[X,\frac{\partial Y}{\partial p_i}\right]+\left[\frac{\partial X}{\partial p_i},Y\right]\right\}\Bigg)$$

$$=-\sum_i\left\{-\frac{\partial Z}{\partial p_i}\cdot\frac{\partial}{\partial q_i}[X,Y]+\frac{\partial Z}{\partial q_i}\frac{\partial}{\partial p_i}[X,Y]\right\}=-[Z,[X,Y]]$$

or, $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$

Hence the result.

5.4.2 Hamilton's equations in terms of Poisson bracket

We have,

$$[q_i,H]=\sum_j\left(\frac{\partial q_i}{\partial q_j}\frac{\partial H}{\partial p_i}-\frac{\partial q_i}{\partial p_j}\frac{\partial H}{\partial q_i}\right)=\sum_j\left(\frac{\partial q_i}{\partial q_j}\frac{\partial H}{\partial p_j}-0\right)$$

$$=\sum_j\frac{\partial H}{\partial p_j}\delta_{ij}=\frac{\partial H}{\partial p_i}.$$

Similarly, $[p_i,H]=-\dfrac{\partial H}{\partial q_i}.$

Hence the Hamilton's equations are

$$\dot{q}_i=\frac{\partial H}{\partial p_i}=[q_i,H] \qquad (5.48)$$

and $\dot{p}_i = -\dfrac{\partial H}{\partial q_i} = [p_i, H].$ (5.49)

Equation (5.48) and (5.49) thus can be referred to as the Hamilton's equations of motion in Poisson bracket form.

5.4.3. Constant of motion

If $F(q_j, p_j, t)$ is a dynamical variable then

$$\frac{dF}{dt} = \sum_j \frac{\partial F}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial F}{\partial p_j}\dot{p}_j + \frac{\partial F}{\partial t}$$

$$= \sum_j \left\{ \frac{\partial F}{\partial q_j}\left(\frac{\partial H}{\partial p_j}\right) + \frac{\partial F}{\partial p_j}\left(-\frac{\partial H}{\partial q_j}\right) \right\} + \frac{\partial F}{\partial t}$$

$$= [F, H]_{q,p} + \frac{\partial F}{\partial t}$$

If F does not involve t explicitly, then $\dfrac{\partial F}{\partial t} = 0.$

In that case,

$$\frac{dF}{dt} = [F, H]_{q,p}. \qquad (5.50)$$

Now, if the Poisson bracket of F with H vanishes then from the above equation

$$\frac{dF}{dt} = [F, H] = 0 \text{ or } F = \text{constant.} \qquad (5.51)$$

That is, F becomes a constant of motion. Thus all the quantities having zero value their Poisson bracket with H are therefore constant of motion.

Conversely, the Poisson bracket of a constant of motion with H will be zero. This properties can be used to identified, i.e. to find the constant of motion.

Theorem 5.4.1. (Poisson's theorem) The Poisson bracket of two constants of motion is itself a constant of motion.

Proof. We have, from Jacobi's identity,

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

Let $Z = H$ then

$$[X,[Y,H]]+[Y,[H,X]]+[H,[X,Y]]=0$$

Now, if X, Y are both constant of motion, then $[X,H]=0$ and $[Y,H]=0$. In this case, the above experssion reduces to

$$[H,[X,Y]]=0 \text{ i.e., } [[X,Y],H]=0.$$

This indicates that $[X, Y]$ is a constant of motion. Hence the Poisson bracket of two constant of motion is itself a constant of motion.

5.5 Hamilton-Jacobi Theory

We know that canonical transformations can be used to provide a general procedure for solving mechanical problems. Two methods have been suggested. If the Hamiltonian is conserved then the solution can be obtained by transforming to new canonical coordinations which are all cyclic, since the integrations of the new equations of motion becomes trivial. An alternative technique is to seek a canonical transformation from the coordinates and momenta, (q, p) at the time t to a new set of constant quantities which may be the 2n initial values, (q_o, p_o), at $t=0$ with such a transformation, the equations of transformation relating to the old and new canonical variables are then exactly the desired solution of the mechanical problem.

$$q = q(q_o, p_o, t),\ p = p(q_o, p_o, t).$$

These give the coordinates and momenta as a function of their initial values and the time.

5.6 Hamilton-Jacobi's Equation

For canonical transformation from the set (q_j, p_j, t) to the set (Q_j, P_j, t), we have

$$\dot{Q}_j = \frac{\partial \overline{H}}{\partial P_j},\ \dot{P}_j = -\frac{\partial \overline{H}}{\partial Q_j} \text{ and } \overline{H} = H + \frac{\partial G}{\partial t}, \tag{5.52}$$

where $\overline{H}, H$ and G are transform Hamiltonian, old Hamiltonian and generating function respectively.

The set of new variables Q_j, P_j are constant in time and then the equations of motion are

$$\dot{Q}_j = \frac{\partial \overline{H}}{\partial P_j} = 0, \quad \dot{P}_j = -\frac{\partial \overline{H}}{\partial Q_j} = 0. \tag{5.53}$$

We can take $\overline{H} = 0$ and hence $H + \frac{\partial G}{\partial t} = 0$. (5.54)

It is convenient to take G as a function of q_j, P_j, t i.e., in our earlier notation we designate the generating function as the transformation equation for the generating function G_2 are

$$P_j = \frac{\partial G_2}{\partial q_j}, \; Q_j = \frac{\partial G_2}{\partial P_j}. \tag{5.55}$$

Using the result of (5.55), we can write the expression (5.54) as

$$H(q_1, q_2, \ldots q_n, p_1, p_2, \ldots p_n, t) + \frac{\partial G}{\partial t} = 0$$

or, $$H\left(q_1, q_2, \ldots q_n, \frac{\partial G_2}{\partial q_1}, \frac{\partial G_2}{\partial q_2}, \ldots \frac{\partial G_2}{\partial q_n}, t\right) + \frac{\partial G_2}{\partial t} = 0 \tag{5.56}$$

The equation (5.56) is a partial differential equation in $(n+1)$ variables while equation (5.54) is in $(2n+1)$ variables, this shows that the above substitutions has reduce the number of variables by n. The equation (5.51) is called **Hamilton-Jacobi's** equation.

The solution of this equation is called **Hamilton principle function** and is denoted by S. In that case Hamilton-Jacobi's equation is expressed as

$$H\left(q_1, q_2, \ldots q_n, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \ldots \frac{\partial S}{\partial q_n}, t\right) + \frac{\partial S}{\partial t} = 0$$

or, simply $H + \frac{\partial S}{\partial t} = 0$.

5.6.1 Physical significance of S

Taking total derivative on S

$$\frac{dS}{dt} = \sum_j \frac{\partial S}{\partial q_j} \dot{q}_j + \frac{\partial S}{\partial t} = \sum_j p_j \dot{q}_j + \frac{\partial S}{\partial t}$$

$$= \sum_j p_j \dot{q}_j - H \quad \left[as\ H + \frac{\partial S}{\partial t} = 0 \right]$$

$$= L$$

or, $S = \int L\, dt + \text{constant}$ (5.57)

So that Hamilton's principle function differ at most from the infinite time integral of the Lagrangian by a constant.

5.7 Separation of Variables in Hamilton-Jacobi Equation

The Hamilton-Jacobi equations are employed as a useful technique if it is possible to separate the variables in the Hamilton-Jacobi equation. Consider a case in which the Hamiltonian is a constant of motion, it may or may not be the total energy. Let us then consider the canonical transformation generated by the Hamilton's characteristic function F and its corresponding Hamilton-Jacobi equation. The variables q_j contained by the equation are only separable if a solution of the type $F = \sum_j F_j(q_j, \alpha_1, \alpha_2, \ldots, \alpha_n)$ (5.58)

exists such that it splits up the Hamilton-Jacobi equation into n equations of the type

$$H_j\left(q_j, \frac{\partial F_j}{\partial q_j}, \alpha_1, \alpha_2, \ldots, \alpha_n\right) = \alpha_j. \tag{5.59}$$

All of the n equations given by (5.59) are of the first order and each of them involves only one of the coordinates q_j and the corresponding partial derivatives of F_j with respect to q_j. Their solutions mainly require them expressible for $\frac{\partial F_j}{\partial q_j}$ and then integrated w.r.t. q_j.

If we take the trial solution for Hamilton-Jacobi equation of the form

$$S(q_j, \alpha_j, t) = F(q_j, \alpha_j) + \bar{S}(t, \alpha_j),$$

then the Hamilton-Jacobi equation yields

$$\frac{\partial S}{\partial t} + H\left(q_j, \frac{\partial F}{\partial q_j}\right) = 0.$$

Here the first term involves only the time and the second one involves only the position coordinates q_i;

therefore it will hold for all values of the variables only if the two terms are equal and opposite constants i.e., say

$$\frac{\partial \bar{S}}{\partial t} = -\alpha_1, \quad H\left(q_j, \frac{\partial F}{\partial q_j}\right) = \alpha_1. \tag{5.60}$$

The first equation gives on integration

$$\bar{S} = -\alpha_1 t.$$

The second equation is the Hamilton-Jacobi equation for F.

Example 5.7.1. Solve the Harmonic oscillation problem by Hamilton-Jacobi method.

Solution. Let us consider one dimensional simple harmonic oscillator. For such a system force is conservative i.e.,

$F = -kq$ where k is spring constant.

Then P.E. $V = -\int F dq = -\int -kq\, dq = \frac{kq^2}{2}$

and K.E. $T = \frac{1}{2}mv^2 = \frac{p^2}{2m}$ where $p = mv$.

$$\therefore H(q,p,t) = T + V = \frac{p^2}{2m} + \frac{kq^2}{2}.$$

Putting $p = \frac{\partial S}{\partial q}$.

We have $H\left(q, \frac{\partial S}{\partial q}, t\right) = \left(\frac{\partial S}{\partial q}\right)^2 \frac{1}{2m} + \frac{kq^2}{2}$.

Hence Hamilton-Jacobi equation is given by

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0$$

i.e., $$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{kq^2}{2} + \frac{\partial S}{\partial t} = 0. \tag{i}$$

Since the explicit dependence S on t is involve only in the last term the solution of (i) can be found in the form

$$S(q,\alpha,t) = W(q,\alpha) - \alpha t, \tag{ii}$$

where α is the constant of integrating to be determine latter as the transformation

$$\frac{\partial S}{\partial q}=\frac{\partial W}{\partial q}\;\frac{\partial S}{\partial t}=-\alpha.$$

Substituting these values in (i), we have

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2+\frac{kq^2}{2}=\alpha$$

or, $$\frac{\partial W}{\partial q}=\sqrt{2m\alpha-kmq^2}=\sqrt{mk}\sqrt{\frac{2\alpha}{k}-q^2}$$

or, $$W=\sqrt{mk}\int\sqrt{\frac{2\alpha}{k}-q^2}\,dq+C \qquad \text{(iii)}$$

where C is a constant of integration.

Then from (ii), we have

$$S=W-\alpha t=\sqrt{mk}\int\sqrt{\frac{2\alpha}{k}-q^2}\,dq-\alpha t+C. \qquad \text{(iv)}$$

Here, C an additive constant does not effect the transformation because taking partial derivative w.r.t. α (new momenta P) to give β (the new position coordinate Q) it reduces to zero.

The value of β (constant) corresponding to new position coordinates can be evaluated as

$$\beta=\frac{\partial S}{\partial \alpha}=\sqrt{mk}\int\frac{dq}{\sqrt{2\alpha/k-q^2}}-t$$

or, $$\beta+t=\sqrt{\frac{m}{2}}\int\frac{dq}{\sqrt{\alpha-\frac{k}{2}q^2}}=\sqrt{\frac{m}{2}}\sqrt{\frac{2}{k}}\sin^{-1}\left(\frac{q\sqrt{k}}{\sqrt{2\alpha}}\right)$$

or, $$\sqrt{\frac{k}{m}}(t+\beta)=\sin^{-1}\left(q\sqrt{\frac{k}{2\alpha}}\right)$$

or, $$q=\sqrt{\frac{2\alpha}{k}}\sin\left\{\sqrt{\frac{k}{m}}(t+\beta)\right\}.$$

Putting $w=\sqrt{\frac{k}{m}}$, $q=\sqrt{\frac{2\alpha}{k}}\sin\{w(t+\beta)\}$,

which is a familier solution of a harmonic oscillator problem.

Now, $p = \frac{\partial S}{\partial q} = \frac{\partial w}{\partial q} = \sqrt{2m\alpha - mkq^2}$ [from (iv)]

$= \sqrt{2m\alpha - 2m\alpha \sin^2\{w(t+\beta)\}}$ [using (v)]

$= \sqrt{2m\alpha}\cos\{w(t+\beta)\}.$ (vi)

Let at time $t = 0$ the particle has the values q_0 as the initial displacement and p_0 as the initial momenta then from (v) and (vi), we have

$$q_0 = \sqrt{\frac{2\alpha}{k}}\sin(w\beta) \qquad \text{(vii)}$$

$$p_0 = \sqrt{2m\alpha}\cos(w\beta). \qquad \text{(viii)}$$

$$\therefore \frac{q_0^2 k}{2\alpha} + \frac{p_0^2}{2m\alpha} = 1 \quad \text{or, } 2m\alpha = p_0^2 + mkq_0^2.$$

Now, from (vii) and (viii) the β is related to q_0 and p_0 as

$$\tan(w\beta) = \sqrt{km}\left(\frac{q_0}{p_0}\right) \qquad \text{(ix)}$$

and α is related to p_0 and q_0 by the relation

$$2m\alpha = p_0^2 + mkq_0^2. \qquad \text{(x)}$$

Let the particles start from rest at $t = 0,\ p_0 = 0$ from (ix) we have

$\tan(w\beta) =$ undefined

or, $w\beta = \pi/2$

From (x), $2\alpha = kq_0^2$ and from (v)

$$q = \sqrt{\frac{2\alpha}{k}}\sin\{w(t+\beta)\} = q_0\sin(wt + \pi/2)$$

$$= q_0\cos(wt).$$

$$= q_0\cos\left(\sqrt{\frac{k}{m}}t\right). \qquad \text{(xi)}$$

This is a particular solution.

Evaluation of S and L

From $S = \sqrt{mk}\sqrt{\frac{2\alpha}{k} - q^2}\,dq - \alpha t + c.$

Substituting the value of q

$$S = \sqrt{2m\alpha}\sqrt{\frac{2\alpha}{k}}\,w\int \cos^2\{w(t+\beta)\}dt - \alpha t + c$$

$$= 2\alpha\int \cos^2\{w(t+\beta)\}dt - \alpha t + c$$

$$\therefore L = \frac{dS}{dt} = 2\alpha\cos^2\{w(t+\beta)\} - \alpha.$$

5.8 Liouville's Theorem

Statement. The phase volume (i.e., the volume in phase) occupied by a set of particles is constant.

In other words, the number of particles per unit volume in phase space is constant, i.e., the density is constant.

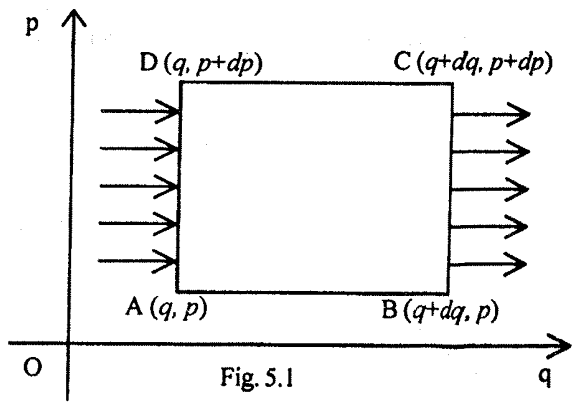


Fig. 5.1

Proof. Case I. For one degree of freedom

In the case of one degree of freedom, we have a two dimensional phase space and the volume element reduces to the area element $dp\,dq$.

Let $\rho = \rho(p,q,t)$ be the number of particles per unit area.

If $\dot{q}$ is the velocity of the points entering through AD, then the number of points which enter through AD in unit time is $\rho\dot{q}\,dq$.

The number of points leaving through

$$AC = \left[\rho\dot{q} + \frac{\partial}{\partial q}(\rho\dot{q})dq\right]dp.$$

Therefore, increase in the number of particles which remain in the element ABCD.

$$= \rho\dot{q}\,dp - \left\{\rho\dot{q} + \frac{\partial}{\partial q}(\rho\dot{q})dq\right\}dp$$

$$= -\frac{\partial}{\partial q}(\rho \dot{q}) dq dp. \tag{5.61}$$

Similarly, the number of particles which enter through $AB = \rho \dot{q}\ dq$.

The number of particles which leave through CD

$$= \left\{ \rho \dot{q} + \frac{\partial}{\partial q}(\rho \dot{p}) dp \right\} dq.$$

Therefore, increase in the number of particles which remain in the element ABCD

$$= \rho \dot{p}\, dq - \left\{ \rho \dot{q} + \frac{\partial}{\partial p}(\rho \dot{p}) dp \right\} dq = -\frac{\partial}{\partial p}(\rho \dot{p}) dq\, dp. \tag{5.62}$$

Total increase in number of particles in the element ABCD from (5.61) and (5.62) is

$$-\left[\frac{\partial}{\partial q}(\rho \dot{q}) + \frac{\partial}{\partial p}(\rho \dot{p}) \right] dq dp.$$

This number must be equal to $\frac{\partial \rho}{\partial t} dp dq$.

Therefore, we must have

$$\frac{\partial \rho}{\partial t} dp dq = -\left[\frac{\partial}{\partial q}(\rho \dot{q}) + \frac{\partial}{\partial p}(\rho \dot{p}) \right] dq dp$$

or, $$\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q}(\rho \dot{q}) + \frac{\partial}{\partial p}(\rho \dot{p}) \right] dp dq = 0$$

or, $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q}(\rho \dot{q}) + \frac{\partial}{\partial p}(\rho \dot{p}) = 0 \quad [as\ dp\, dq \neq 0]$$

or, $$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \dot{q}}{\partial q} + \dot{q} \frac{\partial \rho}{\partial q} + \rho \frac{\partial \dot{p}}{\partial p} + \dot{p} \frac{\partial \rho}{\partial p} = 0. \tag{5.63}$$

From Hamilton's equations, we have

$$\frac{\partial H}{\partial q} = -\dot{p} \text{ and } \frac{\partial H}{\partial p} = \dot{q}.$$

Then $$\frac{\partial \dot{p}}{\partial p} = -\frac{\partial^2 H}{\partial p \partial q} \text{ and } \frac{\partial \dot{q}}{\partial q} = \frac{\partial^2 H}{\partial q \partial p}. \tag{5.64}$$

Since Hamilton's has continuous second order derivatives, then from (5.64) we have

$$\frac{\partial \dot{p}}{\partial p} = -\frac{\partial \dot{q}}{\partial q}. \tag{5.65}$$

Using (5.65), equation (5.63) may be written as

$$\frac{\partial \rho}{\partial t} + \dot{q}\frac{\partial \rho}{\partial q} + \rho\frac{\partial \dot{p}}{\partial p} = 0. \tag{5.66}$$

or, $\quad \frac{d}{dt}\{\rho(p,q,t)\} = 0$

or, $\quad \rho(p,q,t) = \text{constant}.$

Hence the density in phase space is constant which is Liouville's theorem.

Case II. For more than one degrees of freedom

In this case, the phase volume is given by

$$dV = dq_1 dq_2 \ldots dq_k \ldots dp_1 dp_2 \ldots dp_k \ldots \tag{5.67}$$

As in the first case the increase in number of particles in the volume element

$$dV = -\left\{\frac{\partial(\rho\dot{q}_1)}{\partial q_1} + \frac{\partial(\rho\dot{q}_2)}{\partial q_2} + \ldots + \frac{\partial(\rho\dot{q}_k)}{\partial q_k} + \ldots \right.$$

$$\left. + \frac{\partial(\rho\dot{p}_1)}{\partial p_1} + \frac{\partial(\rho\dot{p}_2)}{\partial p_2} + \ldots + \frac{\partial(\rho\dot{p}_k)}{\partial p_k} + \ldots \right\} dV. \tag{5.68}$$

But, this must be equal to $\frac{\partial \rho}{\partial t} dV$, so that we have

$$\frac{\partial \rho}{\partial t} dV = -\left\{\frac{\partial(\rho\dot{q}_1)}{\partial q_1} + \ldots + \frac{\partial(\rho\dot{q}_k)}{\partial q_k} + \ldots + \frac{\partial(\rho\dot{p}_1)}{\partial p_1} + \ldots + \frac{\partial(\rho\dot{p}_k)}{\partial p_k} + \ldots \right\} dV$$

$$= -\sum_k \left\{\frac{\partial(\rho\dot{q}_k)}{\partial q_k} + \frac{\partial(\rho\dot{p}_k)}{\partial p_k}\right\} dV$$

or, $$\frac{\partial \rho}{\partial t} + \sum_k \left\{\frac{\partial \rho}{\partial q_k}\dot{q}_k + \rho\frac{\partial \dot{q}_k}{\partial q_k} + \frac{\partial \rho'}{\partial p_k}\dot{p}_k + \rho\frac{\partial \dot{p}_k}{\partial p_k}\right\} = 0. \tag{5.69}$$

Using Hamilton's equations $\frac{\partial H}{\partial q_k} = -\dot{p}_k$ and $\frac{\partial H}{\partial p_k} = \dot{q}_k$ this give

$$\frac{\partial \dot{p}_k}{\partial p_k} = -\frac{\partial^2 H}{\partial p_k \partial q_k} \text{ and } \frac{\partial \dot{q}_k}{\partial q_k} = \frac{\partial^2 H}{\partial q_k \partial p_k}. \qquad (5.70)$$

Since Hamiltonian has second order continuous derivatives equation (5.70) give

$$\frac{\partial \dot{p}_k}{\partial p_k} = -\frac{\partial \dot{q}_k}{\partial q_k}. \qquad (5.71)$$

Using (5.71), equation (5.69) can be written as

$$\frac{\partial \rho}{\partial t} + \sum_k \left\{ \frac{\partial \rho}{\partial q_k} \dot{q}_k + \frac{\partial \rho}{\partial p_k} \dot{p}_k \right\} = 0$$

or, $\frac{d}{dt}\{\rho(q_1, q_2, \ldots q_k, \ldots, p_1, p_2, \ldots p_k, \ldots, t)\} = 0$

i.e. $\rho(q_1, q_2, \ldots q_k, \ldots, p_1, p_2, \ldots p_k, \ldots, t) = \text{constant}.$

Hence the density in phase space is constant which is Liouvlle's theorem.

5.9 Worked out Examples

Example 5.9.1 Show that the transformation

$$P = \frac{1}{2}(p^2 + q^2),\ Q = \tan^{-1}\frac{q}{p}$$

is canonical.

Solution. The transformation will be canonical if the expression $pdq - PdQ$ is an exact differential.

Here $pdq - PdQ$

$$= pdq - \frac{1}{2}(p^2 + q^2).\frac{pdq - qdp}{p^2 + q^2}$$

$$= pdq - \frac{1}{2}(pdq - qdp)$$

$$= \frac{1}{2}(pdq + qdp) = d\left(\frac{1}{2}pq\right), \text{ an exact differential.}$$

Hence the given transformation is canonical.

Example 5.9.2 Prove that the transformation

$$Q = \log\left(\frac{1}{q}\sin p\right), \quad P = q\cot p$$

is canonical.

Solution. The transformation will be canonical if the expression $pdq - PdQ$ is an exact differential.

Here $pdq - PdQ = pdq - q\cot p.d\left[\log\left(\frac{1}{q}\sin p\right)\right]$

$$= pdq - q\cot p.\frac{1}{\frac{1}{q}\sin p}d\left(\frac{\sin p}{q}\right)$$

$$= pdq - q^2\frac{\cot p}{\sin p}.\frac{q\cos p.dp - \sin p.dq}{q^2}$$

$$= pdq - \cot p(q\cot p.dp - dq)$$

$$= q(1 - \mathrm{cosec}^2 p)dp + (p + \cot p)dq$$

$= d\{q(p + \cot p)\}$, which is exact differential.

Hence the given transformation is canonical.

5.10 Unit Summary

In this unit canonical transformation is defined and its utility to solve physical problems. Legendra transformation is also introduced. Different types of generating functions are deduced. The necessary and sufficient condition for canonical transformation is established. Poincaré theorem is proved in this unit. The one of the most fundamental bracket, the Poisson bracket is defined and several properties are investigated. Poisson bracket is used to represent Hamilton's equations. Hamilton-Jacobi equation is deduced and using it one dimensional harmonic oscillator problem is solved. An exercise is supplied with this unit.

5.11 Self Assessment Questions

5.1 What is a canonical transformation?

Prove that the following transformations are canonical.

(i) $Q = P, P = -q,$

(ii) $Q = q \tan p, \quad P = \log \sin p.$

5.2 Write a note on canonical transformation and the relation between old and new Hamiltonians.

5.3 Show that the transformation

$$Q = \log\left(\frac{\sin p}{q}\right), \quad P = q \cot p$$

is canonical. Find the generating function $G(q, Q)$.

5.4 Show that the transformation

$$Q = \log\left(1 + \sqrt{q}\cos p\right), \quad P = 2\sqrt{q}\left(1 + \sqrt{q}\cos p\right)\sin p.$$

is canonical. Find the generating function $G(q, Q)$.

5.5 Derived the necessary and sufficient condition for a transformation to be canonical.

5.6 Find the values of α and β so that the equation $Q = 2q^{\alpha}\cos\beta p, \; P = q^{\alpha}\sin\beta p$ represents a canonical transformation. What is the form of generating function G_3 in this case.

5.7 Find the condition that the transformation

$$P = ap + bq, \; Q = cp + dq$$

is canonical.

5.8 Show that the transformation

$Q = \dfrac{1}{p}, \; P = qp^2$ is canonical.

5.9 Prove that the transformation

$$q = \sqrt{\frac{P}{\sqrt{k}}}\sin Q, \; p = \sqrt{mP\sqrt{k}}\cos Q$$

is canonical and find its generating function $G(p, Q)$.

5.10 Prove that the Poisson bracket of two constants of motion is itself a constant even when the constants depend on time explicitly.

5.11 Show that if the Hamiltonian and a quantity G are constants of the motion, then $\frac{\partial G}{\partial t}$ must also be a constant.

5.12 Derive Hamilton's equations of motion in terms of Poisson bracket.

5.13 Prove Jacobi identity.

5.14 Show that the Poisson bracket is invariant under canonical transformation.

5.15 If H is the Hamiltonian and f is any function depending on position, momenta and time show that

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+[H,f].$$

5.16 Outline Hamilton-Jacobi theory and apply it to solve the problem of one dimensional harmonic oscillator.

5.17 Use Hamilton-Jacobi method to determine the motion of a particle falling vertically in a uniform gravitational field.

5.18 Apply the Hamilton-Jacobi theory to one dimensional Hamiltonian $H=\frac{1}{2}\left(q^2+p^2\right)$ to deduce the motion of a particle.

5.12 Suggested Further Readings

1. H. Goldstein, *Classical Mechanics*, Addison-Wesley, Cambridge, 1950.
2. T.W.B. Kibble, *Classical Mechanics*, Orient Longman London, 1985.
3. L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed., Pergamon Press, Oxford, 1976.
4. A. Sommerfeld, *Mechanics*, Academic Press, New York, 1964.
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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-IV **Group-A**

Module No. - 42

PRINCIPLE OF MECHANICS

(Motion of a Symmetrical Top, Small Oscillation and Special Theory of Relativity)

Content :

In this unit we have introduced three important topics of mechanics - motion of symmetric top, small oscillation of particles and strings and the special theory of relativity.

Objectives

- Equation of motion of symmetrical top.
- Steady motion is stable.
- Oscillation about equilibrium.
- Normal mode and normal frequencies.
- Vibration of a linear triatomic molecule.
- Oscillation of double pendulum.
- Motion of vibrating string.
- Galilean transformation.
- Basic postulates of special relativity.
- Lorentz transformations.
- Force and mass-energy equation.
- Lagrangiam formulation.
- Exercise.

6.1 Motion of a Symmetrical Top

A top is defined as a material body which is symmetrical about an axis and terminates at one end of the axis in a sharp point called the vertex or apex.

In figure 6.1, O is the vertex and OC is the axis of the top. Since the top is symmetrical about its axis, therefore centre of gravity G lies on the axis OC.

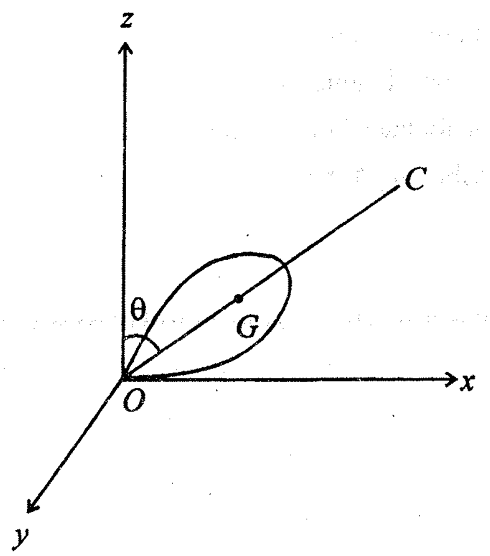

Fig. 6.1

6.1.1 Equations of motion of a top (Derived from Euler's equations)

Suppose a top spins with its vertex *O* fixed in contact with a floor rough enough to prevent slipping. Let *OC* be the axis of the top and *G* its centre of gravity where *OG=h*.

Let *Ox, Oy, Oz* be the fixed axes and *OA, OB, OC* the principle axes at *O*. Let *C* denote the moment of inertia about *OC* and *A* that of about *OA*, and as the top is symmetrical about *OC*, therefore *B=A*.

Initially, *OC* is in the plane *zOx*. At an instance *t*, let *OC* be inclined at an angle θ to *Oz* and the plane *zOC* making the angle ψ with the fixed plane *zOx*. Let $\omega_1, \omega_2, \omega_3$ be the angular velocities of the top about *OA, OB, OC* respectively and *L, M, N* the moments of the external forces about these axes.

The external forces acting on the top are its weight *mg* acting vertically downwards through *G* in the direction parallel to *zO*, and the reactions acting at *O*.

The direction cosines of *Oz* with respect to *OA, OB, OC* are

$$-\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta \text{ or } -\sin\theta, 0, \cos\theta$$

as in case of top $\phi=0$.

Hence the weight $(-mg)$ acting at *G* (0, 0, *h*), has its resolved parts

$mg \sin\theta, 0, -mg\cos\theta$

parallel to *OA, OB, OC* respectively.

The reactions act at the origin *O* (0, 0, 0).

Hence using the formula $L = \sum (y_1 Z_1 - z_1 Y_1)$

and similarly using the formulae of *M* and *N*

We find that

$L = 0, M = hmg \sin\theta, N = 0.$

Hence Euler's equations are

$$A\dot{w}_1 - (B - C)w_2 w_3 = 0 \quad (6.1a)$$

$$B\dot{w}_2 - (C - A)w_3 w_1 = hmg\sin\theta \quad (6.1b)$$

$$C\dot{w}_3 - (A - B)w_1 w_2 = 0. \quad (6.1c)$$

The Euler's geometrical relations are

$$w_1 = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi$$
$$w_2 = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi$$
$$w_3 = \dot{\phi} + \dot{\psi}\cos\theta.$$

Since for a top $\dot{\phi} = 0$, the relations reduce to

$$w_1 = -\dot{\psi}\sin\theta \tag{6.2a}$$
$$w_2 = \dot{\theta} \tag{6.2b}$$
$$w_3 = \dot{\psi}\cos\theta. \tag{6.2c}$$

Here, $B = A$, therefore equation (6.1c) reduces to

$$\dot{w}_3 = 0 \quad \text{or,} \quad w_3 = n \text{ (constant)}$$

or, $\dot{\psi}\cos\theta = n.$ [Using (6.2c)] (6.3)

Equation (6.1a) be comes

$$A\frac{d}{dt}[-\dot{\psi}\sin\theta] - (A - C)\dot{\theta}\,\dot{\psi}\cos\theta = 0 \text{ [Using (6.2a)]}$$

or, $-A\ddot{\psi}\sin\theta - 2A\dot{\theta}\dot{\psi}\cos\theta + C\dot{\theta}\dot{\psi}\cos\theta = 0$

or, $-A\ddot{\psi}\sin\theta - 2A\dot{\theta}\dot{\psi}\cos\theta + Cn\dot{\theta} = 0$ (6.4)

Equation (6.1b) reduces to

$$A\frac{d}{dt}[\dot{\theta}] + (C - A)\,\dot{\psi}^2\sin\theta\cos\theta = mgh\sin\theta$$

or, $A\ddot{\theta} - A\dot{\psi}^2\sin\theta\cos\theta + C\dot{\psi}^2\sin\theta\cos\theta = mgh\sin\theta$

or, $A\ddot{\theta} - A\dot{\psi}^2\sin\theta\cos\theta + Cn\dot{\psi}\sin\theta = mgh\sin\theta.$ (6.5)

Equations (6.4) and (6.5) are the equations of motion of the top and are of second order.

Further, we proceed to reduce from them the first order equations :

Multiplying (6.4) by $\sin\theta$ and integrating, we get

$$A\dot{\psi}\sin^2\theta + Cn\cos\theta = D \text{ (Constant)}. \tag{6.6}$$

Also, multiplying (6.4) by $2\dot{\psi}\sin\theta$ and (6.5) by $2\dot{\theta}$ and substracting, then we get

$$2A\ddot{\theta}\dot{\theta} + 2A\dot{\theta}\dot{\psi}^2\sin\theta\cos\theta + 2A\dot{\psi}\ddot{\psi}\sin^2\theta = 2mgh\sin\theta\cdot\dot{\theta}.$$

Integrating,

$$A\left(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta\right) + 2mgh\cos\theta = \text{E (Constant)}. \tag{6.7}$$

Equations (6.6) and (6.7) are also the equations of motion of the top but are of the first order.

Equations (.6.4), (6.5), (6.6) and (6.7) are known as the equations of motion of the top. While (6.4) and (6.5) are of the second order, (6.6) and (6.7) are of the first order.

Definition :

The motion due to the change in θ is called nutation and due to the change in ψ is called precession. The general motion of the top about its fixed vertex O is a combination of these two motion.

6.1.2 Equations of motion of a top (Deduce from Lagrange's equations)

In the case of the motion of a top

$$B = A, \phi = 0 \text{ and } w_1 = -\dot{\psi}\sin\theta, w_2 = \dot{\theta}, w_3 = n = \dot{\psi}\cos\theta.$$

Hence the K.E. is

$$2T = \left[A\left(w_1^2 + w_2^2\right) + Cw_3^2\right] \quad [\text{since } B = A]$$

$$= \left[A\left(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2\right) + C\dot{\psi}^2\cos^2\theta\right]$$

The P.E. V is

$$V = -mgh\cos\theta$$

where h is distance of C.G. from the vertex O, and θ is the inclination of OC to the vertical Oz.

Here the generalised coordinates are θ and ψ.

Equation for θ

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial V}{\partial \theta}$$

or, $$\frac{d}{dt}\left(A\dot{\theta}\right) - \left\{A\dot{\psi}^2\sin\theta\cos\theta - C\dot{\psi}^2\sin\theta\cos\theta\right\} = mgh\sin\theta$$

or, $$A\ddot{\theta} - A^2\dot{\psi}\sin\theta\cos\theta + Cn\dot{\psi}\sin\theta = mgh\sin\theta. \tag{6.8}$$

Equation for ψ

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right)-\frac{\partial T}{\partial \psi}=\frac{\partial V}{\partial \psi}$$

or, $$\frac{d}{dt}\left(A\dot{\psi}\sin^2\theta+C\dot{\psi}\cos^2\theta\right)=0$$

Integrating,

or, $$A\dot{\psi}\sin^2\theta+Cn\cos\theta=D \text{ (Constant)} \qquad (6.9)$$

Equations (6.8) and (6.9) are well equations of motion of a top.

6.1.3 Steady motion

The motion of a top is said to be steady if it spins about its vertex O in such a way that its axis OC makes the same angle with the vertical OZ throughout the motion.

In case of steady motion $\theta=\alpha$ (Constant), hence from equation $\dot{\psi}\cos\theta=n$ we have $\dot{\psi}=w$ (Constant). This shows that the angular velocity about the vertical OZ is constant.

So when motion is steady the axis of the top describes a cone about the vertical with uniform angular velocity. Thus precessional velocity $\dot{\psi}$ is constant.

Putting $\theta=\alpha, \ddot{\theta}=0$ and $\dot{\psi}=w$ in equation

$$A\ddot{\theta}-A\dot{\psi}^2\sin\theta\cos\theta+Cn\dot{\psi}\sin\theta-mgh\sin\theta=0$$

We have

$$w^2A\cos\alpha-Cnw+mgh=0. \qquad (6.10)$$

or, $$w=\frac{Cn\pm\sqrt{C^2n^2-4Amgh\cos\alpha}}{2A\cos\alpha}. \qquad (6.11)$$

Thus in general there are two values of the precessional velocity w, the two precessional velocities w_1, w_2 (say) are real and distinct provided that

$$C^2n^2>4Amgh\cos\alpha,$$

which is the necessary condition for the existance of steady motion.

A particular case :

If $\alpha = \frac{\pi}{2}$ then from (6.10), $w = \frac{mgh}{Cn}$. This implies that if the top is given an angular velocity n about OC, and OC when horinzontal is given an angular velocity $\frac{mgh}{Cn}$ about OZ, then OC will continue to revolve uniformly in a horizontal plane round the vertical OZ.

6.1.4 Steady motion is stable (when axis is vertical)

A top is executing steady motion with angular velocity n about its axis which is vertical, to show that the motion is stable.

We are required to show that that axis, of the top which is vertical in the state of steady motion, will perform simple harmonic motion about the vertical, if disturbed slightly.

General equations of motion of the top are

$$A\ddot{\theta} - A\dot{\psi}^2 \sin\theta\cos\theta + Cn\dot{\psi}\sin\theta = mgh\sin\theta$$

and
$$A\dot{\psi}\sin^2\theta + Cn\cos\theta = D.$$

Since motion is steady with axis vertical,

$$\theta = 0, \ddot{\theta} = 0, \dot{\psi} = n. \therefore D = Cn.$$

Also, $C^2n^2 > 4Amgh$ condition for steady motion when α=0. Let the motion be slightly disturbed. The disturbed motion is a general motion of the top, hence its equations are

$$A\ddot{\theta} - A\dot{\psi}^2 \sin\theta\cos\theta + Cn\dot{\psi}\sin\theta = mgh\sin\theta. \quad (6.12)$$

$$A\dot{\psi}\sin^2\theta + Cn\cos\theta = Cn. \quad (6.13)$$

From (6.13), we have

$$A\dot{\psi} = \frac{Cn(1-\cos\theta)}{\sin^2\theta} = \frac{Cn}{1+\cos\theta}.$$

Substituting the value of $\dot{\psi}$ in (6.12), we have

$$A^2\ddot{\theta} - \frac{C^2n^2}{(1+\cos\theta)^2}\sin\theta\cos\theta + \frac{C^2n^2}{1+\cos\theta}\sin\theta = A\,mgh\sin\theta.$$

Since θ is small, $\sin\theta \cong \theta, \cos\theta \cong 1$, the above equation becomes

$$A^2\ddot{\theta} - \frac{C^2n^2\theta}{4} + \frac{C^2n^2\theta}{2} = Amgh\theta$$

or, $$\ddot{\theta} = -\left(\frac{C^2n^2 - 4Amgh}{4A^2}\right)\theta. \qquad (.6.14)$$

The coefficient of θ is negative since $C^2n^2 > 4Amgh.$ This is the equation of S.H.M.

Hence the motion of the axis is SHM about the vertical *OZ* from which θ is measured. In other words, axis of the top, if disturbed slightly from its vertical position in steady motion, will tend to come back to its pre-disturbance position. This implies that the steady motion, in which axis of the top is vertical, is stable.

The period of oscillation is

$$2\pi\sqrt{\frac{4A^2}{(C^2n^2 - 4Amgh)}} = \frac{4\pi A}{(C^2n^2 - 4Amgh)^{1/2}}. \qquad (6.15)$$

6.1.5 Steady motion is stable (when axis is not vertical)

Let the axis of the top is inclined at a constant angle α to the vertical, and precessional velocity w. We show that, when the axis of the top disturbed slightly, from its position in steady motion, then top will come back to the original position.

Equations of motion of the top are

$$A\ddot{\theta} - A\dot{\psi}^2\sin\theta\cos\theta + Cn\dot{\psi}\sin\theta = mgh\sin\theta$$

and $$A\dot{\psi}\sin^2\theta + Cn\cos\theta = D.$$

Since the motion, originally, is steady with $\theta = \alpha, \dot{\psi} = w.$ Hence, the above equations reduce to $Cnw - Aw^2\cos\alpha = mgh$ and $D = Aw\sin^2\alpha + Cn\cos\alpha.$

Let the steady motion be slightly disturbed, so that motion is then a general one and so its equations are

$$A\ddot{\theta} - A\dot{\psi}^2\sin\theta\cos\theta + Cn\dot{\psi}\sin\theta = mgh\sin\theta. \qquad (6.16)$$

and $$A\dot{\psi}\sin^2\theta + Cn\cos\theta = Aw\sin^2\alpha + Cn\cos\alpha. \qquad (6.17)$$

Eliminating $\dot{\psi}$ between (6.16) and (6.17), we have

$$A^2\ddot{\theta}\sin^3\theta - \{Aw\sin^2\alpha + Cn(\cos\alpha - \cos\theta)\}^2\cos\theta$$
$$+Cn\sin^2\theta\{Aw\sin^2\alpha + Cn(\cos\alpha - \cos\theta)\} = Amgh\sin^4\theta.$$

Since the disturbance is small, therefore we may write $\theta = \alpha + \xi$, where ξ is small.

The above equation becomes

$$A^2\ddot{\xi}\sin^3(\alpha+\xi) - \left[Aw\sin^2\alpha + Cn\{\cos\alpha - \cos(\alpha+\xi)\}\right]^2\cos(\alpha+\xi)$$
$$+Cn\sin^2(\alpha+\xi)\left[Aw\sin^2\alpha + Cn\{\cos\alpha - \cos(\alpha+\xi)\}\right]$$
$$= Amgh\sin^4(\alpha+\xi).$$

Since ξ is small, neglecting small quantitites of second and higher order of terms of ξ.

$$A^2\ddot{\xi}(\sin\alpha + \xi\cos\alpha)^2 - \left[Aw\sin^2\alpha + Cn\{\cos\alpha - (\cos\alpha - \xi\sin\alpha)\}\right]^2$$
$$\times(\cos\alpha - \xi\sin\alpha) + Cn(\sin\alpha + \xi\cos\alpha)^2\left[Aw\sin^2\alpha\right.$$
$$\left.+Cn\{\cos\alpha - (\cos\alpha - \xi\sin\alpha)\}\right] = Amgh(\sin\alpha + \xi\cos\alpha)^4$$

or, $$A^2\ddot{\xi}\sin^3\alpha - (Aw\sin^2\alpha + Cn\xi\sin\alpha)^2(\cos\alpha - \xi\sin\alpha)$$
$$+Cn(\sin\alpha + \xi\cos\alpha)^2(Aw\sin^2\alpha + Cn\xi\sin\alpha)$$
$$= Amgh(\sin^4\alpha + 4\xi\cos\alpha\sin^3\alpha)$$

or, $$A^2\ddot{\xi}\sin^3\alpha - (A^2w^2\sin^4\alpha + 2\xi A\,Cn\,w\sin^3\alpha)(\cos\alpha - \xi\sin\alpha)$$
$$+Cn(\sin^2\alpha + 2\xi\sin\alpha\cos\alpha)(Aw\sin^2\alpha + Cn\xi\sin\alpha)$$
$$= Amgh(\sin^1\alpha + 4\xi\cos\alpha\sin^3\alpha)$$

or, $$A^2\ddot{\xi}\sin^3\alpha - (A^2w^2\sin^4\alpha + 2\xi A\,Cn\,w\sin^3\alpha)\cos\alpha$$
$$+\xi A^2w^2\sin^5\alpha + Cn(\sin^2 + 2\xi\sin\alpha\cos\alpha)A\,w\sin^2\alpha$$
$$+\xi C^2n^2\sin^3\alpha = Amgh(\sin^4\alpha + 4\xi\cos\alpha\sin^3\alpha)$$

or, $$A^2\ddot{\xi} - A^2w^2\sin\alpha\cos\alpha + Cn\,w\,A\sin\alpha + \xi A^2w^2\sin^2\alpha + \xi C^2n^2$$
$$= Amgh(\sin\alpha + 4\xi\cos\alpha)$$

or, $$A^2\ddot{\xi} - A^2w^2\sin\alpha\cos\alpha + \left(Aw^2\cos\alpha + mgh\right)A\sin\alpha + \xi A^2w^2\sin^2\alpha$$

$$+\xi\left(Aw^2\cos\alpha + mgh\right)^2\left(1/w^2\right) = A\,mgh\left(\sin\alpha + 4\xi\cos\alpha\right)$$

or, $$A^2\ddot{\xi} + \xi\left[A^2w^2\left(\sin^2\alpha + \cos^2\alpha\right) + 2A\,mgh\,w^2\cos\alpha + m^2g^2h^2\right]\left(\frac{1}{w^2}\right)$$

$$= 4\xi\,A\,mgh\cos\alpha$$

or, $$A^2\ddot{\xi} + \xi\left[A^2w^4 - 2A\,mgh\,w^2\cos\alpha + m^2g^2h^2\right]\left(\frac{1}{w^2}\right) = 0$$

or, $$\ddot{\xi} = -\frac{A^2w^4 - 2A\,mgh\,w^2\cos\alpha + m^2g^2h^2}{A^2w^2}\xi$$

$$= -\frac{\left(Aw^2 - mgh\cos\alpha\right)^2 + m^2g^2h^2\sin^2\alpha}{A^2w^2}\xi \qquad (6.18)$$

(The coefficient of ξ is negative)

This is the equation of S.H.M.

This shows that axis of the top, if disturbed slightly from its position in steady motion, will tend to come back to the pre disturbed position.

Hence the steady motion, with axis inclined at α to the vertical is stable.

The period of oscillation is

$$\frac{2\pi\cdot Aw}{\left(A^2w^4 - 2A\,mgh\,w^2\cos\alpha + m^2g^2h^2\right)^{1/2}}$$

If the top is set in motion in usual manner, then n is very large and the two values of the precessional velocity are approximately

$$w = \frac{Cn}{A\cos\alpha}, \frac{mgh}{Cn}.$$

The first value is large and second is small when n is very large. For the first value of w the period of oscillation is

$$\frac{2\pi Aw}{Aw^2} = \frac{2\pi A\cos\alpha}{Cn} \text{ (approximately).} \qquad (6.19)$$

For second value of w the period of oscillation is

$$\frac{2\pi Aw}{mgh} = \frac{2\pi A}{Cn} \text{ (approximately).} \tag{6.20}$$

6.2 Small Oscillations

A system of particles is said to be in static equilibrium if the particles constituting the system are at rest and the total force on each of these particles is constantly zero. We consider the small oscillations about the position of stable or static equilibrium in cases where these small oscillation are regarded of such a small amplitude, which may cause only the fundamental frequencies excited and none of the harmonics. This consideration of small oscillation has very wide applications in molecular spectra, coupled oscillator, etc.

6.2.1 Oscillations about equilibrium

Suppose the system is fully conservative, so that the generalised coordinates $q_1, q_2, ..., q_n$ of the system are time-independent. The system will be in equilibrium if the generalised forces Q_k acting on the system are zero, i.e.

$$Q_k = \left(\frac{\partial V}{\partial q_k}\right)_0 = 0. \tag{6.21}$$

Here the subscript zero denotes that the derivative is to be evaluated at $q_k = q_{0k}, k = 1,2,...,n.$

Thus the potential energy has an extremum at the equilibrium position of the system. The equilibrium is said to be stable when a small oscillation of the system from the position ofthe equilibrium causes it in small bounded motion about the position of rest and it is unstable when such a small oscillation causes it in unbounded motion.

Let $q_k - q_{0k} = \eta_k$, (6.22)

represents the deviation from q_{0k}.

Expanding potential in Taylor's series about q_{0i}, we get

$$V(q_1, q_2, ..., q_n) = V(q_{01}, q_{02}, ...q_{0k}) + \sum_k \left(\frac{\partial V}{\partial q_k}\right)_0 \eta_k$$

$$+\frac{1}{2}\sum_{k,j}\left(\frac{\partial^2 V}{\partial q_k \partial q_j}\right)_0 \eta_k \eta_j + ...$$

or, $$V = V_0 + \sum_k \left(\frac{\partial V}{\partial q_k}\right)_0 \eta_k + \frac{1}{2}\sum_{k,j}\left(\frac{\partial^2 V}{\partial q_k \partial q_j}\right)_0 \eta_k \eta_j + \ldots \quad (6.23)$$

The first terms on the right being a constant may be taken as zero and the second term vanishes by virtue of (6.21).

Then (6.23) becomes

$$V = \frac{1}{2}\sum_{k,j}\left(\frac{\partial^2 V}{\partial q_k \partial q_j}\right)_0 \eta_k \eta_j = \frac{1}{2}\sum_{k,j} V_{kj}\eta_k \eta_j, \quad (6.24)$$

where

$$V_{kj} = \left(\frac{\partial^2 V}{\partial q_k \partial q_j}\right)_0 = V_{jk}.$$

Also, we know that the K.E. T is

$$T = \frac{1}{2}\sum_{k,j} T_{kj}\dot{\eta}_k \dot{\eta}_j. \quad (6.25)$$

Thus $$L = T - V = \frac{1}{2}\sum_{k,j}\left[T_{kj}\dot{\eta}_k\dot{\eta}_j - V_{kj}\eta_k\eta_j\right]. \quad (6.26)$$

[by (6.24) and (6.25)]

The n equations of motion derived from the Lagrangian equation are

$$\sum_j T_{kj}\ddot{\eta}_j + \sum_j V_{kj}\eta_j = 0, k = 1,2,\ldots,n, \quad (6.27)$$

or, $$T\ddot{\eta} + V\eta = 0, \quad (6.28)$$

where $T = [T_{kj}], V = [V_{kj}]$ are two symmetric and constant matrices.

The equations (6.28) forms a set of simultaneous differential equations with constant coefficients representing the motion of a set of coupled oscillations.

Let $T^{-1}V = A.$ (6.29)

Then (6.28) becomes $\ddot{\eta} + A\eta = 0.$ (6.30)

Let $A = w^2 I$ be a trial solution of (6.30), then

$$\ddot{\eta} + w^2\eta = 0. \tag{6.31}$$

Here η may be regarded as a displacement vector with n components $\eta_1, \eta_2, \ldots\ldots, \eta_n$.

Hence from (6.30) and (6.31) we can interpret the result in the form that a S.H.M. of the vector η with angular velocity w is possible if

$$A\eta = w^2\eta. \tag{6.32}$$

or, $\quad if\left(A - w^2 I\right)\eta = 0.$

This implies that η is a eiger vector of A and w^2 be the corresponding eigen value or eigen frequencies. The eigen values are given by

$$|A - w^2 I| = 0 \quad \text{or,} \ |V - w^2 T| = 0.$$

The matrices V and T being symmetric, the eigen values w^2 are real.

Each oscillation with a definite frequency is said to be an eigen vibration or normal mode of the system. It has a characteristic feature that during such a vibration, the corresponding vector η does not change in direction but in magnitude only and consequently the mode consists of the simultaneous oscillations of several degrees of freedom.

In case A is degenrate (i.e., having two or more commensurable frequencies) there is an arbitrary choice of the normal modes which correspondence to the same eigen frequencies.

If we introduce the coordinates x_k along the directions of the eigen vectors η_k of A to describe the system, the equation of motion for the x_k may be written by (6.30) as

$$\ddot{x}_k + w_k^2 x_k = 0 \ \text{or,} \ x_k = c_k \bar{e}^{iw_k t}.$$

Here the coordinates x_k are said to be normal coordinates of the system.

6.2.2 Free vibration of a linear triatomic molecule

Consider the equilibrium configuration of the molecule such that two of its atoms of each of mass m are symmetrically placed on each side of the third atom of mass M. All three atoms are collinear. Assuming the motion along the line of molecules and there being no interaction between the end atoms, let k be the force constant approximated by two strings of free joining the three atoms.

Taking ξ_1, ξ_2, ξ_3 as the displacements of each atom from the equilibrium configuration, the K.E. and P.F. are given by

$$T=\frac{1}{2}m\left(\dot{\xi}_1^2+\dot{\xi}_3^2\right)+\frac{1}{2}M\dot{\xi}_2^2. \tag{6.33}$$

and

$$V=\frac{1}{2}k(\xi_2-\xi_1)+\frac{1}{2}k(\xi_3-\xi_2)^2$$

$$=\frac{k}{2}\left(\xi_1^2+2\xi_2^2+\xi_3^2-2\xi_1\xi_2-2\xi_2\xi_3\right). \tag{6.34}$$

In this case, T and V are

$$T=\frac{1}{2}\begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}, V=\frac{1}{2}\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}. \tag{6.35}$$

The eigen frequencies are given by

$$\left|V-w^2T\right|=0$$

or, $$\begin{vmatrix} k-w^2m & -k & 0 \\ -k & 2k-w^2M & -k \\ 0 & -k & k-w^2m \end{vmatrix}=0. \tag{6.36}$$

or, $$w^2\left(k-w^2m\right)\left[k(M+2m)-w^2mM\right]=0$$

or, $$w^2=0,\frac{k}{m},\frac{k(M+2m)}{mM}.$$

Thus the three normal frequencies are

$$w_1=0, w_2=\sqrt{\frac{k}{m}}, w_3=\sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)}. \tag{6.37}$$

Here the first eigen value $w_1=0$ does not correspond to an oscillatory motion, since the equation of motion for the corresponding normal coordinates is $\ddot{x}_k=0$, which actually produces a uniform translatory motion. Such a vanishing frequency is caused by the molecule translated rigidly along its axis without any change in the potential energy, since restoring force against such motion is zero.

Let η_1,η_2,η_3 be the eigen vectors corresponding to w_1,w_2,w_3 respectively. Let components of η_1 be $\eta_{11},\eta_{12},\eta_{13}$ those of η_2 be $\eta_{21},\eta_{22},\eta_{23}$ those of η_3 be $\eta_{31},\eta_{32},\eta_{33}$.

Therefore, $\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} \eta_{11} \\ \eta_{12} \\ \eta_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or, $k\eta_{11} - k\eta_{12} = 0$

$-k\eta_{11} + 2k\eta_{12} - k\eta_{13} = 0$

$-k\eta_{12} + k\eta_{13} = 0.$

Solving, we get $\eta_{11} = \eta_{12} = \eta_{12} = \alpha$ (say), so

$$\eta_1 = \alpha(1,1,1) = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad (6.38)$$

which follows that each atom is equally displaced (see Figure 6.2).

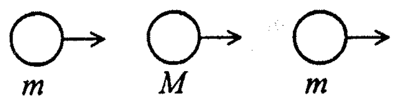

Fig. 6.2

Now, $w_2 = \sqrt{\left(k/m\right)}$ gives

$$\begin{bmatrix} 0 & -k & 0 \\ -k & 2k - k\dfrac{M}{m} & -k \\ 0 & -k & 0 \end{bmatrix} \begin{bmatrix} \eta_{21} \\ \eta_{22} \\ \eta_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, $\eta_{22} = 0,$

$\eta_{21} + \eta_{23} = 0$

or, $\eta_{21} = -\eta_{23} = \beta$ (say)

So, $$\eta_2 = \beta(1,0-1) = \beta \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}. \qquad (6.39)$$

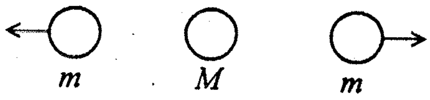

Fig. 6.3

This follows that the central atom remains at rest while the end atoms are displaced equally in opposite sense (see Figure 6.3).

Again, $w_3 = \sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)}$ gives

$$\begin{bmatrix} -2m/M & -k & 0 \\ -k & -M/m & -k \\ 0 & -k & -2m/M \end{bmatrix}\begin{bmatrix} \eta_{31} \\ \eta_{32} \\ \eta_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, $\eta_{31} = \eta_{33} = \gamma$ (say) and $\eta_{32} = -\frac{2m}{M}\gamma$.

Therefore, $$\eta_3 = \gamma\left(1, \frac{-2m}{M}, 1\right) = \gamma\begin{bmatrix} 1 \\ -2m/M \\ 1 \end{bmatrix} \qquad (6.40)$$

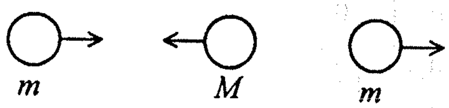

Fig. 6.4

This follows that the displacement of end atoms is equal while that of the central atom is equal to neither of end one's and moreover differs in phase (see Figure 6.4).

The matrices η and T are

$$\eta = \begin{bmatrix} \alpha & \alpha & \alpha \\ \beta & 0 & -\beta \\ \gamma & \frac{-2m}{M}\gamma & \gamma \end{bmatrix} \text{ and } T = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}.$$

The matrix η will be orthogonal, if $\eta' T\eta = I$,

or, $$\begin{bmatrix} \alpha & \alpha & \alpha \\ \beta & 0 & -\beta \\ \gamma & \frac{-2m}{M}\gamma & \gamma \end{bmatrix}\begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \alpha & \beta & \gamma \\ \alpha & 0 & \frac{-2m}{M}\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or, $$\begin{bmatrix} (2m+M)\alpha^2 & 0 & 0 \\ 0 & 2m\beta^2 & 0 \\ 0 & 0 & 2m\gamma^2 + \frac{4m^2}{M}\gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or, $(2m+M)\alpha^2 = 1$ i.e. $\alpha = \frac{1}{\sqrt{2m+M}}$

$2m\beta^2 = 1$ i.e. $\beta = \frac{1}{\sqrt{2m}}$

and $\left(2m + \frac{4m^2}{M}\right)\gamma^2 = 1$ or, $\gamma = \frac{1}{\sqrt{\left\{2m\left(1+\frac{2m}{M}\right)\right\}}}$.

6.2.3 Ex‥mple of the double pendulum as a couple oscillator

Suppose there are two pendulum of equal mars and length suspended from a non-rigid support so that the !acement of one pendulum affects the potential energy of the other and vice-versa.

Let us first c‥ısider the motion of a simple pendulum of mass m and length l inclined at θ_1 to the vertical. At this instant let s_1 be the arc length along the circular path. Then

$$s_1 = l\theta_1. \tag{6.41}$$

Let T_1 and V_1 be the K.E. and P.E. respectively.

$$\therefore T_1 = \frac{1}{2}m\dot{s}_1^2 = \frac{1}{2}ml\dot{\theta}_1^2 \tag{6.42}$$

and $$V_1 = mg(l - l\cos\theta_1) = mgl(1-\cos\theta_1). \tag{6.43}$$

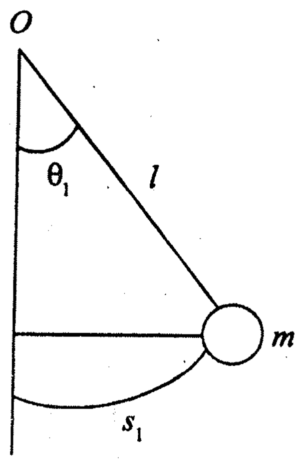

Fig. 6.5

Using the natural unit such that m, l, g all can be made equal to unity, i.e. $m = l = g = 1$, then (6.42) and (6.44) gives

$$T_1 = \frac{1}{2}\dot{\theta}_1^2 \tag{6.44}$$

and

$$V_1 = 1 - \cos\theta_1 = 2\sin^2\frac{\theta_1}{2} = 2\left[\frac{\theta_1}{2} - \frac{1}{3!}\left(\frac{\theta_1}{2}\right)^3 + ...\right]^2.$$

But we are considering small oscillations, therefore cubes and higher powers of θ can be neglected.

Thus we have

$$V_1 = 2\left(\frac{\theta_1}{2}\right)^2 = \frac{1}{2}\theta_1^2. \tag{6.45}$$

Let us now assume that T is the K.E. and V the P.E. of the coupled oscillator, i.e. double pendulum, then the results (6.44) and (6.45) derived for a single pendulum will appear in the following form for a double pendulum,

$$T = \frac{1}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) \tag{6.46}$$

$$V = \frac{1}{2}\left(\theta_1^2 + \theta_2^2 - 2k\,\theta_1\theta_2\right) \tag{6.47}$$

where k is a coupling constant dependent on the actual structure of the pendulum support and θ_2 is the angle of the second pendulum from the vertical.

The two symmetric matrices T and V are

$$T=\frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad \text{and} \quad V=\frac{1}{2}\begin{bmatrix}1 & -k\\ -k & 1\end{bmatrix} \tag{6.48}$$

and the eigen frequencies will be given by

$$\left|V-w^2T\right|=0 \;\text{ or, }\; \begin{vmatrix}1-w^2 & -k\\ -k & 1-w^2\end{vmatrix}=0. \tag{6.49}$$

i.e. $$\left(1-w^2\right)^2-k^2=0 \quad \text{or,} \quad w^2=1\pm k, \tag{6.50}$$

or, $w=\left(1\pm k\right)^{1/2}$.

If k is small then

$$w=1\pm\frac{1}{2}k, \tag{6.51}$$

neglecting higher powers of k.

The normal modes of oscillations are given by

$$\begin{bmatrix}1-w^2 & -k\\ -k & 1-w^2\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

or, $$\left(1-w^2\right)\theta_1-k\theta_2=0$$

$$-k\theta_1+\left(1-w^2\right)\theta_2=0$$

or, $$\left.\begin{aligned}\pm k\theta_1-k\theta_2=0\\ -k\theta_1\pm k\theta_2=0\end{aligned}\right\} \tag{6.52}$$

These give $\dfrac{\theta_1}{\theta_2}=\pm 1$.

Case I. If $\dfrac{\theta_1}{\theta_2}=1$ and $k>0$, the pendulums oscillate together with an angular frequency less than that of one of the pendulum simply.

Case II. If $\dfrac{\theta_1}{\theta_2}=-1$, $k>0$, the two pendulums oscillate in opposite directions at a frequency larger than the uncoupled frequency.

In general the combined oscillation will be the superposition of the two modes of oscillations and the phenomenon of beats is shown by the oscillations of either pendulum.

6.2.4 The equation of motion for the vibrating string

A string is a wire whose length is very large as compared to its diameter and which perfectly uniform and flexible. When a string streatched between two points with a large tension, is plucked transverse vibrations are produced in it. In order to somplify the problem, let us assume that the string vibrates only in vertical plane.

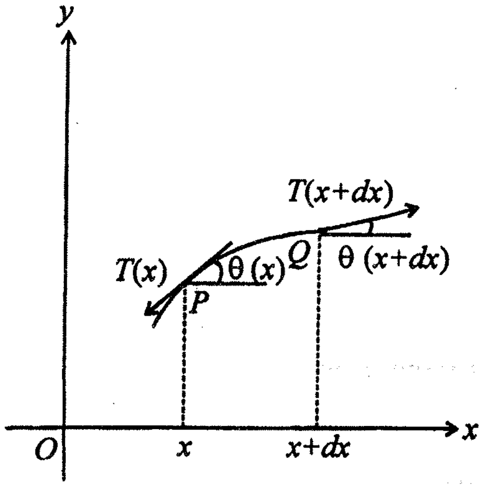

Fig. 6.6

Consider the motion of an element PQ of the string of length dl. Let O be the origin and $P(x, y)$ and $Q(x+dx,$ $+dy)$ be any two points. Let $T(x)$ and $T(x+dx)$ be the tensions at P and Q respectively and $\theta(x)$ and $\theta(x+dx)$ the angles which the tangents at P and Q make with x-axis. The total horizontal force along x-axis on the element PQ is

$$T(x+dx)\cos\theta(x+dx) - T(x)\cos\theta(x). \tag{6.53}$$

The total vertical force along y axis acting on the element PQ is

$$T(x+dx)\sin\theta(x+dx) - T(x)\sin\theta(x). \tag{6.54}$$

Assuming that the vibrations take place in the vertical plane only, the horizontal motion is negligible and hence, force represented by (6.53) is zero.

If m is the mass per unit length of the string, then the mass of the element PQ is $m\,dl$.

According to Newton's law, the equation of motion, neglecting all other forces, is given by

$$m\,dl\frac{\partial^2 y}{\partial t^2} = T(x+dx)\sin\theta(x+dx) - T(x)\sin\theta(x).$$

Dividing by dx, we get

$$m\frac{dl}{dx}\frac{\partial^2 y}{\partial t^2} = \frac{T(x+dx)\sin\theta(x+dx) - T(x)\sin\theta(x)}{dx}.$$

But, $(dl)^2 = (dx)^2 + (dy)^2$.

$$\therefore m\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}\frac{\partial^2 y}{\partial t^2} = \frac{T(x+dx)\sin\theta(x+dx) - T(x)\sin\theta(x)}{dx}$$

Taking the limit $dx \to 0$, the above equation reduces to

$$m\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}[T(x)\sin\theta(x)] = \frac{\partial}{\partial x}(T\sin\theta) \qquad (6.55)$$

But, $\sin\theta = \dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \dfrac{\frac{\partial y}{\partial x}}{\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}}$

Then equation (6.55) reduces to

$$m\frac{\partial^2 y}{\partial t^2}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} = \frac{\partial}{\partial x}\left[T\frac{\frac{\partial y}{\partial x}}{\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}}\right].$$

If we consider that the vibration is small then the slope $\frac{\partial y}{\partial x}$ is small compared to unity, then we have

$$m\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left[T\frac{\partial y}{\partial x}\right].$$

Let us further assume that the tension T is constant throughout the string, then

$$m\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}$$

or, $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2} \qquad (5.56)$$

or, $$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{dx^2}$$

where $c = (T/m)^{1/2}$. (5.57)

The constant c has the dimensions of velocity and actually it is the velocity with which the wave travels along the string. Equation (5.57) represents the equation of motion of the vibrating string.

It may be noted that the motion of the string is described by the function $y(x, t)$ locating each point x on the string at every instant of time just like the motion of the system of particles is described by the functions $x(t)$, $y(t)$, $z(t)$ locating each particle at every instant of time.

If in addition a vertical force f per unit length acts on the string, then the equation of motion of the vibrating string is given by

$$mdl\frac{\partial^2 y}{\partial t^2} = T(x+dx)\sin\theta(x+dx) - T(x)\sin\theta(x) + f\,dl$$

which gives, as before,

$$m\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial^2} + f. \qquad (6.58)$$

This is the equation of motion of the vibrating string for small amplitude of vibration, where tarsion T is tant and f is the external force applied per unit length of the string.

If f is the gravitational force, then $f = -m\,dlg$, and the equation of motion of the vibrating string will be

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} - g. \qquad (6.59)$$

Initial and boundary conditions :

The initial condition may be taken as initial position

$$y_0(x) = y(x,0) \qquad (6.60)$$

and initial velocity $v_0(x) = \left[\frac{\partial y}{\partial t}\right]_{t=0}$. (6.61)

If the string is fixed at its ends, then the boundary conditions are

$$y(0,t) = y(l,t) = 0. \tag{6.62}$$

6.2.5 Normal modes for the vibrating string

To find the solution of (5.57), we use the method of separation of variables. Let us consider the solution of the form

$$y(x,t) = X(x)\phi(t) \tag{6.63}$$

where X is the function of x only and ϕ is the function of t only.

Then we have $\frac{\partial y}{\partial x} = \phi\frac{\partial X}{\partial x}, \frac{\partial^2 y}{\partial x^2} = \phi\frac{\partial^2 X}{\partial x^2}$

and $\frac{\partial y}{\partial t} = X\frac{\partial \phi}{\partial t}, \frac{\partial^2 y}{\partial t^2} = X\frac{\partial^2 \phi}{\partial t^2}$.

Substituting these values in (5.57), we get

$$X\frac{\partial^2 \phi}{\partial t^2} = c^2\phi\frac{\partial^2 X}{\partial x^2}.$$

Dividing throughout by ϕX, we get

$$\frac{1}{\phi}\frac{\partial^2 \phi}{\partial t^2} = \frac{c^2}{X}\frac{\partial^2 X}{\partial x^2}. \tag{6.64}$$

In the above equation L.H.S. is independent of x and R.H.S. is independent of t. Therefore, if above equation is to be satisfied both sides must be equal to a constant $-w^2$ (say).

Then $\frac{1}{\phi}\frac{\partial^2 \phi}{\partial t^2} = -w^2$ (6.65)

and $\frac{c^2}{X}\frac{\partial^2 X}{\partial x^2} = -w^2$. (6.66)

From (6.65),

$$\frac{\partial^2 \phi}{\partial t^2} = -w^2\phi. \tag{6.67}$$

The general solution of this equation is

$$\phi = A_1 \cos wt + B_1 \sin wt, \tag{6.68}$$

where A_1 and B_1 are constants to be determined.

From (6.66),

$$\frac{\partial^2 X}{\partial x^2} = -\frac{w^2}{c^2} X. \tag{6.69}$$

The solution of this equation is

$$X = A_2 \cos\frac{wx}{c} + B_2 \sin\frac{wx}{c}, \tag{6.70}$$

where A_2 and B_2 are constants to be determined.

Applying the boundary conditions

$y(0,t) = y(l,t) = 0$, because the ends of the string are fixed and they do not vibrate.

Then $X(0) = A_2 = 0$

and $X(l) = A_2 \cos\frac{wl}{c} + B_2 \frac{\sin wl}{c} = 0.$

$$B_2 \sin\frac{wl}{c} = 0 \text{ or, } \sin\frac{wl}{c} = 0 \text{ as } B_2 \neq 0.$$

This equation holds iff

$$\frac{wl}{c} = n\pi, \text{ where } n = 1, 2, 3, ...$$

$$\text{or, } \quad w_n = \frac{n\pi c}{l}. \tag{6.71}$$

The frequencies $\nu_n = \frac{w_n}{2\pi} = \frac{n\pi c}{2\pi l} = \frac{nc}{2l}, n = 1,2,3....$ are called the normal frequencies of vibration of the string.

Substituting the values of A_2 and w_n in equation (6.70) and (6.68), we have

$$X = B_2 \sin\frac{n\pi x}{l} \tag{6.72}$$

$$\text{and } \quad \phi = A_1 \cos\frac{n\pi ct}{l} + B_1 \sin\frac{n\pi ct}{l}. \tag{6.73}$$

Hence the solution of (6.57) is given by

$$y(x,t) = B_2 \sin\frac{n\pi x}{l}\left[A_1 \cos\frac{n\pi ct}{l} + B_1 \frac{n\pi ct}{l}\right]$$

$$= A_1 B_2 \sin\frac{n\pi x}{l} + \cos\frac{n\pi ct}{l} + B_1 B_2 \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}$$

$$= A \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l} + B \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}. \tag{6.74}$$

where $A = A_1 B_2$ and $B = B_1 B_2$ are constants.

This solution is called a normal mode of vibration of the string.

The initial position and velocity of the nth mode of vibration of the string given by equation (6.74) are

$$y_0(x) = y(x,0) = A \sin\frac{n\pi x}{l} \tag{6.75}$$

and $$v_o(x) = \left[\frac{\partial y}{\partial t}\right]_{t=0} = \frac{n\pi cB}{l} \sin\frac{n\pi x}{l}. \tag{6.76}$$

The most general solution of the equation (6.57) is given by

$$y(x,t) = \sum_{n=1}^{\infty}\left[A_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}\right]. \tag{6.77}$$

The constants A_n and B_n are given by

$$y_0(x) = y(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \tag{6.78}$$

and $$v_0(x) = \sum_{n=1}^{\infty} \frac{n\pi c\, B_n}{l} \sin\frac{n\pi x}{l}. \tag{6.79}$$

Evaluation of constants A_n and B_n.

Using Fourier's theorem one can computed the constant A_n and B_n using (6.78) and (6.79)

Multiplying (6.78) by $\sin\frac{n\pi x}{l}$ and integrating within the limit from x=0 to $x = \ell$, we have

$$\int_0^{\ell} y_0(x) \sin\frac{n\pi x}{l} dx = \int_0^{\ell} \sum_{n=1}^{\infty} \sin^2 \frac{n\pi x}{l} dx$$

$$= A_n \int_0^{\ell} \sin^2 \frac{n\pi x}{l} dx \text{ (other terms vanish)}$$

$$= \frac{A_n}{2} \int_0^{\ell} \left(1 - \cos\frac{2n\pi x}{l}\right) dx$$

$$= \frac{A_n}{2} \cdot l$$

$$\therefore A_n = \frac{2}{l} \int_0^{\ell} y_0(x) \sin\frac{n\pi x}{l} dx. \tag{6.80}$$

Multiplying (6.79) by $\sin\frac{n\pi x}{l}$ and integrating within the limits 0 and ℓ, we get

$$\int_0^l v_0(x) \sin\frac{n\pi x}{l} dx = \int_0^l \sum_{n=1}^{\infty} \frac{n\pi c\, B_n}{l} \sin^2 \frac{n\pi x}{l} dx$$

$$= \sum_{n=1}^{\infty} \frac{n\pi c\, B_n}{2l} \int_0^l \left(1 - \cos\frac{2n\pi x}{l}\right) dx$$

$$= \frac{n\pi c\, B_n}{2l} \cdot l = \frac{n\pi c\, B_n}{2}$$

$$\therefore B_n = \frac{2}{n\pi c} \int_0^l v_0(x) \sin\frac{n\pi x}{l} dx. \tag{6.81}$$

Thus the complete solution of (6.57) is given by (6.77) where A_n and B_n are available in (6.80) and (6.81).

6.2.6 Lagrange's equations for vibrating string

Let us take $y(x)$ as a set of generalised coordinates analogous to q_k. In place of subscript q denoting the ..rious degrees of freedom, we have the position coordinate x denoting the various points on the string. For an ideal continuous string the number of degrees of freedom is infinite.

Let the string be fixed at $x=0$ and $x = l$, then its position $y(x)$ can be represented by Fourier series from (6.78) as

$$y(x) = \sum_{k=1}^{\infty} q_k \sin\frac{k\pi x}{l}. \tag{6.82}$$

Multiplying both sides by $\sin\frac{k\pi x}{l}$ and integrating between the limits $x=0$ to $x = l$ the coefficients q_k are given by

$$q_k = \frac{2}{l}\int_0^l y(x)\sin\frac{k\pi x}{l}dx, k = 1,2,.... \tag{6.83}$$

The coefficients q_k represent a suitable set of generalised coordinates because they give a complete description of the position of vibrating string. When the string vibrates, the coordinates q_k become functions of t, i.e.,

$$y(x,t) = \sum_{k=1}^{\infty} q_k(t)\sin\frac{k\pi x}{l}. \tag{6.84}$$

Now, to find Lagrange's equation we have to determine Lagrangian in terms of coordinates q_k.

$$\text{The K.E.T.} = \int_0^l \frac{1}{2}m\left(\frac{\partial y}{\partial t}\right)^2 dx. \tag{6.85}$$

Differentiating equation (6.84) w.r.t. t and squaring, we get

$$\left(\frac{\partial y}{\partial t}\right)^2 = \sum_{k=1}^{\infty}\sum_{r=1}^{\infty}\dot{q}_k\dot{q}_r \sin\frac{k\pi x}{l}\sin\frac{r\pi x}{l}. \tag{6.86}$$

$$\therefore T = \int_0^l \frac{1}{2}m\sum_{k=1}^{\infty}\sum_{r=1}^{\infty}\dot{q}_k\dot{q}_r \sin\frac{k\pi x}{l}\sin\frac{r\pi x}{l}dx$$

$$= \sum_{k=1}^{\infty}\frac{1}{4}lm\dot{q}_k^2 \tag{6.87}$$

$$\left[\because \int_0^l \sin\frac{k\pi x}{l}\sin\frac{r\pi x}{l}dx = \begin{cases} l/2 & for\ r = k \\ 0, & for\ r \neq k. \end{cases}\right.$$

Now, calculate the P.E.V. directly calculating the work done against the tension T in moving the string from its equilibrium position to the position $y(x)$. Let $y(x, t)$ be the position of the string at any time t while the string is being moved to $y(x)$.

At time t=0, the string is in the equilibrium position, i.e. $y(x, 0)$=0.

It $t = t_1$ is the time at which the string arrives at its final positions then

$$y(x,t_1) = y(x).$$

The work done against the vertical components of tension T during the time interval dt is given by

$$dV = -\int_0^l \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}dt\right)dx.$$

Integrating by parts and keeping in mind that y and $\frac{\partial y}{\partial t}$ are 0 at $x=0$ and $x = l$, we get

$$dV = \int_0^l T \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x \partial t} dx dt$$

$$= \frac{\partial}{\partial t}\left[\int_0^l \frac{1}{2} T \left(\frac{\partial y}{\partial x}\right)^2 dx\right] dt.$$

The total work done is given by

$$V = \int_0^{t_1} dV = \int_0^l \left[\frac{1}{2} T \left(\frac{\partial y}{\partial x}\right)^2 dx\right]_{t=0}^{t_1}$$

$$= \int_0^l \frac{1}{2} T \left(\frac{\partial y}{\partial x}\right)^2 dx. \qquad (6.88)$$

This work done is stored as the P.E. in the stretched string.

Substituting the value of $\frac{\partial y}{\partial x}$ from (6.82) in (6.88), we get

$$V = \int_0^l \frac{1}{2} T \sum_{k=1}^{\infty} \left(\frac{k\pi}{l}\right)^2 q_k^2 \cos^2 \frac{k\pi x}{l} dx$$

$$= \frac{1}{4} T \int_0^l \sum_{k=1}^{\infty} \left(\frac{k\pi}{l}\right)^2 q_k^2 \left(1 + \cos \frac{2k\pi x}{l}\right) dx$$

$$= \sum_{k=1}^{\infty} \frac{1}{4} Tl \left(\frac{k\pi}{l}\right)^2 q_k^2.$$

The Lagrangian function for the vibrating string may be written as

$$L = T - V = \sum_{k=1}^{\infty} \left[\frac{1}{4} lm \dot{q}_k^2 - \frac{1}{4} Tl \left(\frac{k\pi}{l}\right)^2 q_k^2\right]. \qquad (6.89)$$

Thus $\frac{\partial L}{\partial \dot{q}_k} = \frac{1}{2} lm \dot{q}_k, \frac{\partial L}{\partial q_k} = -\frac{1}{2} Tl \left(\frac{k\pi}{l}\right)^2 q_k.$ (6.90)

The Lagrangian equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

or, $\frac{d}{dt}\left(\frac{1}{2}lm\dot{q}_k\right)+\frac{1}{2}Tl\left(\frac{k\pi}{l}\right)^2 q_k = 0$ [Using (6.90)]

or, $\frac{1}{2}lm\ddot{q}_k+\frac{1}{2}Tl\left(\frac{k\pi}{l}\right)^2 q_k = 0$

or, $\ddot{q}_k+\frac{T}{m}\left(\frac{k\pi}{l}\right)^2 q_k = 0,$ (6.91)

which is required Lagrange's equation.

Substituting

$$w_k^2=\frac{T}{m}\left(\frac{k\pi}{l}\right)^2=\left(\frac{k\pi c}{l}\right)^2 \text{ since } \sqrt{\left(\frac{T}{m}\right)}=c.$$

or, $w_k=\frac{k\pi c}{l}.$ (6.92)

Then (6.91) becomes

$$\ddot{q}_k+w_k^2 q_k=0. \tag{6.93}$$

The general solution of the above equation is given by

$$q_k = A_k\cos w_k t + B_k \sin w_k t.$$

Substituting this result in equation (6.82), we have

$$y(x,t)=\sum_{k=1}^{\infty}\left[A_k\sin\frac{k\pi x}{l}\cos w_k t + B_k\sin\frac{k\pi x}{l}\sin w_k t\right] \tag{6.94}$$

If initial conditions are such that

$$\left.\begin{aligned} &y=y_0(x)\\ \text{and } &\frac{\partial y}{\partial t}=v_0(x) \text{ at } t=0.\end{aligned}\right\} \tag{6.95}$$

Then we have as before

$$y_0(x)=\sum_{k=1}^{\infty}A_k\sin\frac{k\pi x}{l} \tag{6.96}$$

and $v_0(x)=\sum_{k=1}^{\infty}B_k w_k\sin\frac{k\pi x}{l}.$ (6.97)

Multiplying (6.96) and (6.97) by $\sin\frac{k\pi x}{l}$ and integral between the limits x=0 to $x = \ell$ we have

$$A_k = \int_0^l y_0(x)\sin\frac{k\pi x}{l}dx, \tag{6.98}$$

and $$B_k = \frac{2}{w_k l}\int_0^l v_0(x)\sin\frac{k\pi x}{l}dx. \tag{6.99}$$

The coordinates q_k defined by equation (6.83) are called normal coordinates for the vibrating string. Each normal coordinate represents one normal mode of vibration. It is clear from (6.89) that the Lagrangian may be represented by the sum of terms, each term containing one normal coordinate or one degree of freedom. Thus by the use of normal coordinate a problem of many degrees of freedom is sub-divided into separate problems, one for each degree of freedom.

6.3 Special Theory of Relativity

At the end of the nineteenth century, the physics community had two incompatible descriptions of phenomena, Newtonian mechanics and Maxwellian electromagnetic theory. Newtonian mechanics assumed that all inertial frames were equivalent, while Maxwell's wave equations gave a universal speed of light that was the same in all inertial frames. Albert Einstein developed the special theory of relativity to replace Newtonian mechanics with a theory that was consistent with electromagnetic theory.

In Newtonian mechanics, a set of well-verified laws applies in an inertial frame of reference defined by the first law. Any frame moving at constant velocity with respect to an inertial frame is also an inertial frame. Consider frames denoted by S and S' with (t, x, y, z) and (t', x', y', z') the coordinates in S and S' respectively. ...hout loss of generality, we assume the coordinates axes are aligned, x along x' and so on. Let S' be moving ...ative to S in the positive x-direction at a speed v, as show in Figure 7.1

Nowtonian mechanics assumes the spacetime coordinates in S are related to those in S' by the simple expressions

$$\left.\begin{aligned} t' &= t \\ x' &= x - vt \\ y' &= y \\ z' &= z. \end{aligned}\right\} \tag{7.1}$$

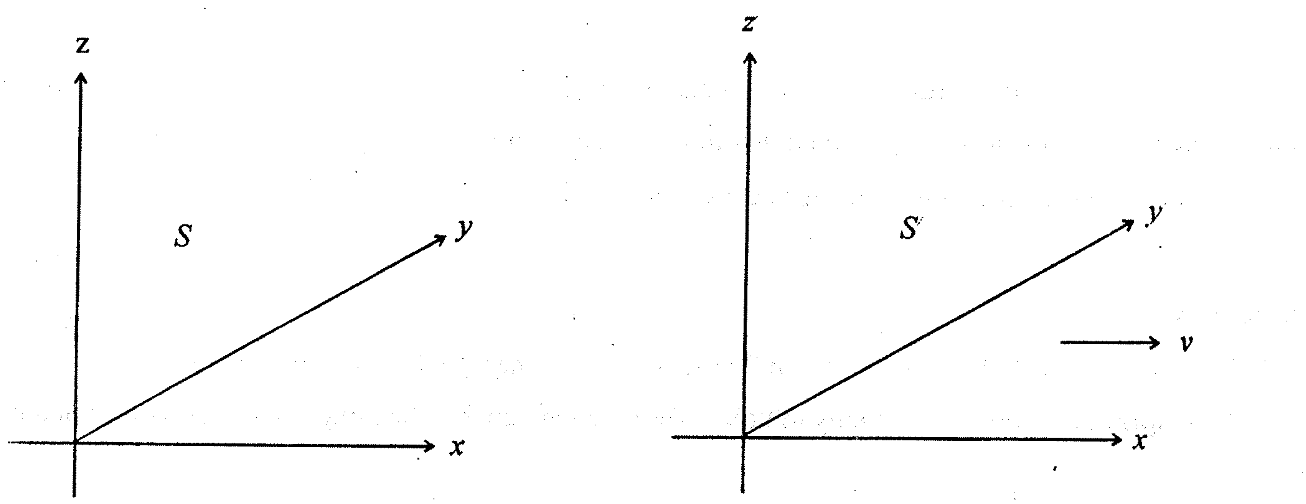

Fig. 7.1 : Galilean transformation from S to S' by a velocity v in the positive x-direction.

Transformations of this type are called Galilean transformations. Under this assumption, it follows that Newton's second law,

$$\vec{F} = \frac{d}{dt}(\vec{p}),$$

relating the applied force, $\vec{F}$ and the momentum, $\vec{p}$ remains invariant, and

$$\vec{F} = \vec{F}', \ t = t' \text{ and } \vec{p} = \vec{p}'. \qquad (7.2)$$

The time in both the S and S' frames is assumed to be $(t = t')$. The Newtonian world view is that the universe consists of three spatial directions and one time direction. All observers agree on the time direction upto ..ssible choice of units. Under these assumptions, there are no universal velocities. It $\vec{u}$ and $\vec{u}'$ are the velocities .. particle as measured in two frames moving with relative velocity $\vec{v}$ as shown by Figure 7.1, then

$$\vec{u}' = \vec{u} - \vec{v}. \qquad (7.3)$$

Maxwell's electromagnetic equations, on the other hand, have a universal constant (denoted by c), which is interpreted as the speed of light. Since this is inconsistent with Newtonian mechanics, either Newtonian or Maxwellian mechanics would have to be modified. After carefully thinking about how the universe would appear to an observer traveling at the speed of light, Albert Einstein decided that Maxwell's equations are correct to all inertial observers and the assumed transformations for Newtonian mechanics are incorrect. The correct transformations make the speed of light the sance to all inertial observers.

Basic postulates :

There are two basic postulates used in special theory of relativity. One of them is the extension of the conclusion drawn from Newtonian mechanics and is called the principle of relativity, while the second is an experimental fact and is called the principle of constancy of velocity of light.

Postulates :

1. **Principle of relativity :** All the laws of Physics have the <u>same form</u> for all <u>inertial frames</u>.
2. **Principle of constancy of velocity of light :** The speed of light has the <u>same value</u> in every inertial frame.

6.3.1 Lorentz transformations

Let *S* and *S′* be inertial systems, the latter is moving with velocity *v* relative to former along positive direction of *x*-axis. Let there be two observers *O* and *O′* situated at the origins of *S* and *S′* respectively. Let us suppose that the axes of two systems coincide at $t=t'=0$. Let us further suppose that a light signal spreads out from the origin at $t=0$. When the same signal reaches at *P*, let the positions and times measured by observers *O* and *O′* be (x, y, z, t) and (t', x', y', z') respectively. The light pulse produced at $t=0$ will spread out as a sphere whose radius will increase with speed *c*. Then the equation of the spherical surface when the pulse reaches at *P*, relative to *O*, will be

$$x^2 + y^2 + z^2 = c^2t^2$$

or, $$x^2 + y^2 + z^2 - c^2t^2 = 0. \tag{7.4}$$

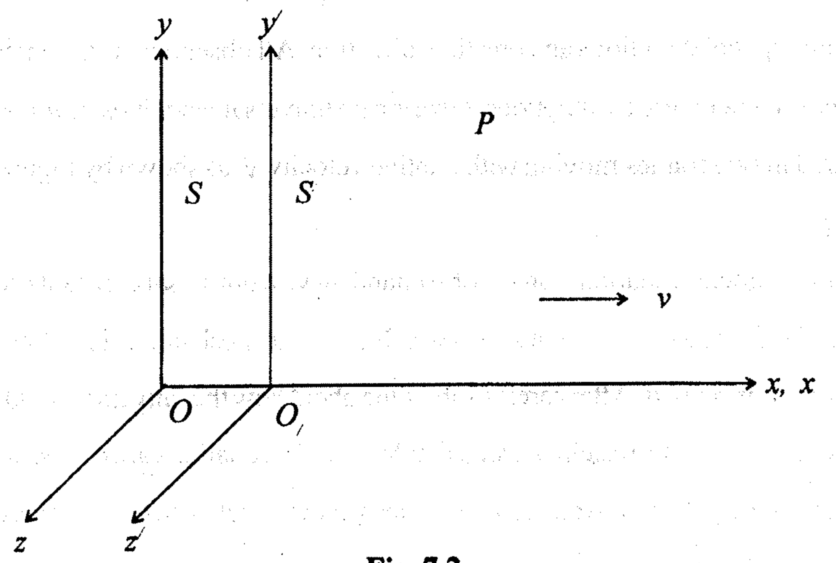

Fig. 7.2

Similarly, the equation of spherical surface relative to O' is

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

or, $$x'^2 + y'^2 + z'^2 - c^2 t'^2 = 0. \tag{7.5}$$

It may be noted that c is not primed since according to second postulate it is constant in both the systems.

We know that the velocity of S' is only along x-axis, therefore from symmetry

$$\left.\begin{aligned} y' &= y \\ z' &= z \end{aligned}\right\}. \tag{7.6}$$

Using (7.6), we get from (7.4) and (7.5)

$$x^2 - c^2 t^2 = x'^2 - c^2 t'^2. \tag{7.7}$$

Let the transformation equations relating to x and x' be

$$x' = k(x - vt), \tag{7.8}$$

where k is independent of x and t.

The reasons for consider this relation are (*i*) the transformations must be linear and simplest, (*ii*) it must reduce to Galilean transformations for low velocities, because Galilean transformations are correct for low velocities.

Equation (7.8) in sufficiently general to satisfy above conditions. Since the motion is relative we may assume that the system S is moving relative to S' with velocity $-v$ along positive x-axis, therefore we may write

$$x = k'(x' + vt'). \tag{7.9}$$

Substituting the value of x' from (7.8) in (7.9), we get

$$x = k'\left[k(x - vt) + vt'\right]$$

or, $$\frac{x}{k'} = kx - kvt + vt'$$

or, $$vt' = \frac{x}{k'} - kx + kvt$$

$$= k\left[\frac{x}{kk'} + vt - x\right]$$

or, $$t' = \frac{k}{v}\left[\frac{x}{kk'} + vt - x\right]$$

$$= k\left[t - \frac{x}{v}\left(1 - \frac{1}{kk'}\right)\right]. \tag{7.10}$$

Putting the value of x' from (7.8) and t' from (7.10) in equation (7.7), we get

$$x^2 - c^2t^2 = k^2(x - vt)^2 - c^2k^2\left[t - \frac{x}{v}\left(1 - \frac{1}{kk'}\right)\right]^2$$

or, $$x^2 - c^2t^2 - k^2\left(x^2 - 2vxt + v^2t^2\right) + c^2k^2\left[t - \frac{x}{v}\left(1 - \frac{1}{kk'}\right)\right]^2 = 0$$

or, $$x^2 - c^2t^2 - k^2\left(x^2 - 2vxt + v^2t^2\right) + c^2k^2\left[t^2 - \frac{2xt}{v}\left(1 - \frac{1}{kk'}\right) + \frac{x^2}{v^2}\left(1 - \frac{1}{kk'}\right)^2\right] = 0. \tag{7.11}$$

As the equation is an identity the coefficients of various powers of x and t must vanish separately.

Equating the coefficient of xt in (7.11) to zero, we have

$$2k^2v + c^2k^2\left\{-\frac{2}{v}\left(1 - \frac{1}{kk'}\right)\right\} = 0$$

or, $$2k^2v - \frac{2c^2k^2}{v}\left(1 - \frac{1}{kk'}\right) = 0$$

or, $$c^2 - \left(c^2 - v^2\right)kk' = 0. \tag{7.12}$$

Equating the coefficient of t^2 in (7.11) to zero, we get

$$-c^2 - k^2v^2 + c^2k^2 = 0$$

or, $$c^2 - \left(c^2 - v^2\right)k^2 = 0. \tag{7.13}$$

Comparing (7.12) and (7.13), we get

$$k = k'. \tag{7.14}$$

Therefore, equation (7.13) gives

$$k^2 = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - v^2/c^2}$$

or, $$k = \frac{1}{\sqrt{\left(1 - v^2/c^2\right)}}. \tag{7.15}$$

Using (7.14), equation (7.10) becomes

$$t' = k\left[t - \frac{x}{v}\left(1 - \frac{1}{k^2}\right)\right] = k\left[t - \frac{x}{v}\left\{1 - \left(1 - \frac{v^2}{c^2}\right)\right\}\right] \text{ (Using (7.15))}$$

$$= k\left[t - \frac{vx}{c^2}\right]. \tag{7.16}$$

Substituting the value of k from (7.15) in (7.8) and (7.16), we get

$$\left.\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}\right\} \tag{7.17}$$

Combining (7.6) and (.7.17) and using standard notations

$$\left.\begin{aligned} \beta &= \frac{v}{c} \\ \gamma &= \frac{1}{\sqrt{1 - v^2/c^2}} \end{aligned}\right\} \tag{7.18}$$

we get

$$\left.\begin{aligned} x' &= r(x + \beta ct) \\ y' &= y \\ z' &= z \\ t &= \gamma\left(t + \frac{\beta x}{c}\right) \end{aligned}\right\} \tag{7.19}$$

These are called <u>Lorentz transformations</u> of space and time.

In deriving the above transformations we have assumed that system S' is moving with velocity v relative to S along positive x-axis, but if we state that system S is moving with velocity $-v$ relative to S along positive x-axis, then in above transformations we may interchange x and x', y and y', z and z', t and t' respectively and obtain

$$\left.\begin{aligned} x &= \gamma(x' + \beta ct') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{\beta x'}{c}\right) \end{aligned}\right\} \tag{7.20}$$

These transformations are known as the inverse Lorentz transformations.

It may be observed that if $v \to 0, \gamma \to 1$, then the Lorentz transformations approach Galilean transformations.

Example 6.3.1. Calculate the length of the rod moving with velocity 0.8C. Given proper length of the rod is 100 cm.

Solution : If l is the length of the rod when it is moving with velocity v, then

$$l = l_0\sqrt{\left(1-\frac{v^2}{c^2}\right)},$$

where l_0 is the actual length of the rod.

Given l_0 = proper length = 100cm, v=0.8c.

$$\therefore l = 100\sqrt{1-\left(\frac{0.8c}{c}\right)^2} = 60cm.$$

Example 6.3.2 : Prove that the four dimensional volume element $dx\,dy\,dz\,dt$ is invariant under Lorentz transformations.

Solution : According to Lorentz transformations

$$dx' = dx\sqrt{\left(1-{v^2}/{c^2}\right)}$$

$$dy' = dy$$

$$dz' = dz$$

and $$dt' = \frac{dt}{\sqrt{\left(1-v^2/c^2\right)}}$$

Therefore, the volume element in system S' is $dx'dy'dz'dt'$

$$= dx\sqrt{\left(1-\frac{v^2}{c^2}\right)}\,dy\,dz\,\frac{dt}{\sqrt{\left(1-v^2/c^2\right)}} = dx\,dy\,dz\,dt$$

= volume element in system S.

That is, four dimensional volume element is invariant under Lorentz transformations.

6.3.2 Equation of force in relativistic mechanics, variation of mass with velocity

According to the Newton's second law, the force acting on a body is given by

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}), \tag{7.21}$$

where $\vec{p}$ is the linear momentum, m is the mass and $\vec{v}$ is the velocity of the body.

In order to find a relation between mass and velocity consider the hypothetical experiment first devised by Tolman and Lewis. Let there be two systems S and S', the latter moving with velicity v relative to former along the positive x-axis. Let there be two similar elastic balls A and B placed in system S and S' respectively. Let the velocity of the ball A be v along positive y-axis and that of B along negative y'-axis.

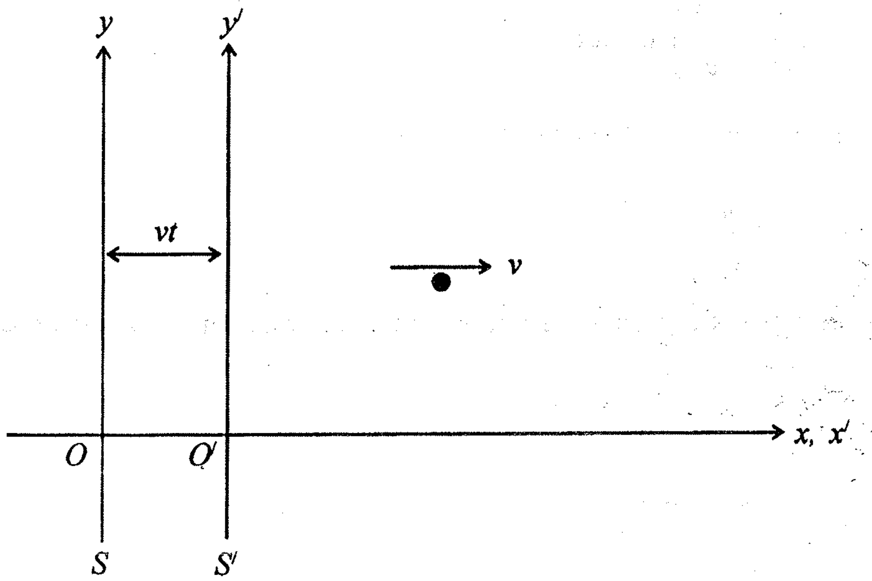

Fig. 7.2

Let the balls be such that their mass are equal when measured in the same system. Let the velocities of the balls and that of frame S' be such that the balls collide when the line of their centres is along y-axis, so that the x and z components of velocities do not change as a result of impalt. If u_{ax}, u_{ay} and u_{az} are the components of velocity of the ball A along x, y and z axes respectively as observed from system S before collision, we have

$$u_{ax} = 0, u_{ay} = u \text{ and } u_{az} = 0.$$

Also, $u_{bx} = v, u_{by} = -u\sqrt{\left(1 - \frac{v^2}{c^2}\right)}, u_{bz} = 0,$

[Using Lorentz transformation]

where u_{bx}, u_{by}, u_{bz} are the velocity components of ball as observed in *S* before collision.

Thus the resultant momentum of the whole system before collision as observed from system *S* is

$$m_0 u \vec{j} + mv\vec{i} - mu\sqrt{\left(1-\frac{v^2}{c^2}\right)}\vec{j} \tag{7.22}$$

where *m* is the mass of the particle moving with velocity *v* while m_o, the mass of the particle at rest.

After impact each ball reverses its *y* components of velocity.

Therefore, the components of velocities of both the balls after impact as observed from *S* are given by

$$w_{ax} = 0, w_{ay} = -u, w_{1z} = 0$$

and $$w_{bx} = v, w_{by} = u\sqrt{\left(1-\frac{v^2}{c^2}\right)}, w_{bc} = 0.$$

Thus the resultant momentum from the system *S* after impact is

$$-m_0 u \vec{j} + mv\vec{i} + mu\sqrt{\left(1-\frac{v^2}{c^2}\right)}\vec{j}. \tag{7.23}$$

According to the principle of conservation of momentum, i.e., momentum before impact is equal to the momentum after impact.

That is,

$$m_0 u \vec{j} + mv\vec{i} - mu\sqrt{\left(1-\frac{v^2}{c^2}\right)}\vec{j}$$

$$= -m_0 u \vec{j} + mv\vec{i} + mu\sqrt{\left(1-\frac{v^2}{c^2}\right)}\vec{j}$$

or, $$2m_0 u = 2mu\sqrt{\left(1-v^2/c^2\right)}$$

or, $$m = \frac{m_0}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}. \tag{7.24}$$

Thus the law of conservation of momentum leads to a very important conclusion, i.e., mass of a body increases with increase of velocity.

The momentum is $\vec{p} = \dfrac{m_0\vec{v}}{\sqrt{\left(1-\dfrac{v^2}{c^2}\right)}}$ (7.25)

and the force

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left\{\frac{m_0\vec{v}}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}\right\}$$

$$= m_0\frac{d}{dt}\left\{\frac{\vec{v}}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}\right\}. \tag{7.26}$$

Equation (7.26) represents the required equation of the force in relativistic mechanics.

Equation (7.26) is invariant under Lorentz transformation.

6.3.3 Equation of energy in relativistic mechanics : Mass energy relation

The force F is given by Newton's second law as

$$\vec{F} = \frac{d}{dt}(m\vec{v}). \tag{7.27}$$

According to the definition of K.E. we know that K.E. of a moving body in equal to work done by the force that imparts the velocity to the body from rest, therefore, K.E

$$T = \int_{v=0}^{v} F ds = \int_0^v \frac{d}{dt}(mv)ds$$

$$= \int_0^v \frac{d}{dt}(mv)\frac{ds}{dt}dt = \int_0^v v\frac{d}{dt}(mv)dt = \int_0^v vd(mv). \tag{7.28}$$

According to the theory of relativity, from (7.24) we have

$$m = \frac{m_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

Then (7.28) becomes

$$T = \int_0^v v\,d\left\{\frac{m_0 v}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}\right\}$$

$$= m_0 \int_0^v v\left[\frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} + \frac{v^2}{c^2\left(1 - \frac{v^2}{c^2}\right)^{3/2}}\right]dv$$

$$= m_0 \int_0^v \frac{v}{\left(1 - v^2/c^2\right)^{3/2}}\,dv. \tag{7.29}$$

Putting $1 - \frac{v^2}{c^2} = z$. Then $-\frac{2v}{c^2}dv = dz$.

$$\therefore T = -\frac{m_0 c^2}{2}\int_1^{1-v^2/c^2} z^{-3/2}dz = c^2\left[\frac{m_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} - m_0\right]$$

$$= c^2(m - m_0). \tag{7.30}$$

This is the K.E. equation in relativistic mechanics.

For the low velocity (7.30) reduces to ordinary expression for K.E. i.e.,

$$T = \frac{1}{2}m_0 v^2, \text{ for } v << c. \tag{7.31}$$

Equation (7.30) represents that the K.E. of a moving body is equal to gain in mass due to its motion times c^2.

This suggests that the increase in energy may be considered as the actual case of the increase in mass. Then we may suppose that the rest mass m_0 is due the presence of an internal store of energy of amount m_0c^2, this is called rest energy of the body.

Total energy

$$E = \text{K.E.} + \text{rest energy}$$

$$= (m - m_0)c^2 + m_0c^2$$

or, $$E = mc^2. \tag{7.32}$$

This is Einstein's famous mass-energy relations and states a universal equivalence between mass and energy.

Example 6.3.3. A particle of rest mass *m* is moving with a velocity 0.9c, calculate (*i*) its relativistic mass, (*ii*) its K.E., (*iii*) why it is incorrect to use expression $\frac{1}{2}m_0v^2$ for K.E. in this case?

Solution : The relativistic mass

$$m = \frac{m_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

Given $v = 0.9$c.

$$\therefore m = \frac{m_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = \frac{m_0}{\sqrt{\left\{1 - \left(\frac{0.9c}{c}\right)^2\right\}}} = 2.3m_0.$$

(ii) The K.E. is given by

$$T = (m - m_0)c^2 = (2.3m_0 - m_0)c^2 = 1.3m_0c^2.$$

(iii) The expression $\frac{1}{2}m_0v^2$ is correct only for low velocities which may be verified as follows:

$$T = (m - m_0)c^2 = \left[\frac{m_0}{\sqrt{(1 - v^2/c^2)}} - m_0\right]c^2$$

$$=\left\{\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}-1\right\}m_0c^2$$

$$=\left(1+\frac{v^2}{2c^2}-1\right)m_0c^2 \text{ for } v<<c$$

$$=\frac{1}{2}m_0v^2.$$

In this case we do not have $v<<c$, therefore it is incorrect to use $\frac{1}{2}m_0v^2$.

6.3.4. The Lagrangian formulation of relativistic mechanics

The Newton's equation of motion is generalised for special relativity, we now establish a Lagrangian formulation of the resulting relativistic mechanics. Here we find a Lagrangian that leads to the relativistic equations of motion in terms of the coordinates of some particular inertial system. Within these limitations there is no great difficulty in constructing a suitable Lagrangian. It is true that the method of deriving the Lagrangian from D'Alembert's principle, will not work here. While the principle itself remains valid in any given Lorentz frame, the derivation there is based on $\vec{p}_i = m_i\vec{v}_i$, which is no longer valid relativistically. But, we may also approach the Lagrangian formulation from the alternative route of Hamilton's principle and attempt simply to find a function L for which the Euler-Lagrange equations, as obtained from the variational principle

$$\delta I = \delta\int_{t_1}^{t_2} L\,dt = 0, \tag{7.33}$$

agree with the known relativistic equations of motion.

A suitable relativistic Lagrangian for a single particle acted on by conservative forces independent of velocity would be

$$L = -mc^2\sqrt{1-\beta^2} - V, \tag{7.34}$$

where V is the potential, depending only upon position and $\beta^2 = v^2/c^2$, with v the speed of the particle in the Lorentz frame under consideration. That this is the correct Lagrangian can be shown by demonstrating that the resultant Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right)-\frac{\partial L}{\partial x_i}=0, \tag{7.35}$$

Since the potential is velocity independent v_i occurs only in the first term of (7.34) and therefore,

$$\frac{\partial L}{\partial v_i}=\frac{mv_i}{\sqrt{1-\beta^2}}=P_i. \tag{7.36}$$

The equations of motion derived from the Lagrangian (7.34) are then

$$\frac{d}{dt}\left(\frac{mv_i}{\sqrt{1-\beta^2}}\right)=-\frac{\partial V}{\partial x_i}=F_i.$$

Note that the Lagrangian is no longer $L = T-V$ but that the partial derivative of L with velocity is still the momentum.

We can readily extend the Lagrangian (7.34) to systems of many particles and change from cartesian to any desired set of generalised coordinates q. The canonical momenta, P, will still be defined by

$$P_i=\frac{\partial L}{\partial \dot{q}_i} \tag{7.37}$$

so that the connection between cyclic coordinates and conservation of the corresponding momenta remains just as in the nonrelativistic theory. If L does not contain the time explicitly, there exists a constant of the motion

$$h=\dot{q}_i\,P_i-L. \tag{7.38}$$

However, the identification of h with the energy for, say, a Lagrangian of the form of (7.34) cannot proceed along the same route. Note that L in (7.34) is not at all a homogeneous function of the velocity components. The direct evaluation of (7.38) from (7.34) shows that in this case h is indeed the total energy :

$$h=\frac{mv_i^2}{\sqrt{1-\beta^2}}+mc^2\sqrt{1-\beta^2}+V$$

which, on collecting terms reduces to

$$h=\frac{mc^2}{\sqrt{1-\beta^2}}+V=T+V=E. \tag{7.39}$$

The quantity h is thus again seen to be the total energy E, which is therefore a constant of the motion under these conditions.

Application : Motion under a constant force

It will be no loss of generality to take the x-axis as the direction of the constant force. The Lagrangian is therefore

$$L = -mc^2\sqrt{1-\beta^2} - max, \tag{7.40}$$

where $\beta = \dot{x}/c$ and a is a constant magnitude of the force per unit mass. From (7.40) the equation of motion is easily found to be

$$\frac{d}{dt}\left(\frac{\beta}{\sqrt{1-\beta^2}}\right) = \frac{a}{c}.$$

The first integration leads to

$$\frac{\beta}{\sqrt{1-\beta^2}} = \frac{at+\alpha}{c}$$

or, $\beta = \dfrac{at+\alpha}{\sqrt{C^2+(at+\alpha)^2}}$, where α is a constant of integration. A second integration over t from 0 to t and x from x_0 to x,

$$x - x_0 = c\int_0^t \frac{(at'+\alpha)dt'}{c^2(at'+\alpha)^2}$$

leads to the complete solution

$$x - x_0 = \frac{c}{a}\left[\sqrt{c^2+(at+\alpha)^2} - \sqrt{c^2+\alpha^2}\right]. \tag{7.41}$$

If the particle starts at rest from the origin so that $x_0 = 0$ and $v_0 = 0 = \alpha$, then (7.41) can be written as

$$\left(\dot{x}+\frac{c^2}{a}\right)^2 - c^2t^2 = \frac{c^4}{a^2},$$

which is the equation of a hyperbola in the xt plane. The non-relativistic limit is obtained from (7.41) by considering $(at+\alpha)$ small compared to c.

6.4 Unit Summary

In the first section of this unit we have deduced the equations of motion of a symmetical top using Euler's and Lagrange's equations. We discuss about the steady motion of the top when the axis is vertical and is not vertical.

In the second section, we discussed about the small oscillation about equilibrium and we solved two problems using this concept. The vibration of strings are also studied here.

In the third section we introduced the special theory of relativity. The postulates of its are stated. The expressions for Lorentz transformation are deduced. The force and mass-energy equations for relativistic mechanics are deduced.

6.5 Self Assessment Questions

6.1 A top has an axis of symmetry *OG*, where *G* is the centre of mass, and it spins with the end *O* on a rough horizontal table. The mass of the top is *M* and its moment of inertia about *OG* and any axis through *O* perpendicular to *OG* are *C* and *A* respectively. Initially, *OG* is vertical and the top is set spinning with spin *n* about its axis. It is then slightly displaced. If in the subsequent motion θ is the angle *OG* makes with the vertical and $\dot{\phi}$ is the angular velocity about the vertical, show that

$$A\dot{\phi}\sin^2\theta = Cn(1-\cos\theta) \text{ and}$$

$$A(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) = 2Mgh(1-\cos\theta)$$

where $OG = h$.

Consider the equilibrium configuration of the molecule such that two of its atoms of each of mass *M* are symmetrically placed on each side of the third atom of mass *m*. All three atoms are collinear. Assume the motion along the line of molecules and there being no interaction between the ends atoms. Compute the K.E. and P.E. of the system and discuss the motion of the atoms.

6.3 Find the Lagrange's equation for the vibrating string fixed at end points and solve it.

6.4 Prove that if the length of the string is held constant and number of particles in the string is assumed to be increasing, the equatin of motion approaches

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}.$$

6.5 Show that under one dimensional Lorentz's transformation $x^2 - c^2t^2 = x'^2 - c^2t'^2$.

6.6 State the fundamental postulates of special theory of relativity and deduce the Lorentz transformations.

6.7 Show by direct application of Lorentz transformation that $x^2 + y^2 + z^2 - c^2t^2$ is invariant.

6.8 Show that for low velocities Lorentz transformation approaches to Galilean transformation.

6.9 State and prove force and energy equations in relativistic mechanics.

6.6 Suggested Further Readings

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-IV

Group – B

Module No. - 43

Partial Differential Equations of First Order

1. Partial Differential Equations

In most of the physical problems in science and technology, there involve two or more independent variables, as a result of which the dependent variable becomes a function of more than one variable and possesses partial derivatives with respect to several variables. For example, the process of diffusion in physical chemistry leads to the equalization of concentration u with a single phase and is governed by the laws connecting the rate of flow of the diffusing substance with concentration gradient. As a consequence, the derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial t}$$

will, in general, be non zero. Moreover, higher derivatives of the types

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^3 u}{\partial t^2 \partial z}, \text{etc.}$$

may be of physical significance in a particular problem. Thus u is a function of x, y, z and t, i.e. u = u (x, y, z, t). Hence, for such a situation, we can obtain a relation between the derivatives of u in the form

$$F\left(x, y, z, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^2 u}{\partial z \partial t}, \ldots\right) = 0 \qquad (1.1)$$

Such as equation relating partial derivatives is known as *partial differential equation*.

Some well known examples are :-

Laplace's equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Heat conduction or diffusion equation: $\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ (1.2)

Wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

Korteweg de Vries (KDV) equation: $\frac{\partial u}{\partial t} + cu\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$

For two independent variables x and y, if z is the dependent variable, i.e. if z = z(x,y), then usually we adopt the following notations:

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2} \tag{1.3}$$

The higher order derivative occurring in a partial differential equation is its *order*. For example, the order of the first three equations of (1.2) is 2, but it is 3 for the last equation.

2. Origin of First-Order Partial Differential Equation

In the study of a physical or social phenomenon, partial differential equations originate in many ways. Let us now demonstrate how partial differential equation of first order occur.

Case I : Elimination of arbitrary constants

Consider a relation f(x,y,z,a,b)=0 involving two independent variables x, y and one dependent variable z such that z = z(x,y), a and b being arbitrary constants. Differentiation this equation partially with respect to x and y respectively, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0$$

Eliminating a and b from these two relations and the given relation, we get an equation of the form

$$F(x, y, z, p.q) = 0 \tag{2.1}$$

which is a first-order partial differential equation.

As an example, consider the equation $(x-a)^2 + (y-b)^2 + z^2 = 1$, where a and b are arbitrary constants. Differentiating thing equation partially with respect to x and y we get respectively

$$z(x-a) + 2zp = 0 \text{ and } z(y-b) + 2zq = 0 \Rightarrow a = x + zp, b = y + zq$$

Substituting the values of a and b in the equation we get $z^2\left(p^2+q^2+1\right)=1$ which is the desired first-order equation.

Case II : Elimination of functions

Suppose that two given functions u = u(x,y,z) and v = v(x,y,z), where z=z(x,y), are connected by the relation ϕ(u,v)=0. Differentiating this relation partially with respect to x and y we obtain

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

and $$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0$$

respectively. Eliminant of $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ from these relations leads to

$$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

or, $$\frac{\partial(u,v)}{\partial(y,z)}p+\frac{\partial(u,v)}{\partial(z,x)}q=\frac{\partial(u,v)}{\partial(x,y)}$$

or, $$Pp+Qq=R \tag{2.2}$$

where, $$P=\frac{\partial(u,v)}{\partial(y,z)},\ Q=\frac{\partial(u,v)}{\partial(z,x)},\ R=\frac{\partial(u,v)}{\partial(x,y)} \tag{2.3}$$

The first-order equation (2.2) is called *Lagrange's equation of the first order*.

As an example, consider the equation $z=f\left(\frac{xy}{z}\right)$. Differentiating this equation partially with respect to x and y, we get respectively

$$p=f'\left(\frac{xy}{z}\right)\left(\frac{y}{z}-\frac{xy}{z^2}p\right) \text{ and } q=f'\left(\frac{xy}{z}\right)\left(\frac{x}{z}-\frac{xy}{z^2}q\right)$$

Eliminating $f'\left(\frac{xy}{z}\right)$ between these two relations we obtain $xp(z-yq)=qy(z-px)$ which is a first-order partial differential equation.

3. Classification of First-Order Partial Differential Equations

First-order partial differential equations are classified acording to the nature given below:

(a) Linear equation: A first-order partial differential equation is said to be linear, if it is linear in x, y and z, i.e, if it is of the form.

$$P(x,y)p+Q(x,y)q=R(x,y,z)+S(x,y) \tag{3.1}$$

For example, the equation $px(x+y)-qy(x+y)=(y-x)z+2(y^2-x^2)$ is linear.

(b) Semi-linear equation: If a first-order partial differential equation is linear in p and q and the coefficients of p and q are functions of x and y only, i.e. if the equation is of the form

$$P(x,y)p+Q(x,y)q=R(x,y,z), \tag{3.2}$$

then the equation is called a semi-linear first-order partial differential equation.

For example, the equation $y^2p-xyq=x(z-2y)$ is a semi-linear partial differential equation.

(c) Quasi-linear equation: A first-order partial differential equation is said to be quasi-linear, if it is linear in p and q, i.e. if the equation is of the form

$$P(x,y,z)p+Q(x,y,z)q=R(x,y,z) \tag{3.3}$$

For example, the equation $x(z-2y^2)p+y(z-y^2-2x^2)q=z(z-y^2-2x^2)$ is quasi-linear.

(d) Nonlinear equation: A partial differential equation which does not belong to the above three types is called non-linear equation.

For example, the equation $2xz-px^2-2qxy+pq=0$ is nonlinear first order partial differential equation.

Existence of Solutions: Cauchy-Kowalewski Theorem

The existence of solution of a partial differential equation is not guaranteed. However, if the equation satisfies a set of conditions (to be stated later on), then its solution does exist. Before the discussions of the existence of the solution, we first define a solution and its various types associated with a partial differential equation.

We have already seen in Section 2 that a relation of the form $f(x,y,z,a,b)=0$ leads to a first-order partial differential equation. Such a relation containing two arbitrary constants a and b is a *solution of* that first-order equation and is called a *complete solution* or *complete integral* of it.

On the other hand, any relation of the type $f\{u(x,y,z),v(x,y,z)\}=0$ providing a solution of the first-order partial differential equation is known as a *general solution* or *general integral* of the equation. We can also obtain the general solution as the locus of a parametric family of curves, called *characteristics* of the envelope of the family $f(x,y,z,a,\Psi(a))=0$, where b is supposed to be a function of a. The general solution of a first-order partial differential equation is a parametric family of surfaces, called *integral surfaces*.

Eliminating the arbitrary constants from the complete integral, we obtain the *singular solution* or *singular integral*. Thus, if the equation $F(x,y,z,p,q)=0$ possesses the complete solution $F(x,y,z,a,b)=0$, then the a, b-eliminant from the relations $f=0, \frac{\partial f}{\partial a}=0, \frac{\partial f}{\partial b}=0$ is the singular solution. However, singular solution can also be obtained by eliminating p and q from the differential equation $F(x,y,z,p,q)=0$ itself and $\frac{\partial F}{\partial p}=0, \frac{\partial F}{\partial q}=0$.

Example 4.1: The equation $z^2(p^2+q^2+1)=c^2$ has a complete integral of the form $(x-a)^2+(y-b)^2+z^2=c^2$, where a and b are constants. Find the singular and a generation integral assuming $b=a$.

Solution : Differentiating the relation $(x-a)^2+(y-b)^2+z^2=c^2$ partially with respect to a and b we get respectively $-2(x-a)=0, -2(y-b)=0 \Rightarrow a=x, b=y$. Thus eliminating a and b from the above relation gives the singular integral as $z=\pm c$.

When $z=\pm c,\ p=\frac{\partial z}{\partial x}=0,\ q=\frac{\partial z}{\partial y}=0$ which satisfy the equation $z^2(p^2+q^2+1)=c^2$.

Again making $b=a$ in $(x-a)^2+(y-b)^2+z^2=c^2$ and then differentiating w.r.t. a, we get $-2(x-a)-2(y-a)=0 \Rightarrow a=\frac{1}{2}(x+y)$. Eliminating a, the general solution is $(x-y)^2+2z^2=2c^2$.

Cauchy problem:

In order to find the existence of solution of a first-order partial differential equation, the conditions to be satisfied are given by *Cauchy problem* which we state as follows:

Let us suppose that

(a) the functions $x_0(\mu), y_0(\mu), z_0(\mu)$ along with their first partial derivatives with respect to μ are continuous in the interval $M: \mu_1 < \mu_1 < \mu_2$ and

(b) the function $F(x,y,z,p,q)$ is continuous in *x, y, z, p, q* in a region *U* of the *xyzpq* space.

Then it is required to establish the existience of a function $\phi(x,y)$ possessing the following properties:

(i) $\phi(x,y)$ and its partial derivatives w.r.t. *x* and *y* are continuous functions of *x* and *y* in a region *R* of the *xy*-space.

(ii) the point $\{x, y, \phi(x,y), \phi_x(x,y), \phi_y(x,y)\} \in U$ and

$F\{x, y, \phi(x,y), \phi_x(x,y), \phi_y(x,y)\} = 0, \ \forall x, y \in R$,

(iii) the point $\{x_0(\mu), y_0(\mu)\} \in R$ and $\phi\{x_0(\mu), y_0(\mu)\} = z_0, \forall \mu \in M$.

Cauchy problem can be stated geometrically as: To prove the existence of a surface $z = \phi(x, y)$ passing through the curve Γ with parametric equations

$$x = x_0(\mu), y = y_0(\mu), z = z_0(\mu) \tag{4.1}$$

and at every point of which the direction $(p, q, -1)$ of the normal is such that

$$F(x, y, z, p, q) = 0 \tag{4.2}$$

We have to make some other assumptions regarding the function F and the curve Γ to prove the existence of a solution of (4.2) passing through the curve Γ. The existence theorem depends on the nature of these assumptions. We now state the existence theorem (without proof) which is due to S.Kowalewski and is known as Cauchy-Kowalewski therorem.

Cauchy - Kowalewski theorem:

Suppose a function g(y) along with its derivatives are continuous for $|y - y| < \delta, x_0$ *be a given number and* $z_0 = g(y_0), q_0 = g'(y_0)$. *Also we assume that the function* $f(x, y, z, q)$ *and all its partial derivatives are continuous in the region* $S : |x - x_0| < \delta, |y - y_0| < \delta, |z - z_0| < \delta$. *Then there exists a unique function* $\phi(x, y)$ *such that*

(i) $\phi(x, y)$ *and all its partial derivatives are continuous in a region* $R : |x - x_0| < \delta_1$, $|y - y_0| < \delta_2$;

(ii) $z = \phi(x, y)$ *is a solution of the equation* $\frac{\partial z}{\partial x} = f\left(x, y, z, \frac{\partial z}{\partial y}\right), \forall x, y \in R$ *and*

(iii) $\phi(x_0, y) = g(y), \forall y$ *of in the interval* $|y - y_0| < \delta_1$.

5. Quasi-linear Equations of First Order

We have seen in Section 2 that Lagrange's equation is given by

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z) \tag{5.1}$$

This equation can be generalised to *n* independent variable as

$$P_1 p_1 + P_2 p_2 + ... + P_n p_n = R \tag{5.2}$$

Where each of $P_i, (i=1,2,...,n)$, and R is a function of n independent variables $x_1, x_2, ..., x_n$ and the dependent variable z and $p_i = \frac{\partial z}{\partial x_i}, (i=1,2,...,n)$.

The method of solving the equation (5.1) lies on the following theorem:

Theorem 5.1: *The equation $Pp+Qq=R$ has the general solution $\phi(u,v)=0$ where ϕ is an arbitrary function of u and v and $u(x,y,z)=c_1$, $v(x,y,z)=c_2$ are the solutions of the equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, c_1 and c_2 being arbitrary constants.*

Proof. Since $u(x,y,z)=c_1$ is a solution of $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, so this equation and the equation $ux\,dx+uy\,dy+uz\,dz=0$ are compatible to each other so that

$$Pu_x+Qu_y+Ru_z=0$$

Similarly, we have $Pv_x+Qv_y+Rv_z=0$.

Solving for P, Q and R, we obtain

$$\frac{P}{\frac{\partial(u,v)}{\partial(y,z)}}=\frac{Q}{\frac{\partial(u,v)}{\partial(z,x)}}=\frac{R}{\frac{\partial(u,v)}{\partial(x,y)}} \tag{5.3}$$

Now we have already seen in Section 2 that the relation $\phi(u,v)=0$ leads to the partial differential equation

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)}$$

Substituting from (5.3), we find that $\phi(u,v)=0$ is a solution of the equation $Pp+Qq=R$.

The equation $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ is known as *Lagrange's auxiliary equations*.

Geometrical interpretation of the equation of $Pp+Qq=R$:

Noting that the direction cosines of the normal to the surface $z=f(x,y)$ at a point are proportional to $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1$ i.e. $p, q, -1$, Lagrange's equation (5.1) can be written as .

$$Pp+Qq+R(-1)=0 \tag{5.4}$$

Thus the normal at any point to the surface $z = f(x, y)$ is perpendicular to a straight line with direction cosines in the ratio $P:Q:R$. On the other hand, the equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ represent a family of curves, the tangent at any point of which has direction cosines in the ratio $P:Q:R$. Also the relation $\phi(u, v) = 0$ where $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two particular integrals of the equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, represents a family of surfaces passing through such curves. Now since a curve of this family through any point on the surface lies entirely on the surface, so the normal to this surface at that point is at right angles to the tangent at the point to the curve. In other words, it is perpendicular to the straight line with direction cosines proportional to $P:Q:R$.

Since the equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ and $Pp + Qq = R$ define the same set of surfaces, so they are equivalent and, therefore, the relation $\phi(u, v) = 0$ is an integral of the equation $Pp + Qq = R$ in which $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two independent solutions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ and ϕ is an arbitrary function of u and v.

The method of solving the general equation (5.2) is given by the following theorem which we state below. The proof the theorem lies along the same lines as in Theorem 5.1.

Theorem 5.2: Let $u_i(x_1, x_2, ..., x_n), (i = 1, 2, ..., n)$, be n independent solutions of the equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = ... = \frac{dx_n}{P_n} = \frac{dz}{R},$$

where each of $P_i, (i = 1, 2, -, n)$ and R is a function of $x_1, x_2, ..., x_n$ and z. Then the relation $\phi(u_1, u_2, ..., u_n) = 0$, here ϕ is arbitrary, is a general solution of the partial differential equation $P_1p_1 + P_2p_2 + ... + P_np_n = R$, where $p_i = \frac{\partial z}{\partial x_i}, (i = 1, 2, ..., n)$.

Example 5.1: Solve $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$

Solution: We rewrite the given equation in the form

$$px(z - 2y^2) + qy(z - y^2 - 2x^3) = z(z - y^2 - 2x^3)$$

Lagrange's auxiliary equations are

$$\frac{dx}{x(z - 2y^2)} = \frac{dy}{y(z - y^2 - 2x^3)} = \frac{dz}{z(z - y^2 - 2x^3)} \quad (1)$$

From the second and third equations of (1) we get $\frac{dy}{y} = \frac{dz}{z} \Rightarrow \frac{y}{z} = c_1$

Also, from (1) we have

$$\frac{dx}{x(z-2y^2)} = \frac{2ydy - dz}{(2y^2 - z)(z - y^2 - 2x^3)}$$

or, $\quad \frac{dx}{x} = \frac{d(y^2 - z)}{y^2 - z + 2x^3}$

or, $\quad \frac{dx}{x} = \frac{dv}{v + 2x^3}$ where $v = y^2 - z$

or, $\quad -\frac{2dv - vdx}{x^2} + 2xdx = 0$

Integrating $-\left(\frac{v}{x}\right) + x^2 = \text{cross} = c_2$ i.e. $-\frac{y^2}{x} + \frac{z}{x} + x^2 = c_2$.

Thus the required solution is $\phi\left(\frac{y}{z}, \frac{z}{x} - \frac{y^2}{x} + x^3\right) = 0$.

Example 5.2: Solve $(t + y + z)\frac{\partial t}{\partial x} + (t + z + x)\frac{\partial t}{\partial y} + (t + x + y)\frac{\partial t}{\partial z} = x + y + z$

Solution. In the given problem, the auxiliary equations are

$$\frac{dx}{t + y + z} = \frac{dy}{t + z + x} = \frac{dz}{t + x + y} = \frac{dt}{x + y + z}$$

$$\Rightarrow \frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(z-t)}{-(z-t)} = \frac{d(x+y+z+t)}{3(x+y+z+t)}$$

The first two terms give on integration $\frac{x-y}{y-z} = c_1$, a constant, the second and third terms lead to $\frac{y-z}{z-t} = c_2$, a constant while the last two terms give $(x+y+z+t)^{1/3}(z-t) = c_3$, a constant.

Hence the required solution is $\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-t}, (x+y+z+t)^{\frac{1}{3}}(z-t)\right) = 0$.

6. Integral Surface Passing Through a Given Curve

We have seen in Section 4 that if the auxiliary equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \tag{6.1}$$

possess solutions $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$, then the solution of the corresponding quasi-linear equation

$$Pp+Qq=R \tag{6.2}$$

is of the form

$$\phi(u,v)=0 \tag{6.3}$$

arising from a relation

$$\phi(c_1,c_2)=0 \tag{6.4}$$

between the constants c_1 and c_2.

Now to find the integral surface passing through a given curve Γ having parametric equations $x=x(t), y=y(t), z=z(t), t$ being a parameter, the solutions $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ must be such that

$$u\{x(t),y(t),z(t)\}=c_1 \text{ and } v\{x(t),y(t),z(t)\}=c_2$$

Eliminating t between these two relations, a relation of the form (6.4) is obtained and then the desired solution will be given by (6.3).

Example 6.1: Find the integral surface of the equation

$$(x-y)y^2p+(y-x)x^2q=(x^2+y^2)z.$$

through the curve $xz=a^3, y=0$

Solution. Here the auxiliary equations

$$\frac{dx}{(x-y)y^2}=\frac{dy}{(y-x)x^2}=\frac{dz}{(x^2+y^2)z}$$

lead from the first two terms on integration $x^3+y^3=c_1$, a constant. Also from the above equations

$$\frac{d(x-y)}{(x-y)(x^2+y^2)}=\frac{dz}{(x^2+y^2)z} \Rightarrow \frac{x-y}{z}=c_2 \text{ a constant.}$$

Thus the general solution is $\phi\left(x^3+y^3,\frac{x-y}{z}\right)=0$.

We take the parametric equations the curve in the form $x = t, y = 0, z = \frac{a^3}{t}$. Then $t^3 = c_1$ and $t^2 = a^3 c_2$, i.e. $c_1^{\frac{1}{3}} = t = a^{\frac{3}{2}} c_2^{\frac{1}{2}} \Rightarrow c_1^2 = a^9 c_2^3$. Hence the required integral surface is $\left(x^3 + y^3\right)^2 = a^9 \frac{(x-y)^3}{z^3}$, i.e. $\left(x^3 + y^3\right)^2 z^3 = a^9 (x-y)^3$.

7. Surfaces Orthogonal to a Given System of Surfaces

Consider a one-parameter family of surfaces

$$F(x,y,z) = c \tag{7.1}$$

c being a parameter. Now the normal at any point (x,y,z) to this surface has direction ratios $(P,Q,R) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$. Also, if the surface $z = f(x,y)$ cuts the system of surfaces (7.1) orthogonally, then its normal at the point (x,y,z) lies along the direction $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$ i.e. $(p,q,-1)$ and is, therefore, perpendicular to the direction (P,Q,R) of the normal at that point. Thus, we have

$$P.p + Q.q + R(-1) = 0, \text{ i.e. } Pp + Qq = R \tag{7.2}$$

Conversely, we note that (7.2) is perpendicular to the normal to the system (7.1) at the same point and, therefore, any solution of the partial differential equation (7.2) is orthogonal to every surface of the system (7.1).

Thus the equation (7.2) represents the general partial differential equation determining surfaces orthogonal to the system (7.1) and these surfaces are generated by integral curves of the equations

$$\frac{dx}{\frac{\partial F}{\partial x}} = \frac{dy}{\frac{\partial F}{\partial y}} = \frac{dz}{\frac{\partial F}{\partial z}} \tag{7.3}$$

Example 7.1: Find the equation of the system of surfaces which cut orthogonally the cones of the system $x^2 + y^2 + z^2 = cxy$.

Solution. We have $F(x,y,z) = \frac{x^2 + y^2 + z^2}{xy} = c$, c being a parameter.

The auxiliary equations are

$$\frac{dx}{\frac{\partial F}{\partial x}}=\frac{dy}{\frac{\partial F}{\partial y}}=\frac{dz}{\frac{\partial F}{\partial z}}\Rightarrow\frac{xdx}{x^2-y^2-z^2}=\frac{ydy}{y^2-z^2-x^2}=\frac{dz}{2z} \qquad (1)$$

From (1), we have $\frac{xdx+ydy}{-2z^2}=\frac{dz}{2z}\Rightarrow x^2+y^2+z^2=c_1$, a constant.

Also the equation (1) gives $\frac{xdx-ydy}{2\left(x^2-y^2\right)}=\frac{dz}{2z}\Rightarrow\frac{x^2-y^2}{z^2}=c_2$, a constant.

Hence the required orthogonal surface is $\phi\left(x^2+y^2+z^2,\frac{x^2-y^2}{z^2}\right)=0$, or $x^2+y^2+z^2=f\left(\frac{x^2-y^2}{z^2}\right)$.

Example 7.2: Find the surface which interest the surface of the system $z(x+y)=c(3z+1)$ orthogonally and which passes through the circle $x^2+y^2=1, z=1$.

Solution. Let $F(x,y,z)=\frac{z(x+y)}{3z+1}=c$, c being a parameter. The auxiliary equations are

$$\frac{dx}{\frac{\partial F}{\partial x}}=\frac{dy}{\frac{\partial F}{\partial y}}=\frac{dz}{\frac{\partial F}{\partial z}}\Rightarrow\frac{dx}{\frac{z}{3z+1}}=\frac{dy}{\frac{z}{3z+1}}=\frac{dz}{\frac{x+y}{(3z+1)^2}}$$

$$\frac{dx}{z}=\frac{dy}{z}=\frac{(3z+1)dz}{x+y} \qquad (2)$$

The first two relations of (2) give $x-y=c_1$, a constant Also from (2) we have

$$\frac{dx+dy}{2z}=\frac{(3z+1)dz}{x+y}\Rightarrow(x+y)d(x+y)=2\left(3z^2+z\right)dz$$

$$\Rightarrow(x+y)^2-\left(4z^3+2z^2\right)=c_2 \text{ a const.}$$

Now the given circle has parametric equations $x=\cos t, y=\sin t, z=1$.

Then $\cos t-\sin t-c_1$ and $(\cos t+\sin t)^2-6=c_2$, i.e. $(\cos t-\sin t)^2=c_1^2$ and $(\cos t+\sin t)^2=6+c_2$. Eliminating t between these two relations we get

$$2=6+c_2+c_1^2,\text{ i.e. } c_1^2+c_2-4=0\Rightarrow(x-y)^2+(x+y)^2-2\left(2z^3+z^2\right)-4=0$$

i.e. $x^2+y^2=2z^3+z^2-2$ is required equation of the orthogonal surface.

8. Compatible Systems

Tar first-order partial differential equations

$$F(x,y,z,p,q)=0 \text{ and } G(x,y,z,p,q)=0 \quad (8.1)$$

are said to be compatible if the solution of any one equation satisfies the other.

We assume the Jacobian $J=\dfrac{\partial(F,G)}{\partial(p,q)}\neq 0$. Then the equations (8.1) can be solved in the form $p=p(x,y,z)$ and $q=q(x,y,z)$.

Now we know that *a necessary and sufficient condition for the Pfaffian differential equation $X.dr=0$ to be integrable is that $X.curl\,X=0$, where $X=(P,Q,R), r=(x,y,z)$, and each of P, Q, R is a function of x,y,z.*

Hence the condition for the equations (8.1) to be compatible is that $dz=pdx+qdy$ must be integrable and, therefore,

$$p\frac{\partial q}{\partial z}-q\frac{\partial p}{\partial z}+\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}=0$$

i.e., $q_x+pq_z=p_y+qp_z$ (8.2)

Differentiating the first equation of (8.2) with respect to x and z, we get

$$F_x+F_pp_x+F_qq_z=0 \text{ and } F_z+F_pp_z+F_qq_z=0 \quad (8.3)$$

respectively. Multiplying the second equation of (8.3) by p and adding the result to the first equation, we obtain

$$F_x+pF_z+F_p(p_z+pp_z)+F_q(q_x+pq_z)=0 \quad (8.4)$$

Similarly from the second equation of (8.1), we have

$$G_x+pG_z+G_p(p_x+pp_z)+G_q(q_x+pq_z)=0. \quad (8.5)$$

Eliminating p_x+pp_z from (8.4) and (8.5), it follows that

$$\frac{\partial(F,G)}{\partial(x,p)}+p\frac{\partial(F,G)}{\partial(z,p)}-\frac{\partial(F,G)}{\partial(p,z)}(q_x+pq_z)=0$$

or, $$q_x+pq_z=\frac{1}{J}\left[\frac{\partial(F,G)}{\partial(x,p)}+p\frac{\partial(F,G)}{\partial(z,p)}\right] \quad (8.6)$$

In a similar way, if we differentiate each of (18.1) with respect to y and z and then proceed as above, we get the result

$$p_y + qp_z = -\frac{1}{J}\left[\frac{\partial(F,G)}{\partial(y,q)} + q\frac{\partial(F,G)}{\partial(z,q)}\right] \tag{8.7}$$

Substitution (8.6) and (8.7) into (8.2) leads to

$$\frac{\partial(F,G)}{\partial(x,p)} + p\frac{\partial(F,G)}{\partial(z,p)} + \frac{\partial(F,G)}{\partial(y,q)} + q\frac{\partial(F,G)}{\partial(z,q)} = 0 \tag{8.8}$$

which, in short, is written as

$$[F,G] = 0 \tag{8.9}$$

(8.8) or (8.9) is the required condition for the equations (8.1) to be compatible.

Example 8.1: Shows that the equations $xp = yq$ and $z(xp + yq) = 2xy$ are compatible and solve them.

Solution. Let $F(x,y,z,p,q) = xp - yq = 0, G(x,y,z,p,q) = z(zp + yq) - 2xy = 0$

Then, $\quad \dfrac{\partial(F,G)}{\partial(x,p)} = 2xy, \dfrac{\partial(F,G)}{\partial(y,q)} = -2xy, \dfrac{\partial(F,G)}{\partial(z,p)} = px^2 + 2xy$

and $\quad \dfrac{\partial(F,G)}{\partial(z,q)} = -xyp - y^2q$

Hence, $\quad [F,G] = 2xy + p(px^2 + qxy) - 2xy + q(-xyp - y^2q)$

$\quad = p^2x^2 - q^2y^2 = p^2x^2 - p^2x^2 = 0. (\because px = qy)$

Thus the given equations are compatible.

Now, the equation $F(x,y,z,p,q) = 0$, that is, $px - qy = 0$ leads to the Lagrange's auxiliary equations as $\dfrac{}{x} = \dfrac{dy}{-y} = \dfrac{dz}{0}$ giving solutions $xy = c_1, z = c_2$, where c_1 and c_2 are constants.

9. Nonlinear First Order Partial Differential Equations

Consider partial differential equations

$$F(x,y,z,p,q) = 0 \tag{9.1}$$

in which F is not necessarily linear. It has already been noted in Section-8 that the two-parameter system of surfaces

$$f(x,y,z,a,b) = 0 \tag{9.2}$$

leads to partial differential equations of the type (9.1). The converse is also true as will be seen later. In fact, any envelope of the system (9.2) touches at each of its points a member of the system, as a result of which one gets the same set of values (x, y, z, p, q) as the particular surface leading to a solution of the differential equation. Hence we are led to three classes of integrals of a partial differential equation of the form (9.1):

(a) For the two-parameter system of surfaces $f(x, y, z, a, b) = 0$, the integral is called a *complete integral.*

(b) If there exists a relation between the parameters a and b of the form $b = \phi(a), \phi$ being arbitrary, then the one-parameter subsystem $f(x, y, z, a, \phi(a)) = 0$ of (9.2) forms its envelope and is known as the *general integral* of (9.1).

(c) If the envelope of the two-parameter system of surfaces (9.2) exists, then it also leads to a solution of (9.1) and is termed as *singular integral.*

Example 9.1: Verify that $z = ax + by + a + b - ab$ is a complete integral $z = px + qy + p + q - pq$, where a and b are arbitrary constants.

Show that the envelope of all planes corresponding to complete integrals provides singular integral of the differential equation and determine a general integral by finding the envelope of those planes that pass through the origin.

Solution. Let $f(x, y, z, a, b) = z - (ax + by + a + b - ab) = 0$, (1)

giving $p = a, q = b$ and, therefore, the (1) is a complete integral of the differential equation

$$z = px + qy + p + q - pq \tag{2}$$

Now $\frac{\partial f}{\partial a} = -(x + 1 - b) = 0$ and $\frac{\partial f}{\partial b} = -(y + 1 - a) = 0$ so that $a = y + 1$ and $b = x + 1$. The envelope of the two-parameter system (1) is obtained by eliminating a and b from (1) as $z = (x+1)(y+1)$ which is the required singular integral.

Again, putting $b = \phi(a)$ in (1), the one-parameter system is

$$f(x, y, z, a, \phi(a)) = z - ax - \phi(a)y - a - \phi(a) + a\phi(a) = 0 \tag{3}$$

so that $$\frac{\partial f}{\partial a} = -x - \phi'(a)y - 1 - \phi'(a) + \phi(a) + a\phi'(a) = 0 \tag{4}$$

Since the envelope of the planes passes through the origin, so from (3) we get

$$-a - \phi(a) + a\phi(a) = 0 \text{ i.e. } \phi(a) = \frac{a}{a-1} \Rightarrow \phi'(a) = -\frac{1}{(a-1)^2}$$

Then from the relation(4), we get

$$-x+\frac{y}{(a-1)^2}-1+\frac{1}{(a-1)^2}+\frac{a}{a-1}-\frac{a}{(a-1)^2}=0$$

Solving for a, we have $a=\sqrt{\frac{y}{x}}+1$ and so $\phi(a)=\frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}}$. Substituting the values of a and $\phi(a)$ in (3), we get the general integral as $z=x+y+2\sqrt{xy}$, i.e. $(x+y-z)^2=4xy$.

10. Cauchy's Method of Characteristics

Cauchy introduced a geometrical method of solving nonlinear partial differential equations of the form

$$F(x,y,z,p,q)=0 \tag{10.1}$$

A plane through the point $P(x,y,z)$ and having normal parallel to $\hat{n}$ with direction ratios $(p_0,q_0,-1)$ is uniquely specified by the set of numbers (x_0,y_0,z_0,p_0,q_0). Conversely, any set of five real numbers defines a plane in three-dimensional space. We call such a set of five numbers (x,y,z,p,q) a *plane element* of the space and a plane element (x_0,y_0,z_0,p_0,q_0) satisfying (10.1) is known as an *integral element* of the equation of the point (x_0,y_0,z_0).

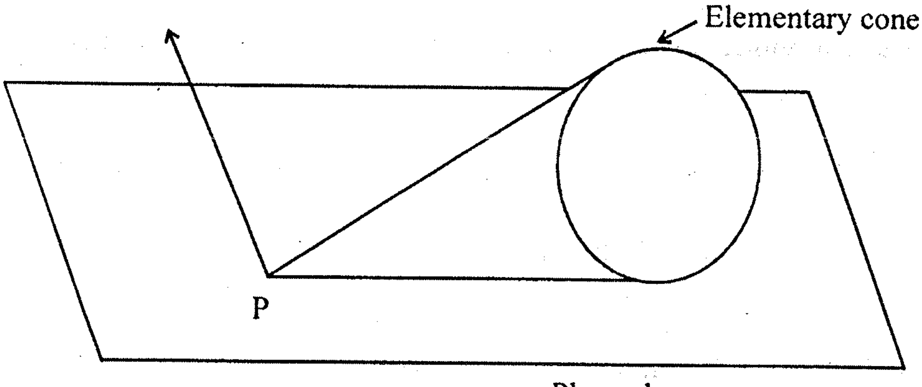


Fig. 1

Let us rewrite the equation (10.1) in the form

$$q=G(x,y,z,p) \tag{10.2}$$

and keep x,y,z fixed, but p to vary. The set of plane elements $\{x_0,y_0,z_0,p,G(x_0,y_0,z_0,p)\}$ depends on the single parameter p and passes through the point P. Thus, the planar elements envelope a cone with P as vertex and this cone is called the *elementary cone* of the equation (10.1) at the point P.

Now consider a surface S given by the equation

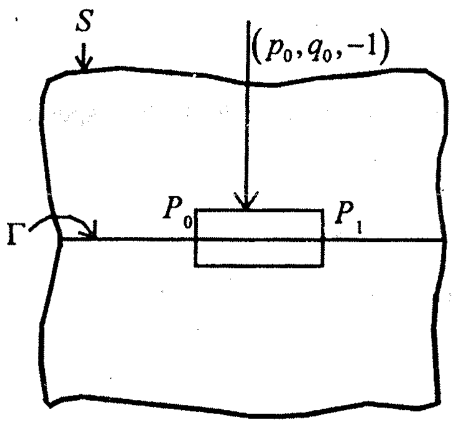
$$z = g(x, y) \tag{10.3}$$

where the function $g(x, y)$ along with its first partial derivatives w.r.t. z and y are assumed to be continuous in a region R of the xy-plane. Then the tangent plane of each point of S defines a plane element $\{x_0, y_0, g(x, y), g_x(x_0, y_0), g_y(x_0, y_0)\}$ which is called the tangent element of the surface S at the point $\{x_0, y_0, g(x_0, y_0)\}$.

Thus we have the following theorem:

Theorem - 10.1: *A surface is an integral surface of a partial differential equation if and only if at each point its tangent element touches the elementary cone of the equation.*

Now a curve with parametric equations $x = x(t), y = y(t), z = z(t)$ lies on the surface (10.3) if $z(t) = g\{x(t), y(t)\}$, for all $t \in I$, I being the given interval. Therefor a point P_0 of Γ determined by the parameter t_0, the direction cosines of the tangent line P_0P_1 are $\{x'(t_0), y'(t_0), z'(t_0)\}$ where $x'(t_0) = \left(\frac{dx}{dt}\right)_{t=t_0}$ etc.



and this direction is perpendicular to the direction $(p_0, q_0, -1)$ if

$$p_0x'(t_0) + q_0y'(t_0) + (-1)z'(t_0) = 0, \text{ i.e. } z'(t_0) = p_0x'(t_0) + q_0y'(t_0).$$

Hence any set

$$\{x(t), y(t), z(t), p(t), q(t)\} \tag{10.4}$$

of five real functions satisfying the condition

$$z'(t) = p(t)x(t) + q(t)y(t) \tag{10.5}$$

defines a strip of the curve Γ at the point (x, y, z). The strip is called an *integral strip* of the equation (10.1) if it is an integral element of the equation (10.1). Thus the set of functions (10.4) is an integral strip if, in addition to satisfying (10.6), it also satisfies the condition

$$F\{x(t), y(t), z(t), p(t), q(t)\} = 0, \forall t \in I \tag{10.6}$$

If at each point the curve Γ touches a generator of the elementary cone, then the corresponding strip is called a *characteristic strip*. For a point $(x + dx, y + dy, z + dz)$ on the tangent plane to the elementary cone we have

$$dz = pdx + qdy \tag{10.9}$$

where p and q satisfy the equation (10.1). Now differentiating (10.1) and (10.9) with respect to p, we get

$$\frac{\partial F}{\partial p} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial p} = 0 \text{ and } 0 = dx + \frac{\partial q}{\partial p}dy$$

Elimination of $\frac{\partial q}{\partial p}$ between these two equations and then use of (10.9) leads to

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + zF_q} \tag{10.10}$$

Thus $x'(t), y'(t), z'(t)$ are proportional to $F_p, F_q, pF_p + qF_q$ respectively along a characteristic strip. We choose the parameter t in such a way that

$$x'(t) = F_p, y'(t) = F_q, z'(t) = pF_p + qF_q \tag{10.11}$$

Now, since p is a function of t along a characteristic strip, we have

$$p'(t) = \frac{\partial p}{\partial x}x'(t) + \frac{\partial p}{\partial y}y'(t)$$

$$= \frac{\partial p}{\partial x}F_p + \frac{\partial q}{\partial x}F_p \cdot \left(\because \frac{\partial p}{\partial y} = \frac{\partial^2 z}{\partial y \partial z} = \frac{\partial q}{\partial x}\right) \tag{10.11}$$

Also differentiating (10.1) w.r.t. x, we get

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0$$

or, $\quad F_x + pF_z + p'(t) = 0$.

Thus along a characteristic strip $q'(t) = -(F_x + pF_z)$. Similarly, we have $q'(t) = -(F_y + qF_z)$.

From the above discussions, it follows that for the determination of the characteristic strip, we have the following set of five ordinary differential equations:

$$x'(t) = F_p, y'(t) = F_q, z'(t) = pF_p + qF_q,$$
$$p'(t) = -F_x - pF_z, q'(t) = -F_y - qF_z. \qquad (10.12)$$

The equations (10.12) are called *Cauchy's characteristic equations* of the partial differential equation (10.1).

Theorem 10.2: *The function* $F(x, y, z, p, q)$ *remains constant along every characteristic strip of the equation* $F(x, y, z, p, q) = 0$.

Proof. Along a characteristic strip, we have

$$\frac{d}{dt}\left[F\{x(t), y(t), z(t), p(t), q(t)\}\right]$$
$$= F_x x'(t) + F_y y'(t) + F_z z'(t) + F_p p'(t) + F_q q'(t)$$
$$= F_x F_p + F_y F_q + F_z(pF_p + qF_q) - F_p(F_x + pF_z) - F_q(F_y + qF_x) \text{ (using 10.12)}$$
$$= 0$$

so that $F(x, y, z, p, q) = \text{constant}$

We have the following result as a corollary:

Corollary: If a characteristic strip contains at least one integral element of $F(x, y, z, p, q) = 0$, then it is an integral strip of this equation.

We are now in a position of solving Cauchy problem stated earlier. Suppose we are to fund the solution of the equation $F(x, y, z, p, q) = 0$ such that the integral surface passes through a curve Γ with parametric equations:

$$x = \phi(u), y = \psi(u), z = \chi(u).$$

Then, in the solution

$$x = x(x_0, y_0, z_0, p_0, q_0, t_0, t) \text{ etc.} \qquad (10.13)$$

of the characteristic equations (10.12), we may take the initial values of x, y, z as $x_0 = \phi(u), y_0 = \psi(u), z_0 = \chi(u)$.

The corresponding intial values of p_0 and q_0 are obtained by the relations

$$x'(u) = p_0\phi'(u) + q_0\psi'(u) \text{ and } F\{\phi(u), \psi(u), \chi(u), p_o, q_0\} = 0.$$

Substituting the values of x_0, y_0, z_0, p_0, q_0 and the appropriate value of t_0 in (10.13), x, y and z can be expressed in terms of two parameters u and t in the form

$$x = X(u,t), y = Y(u,t), z = Z = (u,t).$$

Elimination of u and t from these equations leads to an equation of the form $\theta(x,y,z) = 0$, which is the required equation of the integral surface of the equation $F(x,y,z,p,q) = 0$ through the curve Γ.

Example 10.1: FInd the characteristics of the equation $pq = z$ and determine the integral surface which passes through the parabola $x = 0, y^2 = z$.

Solution: Let $F(x,y,z,p,q) = z - pq = 0$. Then the characteristic equations are

$$x'(t) = F_p = -q, y'(t) = F_q = -p, z'(t) = pF_p + qF_q = -2pq,$$

$$p'(t) = -(F_x + pF_z) = -p, q'(t) = -(F_y + qF_z) = -q$$

Now the given curve is $x = 0, y^2 = z$. We choose the initial values as $x_0 = 0, y_0 = u, z_0 = u^2$. Since $z_0' = p_0x_0' + q_0y_0'$, we have $q_0 = 2u$ and from the given equation $p_0 = \frac{u}{2}$.

Now the equations $x'(t) = -q$ and $q'(t) = -q$ gives $dx = dq$ giving $x = q + c_1$, where c_1 is constant. Similarly, from the equations $y'(t) = -p$ and $p'(t) = -p$, we get $y = p + c_2$, c_2 being constant. Using the initial conditions, we have $c_1 = -2u, c_2 = \frac{u}{2}$. Hence

$$x = q - 2u, y = p + \frac{u}{2}$$

Again the conditions $p'(t) = -p$ and $q'(t) = -q$ imply that $p = c_3e^{-t}$ and $q = c_4e^{-t}$ so that the use of the itial conditions give $c_3 = \frac{u}{2}, c_4 = 2u$. Thus $p = \frac{u}{2}e^{-t}$ and $q = 2ue^{-t}$ and, therefore

$$x(t) = 2u(e^{-t} - 1), y(t) = \frac{u}{2}(e^{-t} + 1)$$

Putting the values of p and q in the characteristic equation $z'(t) = -2pq$, we get

$$z'(t) = -2u^2e^{-2t} \Rightarrow z(t) = u^2e^{-2t} + c_5 \Rightarrow z(t) = u^2e^{-2t}$$

since the initial condition $z_0 = u^2$ gives $c_5 = 0$.

Thus the required characteristics of the given equation are

$$x(t) = 2u(e^{-t} - 1), y(t) = \frac{u}{2}(e^{-t} + 1), z(t) = u^2e^{-2t}$$

From the first two relations of these, we get $e^{-t} = \frac{x+4y}{4y-n}$ and $u = \frac{4y-x}{4}$ which when substituted in the third characteristic leads to the required equation of the integral surface as $(x+4y)^2 = 16z$.

11. Charpit's Method

Based upon the considerations of compatible systems discussed in Section-8, Charpit introduced a method of solving partial differential equations of the type

$$F(x,y,z,p,q)=0 \tag{11.1}$$

In this method, another first-order partial differential equation

$$G(x,y,z,p,q,a)=0 \tag{11.2}$$

a being an arbitrary constant, is introduced, so that

(i) equations (11.1) and (11.2) are solvable for p and q to give

$$p = p(x,y,z,a), q = q(x,y,z,a)$$

and (ii) the equation

$$dz = p(x,y,z,a)\,dx + q(x,y,z,a)\,dy \tag{11.3}$$

is integrable. If we can find such a function $G(x,y,z,p,q,a)$ then the equation (11.3) can be solved to have a solution of the form

$$f(x,y,z,a,b)=0 \tag{11.3}$$

which contains two constants a and b and this solution will then be a solution of the given equation (11.1). It also follows from Section-9 that (11.3) is a complete integral of (11.1).

To find the equation (11.2) compatible with (11.1) we must have (see section-8)

$$J = \frac{\partial(F,G)}{\partial(p,q)} \neq 0 \text{ and } [F,G]=0.$$

Expansion of the last equation gives the equivalent linear partial differential equation

$$F_p\frac{\partial G}{\partial x} + F_q\frac{\partial G}{\partial y} + \left(pF_p + qF_q\right)\frac{\partial G}{\partial z} - \left(F_x + pF_z\right)\frac{\partial G}{\partial p} - \left(F_y + qF_z\right)\frac{\partial G}{\partial q} = 0$$

which determines G. We can obtain a solution of this equation by finding an integral of the subsidiary equation

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{dp}{-\left(F_x + pF_z\right)} = \frac{dq}{-\left(F_y + qF_z\right)} \tag{11.4}$$

in accordance with Theorem 5.2. The equations (11.4) are known as Charpit's equations and they are equivalent to the characteristic equations (10.12).

If the function $G(x,y,z,p,q)$ can be found, then the problem reduces to that of solving for p and q and to integrate the equation (11.3). It may be noted that all equations of (11.4) are not necessary, but p or q must be involved in the obtained solution.

Example 11.1: Solve the equation $2xz - px^2 - 2qxy + pq = 0$.

Solution: Let $F(x,y,z,p,q) = 2xz - px^2 - 2qxy + pq = 0$. Then

$$F_x = 2z - 2px - 2qy, F_y = -2qx, F_z = 2x, F_p = -x^2 + q, F_q = -2xy + p.$$

Thus Charpit's equations are

$$\frac{dx}{-x^2+q} = \frac{dy}{-2xy+p} = \frac{dz}{-px^2-2qxy+2pq} = \frac{dp}{-2z+2qy} = \frac{dz}{0}$$

so that $dq = 0 \Rightarrow q = \text{constant} = a$ (say). The given equation then gives $p = \dfrac{2x(ay-z)}{a-x^2}$. Hence from the relation

$dz = pdx + qdy$,

We get

$$dz = \frac{2x(ay-z)}{a-x^2}dx + ady$$

or,
$$\frac{d(z-ay)}{z-ay} = \frac{2xdx}{x^2-a}$$

Integrating, we have $\log(z-ay) = \log(x^2-a) + \log b$, b being const. Hence the required solution is $z - ay = b(x^2-a)$.

Example 11.2: Solve the equation $px + qy = z(1+pq)^{\frac{1}{2}}$.

Solution. Let $F(x,y,z,p,q) = px + qy - z(1+pq)^{\frac{1}{2}} = 0$. Then

$$F_x = p, F_y = q, F_z = -(1+pq)^{\frac{1}{2}}, F_p = x - \frac{1}{2}qz(1+pq)^{-\frac{1}{2}}, F_q = y - \frac{1}{2}pz(1+pq)^{-\frac{1}{2}}$$

Thus Charpit's equations are

$$\frac{dx}{x-\frac{1}{2}qz(1+pq)^{-\frac{1}{2}}}=\frac{dy}{y-\frac{1}{2}pz(1-pq)^{-\frac{1}{2}}}=\frac{dz}{px+qy-pqz(1+pq)^{-\frac{1}{2}}}$$

$$=\frac{dp}{-p\left\{1-(1+pq)^{-\frac{1}{2}}\right\}}=\frac{dq}{-p\left\{1-(1+pq)^{-\frac{1}{2}}\right\}}$$

The last two equations give $\frac{dp}{p}=\frac{dq}{q}\Rightarrow p=aq$, a being constant.

The given equation then gives $q=\frac{z}{\left\{(ax+y)^2-az^2\right\}^{\frac{1}{2}}}$. Then the relation $dz=pdx+qdy=q(adx+dy)$

leads to

$$dz=\frac{z(adx+dy)}{\left\{(ax+y)^2-az^2\right\}^{\frac{1}{2}}}$$

or, $\sqrt{a}\left(t^2-z^2\right)^{\frac{1}{2}}dz=z\sqrt{a}dt$ [puting $\sqrt{a}t=ax+y$]

or, $\frac{dz}{z}=\frac{du}{\left(u^2-1\right)^{\frac{1}{2}}-u}$ [Putting $t=uz$]

or, $\frac{dz}{z}=-\left\{u+\left(u^2-1\right)^{\frac{1}{2}}\right\}du$

Integrating $\log z+\frac{1}{2}u^2+\frac{1}{2}u\left(u^2-1\right)^{\frac{1}{2}}+\frac{1}{2}\log\left\{u+\left(u^2-1\right)^{\frac{1}{2}}\right\}=$ constant $=b$, say

Putting $u=\frac{t}{z}=\frac{ax+y}{\sqrt{a}.z}$, the required solution is

$$\log z+\frac{ax+y}{z^2}\left\{(ax+y)+(ax+y-az)^{\frac{1}{2}}\right\}$$

$$+\log\left\{(ax+y)+\left(ax+y-\sqrt{a}z\right)^{\frac{1}{2}}\right\}=b.$$

Some special cases

Charpit's method can easily be applied to solve some special types of nonlinear partial differential equations of the first order as shown below:

(a) Equations involving only p and q

If the given equation is of the type

$$F(p,q)=0 \tag{11.5}$$

then Charpit's equations (11.4) give

$$\frac{dx}{F_p}=\frac{dy}{F_q}=\frac{dz}{pF_p+qF_q}=\frac{dp}{0}=\frac{dq}{0}$$

An obvious solution of this equation is p = const. = a, say. Then the given equation (11.5) gives the value of q in the form $q=\phi(a)=$ const. Hence the equation $dz=pdx+qdy=adx+\phi(a)dy$ leads to the solution $z=ax+\phi(a)y+b$, where b is another constant.

It is to be noted that in the above $dp=0$ has been chosen as the second equation. However, sometimes it is more convenient to use the equation $dq=0$ as the second equation for which q = const.

Example 11.3: Solve the equation $p+q=pq$

Solution. Let $F(p,q)=p+q-pq=0$. Then Charpit's equations are

$$\frac{dx}{1-q}=\frac{dy}{1-p}=\frac{dy}{p+q-pq}=\frac{dp}{0}=\frac{dq}{0}$$

so that p = const. = a, say and then the given equation produces $q=\frac{a}{a-1}$. Thus the equation $dz=pdx+qdy$ gives $dz=adx+\frac{a}{a-1}dy$ which, on integration, leads to the desired solution as $z=ax+\frac{a}{a-1}y+b$, where b is constant.

(b) Equations which do not involve independent variables x and y

In this case, the equations are of the form

$$F(z,p,q)=0 \tag{11.6}$$

and then teh Charpit's equations are

$$\frac{dx}{F_p}=\frac{dy}{F_q}=\frac{dz}{pF+qF_q}=\frac{dp}{-pF_z}+\frac{dq}{-qF_z}$$

The last two relations, on integration, gives $p=aq$, a being constant. This, with the use of the equation (11.6) gives the expressions for p and q. The complete integrating the equation $dz=pdx+qdy$.

Example 11.4: Solve $p^2z^2 + q^2 = 1$.

Solution. Let $F(z, p, q) = p^2z^2 + q^2 - 1 = 0$. Then Charpit's equations are

$$\frac{dx}{2pz^2} = \frac{dy}{2q} = \frac{dz}{2(p^2z^2 + q^2)} = \frac{dp}{-2p^3z} = \frac{dq}{-2p^2qz}$$

The last two relations give $p = aq$, where a is constant. Hence the use of the given equation leads to

$p = \frac{a}{\sqrt{a^2z^2 + 1}}$ and $q = \frac{1}{\sqrt{a^2z^2 + 1}}$, so that from the equation $dz = pdx + qdy$, we have

$$dz = \frac{adx + dy}{\sqrt{a^2z^2 + 1}}, \text{ i.e. } \sqrt{a^2z^2 + 1}\ dz = adx + dy$$

Integrating, the required solution is

$$az\left(a^2z^2 + 1\right) + \log\left\{az + \left(a^2z^2 + 1\right)^{\frac{1}{2}}\right\} = 2a(ax + y + b)$$

where b is constant.

(c) Separable equations

If a partial differential equation $F(x, y, z, p, q) = 0$ can be written in the form

$$F(x, y, z, p, q) = \phi(x, p) - \psi(y, q) = 0 \qquad (11.7)$$

then it is said to be separable. Here Charpit's equations are

$$\frac{dx}{\phi_p} = \frac{dy}{-\psi_q} = \frac{dz}{p\phi_p - q\psi_q} = \frac{dp}{-\phi_x} = \frac{dq}{-\psi_y}$$

It is seen that the first and the fourth equations produce an ordinary differential equation $\frac{dp}{dx} + \frac{\phi_x}{\phi_p} = 0$, i.e. $\phi_p dp + \phi_x dx = 0$ in x and p and the solution of this equation can be obtained in the form $\phi(x, p) = \text{const.} = a$, say, so that $\psi(y, q) = a$. Then we proceed as in the general theory.

Example 11.5: Solve the equation $p^2y(1 + x^2) = qx^2$.

Solution. In this case we can write the equation in the form

$$F(x, y, z, p, q) = \frac{p^2(1 + x^2)}{x^2} - \frac{q}{y} = 0 = \phi(x, p) - \psi(y, q) = 0$$

Where $\phi(x,p)=\frac{p^2(1+x^2)}{x^2}, y(y,q)=\frac{q}{y}$. Thus using Charpit's equation as above, we derive the ordinary differential equation $\phi_p dp+\phi_x dx=0$ leading to $\phi(x,p)=$ const. $=a$, say. Hence $\frac{p^2(1+x^2)}{x^2}=a=\frac{q}{y}$, so that $p=\frac{\sqrt{a}x}{\sqrt{1+x^2}}$ and $q=ay$. Then from the equation $dz=pdx+qdy$, we get

$$dz=\sqrt{a}.\frac{xdx}{\sqrt{1+x^2}}+aydy.$$

Integrating, the desired solution is

$$z=\sqrt{a(1+x^2)}+\frac{1}{2}ay^2+b.$$

where b is constant.

(d) Clairaut's equations

If a first-order partial differential equation $F(x,y,z,p,q)=0$ can be written in the form

$$z=px+qy+f(p,q) \tag{11.8}$$

then it is said to be in Clairaut's form. Here Charpit's equations are

$$\frac{dx}{x+f_p}=\frac{dy}{y+f_q}=\frac{dz}{px+qy+pf_p+qf_q}=\frac{dp}{0}=\frac{dq}{0}$$

that $p=a, q=b$, where a and b are constants. Putting these values of p and q in the given equation, we get its complete integral as $z = ax + by + f(a,b)$.

Example 11.6 : Find the complete integral of the equation.

$$pqz=p^2(xq+p^2)+q^2(yp+q^2)$$

Solution. We can write the given equation in the form $z=px+qy+\frac{p^4+q^4}{pq}$ which is in Clairaut's form and, therefore, the complete integral is $z=ax+by+\frac{a^4+b^4}{ab}$, where a and b are arbitrary constants.

12. Jacobi's Method

Jacobi's method of solving first-order partial differential equations

$$F(x,y,z,p,q)=0 \qquad (12.1)$$

lies on the fact that if there exists a relation

$$u(x,y,z)=0 \qquad (12.2)$$

between x,y,z then on substituting

$$p=-\frac{u_1}{u_3}, q=-\frac{u_2}{u_3} \qquad (12.3)$$

where $u_i=\frac{\partial u}{\partial x_i}, (i=1,2,3)$, into (12.1), we get a partial differential of the type

$$f(x,y,z,u_1,u_2,u_3)=0 \qquad (12.4)$$

in which the new dependent variable u does not appears.

In this method, we are to introduce two first-order partial differential equations of the type

$$g(x,y,z,u_1,u_2,u_3,a)=0, h(x,y,z,u_1,u_2,u_3,b)=0 \qquad (12.5)$$

such that

(i) equations (12.4) and (12.5) are sovable for u_1,u_2,u_3 and

(ii) the equation

$$du=u_1dx+u_2dy+u_3dz \qquad (12.6)$$

is integrable.

Since (12.4) and (12.5) must be mutually compatible, we have

$$[f,g]=0,[g,h]=0,[h,f]=0.$$

ow the equation $[f,g]=0$ gives

$$\frac{\partial(f,g)}{\partial(x,u_1)}+\frac{\partial(f,g)}{\partial(y,u_2)}+\frac{\partial(f,g)}{\partial(z,u_3)}=0$$

i.e. $$f_{u_1}\frac{\partial g}{\partial x}+f_{u_2}\frac{\partial g}{\partial y}+f_{u_3}\frac{\partial g}{\partial z}-f_x\frac{\partial g}{\partial u_1}-f_y\frac{\partial g}{\partial u_2}-f_z\frac{\partial g}{\partial u_3}=0$$

which has subsidiary equations

$$\frac{dx}{f_{u_1}}=\frac{dy}{f_{u_2}}=\frac{dz}{f_{u_3}}=\frac{du_1}{-f_x}=\frac{du_2}{-f_y}=\frac{du_3}{-f_z} \qquad (12.7)$$

And two solutions of (12.7) containing u_1,u_2 and u_3 serve the purpose of equations (12.5) provided that the compatibility conditions are satisfied. Then solving (12.4) and (12.5) for u_1, u_2 and u_3 and putting these values in (12.6), the required solution is obtained after integration.

Jacobi's method is more advantageous than Charpit's method in the sense that it can be generalised to any number of variables. Thus, in order to solve the partial differential equation.

$$f(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_n) = 0 \tag{12.8}$$

where $u_i = \dfrac{\partial u}{\partial x_i}, (i = 1, 2, \ldots, n)$, the auxiliary equation is

$$\frac{dx_1}{f_{u_1}} = \frac{dx_2}{f_{u_2}} = \ldots = \frac{dx_n}{f_{u_n}} = \frac{du_1}{-f_{x_1}} = \frac{du_2}{-f_{x_2}} = \ldots = \frac{du_n}{-f_{x_n}} \tag{12.9}$$

which involves $n-1$ arbitrary constants. Solving these equations for $u_i, (i = 1, 2, \ldots, n)$, we determine u by integrating the equation

$$du = \sum_{i=1}^{n} u_i dx_i$$

the solution containing n arbitrary constants.

Example 12.1: Solve the equation $p^2x + q^2y = z$ by Jacobi's method.

Solution. Putting $p = -\dfrac{u_1}{u_3}, q = -\dfrac{u_2}{u_3}$ in the given equation, we get

$$f(x, y, z, u_1, u_2, u_3) = u_1^2 x + u_2^2 y - u_3^2 z = 0 \tag{1}$$

Jacobi's equations (12.7) are

$$\frac{dx}{2u_1 x} = \frac{dy}{2u_2 y} = \frac{dz}{-2u_3 z} = \frac{du_1}{-u_1^2} = \frac{du_2}{-u_2^2} = \frac{du_3}{-u_3^2}$$

The first and the fourth equations give

$$\frac{dx}{x} + \frac{2du_1}{u_1} = 0 \Rightarrow xu_1^2 = a, \text{ i.e. } u_1 = \sqrt{\frac{a}{x}}, \text{ a being constant.}$$

Also the second and fifth equations produce in a similar way $u_2 = \sqrt{\dfrac{b}{y}}$, where b is another constant. Then from (1), we have $u_3 = \sqrt{\dfrac{a+b}{z}}$.

Thus from the relation $du = u_1 dx + u_2 dy + u_3 dz$, we get

$$du = \sqrt{\frac{a}{x}}dx + \sqrt{\frac{b}{y}}dy + \sqrt{\frac{a+b}{z}}dz$$

which, on integration leads to

$$u = 2\sqrt{ax} + 2\sqrt{by} + 2\sqrt{(a+b)z} + 2c$$

where $2c$ is constant. Hence the required solution is $u(x,y,z)=0$ i.e. $\sqrt{ax}+\sqrt{by}+\sqrt{(a+b)z}+c=0$.

14. Solutions of Partial Differential Equations Satisfying Given Conditions

Suppose we are to determine the equations of surfaces satisfying the partial differential equation

$$F(x,y,z,p,q)=0 \tag{14.1}$$

subject to some conditions like passing through a given curve or circumscribing a given surface and to derive one complete integral from the other.

First we consider the solution of (14.1) and determine the integral surface passing through a given curve Γ with parametric equations $x=x(t), y=y(t), z=z(t), t$ being parameter. Then it is either

(a) a particular case of the complete integral

$$f(x,y,z,a,b)=0 \tag{14.2}$$

which is obtained by giving particular values to a or b; or,

(b) the envelope of a one-parameter subsystem of (14.2), i.e. a particular case of the general integral corresponding to (14.2); or,

(c) the envelope of the two-parameter system (14.2).

It is unlikely that the solution falls into categories (a) or (c). So we consider the case (b) only.

Suppose a surface S passes through the curve Γ. Then at its every point, the envelope of S is touched by a member of its subsystem. Let the curve Γ is touched at a point P on it by S_p, a member of the subsystem and since S_p touches S at P, it also touches Γ at P. Thus S is the envelope of a one-parameter subsystem of (14.2), each member of which touches the curve, provided such a subsystem exists. Let us suppose that the subsystem is made up of those members of the family (14.2) which touch the curve Γ. Then the points of intersection of the surface (14.2) and the curve Γ are obtained by the equation

$$f\{x(t),y(t),z(t),a,b\}=0 \tag{14.3}$$

in terms of the parameter t. The curve Γ touches the surface (14.2) provided the equation (14.3) has two equal roots, i.e. the equations (14.3) and

$$\frac{\partial}{\partial t}\left[f\{x(t),y(t),z(t),a,b\}\right]=0 \tag{14.4}$$

have a common root, the condition for which is obtained by eliminating t between (14.3) and (14.4) in the form

$$\psi(a,b)=0 \tag{14.5}$$

which can be factorised to give

$$b=\psi_1(a), b=\psi_2(a),..., \tag{14.6}$$

each of which being a one-parameter sub system. The envelope of these one-parameter subsystems give the solution of the problem.

Example 14.1: Find a complete integral of the equation $p^2x+qy=z$ and hence find the solution of an integral surface of which the line $y=1, x+z=0$ is a generator.

Solution. Let $F(x,y,z,p,q)=z-p^2q-qy=0$. Then Charpit's equations are

$$\frac{dx}{-2px}=\frac{dy}{-q}=\frac{dz}{-2p^2x-qy}=\frac{dp}{p^2+p}=\frac{dq}{0}$$

the last equation of which gives q = const. = a, say. So the given equation leads to $p=\left(\frac{z-ay}{x}\right)^{\frac{1}{2}}$. Thus, from the equation $dz=pdx+qdy$, we have

$$dz=\left(\frac{z-ay}{x}\right)^{\frac{1}{2}}dx+ady \Rightarrow \frac{d(z-ay)}{(z-ay)^{\frac{1}{2}}}=\frac{dx}{x^{\frac{1}{2}}},$$

the integration of which gives $(z-ay)^{\frac{1}{2}}=x^{\frac{1}{2}}+b^{\frac{1}{2}}$, i.e. $(x+ay-z+b)^2=4bx$ which is the complete integral, b being constant.

Now the parametric equations of the given line are $x=t, y-1, z=-t$. The intersection of this line with the above equation is $(2t+a+b)^2=4bt$, i.e. $4t^2+4ab+(a+b)^2=0$ which has equal roots if $a^2=(a+b)^2$, i.e. $b=-2a$ (neglecting $b=0$). Hence the one-parameter subsystem is

$$(x+ay-z-2a)^2=-8ax, \text{ or } a^2(y-2)^2+2a\{x(y+4)-z(y+z)\}+(x-z)^2=0$$

which has the envelope $\{x(y+4)-z(y+z)\}^2=(y-2)^2(x-z)^2$, i.e. $xy=z(y-2)$.

The function z defined by this equation is the solution of the problem.

Next let us consider the case of finding one complete integral from the other. For this, we suppose that (14.2) i.e. $f(x,y,z,a,b)=0$ is the complete integral. We now show that there also exists another relation of the form

$$g(x,y,z,c,d)=0 \tag{14.7}$$

involving two arbitrary constants c and d and this is also a complete integral. On this surface (14.7) we choose a curve Γ whose equations contain constants as independent parameters and then the envelope of the one-parameter subsystem of (14.2) touching the curve Γ is found out. Since this solution contains two arbitrary constants, so this is also a complete integral of the given equation.

Example 14.2: Show that the equation $xpq+yq^2=1$ has complete integrals (a) $(z+b)^2=4(ax+y)$ and (b) $dx(z+c)=d^2y+x^2$ and deduce (b) from (a).

Solution. Consider the curve $y=0, x=d(z+c)$ on the surface (b). The intersection of this curve with (a) is $(z+b)^2-4ad(z+b)+4ad(b-c)=0$ which has equal roots if $a^2d^2=ad(b-c)$, i.e. if $ad=0$, or $b=c+ad$.

Since the envelope of the subsystem for $a=0$ does not depend on c and d, so this cannot be desired one. So the required subsystem has equation

$$(z+c+ad)^2=4(ax+y), \text{ i.e. } d^2a^2+2a\{d(z+c)-2x\}+(z+c)^2-4y=0$$

which has the envelope

$$\{d(z+c)-2x\}^2=\{(z+c)^2-4y\}d^2, \text{ i.e. } dx(z+c)=d^2y+x^2.$$

Lastly, we give a sketch of the procedure to obtain an integral surface circumscribing a given surface. Two surfaces circumscribe each other if they touch along a given curve. For example, *a* coincoid

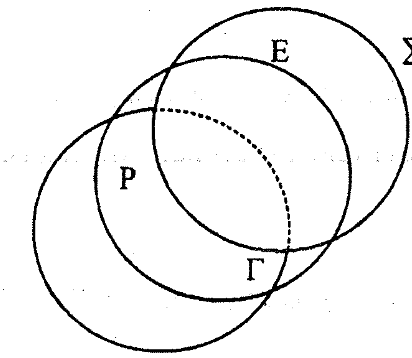


Fig. 3

and its enveloping cylinder touch along a curve. It is to be noted that this curve needs not be a plane curve.

Suppose now that the partial differential equation (14.1): $F(x,y,z,p,q)=0$ has a complete integral (14.2): $f(x,y,z,a,b)=0$. Our object is to find an integral surface of (14.1) which circumscribes the surface

$$\psi(x,y,z)=0 \tag{14.8}$$

by the use of (14.2). Suppose the surface

$$E: u(x,y,z)=0 \tag{14.9}$$

is of the required kind. Then this falls into either of the category (a), (b) or (c) listed above. Owing to frequent occurrence, we consider the possibility of (b). Since E is the envelope of a one-parameter subsystem S of (14.2), so it is touched at each of its points and, in particular, at each point P of Γ by a member S_p of S. Now S_p touches Σ at P as it touches E at P. Thus the equation (14.9) represents the equation of a set of surfaces (14.2) which touch the surface (14.8).

Let us now find the surface (14.2) which touch E and see whether they provide a solution to the problem. The surface (14.2) touches the surface (14.9) provided that the equations (14.2), (14.8) and

$$\frac{f_x}{\psi_x}=\frac{f_y}{\psi_y}=\frac{f_z}{\psi_z} \tag{14.10}$$

are consistent. The condition for this is the eliminant of x, y, z from these four equations of the type.

$$x(a,b)=0. \tag{14.11}$$

This equation can be factorised into a set of relations of the form

$$b=x_1(a), b=x_2(a),... \tag{14.12}$$

each of which defines a subsystem of (14.2) whose members touch (14.8). The points of contact lie on the surface whose equation is obtained by eliminating a and b from (14.10) and (14.12). The intersection of this surface with Σ is the curve Γ. Then each of the relations (14.12) defines a subsystem whose envelope E touches the surface Σ along the curve Γ.

Example 14.3: Show that the integral surface of the equation $2q(z-px-qy)=1+q^2$ which is circumscribed about the paraboloid $2x=y^2+z^2$ is the enveloping cylinder which touches it along its section by the plane $y+1=0$.

Solution. The equation $2q(z-px-qy)=1+q^2$ can be written in Clarauit's form: $z=px+qy+\dfrac{1+q^2}{2q}$ which has the complete integral

$$z = ax + by + \frac{1+b^2}{2b}.$$

Let $f(x,y,z) = ax + by + \frac{1+b^2}{2b} - z = 0$ and $\psi(x,y,z) = 2x - y^2 - z^2 = 0$. Then using the equation (14.10) we get

$$\frac{a}{2} = \frac{b}{-2y} = \frac{-1}{-2z} \Rightarrow y = -\frac{b}{a}, z = \frac{1}{a}$$

Eliminating x between the equations $f(x,y,z) = 0$ and $\psi(x,y,z) = 0$, we have

$$aby^2 + 2b^2y + abz^2 - 2bz + b^2 + 1 = 0.$$

Substituting the values of y and z in this equation, it follows that

$$(b-a)(b^2+1) = 0$$

so that the relation $b = a$ gives a subsystem whose envelope is a surface of the required kind. The envelope of the system $\{2(x+y)+1\}a^2 - 2az + 1 = 0$ is $z^2 = 2(x+y) = 1$. The surface $2x = y^2 + z^2$ touches this envelope where $(y+1)^2 = 0$, i.e. $y+1 = 0$.

Exercises

1. Formulate the partial differential equations by eliminating arbitrary constants or functions from the following:
   (i) $2z = (ax+y)^2 + b$; Ans. $px + qy = q^2$.
   (ii) $f(x+y+z, x^2+y^2-z^2) = 0$; Ans. $(y+z)p - (x+z)q = x - y$.
   (iii) $z = (x+a)(y+b)$; Ans. $pq = z$.
   (iv) $ax^2 + by^2 + z^2 = 1$; Ans. $z(px+qy) = z^2 - 1$.
   (v) $(x-a)^2 + (y-b)^2 + z^2 = 1$; Ans. $z^2(1+p^2+q^2) = 1$.
   (vi) $z = xy + f(x^2+y^2)$; Ans. $x^2 - y^2 = qx - py$.
   (vii) $z = x + y + f(xy)i$ Ans. $px - qy = x - y$
   (viii) $z = f(x-y)$; Ans. $p + q = 0$.
2. Find the general integrals of the linear partial differential equations:
   (i) $p(y+zx) - q(x+yz) = x^2 - y^2$; Ans. $f(z^2+y^2-z^2, xy+z) = 0$.

(ii) $z(px-qy)=y^2-x^2$; Ans. $f(x^2+y^2+z^2,xy)=0$.

(iii) $px(x+y)=qy(x+y)-(x-y)(2x+2y+z)$; Ans. $f\{(x+y)(x+y+z),xy\}=0$.

(iv) $x(x^2+3y^2)p-y(3x^2+y^2)q=2z(y^2-x^2)$; Ans. $f\left\{\frac{2y}{z},z(x^2+y^2)\right\}=0$.

(v) $y^2p-xyq=x(z-2y)$; Ans. $f(x^2+yz,x^2+y^2)=0$.

(vi) $p\cos(x+y)=q\sin(x+y)=z$;

$$\text{Ans. } f\left[\{\cos(x+y)+\sin(x+y)\}e^{y-x},\tan\left\{\frac{1}{2}(x+y)+\frac{\Pi}{8}\right\}.z^{-\sqrt{2}}\right]=0.$$

3. Find the integral surface of the linear partial differential equation

$$x(y^2+z)p-y(x^2+z)q=(x^2-y^2)z$$

which contains the straight line $x+y=0, z=1$. [Ans. $f(x^2+yz,x^2+y^2)=0$].

4. Find the integral surface of the differential equation

$$2y(z-3)p+(2x-z)q=y(2x+3)$$

which passes through the circle $z=0, x^2+y^2=2x$. [Ans. $x^2+y^2-2x=z^2-4z$].

5. Find the general integral of the differential equation

$$(2xy-1)p+(z-2x^2)q=2(x-yz)$$

and also the particular integral which passes through the line

$x=1, y=0$. [Ans. $x^2+y^2-xz-y+z-1=0$].

6. Find the general integral of the equation $(x-y)p+(y-z-x)q=z$ and equation of the integral surface of this equation which passes through the circle $x^2+y^2=1, z=1$.

$$\left[\text{Ans. } f\left(x+y+z,\frac{x-y+z}{z^2}\right)=0, z^4(x+y+z)(x+y+z-2)+(x-y+z)(x-y+z-2z^2)=0\right].$$

7. Find the surface which is orthogonal to the one-parameter system $z=cxy(x^2+y^2)$ and which passes through the hyperbola $x^2-y^2=a^2, z=0$.

$$\left[\text{Ans. } (x^2+y^2+4z^2)(x^2-y^2)^2=a^4(x^2+y^2)\right].$$

8. Find a complete integral of the partial differential equation $(p^2+q^2)x=qz$ and deduce the solution which

passes through the curve $x = 0, z^2 = 4y$.

$\left[\text{Ans. } z^2 = a^2x^2 + (ay+b)^2, (2y - z^2)^2 = 4(x^2 + y^2)\right]$

9. Show that the integral surface of the equation $z(1-q^2) = 2(px + pq)$ which passes through the line $x = 1, y = hz + k$ has equation $(y - kx)^2 = z^2\{(1+h^2)x - 1\}$.

10. Show that the differential equation $2xz + q^2 = x(xp + yp)$ has a complete integral $z + a^2x = axy + bx^2$ and deduce that $x(y + cx^2 = 4(z - d.x^2))$ is also a complete integral.

11. Show that the integral surface of the equation $2y(1+p^2) = pq$ which is circumscribed about the cone $x^2 + z^2 = y^2$ has the equation $x^2 = y^2(4y^2 + 4x + 1)$.

12. Find the complete integral of the differential equation $(y + zq)^2 = z^2(1 + p^2 + q^2)$ circumscribed about the surface $x^2 - z^2 = 2y$.

$\left[\text{Ans. } (x-a)^2 + y^2 + z^2 = 2by, (y^2 + 4y + 2z^2)^2 = 8x^2y^2\right]$

13. Show that the equations $xp - yq = x, x^2p + q = xz$ are compatible and find their solution.

$\left[\text{Ans. } f(z - x, xy) = 0\right]$

14. Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0, y = 0$.

$\left[\text{Ans. } x = 2u(2 - c^{-t}), y = 2\sqrt{2}u(1 - e^{-t}), z = -u^2e^{-2t}; (x - \sqrt{2}y)^2 + 4z = 0\right]$

15. Find the solution of the equation $2z = (p^2 + q^2)(p - x)(q - y)$ which passes through the x-axis.

$\left[\text{Ans. } 2z = y(4x - 3y)\right]$

16. Solve the following equations by Charpit's or Jacobi's method;

(i) $px + pq = pq; \left[\text{Ans. } 2az = (ax + y)^2 + 2b\right]$

(ii) $(p^2 + q^2)y = qz; \left[\text{Ans. } (z^2 - a^2y^2)^{\frac{1}{2}} = ax + b\right]$

(iii) $z - px - qy = p^2 + q^2; \left[\text{Ans. } z = ax + by + a^2 + b^2\right]$

(iv) $p^2+q^2-2px-2q+1$; $\left[\text{Ans. } (a^2+1)z=b+\frac{1}{2}u^2+\frac{1}{2}u\sqrt{(u^2-a^2-1)}\right.$

$\left.-\frac{1}{2}(a^2+1)\log\left\{u+\sqrt{(u^2+a^2-1)}\right\}\right]$, where $u=ax+y]$.

(v) $q+px=p^2$; $\left[\text{Ans. } z=\frac{1}{4}\left[x^2\pm\left\{x\sqrt{x^2+4a}+a\log\left(x+\sqrt{x^2+4a}\right)\right\}+ay+b\right]\right.$

(vi) $z=pq$; $\left[\text{Ans. } 2\sqrt{az}=ax+y+b\right]$

(vii) $(p+q)(px+qy)-1=0$; $\left[\text{Ans. } z=\frac{2}{\sqrt{1+a}}\sqrt{(ax+y)}+b\right]$

(viii) $p=(qy+z)^2$; $\left[\text{Ans. } yz=ax+z\sqrt{ay}+b\right]$

(ix) $pxy+pq+qy-yz=0$; $\left[\text{Ans. } (z-ax)(y+a)^a=be^y\right]$

(x) $(p^2+q^2)x=pz$; $\left[\text{Ans. } z=bx^a a^{1/a}\right]$

(xi) $(p^2+q^2)y=qz$; $\left[\text{Ans. } (x+b)^2+y^2=az^2\right]$

(xii) $z^2=pqxy$; $\left[\text{Ans. } z=bx^a y^{1/a}\right]$

(xiii) $2(z+px+qy)=p^2y$; $\left[\text{Ans. } z=\frac{ax}{y^2}+\frac{b}{y}-\frac{a^2}{4y^2}\right]$

(xiv) $pq=1$; $\left[\text{Ans. } z=ax+\frac{y}{a}+b\right]$

(xv) $(p+q)(z-px-qy)=1$; $\left[\text{Ans. } z=ax+by+\frac{1}{a+b}\right]$

(xvi) $zpq=p+q$; $\left[\text{Ans. } z^2=2(a+1)\left(x+\frac{y}{a}\right)+b\right]$

(xvii) $p^2q^2+x^2y^2=x^2q^2(x^2+y^2)$; $\left[\text{Ans. } z=\frac{1}{3}(x^2+y^2)^{3/2}+(y^2-a^2)^{1/2}+b\right]$

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-IV

Group–B

Module No. - 44

Partial Differential Equations of Second Order

1. Introduction :

In this module we shall consider preliminary discussions of second order partial differential equations along with some higher order equations with constant coefficients. In fact, three main types of second order partial differential equations, called elliptic, parabolic and hyperbolic equations, are of tremendous use from the applications point of view and need broad discussions. As a result, these are elaborated in the following three chapters.

2. Origin of Second-Order Equations

Consider a function z of independent variables x and y defined by

$$z=f(u)+g(v)+w \tag{2.1}$$

where f and g are arbitrary and each of u, v, w is a function of x and y.

We use the symbols

$$p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}, r=\frac{\partial^2 z}{\partial x^2}, s=\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial^2 z}{\partial y\,\partial x}, t=\frac{\partial^2 z}{\partial y^2}. \tag{2.2}$$

Differentiating both sides of (2.1) w.r.t. x and y partially, we get

$$p=f'(u)u_x+g'(v)v_x+w_x,$$

$$q=f'(u)u_y+g'(v)v_y+w_y$$

respectively, so that

$$r=f''(u)u_x^2+g''(v)v_x^2+f'(u)u_{xx}+g'(v)v_{xx}+w_{xx},$$

$$s = f''(u)u_x u_y + g''(v)v_x v_y + f'(u)u_{xy} + g'(v)v_{xy} + w_{xy},$$
$$t = f''(u)u_y^2 + g''(v)v_y^2 + f'(u)u_{yy} + g'(v)v_{yy} + w_{yy}.$$

Eliminating the four arbitrary quantities f', g', f'' and g'' from the above five equations, we obtain

$$\begin{vmatrix} p - w_x & u_x & v_x & 0 & 0 \\ q - w_y & u_y & v_y & 0 & 0 \\ r - w_{xx} & u_{xx} & v_{xx} & u_x^2 & v_x^2 \\ s - w_{xy} & u_{xy} & v_{xy} & u_x u_y & v_x v_y \\ t - w_{yy} & u_{yy} & v_{yy} & u_y^2 & v_y^2 \end{vmatrix} = 0 \tag{2.3}$$

This is an equation involving only the derivatives p, q, r, s, t and known functions of x and y and is, therefore, a second-order partial differential equation. Expanding the determinant on the left side of (2.3) in terms of elements in the first column, an equation of the form

$$Rr + Ss + Tt + Pp + Qq = W \tag{2.4}$$

is obtained in which each of R, S, T, P, Q and W is a function of x and y. Thus (2.1) is a solution of the second-order partial differential equation (2.4).

Example-2.1 : If $z = f(x^2 - y) + g(x^2 + y)$, where f and g are arbitrary functions, prove that

$$\frac{\partial^2 z}{\partial x^2} - \frac{1}{x}\frac{\partial z}{\partial x} = 4x^2 \frac{\partial^2 z}{\partial y^2}.$$

Solution. We have

$$\frac{\partial z}{\partial x} = 2xf'(x^2 - y) + 2xg'(x^2 + y), \frac{\partial z}{\partial y} = -f'(x^2 - y) + g'(x^2 + y),$$

$$\frac{\partial^2 z}{\partial x^2} = 2f'(x^2 - y) + 4x^2 f''(x - y) + 2g'(x^2 + y) + 4x^2 g''(x^2 + y),$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x^2 - y) + g''(x^2 + y).$$

Thus $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{1}{x}\dfrac{\partial z}{\partial x} = 2f'(x^2 - y) + hx^2 f''(x - y) + 2g'(x^2 + y) + 4x^2 g''(x^2 + y)$

$$-\frac{1}{x}\left[2x f'(x^2 - y) + 2x g'(x^2 + y)\right]$$

$$= 4x^2\left[f''(x - y) + g''(x^2 - y)\right]$$

$$\therefore \frac{\partial^2 z}{\partial x^2} - \frac{1}{x}\frac{\partial z}{\partial x} = 4x^2 \frac{\partial^2 z}{\partial y^2}.$$

3. Linear Partial Differential Equations in Two Independent Variables and with Constant Coeffiients

Consider a differential operator of the form

$$F(D, D') = \sum_r \sum_s C_{rs} D^r D'^s \qquad (3.1)$$

where $D \equiv \frac{\partial}{\partial x}, D' \equiv \frac{\partial}{\partial y}$ and the coefficients c_{rs} are constants. Then an equation of the type

$$F(D, D')z = f(x, y) \qquad (3.2)$$

is called a linear partial differential equation in two independent variables x and y with constant coefficients.

The most general solution, that is, the solution containing the exact number of arbitratory functions of the corresponding linear homogeneous equation

$$F(D, D')z = 0 \qquad (3.3)$$

is called the complementary function of the equation (3.2) and any other solution of (3.2) is its particular integral. Thus if u be the complementary function and z_1 be a particular integral of (3.2) so that $F(D, D')u = 0$ and $F(D, D')z_1 = f(x, y)$, then $F(D, D')(u + z_1) = F(D, D')u + (F, D')z_1 = F(D, D')z_1 = f(x, y)$, showing that $u + z_1$ is a general integral of (3.2).

Also if u_i $(i = 1, 2, ..., n)$, be n solutions of the n-th order linear homogeneous partial differential equation $F(D, D')z = 0$, then $\sum_{r=1}^{n} C_r u_r$, where $C_r's$ are arbitrary constants, is also a soluble of the equation. For, then we have,

$$F(D, D')(C_r u_r) = C_r F(D, D')u_r \text{ and } F(D, D')\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} F(D, D')u_r$$

for any set of functions and consequently, it follows that

$$F(D, D')\sum_{r=1}^{n} C_r u_r = \sum_{r=1}^{n} C_r F(D, D')u_r = 0,$$

i.e. $\sum_{r=1}^{n} C_r u_r$ is a solution of $F(D, D')z = 0$.

The linear differential operators can be classified into two types :

(a) $F(D,D')$ *is reducible* if it can be expressed as the product of linear factors of the form $\alpha D+\beta D'+\gamma$, where α,β and γ are constants.

For example, the differential operator $D^3-3D^2D'-DD'^2-D'^3-6D^2-4DD'-2D'^2+11D+5D'-6$ is reducible as it can be expressed in the form $(D+D'-1)(D+D'-3)(D-D'-2)$.

(b) $F(D,D')$ *is irreducible* if it cannnot be written as the product of linear factors.

For example, the differential operator $D^2D'+D'^2-2$ is irreducible.

Rules for finding complementary functions.

I. Reducible Operator with non-repeated linear factors

Theorem - 3.1 : For reducible operator $F(D,D')$, the order of the linear factors is unimportant.

Proof. Since for reducible operator, we have

$$(\alpha_r D+\beta_r D'+\gamma_r)(\alpha_s D+\beta_s D'+\gamma_s)$$

$$=\alpha_r\alpha_s D^2+(\alpha_r\beta_s+\alpha_s\beta_r)DD'+\beta_r\beta_s D'^2+(\alpha_r\gamma_s+\alpha_s\gamma_r)D$$

$$+(\beta_r\gamma_s+\beta_s\gamma_r)D'+\gamma_r\gamma_s$$

$$=(\alpha_s D+\beta_s D'+\gamma_s)(\alpha_r D+\beta_r D'+\gamma_r),$$

so we can write $F(D,D')=\prod_{r=1}^{n}(\alpha_r D+\beta_r D+\gamma_r)$.

Theorem -3.2 : If $\alpha_r D+\beta_r D'+\gamma_r$ be a linear factor of $F(D,D')$ and $\phi_r(z)$ is an arbitrary function of the single variable ξ, then a solution of the equation $F(D,D')z=0$ is given by

$$u_r=\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x-\alpha_r y),$$

provided that $\alpha_r\neq 0$.

Proof. Differentiating the expression $u_r=\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x-\alpha_r y)$ partially with respect to x and y, we get respectivey

$$Du_r=-\frac{\gamma_r}{\alpha_r}u_r+\beta_r\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi'(\beta_r x-\alpha_r y)$$

and $D'u_r=-\alpha_r\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi'(\beta_r x-\alpha_r y)$

so that $(\alpha_r D+\beta_r D'+\gamma_r)u_r=0$ and hence by Theorem 3.1, we have

$$F(D,D')u_r=\left\{\prod_{\substack{s=1\\s\neq r}}^{n}(\alpha_s D+\beta_s D'+\gamma_s)\right\}(\alpha_r D+\beta_r D'+\gamma_r)u_r=0.$$

Theorem 3.3 : If $\beta_r D'+\gamma_r,(\beta_r\neq 0)$, be a factor of $F(D,D')$ and $\phi_r(\varepsilon)$ is an arbitrary function of the single variable ξ, then

$$u_r=\exp\left(-\frac{\gamma_r y}{\beta_r}\right)\phi_r(\beta_r x)$$

is a solution of the equation $F(D,D')z=0$.

Proof. The proof follows immediately along the same lines as in Theorem -3.9.

Example-3.1 : Solve the equation $(D^3+3D^2D'+3DD'^2+D'^3-4D^2-8DD'-4D'^2+3D+3D')z=0.$

Solution. The equation can be written in factorised form as

$$(D+D'-1)(D+D'-3)(D+D')z=0.$$

Hence by Theorem-3.2, the solution is

$$z=e^x\phi_1(x-y)+e^{3x}\phi_2(x-y)+\phi_3(x-y).$$

Example-3.2 : Solve the equation $(5D+2D'+3)(2D'+3)z=0.$

Solution By Theorem-3.2 and 3.3, the required solution is

$$z=e^{-\frac{3x}{5}}\phi_1(2x-5y)+e^{-\frac{3}{2}y}\phi_2(2x)$$

Reducible operator with repeated factors

For repeated factors of the form $(\alpha_r D+\beta_r D'+\gamma_r)^k,(k>1,)$ of the differential operator $F(D,D')$, the solution corresponding to a factor of that type can be obtained by applying Theorems 3.2 and 3.3 For example, if $k=2$, then for the solution of the equation

$$(\alpha_r D+\beta_r D'+\gamma_r)^2 z=0, \tag{3.4}$$

We put $(\alpha_r D+\beta_r D'+\gamma_r)z=Z$ to get $(\alpha_r D+\beta_r D'+\gamma_r)Z=0$ and his solution, by Theorem-3.2 is

$$Z=\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x-\alpha_r y),\text{ provided }\alpha_r\neq 0.$$

Thus we have,

$$\alpha_r \frac{\partial z}{\partial x} + \beta_r \frac{\partial z}{\partial y} + \gamma_r z = \exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x - \alpha_r y) \tag{3.5}$$

i.e. $\alpha_r p + \beta_r q = -\gamma_r z + \exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x - \alpha_r y)$.

Lagrange's auxiliary equations are then

$$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r} = \frac{dz}{-\gamma_r z + \exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x - \alpha_r y)}$$

the first two of which leads, on integration, to $\beta_r x - \alpha_r y = \text{cosntant} = c_1$, say.

$$\frac{dx}{\alpha_r} = \frac{dz}{-\gamma_r z + \exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(c_1)}$$

i.e. $\frac{dz}{dx} + \frac{\gamma_r}{\alpha_r} z = \frac{\phi_r(c_1)}{\alpha_r}\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)$,

which, on integration, gives the solution

$$z = \frac{1}{\alpha_r}\{\phi_r(c_1)xt\, c_2\}\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)$$

Hence the solution of (3.5) and, therefore, if (3.4) is

$$z = \{x\phi_r(\beta_r x - \alpha_r y) + \psi_r(\beta_r x - \alpha_r y)\}\exp\left(-\frac{\gamma_r x}{\alpha_r}\right),$$

where ϕ_r and ψ_r are arbitrary functions.

The above result can easily be generalised by the method of induction and, therefore, we have the following theorem:

Theorem-3.4 : If $(\alpha_r D + \beta_r D' + \gamma_r)^k, (\alpha_r \neq 0)$, be a factor of $F(D, D')$ and the functions $\phi_{r_0}, \phi_{r_1}, \ldots, \phi_{r_k}$ are arbitrary, then a solution of $F(D, D')z = 0$ is $\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\sum_{s=1}^{k} x^{s-1}\phi_{rs}'(\beta_p x - \alpha_r y)$.

Similarly, the generatlisation of Theorem-3.3 given as following :

Theorem-3.5 : If $(\beta_r D' + \gamma_r)^m, (\beta_r \neq 0)$, be a factor of $F(D, D')$ and the function $\psi_{r_1}, \psi_{r_2}, ..., \psi_{r_m}$ are arbitrary, then a solution of $F(D, D')z = 0$ is $\exp\left(-\frac{\gamma_r y}{\beta_r}\right)\sum_{s=1}^{m} x^{s-1}\psi_{rs}(\beta_r x)$.

Example-3.3 : Solve the equation $(2D + 3D' - 5)^2 (D' - 3)^3 z = 0$.

Solution. Using Theorem-3.4 and 3.5, the required solution is

$$z = e^{\frac{5}{2}x}\{\phi_1(3x - 2y) + x\phi_2(3x - 2y)\} + e^{3y}\{\psi_1(x) + x\psi_2(x) + x^2\psi_3(x)\}$$

III. Irreducible operator

If the operator $F(D, D')$ is irreducible, then it may not always be possible to find a solution containing full number of arbitrary functions. However, it is possible to obtain solutions containing arbitrary constants as many as we desire. Before discussions of dealing with such a case we note the following proposition :

Proposition-3.1 : $F(D, D')e^{ax+by} = F(a, b)e^{ax+by}$, where a and b are constants.

Proof. Since the operator $F(D, D')$ is made up of terms of the type $C_{r_s} D^r D'^s$ and $D^r(e^{ax+by}) = a^r e^{ax+by}$ and $D'^s(e^{ax+by}) = b^s e^{ax+by}$ we have $(C_{r_s} D^r D'^s)(e^{ax+by}) = C_{r_s} a^r b^s e^{ax+by}$ so that $F(D, D')e^{ax+by} = F(a, b)e^{ax+by}$.

This proves the proposition.

Now to find the complementary function of the equation $F(D, D')z = 0$ we first split the operator $F(D, D')$ into factors. The reducible factors are treated by the methods discussed in I and II above. For irreducible factors, we note by the above proposition that e^{ax+by} is a solution of the equation $F(D, D')z = 0$, provided $F(a, b) = 0$.

Hence

$$z = \sum_r C_r \exp(a_r x + b_r y) \tag{3.6}$$

where a_r, b_r and c_r are constants, is also a solution of $F(D, D')z = 0$, provided that $F(a_r, b_r) = 0$.

It is to be noted that the series (3.6) may not be finite. In case of infinite series, it is to be uniformly convergent for a solution of the equation.

Example-3.4 : Solve $(D - 2D' + 3)(D^2 - 3D' + 5)z = 0$.

Solution. The required solution is $z = e^{-3x}\phi(2x + y) + \sum_r c_r e^{a_r x + b_r y}$, where $a_r^2 - 3b_r + 5 = 0$.

Rules for finding particular integrals

For the equation (3.2), that is

$$F(D,D')z = f(x,y)$$

the particular integral is

$$P.I. = \frac{1}{F(D,D')}f(x,y). \qquad (3.7)$$

We shall consider the following cases to find the particular integral for different forms of the function $f(x,y)$.

I. $f(x,y)$ is a polynomial in x and y

Let $f(x,y) = \sum_{k,l} a_{kl}x^k y^l$, where k, l are positive constants or zero and a_{kl} are constants. Here if m be the highest power of x, then

$$P.I. = \frac{1}{D^m f(D'/D)}f(x,y)$$

and $\{f(D'/D)\}^{-1}$ is expanded by binomial theorem retaining upto the term n, the highest power of y. The particular integral is obtained by integrating the obtained expresion for m times.

Example-3.5 : Solve the equation $(D+D')^2 z = x^2 + xy + y^2$.

Solution. The complementary function is $u = \phi_1(x-y) + x\phi_2(x-y)$.

The P.I. is $z_1 = \frac{1}{(D+D')^2}(x^2+xy+y^2) = \frac{1}{D^2}\left(1+\frac{D'}{D}\right)^{-2}(x^2+xy+y^2)$

$$= \frac{1}{D^2}\left(1 - \frac{2D'}{D} + \frac{3D'^2}{D^2}\ldots\right)(x^2+xy+y^2)$$

$$= \frac{1}{D^2}\left[(x^2+xy+y^2) - \frac{2}{D}(x+2y) + \frac{3}{D^2}(2)\right]$$

$$= \frac{1}{D^2}(x^2+xy+y^2) - \frac{2}{D^3}(x+2y) + \frac{6}{D^4}.1$$

$$= \left(\frac{x^4}{12} + \frac{x^3y}{6} + \frac{x^2y^2}{2}\right) - 2\left(\frac{x^4}{24} + \frac{x^3y}{3}\right) + \frac{x^4}{4}$$

$$= \frac{1}{4}\left(x^4 - 2x^3y + 2x^2y^2\right).$$

Hence the complete solution of the given equation is $z = u + z_1$, i.e.

$$z = \phi_1(x-y) + x\phi_2(x-y).$$

III. $f(x,y) = e^{ax+by}$

Since by proposition-3.1, $\frac{1}{F(D,D')}e^{ax+by} = \frac{1}{F(a,b)}e^{ax+by}$ so $\frac{1}{F(D,D')}e^{ax+by} = \frac{1}{F(a,b)}e^{ax+by}$ provided $F(a,b) \neq 0$.

Example-3.6 : Solve $(D^3 - 7DD'^2 - 6D'^3)z = e^{2x+y}$.

Solution. The given equation can be written as

$$(D+D')(D+2D')(D-3D')z = e^{2x+y}$$

The complementary function is $u = \phi_1(x-y) + \phi_2(2x-y) + \phi_3(3x+y)$ and the particular integral is

$$z_1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3}e^{2x+y}$$

$$= \frac{1}{2^3 - 7\cdot 2\cdot 1^2 - 6\cdot 1^3}e^{2x+y}$$

$$= \frac{1}{-12}e^{2x+y}$$

Hence the required solution is $z = u + z_1 = \phi_1(x-y) + \phi_2(2x-y) + \phi_3(3x+y) + \frac{1}{12}e^{2x+y}$.

III. $f(x, y) = \sin(ax + by)$ **or** $\cos(ax + by)$

Suppose $F(D,D') = \phi(D^2, DD', D'^2)$. Then we have

$$\phi(D^2, DD', D'^2)\sin(ax+by) = \phi(-a^2, -ab, -b^2)\sin(ax+by)$$

so that

$$P.I. = \frac{1}{F(D,D')}\sin(ax+by) = \frac{1}{\phi(D^2, DD', D'^2)}\sin(ax+by)$$

$$= \frac{\sin(ax+by)}{\phi(-a^2-ab,-b^2)}, \text{ provided } F(a,b) = \phi(-a^2,-ab,-b^2) \neq 0.$$

Similarly, for $f(x, y) = \cos(ax + by)$,

$$P.I. = \frac{\cos(ax+by)}{\phi(-a^2,-ab,-b^2)}, \text{ provided } F(a,b) = \phi(-a^2,-ab,-b^2) \neq 0.$$

Example-3.7 : Solve the euqation $(D^2+5DD'+5D'^2)z = \cos(3x-2y)$.

Solution. We can write the given equation in the form

$$\left(D+\frac{5+\sqrt{5}}{2}D'\right)\left(D+\frac{5-\sqrt{5}}{2}D'\right)z = \cos(3x-2y)$$

Thus the complementary function is $u = \phi_1\left(\frac{5+\sqrt{5}}{2}x-y\right)+\phi_2\left(\frac{5-\sqrt{5}}{2}x-y\right)$ and the particular integral is

$$z_1 = \frac{1}{-3^3-5\cdot3\cdot(-2)-5\cdot(-2)^2}\cos(3x-2y) = \frac{\cos(3x-2y)}{1}$$

Hence the required solution is $z = u + z_1$, i.e.

$$u = \phi_1\left(\frac{5+\sqrt{5}}{2}x-y\right)+\phi_2\left(\frac{5-\sqrt{5}}{2}x-y\right)+\cos(3x-2y)$$

IV. $F(a, b) = 0$

If $F(a, b) = 0$, the above methods fail. In such a case, $(bD-aD')$ is a factor of $F(D,D')$ and, therefore, we can write $F(D,D') = (bD-aD')G(D,D')$ where $G(D,D') \neq 0$.

Suppose $(bD-aD')z = f(ax+by)$, i.e. $bp-aq = f(ax+by)$. Then Lagrange's auxiliary equations are

$$\frac{dx}{b} = \frac{dy}{-a} = \frac{dz}{f(ax+by)}$$

The first two relations give $ax+by = const. = c$, say. Thus the first and third relations, viz. $\frac{dx}{b} = \frac{dz}{f(c)}$ gives $z = \frac{x}{b}f(c) = \frac{x}{b}f(ax+by)$. Hence the particular integral is now

$$\frac{1}{F(D,D')}f(ax+by) = \frac{x}{b}\frac{1}{G(D,D')}f(ax+by) = \frac{x}{bG(a,b)}\phi(ax+by)$$

where $\phi(ax+by)$ is obtained after integration of $f(ax+by)$ and it is supposed that $G(a,b) \neq 0$.

Next consider the relation $F(D,D') = (bD - aD')G(D,D')$. Differentiating w.r.t. D, we get

$$F'(D,D') = bG(D,D') + (bD - aD')G'(D,D')$$

so that $F'(a,b) = bG(a,b)$. Thus $P.I. = \frac{x}{F'(a,b)}\phi(ax+by)$.

Generally, if $F(a,b) = 0$, we write

$$F(D,D') = \left(D - \frac{a}{b}D'\right)^r G(D,D'), \text{ provided } G(a,b) \neq 0.$$

Then,

$$P.I. = \frac{1}{F(D,D')}f(ax+by) = \frac{1}{G(a,b)} \cdot \frac{1}{\left(D - \frac{a}{b}D'\right)^r} f(ax+by)$$

Example-3.8 : Solve the equation $2r - 5s + 2t = 5\sin(2x+y)$

Solution. The given equation is $(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y)$

i.e. $(2D - D')(D - 2D')z = 5\sin(2x+y)$

Thus the complementary function is $u = \phi_1(x+2y) + \phi_2(2x+y)$

Here $F(D,D') = (2D^2 - 5DD' + 2D^2)$ so that $F(2,1) = 2\cdot 2^2 - 5\cdot 2\cdot 1 + 2\cdot 1^2 = 0$

Now $F'(D,D') = 4D - 5D'$ so that $F'(2,1) = 3$. Hence the particular integral is

$$z_1 = 5x \cdot \frac{1}{F'(2,1)}\{-\cos(2x+y)\} = -\frac{5}{3}x\cos(2x+y).$$

Thus the required solution is $z = u + z_1$, i.e. $z = \phi_1(x+2y) + \phi_2(2x+y) - \frac{5x}{3}\cos(2x+y)$

V. Let $f(x,y) = e^{ax+by}V(x,y)$. In this case, it is easy to find that

$$\frac{1}{F(D,D')}\{e^{ax+by}V(x,y)\} = \frac{1}{F(D+a,D'+b)}v(x,y).$$

Example-3.9 : Solve $D(D-2D')(D+D')z = e^{x+2y}(x^2+4y^2)$.

Solution. The complementary function is $u = \phi_1(g) + \phi_2(2x+y) + \phi_3(x-y)$.

The particular integral is

$$z_1=\frac{1}{D(D-2D')(D+D')}e^{x+2y}\left(x^2+2y^2\right)$$

$$=e^{x+2y}\frac{1}{(D+1)(D+1-2D'-4)(D+1+D'+2)}\left(x^2+2y^2\right)$$

$$=e^{x+2y}\frac{1}{(D+1)(D-2D'-3)(D+D'+3)}\left(x^2+2y^2\right)$$

$$=-\frac{1}{3}e^{x+2y}\frac{1}{(1+D)(3-D+2D')}\left(1+\frac{D+D'}{3}\right)\left(x^2+4y^2\right)$$

$$=-\frac{1}{3}e^{x+2y}\frac{1}{(1+D)(3-D+2D')}\left(1-\frac{D+D'}{3}+\frac{D^2+2DD'+D^2}{9}...\right)\left(x^2+4y^2\right)$$

$$=-\frac{1}{9}e^{x+2y}\frac{1}{(1+D)}\left(1-\frac{D-2D'}{3}\right)^{-1}\left(x^2+4y^2-\frac{2x}{3}-\frac{8y}{3}+\frac{10}{9}\right)$$

$$=-\frac{1}{9}e^{x+2y}\frac{1}{1+D}\left(1+\frac{D-2D'}{3}+\frac{D^2-4DD'+4D'^2}{9}+...\right)\left(x^2+4y^2-\frac{2x}{3}-\frac{8y}{3}+\frac{10}{9}\right)$$

$$=-\frac{1}{9}e^{x+2y}\left(1-D+D^{12}...\right)\left(x^2+4y^2-8y+\frac{58}{9}\right)$$

$$=-\frac{1}{9}e^{x+2y}\left(x^2+4y^2-8y+\frac{58}{9}-2x+2\right)$$

Hence the general solution of the given equation is

$$z=\phi_1(y)+\phi_2(2x+y)+\phi_3(x-y)-\frac{1}{81}e^{x+2y}\left(9x^2+36y^2-18x-72y+76\right).$$

Example-3.10 : Solve $\left(3D^2-2D'^2+D-1\right)z=3e^{x+y}\sin(x+y)$

Solution. The complementary function is $u=\sum_{r=1}^{\infty}C_r e^{a_r x+b_r y}$, where $3a_r^2-2b^2+a_r-1=0$.

The particular integral is

$$z_1=\frac{1}{3D^2-2D'^2+D-1}3e^{x+y}\sin(x+y)$$

$$= 3e^{x+y} \frac{1}{3(D+1)^2 - 2(D'+1)^2 + (D+1) - 1} \sin(x+y)$$

$$= 3e^{x+y} \frac{1}{3D^2 - 2D'^2 + 7D - 4D' + 1} \sin(x+y)$$

$$= 3e^{x+y} \frac{1}{3(-1^2) - 2\cdot(-1^2) + 7D - 4D' + 1} \sin(x+y)$$

$$= 3e^{x+y} \frac{1}{7D - 4D'} \sin(x+y)$$

$$= 3e^{x+y} \frac{7D + 4D'}{49D^2 - 16D'^2} \sin(x+y)$$

$$= 3e^{x+y} \frac{7D + 4D'}{49(-1^2) - 16(-1^2)} \sin(x+y)$$

$$= -\frac{3e^{x+y}}{33}(7D + 4D')\sin(x+y) = -e^{x+y}\cos(x+y)$$

Thus the required solution is

$$z = \sum_{r=1}^{\infty} C_r e^{a_r x + b_r y} - e^{x+y}\cos(x+y),$$

where $3a_r^2 + a_r - 1 = 0$.

Example-3.11 : Solve $(D^2 + DD' - 6D'^2)z = x^2 \sin(x+2y)$

Solution. The given equation is $(D - 2D')(D + 3D')z = x^2 \sin(x+2y)$.

The complementary function is $u = \phi_1(2x+y) + \phi_2(3x-y)$

Nothing that $\sin(x+2y) = \text{Im}\, e^{i(x+2y)}$, we get the particular integral as

$$z_1 = \text{Im}\left[\frac{1}{(D-2D')(D+3D')} x^2 e^{i(x+2y)}\right]$$

$$= \text{Im}\left[e^{i(x+2y)} \frac{1}{(D+i-2D'-4i)(D+i+3D'-6i)} x^2\right]$$

$$= \text{Im}\left[e^{i(x+2y)} \frac{1}{(D-2D'-3i)(D+3D'+7i)} x^2\right]$$

$$= \text{Im}\left[\frac{1}{7i}e^{i(x+2y)}\frac{1}{(D-2D'-3i)}\left(1+\frac{D+3D'}{7i}\right)x^2\right]$$

$$= \text{Im}\left[\frac{1}{7i}e^{i(x+2y)}\frac{1}{D-2D'-3i}\left(1-\frac{D+3D'}{7i}-\frac{D^2+6DD'+9D'^2}{49}\cdots\right)x^2\right]$$

$$= \text{Im}\left[\frac{1}{7i}e^{i(x+2y)}\frac{1}{D-2D'-3i}\left(x^2-\frac{2}{7i}x-\frac{2}{49}\right)\right]$$

$$= \text{Im}\left[\frac{1}{21}e^{i(x+2y)}\left(1-\frac{D-2D'}{3i}\right)^{-1}\left(x^2+\frac{2i}{7}x-\frac{2}{49}\right)\right)$$

$$= \text{Im}\left[\frac{1}{21}e^{i(x+2y)}\left(1+\frac{D-2D'}{3i}-\frac{D^2-4DD'+4D'^2}{9}+\ldots\right)\left(x^2+\frac{2i}{7}x-\frac{2}{49}\right)\right]$$

$$= \text{Im}\left[\frac{1}{21}e^{i(x+2y)}\left(x^2+\frac{2i}{7}x-\frac{2}{49}+\frac{2x}{3i}+\frac{2i}{7}\cdot\frac{1}{3i}-\frac{2}{9}\right)\right]$$

$$= \frac{1}{21}\text{Im}\left[e^{i(x+2y)}\left(x^2+\frac{2i}{7}x-\frac{2}{49}-\frac{2xi}{3}+\frac{2}{7}-\frac{2}{9}\right)\right]$$

$$= \frac{1}{21}\text{Im}\left[\{\cos(x+2y)+i\sin(x+2y)\}\left\{\left(x^2-\frac{116}{441}\right)-\frac{8ix}{21}\right\}\right]$$

$$= \frac{1}{21}\text{Im}\left\{\left(x^2-\frac{116}{441}\right)\sin(x+2y)-\frac{8x}{21}\cos(x+2y)\right\}$$

Thus the required solution is

$$z = \phi_1(2x+y)+\phi_2(3x-y)+\frac{1}{21}\left\{\left(x^2-\frac{116}{441}\right)\sin(x+2y)-\frac{8x}{21}\cos(x+2y)\right\}.$$

VI. General method

If the function $f(x, y)$ is not of above discussed form or even the above methods fail, then we adopt the following procedure to find the particular integral

$$z_1 = \frac{1}{F(D, D')}f(x, y).$$

Let us consider the equation $(D-mD')z=f(x,y)$, i.e. $p-mq=f(x,y)$, so that Lagrange's subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-m}=\frac{dz}{f(x,y)}.$$

The first two relations lead to $y+mx=\text{constant}=c$, say and the equations

$$\frac{dx}{1}=\frac{dz}{f(x,y)}$$

give $z=\int f(x,y)dx=\int f(x,c-mx)dx.$

Thus

$$z_1=\frac{1}{D-mD'}f(x,y)=\int f(x,c-mx)dx$$

in which we have to replace c by $y+mx$ after integration.

If the given equation is of the form $F(D,D')=f(x,y)$, where

$F(D,D')=(D-m_1D')(D-m_2D')...(D-m_nD')$, then the particular integral

$$z_1=\frac{1}{(D-m_1D')(D-m_2D')...(D-m_nD')}f(x,y)$$

is evaluated by the repeated application of the above method.

Example-3.12 : Solve $(D^2-4D'^2)z=\frac{4x}{y^2}-\frac{y}{x^2}$.

Solution. The complementary function is $u=\phi_1(2x+y)+\phi_2(2x-y)$.

The particular integral is

$$z_1=\frac{1}{(D+2D')(D-2D')}\left(\frac{4x}{y^2}-\frac{y}{x^2}\right)$$

$$=\frac{1}{D+2D'}\int\left\{\frac{kx}{(c-2x)^2}-\frac{c-2x}{x^2}\right\}dx$$

[as corresponding to $(D-2D')z=0, y+2x=c$]

$$=\frac{1}{D+2D'}\int\left\{-\frac{2}{c-2x}+\frac{2c}{(c-2x)^2}-\frac{c}{x^2}+\frac{2}{x}\right\}dx$$

$$=\frac{1}{D+2D'}\left\{\log(c-2x)+\frac{2}{c-2x}+\frac{c}{x}+2\log x\right\}$$

$$=\frac{1}{D+2D'}\left\{\log y+\frac{y+2x}{y}+\frac{y+2x}{x}+2\log x\right\}$$

(replacing c by $y+2x$)

$$=\int\left\{\log(c'+2x)+\frac{c'+4x}{c'+2x}+\frac{c'+4x}{x}+2\log x\right\}dx$$

[corresponding to $(D+2D')z=0, y-2x=c'$]

$$=\int\left\{\log(c'+2x)+2-\frac{c'}{c'+2x}+\frac{c'}{x}+4+2\log x\right\}dx$$

$$=\int\left\{\log(c'+2x)+6-\frac{c'}{c'+2x}+\frac{c'}{x}+2\log x\right\}dx$$

$$=x\log(c'+2x)-\int\frac{2x}{c'+2x}dx+6x-\frac{1}{2}c'\log(c'+2x)+c'\log x+2x\log x-2x$$

$$=x\log(c'+2x)-\int\frac{c'+2x-c'}{c'+2x}dx+4x-\frac{1}{2}c'\log(c'+2x)+c'\log x+2x\log x$$

$$=x\log(c'+2x)-x+\frac{1}{2}c'\log(c'+2x)+4x-\frac{1}{2}c'\log(c'+2x)+c'\log x+2x\log x$$

$$=x\log(c'+2x)+(c'+2x)\log x+3x$$

$=x\log y+y\log x+3x$ (replacing c' by $y-2x$)

Hence the required solution is

$$z=\phi_1(2x+y)+\phi_2(2x-y)+x\log y+y\log x+3x.$$

4. Homogeneous Equations with Variable Coefficients : Cauchy - Euler Equations.

Sometimes it is possible to reduce a partial differential equation with variable coefficients into an equation with constant coefficients by suitable transformations. One such type is a homogeneous partial differential equation, valled Cauchy-Euler equation, of the form $F(xD, yD')z = f(x,y)$. Here to find the solution, we put $x = e^u, y = e^v$, i.e. $u = \log x, v = \log y$.

Then, if $D \equiv \frac{\partial}{\partial x}, D' \equiv \frac{\partial}{\partial y}, \theta \equiv \frac{\partial}{\partial u}, \theta' \equiv \frac{\partial}{\partial v}$, we have

$$Dz \equiv \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial u}, \text{ i.e. } xD \equiv \theta,$$

$$D^2 z \equiv \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial u}\right) = -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x^2}\frac{\partial^2 z}{\partial u^2}, \text{i.e.} (xD)^2 \equiv \theta(\theta - 1)$$

so that, in general, $(xD)^m = \theta(\theta-1)(\theta-2)...(\theta-m+1)$.

Similarly, $(yD')^n = \theta'(\theta'-1)(\theta'-2)...(\theta'-n+1)$,

Substituting these into the given equation, we get an equation with constant coefficeints and solution can be found by the method discussed in Section-3.

Example-4.1 : Solve the equation

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = \left(x^2 + y^2\right)^{n/2}$$

Solution.

Let $x = e^u, y = e^v, D \equiv \frac{\partial}{\partial x}, D' \equiv \frac{\partial}{\partial y}, \theta \equiv \frac{\partial}{\partial u}, \theta' \equiv \frac{\partial}{\partial v}$

Then $(xD)^2 = \theta(\theta-1), (xD)(yD') = \theta\theta', (yD')^2 = \theta'(\theta'-1)$ and the given equation reduces to

$$\{\theta(\theta-1) + \theta\theta' + \theta'(\theta'-1)\}z = \left(e^{2u} + e^{2v}\right)^{n/2}$$

or, $\quad (\theta+\theta')(\theta+\theta'-1)z = \left(e^{2u} + e^{2v}\right)^{n/2}$

Here the complementary function is $u_1 = \phi_1(v-u) + e^u\phi_2(v-u)$

i.e. $u_1 = f_1\left(\frac{y}{x}\right) + x f_2\left(\frac{y}{x}\right)$

and the particular integral is

$$z_1 = \frac{1}{(\theta+\theta')(\theta+\theta'-1)}\left(e^{2u}+e^{2v}\right)^{n/2}$$

$$= \frac{1}{(\theta+\theta')(\theta+\theta'-1)}e^{nu}\left[1+e^{2(v-u)}\right]^{n/2}$$

$$= \frac{1}{(\theta+\theta')(\theta+\theta'-1)}\left[e^{nu}+\frac{1}{2}ne^{(n-2)u+2v}+\frac{\frac{1}{2}n\left(\frac{1}{2}n-1\right)}{2!}e^{(n-4)u+4v}+\dots\right]$$

$$= \frac{e^{nu}}{n^2-n}\left[1+\frac{1}{2}ne^{2(v-u)}+\dots\right]$$

$$= \frac{e^{nu}\left[1+e^{2(v-u)}\right]^{n/2}}{n(n-1)}$$

$$= \frac{\left(e^{2u}+e^{2v}\right)^{n/2}}{n(n-1)}$$

$$= \frac{\left(x^2+y^2\right)^{n/2}}{n(n-1)}.$$

Thus the complete solution of the given equation is

$$z = f_1\left(\frac{y}{x}\right)+x f_2\left(\frac{y}{x}\right)+\frac{\left(x^2+y^2\right)^{n/2}}{n(n-1)}.$$

5. Classification of Second Order Partial Differential Equations. Canonical Forms.

Let us consider partial differential equations of the form

$$Rr+Ss+Tt+f(x,y,z,p,q)=0 \qquad (5.1)$$

where

$$r=\frac{\partial^2 z}{\partial x^2},\ s=\frac{\partial^2 z}{\partial x\partial y},\ t=\frac{\partial^2 z}{\partial y^2}$$

and each of R, S, T is a continuous function of x and y possessing continous partial derivatives of as high order as necessary with respect to x and y.

A second order partial differential equation which is linear with respect to the second order derivatives is said to be a quasi-linear partial differential equation.

We now proceed to show that by suitable change of independent variable the equation (5.1) can be transformed into one of three canonical forms which can be easily integrated. The equation (5.1) is said to be hyperbolic, parabolic or elliptic according as $S^2-4RT>0, S^2-4RT=0$ or $S^2-4RT<0$ at a point (x_0, y_0) within a domain Ω. If this is true for all points within the domain Ω, then the equation is hyperbolic, parabolic or elliptic in Ω. Canonical (or normal) form:

In order to reduce the equation (5.1) into canonical form, let us introduce transformations

$$\xi=\xi(x,y), \eta=\eta(x,y) \tag{5.2}$$

where ξ and η are supposed to be continuously differentiable and the Jacobian

$$J=\frac{\partial(\xi,\eta)}{\partial(x,y)}=\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix}=\xi_x\eta_y-\xi_y\eta_x\neq 0. \tag{5.3}$$

Also we write the dependent variable ξ in the transformed space instead of z.

Now

$$p=\frac{\partial z}{\partial x}=\frac{\partial\zeta}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial\zeta}{\partial\eta}\frac{\partial\eta}{\partial x}=\zeta_\xi\xi_x+\zeta_\eta\eta_x.$$

Similarly, $q=\zeta_\xi\xi_y+\zeta_\eta\eta_y$.

Also

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\zeta_\xi\xi_x+\zeta_\eta\eta_x\right)=\zeta_{\xi\xi}\xi_x^2+2\zeta_{\xi\eta}\xi_x\eta_x+\zeta_{\eta\eta}\eta_x^2+\zeta_\xi\xi_{xx}+\zeta_\eta\eta_{xx}.$$

Similarly, $s=\dfrac{\partial^2 z}{\partial x\partial y}=\zeta_{\xi\xi}\xi_x\xi_y+\zeta_{\xi\eta}\left(\xi_x\eta_y+\xi_y\eta_x\right)+\zeta_{\eta\eta}\eta_x\eta_y+\zeta_\xi\xi_{xy}+\zeta_\eta\eta_{xy}$

and $t=\dfrac{\partial^2 z}{\partial y^2}=\zeta_{\xi\xi}\xi_y^2+2\zeta_{\xi\eta}\xi_y\eta_y+\zeta_{\eta\eta}\eta_y^2+\zeta_\xi\xi_{yy}+\zeta_\eta\eta_{yy}$

Substitutions of the above values of p, q, r, s and t in (5.1) lead to the following form :

$$A(\xi_x,\xi_y)\zeta_{\xi\xi}+2B(\xi_x,\xi_y,\eta_x,\eta_y)\zeta_{\xi\eta}+A(\eta_x,\eta_y)\zeta_{\eta\eta}=F(\xi,\eta,\zeta_\xi,\zeta_\eta) \tag{5.4}$$

where

$$A(u,v)=Ru^2+Suv+Tv^2,$$

$$B(u_1,v_1,u_2,v_2) = Ru_1u_2 + \frac{1}{2}S(u_1v_2 + u_2v_1) + Tv_1v_2, \tag{5.5}$$

$$F(\xi,\eta,\zeta,\zeta_\xi,\zeta_\eta) = -\left[\zeta_\xi\left(R\xi_{xx} + S\xi_{xy} + T\xi_{yy}\right) + \zeta_\eta\left(R\eta_{xx} + S\eta_{xy} + T\eta_{yy}\right) + f(\xi,\eta,\zeta,\zeta_\xi,\zeta_\eta)\right]$$

It is easy to verify that

$$B^2(\xi_x,\xi_y,\eta_x,\eta_y) - A(\xi_x,\xi_y)A(\eta_x,\eta_y) = (S^2 - 4RT)J^2. \tag{5.6}$$

There are the following three different cases :

Case I. $S^2 - 4RT > 0.$

If $S^2 - 4RT > 0$, then $R\lambda^2 + S\lambda + T = 0$, called characteristic equation, has two real and distinct roots λ_1 and λ_2, say. We choose ξ and η in such a way that $\xi_x = \lambda_1\xi_y, \eta_x = \lambda_2\eta_y$.

Now the equation $\xi_x - \lambda_1\xi_y = 0$ gives $\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{d\xi}{0}$ so that $\xi(x,y) =$ constant and

$$\frac{dy}{dx} + \lambda_1(x,y) = 0. \tag{5.7a}$$

Similarly, we have $\eta(x,y) =$ constant and

$$\frac{dy}{dx} + \lambda_2(x,y) = 0 \tag{5.7b}$$

Let the solutions of the equations (5.7a, b) be

$$\xi = f_1(x,y), \eta = f_2(x,y). \tag{5.8}$$

Now it follows that

$$A(\xi_x,\xi_y) = R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2 = \xi_x^2\left(R\lambda_1^2 + S\lambda_1 + T\right) = 0.$$

Similarly $A(\eta_x,\eta_y) = 0$. Thus (5.6) shows that $B^2 = (S^2 - 4RT)J^2 > 0$.

Hence the equation (5.4) reduce to the form

$$\zeta_{\xi\eta} = \phi(\xi,\eta,\zeta,\zeta_\xi,\zeta_n) \tag{5.9}$$

which is the required canonical form of hyperbolic partial differential equation.

Case II. $S^2 - 4RT = 0.$

In this case the equation $R\lambda^2 + S\lambda + T = 0$ has equal roots $\lambda, \lambda;$ say. Let us take $\xi = f_1(x,y)$, where $f_1(x,y) =$ const., as a solution of the equation

$$\frac{dy}{dx}+\lambda(x,y)=0.$$

Noting that $A(\xi_x,\xi_y)=R\xi_x^2+S\xi_x\xi_y+T\xi_y^2=\xi_y^2(R\lambda^2+S\lambda+T)=0$ and $S^2-4RT=0$,

We have from (5.6) that $B=0$. However, $A(\eta_x,\eta_y)\neq 0$, otherwise η will depend on ξ. Then the equation (5.4) leads to

$$\zeta_{\eta\eta}=\psi(\xi_x,\eta,\zeta,\zeta_\xi,\zeta_\eta) \qquad (5.10)$$

This is the canonical forms of parabolic partial differential equation.

Note : In the above, the function $\eta(x,y)$ can be chosen arbitrarily provided that the Jacobian J of the transformation does not vanish.

Case III. $S^2-4RT<0$.

Here the roots λ_1 and λ_2 of the equation $R\lambda^2+S\lambda+T=0$ are complex and so ξ and η are complex conjugate. We put $\xi=\alpha+i\beta,\eta=\alpha-i\beta$ so that $\alpha=\frac{1}{2}(\xi+\eta),\beta=\frac{i}{2}(\eta-\xi)$. Then

$$\frac{\partial\zeta}{\partial\xi}=\frac{1}{2}\left(\frac{\partial\zeta}{\partial\alpha}-i\frac{\partial\zeta}{\partial\beta}\right),\frac{\partial\zeta}{\partial\eta}=\frac{1}{2}\left(\frac{\partial\zeta}{\partial\alpha}+i\frac{\partial\zeta}{\partial\beta}\right),\frac{\partial^2\zeta}{\partial\xi\partial\eta}=\frac{1}{4}\left(\frac{\partial^2\zeta}{\partial\alpha^2}+\frac{\partial^2\zeta}{\partial\beta^2}\right).$$

It is easily seen that $A(\xi_x,\xi_y)=A(\eta_x,\eta_y)=0$ and, therefore, from (4.6) $B^2<0$.

Thus the equation (5.4) reduces to

$$\frac{\partial^2\zeta}{\partial\alpha^2}+\frac{\partial^2\zeta}{\partial\beta^2}=\chi(\alpha,\beta,\zeta,\zeta_\alpha,\zeta_\beta). \qquad (5.11)$$

which is the canonical form of elliptic partial differential equation.

Example-5.1 : Reduce the equation

$$xyr-(x^2-y^2)s-xyt+py-qx=2(x^2-y^2)$$

into canonical form and hence solve it.

Solution. Here $R=xy,S=-(x^2-y^2),T=-xy,f(x,y,z,p,q)=py-qx-2(x^2-y^2)$.

Since $S^2-4RT=(x^2-y^2)^2+4x^2y^2=(x^2+y^2)^2>0$, so the given equation is of hyperbolic type.

Let $\xi=\xi(x,y),\eta=\eta(x,y)$ and the dependent variable is ζ in the transformed space. Transforming the given equation in new variables, we have

$$A(\xi_x,\xi_y)\zeta_{\xi\xi}+2B(\xi_x,\xi_y,\eta_x,\eta_y)\zeta_{\xi\eta}+A(\eta_x,\eta_y)\zeta_{\eta\eta}=F(\xi,\eta,\zeta,\zeta_\xi,\zeta_\eta) \quad (1)$$

where A, B and F are obtained from (5.5) as

$$A(u,v)=xyu^2-(x^2-y^2)uv-xyu^2$$

$$B(\xi_x,\xi_y,\eta_x,\eta_y)=xy\,\xi_x\eta_x-\frac{1}{2}(x^2-y^2)(\xi_x\eta_y+\xi_y\eta_x)-xy\,\xi_y\eta_y \quad (2)$$

$$F(\xi,\eta,\zeta,\zeta_\xi,\zeta_\eta)=-\left[\zeta_\xi\left\{xy\xi_{xx}-(x^2-y^2)\xi_{xy}-xy\xi_{yy}\right\}+\zeta_\eta\left\{xy\,\eta_{xx}-(x^2-y^2)\eta_{xy}-xy\,\eta_{xy}\right\}\right.$$
$$\left.+(\zeta_\xi\xi_x+\zeta_\eta\eta_x)y-(\zeta_\xi\xi_y+\zeta_\eta\eta_y)x-2(x^2-y^2)\right]$$

Now consider the equation $R\lambda^2+S\lambda+T=0$, i.e. $xy\lambda^2-(x^2-y^2)\lambda-xy=0$ whose roots are $\lambda_1=\frac{x}{y},\lambda_2=-\frac{y}{x}\cdot$ We, therefore, have

$$\frac{dy}{dx}+\frac{x}{y}=0,\ \frac{dy}{dx}-\frac{y}{x}=0$$

whose solutions are $x^2+y^2=$ const. and $\frac{y}{\alpha}=$ const. We choose $\xi(x,y)=x^2+y^2,\eta(x,y)=\frac{y}{x}$, so that

$$\xi_x=2x,\xi_y=2y,\xi_{xx}=2,\xi_{xx}=0,\xi_{yy}=2;$$

$$\eta_x=-\frac{y}{x^2},\eta_y=\frac{1}{x},\eta_{xx}=\frac{2y}{x^3},\eta_{xy}=-\frac{1}{x^2},\eta_{yy}=0.$$

If follows from (2) that $A(\xi_x,\xi_y)=xy\cdot 4x^2-(x^2-y^2)\cdot 4xy-xy\cdot 4y^2=0$ and

$$A(\eta_x,\eta_y)=xy\cdot\frac{y^2}{x^4}-(x^2-y^2)\cdot\left(-\frac{y}{x^3}\right)-xy\cdot\frac{1}{x^2}=0.$$

$$B(\xi_x,\xi_y,\eta_x,\eta_y)=xy\cdot(2x)\left(-\frac{y}{x^2}\right)-\frac{1}{2}(x^2-y^2)\left(2x\cdot\frac{1}{x}-2y\cdot\frac{y}{x^2}\right)-xy\cdot 2y\cdot\frac{1}{x}=-\frac{(x^2+y^2)^2}{x^2}$$

and $F(\xi,\eta,\zeta,\zeta_\xi,\zeta_\eta)=-\left[\zeta_\xi(xy\cdot 2-xy\cdot 2)+\zeta_\eta\left\{xy\cdot\frac{2y}{x^3}+(x^2-y^2)\cdot\frac{1}{x^2}-xy\cdot 0\right\}\right.$

$$\left.+\left(\zeta_\xi\cdot 2x-\zeta_\eta\cdot\frac{y}{x^2}\right)\cdot y-\left(\zeta_\xi\cdot 2y+\zeta_x\cdot\frac{1}{x}\right)x-2(x^2-y^2)\right.$$

$$=2(x^2-y^2)$$

Thus from (1) we get

$$-2\frac{\left(x^2+y^2\right)^2}{x^2}\frac{\partial^2\zeta}{\partial\xi\partial\eta}=2\left(x^2-y^2\right),\ i.e.\ \frac{\partial^2\zeta}{\partial\xi\partial\eta}=\frac{\left(y^2-x^2\right)x^2}{\left(x^2+y^2\right)^2}=\frac{\frac{y^2}{x^2}-1}{\left(\frac{y^2}{x^2}+1\right)^2}$$

and hence

$$\frac{\partial^2\zeta}{\partial\xi\partial\eta}=\frac{\eta^2-1}{\left(\eta^2+1\right)^2}. \tag{3}$$

This is the required canonical form of the given equation.

Integration of (3), first w.r.t. η and ξ then w.r.t. ξ leads to

$$\zeta=-\frac{\xi\eta}{\eta^2+1}+\psi_1(\xi)+\psi_2(\eta)$$

so that the solution of the given equation is

$$z=-\frac{\left(x^2+y^2\right)\cdot\frac{y}{x}}{\frac{y^2}{x^2}+1}+\psi_1\left(x^2+y^2\right)+\psi_2\left(\frac{y}{x}\right)$$

i.e. $z=-xy+\psi_1\left(x^2+y^2\right)+\psi_2\left(\frac{y}{x}\right).$

Example-5.2 : Reduce the equation

$$x^2r-2xys+y^2t-xp+3yq=8\cdot\frac{y}{x}.$$

into canonical form and hence solve it.

Solution. Here $R=x^2, S=-2xy, T=y^2, f(x,y,z,p,q)=-xp+3yq-8\cdot\frac{y}{x}.$

Since $S^2-4RT=4x^2y^2-4\cdot x^2\cdot y^2=0,$ so the given equation is of parabolic type.

Now, the equation $R^2\lambda^2+S\lambda+T=0,$ i.e. $x^2\lambda^2-2xy\lambda+y^2=0$ gives equal roots $\lambda,\lambda=\frac{y}{x}.$ Then from the equation $\frac{dy}{dx}+\frac{y}{x}=0,$ we get the solution as $xy=$ const. Therefore, we take $\xi(x,y)=xy.$ We choose

$\eta(x,y)=\frac{y}{x}$. Thus

$$\xi_x = y, \xi_y = x, \xi_{xx} = 0, \xi_{xy} = 1, \xi_{yy} = 0$$

$$\eta_x = -\frac{y}{x^2}, \eta_y = \frac{1}{x}, \eta_{xx} = \frac{2y}{x^3}, \eta_{xy} = -\frac{1}{x^2}, \eta_{yy} = 0.$$

Hence $A(\xi_x,\xi_y) = R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2 = x^2 \cdot y^2 - 2xy \cdot y \cdot x + y^2 \cdot x^2 = 0$

$$A(\eta_x,\eta_y) = R\eta_x^2 + S\eta_x\eta_y + T\eta_y^2 = x^2 \cdot \frac{y^2}{x^4} - 2xy \cdot \left(-\frac{y}{x^2}\right) \cdot \frac{1}{x} + y^2 \cdot \frac{1}{x^2} = \frac{4y^2}{x^2} = 4\eta^2$$

$$B(\xi_x,\xi_y,\eta_x,\eta_y) = x^2 \cdot y \cdot \left(-\frac{y}{x^2}\right) + \frac{1}{2}(-2xy)\left(y \cdot \frac{1}{x} - x \cdot \frac{y}{x^2}\right) + y^2 \cdot x \cdot \frac{1}{x} = 0$$

$$F(\xi,\eta,\zeta,\zeta_\xi,\zeta_\eta) = -\left[\zeta_\xi\left(x^2 \cdot 0 - 2xy \cdot 1 + y^2 \cdot 0\right) + \zeta_\eta\left(x^2 \cdot \frac{2y}{x^3} + 2xy \cdot \frac{1}{x^2} + y^2 \cdot 0\right)\right.$$

$$\left. -x \cdot \left(\zeta_\xi y - \zeta_\eta \cdot \frac{y}{x^2}\right) + 3y\left(\zeta_\xi \cdot x + \zeta_\eta \cdot \frac{1}{x}\right) - \frac{8y}{x}\right]$$

$$= -8 \cdot \frac{y}{x}\zeta_\eta + \frac{8y}{x} = -8\eta\zeta_\eta + 8\eta.$$

Thus the required canonical form of the given equation is

$$4\eta^2\zeta_{\eta\eta} = -8\eta\zeta_\eta + 9\eta$$

i.e. $\zeta_{\eta\eta} + \frac{2}{\eta}\zeta_\eta = \frac{2}{\eta}$.

Let $\zeta_\eta = Z$, so that this equation is transformed to $Z_\eta + \frac{2}{\eta}Z = \frac{2}{\eta}$

This is a linear differential equation with integrating factor $e^{\int \frac{2}{\eta}d\eta} = \eta^2$ and so the solution of this equation is

$$\zeta_\eta = Z = 1 + \frac{1}{\eta^2}\phi_1(\xi)$$

Integrating again, we get

$$\zeta = \eta - \frac{1}{\eta}\phi_1(\xi) + \phi_2(\xi).$$

Thus the solution of the given equation is

$$z=\frac{y}{x}-\frac{x}{y}\phi_1(xy)+\phi_2(xy).$$

Example-5.3 : Reduce the equation

$$e^x z_{xx}+e^{y} z_{yy}=z$$

to canonical form.

Solution. Here $R=e^x, S=0, T=e^y$ $f(x,y,z,p,q)=-z.$

$$\therefore S^2-4RT=-4e^{x+y}<0.$$

Thus the given equation is of elliptic type.

Now the equation $R\lambda^2+S\lambda+T=0$ i.e. $e^x\lambda^2+e^y=0$ gives $\lambda=\pm ie^{\frac{y-x}{2}}$. The characteristic equations are

$$\frac{dy}{dx}-ie^{\frac{y-x}{2}}=0 \text{ and } \frac{dy}{dx}+ie^{\frac{y-x}{2}}=0$$

i.e. $e^{-\frac{y}{2}}dy-ie^{-\frac{x}{2}}=0$ and $e^{-\frac{y}{2}}dy-e^{-\frac{x}{2}}=0$

whose solutions are $e^{-\frac{y}{2}}-ie^{-\frac{x}{2}}=$ consts. and $e^{-\frac{y}{2}}+ie^{-\frac{x}{2}}$=consts. We take

$$\xi(x,y)=e^{-\frac{y}{2}}-ie^{-\frac{x}{2}} \text{ and } \eta(x,y)=e^{\frac{y}{2}}+ie^{-\frac{x}{2}}$$

Introducing the second transformation $\xi=\alpha+i\beta$ and $\eta=\alpha-i\beta$, we have

$$\alpha=e^{-\frac{y}{2}} \text{ and } \beta=e^{-\frac{x}{2}}$$

Proceeding exactly along the same lines as given in Examples 5.1 and 5.2 we find that

$$A(\xi_x,\xi_y)=A(\eta_x,\eta_y)=0,$$

$$B(\xi_x,\xi_y,\eta_{x_1},\eta_y)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$$F(\xi,\eta,z,\zeta_x,\zeta_y)=-\left[\zeta_\xi\left\{e^x\cdot\left(-\frac{1}{4}i\right)e^{-\frac{x}{2}}+e^y\cdot\frac{1}{4}e^{-\frac{y}{2}}\right\}+\zeta_\eta\left\{e^x\left(\frac{1}{4}ie^{-\frac{x}{2}}+e^y\cdot\frac{1}{4}e^{-\frac{y}{2}}\right)\right\}-\zeta\right]$$

$$=\frac{1}{4}\left[4\zeta-\frac{1}{\alpha}\zeta_\alpha-\frac{1}{\beta}\zeta_\beta\right]$$

Thus the required canonical form of the given equation is

$$2\cdot\frac{1}{2}\cdot\frac{1}{4}\left(\frac{\partial^2\zeta}{\partial\alpha^2}+\frac{\partial^2\zeta}{\partial\beta^2}\right)=\frac{1}{4}\left(4\zeta-\frac{1}{\alpha}\zeta_\alpha-\frac{1}{\beta}\zeta_\beta\right)$$

or, $$\frac{\partial^2\zeta}{\partial\alpha^2}+\frac{\partial^2\zeta}{\partial\beta^2}=4\zeta-\frac{1}{\alpha}\zeta_\alpha-\frac{1}{\beta}\zeta_\beta.$$

6. Solution of Linear Hyperbolic Equations (Riemann's Method)

Consider the Linear hyperbolic partial differential equation

$$\frac{\partial^2 z}{\partial x\partial y}+a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}+cz=f(x,y)$$

or, $$L(z)=f(x,y) \tag{6.1}$$

where $L\equiv\frac{\partial^2}{\partial x\partial y}+a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}+c$

and each of a, b, c is a function of x and y having continuous first order derivatives with respect to x and y. Suppose w is another function of x and y having continuous first order partial derivatives. Now it follows than

$$w\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 w}{\partial x\partial y}=\frac{\partial}{\partial y}\left(w\frac{\partial z}{\partial x}\right)-\frac{\partial}{\partial x}\left(z\frac{\partial w}{\partial y}\right)$$

$$wa\frac{\partial z}{\partial x}+z\frac{\partial}{\partial x}(aw)=\frac{\partial}{\partial x}(awz)$$

$$wb\frac{\partial z}{\partial y}+z\frac{\partial}{\partial x}(bw)=\frac{\partial}{\partial y}(bwz)$$

Then it is evident from these relations that

$$wL(z)-zL^*(w)=\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y} \tag{6.2}$$

where

$$L^*(w)=\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial}{\partial x}(aw)-\frac{\partial}{\partial y}(bw)+cw,$$

$$U=awz-z\frac{\partial z}{\partial y},\ V=bwz+w\frac{\partial z}{\partial x}.$$

The operator L^* defined by (6.3) is known as adjoint operator of L and the equation (6.2) is called the Lagrarge identity. If $L^* = L$, then the operator L is self-adjoint operator.

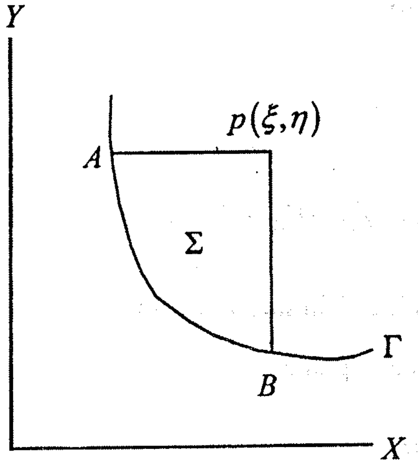

Fig. 1

Now consider a curve Γ and $p(\xi,\eta)$ a point on the xy - plane and let AB be an are of Γ. Draw PA and PB parallel to the x and y-axes interesting Γ at the points A and B respectively. Also suppose that Σ be the area enclosed by the contour $ABPA$. Then by Green's theorem

$$\iint_{\Sigma}\left[wL(z)-zL^*(w)\right]dx\,dy=\iint_{\Sigma}\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)dx\,dy$$

$$=\int_{ABPA}(U\,dy-V\,dx)=\int_A^B(U\,dy-V\,dx)+\int_B^P U\,dy-\int_P^A V\,dx \qquad (6.4)$$

Now $\int_P^A V\,dx=\int_P^A\left(bwz+w\frac{\partial z}{\partial x}\right)dx=\int_P^A z\left(bw-\frac{\partial w}{\partial x}\right)dx+\int_P^A\frac{\partial}{\partial x}(zw)\,dx$

$$=\int_P^A z\left(bw-\frac{\partial w}{\partial x}\right)dx+[zw]_A-[zw]_P$$

so that

$$[zw]_P=[zw]_A+\int_P^A z\left(bw-\frac{\partial w}{\partial x}\right)dx-\int_P^A V\,dx$$

$$=[zw]_A+\int_P^A z\left(bw-\frac{\partial w}{\partial x}\right)dx-\int_B^P U\,dy-\int_A^B(U\,dy-V\,dx)$$

$$+\iint\limits_{\Sigma}\left[wL(z)-zL^{*}(w)dx\,dy\right] \qquad \text{by using (6.4)}$$

$$=[zw]_A+\int_P^A z\left(bw-\frac{\partial w}{\partial x}\right)dx-\int_B^P z\left(aw-\frac{\partial w}{\partial y}\right)dy$$

$$-\int_A^B wz(ady-bdz)+\int_A^B z\left(z\frac{\partial w}{\partial y}dy+w\frac{\partial z}{\partial z}dx\right)$$

$$+\iint\limits_{\Sigma}\left[wL(z)-zL^{*}(w)\right]dx\,dy. \tag{6.5}$$

Let us choose the arbitrary function $w(x,y)$ in such a way that

(i) $L^{*}(w)=0$ throughout the xy - plane,

(ii) $\frac{\partial w}{\partial x}=b(x,y)$ when $y=\eta$,

(iii) $\frac{\partial w}{\partial x}=a(x,y)$ when $x=\xi$ (6.6)

and (iv) $w=1$ when $x=\xi, y=\eta$.

Such a function w is called Green's function or Riemann-Green's function.

Thus, noting that $L(z)=f(x,y)$, we have from (6.5)

$$[z]_p=[zw]_A-\int_A^B wz(a\,dy-b\,dx)+\int_A^B\left(z\frac{\partial w}{\partial y}dy+w\frac{\partial z}{\partial x}dx\right)$$

$$+\iint\limits_{\Sigma} w(x,y)\,dx\,dy \tag{6.7}$$

Then, if the values of z and $\frac{\partial z}{\partial x}$ are given on the curve AB, the value of z at any point P can be obtained from (6.7).

On the other hand, if the values of z and $\frac{\partial z}{\partial y}$ are prescribed on the curve, then we reawrite the equation (6.7) in the form

$$[z]_p=[zw]_B-\int_A^B wz(a\,dy-b\,dx)+\int_A^B\left(z\frac{\partial w}{\partial y}dx+w\frac{\partial z}{\partial y}dy\right)$$

$$+\iint\limits_{\Sigma} wf(x,y)\,dx\,dy \tag{6.8}$$

Adding (6.7) and (6.8) and then dividing by 2, we obtain

$$[z]_p = \frac{1}{2}\{[zw]_A + [zw]_B\} - \int_A^B wz(a\,dy - b\,dx) - \frac{1}{2}\int_A^B w\left(\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right)$$

$$-\int_A^B z\left(\frac{\partial w}{\partial x}dx - \frac{\partial w}{\partial y}dy\right) + \iint_\Sigma wf(x,y)\,dx\,dy \qquad (6.9)$$

Hence the solution of equation (6.1) at any point in Σ is obtained in terms of the given values of $z, \frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$ along a curve Γ by by means of either of the formulas (6.7), (6.8) or (6.9).

Example-6.1 : Construct the adjoint of the differential equation $L(z) = c^2 z_{xx-} z_{tt} - z_{tt}$.

Solution. Let w be a function of x and t having continuous first order partial derivatives. Then we have

$$w\left(\frac{\partial^2 z}{\partial x^2}\right) - z\left(\frac{\partial^2 w}{\partial x^2}\right) = \frac{\partial}{\partial x}\left(w\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial x}\left(z\frac{\partial w}{\partial x}\right)$$

and $$w\left(\frac{\partial^2 z}{\partial t^2}\right) - z\left(\frac{\partial^2 w}{\partial t^2}\right) = \frac{\partial}{\partial t}\left(w\frac{\partial z}{\partial t}\right) - \frac{\partial}{\partial t}\left(z\frac{\partial w}{\partial t}\right)$$

Then if we define $L^*(w) = c^2 w_{xx} - w_{tt}$, then

$$wL(z) - zL^*(w) = wc^2 z_{xx} - wz_{tt} - zc^2 w_{xx} + zw_{tt}$$

$$= \frac{\partial}{\partial x}\left[c^2 w\frac{\partial z}{\partial x} - c^2 z\frac{\partial w}{\partial x}\right] - \frac{\partial}{\partial t}\left[w\frac{\partial z}{\partial t} - z\frac{\partial w}{\partial t}\right]$$

$$= \frac{\partial U}{\partial x} + \frac{\partial V}{\partial t}$$

where $U = c^2\left(w\frac{\partial z}{\partial x} - z\frac{\partial w}{\partial x}\right)$ and $V = z\frac{\partial w}{\partial t} - w\frac{\partial z}{\partial t}$. Thus $L^* \equiv c^2\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}$ is the adjoint operator of L.

Since $L^* = L$, so L is self-adjoint operator.

Example-6.2 : Prove that for the equation $\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{4}z = 0$, the Green's function is given by $w(x,y,\xi,\eta) = J_0\left\{\sqrt{(x-\xi)(y-\eta)}\right\}$, where $J_0(z)$ is the Bessel function of the first kind of order zero.

Proof. The given equation is $L(z) = \frac{\partial^2 z}{\partial x \partial y} + \frac{1}{4}z = 0$ and its adjoint equation is (e.f. equation (6.3))

$$L^{*}(w)=\frac{\partial^2 w}{\partial x\partial y}+\frac{1}{4}w=0,$$

such that

(i) $L^{*}(w)=0$ throughout the xy - plane.

(ii) $\dfrac{\partial w}{\partial x}=0$ on $y=\eta,$

(iii) $\dfrac{\partial w}{\partial y}=0$ on $x=\xi,$

and (iv) $w=1$ at (ξ,η).

Now, if

$$w=J_0\left\{\sqrt{(x-\xi)(y-\eta)}\right\}$$

then

$$\frac{\partial w}{\partial x}=-J_1\left\{\sqrt{(x-\xi)(y-\eta)}\right\}\cdot\sqrt{y-\eta}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x-\xi}}$$

$$\frac{\partial w}{\partial y}=-J_1\left\{\sqrt{(x-\xi)(y-\eta)}\right\}\cdot\sqrt{y-\xi}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{y-\eta}}$$

$$\frac{\partial^2 w}{\partial x\partial y}=-\frac{1}{4}J_1'\left\{\sqrt{(x-\xi)(y-\eta)}\right\}\cdot\sqrt{x-\xi}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{y-\eta}}\cdot\sqrt{y-\eta}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x-\xi}}$$

$$-J_1\left\{\sqrt{(x-\xi)(y-\eta)}\right\}\cdot\frac{1}{4}\cdot\frac{1}{\sqrt{(x-\xi)(y-\eta)}}$$

$$=-\frac{1}{4\sqrt{(x-\xi)(y-\eta)}}\left[\sqrt{(x-\xi)(y-\eta)}J_1'\left\{\sqrt{(x-\xi)(y-\eta)}\right\}+J_1\left\{\sqrt{(x-\xi)(y-\eta)}\right\}\right]$$

$$=-\frac{1}{4\sqrt{(x-\xi)(y-\eta)}}\cdot\sqrt{(x-\xi)(y-\eta)}\cdot J_0\left\{\sqrt{(x-\xi)(y-\eta)}\right\}$$

$$=-\frac{1}{4}w$$

so that $L^{*}(w)=\dfrac{\partial^2 w}{\partial x\partial y}+\dfrac{1}{4}w=0$ throughout the xy-plane. Also it is easy to verify that all the conditions *(ii)*, (*iii*) and (*iv*) are satisfied by $w = J_0\left\{\sqrt{(x-\xi)(y-\eta)}\right\}$. Hence the given functiion $J_0\left\{\sqrt{(x-\xi)(y-\eta)}\right\}$ is the Green's function of the given equation.

Exercise

1. Show that if f and g are arbitrary functions of their respective arguments, then $u = f(x+i\alpha y - vt) + g(x - i\alpha y - vt)$ is a solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$, where is satisfied by where $\alpha^2 = 1 - \frac{v^2}{c^2}$.

2. Verify that the partial differential equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{2z}{x^2}$ is satisfied by $z = \frac{1}{x}\phi(y-x) + \phi'(y-x)$, where ϕ is an arbitrary function.

3. If $u = f(x+iy) + g(x-iy)$, where f and g are arbitrary functions, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

4. Solve the following equations :

(*i*) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y \left[Ans. z = \phi_1(x+y) + \phi_2(x-y) + \frac{1}{4}x(x-y)^2 \right]$

(*ii*) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = 2y - x^2 \left[\text{Ans.} z = \sum_r c_r e^{a_r x + b_r y} + x^2 y, \text{where } a_r^2 - b_r = 0 \right]$

(*iii*) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = e^{x+y} \left[\text{Ans.} z = \phi_1(x+y) + \phi_2(x-2y) + \frac{1}{3}x e^{x+y} \right]$.

(*iv*) $r - 4s + 4t = e^{x+y} \left[\text{Ans.} z = \phi_1(2x+y) + x\phi_2(2x+y) + \frac{1}{2}x^2 e^{2x+y} \right]$

(*v*) $r + s - 6t = y\cos x \left[\text{Ans.} z = \phi_1(3x-y) + \phi_2(2x+y) + \sin x - x\cos x + \frac{10}{9}\sin(2x+y) \right]$

(*vi*) $r + 3s + 2t = x + y \left[\text{Ans.} z = \phi_1(x-y) + \phi_2(2x-y) + \frac{x^2 y}{2} - \frac{x^3}{3} \right]$

(*vii*) $r - s = \sin x \cos 2y \left[\text{Ans.} z = \phi_1(y) + \phi_2(x+y) + \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y) \right]$

(viii) $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$ $\left[\text{Ans.} z = \phi_1(x+y) + \phi_2(x-2y) + \frac{1}{3}xe^{x+2y}\right]$

(ix) $(D^3 - 7DD' - 6D'^3)z = \sin(x+2y) + e^{2x+y}$

$$\left[\text{Ans.} z = \phi_1(x-y) + \phi_2(3x+y) + \phi_3(2x-y) - \frac{1}{75}\cos(x+2y) - \frac{1}{12}e^{2x+y}\right]$$

(x) $(D^3 - 7DD' - 6D'^3)z = \cos(x+y) + (x^2 + xy + y^2)$

$$\left[\text{Ans.} z = \phi_1(x-y) + \phi_2(2x-y) + \phi_3(3x+y) + \frac{1}{4}x\cos(x-y) + \frac{5}{72}x^6 + \frac{1}{60}x^5(1+21y) + \frac{1}{24}x^4y^2 + \frac{1}{6}x^3y^3\right]$$

(xi) $\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^3 z}{\partial x \partial y} + 5\frac{\partial^3 z}{\partial y^3} = x\sin(3x-2y)$ $\left[z = \phi_1\left(\frac{5-\sqrt{5}}{2}x - y\right) + \phi_2\left(\frac{5+\sqrt{5}}{2}x - y\right) + x\sin(3x-2y) + 4\cos(3x-2y)\right]$

(xii) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x^2 y$

$$\left[\text{Ans.} z = \sum_r c_r e^{a_r x + b_r y} + \frac{1}{360}(x^6 - 6x^5 + 15x^4 + 180x^2 y) \text{ where } a_r^2 + a_r + b_r = 0\right]$$

(xiii) $(D - 2D' + 5)(D^2 + D' + 3)z = \sin(2x+3y)$

$$\left[\text{Ans.} z = e^{-5x}\phi(2x+y) + \sum_r c_r e^{a_r x + b_r y} + \frac{1}{410}\{7\sin(2x+3y) - 19\cos(2x+3y)\}\right]$$

where $a_r^2 + b_r + 3 = 0$

(xiv) $\frac{\partial^3 z}{\partial x^2 \partial y} + \frac{\partial^2 z}{\partial y^2} - 2z = e^{2y}\cos 3x + e^x \sin 2y$

$$\left[\text{Ans.} z = \sum_r c_r e^{a_r x + b_r y} - \frac{1}{16}e^{2y}\cos 3x - \frac{1}{2}e^x(\cos 2y + 3\sin 2y)\right]$$

where $a_r^2 b_r + b_r^2 - 2 = 0$]

(xv) $(D+D'-1)(D+D'-3)(D+D')z = e^{x+y+2}\cos(2x-y)$

$\left[\text{Ans.}\, z = e^{x}\phi_1(x-y) + e^{3x}\phi_2(x-y) + \phi_3(x-y) - \frac{1}{10}e^{x+y+2}\left\{\sin(2x-y) + 2\cos(2x-y)^2\right\}\right]$

5. Solve the following equations :

(i) $(xy\,DD' - y^2D'^2 - 3xD + 2yD')z = 0$ $\left[\text{Ans.}\, z = \phi_1(xy) + y^3\phi_2(x)\right]$

(ii) $(x^2D^2 + xyDD' - 2y^2D'^2 - xD - 6yD')z = 0$ $\left[\text{Ans.}\, z = \phi_1\left(\frac{y}{x^2}\right) + \phi_2(xy)\right]$

(iii) $(x^3y^2D^2D'^2 - x^2y^3D^2D'^3)z = 0$ $\left[\text{Ans.}\, z = \phi_1(x) + \phi_2(y) + y\phi_3(x) + x\phi_4(y) + \phi_5(xy)\right]$

(iv) $\left\{x^2D^2 - 2xyDD' + y^2D'^2 - n(xD + yD') + n\right\}z = x^2 + xy + x^3$

$\left[\text{Ans.}\, z = x\phi_1\left(\frac{y}{x}\right) + x^n\phi_2\left(\frac{y}{x}\right) - \frac{x^2+y^2}{n-2} - \frac{x^3}{2(n-3)}\right]$

(v) $(x^2D^2 - 2xyDD' + 3y^2D'^2 + xD - 3yD')z = -x^2y\sin(\log x^2)$

$\left[\text{Ans.}\, z = \phi_1(x^3y) + \phi_2\left(\frac{y}{x}\right) - \frac{1}{65}x^2y\left\{4\cos(\log x^2) + 7\sin(\log x^2)\right\}\right]$

(vi) $(x^2D^2 - xyDD' - 2y^2D'^2 + xD - 2yD')z = \log\left(\frac{y}{x}\right) - \frac{1}{2}$

$\left[\text{Ans.}\, z = \phi_1(x^2y) + \phi_2\left(\frac{y}{x}\right) + \frac{1}{2}(\log x)^2\log y + \frac{1}{2}\log x.\log y\right]$

(vii) $(x^2yD^2D' - xy^2DD'^2 - x^2D^2 + y^2D'^2)z = \frac{x^3+y^3}{xy}$

$\left[\text{Ans.}\, z = x\phi_1(y) + y\phi_2(x) + \phi_3(xy) - \frac{x^3-y^3}{6xy}\right]$

(viii) $(x^2D^2 - 4xyDD' + 4y^2D'^2 + 6yD')z = x^3y^4$

$\left[\text{Ans.}\, z = \phi_1(x^2y) + x\phi_2(x^2y) + \frac{1}{30}x^3y^4\right]$

(ix) $(x^2D^2 - 4y^2D'^2 - 4yD' - 1)z = x^2y^3\log y$

$$\left[\text{Ans.}\, z = \sum_r c_r x^{a_r} y^{b_r} - \frac{1}{1225} x^2 y^3 (35 \log y - 24), \text{where}\, a_r^2 - 4b_r^2 - a_r - 1 = 0\right]$$

(x) $\left(\frac{1}{x^2} D^2 - \frac{1}{x^3} D\right) z = \left(\frac{1}{y^2} D'^2 - \frac{1}{y^3} D'\right) z.$ $\left(\text{Hints : Put}\, u = \frac{1}{2} x^2 v = \frac{1}{2} y^2\right)$

$\left[\text{Ans.}\, z = \phi_1\left(x^2 + y^2\right) + \phi_2\left(x^2 - y^2\right)\right].$

6. Solve the following equatins by reducing them into canonical form :

(i) $yr + (x+y)s + xt = 0 \left[\text{Ans.}\, z = f_1(x-y) + \frac{1}{y-x} f_2\left(x^2 - y^2\right)\right]$

(ii) $3r + 10s + 3t = 0 \left[\text{Ans.}\, z = f_1(3x - y) + f_2\left(y - \frac{x}{3}\right)\right]$

(iii) $\left(D^2 + 2DD' + D'^2\right) z = 0 \left[\text{Ans.}\, z = (x+y) f_1(x-y) + f_2(x-y)\right]$

(iv) $\left(D^2 + 2DD' + D'^2\right) z = 0 \left[\text{Ans.}\, z = (x+y) f_1(x-y) + f_2(x-y)\right]$

(v) $\left(D^2 - 2\sin x\, DD' - \cos^2 x\, D'^2\right) z = 0 \left[\text{Ans.}\, z = f_1(y - \cos x + x) + f_2(y - \cos x - x)\right]$

7. Reduce the following equations into canonical form :

(i) $z_{xx} + x z_{yy} = 0 \left[\text{Ans.}\ \zeta_{\xi-\eta}\left(\zeta_\xi - \zeta_\eta\right),\right.$ hyperbolic if $n < 0$

$\zeta_{\alpha\alpha} + \zeta_{\beta\beta} + \frac{1}{3\beta} \zeta_\beta = 0,$ elliptic if $x > 0]$

ii) $\left(\sin^2 x\, D^2 + \sin 2x\, DD' + \cos^2 x\, D'^2\right) = x \left[\text{Ans.} \left\{1 - e^{2(\eta-\xi)}\right\} \zeta_{\eta\eta} = \sin^{-1}\left(e^{\eta-\xi}\right) - \zeta_\xi, \text{parabolic}\right]$

(iii) $\left(D^2 + 2DD' + 4D'^2 + 2D + 3D'\right) z = 0 \left[\text{Ans.}\, \zeta_{\alpha\alpha} + \zeta_{\beta\beta} = -\frac{1}{3}\left(\zeta_\alpha + 2\sqrt{3}\zeta_\beta\right), \text{parabolic}\right]$

(iv) $y^2 z_{xx} - x^2 z_{yy} = 0; x > 0, y > 0 \left[\text{Ans.}\, 2\left(\xi^2 - \eta^2\right)\zeta_{\xi\eta} - \eta\zeta_\xi + \xi\zeta_\eta = 0, \text{hyperbolic}\right]$

(v) $4\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2 \left[\text{Ans.}\ \zeta_{\xi\eta} = \frac{1}{3}\zeta_\eta - \frac{8}{9}, \text{hyperbolic}\right]$

(vi) $\left\{\left(1 + x^2\right) D^2 + \left(1 + y^2\right) D'^2 + xD + yD'\right\} z = 0 \left[\text{Ans.}\ \zeta_{\alpha\alpha} + \zeta_{\beta\beta} = 0, \text{elliptic}\right]$

8. Construct the adjoints of the following differential equations :

(i) $L(z) = z_{xn} + z_{yy}$ $\left[\text{Ans. } L^{*}(w) = w_{xx} + w_{yy}, L \text{ is self-adjoint}\right]$

(ii) $L(z) = z_{xn} - z_{t} \left[\text{Ans. } L^{*}(w) = w_{xx} + w_{t}\right]$

(iii) $L(z) = Az_{xx} + Bz_{xy} + cz_{yy} + Dz_{x} + Ez_{y} + Fz$, where each of A, B, C, D, E and F is a function of x and y only.

$$\left[\text{Ans. } L^{*}(w) = \frac{\partial^2}{\partial x^2}(Aw) + \frac{\partial^2}{\partial x \partial y}(Bw) + \frac{\partial^2}{\partial y^2}(Cw) - \frac{\partial}{\partial x}(Dw) - \frac{\partial}{\partial y}(Ew) + Fw\right]$$

9. Prove that for the equation

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{2}{x+y}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = 0,$$

the Riemann-Green's function is given by

$$w(x, y; \xi, \eta) = \frac{(x+y)\{2xy + (\xi - \eta)(x-y) + 2\xi\eta\}}{(\xi + \eta)^3}.$$

Hence find the solution of the differential equation which satisfies the conditions $z = 0, \frac{\partial z}{\partial x} = 3x^2$ on $y = x$.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-IV

Group–B

Module No. - 45
ELLIPTIC EQUATIONS

1. Introduction :

We have seen in Module-44 that second order linear partial differential equations can be classified into three types, viz. elliptic, parabolic and hyperbolic. In this module, we shall consider elliptic equation in the form of Laplace and Poison equation, as these are frequently occur in practical situations.

2. Occurrence of Laplace and Poisson Equations

As mentioned above, the frequence occurence of Laplace and Poisson equations, we illustrate some branches about this

(a) **Gravitation :** The force of attraction $\vec{F}$ at a point inside or outside the gravitating matter can be expressed in terms of a potential function U as $\vec{F} = \vec{\nabla}U$. In empty U satisfies Laplace's equation $\nabla^2 U = 0$ while at a point inside the matter, it satisfies Poisson's equation $\nabla^2 U = -4\pi\rho, \rho$ being the density of the matter.

(b) **Irrotational motion of an ideal liquid :** The velocity v of an ideal liquid for irrotational motion can be expressed in terms of a potential function ϕ in the form $v = -\vec{\nabla}\phi$ in the absence of source, sink etc. and this function ϕ satisfies Laplace's equation $\nabla^2\phi = 0$.

(c) **Torsion problem in solid mechanics :** In the case of torsion problems of cylindrical bars in solid mechanics, the stress function $\bar{\psi}$ satisfies Poisson's euqation of the form $\nabla^2\bar{\psi} = -2$.

(d) **Steady currents :** The conduction current vector $\vec{i}$ can be derived from a potential function ψ by the formula $\vec{j} = -\sigma\vec{\nabla}\psi, \sigma$ being the conductivity. This function satisfies the equation $\vec{\nabla}\cdot\left(\sigma\vec{\nabla}\psi\right) = 0$ as $\vec{\nabla}\cdot\vec{j} = 0$ so that for constant conductivity, this reduces to Laplace equation $\nabla^2\psi = 0$.

(e) **Steady flow of heat :** If there is steady heat flow, then the temperature T satisfies the equation $\vec{\nabla}\cdot\left(\kappa\vec{\nabla}T\right) = 0, \kappa$ being the thermal conductivity. Thus, for constant κ, T satisfies Laplace equation $\nabla^2 T = 0$.

3. Boundary Value Problems

Suppose that for a given problem the function U is to be determined such that U satisfies either Laplace or Poisson equation in a bounded region V and also satisfies some conditions on the surface S enclosing V. Any such problem in which we require such a function is called boundary value problem (BVP) for Laplace or Poisson equation. In general, there are two main types of BVPs associated with the names of Dirichlet and Neumann.

(a) **Dirichlet problme :**

By the *interior Diirchlet problem* we mean-Let f be a continuous function prescribed on the boundary S of a bounded region V. Then the problem is to determine a function $U(x, y, z)$ which satisfies Laplace's equation $\nabla^2 U = 0$ at all points within V and the condition $U(x, y, z) = f$ on S.

On the other hand, if we are to determine $U(x, y, z)$ such that $\nabla^2 U = 0$ outside the region V and $U(x, y, z) = f$ on S, then the problem is called *exterior* Dirichlet problem.

Dirichlet problem is also known as first boundary value problem.

(b) **Neumann problem :**

By the *interior Neumann problem* we mean – Let f be a continuous function prescribed on the boundary S of a bounded region V. Then the problem is to determine a function $U(x, y, z)$ which satisfies Laplace's equation $\nabla^2 U = 0$ at any point within V and the normal derivative $\frac{\partial U}{\partial n} = f$ on S.

In a similar way, if $U(x, y, z)$ is such that $\nabla^2 U = 0$ outside V and $\frac{\partial U}{\partial n} = f$ on S, then we have exterior Neumann problem.

Neumann problem is also called *second boundary value problem.*

In addition to the above two main boundary value problems, there exists another boundary value problem, called Churchill or mixed or third boundary value problem given below.

(c) Churchill problem :

By the interior (exterior) Churchill we mean – Let f be a continous function prescribed on the boundary S of a bounded region V. Then the problem is to determine a function $U(x, y, z)$ which satisfies Laplace equation $\nabla^2 U = 0$ at all points within (outside) V and $\frac{\partial U}{\partial n} + (k+1)U = f$, ($k$ being a constant), at every point of S.

In the above, we have stated boundary value problems for Laplace's equation. The same can be formulated for Poisson's equation.

4. Some Mathematicsl Results :

Let us recall some mathematical results which are of much use in our discussion in the sequel.

Suppose τ is a closed regular region bounded by a closed surface S. Let U and V be two functions of x,y,z and continuous in $V+S$ together with their first order partial derivatives. In addition, V has second order partial derivatives in $V+S$. Then using Gauss' divergence theorem

$$\iiint_\tau \nabla \cdot F \, d\tau = \iint_S F \cdot \hat{n} \, dS$$

by putting $F = U \nabla V$, we get

$$\iiint_\tau \nabla \cdot (U \nabla V) \, d\tau = \iint_S U \nabla V \cdot \hat{n} \, dS$$

i.e. $$\iiint_\tau U \nabla^2 V \, d\tau + \iint_S \nabla U \cdot \nabla V \cdot d\tau = \iint_S U \frac{\partial V}{\partial n} dS \qquad \textbf{(4.1)}$$

This is know as *Green's first identity.*

On the other hand, if the function also possesses continuous second order partial derivatives in $V+S$, then interchanging ϕ and ψ in (4.1) we have

$$\iiint_V \psi \nabla^2 \phi \, dv + \iiint_V \nabla \psi \cdot \nabla \phi \, dv + \iint_S \psi \frac{\partial \phi}{\partial n} ds \qquad (4.2)$$

Subtracting (4.2) from (4.1) we get

$$\iiint_V \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) dv = \iint_s \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) ds \qquad \textbf{(4.3)}$$

This is known as Green's second identity.

5. Harmonic Function :

A function $U(x, y, z)$ is said to be harmonic at a point (x, y, z) if it possesses continuous second order partial derivatives and satisfies Laplace's equation $\nabla^2 U = 0$ throughout some neighbourhood of that point. U is harmonic in a domain or open continuum, if it is harmonic at all points in that domain. It is said to be harmonic in a closed region, if U is continuous at all interior points of the region.

A function $U(x,y,z)$ is said to be regular at infinity if $rU, r^2\frac{\partial U}{\partial x}, r^2\frac{\partial U}{\partial y}, r^2\frac{\partial U}{\partial z}$ are bounded for sufficiently large r, where $r^2 = x^2 + y^2 + z^2$. If a function is harmonic in an unbounded region, then it must be regular at infinity.

Some properties of harmonic function :

Theorem 4.1 : If a harmonic function vanishes at all points on the boundary, then it is identically zero everywhere.

Proof. Suppose the function $U(x, y, z)$ is harmonic in a region τ so that $\nabla^2 U = 0$ in τ. Then by the given condition $U = 0$ on the surface S of τ. Now putting $V = U$ in Green's first identity (4.2) we get

$$\iiint_V U\nabla^2 U\, d\tau + \iiint_V (\nabla U)^2 d\tau = \iint_S U\frac{\partial U}{\partial n} dS.$$

Using the conditions $\nabla^2 U = 0$ in τ and $U = 0$ on S, it follows that

$$\iiint_V (\nabla U)^2\, d\tau = 0.$$

which is satisfied provided $\nabla U = 0$ in τ, i.e. U = const. in τ. Since U is continuous in $\tau + S$ and $U = 0$ on S, we must have $U = 0$ in τ.

Theorem 4.2 : If a function U is harmonic in τ and its normal derivative $\frac{\partial U}{\partial n}$ vanishes at all points on the boundary S of τ, then U is constant in τ.

Proof. The proof is left as an exercise.

Theorem 4.3 : Dirichlet problem for a bounded region possesses a unique solution.

Proof. If possible, suppose that U_1 and U_2 be two solutions for the interior Dirichlet problem. Then $\nabla^2 U_1 = \nabla^2 U_2 = 0$ in τ and $U_1 = U_2 = f$ on S. Let $U = U_1 - U_2$ so that $\nabla^2 U = 0$ in τ and $U = 0$ on S. Then by Theorem 4.1, $U = 0$ on $\tau + S$ i.e. $U_1 = U_2$ on $\tau + S$. Thus Dirichlet interior problem has a unique solution.

Similarly, Dirichlet exterior problem also possesses a unique solution.

Theorem 4.4 : Neumann problem for a bounded region either possesses a unique solution or solutions differing from one another by a constant.

Proof. Suppose U_1 and U_2 be two different solutions of Neumann problem. Then $\nabla^2 U_1 = \nabla^2 U_2 = 0$ in τ and $\frac{\partial U_1}{\partial n} = \frac{\partial U_2}{\partial n} = f$ on S. Let $U = U_1 - U_2$ so that $\nabla^2 U = 0$ in τ and $\frac{\partial U}{\partial n}$ =0 on S. Hence by Theorem 4.2, U= constant in $\tau + S$.

If the constant is zero, the $U_1 - U_2 = U = 0$, i.e. $U_1 = U_2$ in $\tau + S$. Thus, the solution is unique. On the other hand, for non-zero constant, $U_1 = U_2$ + const. i.e. solutions differ by constant.

Spherical mean :

Let $P(x, y, z)$ be any point in a bounded region τ enclosed by a closed surface S. Also, suppose that Σ be a sphere with centre at P and readius r such that Σ lies entirely within τ. We define a function defined $P(x, y, z)$ and continuous in τ. Then the spherical mean of U is defined by

$$\bar{U}(r) = \frac{1}{4\pi r^2} \iint_{\Sigma} U(Q)\, d\Sigma, r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 \tag{5.1}$$

where $Q(\xi, \eta, \zeta)$ is a variable point on Σ and dΣ is the surface element of integration.

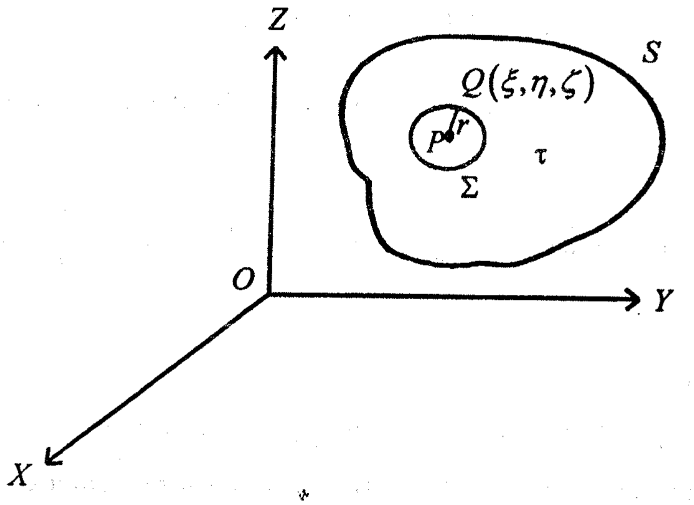


Fig. 3.1

Taking the origin at P, we have

$$\xi = x + r\sin\theta\cos\phi, \eta = y + r\sin\theta\sin\phi, \zeta = z + r\cos\theta.$$

Since U is continuous on $\Sigma, \bar{U}$ is also a continuous function of r on some interval $0 \le r \le R$, because

$$\bar{U}(r) = \frac{1}{4\pi} \iint_{\Sigma} U(Q)\sin\theta\, d\theta\, d\phi = \frac{U(Q)}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta . d\theta\, d\phi = U(Q).$$

and $Q \to P$ as $r \to 0$ so that $\bar{U}(r) \to U(P)$. Hence U is continuous in $0 \le r \le R$.

Theorem 4.5 : (Mean value theorem for harmonic function). Let a function U be harmonic in a region τ and $U(x, y, z)$ be a given point in τ. Also suppose that Σ is a sphere with centre at P and radius r such that Σ lies entirely in τ. Then

$$U(P) = \bar{U}(r) = \frac{1}{4\pi r^2} \iint_{\Sigma} U(Q)\, d\Sigma$$

Proof. Since U is harmonic in τ, so its spherical mean $\bar{U}(r)$ is continuous and is given by

$$\bar{U}(r) = \frac{1}{4\pi r^2} \iint_{\Sigma} U(Q)\, d\Sigma = \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} U(\xi, \eta, \zeta) \sin\theta \cdot d\theta\, d\phi$$

so that $$\frac{d\bar{U}(r)}{dr} = \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left(U_\xi \xi_r + U_\eta \eta_r + U_\zeta \zeta_r\right) \sin\theta \cdot d\theta\, d\phi$$

$$= \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left(U_\xi \sin\theta \cos\phi + U_\eta \sin\theta \sin\phi + U_\zeta \cos\theta\right) \sin\theta \cdot d\theta\, d\phi$$

Since the normal $\hat{n}$ on Σ has direction cosines $\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta$, we have

$$\frac{d\bar{U}}{dr} = \frac{1}{4\pi r^2} \iint_{\Sigma} \nabla U \cdot \hat{n}\, r^2 \sin\theta\, d\theta\, d\phi \left[\because \nabla U = \hat{i}\, U_\xi + \hat{j}\, U_\eta + \hat{k}\, U_\zeta\right]$$

$$= \frac{1}{4\pi r^2} \iint_{\Sigma} \nabla U \cdot \hat{n}\; dS$$

$$= \frac{1}{4\pi r^2} \iiint_{\tau} \nabla \cdot \nabla U\, d\tau$$

$$= \frac{1}{4\pi r^2} \iiint_{\tau} \nabla^2 U\, d\tau = 0.$$

Thus U is constant, the continuity of U at $r = 0$, therefore, proves the result (5.2)

Dirichlet principle

Theorem 4.6 : Let f be a continuous function prescribed on the boundary S of a bounded region τ. Then among all functions U that satisfy the Dirichlet condition $U = f$ on S, the lowest energy defined by

$$E(U)=\frac{1}{2}\iiint_{\tau}|\nabla U|^2 d\tau$$

is attained by a harmonic function satisfying $U=f$ on S.

Proof. Let V be the unique harmonic function and W be a function satisfying respectively the Dirichlet condition $V=f$ on S and $W=0$ on S. Put $U=V-W$. Then

$$E(U)=E(V-W)=\frac{1}{2}\iiint_{\tau}|\nabla(V-W)|^2 d\tau=\frac{1}{2}\iiint_{\tau}\left(|\nabla V|^2-2\nabla V\cdot\nabla W+|\nabla W|^2\right)d\tau$$

$$=\frac{1}{2}\iiint_{\tau}|\nabla V|^2 d\tau+\frac{1}{2}\iiint_{\tau}|\nabla W|^2 d\tau-\iiint_{\tau}\nabla V\cdot\nabla W$$

$$=E(V)+E(W)-\iiint_{\tau}\nabla V\cdot\nabla W\,d\tau$$

Now considering $U\to W$ and $V\to V$ in Green's first identity (5.1) we have

$$\iiint_{\tau}\nabla V\cdot\nabla W\,d\tau=\iint_{S}W\frac{\partial V}{\partial n}dS-\iiint_{\tau}W\nabla^2 V\,d\tau,$$

so that from (5.3), it follows that

$$E(U)=E(V)+E(W)-\iint_{S}W\frac{\partial V}{\partial n}dS+\iiint_{\tau}W\,\nabla^2 V\,d\tau$$

Since $W=0$ on S and $\nabla^2 V=0$ in τ, we get

$$E(U)=E(V)+E(W)\geq E(V)$$

which completes the proof.

5. Solution of Two-dimensional Laplace's Equation (Separation of Variable Method)

I. Cartesian coordinates (x,y) :

To solve two-dimensional Laplace's equation in Cartesian coordinates given by

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0 \tag{5.1}$$

we put $U(x,y)=X(x)Y(y)$ **(5.2)**

in (5.1) and get

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = k \qquad (5.3)$$

where x is separation constant. Then we have the following cases :

Case (i): Let $k = p^2 > 0$, p being req. Thus (5.3) gives

$$\frac{d^2X}{dx^2} - p^2X = 0 \text{ and } \frac{d^2Y}{dy^2} - p^2Y = 0$$

whose solutions are

$$X(x) = c_1e^{px} + c_2e^{-px} \text{ and } Y(y) = c_3\cos py + c_4\sin py,$$

where c_1, c_2, c_3 and c_4 are constants. Hence by using (5.2), the solution of the Laplace's equation (5.1) is

$$U(x, y) = \left(c_1e^{px} + c_2e^{-px}\right)\left(c_3\cos py + c_4\sin py\right) \qquad (5.4)$$

Case (ii) : Let $k = 0$ so that (5.3) leads to

$$\frac{d^2X}{dx^2} = 0,\ \frac{d^2Y}{dy^2} = 0$$

whose solutions are

$$X(x) = \xi x + c_6, Y(y) = c_7xy + c_8.$$

Thus the solution of (5.1) is

$$U(x) = (\xi x + c_6)(c_7y + c_8)$$

Case (iii) : Let $k = -p^2 < 0$. Then proceeding as in case (i), the solution of (5.1) is obtained as

$$U(x, y) = \left(c_9\cos px + c_{10}\sin px\right)\left(c_{11}e^{px} + c_{12}e^{-px}\right). \qquad (5.6)$$

In all the above cases, the constants $c_1, c_2, ..., c_{12}$ are determined by the use of boundary conditions. We now illustrate the above results by some specific problems.

Dirichlet interior problem for a rectangle

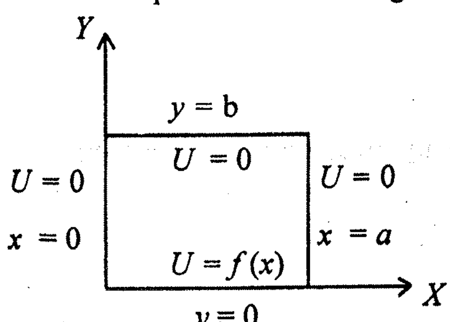


Fig. 3.2

Dirichlet interior problem for a rectangle is defined as follows: To solve Laplace's equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0$$

at any point interior to the rectangle $0 \le x \le a, 0 \le y \le b$ subject to other boundary conditions

$$U(x,b)=U(0,y)=U(a,y)=0, U(x,0)=f(x)$$

in which the function $f(x)$ is supposed to be expansible in Fourier since series.

Now consider the solution (5.4). Using the boundary conditions and noting that $c_3 \cos py + c_4 \sin py \ne 0$, simply we get

$$c_1+c_2=0, c_1e^{ap}+c_2e^{-ap}=0$$

leading to $c_1 = c_2 = 0$ so that $U(x,y)=0$ is the only non-trivial solution. Thus, the solution (5.4) is ruled out.

It is easy to see that the solution (5.5) also yields the non-trivial solution. So this solution cannot be accepted.

Therefore, the only possible solution is given by (5.6). Here the boundary condition $U(a,y)=0$ gives $\sin pa = 0, i.e. p=\frac{n\pi}{a}, (n=1,2,...)$. Hence the possible non-trivial solution is given by using superposition principle, as

$$U(x,y)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\left[A_n e^{n\pi/a}+B_n e^{-n\pi/a}\right].$$

Again, the boundary condition $U(x,b)=0$ gives

$$A_n e^{n\pi b/a}+B_n e^{n\pi b/a}=0 \Rightarrow B_n=-A_n\frac{e^{n\pi b/a}}{e^{-n\pi b/a}}$$

so that

$$U(x,y)=\sum_{n=1}^{\infty}\frac{A_n}{e^{-n\pi b/a}}\cdot\sin\frac{n\pi x}{a}\left[\exp\left\{\frac{n\pi(y-b)}{a}\right\}-\exp\left\{\frac{n\pi(y-b)}{a}\right\}\right]$$

i.e. $U(x,y)=\sum_{n=1}^{\infty}c_n \sin\frac{n\pi x}{a}\cdot\sinh\left\{\frac{n\pi(y-b)}{a}\right\}$

where $c_n' = 2A_n e^{n\pi b/a}$. Finally, the non-homogeneous boundary condition $U(x,0)=f(x)$ gives

$$\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a}\sinh\left(-\frac{n\pi b}{a}\right)=f(x)$$

which is a half-range Fourier since series so that

$$c_n \sinh\left(-\frac{n\pi b}{a}\right)=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

Thus the required solution of the given Dirichlet interior problem is

$$U(x,y)=\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left\{\frac{n\pi(y-b)}{a}\right\} \tag{5.7}$$

where

$$c_n=-\frac{2}{a\sin h\frac{n\pi b}{a}}\int_0^a f(x)\sin\frac{n\pi x}{a}dx. \tag{5.8}$$

Neumann interior problem for a rectangle.

For this problem, the boundary conditions of Dirichlet's interior problem are to be replaced by

$$U_x(0,y)=U_x(a,y)=U_y(x,y)=0, U_y(x,b)=f(x)$$

By arguing as in the preceding problem, it can be easily verified that the only suitable solution of Laplace's equation is given by (5.4), i.e.

$$U(x,y)=(c_1\cos px+c_2\sin px)(c_3e^{py}+c_4e^{-py}).$$

Now the boundary conditions $U_x(0,y)=0$ and $U_x(a,y)=0$ give respectively $c_2=0$ and $\sin pa=0$, i.e. $p=n\pi/a,(n=0,1,n,...)$. Thus, we have $U(x,y)=\cos\frac{n\pi x}{a}\left(Ae^{n\pi y/a}+Be^{-n\pi y/a}\right)$,

where $A=c_1c_2, B=c_1c_4$. The boundary condition $U_y(x,0)=0$ gives $B=A$. The defining $2A=A_n$ and using the superposition principle, we obtain

$$U(x,y)=\sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{a}\cosh\frac{n\pi y}{a}.$$

Finally, the boundary condition $U_y(x,b)=f(x)$ gives

$$f(x)=\sum_{n=1}^{\infty} A_n\cdot\frac{n\pi}{a}\cos\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}$$

which is the half-range Fourier cosine series, so that

$$A_n \cdot \frac{n\pi}{a} \cdot \sin h \frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx.$$

Hence, the solution of Neumann interior problem for a rectangle is

$$U(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cdot \cos\frac{n\pi x}{a} \cdot \cosh\frac{n\pi y}{a} \qquad (5.9)$$

where A_0 is arbitrary and

$$A_n = \frac{2}{n\pi \sin h\frac{n\pi b}{a}} \int_0^a f(x)\cos\frac{n\pi x}{a}\,dx.$$

Example 5.1 : Consider the Cauchy problem for the Laplace equation $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$, subject to $U(x,0) = 0$, $U_y(x,0) = \frac{1}{n}\sin nx$, where n is a positive integer. Show that its solution is $U(x,y) = \frac{1}{n^2}\sinh ny \cdot \sin nx$.

Solution. It can be readily seen that the solution of the Laplace equation consistent with the given boundary conditions is

$$U(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}).$$

The condition $U(x,0) = 0$ gives $c_4 = -c_3$ so that

$$U(x,y) = (A\cos px + B\sin px)\sin h\, py$$

Also using the condition $U_y(x,0) = \frac{1}{n}\sin nx$, we have

$$p(A\cos px + B\sin px) = \frac{1}{n}\sin px, \text{ for all } x.$$

So, it follows that $pA = 0, pB = \frac{1}{n}$ and $p = n$ leading to $A = 0, B = \frac{1}{n^2}$.

Hence the required solution is

$$U(x,y) = \frac{1}{n^2}\sin nx \sin h\, ny.$$

Example 5.2 : Solve Laplace's equation $\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0$, in the semi-infinite region $x \geq 0, 0 \leq y \leq 1$ subject to the boundary conditions

$$U_x(0,y)=U_y(x,0)=0, U(x,1)=f(x).$$

Solution. Solution of the Laplace's equation consistent with the given conditions is given by

$$U(x,y)=(c_1 \cos px + c_2 \sin px)(c_3 e^{py}+c_4 e^{-py}).$$

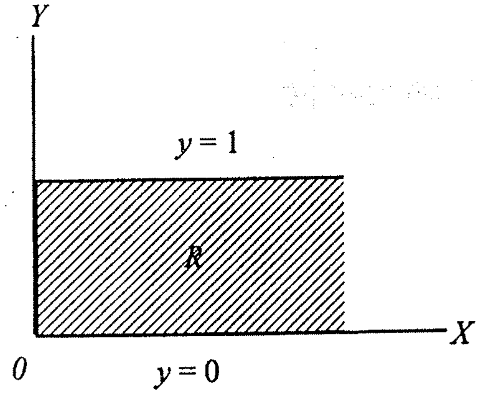


Fig. 3.3

The condition $U_x(0,y)=0$ gives $c_2=0$ while the condition $U_y(x,0)$ leads to $c_3-c_4=0, i.e.\ c_4=c_3$. Thus

$$U(x,y)=A\cos px \cosh py$$

where $A=2c_1c_3$.

Now, since all real positive values of p are permissible, so the general solution of the Laplace's equation subject to the first two given boundary conditons is

$$U(x,y)=\int_0^\infty A(p)\cos px \cosh py\, dp$$

where $A(p)$ is an arbitrary function of p. Putting $y=1$, we get

$$f(x)=\int_0^\infty A(p)\cos px \cosh p\, dp$$

Using the Fourier cosine integral formula, we have

$$A(p)\cos hp=\frac{2}{\pi}\int_0^\infty f(\xi)\cosh p\xi\, d\xi$$

so that

$$A(p)=\frac{2}{\pi\cosh p}\int_0^\infty f(\xi)\cosh p\xi\, d\xi$$

and hence

$$U(x,y)=\frac{2}{\pi}\int_0^\infty \frac{\cos px\cosh py}{\cosh p}\left[\int_0^\infty f(\xi)\cosh p\xi d\xi\right]dp.$$

[In particular, if we take

$$f(x)=\begin{cases}1, 0\le x\le 1\\ 0, x>1\end{cases}$$

Then

$$\int_0^\infty f(\xi)\cos p\xi d\xi=\int_0^1 \cos p\xi d\xi=\frac{\sin p}{p}$$

and so

$$U(x,y)=\frac{2}{\pi}\int_0^\infty \frac{\cos px\cdot\cosh py\cdot\sin p}{p\cosh p}dp$$

Example 5.3 : In the theory of elasticity, the stress function U, in the problem of torsion of a beam satisfies the Poisson

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=-2,\ 0\le x\le 1, 0\le y\le 1$$

with the boundary conditions $U=0$ on sides $x=0, x=1, y=0, y=1$. Find the stress function.

Solution. Let us assume the solution in the form

$$U=V+W$$

where V is a particular solution of the Poisson equation while W represents the solution of the homogeneous Laplace equation, i.e.

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=-2 \text{ and } \frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}=0.$$

In general, v is assumed in the form

$$V(x,y)=a+bx+cy+dx^2+exy+fy^2$$

so that $2d+2f=-2$. Taking $f=0$, we have $d=-1$. The remaining coefficients can be chosen arbitrarily. Thus, we have

$$V(x,y)=x-x^2.$$

which satisfies the boundary conditions $V=0$ on the sides $x=0$ and $x=1$.

Now to find W from

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}=0, 0\le x\le 1, 0\le y\le 1$$

satisfying

$$W(0,y)=-V(0,y)=0,$$

$$W(1,y)=-V(1,y)=0,$$

$$W(x,0)=-V(x,0)=x^2-x,$$

$$W(x,1)=-W(x,1)=x^2-x,$$

it can be easily seen that the solution is given by

$$W(x,y)=(c_1\cos px+c_2\sin px)(c_3\cosh py+c_4\sinh py)$$

The condition $W(0,y)=0$ gives $c_1=0$ while the condition $W(1,y)=0$ leads to $\sin p=0, i.e. p=n\pi,(n=1,2,...)$. Thus, by the superposition principle, we have

$$W(x,y)=\int_0^\infty \sin(n\pi x)\{a_n\cosh(n\pi y)+b_n\sinh(n\pi y)\}.$$

Again, using the non-homogeneous boundary conditions $W(x,0)=W(x,1)=x^2-x$, we have

$$x^2-x=\sum_{n=1}^{\infty}a_n\sin(n\pi x)$$

and $x^2-x=\sum_{n=1}^{\infty}\sin(n\pi x)\{a_n\cosh(n\pi)+b_n\sinh(n\pi)\}$

so that the first relation leads to

$$a_n = 2\int_0^1 \left(x^2 - x\right)\sin\left(n\pi x\right)dx = \frac{4}{n^3\pi^3}\left\{(-1)^n - 1\right\}$$

$$= \begin{cases} -\dfrac{8}{n^3\pi^3}, \text{if } n \text{ is odd} \\ 0, \text{if } n \text{ is even} \end{cases}$$

while the second relation gives

$$a_n \cosh\left(n\pi\right) + b_n \sinh\left(n\pi\right) = 2\int_0^1 \left(x^2 - x\right)\sin\left(n\pi x\right)dx = a_{n_y}$$

and, therefore,

$$b_n = a_n \cdot \frac{1 - \cosh\left(n\pi\right)}{\sinh\left(n\pi\right)}$$

Hence

$$W\left(x, y\right) = \sum_{n=1}^{\infty} \frac{a_n \cdot \sin\left(n\pi x\right)}{\sinh\left(n\pi\right)}\left\{\cosh\left(n\pi y\right)\sinh\left(n\pi\right) + \sinh\left(n\pi y\right) - \sinh\left(n\pi y\right)\cosh\left(n\pi\right)\right\}$$

$$= \sum_{n=1}^{\infty} \frac{a_n \sin\left(n\pi x\right)}{\sinh\left(n\pi\right)}\left\{\sinh n\pi\left(1 - y\right) + \sinh\left(n\pi y\right)\right\}$$

$$= -\frac{8}{\pi^3}\sum_{n=add} \frac{\sin\left(n\pi x\right)}{n^3 \sinh\left(n\pi\right)}\left\{\sinh n\pi\left(1 - y\right) + \sinh\left(n\pi y\right)\right\}$$

$$= -\frac{8}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin\left(2n-1\right)\pi x}{\left(2n-1\right)^3 \sinh\left(2n-1\right)\pi}\left\{\sinh\left(2n-1\right)\pi\left(1 - y\right) + \sinh\left(2n-1\right)\pi y\right\}$$

Thus the required solution of the given Poisson equation is

$$U\left(x, y\right) = V + W = x - x^2 - \frac{8}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin\left(2n-1\right)\pi x}{\left(2n-1\right)^2 \sinh\left(2n-1\right)\pi} \times \left\{\sinh\left(2n-1\right)\pi\left(1 - y\right) + \sinh\left(2n-1\right)\pi y\right\}.$$

II. Plane polar coordinates (r, θ):

In plane polar coordinates (r, θ), the Laplace equation is

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0. \qquad (5.11)$$

To solve this equation by separation of variable technique, we put

$$U(r,\theta)=R(R)\Theta(\theta)$$

in (5.11) and get (5.12)

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}\right)=-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=k\,(say), \tag{5.13}$$

where k is separation constant. Then, we have the following cases :

Case (i) : Let $k=p^2>0$ where p is real. In this case, we have from (5.13)

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-p^2R=0 \text{ and } \frac{d^2\Theta}{d\theta^2}+p^2\Theta=0$$

whose solutions are

$$R(r)=c_1r^p+c_2r^{-p} \text{ and } \Theta(\theta)=c_3\cos p\theta+c_4\sin p\theta$$

respectively, so that the solution of (5.11) is given by using (5.12) as

$$U(r,\theta)=\left(c_1r^p+c_2r^{-p}\right)\left(c_3\cos p\theta+c_4\sin p\theta\right) \tag{5.14}$$

Case (ii) : Let $k=0$. Then from (5.13) we get

$$r\frac{d^2R}{dr^2}+\frac{dR}{dr}=0 \text{ and } \frac{d^2\Theta}{d\theta^2}=0$$

having solutions $R(r)=c_5lr\,r+c_6,\ \Theta(\theta)=c_7\theta+c_8$ respectively, so that the solution of (3.11) is

$$U(r,\theta)=(c_5\ln r+c_6)(c_7\theta+c_8). \tag{5.15}$$

(iii) : Let $k=-p^2<0$. Here the equations (5.13) give

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+p^2R=0 \text{ and } \frac{d^2\Theta}{d\theta^2}-p^2\Theta=0$$

whose solutions are

$$R(r)=c_9\cos(p\ln r)+c_{10}\sin(p\ln r),\Theta(\theta)=c_{11}e^{p\theta}+c_{12}e^{-p\theta}$$

Thus the solution of (5.11) is

Let us now illustrate the above results by some specific problems.

Interior Dirichlet problem for a circle

Here the problem is to determine a function U in terms of its value on the boundary $r=a$ such that U is single-valued and continuous within and on the circular region and satisfies the two-dimensional Laplace's equation (5.11) for $0 \le r \le a, 0 \le \theta \le 2\pi$ subject to the boundary condition $U(a,\theta) = f(\theta), 0 \le \theta \le 2\pi, f(\theta)$ being a continuous function of θ.

Since the function U is single-valued, so it must satisfy the periodicity condition $U(r,\theta+2\pi) = U(r,\theta), 0 \le \theta \le 2\pi$. Now, noting that $r = 0$ is a point of the domain of definition, so *ln*(*r*) is undefined and, therefore, the solutions (5.15) and (5.16) are ruled out and the solution (5.14) is to be taken into account. Also the periodicity condition gives

$$c_3 \cos p\theta + c_4 \sin p\theta = c_3 \cos p(2\pi+\theta) + c_4 \sin p(2\pi+\theta)$$

or, $\quad c_3\left[\cos\theta - \cos p(2\pi+\theta)\right] - c_4\left[\sin p(2\pi+\theta) - \sin p\theta\right] = 0$

or, $\quad \sin p\pi\left[c_3 \sin(p\theta + p\pi) - c_4 \cos(p\theta + p\pi)\right] = 0$

or, $\quad \sin p\pi = 0$

$$\therefore p = n, (n = 0, 1, 2, \ldots)$$

Hence, from (5.14), we get by superposition principle,

$$U(r,\theta) = \sum_{n=1}^{\infty}\left(c_n r^n + D_n r^{-n}\right)\left(A_n \cos n\theta + B_n \sin n\theta\right) \tag{5.17}$$

Since the solution must be finite at $r = 0$, so $D_n = 0$ and we can write the above solution in the form

$$U(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\theta + b_n \sin n\theta\right) r^n \tag{5.18}$$

where $a_0 = 2A_0, a_n = A_n c_n, b_n = B_n c_n, (n > 0)$. The solution (5.18) is a full range Fourier series.

Now the boundary condition $U(a,\theta) = f(\theta)$ gives

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\theta + b_n \sin n\theta\right) a^n$$

giving

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\chi)\,d\chi, a_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\chi)\cos n\chi\,d\chi, b_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\chi)\sin n\chi\,d\chi \tag{5.19}$$

Thus the solution (6.18) can be written as

$$U(r,\theta)=\frac{1}{\pi}\int_0^{2\pi} f(\chi)\left\{\frac{1}{2}\left(\frac{r}{a}\right)^n \cos n(x-\theta)\right\}d\chi$$

Let us now put

$$c=\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\chi-\theta) \text{ and } s=\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \sin n(\chi-\theta)$$

so that

$$c+is=\sum_{n=1}^{\infty}\left\{\frac{r}{a}e^{i(\chi-\theta)}\right\}^b=\frac{\left(\frac{r}{a}\right)e^{i(\chi-\theta)}}{1-\frac{r}{a}e^{i(\chi-\theta)}}\left[\because \frac{r}{a}<1 \text{ and } \left|e^{i(\chi-\theta)}\right|<1\right]$$

Equating real parts, we have

$$c=\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\chi-\theta)=\frac{(r/a)\cos(\chi-\theta)-(r^2/a^2)}{1-(2r/a)\cos(\chi-\theta)+(r^2/a^2)}$$

Hence the required unique solution of Dirichlet interior problem is obtained from (5.20) as

$$U(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(a^2-r^2)f(\chi)}{a^2-2ar\cos(\chi-\theta)+r^2}d\chi, r<a \quad (5.21)$$

This is known as Poisson's integral formula for a circle.

Exterior Dirichlet problem for a circle.

In this case the problem is to find the value of the function U at any point exterior of the circle $r=a$ satisfying Laplace's equation (5.11) for $r\geq a, 0\leq\theta\leq 2\pi$ and the boundary condition $U(a,\theta)=f(\theta), 0\leq\theta\leq 2\pi$, where $f(\theta)$ is a continuous function of Q and U is bounded as $r\to\infty$.

Here the solution is given by (5.14). Noting that U must be bounded as $r\to\infty$, we take the solution as

$$U(r,\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)r^{-n}$$

where

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(\chi)d\chi, a_n=\frac{a^n}{\pi}\int_0^{2\pi}f(\chi)\cos n\chi\cdot d\chi, b_n=\frac{a^n}{\pi}\int_0^{2\pi}f(\chi)\sin n\chi\cdot d\chi$$

Then, proceeding exactly along the same lines as in the interior Disichlet problem, we obtain

$$U(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(r^2-a^2)f(\chi)}{r^2-2ar\cos(\chi-\theta)+a^2}d\chi, r>a. \quad (5.22)$$

Interior Neumann problem for a circle

The problem is to determine a function $U(r,\theta)$ at any point interior to the circle $r=a$ satisfying Laplace's equation (5.11) for $0\le r\le a, 0\le\theta\le 2\pi$ and the boundary condition $\frac{\partial U}{\partial n}=\frac{\partial U}{\partial r}=g(\theta)$ on $r = a$, where $g(\theta), 0\le\theta\le 2\pi$, is a continuous function of θ.

Here the solution of the equation (5.11) in conformity with the problem, as in the case of interior Dirichlet problem is taken in the form

$$U(r,\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)r^n \quad (5.23)$$

Thus at the boundary $r=a$, where $\frac{\partial U}{\partial r}=g(\theta)$ we have

$$g(\theta)=\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)na^{n-1}$$

which is a full range Fourier series in $g(\theta)$ and, therefore,

$$a_n=\frac{1}{na^{n-1}\pi}\int_0^{2\pi}g(\chi)\cos n\chi, d\chi, b_n=\frac{1}{na^{r-1}\pi}\cdot\int_0^{2\pi}g(\chi)\sin n\chi d\chi$$

so that

$$U(r,\theta)=\frac{1}{2}a_0+\int_0^{2\pi}g(\chi)\left\{\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cdot\frac{a}{n\pi}\cos n(\chi-\theta)\right\}d\chi$$

Now we put

$$c=\sum_{n=1}^{\infty}\frac{a}{n\pi}\left(\frac{r}{a}\right)^n\cos n(\chi-\theta) \text{ and } s=\sum_{n=1}^{\infty}\frac{a}{n\pi}\left(\frac{r}{a}\right)^n\sin n(\chi-\theta)$$

so that

$$c+is=\frac{a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\frac{r}{a}e^{i(\chi-\theta)}\right\}^{n}=-\frac{a}{\pi}\ln\left\{1-\frac{r}{a}e^{i(\chi-\theta)}\right\}$$

Equating real parts on both sides, we have

$$c=-\frac{a}{2\pi}\ln\left\{\left(a^2-2ar\cos(\chi-\theta)+r^2\right)/a^2\right\}$$

Hence the required solution of the interior Neumann problem is obtained from (5.28) as

$$U(r,\theta)=\frac{1}{2}a_0-\frac{a}{2\pi}\int_0^{2\pi}\ln\left\{1-2\frac{r}{a}\cos(\chi-\theta)+\frac{r^2}{a^2}\right\}\cdot g(\chi)d\chi. \quad (5.29)$$

Note : The exterior Neumann problem can similarly be formulated.

Example 5.4 : For an infinitely long conducting cylinder of radius a, with its axis coincidence with its axis coincident with the z-axis, the voltage $U(r,\theta)$ obeys the Laplace's equation

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2}=0, 0\le r\le\infty, 0\le\theta\le 2\pi.$$

Find the voltage $U(r,\theta)$ for $r\ge a$ if $\lim_{r\to\infty}U(r,\theta)=0$ subject to the condition $\frac{\partial U}{\partial r}=\frac{1}{a}U_0\sin 3\theta$ at $r=a$.

Solution. The solution of the Laplace's equation in conformity with the given problem is

$$U(r,\theta)=\sum_{n=1}^{\infty}p^{-n}(a_n\cos n\theta+b_n\sin n\theta)$$

Using the condition $\frac{\partial U}{\partial r}=\frac{1}{a}U_0\sin 3\theta$ at $r=a$, we have

$$-\sum_{n=1}^{\infty}na^{-n-1}(a_n\cos n\theta+b_n\sin n\theta)=\frac{1}{2}U_0\sin 3\theta, \text{ for all } \theta.$$

so that $a_n=0$ for all n, $b_n=0$ for all $n\ne 3$ and $b_3=-\frac{U_0a^3}{3}$.

Hence the required solution of the given problem is

$$U(r,\theta)=-\frac{U_0}{3}\left(\frac{a}{r}\right)^3\sin 3\theta, a\le r\le\infty.$$

Example 5.5 : Solve the partial differential equation

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}=0,$$

subject to the conditions

$$\frac{\partial U}{\partial r}=0 \text{ at } r=a,$$

and $\dfrac{\partial U}{\partial r}=U_\infty \cos\theta, \dfrac{1}{r}\dfrac{\partial U}{\partial \theta}=-U_\infty \sin\theta$ *as* $r\to\infty$.

Solution : Let $U(r,\theta)=R(r)\Theta(\theta)$. Then the given Laplace equation gives

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}\right)=-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=n^2 \text{ (say)}$$

so that

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-n^2R=0 \text{ and } \frac{d^2\Theta}{d\theta^2}+n^2\Theta=0$$

whose solutions are : $R(r)=a_nr^n+b_nr^{-n}$ and $\Theta(\theta)=c_n\cos n\theta+d_n\sin n\theta$, so that the solution of Laplace's equation can be written as

$$U(r,\theta)=\sum_{n=1}^{\infty}r^n\left(A_n\cos n\theta+B_n\sin n\theta\right)+\sum_{n=1}^{\infty}r^{-n}\left(C_n\cos n\theta+D_n\sin n\theta\right)$$

provided $n\neq 0$. If $n=0$, then the solution is

$$U(r,\theta)=\left(C_0\ln r+D_0\right)\left(A_0\theta+B_0\right).$$

Thus the solution of the given equation

$$U(r,\theta)=\left(C_0\ln r+D_0\right)\left(A_0\theta+B_0\right)+\sum_{n=1}^{\infty}r^n\left(A_n\cos n\theta+B_n\sin n\theta\right)+\sum_{n=1}^{\infty}r^{-n}\left(C_n\cos n\theta+D_n\sin n\theta\right)$$

Now

$$\frac{\partial U}{\partial r}=\frac{C_0}{r}\left(A_0\theta+B_0\right)+\sum_{n=1}^{\infty}nr^{n-1}\left(A_n\cos n\theta+B_n\sin n\theta\right)-\sum_{n=1}^{\infty}nr^{-n-1}\left(C_n\cos n\theta+D_n\sin n\theta\right)$$

so that to satisfy the condition at infinity, viz. $\dfrac{\partial U}{\partial r}=U_\infty\cos\theta$, we must have $A_1=U_\infty, A_n=0\,(n\geq 2)$ and

$B_n = 0$. Also, the condition $\frac{\partial U}{\partial r} = 0$ at $r = a$ gives

$$\frac{C_0}{a}(A_0\theta + B_0) + U_\infty \cos\theta - \sum_{n=1}^{\infty} na^{-n-1}(C_n \cos n\theta + D_n \sin n\theta) \text{ for all } \theta, \text{ implying and}$$

$$C_0 = 0, C_1 = a^2U_\infty, C_n = 0(n \geq 2) \text{ and } D_n = 0.$$

Hence the required solution of the given problem is

$$U(r,\theta) = k\theta + U_\infty r\left(1 + \frac{a^2}{r^2}\right)\cos\theta.$$

Example 5.6 : Find the steady state temperature distribution in a semi-circular plate of radius a, insulated on both the faces with its curved boundary kept at a constant temperature U_0 and its bounding diameter kept at zero temperature.

Solution. In the steady state, the temperature U is independent of time and satisfies Laplace's equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0.$$

The given boundary conditions are

$$U(a,\theta) = U_0, U(r,0) = U(r,\pi) = 0.$$

The appropriate solution of the above equation in conformity with the given problem is

$$U(r,\theta) = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta),$$

p^2 being the separation constant.

From the boundary condition $U(r, 0) = 0$, we have $c_3 = 0$ and the condition $U(r,\pi) = 0$ gives $\sin p\pi = 0$, i.e. $p = n$ (neglecting the trivial solution). Also, noting that U must be finite at $r = 0$, so $c_2 = 0$. Hence the solution for U, by superposition principle is of the form

$$U(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta$$

Finally, the condition $U(a,\theta) = U_0$ gives

$$U_0 = \sum_{n=1}^{\infty} A_n a^n \sin n\theta$$

leading to $A_n a^n = \frac{2U_0}{\pi}\int_0^{\pi}\sin n\theta d\theta = \begin{cases} \frac{4U_0}{n\pi}, \text{if } n \text{ is odd} \\ 0, \text{if } n \text{ is even.} \end{cases}$

Hence $A_n = \frac{4U_0}{n\pi a^n}$ if n is odd and $A_n = 0$ if n is even.

Thus the required solution of the given problem is

$$U(r,\theta) = \frac{4U_0}{\pi}\sum_{n=odd}\frac{1}{n}\left(\frac{r}{a}\right)^n \sin n\theta = \frac{4U_0}{\pi}\sum_{n=1}^{\infty}\frac{\left(r/a\right)^{2n-1}}{2n-1}\sin(2n-1)\theta$$

III. Spherical polar coordinates (r,θ):

Assuming axial symmetry about the polar axis $\theta = 0$ so that $U(r,\theta,\phi)$ is independent of ϕ and Laplace's equation in spherical polar coordinates is

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) = 0. \tag{5.30}$$

To solve this equation, by separation of variables technique, we put

$$U(r,\theta) = R(r)\Theta(\theta)$$

and get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\cdot\frac{d\Theta}{d\theta}\right) = k \text{ (say)}$$

where k is separation constant. Taking $k = n(n+1)$, we have

$$r^2\frac{\partial^2 R}{\partial r^2} + 2r\frac{\partial R}{\partial r} - n(n+1)R = 0$$

and $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + n(n+1)\Theta = 0$$

i.e. $$\frac{d}{d\mu}\left\{\left(1-\mu^2\right)\frac{d\Theta}{d\mu}\right\} + n(n+1)\Theta = 0$$

where $\mu = \cos\theta$. The solutions of the above equations are

$$R(r) = A \cdot r^n + \frac{B}{r^{n+1}} \text{ and } \Theta(\theta) = C P_n(\mu) + D Q_n(\mu)$$

where $P_n(\mu)$ and $Q_n(\mu)$ are Legendre functions of the first kind and second kind respectively. Hence the solution of (5.30), with the use of superposition principle, is

$$U(r,\theta) = \sum_{n=1}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)\left[C_n P_n(\cos\theta) + D_n Q_n(\cos\theta)\right]. \tag{5.31}$$

Example 5.6 : If U is a harmonic function which is zero on the cone $\theta = \alpha$ and takes the value $\sum \alpha_n r^n$ on the cone $\theta = \beta$, show that, when $\alpha < \theta < \beta$,

$$U = \sum_{n=0}^{\infty} \alpha_n r^n \left\{\frac{P_n(\cos\theta) Q_n(\cos\alpha) - P_n(\cos\alpha) Q_n(\cos\theta)}{P_n(\cos\beta) Q_n(\cos\alpha) - P_n(\cos\alpha) Q_n(\cos\beta)}\right\}.$$

Solution. We consider the solution (5.31). Since U must be finite at $r = 0$, so $B_n = 0$.

Hence, we can write

$$U(r,\theta) = \sum_{n=0}^{\infty} r^n \left[a_n P_n(\cos\theta) + b_n Q_n(\cos\theta)\right]$$

The conditions $U(r,\theta) = 0, U(r,\beta) = \sum \alpha_n r^n$ give

$$\sum_{n=0}^{\infty} r^n \left[a_n P_n(\cos\alpha) + b_n Q_n(\cos\alpha)\right] = 0$$

and $$\sum_{n=0}^{\infty} r^n \left[a_n P_n(\cos\beta) + b_n Q_n(\cos\beta)\right] = \sum_{n=0}^{\infty} \alpha_n r^n$$

leading to

$$a_n P_n(\cos\alpha) + b_n Q_n(\cos\alpha) = 0$$

and $$a_n P_n(\cos\beta) + b_n Q_n(\cos\beta) = \alpha_n$$

So that

$$b_n = -a_n \frac{P_n(\cos\alpha)}{Q_n(\cos\alpha)} \text{ and } a_n = \frac{\alpha_n Q_n(\cos\alpha)}{P_n(\cos\beta) Q_n(\cos\alpha) - P_n(\cos\alpha) Q_n(\cos\beta)}$$

Hence the required solution of the harmonic equation is

$$U(r,\theta)=\sum_{n=0}^{\infty}\alpha_n r^n\left\{\frac{P_n(\cos\theta)Q_n(\cos\alpha)-P_n(\cos\alpha)Q_n(\cos\theta)}{P_n(\cos\beta)Q_n(\cos\alpha)-P_n(\cos\alpha)Q_n(\cos\beta)}\right\}.$$

Example 5.7 : Determine the potential of a grounded conducting sphere of radius a in a uniform field defined by

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)=0, 0\le r\le a, 0<\theta<\pi,$$

with boundary conditions $U(a,\theta)=0$ and $U\to -E_0 r\cos\theta$ as $r\to\infty$.

Solution. The solution of the given equation in conformity with the given conditions is

$$U(r,\theta)=\sum_{n=0}^{\infty}\left(A_n r^n+\frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

The condition $U\to -E_0 r\cos\theta$ as $r\to\infty$ gives

$$\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta)=-E_0 r\cos\theta, \text{ for all } \theta.$$

Thus $A_1=-E_0, A_n=0$ for all $n\neq 1$. Hence

$$U(r,\theta)=-E_0 r\cos\theta+\sum_{n=0}^{\infty}\frac{B_n}{r^{n+1}}P_n(\cos\theta).$$

The boundary condition $U(r,\theta)=0$ gives

$$0=-E_0 a\cos\theta+\sum_{n=0}^{\infty}\frac{B_n}{a^{n+1}}P_n(\cos\theta).$$

Multiplying both sides by $P_m(\cos\theta).\sin\theta$ and integrating between the limits 0 to π, we get

$$E_0 a\int_0^{\pi}P_1(\cos\theta)P_m(\cos\theta)\sin\theta d\theta=\sum_{n=1}^{\infty}\frac{B_n}{a^{n+1}}\int_0^{\pi}P_n(\cos\theta).P_m(\cos\theta)\cdot\sin\theta d\theta$$

Using the orthogonal property of Legendre polynomials, viz.

$$\int_0^{\pi}P_n(\cos\theta)P_m(\cos\theta)\sin\theta d\theta=\begin{cases}\dfrac{2}{2m+1}\text{if } n=m\\ 0 \text{ if } \mathrm{n}\neq\mathrm{m}\end{cases}$$

We get

$$E_0 a \int_0^{\pi} P_1(\cos\theta)\cdot P_m(\cos\theta)\sin\theta d\theta = \frac{B_m}{a^{m+1}}\cdot\frac{2}{2m+1}.$$

But, according to the above orthogonal property, the left hand side vanishes for all m except for $m=1$. Thus

$$E_0 a\cdot\frac{2}{2\cdot 1+1} = \frac{B_1}{a^{1+1}}\cdot\frac{2}{2\cdot 1+1}, \text{ i.e. } B_1 = E_0 a^3.$$

Hence the required potential is

$$U(r,\theta) = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta.$$

6. Solution of Three-dimensional Laplace's Equation (Separation of Variable Method) :

I. Cartesian Coordinates (x, y, z):

In Cartesian coordinates (x, y, z), the Laplace's equation is

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0. \tag{6.1}$$

To solve this equation by separation of variable technique, we put

$$U(x,y,z) = X(x)Y(y)Z(z) \tag{6.2}$$

in (6.1) and get

$$\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = -\frac{1}{X}\frac{d^2X}{dx^2} = \lambda_1^2, \text{(say)} \tag{6.3}$$

λ_1^2 being separation constant. Then we have

$$\frac{d^2X}{dx^2} + \lambda_1^2 X = 0$$

leading to the solution $X(x) = c_1\cos\lambda_1 x + c_2\sin\lambda_1 x.$

Again, from (6.3), we get

$$\frac{1}{Z}\frac{d^2Z}{dz^2} - \lambda_1^2 = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda_2^2, \text{say,}$$

where λ_2^2 is separation constant, from which it follows that

$$\frac{d^2Y}{dy^2} + \lambda_2^2 Y = 0 \text{ and } \frac{d^2Z}{dz^2} - \lambda_3^2 Z = 0$$

where $\lambda_3^2 = \lambda_1^2 + \lambda_2^2$. These equations have solutions

$$Y(y) = c_3 \cos \lambda_2 y + c_4 \sin \lambda_2 y \text{ and } Z(z) = c_5 \cosh \lambda_3 z + c_6 \sinh \lambda_3 z.$$

Hence the general solution of the equation (6.1) is

$$U(x,y,z) = (c_1 \cos \lambda_1 x + c_2 \sin \lambda_1 x)(c_3 \cos \lambda_2 y + c_4 \sin \lambda_2 y)(c_5 \cosh \lambda_3 z + c_6 \sinh \lambda_3 z). \quad (6.4)$$

Example 6.1: Show that $U = \frac{k}{|r-r'|}$, where k is constant, is a solution of Laplace's equation

Solution. We have $U = \frac{k}{|r-r'|} = \frac{k}{\sqrt{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}}}$

Then $\frac{\partial U}{\partial x} = -\frac{k(x-x')}{|r-r'|^3}$ and $\frac{\partial^2 U}{\partial x^2} = -\frac{k}{|r-r'|^3} + \frac{3k(x-x')^2}{|r-r'|^5}$.

Similarly, $\frac{\partial^2 U}{\partial y^2} = -\frac{k}{|r-r'|} + \frac{3k(y-y')^2}{|r-r'|^5}$ and $\frac{\partial^2 U}{\partial z^2} = -\frac{k}{|r-r'|} + \frac{3k(z-z')^2}{|r-r'|^5}$

Hence $\nabla^2 U = -\frac{3k}{|r-r'|^3} + \frac{3k\{(x-x')^2 + (y-y')^2 + (z-z')^2\}}{|r-r'|^5} = 0$

Example 6.2 : Find the potential U in a rectangular box defined by $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$, if the potential is zero on all sides and the bottom, while $U = f(x,y)$ on the top $z = c$ of the box.

Solution. The potential function $U(x,y,z)$ in the rectnagular box satisfies

Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0,$$

the boundary conditions for the given problem being

$$U(0,y,z) = U(a,y,z) = 0,$$

$$U(x,0,z) = U(x,b,z) = 0,$$

$$U(x,y,0) = 0, U(x,y,c) = f(x,y)$$

in which $f(x,y)$ is assumed to be expansible in double Fourier series.

The general solution of the Laplace's equation is given by (6.4).

Now the boundary conditions $U(0, y, z) = 0$ and $U(a, y, z) = 0$ give respectively $c_1 = 0$ and $\lambda_1 = m\pi/a \,(m = 1, 2, ...)$. Also the conditions $U(x, 0, z) = 0$ and $U(x, b, z) = 0$ lead respectively to $c_3 = 0$ and $\lambda_2 = n\pi/b \,(n = 1, 2, ...)$. Also the condition $U(x, y, 0) = 0$ gives $c_5 = 0$. Further, we note that

$$\lambda_3^2 = \lambda_1^2 + \lambda_2^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) = \lambda_{mn}^2 \text{ (say)},$$

so that $\lambda_3 = \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = \lambda_{mn}$.

Thus, by the use of superposition principle, the solution for U is

$$U(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn} \sin\left(\frac{m\pi x}{a}\right) \cdot \sin\left(\frac{n\pi y}{b}\right) \sinh(\lambda_{mn} z) \tag{6.4}$$

Now the boundary condition $U(x, y, c) = f(x, y)$

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sinh(\lambda_{mn} c)$$

which is a double Fourier sine series, so that

$$c_{mn} \sinh(\lambda_{mn} c) = \frac{4}{ab}\int_0^a\int_0^b f(x, y) \sin\left(\frac{m\pi x}{a}\right) \cdot \sin\left(\frac{n\pi y}{b}\right) dx\, dy.$$

Hence the required potential is given by (6.4) where c_{mn} is obtained from the above integral.

II. Cylindrical coordinates (r, θ, z):

Laplace's equation in three-dimensional cylindrical coordinates (r, θ, z) is given by

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = 0. \tag{6.5}$$

Let

$$U(r, \theta, z) = R(r)\Theta(\theta)Z(z) \tag{6.6}$$

Substituting this in (6.5) we get

$$\frac{1}{R}\left(\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0 \tag{6.7}$$

Assume $\frac{1}{Z}\frac{d^2Z}{dz^2}=m^2$ and $\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=-n^2$, i.e.

$$\frac{d^2Z}{dz^2}-m^2Z=0 \text{ and } \frac{d^2\Theta}{d\theta^2}+n^2\Theta=0$$

whose solutions are of the form $Z=e^{\pm mz}$ and $\Theta=e^{\pm in\theta}$ respectively. Thus (6.7) gives

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(m^2-\frac{n^2}{r^2}\right)R=0$$

whose solution is

$$R(r)=c_1J_n(mr)+c_2Y_n(mr).$$

Hence the most general solution of the equation (6.5) is

$$U(r,\theta,z)=\{c_1J_n(mr)+c_2Y_n(mr)\}(c_3\cos n\theta+c_4\cos n\theta)(c_5e^{mz}+c_6e^{-mz}) \tag{6.8}$$

Example 6.3 : Find the potential U inside the cylinder $0\le r\le a, 0\le\theta\le 2\pi, 0\le z\le h,$ if the potential on the top $z=h$ and on the lateral surface $r=a$ is held at zero, while on the base $z=0, U=U_0\left(1-\frac{r^2}{a^2}\right)$, where U_0 is constant.

Solution. In this case, the potential U is single-valued and satisfies Laplace's equation (6.5). The boundary conditions are

$$U(r,\theta,h)=0, U(a,\theta,z)=0, U(r,\theta,0)=U_0\left(1-\frac{r^2}{a^2}\right).$$

Now consider the general solution (6.8). Since $Y_n(mr)\to\infty$ as $r\to 0$, so $c_2=0$.

Also, the face $z=0$ has the potential $U_0\left(1-\frac{r^2}{a^2}\right)$, a function independent of θ, threfore, the potential U must be independent of θ inside the cylinder and this is possible provided $n=0$. Hence, the general solution of Laplace's equation for the given problem is of the form

$$U(r,z)=J_0(mr)\left(Ae^{mz}+Be^{-mz}\right).$$

Now the boundary condition $U(r,\theta,h)=0$ gives

$$Ae^{mh} + Be^{-mh} = 0, i.e. B = -A \cdot \frac{e^{mh}}{e^{-mh}}$$

so that

$$U(r,z) = C J_0(mr) \sinh m(z-h), \text{ where } C = 2A/e^{-mh}$$

The boundary condition $U = 0$ on the lateral surface $r = a$ shows that $J_0(ma) = 0$ which has infinitely many positive roots ξ_p, say, so that $\xi_p = ma$, i.e. $m = \xi_p/a$. Thus the solution of Laplace equation takes the form

$$U(r,z) = \sum_{p=1}^{\infty} A_p J_0(\xi_p r/a) \sinh\{\xi_p (z-h)/a\}.$$

Again the condition $U(r,z) = U_0\left(1 - \frac{r^2}{a^2}\right)$ at $z = 0$ implies

$$U_0\left(1 - \frac{r^2}{a^2}\right) = \sum_{p=1}^{\infty} A_p \sinh\{-\xi_p h/a\} \cdot J_0(\xi_p r/a)$$

which is a Fourier-Bessel series. Multiplying both sides by $rJ_0\left(\xi_p \, r/a\right)$ and then integrating with respect to r between the limits 0 to a we get

$$U_0 \int_0^a \left(1 - \frac{r^2}{a^2}\right) r J_0\left(\xi_q \, r/a\right) dr = \sum_{p=1}^{\infty} A_p \sinh\left\{-\xi_p \, h/a\right\} \int_0^a r J_0\left(\xi_q \, r/a\right) J_0\left(\xi_q \, r/a\right) dr$$

Using the orthogonal property of Bessel function, viz.

$$\int_0^a x J_n(\alpha_i x) J_n(\alpha_j x)\, dx = \begin{cases} \frac{a^2}{2} J_{n+1}^2(\alpha_i), \text{if } i = j \\ 0, \text{if i} \neq \text{j} \end{cases}$$

where α_i and α_i are roots of $J_0(x) = 0$, we get

$$U_0 \int_0^a \left(1 - \frac{r^2}{a^2}\right) r J_0\left(\xi_q \, r/a\right) dr = \frac{a^2}{2} \sum_{p=1}^{\infty} A_p \sinh\left\{-\xi_p \, h/a\right\} \cdot J_1^2(\xi_p)$$

giving

$$A_p = \frac{2U_0}{a^2 \sinh(-\xi_p \, h/a) \cdot} \int_0^a \left(1 - \frac{r^2}{a^2}\right) r J_0\left(\xi_p \, r/a\right) dr.$$

Noting the relations $\int x J_0(x)dx = xJ_1(x)$ and $\int x^2 J_1(x)dx = x^2 J_2(x)$, we obtain after integrating by parts

$$A_p = \frac{4U_0 J_2(\xi_p)}{\xi_p^2 \sinh\left(-\xi_p h/a\right) J_1^2(\xi_p)}$$

Noting the recourrence relation $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$, we have by putting $n = 1, x = \xi_p$ and keeping in mind that $J_0(\xi_p) = 0, J_2(\xi_p) = 2J_1(\xi_p)/\xi_p$ and, therefore,

$$A_p = \frac{8U_0 J_1(\xi_p)}{\xi_p^3 \sinh(-\xi_p h/a) J_1^2(\xi_p)}$$

Hence the required solution for the potential function is

$$U(r,z) = 8U_0 \sum_{p=1}^{\infty} \frac{J_0(\xi_p r/a)\sinh\{\xi_p(z-h)/a\}}{\xi_p^3 J_1(\xi_p)\sinh(-\xi_p h/a)}.$$

III. Spherical polar coordinates (r,θ,ϕ):

In this case, the Laplace equation is

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial U}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2} = 0 \qquad (6.9)$$

To solve this equation, we put $U(r,\theta,\phi) = R(r)F(\theta,\phi)$ in (6.9) we get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{\partial U}{\partial r}\right) = -\frac{1}{F\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\phi^2}\right\} = -n(n+1),\ \text{say} \qquad (6.10)$$

$(n+1)$ being a separation parameter. So we have

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{\partial U}{\partial r}\right) + n(n+1)R = 0 \qquad (6.10)$$

and $$\frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\phi^2}\right\} + n(n+1)F = 0 \qquad (6.11)$$

Solution of the equation (6.10) is

$$R(r) = c_1 r^n + c_2 r^{-(n+1)}$$

For the solution of (6.11) we take $F(\theta,\phi)=\Theta(\theta)\Phi(\phi)$ so that

$$\frac{\sin\theta}{\Theta}\left\{\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+n(n+1)\sin^2\theta\cdot\Theta\right\}=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=m^2,\text{ say} \tag{6.12}$$

m^2 being another separation constant. Then we have

$$\frac{\sin\theta}{\Theta}\left\{\frac{d}{d\theta}\left(\sin\theta\cdot\frac{d\Theta}{d\theta}\right)+n(n+1)\sin^2\theta\cdot\Theta\right\}=m^2$$

i.e. $\left(1-\mu^2\right)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\left\{n(n+1)-\frac{m^2}{1-\mu^2}\right\}\Theta=0$ (putting $\cos\theta=\mu$)

and $\frac{d^2\Phi}{d\phi^2}+m^2\Phi=0.$

Solutions of these equations are given by

$$\Phi(\mu)=c_3P_n^m(\mu)+c_4Q_n^m(\mu)\text{ and }\Phi(\phi)=c_5\cos m\phi+c_6\sin m\phi,$$

for $-1\le\mu\le1$, respectively, where P_n^m and Q_n^m are associated Legendre functions of the first and second kind. Since Q_n^m has a singularity at $\theta=0$, so we take $c_4=0$.

Hence the general solution of the Laplace's equation (6.9) is

$$U(r,\theta,\phi)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(c_1r^n+\frac{c_2}{r^{n+1}}\right)(B_1\cos m\phi+B_2\sin m\phi)P_n^m(\cos\theta), \tag{6.12}$$

where $B_1=c_3c_5, B_2=c_4c_5$ and $0\le\theta\le\pi$

In particular, for the axisymmetric case, the general solution is independent of ϕ and we have

$$U(r,\theta)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(c_1r^n+\frac{c_2}{r^{n+1}}\right)P_n^m(\cos\theta). \tag{6.13}$$

Interior Dirichlet problem for a sphere :

The problem is to find the value of the function U at any point interior of the sphere $r=a$ such the $\nabla^2U=0, 0\le r\le a, 0\le\theta\le\pi, 0\le\phi\le2\pi$ and $U(r,\theta,\phi)=f(\theta,\phi)$

Since $r=0$ is a point within the sphere $r=a$, so we must have $c_2=0$.

Thus from (6.12), we have

$$U(r,\theta,\phi)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}r^n\left[A_{mn}\cos m\phi+B_{mn}\sin m\phi\right]P_n^m(\cos\theta),. \tag{6.14}$$

A_{mn}, B_{mn} being new constants after adjustment. Using the boundary condition $U(r,\theta,\phi)=f(\theta,\phi)$, we get

$$U(\theta,\phi)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a^n\left[A_{mn}\cos m\phi+B_{mn}\sin m\phi\right]P_n^m(\mu) \tag{6.15}$$

where it is assumed that $U(\theta,\phi)$ is expansible in series of associated Legendre function. Multiplying both sides of (6.15) by $P_n^m(\mu)\cos m\phi$ and then integrating w.r.t. μ between -1 to 1 and ϕ between 0 to 2π, we have

$$\int_0^{2\pi}\int_{-1}^{1}f(\theta,\phi)P_n^m(\mu)\cos m\phi\, d\mu\, d\phi$$

$$=a^2A_{mn}\int_0^{2\pi}\cos^2 m\phi\left[\int_{-1}^{1}\left\{P_n^m(\mu)\right\}d\mu\right]d\phi=\frac{2\pi a^n A_{mn}(m+n)!}{(2n+1)(n-m)!}$$

so that

$$A_{mn}=\frac{(2n+1)(n-m)!}{2\pi a^n(m+n)!}\int_0^{2\pi}f(\theta,\phi)P_n^m(\mu)\cos m\phi\, d\mu\, d\phi. \tag{6.16}$$

Similarly, mutliplying both sides of (6.15) by $P_n^m(\mu)\sin m\phi$ and then integrating with respect to μ from -1 to 1 and ϕ from 0 to 2π, we obtain

$$B_{mn}=\frac{(2n+1)(n-m)!}{2\pi a^n(m+n)!}\int_0^{2\pi}f(\theta,\phi)P_n^m(\mu)\sin m\phi\, d\mu\, d\phi. \tag{6.17}$$

Using (6.16) and (6.17) in (6.14), it follows that the required solution is

$$U(r,\theta,\phi)=\frac{1}{2\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2n+1)(n-m)!}{(m+n)!}\left(\frac{r}{a}\right)^n\times$$

$$\int_{-1}^{1}\int_0^{2\pi}f(\eta,\chi)P_n^m(\mu)P_n^m(\mu')\left[\cos m\chi\cdot\cos m\eta+\sin m\chi\cdot\sin m\eta\right]d\eta\, d\chi$$

i.e. $$U(r,\theta,\phi)=\frac{1}{2\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2n+1)(n-m)!}{(m+n)!}\left(\frac{r}{a}\right)^n\times$$

$$\int_{-1}^{1}\int_0^{2\pi}f(\eta,\chi)P_n^m(\cos\chi)P_n^m(\cos\eta)\cos m(\chi-\eta)\, d\eta\, d\chi. \tag{6.18}$$

7. Potential Due to a Continuous Distribution

Consider the function U given by

$$U(x,y,z) = \frac{q}{|r-r'|} = \frac{q}{\sqrt{\{(x-x')^2+(y-y')^2+(z-z')^2\}}} \tag{7.1}$$

where q is a constant and (x',y',z') are the coordinates of a fixed point.

$$\frac{\partial U}{\partial x} = -\frac{q(x-x')}{|r-r'|^3} \text{ etc. and } \frac{\partial^2 U}{\partial x^2} = -\frac{q(x-x')}{|r-r'|^3} + \frac{3q(x-x')^2}{|r-r'|^5} \text{ etc.,}$$

it is readily seen that $\nabla^2 U = 0$ so that (7.1) is a solution of Laplace's equation except possibly at the point (x',y',z') where it is undefined. The function U is called potential function.

The function U given by (7.1) is a possible form for the electrostatic potential corresponding to a space which, apart from the isolated point (x',y',z') is empty of electric charge q. By a simple superposition procedure. We see that for n charges $q_i,(i=1,2,...,r)$,

$$U(x,y,z) = \sum_{i=1}^{n} \frac{q_i}{|r-r'|} \tag{7.2}$$

is the solution of Laplace's equation corresponding to n charges q_i situated at points $r_i,(i=1,2,...,n)$.

In reality, we usually deal with continuous distributions of charge instead of point charges or dipoles. For a continuous distribution of charge filling a region V of space, the corresponding form of the potential function U is given by

$$U(r) = \int_V \frac{\rho(r')dV'}{|r-r'|} \tag{7.3}$$

where $\rho(r')$ denotes the charge density at the point r'.

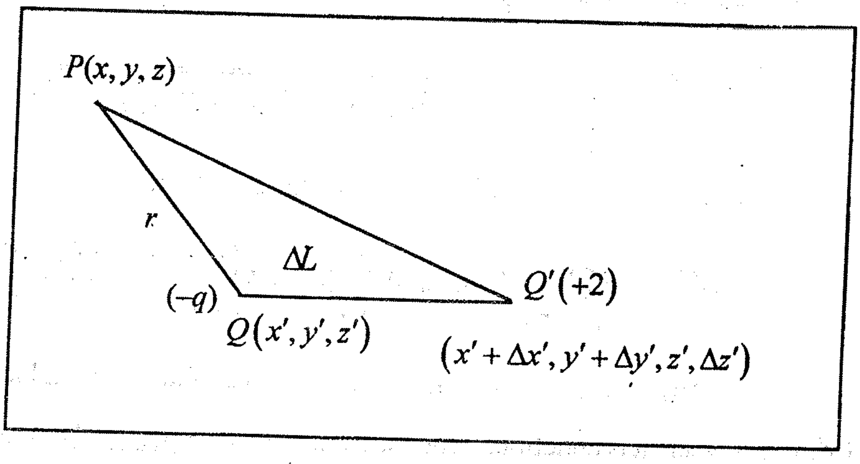
Similarly, it can be shown that the solution of Laplace's equation corresponding to a continuous distribution of charges of density σ on a surface S is

$$U(r) = \int_S \frac{\sigma(r')dS'}{|r-r'|}. \tag{7.4}$$

It is to be noted that the results (7.3) and (7.4) also hold good for continuous distribution of magnetic poles or gravitational masses.

Potential of a double lager

Suppose a charge $-q$ be placed at $Q(x',y',z')$ and a charge $+q$ at $Q'(x'+\Delta x', y'+\Delta y', z'+\Delta z')$ where $QQ'(=\Delta L)$ is very small. If q is very large and Δl is very small such that the product $q\Delta L(=\mu)$ is finite, then such a pair forms a doublet or dipole and the direction of the vector $\overrightarrow{QQ'}$ is called the axis of the doublet.



Now, if $PQ=r$ and $PQ'=r+\Delta r$, then the potential at P due to this doublet is

$$U=\lim_{\Delta L\to 0}\left[\frac{2}{r+\Delta r}-\frac{q}{r}\right]=\lim_{\Delta L\to 0}\frac{q\Delta L}{\Delta L}\left[\left\{\frac{1}{r}+\left(\Delta x'\frac{\partial}{\partial x'}+\Delta y'\frac{\partial}{\partial y'}+\Delta z'\frac{\partial}{\partial z'}\right)\left(\frac{1}{r}\right)+\ldots\right\}-\frac{1}{r}\right]$$

$$=\lim_{\Delta L\to 0}q\Delta L\left[\left(\frac{\Delta x'}{\Delta L}\frac{\partial}{\partial x'}+\frac{\Delta y'}{\Delta L}\frac{\partial}{\partial y'}+\frac{\Delta z'}{\Delta L}\frac{\partial}{\partial z'}\right)\left(\frac{1}{r}\right)+\ldots\right]=\mu\left(l_1\frac{\partial}{\partial x'}+m_1\frac{\partial}{\partial y'}+n_1\frac{\partial}{\partial z'}\right)\left(\frac{1}{r}\right)$$

i.e. $$U(x,y,z)=\mu\frac{\partial}{\partial n}\left(\frac{1}{r}\right), \qquad (7.5)$$

where $\frac{\partial}{\partial n}=l_1\frac{\partial}{\partial x'}+m_1\frac{\partial}{\partial y'}+n_1\frac{\partial}{\partial z'}$ and (l_1,m_1,n_1) are the direction cosines of QQ' and δn is an element of the doublet, μ is called the moment of the doublet.

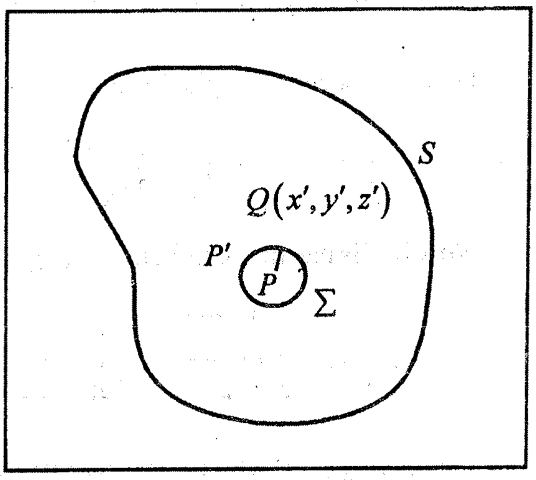
Now let us consider two surfaces S sand S' at a small distance ΔL apart and the charges be distributed thereon in such a way that the negative charges lie on the surface S and positive charges on S', the axis of the charges being everywhere normal to both the surface and is directed from the negative to positive charges. Passing to the limit as $\Delta L\to 0$, we obtain the double layer as a combination of two single layers with opposite charges at a very small distance from one another. The potential U of all such charges distributed on the double layer is obtained in a similar process as above

$$U(x,y,z)=\int_S \mu\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS' \qquad (7.6)$$

where μ is the moment of the double layer.

The result (7.6) is also true for magnetic poles and mass distribution.

Representation of harmonic function as the sum of potentials of a simple and double layers.



Let V be any regular region and $P(x,y,z)$ be any interior point. In Green's second identity (4.3), we put $\phi = U$ and $\psi = \frac{1}{r}$, where $r = \overline{PQ}$ and $Q(x',y',z')$ being a point in V. Since P is interior to V, so this identity cannot be applied to the whole region V. So, we surround P by a small sphere Σ with P as centre and radius ε. For the resulting region V', we have by noting that $\frac{1}{r}$ is harmonic in V

where δn is an element of normal to the boundary of V pointing outward from V' so that on Σ, it has the direction opposite to that of r.

Now, we have

$$\int_{z} U\frac{\partial}{\partial n}\left(\frac{1}{r}\right)d\Sigma = -\int_{\Sigma} U\left(l_1\frac{\partial}{\partial x'}+m_1\frac{\partial}{\partial y'}+n_1\frac{\partial}{\partial z'}\right)\left(\frac{1}{r}\right)d\Sigma$$

$$= \int_{\Sigma} U\left[l_1\cdot\frac{1}{r^2}\frac{\partial r}{\partial x'}+m_1\cdot\frac{1}{r^2}\frac{\partial r}{\partial y'}+n_1\cdot\frac{1}{r^2}\frac{\partial r}{\partial z'}\right]d\Sigma$$

$$\left(\because r^2 = (x'-x)^2+(y'-y)^2+(z'-z)^2\right)$$

$$= \int_{\Sigma} U\left[\frac{x'-x}{r}\cdot\frac{1}{r^2}\cdot\frac{x'-x}{r}+\frac{y'-y}{r}\cdot\frac{1}{r^2}\cdot\frac{y'-y}{r}+\frac{z'-z}{r}\cdot\frac{1}{r^2}\cdot\frac{z'-z}{r}\right]d\Sigma$$

$$= \int_{\Sigma} U\cdot\frac{1}{r^2}d\Sigma = \int_{\Sigma} U d\Omega$$ (where $d\Omega$ is the solid angle subtended at P by $d\Sigma$)

$$= U(P')\int d\Omega = 4\pi U(P')$$ [where P' is Σ]

$$\rightarrow 4\pi U(P) \text{ as } \varepsilon \rightarrow 0.$$

Again $\left|\int_{\Sigma}\frac{1}{r}\frac{\partial U}{\partial n}d\Sigma\right| \le \int_{\Sigma}\frac{1}{r}\left|\frac{\partial U}{\partial n}\right|d\Sigma \le M\int_{\Sigma}\frac{1}{r}d\Sigma$ (where M is the upper bound of $\frac{\partial U}{\partial n}$)

$$= M\cdot\frac{1}{\varepsilon}\int_{\Sigma}d\Sigma = B\cdot\frac{1}{\varepsilon}\cdot 4\pi\varepsilon^2 = 4\pi B\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$

Hence, proceeding to the limit as $\varepsilon \to 0$, we obtain from (7.7)

$$-\int_V \frac{1}{r}\nabla^2 U dV = \int_S\left\{U\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial U}{\partial n}\right\}dS + 4\pi U(P) \tag{7.8}$$

Now if U is harmonic in V, i.e. if $\nabla^2 U = 0$ in V, then (7.8) gives

$$U(P) = \int_S \frac{\frac{1}{4\pi}\frac{\partial U}{\partial n}}{r}dS + \int_S\left(-\frac{U}{4\pi}\right)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS \tag{7.9}$$

Comparing (7.9) with (7.4) and (7.6) we see that the first term on the right hand side of (7.9) represents the potential of a surface distribution of charges on S of density $\frac{1}{4\pi}\frac{\partial U}{\partial n}$ while the second term is the potential of a double layer on S of moment $\left(-\frac{U}{4\pi}\right)$. Hence every regular harmonic function can be represented as the sum of potentials due to a simple surface distribution of charges and due to a double layer on the surface.

Exercise

By separating the variables, show that the equation $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$ has solutions of the form $A\exp(\pm nZ \pm iny)$, where A and n are constants. Deduce that functions of the form

$$U(x,y) = \sum_r A_r \exp\left(-\frac{r\pi x}{a}\right)\cdot\sin\left(-\frac{r\pi y}{a}\right), x \ge 0, y \ge 0,$$

where A_r's are constants, are plane harmonic functinos satisfying the conditioins $U(x,0) = 0, U(x,a) = 0$ and $U(x,y) \to 0$ as $x \to \infty$.

2. A thin rectangular homogeneous thermally conducted plate occupies the region $0 \le x \le a, 0 \le y \le b$. The edge $y = 0$ is held at temperature $Tx(x-a)$, where T is a constant and the other edges are maintained at 0^0.

The other faces are insulated and there is no heat source or sink inside the plate. Find the steady state temperature inside the plate.

$$\left[Ans.\ U(x,y)=\frac{8Ta^2}{\pi^3}\sum_{n=0}^{\infty}\frac{\cos ech\{(2n+1)\pi b/a\}}{(2n+1)^3}\sin\frac{(2n+1)\pi x}{a}\cdot\sinh\frac{(2n+1)(j-b)\pi}{a}\right]$$

3. Solve the Laplace's equation satisfying the boundary conditions

$$U(0,y)=U(x,0)=U(x,b)=0, U_x(a,y)=T\sin^3\frac{\pi y}{a}.$$

$$\left[Ans.\ U(x,y)=\frac{bT}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sec h\left(\frac{n\pi a}{b}\right)\sinh\left(\frac{n\pi x}{b}\right)\sin^3\left(\frac{\pi y}{a}\right)\right]$$

4. Let $f(x)$ and $g(x)$ be analytic and $U_1(x,y)$ be the solution of the Cauchy problem described by

$$\frac{\partial^2 U}{dx^2}+\frac{\partial^2 U}{\partial y^2}=0 \text{ subject to } U(x,0)=f(x), \frac{\partial U(x,0)}{\partial y}=g(x)$$

and let $U_2(x,y)$ be the solution of the above partial differential equation subject to

$U(x,y)=f(x), \frac{\partial U(x,0)}{\partial y}=g(x)+\frac{1}{n}\sin nx.$ Show that $U_2(x,y)-U_1(x,y)=\frac{1}{n^2}\sin nx\cdot\sinh ny.$

5. Let $U(r,\theta)$ satisfies the equation $\frac{\partial^2 U}{dr^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{d\theta^2}=0$ within the region of the plane bounded by $r=a, r=b, \theta=0$ and $\theta=\pi/2$. Its value along the boundary $r=a$ is $\theta\left(\frac{\pi}{2}-\theta\right)$ and along the other boundaries it is zero.

Show that

$$U(r,\theta)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(r/b)^{4n-2}-(b/r)^{4n-2}}{(a/b)^{4n-2}(b/a)^{4n-2}}\cdot\frac{\sin(4n-2)\theta}{(2n-1)^3}$$

6. A long circular cylinder is made of two halves, the upper half-surface is at temperature T_1 while the lower half is at temperature T_2. Find the steady-state distribution of temperature inside the cylinder.

$$\left[Ans. U(r,\theta)=\frac{1}{2}(T_1+T_2)+\frac{2(T_1-T_2)}{\pi}\sum_{n=1}^{\infty}\frac{r^{2n-1}\sin(2n-1)\theta}{(2n-1)a^{2n-1}}\right]$$

7. Solve the partial differential equation $\frac{\partial^2 U}{dr^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{d\theta^2}=0$ subject to the conditions $\frac{\partial U}{\partial r}=0$ at $r=a$ and $\frac{\partial U}{\partial r}\to U\cos\theta, \frac{1}{r}\frac{\partial U}{\partial 0}\to -U\sin\theta$ as $r\to\infty$.

8. A thermally conducting solid, bounded by two concentric spheres of radii a and $b\,(a<b)$, is such that the internal boundary is kept at temperature $f_1(\theta)$ and the outer boundary at $f_2(\theta)$. Find the state temperature in the solid.

$$\left[Ans. U(r,\theta)=\sum_{n=0}^{\infty}\left(A_n r^n+\frac{B_n}{r^{n+1}}\right)P_n(\cos\theta), \text{where } A_n=\frac{C_n a^{n+1}-D_n b^{n+1}}{a^{2n+1}-b^{2n+1}}, B_n=\frac{a^{n+1}b^{n+1}\left(C_n b^n-D_n a^n\right)}{b^{2n+1}-a^{2n+1}}\right]$$

$$\text{and } c_n=\frac{2n+1}{2}\int_0^{\pi} f_1(\theta)\,p_n(\cos\theta)\sin\theta d\theta, D_n=\frac{2n+1}{2}\int_0^{\pi} f_2(\theta)p_n(\cos\theta)\sin\theta d\theta$$

9. Find the steady state temperature distribution in a semi-circular plate of radius a insulated on both the faces with its curved boundary kept at a constant temperature T_0 and its bounding diameter kept at zero temperature.

$$\left[Ans.\ U(r,\theta)=\frac{4T_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{r}{a}\right)^{2n+1}\sin(2n+1)\theta\right]$$

10. In a solid sphere of radius a, the surface is maintained at the temperature given by

$$f(\theta)=\begin{cases} k\cos\theta, 0\le\theta\le\pi/2 \\ 0, \frac{\pi}{2}<\theta<\pi \end{cases}.$$

Prove that the steady state temperature within the solid is

$$U(r,\theta)=k\left[\frac{1}{4}P_0(\mu)+\frac{1}{2}\left(\frac{r}{a}\right)P_1(\mu)+\frac{5}{16}\left(\frac{r}{a}\right)^2 P_2(\mu)-\frac{3}{32}\left(\frac{r}{a}\right)^4 P_4(\mu)+\ldots\right],$$

where $\mu=\cos\theta$

11. Find the electrostatic potential U for the spherical shell bounded by the concentric sphere $r=a, r=b\,(0<a<b)$ if the inner and outer surface are kept at potentials V_1 and $V_2\,(V_1\ne V_2)$.

$$\left[Ans.\ U(r)=\frac{ab}{b-a}\left\{V_1\left(\frac{1}{r}-\frac{1}{b}\right)+V_2\left(\frac{1}{a}-\frac{1}{r}\right)\right\}\right].$$

12. Show that in cylindrical coordinates (r,θ,z), Laplace's euquation has solution of the form $R(r)\exp(\pm mz \pm in\theta)$, where $R(r)$ is a solution of the Besseel's equation

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(m^2-\frac{n^2}{r^2}\right)R=0.$$

If the solution tends to zero as $z \to \infty$ and is finite when $r=0$, show that in the usual notations of Bessel function, the appropriate solutions are made up of terms of the form $J_n(mr)\exp(-mz \pm in\theta)$.

13. A solid right circular cylinder is bounded by the surface $p=a, z=\pm h$, the system of coordinates being (r,θ,z). Find the steady temperature $U(r,z)$ at an internal point (r,θ,z) if $U=0$ on $r=a$, $U=T_1$ on $z=h$ and $U=T_2$ on $z=-h$.

$$\left[Ans.U(r,z)=\frac{2}{a}\sum_{n=1}^{\infty}\frac{\left\{T_2\left(e^{\xi_n(z-h)}-e^{-\xi_n(z-h)}\right)-T_1\left(e^{\xi_n(z+h)}-e^{-\xi_m(z+h)}\right)\right\}}{\xi_n J_1(\xi\cdot na)\left(e^{2\xi_n h}-e^{-2\xi_n h}\right)}\right]$$

ξ_n being the roots of $J_0(\xi_n)=0$

14. A homogeneous thermally conducting cylinder occupies the region $0\le r\le a, 0\le\theta\le 2\pi, 0\le z\le h$, where (r,θ,z) are cylinderical coordinates. The top $z=h$ and the lateral surface $r=a$ are held at 0°C while the base $z=0$ is held at 100°C. Assuming that there are no sources of heat within the cylinder, find the steady temperature distribution within the cylinder.

$\left[Ans.U(r,\theta)=200\sum_{n=1}^{\infty}\frac{J_n(\xi_n r/a)\sinh\{\xi_n(z-h)/a\}}{\xi_n J_1(\xi_n)\sinh(-\xi_n h/a)}\right.$, ξ_n being the roots of the equation $\left.J_0(\xi a)=0\right]$

15. In the theory of elasticity, the stress function $U(x,y)$ in the problem of torsion, satisfies the Poisson's equation $\frac{\partial^2U}{\partial x^2}+\frac{\partial^2U}{\partial y^2}=2, 0\le x\le 4, 0\le y\le 5$, with the boundary conditions $U=0$ on $x=0, 4$ and $y=0, 5$. Find the stress function $U(x,y)$.

$$\left[Ans.U(x,y)=x(x-4)+\frac{64}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{(2n-1)\pi x}{4}}{(2n-1)^3\sinh\frac{5(2n-1)\pi}{4}}\times\left\{\sinh\frac{(2n-1)(5-y)\pi}{4}+\sinh\frac{(2n-1)\pi y}{4}\right\}\right]$$

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-IV

Group–B

Module No. - 46

PARABOLIC EQUATIONS

1. Introduction :

The diffusion phenomena like conduction of heat in solids, diffusion in isotropic media, diffusion of vorticity, slowing down of neutrons in matter etc. are all governed by parabolic equation of the form

$$k\nabla^2 T = \frac{\partial T}{\partial t} \tag{1.1}$$

where k is constant. The equation (1–1) is known as **diffusion or heat conduction equation.** In this unit, we consider various properties and solution of parabolic equation.

2. Occurrence of Diffusion Equation

We now illustrate some examples regarding occurrence of diffusion equation in various fields.

(a) Conduction of heat in solids : If $T(r, t)$ be the temperature at a point in a homogeneous isotropic elastic solid, then the rate of heat flow per unit area across a plane is

$$q = -k\frac{\partial T}{\partial n} \tag{2.1}$$

where k is thermal conductivity of the solid and the operator $\frac{\partial}{\partial n}$ denotes the differentiation in the direction of the normal to the plane. Now if the solid does not undergo radioactive decay or absorbing radiation or if there is no generation or absorption of heat due to chemical reaction, then flow of heat is governed by the equation

$$\rho c\frac{\partial T}{\partial T} = \vec{\nabla}\cdot\left(k\vec{\nabla}T\right) \tag{2.2}$$

where ρ is the density and c is the specific heat. In particular, for constant conductivity κ, equation (2.2) reduces to

$$\frac{\partial T}{\partial t} = k\nabla^2 T \tag{2.3}$$

where $\kappa = \kappa/\rho c$.

(b) **Diffusion in isotropic media :** If c be the concentration of the diffusing substance, then the diffusion current vector $\vec{J}$ is governed by Fick's law of diffusion $\vec{J} = -D\vec{\nabla}c$, D being the diffusion coefficient. The equation of continuity for the diffusing substance is given by

$$\frac{\partial c}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0, \text{ i.e.} \frac{\partial c}{\partial t} = \vec{\nabla}\cdot\left(D\vec{\nabla}c\right)$$

and for constant diffusion coefficient

$$\frac{\partial c}{\partial t} = D\nabla^2 c \tag{2.4}$$

(c) **Diffusion of vorticity :** For the motion of a viscous fluid of density ρ and kinetic viscosity υ, the vorticity $\vec{w}$ satisfies diffusion equation

$$\frac{\partial \vec{w}}{\partial t} = \upsilon\nabla^2 \vec{w} \tag{2.5}$$

(d) **Conducting media :** In the case of propagation of long waves in a good conductor, the electric field vector $\vec{E}$ satisfies the equation of the form

$$\frac{\partial \vec{E}}{\partial t} = \upsilon\nabla^2 \vec{E} \tag{2.6}$$

where $\upsilon = c^2/4\pi\sigma\mu, c$ is the velocity of light in free space, σ is the conductivity and μ is the permeability.

(e) **Slowing down of neutrons in matter :** Under certain conditions, the transport equation for slowing down of neutrons can be reduced to the form

$$\frac{\partial \chi}{\partial \theta} = \frac{\partial^2 x}{\partial z^2} + T(z,\theta)$$

where $\chi(z,\theta)$ is the number of neutrons per unit time which reach age and $T(z,\theta)$ represents the number of neutrons produced per unit time per unit volume.

3. Boundary and Initial Conditions

To solve problems involving diffusion equation, we require specification of boundary and intial conditions.

There are mainly three types of boundary conditions given as follows:

(a) **Dirichlet Condition :** In this case, the temperature T is prescribed over the surface, i.e. $T = f(\vec{r}, t)$ on the boundary, where $G(\vec{r}, t)$ is some prescribed function, which may sometimes be function of position r only, or a function of time t only or a constant. In particular, if $G(\vec{r}, t) = 0$, then it is called a homogeneous boundary condition.

(b) **Neumann Condition :** Here the heat flux $\frac{\partial T}{\partial n}$ is prescribed on the boundary, where $\frac{\partial T}{\partial n}$ is the normal derivative, i.e. $\frac{\partial T}{\partial n} = g(\vec{r}, t)$ on the boundary. In particular, if $g(\vec{r}, t) \equiv 0$, then we have insulated boundary condition.

(c) **Robin's Condition :** If a linear combination of the temperature and its normal derivative is prescribed on the boundary, i.e. if $k\frac{\partial T}{\partial n} + hT = G(\vec{r}, t)$, where k and h are constants, then this condition is called Robin's condition. In tis case, the boundary surface dissipates heat by convection. Using Newton's law of cooling which states that the rate of heat transferred from the body to the surroundings is proportional to the difference of temperature between the body and the surroundings so that we have $k\frac{\partial T}{\partial n} = h(T - T_a), T_a$ being the temperature of the surroundings. In particular, for homogeneous boundary condition, we have $k\frac{\partial T}{\partial n} + hT = 0.$

In addition to the above boundary conditions, the initial condition is to be prescribed to solve the diffusion equation.

4. **Elementary Solution of Diffusion Equation :**

Consider the one-dimensional diffusion equation

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, -\infty < x < \infty, t > 0 \tag{4.1}$$

Now, putting

$$T(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left\{-(x-\xi)^2 / 4kt\right\}, \tag{4.2}$$

so that

$$\frac{\partial^2 T}{\partial x^2}=\frac{1}{\sqrt{4\pi kt}}\left\{\frac{(x-\xi)^2}{4k^2t^2}-\frac{1}{2\pi kt}\right\}\exp\left\{-(x-\xi)^2/4kt\right\}$$

and $\frac{\partial T}{\partial t}=\frac{1}{\sqrt{4\pi kt}}\left\{\frac{(x-\xi)^2}{4kt^2}-\frac{1}{2t}\right\}\exp\left\{-(x-\xi)^2/4kt\right\},$

where ξ is a real constant, we see that (4.2) is a solution of the equation (4.1). The function (4.2), called the kernel, is the elementary / or the fundamentary solution of (4.1) for $-\infty < x < \infty$. This kernel is an analytic function of x and t for t>0 and it is positive for all x. It may be noted that $T \to 0$ as $|x| \to \infty$.

To get an insight into the solution (4.2) of the equation (4.1), we consider the equation for an infinite region $-\infty < x < \infty$ subjected to an initial temperature $f(x)$, i.e. $T(x, 0)$,$f(x)$.

Let $T(x,t)=X(x)\theta(t)$ so that from (4.1), we have

$$\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{k\theta}\frac{d\theta}{dt}=\lambda, \text{ say,} \tag{4.3}$$

where λ is a separation constant.

Now the solution for θ is $\theta = ce^{k\lambda t}$. For $\lambda > 0$, then θ and, therefore, T grows exponentially with time and this is unrealistic from the physical point of view. Thus we assume that $f(x) \to 0$ as $|x| \to \infty$ and $|T(x,t)| < M$ as $|x| \to \infty$. Hence, for $T(x, t)$ to remain bounded, λ must be negative and, therefore, we take $\lambda = -\mu^2$ so that $\theta(t) = ce^{-k\mu^2 t}$.

Again from (4.3) we have

$$\frac{1}{X}\frac{d^2X}{dx^2}+\mu^2 X = 0$$

whose solution is $X(x) = c_1 \cos \mu x + c_2 \sin \mu x$.

Thus the solution of the equation (4.1) is

$$T(x,t;\mu) = (A\cos\mu x + B\sin\mu x)e^{-k\mu^2 t}$$

where $A = cc_1, B = cc_2$ arbitrary constants. Now, it is to be noted that $f(x)$ is, in general, non-periodic and so we may consider Fourier integral instead of Fourier series. Also, A and B being arbitrary, we may consider them as functions of μ. Moreover, as we have no boundary condition which limit our choice of μ, we are to consider all possible values. Thus, by the principle of superposition, it follows that the solution of (4.1) is

$$T(x,t)=\int_0^{\infty}T(x,t;\mu)d\mu=\int_0^{\infty}\left[A(\mu)\cos\mu x+B(\mu)\sin\mu x\right]e^{-k\mu^2 t}d\mu \tag{4.4}$$

The boundary condition $T(x,0)=f(x)$ gives

$$F(x)=\int_0^{\infty}\left[A(\mu)\cos\mu x+B(\mu)\sin\mu x\right]d\mu \tag{4.5}$$

Now, using the Fourier integral theorem, viz.

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty}f(y)\cos\mu(x-y)dy\right]d\mu$$

we get $=\dfrac{1}{\pi}\displaystyle\int_0^{\infty}\left[\cos\mu x\int_{-\infty}^{\infty}f(y)\cos\mu y dy+\sin\mu x\int_{-\infty}^{\infty}f(y)\sin\mu y dy\right]d\mu$

Putting $A(\mu)=\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}f(y)\cos\mu y dy,\ B(\mu)=\dfrac{1}{\pi}\int_{-\infty}^{\infty}f(y)\sin\mu y dy,$ (4.6)

we see from (4.5) that from (4.5) that

$$T(x;0)=f(x)=\frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty}f(y)\cos\mu(x-y)dy\right]d\mu$$

Thus from (4.4), the required solution for $T(x,t)$ is

$$T(x,t)=\frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty}f(y)\cos\mu(x-y)dy\right]e^{-k\mu^2 t}d\mu$$

i.e. $T(x,t)=\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}f(y)\left[\int_0^{\infty}e^{-k\mu^2 t}\cos\mu(x-y)d\mu\right]dy,$

where, we have assumed that the order of integration is interchangeble.

Using the result

$$\int_0^{\infty}e^{-z^2}\cos(2bz)dz=\frac{\sqrt{\pi}}{2}e^{-b^2}$$

we get by putting $z=\mu\sqrt{\kappa t}$ and $2b=\dfrac{x-y}{\sqrt{\kappa t}}$, b being real,

$$\int_0^{\infty}e^{-k\mu^2 t}\cos\mu(x-y)d\mu=\frac{\sqrt{\pi}}{2\sqrt{kt}}e^{-x(x-y)^2/4kt}$$

so that the solution (4.7) can be expressed as

$$T(x,t)=\frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/4kt}dy. \tag{4.8}$$

Particular Case

As a particular case of the above result, we suppose that the initial temperature is constant, say T_0, within the region $a<x<b$ and zero outside this region, i.e.

$$f(x)=\begin{cases} T_0, a<x<b \\ 0, \text{outside the region} \end{cases}$$

Then the solution (4.8) reduces to

$$T(x,t)=\frac{1}{\sqrt{4k\pi t}}\int_a^b e^{-(x-y)^2/4kt}dy.$$

$$=\frac{T_0}{\sqrt{\pi}}\int_{(a-x)/\sqrt{4kt}}^{(b-x)/\sqrt{4kt}} e^{-\xi^2}d\xi \qquad \left(\text{Putting} -\frac{x-y}{\sqrt{4kt}}=\xi\right)$$

$$=\frac{T_0}{2}\left[\frac{2}{\sqrt{\pi}}\int_0^{(b-x)/\sqrt{4kt}} e^{-\xi^2}d\xi-\frac{2}{\sqrt{\pi}}\int_0^{(a-x)/\sqrt{4kt}} e^{-\xi^2}d\xi\right].$$

i.e. $$T(x,t)=\frac{T_0}{2}\left[erf\left(\frac{b-x}{\sqrt{4kt}}\right)-erf\left(\frac{a-x}{\sqrt{4kt}}\right)\right] \tag{4.9}$$

5. Solution of Diffusion Equation in One-dimension (separation of Variables Method)

I. Cartesian Coordinate

In Cartesian coordinates, the solution of the one-dimensional diffusion equation

$$\frac{\partial T}{\partial t}=k\frac{\partial^2 T}{\partial x^2} \tag{5.1}$$

is given by separation of variable method as

$$T(x,t)=(A\cos\mu x+B\sin\mu x)e^{-k\mu^2 t} \tag{5.2}$$

(vide section-4), where $-\mu^2$ is separation constant and A, B are arbitrary constants.

Example 5.1 : A conducting bar of uniform cross-section lies along the x-axis with ends at x=0 and x=L. It is kept at temperature 0^0 and its lateral surface is insulated. There are no heat sources in the bar. The end $x = 0$ is kept at 0^0 and heat is suddenly applied at the end $x = L$ so that there is a constant flux Q_0 at $x = L$. Find the temperature distribution in the bar for $t > 0$.

Solution : The given initial value problem isto solve the diffusion equation

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

subject to the initial condition $T(x,0) = 0, 0 \le x \le L$ and boundary conditions

$$T(0,t) = 0, \frac{\partial T}{\partial x}(L,t) = Q_0 \text{ for } t > 0.$$

Now, prior to applying heat suddenly at the end $x = L$ when $t > 0$, the heat flow in the bar must be independent of time. Hence the temperature $T(x, t)$ consists of two parts, viz. a steady part $T_s(x)$ and an unsteady part $T_s(x,t)$, say, i.e. $T(x,t) = T_\beta(x) + T_u(x,t)$.

For steady state condition, the differential equation is $\frac{d^2Ts}{dx^2} = 0$ and the boundary conditions are $T_s = 0$ at $x = 0$ and $\frac{dTs}{dx} = Q_0$ at $x = L$. Using these conditions, the solution of the above differential equation is $T_s(x) = Q_0 x$.

To find the solution for the unsteady part $T_u(x,t)$, we are to solve the equation

$$\frac{\partial T_u}{\partial t} = k\frac{\partial^2 T_u}{\partial x^2}$$

subject to the initial condition $T_u(x,0) = T(x,0) = T_\mu(x) = -Q_0 x, 0 < x < L$ and boundary conditions :

$T_u(0,t) = T(0,t) - T_s(0) = 0, \frac{\partial T_u(L,t)}{\partial x} = \frac{\partial T(L,t)}{\partial x} - \frac{\partial T_s(L,t)}{\partial x} = 0$ for t>0. Now the solution of the above equation subject to the boundary condition $T_u(0,t) = 0$ is $T_u(x,t) = Be^{-k\mu^2 t} \sin \mu x, -\mu^2$ being separation constant.

The boundary condition $\frac{\partial T_u(L,t)}{\partial x} = 0$ gives $\cos \mu L = 0$ so that $\mu = \frac{(2n+1)\pi}{2L}$ where $n = 0, 1, 2, \ldots$ Thus

$$T_u(x,t)=\sum_{n=0}^{\infty} B_n e^{-k\frac{(2n+1)^2\pi^2}{4L^2}t} \sin\left(\frac{2n+1}{2L}\pi x\right).$$

The initial condition $T_u(x,0)=-Q_0x$ gives

$$-Q_0(x)=\sum_{n=0}^{\infty} B_n \sin\left(\frac{2n+1}{2L}\pi x\right)$$

Multiplying both sides by $\left(\frac{2m+1}{2L}\pi x\right)$ and integrating between 0 to L and then noting that

$$\int_0^L B_n \sin\left(\frac{2n+1}{2L}\pi x\right)\cdot\sin\left(\frac{2m+1}{2L}\pi x\right)dx=\begin{cases}\frac{B_nL}{2}, m=n\\ 0, m\neq n\end{cases}$$

we get on integration by parts on the left hand side

$$-Q_0\cdot\frac{4L^2}{(2n+1)^2\pi^2}(-1)^n=B_n\cdot\frac{1}{2}$$

or, $\quad B_n=\dfrac{8(-1)^{n+1}LQ}{(2n+1)^2\pi^2}.$

Hence the required solution of the given problem is

$$T(x,t)=Q_0x+\frac{8LQ_0}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)^2}\exp\left\{-k\frac{(2n+1)^2\pi^2t}{4L^2}\right\}\times\sin\left(\frac{2n+1}{2L}\pi x\right).$$

Example 5.2 : Solve the diffusion equation $\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}$ satisfying the boundary contion $T=0$ at $x=0$ and $x=1$ and the initial condition

$$T(x,0)=\begin{cases}2x, 0\le x\le\frac{1}{2}\\ 2(1-x), \frac{1}{2}\le x\le 1.\end{cases}$$

Solution : The general solution of the given equation is

$$T(x,t)=(A\cos\mu x+B\sin\mu x)e^{-\mu^2t},$$

$-\mu^2$ being separation constant and A, B are arbitrary constants. Using the boundary conditions we get $A = 0, \sin\mu = 0, i.e. \mu = n\pi$. Thus, by superposition, we have

$$T(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 t} \sin(n\pi x).$$

The given initial conditions lead to

$$B_n = 2\int_0^1 T(x,0)\sin(n\pi x)\,dx$$

$$= 2\left[2\int_0^{1/2} x\sin(n\pi x)\,dx + 2\int_{1/2}^{1}(1-x)\sin(n\pi x)\,dx\right] = \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Hence the required solution is

$$T(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(n\frac{\pi}{2}\right)\sin(n\pi x)e^{-n^2\pi^2 t}.$$

II. Plane polar coordinates

Here the diffusion equation is

$$k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) = \frac{\partial T}{\partial t} \qquad \text{..................... (5.3)}$$

where $T = T(r, t)$

Let us put $T(r,t) = R(r)\Theta(t)$ in (5.3) so that we have

$$\frac{1}{R}\left(\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) = \frac{1}{k\theta}\frac{d\theta}{dt} = -\mu^2,$$

$-\mu^2$ being separation constant. Then it follows that

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \mu^2 R = 0 \text{ and } \frac{d\theta}{dL} + \mu^2 k\theta = 0$$

whose solutions are $R(r) = c_1 J_0(\mu r) + c_2 Y_0(\mu r)$ and $\theta(t) = c_3 e^{-k\mu^2 t}$. Thus the required solution of the equation (5.3) is

$$T(r,t) = \{AJ_0(\mu r) + BY_0(\mu r)\}e^{-k\mu^2 t} \qquad (5.4)$$

where $A = c_1 c_3, B = c_2 c_3$.

Example 5.3 : If $T(r,t)$ satisfies the diffusion equation $\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}=\frac{1}{k}\frac{\partial T}{\partial t}$ in $0 \le r \le a, t>0$ and satisfies the initial condition $T(r,0)=f(r), 0 \le r \le a$ and the boundary condition $\frac{\partial T}{\partial r}+hT=0$ at $r=a, t>0$, show that

$$T(r,t)=\frac{2}{a^2}\sum_{n=1}^{\infty}\left[\frac{\xi_n^2 e^{-k\xi_n^2 t}J_0(\xi_n r)}{(h^2+\xi_n^2)\{J_0(\xi_n a)\}^2}\int_0^a uf(u)J_0(\xi_n u)du\right]$$

where $\xi_n, (n=1,2,...)$, are the roots of the equation $hJ_0(a\xi)=\xi J_1(a\xi)$.

Solution. Noting that $Y_0(\xi r)$ is infinite at $r=0$ so that we have to discard this solution, we get from (5.4),

$$T(r,t)=AJ_0(\xi r)e^{-k\xi^2 t}$$

where $-\xi^2$ is separation constant. The boundary condition gives

$$\xi J_0'(\xi a)+hJ_0(\xi a)=0, \text{ i.e. } \xi J_0(\xi a)=\xi J_1(\xi a).$$

Let the roots of this equation be $\xi_1, \xi_2, ...$ Then, by superposition,

$$T(r,t)=\sum_{n=1}^{\infty}A_n J_0(\xi_n r)e^{-k\xi_n^2 t}$$

The initial condition $T(r,0)=f(r)$ gives

$$f(r)=\sum_{n=1}^{\infty}A_n J_0(\xi_n r)$$

Multiplying both sides by $rJ_0(\xi_m r)$ and integrating both sides with respect r between 0 to a, we have

$$\int_0^a rf(r)J_0(\xi_m r)dr=\sum_{r=1}^{\infty}A_n\int_0^a rJ_0(\xi_n r)J_0(\xi_m r)dr$$

Since

$$\int_0^a rJ_0(\xi_n r)J_0(\xi_m r)dr=\begin{cases}\frac{1}{2}a^2\left[J_0^2(\xi_n a)+J_1^2(\xi_n a)\right], \text{for } m=n\\ 0, \text{for } m\neq n\end{cases}$$

it follows that

$$\int_0^a rf(r)J_0(\xi_n r)dr=\frac{1}{2}a^2A_n\left[J_0^2(\xi_n a)+J_1^2(\xi_n a)\right]$$

$$= \frac{1}{2} a^2 A_n \cdot J_0^2(\xi_n a) \cdot \frac{h^2 + \xi_n^2}{\xi_n^2}$$

and, therefore,

$$A_n = \frac{2\xi_n^2}{a^2\left(h^2 + \xi_n^2\right) J_0^2(\xi_n a)} \int_0^a r f(r) J_0(\xi_n r)\, dr.$$

Thus the reqired solution is

$$T(r,t) = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{\xi_n^2}{\left(h^2 + \xi_n^2\right) J_0^2(\xi_n a)} \int_0^a u f(u) J_0(\xi_n u)\, du.$$

III. **Spherical polar coordinates**

One-dimensional diffusion equation in spherical polar coordinates is

$$k\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) = \frac{\partial T}{\partial t} \tag{5.5}$$

where $T = T(r, t)$.

Putting $T(r,t) = R(r)\theta(t)$ in (5.5) we get

$$\frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right) = \frac{1}{k\theta}\frac{d\theta}{dt} = -\mu^2, \text{ separation const.}$$

so that

$$\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r} + \mu^2 R = 0 \text{ and } \frac{d\theta}{dt} + \mu^2 k\theta = 0$$

whose solutions are $R(r) = c_1 \cos(\mu r) + c_2 \sin(\mu r)$ and $\theta(t) = c_3 e^{-k\mu^2 t}$ and, hence, the solution of the equation (5.5) is

$$T(r,t) = \{A\cos(\mu r) + B\sin(\mu r)\} e^{-k\mu^2 t} \tag{5.6}$$

where $A = c_1 c_3, B = c_2 c_3$.

Example 5.4: A homoegenous solid sphere of radius a has the initial temperature distribution $f(r), 0 \le r \le a$, where r is the distance measured from the centre. The surface temperature is maintained at 0^0. Show that the temperature $T(r, t)$ in the sphere is a solution of

$$\frac{\partial^2 T}{\partial t} = c^2 \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} \right)$$

and the temperature in the sphere for $t > 0$ is given by

$$T(r,t) = \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(n\pi \, r/a\right) e^{-c^2 n^2 \pi^2 t/a^2}$$

whee c^2 is constant and

$$B_n = \frac{2}{a}\int_0^a r f(r) \sin(n\pi \, r/a)\, dr$$

Solution. Let $T(r,t) = \frac{1}{r} U(r,t)$. Then the given equation reduces to

$$\frac{\partial U}{\partial t} = c^2 \frac{\partial^2 U}{\partial r^2} \tag{5.7}$$

and the corresponding initial condition is $U(r,0) = r\, f(r), 0 \le r \le a$ and boundary conditions are $U(0,t) = U(a,t) = 0, t > 0$.

To solve the equation (5.7), we put $U(r,t) = V(r)\theta(t)$ so that (5.7) gives

$$\frac{1}{V}\frac{d^2V}{dr^2} = \frac{1}{c^2\theta}\frac{d\theta}{dt} = -\mu^2 \text{ (say), separation const.}$$

so that

$$\frac{d^2V}{dr^2} + \mu^2 V = 0 \text{ and } \frac{d\theta}{dt} + \mu^2 c^2 \theta = 0$$

whose solutions are $V(r) = c_1 \cos(\mu r) + c_2 \sin(\mu r)$ and $\theta(t) = c_3 e^{-\mu^2 c^2 t}$. Thus

$$U(r,t) = \{A\cos(\mu r) + B\sin(\mu r)\} e^{-\mu^2 c^2 t}$$

where $A = c_1 c_3, B = c_2 c_3$. The boundary condition $U(0,t)=0$ gives $A = 0$ while that $U(a, t)=0$ leads to $\sin(\mu a) = 0$, so that $\mu a = n\pi$, i.e. $\mu = n\pi/a$, is $n = 1, 2, ...$

Hence, by superposition,

$$U(r,t) = \sum_{n=1}^{\infty} B_n \sin\left(n\pi \, r/a\right) e^{-n^2\pi^2 c^2 t/a^2}$$

Using the initial condition $U(r,0)=rf(r)$, we have

$$rf(r)=\sum_{n=1}^{\infty} B_n \sin\left(n\pi \, {}^r\!/_a\right)$$

which is a half-range Fourier series of $rf(r)$ and, therefore

$$B_n = \frac{2}{a}\int_0^a rf(r)\sin\left(n\pi \, {}^r\!/_a\right)dr. \tag{5.8}$$

Hence the solution of the given problem is

$$T(r,t)=\frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi r}{a}\right)e^{-n^2\pi^2c^2t/a^2}$$

where B_n is given by (5.8).

6. Solution of Diffusion Equation in Two-dimensicns (Separation of Variable Method)

I. Cartesian coordinates

The diffusion equation in two-dimensional Cartesian coordinates (x, y) is

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=\frac{\partial T}{\partial t} \tag{6.1}$$

Let $T(x,y,t)=X(x)Y(y)Q(t)$. Then from (5.9) we have

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=\frac{1}{k\theta}\frac{d\theta}{dt}=-\mu^2, \text{ say,} \tag{6.1}$$

where μ^2 is separation constant. Then $\theta(t)=A_1e^{-k\mu^2t}$.

Again from (6.2) it follows that

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\left(\frac{1}{Y}\frac{d^2Y}{dy^2}+\mu^2\right)=-\alpha^2, \text{ say,}$$

α^2 being separation constant. These equations have solutions

$$X(x)=A\cos(\alpha x)+B\sin(\alpha x) \text{ and } Y(y)=c_1\cos(\beta y)+c_2\sin(\beta y) \tag{6.1}$$

respectively, where $\beta^2=\mu^2-\alpha^2$, Thus the general solution of the equation (5.1) is

$$T(x,y,t)=\{A\cos(\alpha x)+B\sin(\alpha x)\}\{C\cos(\beta y)+D\sin(\beta y)\}e^{-k\mu^2t} \tag{6.3}$$

where we have put $C=A_1c_1$ and $D=A_1c_2$.

Example 6.1: The edges $x = 0$, a and $y = b$ of the rectangle $0 \le x \le a, 0 \le y \le b$ are maintained at zero temperature while the temperature along the edge $y = 0$ is made to vary according to the rule $T(x, 0, t) = f(x)$ $0 \le x \le a, t > 0$. If the initial temperature in the rectangle is zero, find the temperature at any subsequent time t and deduce that the steady state temperature is

$$\frac{2}{a}\sum_{n=1}^{\infty}\frac{\sinh\left[n\pi(b-y)/a\right]}{\sinh(n\pi b/a)}\sin(n\pi x/a)\int_0^a f(u)\sin(n\pi u/a)\,du$$

Solution : The given initial boundary value problem is to solve the diffusion equation

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=\frac{\partial T}{\partial t} \text{ within } 0 \le x \le a, 0 \le y \le b \tag{6.4}$$

subject to the initial condition $T(x,y,0) = 0, (0 \le x \le a, 0 \le y \le b)$ and boundary conditions $T(0,y,t) = T(a,y,t) = T(x,b,t) = 0$ and $T(x,0,t) = f(x), 0 \le x \le a, t > 0$.

Now prior to applying heat at the edge $y = 0$, the heat flow within the rectangle is independent of time (steady-state condition). So let us put

$$T(x,y,t) = T_s(x,t) + T_u(x,y,t)$$

where $T_s(x,y)$ is the steady part and $T_u(x,y,t)$ is the unsteady part of the temperature.

It is obvious from (6.4) that $T_s(x,y)$ satisfies the Laplace's equation

$$\frac{\partial^2 T_s}{\partial x^2}+\frac{\partial^2 T_s}{\partial y^2}=0, \quad 0 \le x \le a, 0 \le y \le b \tag{6.5}$$

subject to the boundry conditions

$$T_s(0,y) = T_s(a,y) = T_s(x,b) = 0 \text{ and } T_s(x,0) = f(x), 0 \le x \le a. \tag{6.6}$$

To solve the equation (6.5), we put $T_s(x,y) = X(x)Y(y)$, so that

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -\mu^2, \text{ say,}$$

μ^2 being separation constant. then we have

$$\frac{d^2X}{dx^2}+\mu^2 X = 0 \text{ and } \frac{d^2Y}{dy^2}-\mu^2 Y = 0.$$

whose solutions are $X(x) = c_1\cos(\mu x) + c_2\sin(\mu x)$ and $Y(y) = c_3\cosh(\mu y) + c_4\sinh(\mu y)$ respectively. Hence

$$T_s(x,y) = \{c_1\cos(\mu x) + c_1\sin(\mu x)\}\{c_3\cosh(\mu y) + c_4\sinh(\mu y)\}. \tag{6.7}$$

The first three boundary conditions of (6.6) give $c_1 = 0, \sin(\mu a) = 0$

and $c_3\cosh(\mu b) + c_4\sinh(\mu b) = 0$. Thus $\mu a = n\pi$, i.e. $u = n\pi/a, (n = 1, 2, ...)$ and $c_4 = -\dfrac{\cosh(\mu b)}{\sinh(\mu b)}$.

Thus, the solution (6.7) reduces to the form

$$T_s(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a)\sinh\left\{\frac{n\pi}{a}(b-y)\right\}$$

The last boundary condition of (6.6) gives

$$f(x) = \sum_{n=1}^{\infty} A_n \sinh(n\pi b/a)\sin(n\pi x/a)$$

which is Fourier sine series. Thus

$$A_n \sinh(n\pi b/a) = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)\,dx.$$

Thus, the steady-state solution $T_s(x,y)$ is given by

$$T_s(x,y) = \frac{2}{a}\sum_{n=1}^{\infty}\frac{\sinh\{n\pi(b-y)/a\}}{\sinh(n\pi b/a)}\sin(n\pi x/a)\int_0^a f(u)\sin(n\pi u/a)\,du \tag{6.8}$$

Next let us consider the unstaedy part $T_u(x,y,t)$ which satisfies the diffusion equation

$$k\left(\frac{\partial^2 T_u}{\partial x^2} + \frac{\partial^2 T_u}{\partial y^2}\right) = \frac{\partial T_u}{\partial t}, \quad 0 \le x \le a, 0 \le y \le b, \tag{6.9}$$

the initial condition $T_u(x,y,0) = -T_s(x,y)$ and the boundary conditions

$$T_u(0,y,t) = T_u(a,y,t) = T_u(x,0,t) = T_u(x,b,t) = 0, t > 0.$$

Proceeding along the same lines as in Section –6, we find that the solution of the equation (6.8) is

$$T_u(x,y,t) = \{A\cos(\alpha x) + B\sin(\alpha x)\}\{C\cos(\beta y) + D\sin(\beta y)\}e^{-k\mu^2 t}$$

where α, β are separation constants with $\mu^2 = \alpha^2 + \beta^2$. Now the boundary conditions give $A = 0, \alpha = p\pi/a, c = 0, \beta = q\pi/b$ where $p = 1, 2, ...$ and $q = 1, 2, ...$ Thus

$$T_u(x,y,t)=\sum_p\sum_q A_{pq}\sin(p\pi x/a)\sin(q\pi y/b)\cdot e^{-k\pi^2\left(\frac{p^2}{a^2}+\frac{q^2}{b^2}\right)} \tag{6.10}$$

The intial condition gives

$$-T_s(x,y)=\sum_p\sum_q A_{pq}\sin(p\pi x/a)\sin(q\pi y/b)$$

which is double Foureir sine series so that

$$A_{pq}=-\frac{4}{ab}\int_0^a\int_0^b T_s(x,y)\sin(p\pi x/a)\sin(p\pi y/b)\,dx\,dy. \tag{6.11}$$

Hence the required solution of the given problem is

$$T(x,y,t)=T_s(x,y)+T_u(x,y,t)$$

where $T_s(x,y)$ is given by (6.8) and $T_u(x,y,t)$ by (6.10) with A_{pq} as given in (6.11).

II. Plane polar coordinates.

To solve diffusion equation

$$k\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}\right)=\frac{\partial T}{\partial t} \tag{6.12}$$

in plane polar coordinates (r,θ), we put $T(r,\theta,t)=R(r)Q(\theta)T(t)$. Then this equation gives

$$\frac{1}{R}\frac{d^2R}{dr^2}+\frac{1}{Rr}\frac{dR}{dr}+\frac{1}{r^2Q}\frac{d^2Q}{d\theta^2}=\frac{1}{kT}\frac{dT}{dt}=-\mu^2, \text{ (separation const.)}$$

Then $T(t)=Ee^{-k\mu^2 t}$. Also, we have

$$r^2\left(\frac{1}{R}\frac{d^2R}{dr^2}+\frac{1}{Rr}\frac{dR}{dr}+\mu^2\right)=-\frac{1}{Q}\frac{d^2Q}{d\theta^2}=\lambda^2, \text{ say, (separation const.)}$$

so that

$$\frac{d^2R}{dr^2}+\frac{dR}{dr}+\left(\mu^2-\frac{\lambda^2}{r^2}\right)R=0 \text{ and } \frac{d^2Q}{d\theta^2}+\lambda^2 Q=0$$

which have solutions $R(r)=AJ_\lambda(\mu r)+BY_\lambda(\mu r)$ and $Q(\theta)=c_1\cos(\lambda\theta)+c_2\sin(\lambda\theta)$ respectively.

Thus, the solution of the equation (6.12) is

$$T(r,\theta,t)=\{AJ_\lambda(\mu r)+BY_\lambda(\mu r)\}\{C\cos(\lambda\theta)+D\sin(\lambda\theta)\}e^{-k\mu^2 t} \tag{6.13}$$

where $C = c_1 E$ and $D = c_2 E$.

Example 6.2: Find the temperature in a long cylindrical region bounded by the planes $n = a, \theta = 0$ and $\theta = \pi$ which are maintained at zero temperature and its initial temperature is $f(r,\theta)$.

Solution. Consider the solution (6.13) of the diffusion equation (6.12). Since T must be finite at $r = 0$ where $Y_\lambda(\mu r)$ is undefined, so we must put $B = 0$. Also, the boundary conditions $T = 0$ at $\theta = 0$ and π give $C = 0$ and $\sin \lambda\pi = 0$, i.e. $\lambda = n(n = 1, 2, ...)$ and the condition at $r = a$ leads to $J_n(\mu a) = 0$ which has roots $\mu_1 a, \mu_2 a, ...$ Thus by using the principle of superposition, we have the solution

$$T(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{n,m} J_n(\mu_m r) \cdot \sin(n\theta) e^{-k\mu_m^2 t} \tag{6.14}$$

Again, the initial condition gives

$$f(r,\theta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{n,m} J_n(\mu_m r) \sin(n\theta)$$

Multiplying both sides by $r J_n(\mu_k r) \sin(p\theta)$ and performing double integration w.r.t. r and θ for $0 < r < a, 0 < \theta < \pi$ resectively, we have

$$\int_0^a \int_0^\pi r f(r,\theta) J_n(\mu_k r) \sin(p\theta)\, dr\, d\theta$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{n,m} \int_0^a r J_n(\mu_k r) J_n(\mu_m r) \left\{ \int_0^\pi \sin(p\theta) \sin(n\theta)\, d\theta \right\} dr \tag{6.15}$$

Noting that

$$\int_0^a r J_n(\mu_k r) J_n(\mu_m r)\, dr = \begin{cases} \frac{a^2}{2} J_n^2(\mu_m a) \text{ for } k = m \\ 0 \text{ for } k \neq m \end{cases}$$

and

$$\int_0^\pi \sin(p\theta) \sin(n\theta)\, d\theta = \begin{cases} \frac{\pi}{2} \text{ for } p = n \\ 0 \text{ for } p \neq n \end{cases}$$

we get from (6.15)

$$\int_0^a\int_0^\pi r\, f(r,\theta) J_n(\mu_m r)\sin(n\theta)\,dr\,d\theta = \frac{\pi a^2}{4} J_n^2(\mu_m a)\cdot A_{n,m}$$

so that

$$A_{n,m} = \frac{4}{\pi a^2 J_n^2(\mu_m a)}\int_0^a\int_0^\pi r\, f(r,\theta) J_n(\mu_m r)\sin(n\theta)\,dr\,d\theta \tag{6.16}$$

Hence the required solution of the given problem is obtained from (6.14) and (6.16) as

$$T(r,\theta,t) = \frac{4}{\pi a^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{B_{n,m} J_n(\mu_m r)\sin(n\theta)}{J_n^2(\mu_m a)}\cdot e^{-k\mu_m^2 t} \tag{6.17}$$

where $B_{n,m} = \int_0^a\int_0^\pi r\, f(r,\theta) J_n(\mu_m r)\sin(n\theta)\,dr\,d\theta.$

III. Spherical polar coordinates with axial symmetry

The diffusion equation in spherical polar coordinates with axial symmetry is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) = \frac{1}{k}\frac{\partial T}{\partial t} \tag{6.18}$$

where T is a function of r,θ and t. Putting $T(r,\theta,t) = R(r)Q(\theta)\tau(t)$ in (6.18), we get

$$\frac{1}{r^2 R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2\sin\theta}\cdot\frac{1}{Q}\frac{d}{d\theta}\left(\sin\theta\frac{dQ}{d\theta}\right) = \frac{1}{kT}\frac{dT}{dt} = -\lambda^2,\ \text{say}$$

where $-\lambda^2$ is separation constant. Then $\tau(t) = Ee^{-k\lambda^2 t}$. Also we have from this relation

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \lambda^2 r^2 = -\frac{1}{Q\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dQ}{d\theta}\right) = n(n+1),$$

where $n(n+1)$ is sepration constant. Then we have

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left\{\lambda^2 - \frac{n(n+1)}{r^2}\right\}R = 0 \tag{6.19}$$

and $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dQ}{d\theta}\right) + n(n+1)Q = 0 \tag{6.20}$$

To solve the equation (6.19), we put $R(r) = (\lambda r)^{-\frac{1}{2}} S(r)$. Then it is easily seen that the equation reduces to

$$\frac{d^2S}{dr^2}+\frac{1}{r}\frac{dS}{dr}+\left\{\lambda^2-\frac{\left(n+\frac{1}{2}\right)^2}{r^2}\right\}S=0$$

whose solution is $S(r)=AJ_{n+\frac{1}{2}}(\lambda r)+BY_{n+\frac{1}{2}}(\lambda r)$ so that

$$R(r)=(\lambda r)^{-\frac{1}{2}}\left\{AJ_{n+\frac{1}{2}}(\lambda r)+BY_{n+\frac{1}{2}}(\lambda r)\right\}$$

Again, putting $\cos\theta=\mu$ in (6.20), we have

$$\frac{d}{d\mu}\left\{\left(1-\mu^2\right)\frac{dS}{d\mu}\right\}+n(n+1)\Theta=0$$

which has the solution $Q(\theta)=A_1P_n(\mu)+B_1Q_n(\mu)$ where $P_n(\mu)$ and $Q_n(\mu)$ are Legendre functions of the first and second kind respectively. Thus the solution of the equation (6.18) is given by superposition as

$$T(r,\theta,t)=\sum_{n=1}^{\infty}(\lambda r)^{-\frac{1}{2}}\left[A_nJ_{n+\frac{1}{2}}(\lambda r)+B_nY_{n+\frac{1}{2}}(\lambda r)\right]\times$$

$$\times\left[C_nP_n(\cos\theta)+D_nQ_n(\sin\theta)\right]e^{-k\lambda^2t}. \qquad (6.21)$$

Example 6.3 : Determine the temperature in a sphere of radius a when its surface is at zero temperature and its initial temperature is $f(r,\theta)$.

Solution. Since $Y_{n+\frac{1}{2}}(\lambda r)$ and $Q_n(\cos\theta)$ are unbounded at $r=0$ and $\theta=\pi/2$ respectively, so we must $B_n=0$ and D_n 0. Also the boundary condition $T(a,\theta,t)=0$ leads to $J_{n+\frac{1}{2}}(\lambda a)=0$. Let $\lambda_1 a,\lambda_2 a,...$ be the roots of this equation. Then the solution (6.21) can be rewritten in the form

$$T(r,\theta,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{n,m}(\lambda_m r)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\lambda_m r)P_n(\cos\theta)e^{-k\lambda_m^2t} \qquad (6.22)$$

Applying the initial condition $T(r,\theta,0)=f(r,\theta)$ we get

$$f(r,\theta)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\lambda_m r)^{-\frac{1}{2}}A_{n,m}J_{n+\frac{1}{2}}(\lambda_m r)P_n(\cos\theta)$$

Multiplying both sides by $P_k(\cos\theta)d(\cos\theta)$ and integrating between -1 to $+1$, we have

$$\int_{-1}^{1} f(r,\theta)P_k(\cos\theta)d(\cos\theta)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\lambda_m r)^{-\frac{1}{2}}A_{n,m}J_{n+\frac{1}{2}}(\lambda_m r)\times\int_{-1}^{1}P_n(\cos\theta)P_k(\cos\theta)d(\cos\theta)$$

i.e. $$\int_{-1}^{1} f(r,\theta)P_n(\cos\theta)d(\cos\theta)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{2}{2n+1}(\lambda_m r)^{-\frac{1}{2}}A_{n,m}J_{n+\frac{1}{2}}(\lambda_m r) \qquad (6.23)$$

where we have used the orthogonal property of Legendre functions, viz.

$$\int_{-1}^{1}P_n(\cos\theta)P_k(\cos\theta)d(\cos\theta)=\begin{cases}\frac{2}{2n+1}, k=n\\ 0, k\neq n\end{cases}$$

Again, multiplying both sides of (6.23) by $r^{3/2}J_{n+\frac{1}{2}}(\lambda_j r)$ and integrating w.r.t. r between 0 to a, we have

$$\int_0^a r^{3/2}J_{n+\frac{1}{2}}(\lambda_j r)\left[\int_{-1}^{1}f(r,\theta)P_n(\cos\theta)d(\cos\theta)\right]dr$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{2}{2n+1}(\lambda_m)^{-\frac{1}{2}}A_{n,m}\int_0^a rJ_{n+\frac{1}{2}}(\lambda_m r)J_{n+\frac{1}{2}}(\lambda_j r)dr$$

$$=A_{n,m}(\lambda_m)^{-\frac{1}{2}}\cdot\frac{2}{2n+1}\cdot\frac{a^2}{2}\cdot J'^2_{n+\frac{1}{2}}(\lambda_m a)\text{ for } j=m$$

so that $$A_{n,m}=\frac{2n+1}{a^2}\cdot\frac{(\lambda_m)^{\frac{1}{2}}}{J_{n+\frac{1}{2}}(\lambda_m a)}\int_0^a\int_{-1}^{1}r^{\frac{3}{2}}J_{n+\frac{1}{2}}(\lambda_m r)P_n(\mu)f(r,\theta)dr\,d\mu$$

Thus the required solution of the given problem is

$$T(r,\theta,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(2n+1)B_{n,m}}{a^2J'^2_{n+\frac{1}{2}}(\lambda_m a)}r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\lambda_m r)P_n(\mu)e^{-k\lambda_m^2 t}$$

where $B_{n,m}=\int_0^a\int_{-1}^{1}r^{\frac{3}{2}}J_{n+\frac{1}{2}}(\lambda_m r)P_n(\mu)f(r,\theta)dr\,d\mu$ and $\mu=\cos\theta$.

Note : In section 5 and 6, we have considered solution one and two-dimentionsal diffusion equation in different system of coordinates by the method of separation of variables. Three-dimensional cases can also be dealt with along the same lines.

7. Maximum-Minimum Principle and its Consequences

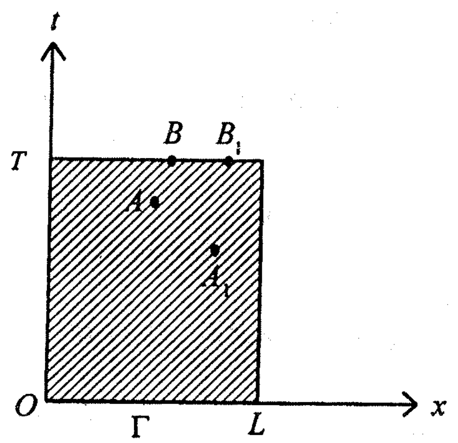
Maximum-Minimum Principle

Theorem 7.1 : Suppose the solution $T(x, t)$ of the diffusion equation

$$k\frac{\partial^2 T}{\partial x^2}=\frac{\partial T}{\partial t} \tag{7.1}$$

be continuous for $0 \le x \le L, 0 \le t \le T$, where $T(>0)$ is a fixed time. Then T attains its maximum and minimum values at time $t=0$ or at the end points $x=0$ and $x=L$ at some time in the interval $0 \le t \le T$.

Proof.



Consider the given region $0 \le x \le L, 0 \le t \le T$, in the (x, t) plane. Then $T(x, t)$ is given by the dark horizontal and vertical lines, the darken portion of the boundary being denoted by Γ. Noting that heat flows from higher to lower temperature, we expect that the temperature $T(x, t)$ in the shaded region attains its maximum on Γ. To prove this, we assume the contrary, that is, we can find a point (x_0, t_0) which is either an interior point A or an upper boundary point B such that $T(x_0, t_0)$ >l.u.b. of $T(x, t)$ on Γ. Define an auxiliary function $\psi(x,t)$ by

$$\psi(x,t)=T(x,t)-\varepsilon(t-t_0)$$

where $\varepsilon > 0$ is constant. Since $\psi(x_0,t_0)=T(x_0,t_0)$ which exceeds by some definite the greatest value of $T(x, t)$ on Γ, we can choose ϵ so small that $\psi(x_0,t_0) > \max.\psi(x,t)$ on Γ. Thus $\psi(x,t)$ attains its maximum not on Γ, say at A_1 or B_1.

At this maximum point, we have

$$\frac{\partial^2 \psi}{\partial x^2} \le 0, \frac{\partial \psi}{\partial t} > 0 \text{ so that } \frac{\partial^2 T}{\partial x^2} \le 0, \frac{\partial T}{\partial t} > 0$$

which is a contradiction that $k\frac{\partial^2 T}{\partial x^2}=\frac{\partial T}{\partial t}$ at this point. Thus the maximum principle is proved.

Similarly, the minimum principle can be proved.

Consequences :

1. Theorem - 7.2 (Uniqueness Theorem) : Let $T(x, t)$ be a solution of the diffusion equation

$$k\frac{\partial^2 T}{\partial x^2}=\frac{\partial T}{\partial t}, 0\le x\le L, 0\le t\le T \qquad (7.2)$$

subject to the initial condition $T(x, 0) = f(x)$ and the boundary conditions $T(0, t) = g(t)$, $T(L, t) = h(t)$, where $f(x)$, $g(t)$ and $h(t)$ are continuous functions on their domains of definitions. Then the solution $T(x, t)$ must be unique.

Proof. If possible, suppose that $T_1(x,t)$ and $T_2(x,t)$ be two solutions of (7.2) both of which satisfy the give initial and boundary conditions. Then $\upsilon(x,t)=T_1(x,t)-T_2(x,t)$ is also a solution of (7.2) and is a continuous function of x and t. Also $\upsilon(x,0)=0$ in $0\le x\le L$ and $\upsilon(0,t)=\upsilon(L,t)=0$ in $0\le t\le T$. Thus $\upsilon(x,t)$ satisfies the conditions of maximum-minimum principle and hence $\upsilon(x,t)=0$ for $0\le x\le L, 0\le t\le T$, so that $T_1(x,t)=T_2(x,t)$, which shows that the equation (7.2) admits a unique solution.

2. Theorem 7.3 (Stability Property) : The solution $T(x, t)$ of (7.2) subject to the conditions of Theorem-7.2 depends continuously on the initial and boundary conditions.

The proof is omitted.

Exercise

1. Solve the one-dimensional diffusion equation in the region $0\le x\le \pi, t\ge 0$ subject to the conditions

   i) $T = 0$ if $x = 0, \pi$ for all $t > 0$

   ii) $T=\begin{cases} x, 0\le x\le \frac{\pi}{2} \\ \pi - x, \frac{\pi}{2}\le x\le \pi \end{cases}$ for $t = 0$

   iii) T remains finite as $t\to\infty$.

$$\left[\text{Ans. } T(x,t) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(nx)\sin\left(\frac{n\pi}{2}\right)\cdot e^{-kn^2t}\right]$$

2. A uniform rod of length L whose surface is thermally insulted is initially at temperature $T = T_0$. At time $t=0$, one end is suddenly cooled to $T=0$ and subsequently maintained at this temperature, the other end remains thermally insulated. Find the temperature distribution $T(x, t)$.

$$\left[\text{Ans. } T(x,t) = \frac{4T_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\frac{2n-1}{2L}nx\right)e^{-k(2n-\frac{1}{2}L)^2\pi^2t}\right]$$

3. Find the solution of the one-dimensional diffusion equation satisfying the initial condition $T(x,0) = x(a-x), 0 < x < a$, the regularity condition that T is bounded as $t \to \infty$ and the boundary condition $\frac{\partial}{\partial x}T(0,t) = \frac{\partial}{\partial x}T(a,t)$ for all $t>0$.

$$\left[\text{Ans. } T(x,t) = \frac{a_0}{6} - \frac{4a^2}{\pi^2}\sum_{n=even}\frac{1}{n^2}\cos(n\pi x/a)\cdot e^{-k\frac{n^2\pi^2t}{a^2}}\right]$$

4. A circular cylinder of radius a has its surface kept at a constant temperature T_0. If the initial temperature is zero throughout the cylinder, prove that for $t>0$

$$T(r,t) = T_0\left\{1 - \frac{2}{a}\sum_{n=1}^{\infty}\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n a)}e^{-k\xi_n^2 t}\right\},$$

where $\xi_n, (n = 1,2,3,...)$ are the roots of the equation $J_0(\xi a) = 0$.

A conducting bar of uniform cross-section lies along the x-axis, with its ends at $x=0$ and $x=l$. The lateral ·e is insulted and there are no heat sources within the body. The ends are also insulated. The initial temperature is $lx - x^2, 0 \le x \le l$. Find the temperature in the bar for $t>0$.

[Ans. Same as Exercise Problem-3 with a = l]

6. The faces $x=0, x=a$ of a finite slab are maintained at zero temperature. The initial distribution of temperature in the slab is given by $T(x,0) = f(x), 0 \le x \le a$. Determine the temperature at subsequent times.

$$\left[Ans.\ T(x,t)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{a}x\right)\exp\left\{-k\frac{n^2\pi^2t}{a^2}\right\}\ where\ A_n=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx\right]=$$

7. Show that the solution of the equation $\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}$ satisfying the conditions :

(i) $T\to 0$ as $t\to\infty$,

(ii) $T=0$ for $x=0$ and $x=a$ for all $t>0$,

(iii) $T=x$ when $t=0$ and $0<x<a$

is $T(x,t)=\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\left(\frac{n\pi x}{a}\right)\exp\left\{-\frac{n^2\pi^2 t}{a^2}\right\}$.

8. Solve the diffusion equation $\frac{\partial T}{\partial t}=k\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{4}\frac{\partial T}{\partial r}\right)$ subject to (*i*) $r=0$, T is finite, $t\ge 0$, (*ii*) $r=a, T=0,$

$t>0$, (*iii*) $T=\frac{T_0}{4\mu}\left(a^2-r^2\right), t=0$

where are T_0, k and μ constants.

$$\left[\text{Ans. } T(r,t)=\frac{2T_0a^2}{\mu}\sum_{n=1}^{\infty}\frac{J_0(\xi_n r/a)}{\xi_n^3 J_1(\xi_n)}\exp\left(-k\xi_n^2\, t/a^2\right)\right]$$

9. If $f(x)$ is bounded for all real values of x, show that

$$T(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}f(\xi)\exp\left\{-(x-\xi)^2/4kt\right\}d\xi$$

is a solution of $\frac{\partial T}{\partial t}=k\frac{\partial^2 T}{\partial x^2}$ such that $T(x,0)=f(x)$

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-IV Group–B

Module No. - 47
HYPERBOLIC EQUATIONS

1. Introduction:

In this unit we shall consider hyperbolic equation of the type

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}$$

or, $\nabla^2 U=\frac{1}{C^2}\frac{\partial^2 U}{\partial t^2}$

which is also called *wave equation*. If we assume a solution of this equation of the form

$$U(x,y,z,t)=f(x,y,z)e^{\pm ikct}$$

t... the function $f(x,y,z)$ must satisfy the equation

$$\left(\nabla^2+k^2\right)f=0$$

This is known as *Helmholtz's equation* or *space form of the wave equation.*

2. Occurrence of Wave Equation

Let us indicate some situations in physics and engineering in which the wave equation is involved.

(a) **Transverse vibrations of a string.** If a string of uniform linear density ρ be stretched to a uniform tension T and the string coireidos with the x-axis in the equilibrium position, then for slight disturbance from this position, of the string, the transverse displacement $y(x,t)$ satisfies the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

where $c^2 = T/\rho$ and $y(a,t) = 0$ for all t at any fixed point $x = a$.

(b) **Longitudinal vibrations of a bar.** Suppose a uniform elastic bar of uniform cross-section with its axis lying along the direction of x-axis be stressed such that each point of a typical cross-section of the bar takes the same displacement $\xi(x,t)$. Then ξ satisfies the wave equation

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2}$$

when $c^2 = E/\rho, E$ being the Young's modulus and P is the density of the material of the bar.

(c) **Transverse vibration of the membrane.** Consider a thin elastic membrane of uniform density σ be stretched to a tension T and the membrane coincides with the xy-plane in the equilibrium position. Then the transverse displacement $z(x,y,t)$ of any point (x,y) at time t is given, for small transverse vibrations of the membrane, by the two-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

where c is the wave velocity defined by $c^2 = T/\sigma$.

(d) **Sound waves in space.** Consider sound waves in a passage and suppose that the velocity of the gas at a point (x,y,z) at time t be given by $V = (u,v,w)$ and p, ρ denote the pressure and density at that point. For irrotational motion of the gas, we have $V = -\Delta\phi$. Then the potential function ϕ satisfies the wave equation

$$\nabla^2 \phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$

where $c^2 = (dp/d\rho)_0$, the suffix zero indicating that the quantity is to be evaluated at the state of equilibrium.

(e) **Elastic waves in solids.** Suppose $V = (u,v,w)$ denote the displacement vector at a point (x,y,z) of the particle at time t of the elastic solid. Then, writing $V = \nabla\phi + \nabla \times \psi$, it is seen that both ϕ and each component of ψ satisfy wave equation.

(f) Electromagnetic wave. Let it be the magnetic field and be the electric field. Then, if we take $H = \nabla \times A$, $E = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\phi$, the Maxwell's equations

$$\nabla{:}E = 4\pi\rho, \nabla.H = 0, \nabla\times E = -\frac{1}{c}\frac{\partial H}{\partial t}, \nabla\times H = \frac{4\pi}{c}i + \frac{1}{c}\frac{\partial E}{\partial t}$$

are identically satisfied, provided that ϕ satisfy the equations

$$\nabla^2 A = \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \frac{4\pi}{c}i, \nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - 4\pi\rho$$

These equations, in the absence of charges ρ or currents, reduce to the wave equations, *c* denoting the velocity of light.

3. Transverse Vibrations of a Tightly Stretched Elastic String

Suppose a string, such as a vislin or piano string, is in equilibrium and extends along the positive x-axis with one end 0 taken as the origin and isstretched to a tension *T* in this direction. Let the string be disturbed from this position by some small transverse impulse so that every particle of the string executes transverse vibration and the transverse displacements of all points in the vibrating plane are at right angles to the undisturbed position. Thus the wave set up is transverse causing a flow of energy. We make the following assumptions:

(i) The string is perfectly elastic and flexible as a result of which it supports only tension and there is no bending or shearing force;

(ii) The density (mass per unit length) ρ is uniform;

(iii) The tension *T* is sufficiently large so that the effect of gravity can be neglected or the string is suppose to vibrate on a smooth horizontal plane. Moreover, it is also supposed that there is no damping and, therefore, the string executes free vibration;

(iv) The motion is entirely transverse and the transverse displacement at any point, at any time *t*, is so small that the angle which tangent to the curve at time *t* makes with the positive x-axis is also small. Thus *T* is independent of *x* and is, therefore, the same throughout the string. We also assume that *T* is independent of time. Hence *T* is constant.

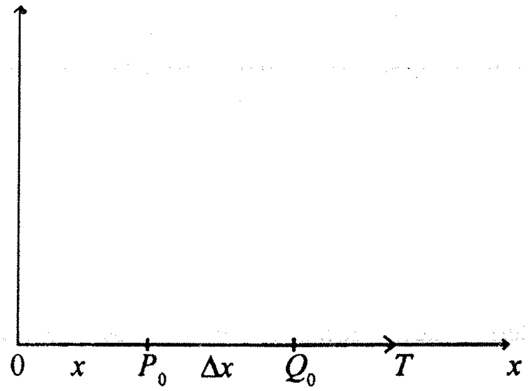

Fig. 3.1

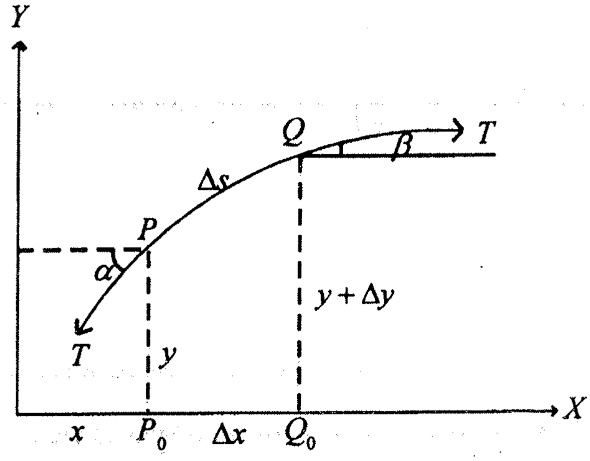

Fig. 3.2

Consider a section of the string as shown in Fig. 3.2. Suppose the points P_0 and Q_0 of the string be at distances x and $x+\Delta x$ from its one end 0 which we take as the origin and these points have transverse displacements $P_0P = y$ and $Q_0Q = y + \Delta y$ respectively at time t, where it is supposed that y is a differentiable function of x and t. Let arc $PQ = \Delta s$ and the angles which the tangents at P and Q make with the positive x-axis are α and β and these angles are small. Then the equation of transverse motion of the element PQ is

$$\rho \Delta s \frac{\partial^2 y}{\partial t^2} = T \sin\beta - T\sin\alpha \tag{3.1}$$

Now, nothing that α and β are small, we have

$$x \approx \sin\alpha \approx \tan\alpha \approx \left(\frac{\partial y}{\partial x}\right)_x = \phi(x,t), \text{say}$$

$$\text{and } \beta \approx \sin\beta \approx \tan\beta \approx \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} = \phi(x+\Delta x, t), \text{ say}$$

Hence from (3.1) we get by dividing throughout by Δx,

$$\rho \frac{\partial^2 y}{\partial t^2}\cdot\frac{\Delta s}{\Delta x} = T\cdot\frac{\phi(x+\Delta x,t)-\phi(x,t)}{\Delta x}$$

and proceeding to the limit as $\Delta x \to 0$, it follows that

$$\rho\frac{\partial^2 y}{\partial t^2}\cdot\frac{ds}{dx} = T\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) \tag{3.2}$$

Since $\frac{ds}{dx} = \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} \approx 1$, so the required equation of motion for transverse vibration of a string is given from (3.2) as

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial x^2} \tag{3.3}$$

where $c = \sqrt{T/\rho}$ has the dimension of the wave velocity. y is called the deflection of the string in the xy-plane.

4. Solution of one-dimensional Wave Equation

We now proceed to find the solution of the one-dimensional wave equation (3.3) by various methods. In the sequal, we shall consider the function $U(x,t)$ in place of $y(x,t)$ i.e.

$$\frac{\partial^2 U}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 U}{\partial x^2} \tag{4.1}$$

I. Solution by canonical reduction

Let us choose the characteristic lines $\xi = x+ct, \eta = x-ct$, so that

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)U$$

and similarly, $\frac{\partial U}{\partial t} = c\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)U.$

Also $\frac{\partial^2 U}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2 U$ and $\frac{\partial^2 U}{\partial t^2} = c^2\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)^2 U.$

Substituting these in (4.1), we have

$$\frac{\partial^2 U}{\partial \xi \partial \eta} = 0$$

whose solution is $U = \phi(\xi) + \psi(\eta)$, ϕ and ψ being arbitrary. Thus the solution of the equation (4.1) is given by

$$U(x,t) = \phi(x+ct) + \psi(x-ct) \tag{4.2}$$

It follows that for an arbitrary real parameter k,

$$U(x,t)=\phi\{k(x+ct)\}+\psi\{k(x-ct)\} \quad (4.3)$$

is also a solution of (4.1). Further, it may be verified that if $w=kc$, then

$$U(x,t)=\phi(kx+wt)+\psi(kx-wt) \quad (4.4)$$

satisfies the wave equation (4.1). The quantity $kx+wt$ is called the phase for the right travelling wave and $x\pm ct$ are the characteristics of the one-dimensional wave equation.

II. 'D'Alembert's solution - Initial value problem

Consider the Cauchy type initial value problem described by

$$\frac{\partial^2 U}{\partial x^2}=c^2\frac{\partial^2 U}{\partial t^2}, -\infty<x<\infty, t\geq 0 \quad (4.5)$$

subject to the initial conditions

$$U(x,0)=\eta(x) \text{ and } \frac{\partial U}{\partial t}(x,0)=\upsilon(x) \quad (4.6)$$

where the curve on which the initial values $\eta(x)$ and $\upsilon(x)$ are prescribed is the x-axis. It is also assumed that the functions $\eta(x)$ and $\upsilon(x)$ are twice continuously differentiable.

Now if $y(x,t)$ be the transverse displacement of a point of the string for any x and t, then $\eta(x)$ and $\upsilon(x)$ are the prescribed values of the initial displacement and velocity respectively. Since the solution of the equation (4.5) is given by (4.2), we have

$$\phi(x)+\psi(x)=\eta(x) \text{ and } C\{\phi'(x)-\psi'(x)\}=\upsilon(x) \quad (4.7)$$

so that the second relation of (4.7) leads to

$$\phi(x)-\psi(x)=\int_0^x \upsilon(\zeta)\,d\zeta$$

This, along with the first relation of (4.7) give

$$\phi(x)=\frac{1}{2}\eta(x)+\frac{1}{2c}\int_0^x \upsilon(\zeta)\,d\zeta \text{ and } \psi(x)=\frac{1}{2}\eta(x)-\frac{1}{2c}\int_0^x \upsilon(\zeta)\,d\zeta$$

Hence the solution (4.2) of the wave equation (4.5) is given by

$$U(x,t)=\frac{1}{2}\{\eta(x+ct)+\eta(x-ct)\}+\frac{1}{2c}\int_{x-ct}^{x+ct}\upsilon(\zeta)d\zeta \qquad (4.8)$$

This is known as D'Alembert's solution of the one-dimensional wave equation.

If the string be released from rest so that $\upsilon(x)=0$, then the equation (4.8) becomes

$$U(x,t)=\frac{1}{2}\{\eta(x+ct)+\eta(x-ct)\} \qquad (4.9)$$

Hence the subsequent displacement of the string is produced by two pulses of shape $U=\frac{1}{2}\eta(x)$, each moving with velocity *c*, one to the right and the other to the left.

We now establish some corollaries of D'Alembert's formula.

(1) Domain of dependence: The value of *U* at (x_0,t_0) is debermined by the restriction of initial functions $\eta(x)$ and $\upsilon(x)$ in the interval $[x_0-ct_0,x_0+ct_0]$ on the x-axis, the end-points of which are cut out by the characteristics: $x-x_0=\pm c(t-t_0)$ through the point (x_0,t_0).

The Characteristic triangle $\Delta(x_0,t_0)$ is defined to be the triangle in $R\times R^2$ having vertices $A_0(x_0-ct_0,0)$, $B_0(x_0+ct_0,0)$ and $P(x_0,t_0)$.

For every $(x_1,t_1)\in\Delta(x_0,t_0)$ we have

$$[x_1-ct_1,x_1+ct_1]\subset[x_0-ct_0,x_0+ct_0]$$

$$\Delta(x_1,t_1)\subset\Delta(x_0,t_0)$$

where $u(x_1,t_1)$ is to be determined by the values of $\eta(x)$ and $\upsilon(x)$ on $[x_1-ct_1,x_1+ct_1]$.

(2) Domain of influence: The point $(x_0,0)$ on the x-axis influences the value of *u* at (x,t) in the wedge-shaped region

$$I(x_0)=\{(x,t):x_0-ct\le x\le x_0+ct,t\ge 0\}$$

For any

$$P_1(x_1,t_1)\in I(x_0), \Delta(x_1,t_1)\cap I(x_0)\neq\phi$$

$$P_1(x_1,t_1)\notin I(x_0), \Delta(x_1,t_1)\cap I(x_0)=\phi.$$

III. Riemann - Volterra Solution

Putting $y = ct$, the equation (4.1) reduces to

$$\frac{\partial^2 U}{\partial x^2}=\frac{\partial^2 U}{\partial y^2} \tag{4.10}$$

Let us assume

$$U=\eta(x,y) \text{ and } \frac{\partial Y}{\partial n}=\upsilon(x,y) \text{ on the strip } C \tag{4.11}$$

and Γ, the projection of C on the xy-plane, is a curve whose equation is $U(x,y)=0$.

Let us now find the value of the wave function $U(x,y)$ at any a point P with coordinates $(\overline{x},\overline{y})$. Then the characteristics of (4.10) through P are

$$x\pm y=\overline{x}\pm\overline{y} \tag{4.12}$$

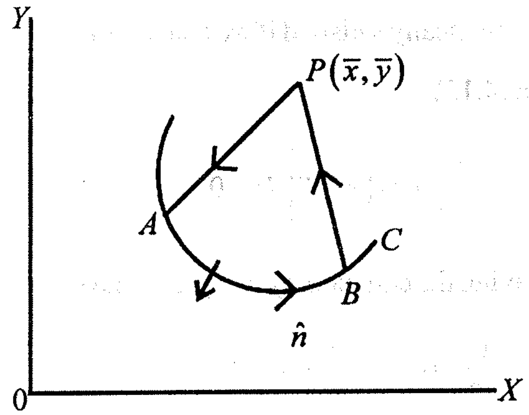

Fig. 3.3

Let the first line of C intersects the curve C at the point A while that the second line at B. Putting $L\equiv\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}$ and noting that L is self-adjoint with respect to the Green's function W, we have

$$\iint_{\Sigma}(WLU-ULW)\,dxdy = \iint_{\Sigma}\left[\frac{\partial}{\partial x}\left(W\frac{\partial U}{\partial x}\right)-\frac{\partial}{\partial x}\left(U\frac{\partial W}{\partial x}\right)-\frac{\partial}{\partial y}\left(W\frac{\partial U}{\partial y}\right)+\frac{\partial}{\partial y}\left(U\frac{\partial W}{\partial y}\right)\right]dxdy$$

$$= \iint_{\Sigma}\left[\frac{\partial}{\partial x}\left(W\frac{\partial U}{\partial x}-U\frac{\partial W}{\partial x}\right)+\frac{\partial}{\partial x}\left(U\frac{\partial W}{\partial y}-W\frac{\partial U}{\partial y}\right)\right]dxdy$$

$$= \int\left[u\cos(n,x)+v\cos(n,y)\right]ds \qquad (4.13)$$

where $u = W\dfrac{\partial U}{\partial x}-U\dfrac{\partial W}{\partial x}$ and $v = U\dfrac{\partial W}{\partial y}-W\dfrac{\partial U}{\partial y}$ (4.14)

and C' denotes the closed path $APBA$ enclosing the area Σ. Noting that the Green's function W satisfies the conditions

(i) $LW = 0$

(ii) $\dfrac{\partial W}{\partial n} = 0$ on AP and BP

and (iii) $W = 1$ at the point P,

we see that these conditions are automatically satisfied if we take $W = 1$.

Again, since $LU = 0$, we have from (4.13)

$$\left(\int_{PA}+\int_{AB}+\int_{BP}\right)\left[\frac{\partial U}{\partial x}\cos(n,x)-\frac{\partial U}{\partial y}\cos(n,y)\right]ds = 0 \qquad (4.15)$$

Now, along the characteristic PA having the equation $xy = \bar{x}\,\bar{y}$, we have

$$\cos(n,x) = \frac{-1}{\sqrt{2}}, \cos(n,y) = \frac{1}{\sqrt{2}}, ds = -\sqrt{2}dx = -\sqrt{2}dy$$

so that

$$\int_{PA}\left[\frac{\partial U}{\partial x}\cos(n,x)-\frac{\partial U}{\partial y}\cos(n,y)\right]ds = \int_{P}^{A}\left(\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy\right) = U_A - U_P$$

Similarly, along the characteristic $PB : x + y = \bar{x} + \bar{y}$, we have

$$\cos(n,x)=\frac{1}{\sqrt{2}},\cos(n,y)=\frac{1}{\sqrt{2}},ds=-\sqrt{2}dx=\sqrt{2}dy$$

so that

$$\int_{BP}\frac{\partial U}{\partial x}\cos(n,x)-\frac{\partial U}{\partial y}\cos(n,y)\,ds=-\int_{B}^{P}\left(\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy\right)=U_B-U_P$$

Hence from (4.15), it follows that

$$U_P=\frac{1}{2}(U_A+U_B)-\frac{1}{2}\int_{AB}\left[\frac{\partial U}{\partial x}\cos(n,x)-\frac{\partial U}{\partial y}\cos(n,y)\right]ds \qquad (4.16)$$

is the solution of the Cauchy problem.

As for example, if we have $U=\eta(x),\frac{\partial U}{\partial y}=\upsilon(x)$ on $y=0$ and P is the point (x,y), then the coordinates of the points A and B are respectively $(x+y,0)$ and $(x-y,0)$. Thus $U_A=\eta(x+y)$ and $U_B=\eta(x-y)$ and

$$\int_{AB}\left[\frac{\partial U}{\partial x}\cos(n,x)-\frac{\partial U}{\partial y}\cos(n,y)\right]ds=-\int_{x-y}^{x+y}\upsilon(\zeta)\,d\zeta$$

Hence in terms of the original variables x and t, the required solution is obtained from (4.16) as

$$U(x,t)=\frac{1}{2}\{\eta(x+ct)+\eta(x-ct)\}+\frac{1}{2c}\int_{x-ct}^{x+ct}\upsilon(\rho)\,d\rho \qquad (4.17)$$

Example 4.2: Solve the Cauchy problem, described by the inhomogeneous wave equation

$$\frac{\partial^2 U}{\partial t^2}-c^2\frac{\partial^2 U}{\partial x^2}=f(x,t)$$

subject to the initial conditions $U(x,0)=\eta(x)$ and $\frac{\partial U}{\partial t}=\upsilon(x)$ at $(x,0)$.

Solution. Proceeding exactly along the same lines as described above by introducing an extra term $\iint_{\Sigma} f(x,y)\,dxdy$, it is easy to see the solution of the given problem is

$$U(x,t)=\frac{1}{2}\{\eta(x+ct)+\eta(x-ct)\}+\frac{1}{2c}\int_{x-ct}^{x+ct}\upsilon(\zeta)d\zeta-\frac{1}{2c}\int_{0}^{t}\int_{x-ct}^{x+ct}f(\xi,\tau)\,d\xi\,d\tau.$$

IV. Solution by separation of variables.

To solve the one-dimensional wave equation

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial t^2}, -\infty < x < \infty, t > 0 \qquad (4.18)$$

by the method of separation of variables, we put $U(x,t) = X(x)T(t)$ in (4.18) and get

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2 T}\frac{d^2T}{dt^2} = \text{K}, \text{ a separation constant.} \qquad (4.19)$$

Case (i) : Let $\text{K} = \lambda^2 > 0$. Then from (4.19), we have

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0 \text{ and } \frac{d^2T}{dt^2} - c^2\lambda^2 T = 0$$

whose solutions are $X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$ and $T(t) = C_3 e^{c\lambda t} + C_4 e^{-c\lambda t}$

so that $$U(x,t) = \left(C_1 e^{\lambda x} + C_2 e^{-\lambda x}\right)\left(C_3 e^{c\lambda t} + C_4 e^{-c\lambda t}\right) \qquad (4.20)$$

is the solution of the wave equation (4.18).

Case (ii) : Let $\text{K} = 0$. Then (4.19) gives $X(x) = C_5 x + C_6, T(t) = C_7 t + C_8$ and, therefore, the solution of the wave equation (4.18) is

$$U(x,t) = (C_5 x + C_6)(C_7 t + C_8) \qquad (4.21)$$

Case (iii) : Let $k = -\lambda^2 < 0$. Then from the equation (4.19), we have

$$X(x) = C_9 \cos\lambda x + C_{10}\sin\lambda x,\ T(t) = C_{11}\cos c\lambda t + C_{12}\sin c\lambda t$$

leading to the solution of (4.18) as

$$U(x,t) = (C_9 \cos\lambda x + C_{10}\sin\lambda x)(C_{11}\cos c\lambda t + C_{12}\sin c\lambda t) \qquad (4.22)$$

Example 4.3. A string of length L is released from rest in the position of $y = f(x)$. SHow that the total energy of the string is $\frac{\pi^2 T}{4L}\sum_{n=1}^{\infty} n^2 A_n^2$, where $A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ and T is the tension in the string.

If the mid-point of a string is pulled aside through a small distance and then released, show that in the subsequent motion the fundamental mode (i.e. $n = 1$) contributes $\frac{8}{\pi^2}$ of the total energy.

Solution. Here the wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are : $y(0,t) = y(L,t) = 0$

The initial conditions are : $y(x,0) = f(x)$ and $\frac{\partial y}{\partial t}(x,0) = 0$

Using the separation of variables method we first consider the solution of the type (4.20), viz.

$$y(x,t) = \left(C_1 e^{\lambda x} + C_2 e^{-\lambda x}\right)\left(C_3 e^{c\lambda t} + C_4 e^{-c\lambda t}\right) \tag{4.23}$$

The given boundary conditions then lead to $C_1 + C_2 = 0$ and $C_1 e^{\lambda L} + C_2 e^{-\lambda L} = 0$ so that $C_1 = C_2 = 0$ and thus we have the trivial solution. Therefore, the solution (4.23) is not acceptable.

Next consider the solution of the form (4.21), viz.

$$g(x,t) = (C_5 x + C_6)(C_7 t + C_8) \tag{4.24}$$

Using the boundary conditions we have $C_6 = 0$ and $C_5 L + C_6 = 0$ i.e., $C_5=0$ so that $y(x,t) = 0$, a trivial solution and hence the solution (4.24) is also omitted.

Lastly, we consider the solution (4.22), viz

$$y(x,t) = (C_9 \cos \lambda x + C_{10} \sin \lambda x)(C_{11} \cos c\lambda t + C_{12} \sin c\lambda t) \tag{4.25}$$

The boundary conditions give $C_9 = 0, \sin \lambda L = 0,$ i.e. $\lambda = n\pi/L, (n = 1, 2, ...)$

Hence $y(x,t) = \sin\frac{n\pi x}{L}\left(A\cos\frac{cn\pi t}{L} + B\sin\frac{cn\pi t}{L}\right)$

where $A = C_{10}C_{11}$ and $B = C_{10}C_{12}$.

Also, the initial condition $\frac{\partial t(x,0)}{\partial t} = 0$ shows that $B = 0$.

Thus using the superposition principle, we have the solution

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{cn\pi t}{L} \tag{4.26}$$

Again the initial condition $y(x,0)=f(x)$ gives

$$f(x)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L}$$

which is half-range Fourier sine series. Hence

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \qquad (4.27)$$

Hence the solution of the above equation is given by (4.26) where A_n is obtained from (4.27).

VI. Uniqueness of solution

Let us consider the forced vibration of a string of length L described by the wave equation

$$\frac{\partial^2 U}{\partial t^2}-c^2\frac{\partial^2 U}{\partial x^2}=F(x,t), 0<x<L, t>0 \qquad (4.28)$$

where $c^2(x)-T_0/\rho(x)$ and $\rho(x)$ is the density of the material of the string. The initial conditions are

$$U(x,0)=\eta(x),\frac{\partial U(x,0)}{\partial t}=\upsilon(x)$$

and the boundary conditions are

$$U(0,t)=f(t),U(L,t)=g(t).$$

Now, if possible, we assume that the above initial boundary value problem admits two solutions U_1 and U_2 for $<T$, T being fixed. Then, if we put $u=U_1-U_2$ it readily follows that

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0, 0<x<L, t>0 \qquad (4.29)$$

subject to the initial conditions

$$u(x,0)=0,\frac{\partial u(x,0)}{\partial t}=0$$

and boundary conditions

$$u(0,t)=0,u(L,t)=0.$$

Multiplying both sides of (4.29) by $\rho\frac{\partial u}{\partial t}$ and then integrating the results w.r.t. x from 0 to L and w.r.t. t from 0 to T, we have

$$\frac{1}{2}\int_0^L\left\{\rho(x)\left[\frac{\partial u(x,T)}{\partial t}\right]^2+T_0\left[\frac{\partial u(x,T)}{\partial x}\right]^2\right\}dx$$

$$-\frac{1}{2}\int_0^L\left\{\rho(x)\left[\frac{\partial u(x,0)}{\partial t}\right]^2+T_0\left[\frac{\partial u(x,0)}{\partial x}\right]^2\right\}dx$$

$$-T_0\int_0^T\frac{\partial u(L,t)}{\partial x}\frac{\partial u(L,t)}{\partial t}dt+T_0\int_0^T\frac{\partial u(0,t)}{\partial x}\frac{\partial u(0,t)}{\partial t}dt=0 \qquad (4.30)$$

The first two integrals in (4.30) represent the difference in the total energy (vide Example 4.3) at times T and 0. On the other hand, the last two integrals represent the work done by the y-components of the tensile force at the ends of the string. It is clear that

$$\frac{\partial u(0,t)}{\partial x}=\frac{\partial u(L,t)}{\partial t}=\frac{\partial u(x,0)}{\partial x}=0.$$

Thus we have from (4.30)

$$\int_0^L\left\{\rho(x)\left[\frac{\partial u(x,T)}{\partial t}\right]^2+T_0\left[\frac{\partial u(x,T)}{\partial t}\right]^2\right\}dx=0 \qquad (4.31)$$

This shows that if the string has no energy initially $(t=0)$, then the energy remains to be zero if no work is done. Since $\rho(x)>0$ and $T>0$, so the integral in (4.31) can never be negative. Hence, assuming that the integrand is continuous, it must identically be zero, so that far any $\tau<T$, we have

$$\frac{\partial u(x,\tau)}{\partial t}=0 \text{ for } 0<\tau<T \text{ and } 0<x<L$$

i.e. $u(x,\tau)=$ constant.

But $u(x,0)=0$ and so $u(x,\tau)=0$ for $\tau<T$. Since τ is arbitrary, it follows that $u(x,t)=0$, i.e. $U_1(x,t)=U_2(x,t)$ for any t. Thus the given equation has a unique solution.

5. Self Assessment Questions

1. A tightly stretched homogeneous string of length L, with its fixed ends at $x=0$ and $x=L$, executes transverse vibrations. Motion is started with zero initial velocity by displacing the string into the form $f(x)=a\sin^3 \pi x$. Find the deflection at any time t.

2. By separating the variables, show that the one-dimensional wave equation $\frac{\partial^2 U}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}$ has a solution of the form $A\exp(\pm inx \pm ict)$, where A and n are constants. Hence show that function of the form

$$U(x,t)=\sum_n\left(A_n\cos\frac{n\pi ct}{a}+B_n\sin\frac{n\pi ct}{a}\right)\sin\frac{n\pi x}{a}$$

where A_n and B_n are constants, satisfy the wave equation and the boundary conditions $U(0,t)=0$, $U(a,t)=0$ for all t.

3. Find the solution of the radio equation $\frac{\partial^2 U}{\partial x^2}=LC\frac{\partial^2 U}{\partial t^2}$ when a periodic e.m.f. $A\cos pt$ is applied at the end $x=0$ of the line, A being constant.

$$\left[Ans. U(x,t)=A\cos\left(pt-p\sqrt{LC}x\right)\right]$$

4. A tightly stretched string with fixed end point $x=0$ and $x=L$ is initially in a position given by

(i) $U=U_0\sin^3(\pi x/L), 0\le x\le L$

$$\left[\text{Ans. } U(x,t)=\frac{1}{4}U_0\left[3\sin(\pi x/L)\cos(\pi ct/L)-\sin(3\pi x/L)\cos(3\pi ct/L)\right]\right]$$

(ii) $U=\mu x(L-x), 0\le x\le L$

$$\left[\text{Ans. } U(x,t)=\frac{8\mu L^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin\frac{(2n+1)\pi x}{L}\cos\frac{(2n+1)\pi ct}{L}\right]$$

(iii) $U=f(x)$ where $f(x)=\begin{cases}\frac{2\lambda x}{L}, 0<x<\frac{L}{2}\\ \frac{2\lambda}{L}(L-x), \frac{L}{2}<x<L\end{cases}$

$$\left[\text{Ans. } U(x,t) = \frac{8\lambda}{\pi^3}\sum_{n=1}^{\infty}\sin(n\pi/2)\cos(nct\pi/L)\sin(n\pi x/L)\right]$$

5. Use D'Alembert's solution to find the deflection $U(x,t)$ of a vibrating string of a unit length having fixed ends with zero initial velocity and initial deflection at any time t:

(i) $f(x) = a(x - x^3)$

$$\left[\text{Ans. } U(x,t) = ax(1 - x^2 - 3c^2t^2)\right]$$

(ii) $f(x) = ax^2(1 - x)$

$$\left[\text{Ans. } U(x,t) = a(x^2 + c^2t^2 - x^3 - 3xc^2t^2)\right]$$

6. Solve the initial-boundary value problem described by $\frac{\partial^2 U}{\partial t^2} = c^2\frac{\partial^2 U}{\partial x^2}$, $x > 0, t > 0$, given that $U(x,0) = 0, \frac{\partial U(x,0)}{\partial t} = 0, x > 0$ and $U(0,t) = \sin t, t > 0$.

$$\left[\text{Ans. } U(x,t) = \begin{cases} 0, x < ct \\ \sin\left(t - \frac{x}{c}\right), x > ct \end{cases}\right]$$

7. Solve the initial value problem described by $\frac{\partial^2 U}{\partial t^2} - c^2\frac{\partial^2 U}{\partial x^2} = e^x$, given that $U(x,0) = 5$ and $\frac{\partial^2 U}{\partial t^2} - c^2\frac{\partial^2 U}{\partial x^2} = e^x$.

$$\left[\text{Ans. } U(x,t) = 5 + x^2t + \frac{1}{3}c^2t^2 + \frac{1}{2t^2}\left(e^{x+ct} + e^{x-ct} - 2e^x\right)\right]$$

8. Solve the initial value problem described by $\frac{\partial^2 U}{\partial t^2} - c^2\frac{\partial^2 U}{\partial x^2} = xe^t$, given that $U(x,0) = \sin x$ and $\frac{\partial U(x,0)}{\partial t} = 0$. $\left[\textit{Ans. } U(x,t) = \sin x.\cos ct + (xt + x)(e^t - 1) - xte^t\right]$

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-IV

Group–B

Module No. - 48

GREEN'S FUNCTION

INTRODUCTION

It is well known in the theory of non-homogeneous ordinary differential equation that the solution of the equation Lu(x) = f(x), when $L \equiv \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\}$ is the Sturm-Liouville operator and p(x) (>0) and q(x) are real-valued functions of x, subject to given boundary conditions at the end points x = a, b of a given interval [a, b], can be obtained in the form of an integral as

$$U(x) = \int_a^b G(x,\xi)f(\xi)\,d\xi$$

where $G(x,\xi)$ known as *Green's function*, satisfies the equation

$$LG(x,\xi) = \delta(x-\xi),$$

$\delta(x-\xi)$ being the Dirac-delta function.

The above concept can be extended to partial differential equation. Let us consider the equation

$$L\{u(X)\} = f(X) \qquad (1.1)$$

where L is some linear partial differential operator in three independent variables x, y, z and X is a vector in three-dimensional space. Then Green's G (X; α) satisfies the equation

$$L\{G(X;\alpha)\} = \delta(X-\alpha) \qquad (1.2)$$

which reads, on expansion

$$L\{G(x,y,z;\xi,\eta,\zeta)\} = \delta(x-\xi)\delta(y-\eta)\delta(x-\zeta) \qquad (1.3)$$

Here G (x; α) represents the effect at the point X due to a source or delta function input at α.

We shall restrict our discussions for the solution of Laplace's equation by means of Green's function.

MODULE - 48

GREEN'S FUNCTION

§ 1. Introduction

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Here $G(\mathbf{X};\alpha)$ represents the effect at the point **X** due to a source or delta function input at α.

We shall restrict our discussions for the solution of Laplece's equation by means of Green's function.

§ 2. Green's Function for Dirichlet Problem.

Let us first construct Green's function for the interior Dirichlet problem, in other words, we like to find u such that $\nabla^2 u = 0$ inside a finitely bounded region V enclosed by a sufficiently smooth surface S and u = f on S

Consider a point P ∈ V and suppose $\overrightarrow{OP} = \mathbf{r}$. Let us construct a small sphere Σ with centre at P and radius ϵ. Take another point Q in $V' = V - \Sigma$, or on the boundary S' of V - Σ such that $\overrightarrow{OQ} = \mathbf{r}'$ and

$$u' = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \tag{2.1}$$

We also suppose that the functions u and u' are twice continuously differentiable in V and

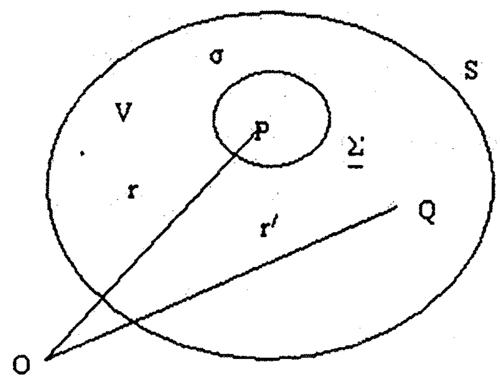


Fig - 6.1

have first order partial derivatives on S. Then by Green's theorem, we have

$$\iiint_{V'} \left(u \nabla^2 u' - u' \nabla^2 u\right) dv = \iint_{S'} \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) dS' \tag{2.2}$$

where n is the outward drawn unit vector normal to ds. Then since $\nabla^2 u = \nabla^2 u' = 0$ within $V' = V - \Sigma$, we have from (2.2)

$$\iint_S \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) dS + \iint_\sigma \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) d\sigma = 0$$

$$\int\int_S \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n} \right) dS + \int\int_\sigma \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n} \right) d\sigma = 0$$

$$\int\int_S \left[u\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial u}{\partial n} \right] dS + \int\int_\sigma \left[u\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial u}{\partial n} \right] d\sigma = 0 \quad (2.3)$$

Now, if Q lies on σ, then $\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\epsilon}\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)$, and also $d\sigma = \epsilon^2 \sin\theta d\theta d\phi$.

Moreover, on σ

$$\begin{aligned} u(\mathbf{r}') &= u(\mathbf{r}) + \mathbf{r}.\nabla u \\ &= u(\mathbf{r}) + \left(x\tfrac{\partial u}{\partial x} + y\tfrac{\partial u}{\partial y} + z\tfrac{\partial u}{\partial z}\right) \\ &= u(\mathbf{r}) + \epsilon\left(\sin\theta\cos\phi\tfrac{\partial u}{\partial x} + \sin\theta\sin\phi\tfrac{\partial u}{\partial y} + \cos\theta\tfrac{\partial u}{\partial z}\right) \\ &= u(\mathbf{r}) + o(\epsilon) \end{aligned}$$

and so $\frac{\partial u}{\partial n}(\mathbf{r}') = \frac{\partial u}{\partial n}(\mathbf{r}) + o(\epsilon)$ on σ.

Thus

$$\begin{aligned} \int\int_\sigma u(\mathbf{r}')\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) d\sigma &= \int\int_\sigma [u(\mathbf{r}) + o(\epsilon)].\frac{1}{\epsilon^2}.\epsilon^2 \sin\theta d\theta d\phi \\ &= u(\mathbf{r})\int\int_\sigma \sin\theta d\theta d\phi + o(\epsilon) \\ &= u(\mathbf{r})\int_{\phi=0}^{2\pi}\int_{\sigma=0}^{\pi} \sin\theta d\theta d\phi + o(\epsilon) \\ &= 4\pi u(\mathbf{r}) + o(\epsilon) \end{aligned} \quad (2.4)$$

and

$$\iint_\sigma \frac{1}{|r-r'|}\frac{\partial u}{\partial n}(r')\,d\sigma = \frac{1}{\varepsilon}\iint_\sigma \left[\frac{\partial u}{\partial n}(r) + o(\varepsilon)\right]\varepsilon^2 \sin\theta d\theta d\phi = o(\varepsilon). \quad (1)$$

Substituting (2.4) and (2.5) into (2.3) and proceeding to the limit as $\varepsilon \to 0$, we get

$$u(r) = \frac{1}{4\pi}\iint_s \left[\frac{1}{|r-r'|}\frac{\partial u}{\partial n}(r') - u(r')\frac{\partial}{\partial n}\left(\frac{1}{|r-r'|}\right)\right] ds. \quad (2)$$

Thus if u and $\frac{\partial u}{\partial n}$ are given on the boundary S of V, then the value of u can be obtained at any interior point of V.

In a similar way, we may consider exterior problem. For such a case, we take the region to be bounded by the surface S, a small sphere σ surrounding the point P and S, a sphere with center at the origin O and large radius R (Fig. 6.2).

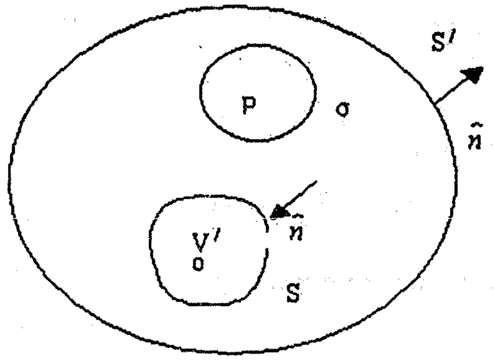


Fig - 6.2

Then considering outward drawn normals as indicated in the figure and proceeding as above we get

$$4\pi u(r)+\iint_{s}\left[\frac{1}{|r-r'|}\frac{\partial u}{\partial n}(r')-u(r')\frac{\partial}{\partial n}\left(\frac{1}{|r-r'|}\right)\right]ds+\iint_{s'}\left[\frac{1}{R}\frac{\partial u}{\partial n}(r')+\frac{u(r')}{R^2}\right]ds'+o(\varepsilon)=0. \quad (3)$$

Assuming that Ru and R^2u are finite as $R\to\infty$ and letting $\varepsilon\to 0$ as $R\to\infty$ in the above, we see that the equation (2.6) is also true for exterior Dirichlet problem.

It is seen that to obtain solution of Dirichlet problem, it is necessary to know the values of u and $\frac{\partial u}{\partial n}$. But we now show that this is not so if we introduce $G(r,r)$ defined by

$$G(r,r')=H(r,r')+\frac{1}{|r-r'|} \quad (4)$$

where $r'=(x',y',z')$ and the function $H(r,r')$ satisfies the relations

$$\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}\right)H(r,r')=0 \ \ within V \quad (5)$$

$$and \quad H(r,r')+\frac{1}{|r-r'|}=0 \ \ on S. \quad (6)$$

now proceeding along the same lines as in the case of derivation of the equation (2.6), it is easy to show that

$$u(r) = \frac{1}{4\pi}\iint_{s}\left[G(r,r')\frac{\partial u}{\partial n}(r') - u(r')\frac{\partial G}{\partial n}(r,r')\right]ds \tag{7}$$

Noting from (2.7) and the second relation of (2.8), it is, therefore, seen that the solution of Dirichlet problem is given by

$$u(r) = -\frac{1}{4\pi}\iint_{s} u(r')\frac{\partial G}{\partial n}(r,r')ds \tag{8}$$

The function $G(r,r)$ is known as *Greens function of the first kind* for V.

3. An important result.

Consider a sphere with center at the origin O and radius a. Since

$\nabla.\nabla\left(\frac{1}{r}\right) = \nabla^2\left(\frac{1}{r}\right)$ and $\nabla\left(\frac{1}{r}\right) = \hat{e}_r\frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\right) + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r}\right)$

in spherical polar coordinates (r,θ,ϕ), we have

$\iiint\limits_{V}\nabla.\nabla\left(\frac{1}{r}\right)dv = \iint\limits_{S}\nabla\left(\frac{1}{r}\right).\hat{e}_r ds$

$= \iint\limits_{S}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)ds$

$= -\frac{1}{a^2}4\pi a^2 = -4\pi$

So that

$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(r)$

where $\delta(r)$ is the Dirac-delta function.

4. Some properties of Greens Function

I. *Green,s function $G(r,r)$ has the symmetric property*

Proof.

Let us define Greens function $G(r,r)$ by the formula (2.7), viz.

$G(r,r') = H(r,r') + \frac{1}{|r-r'|}$

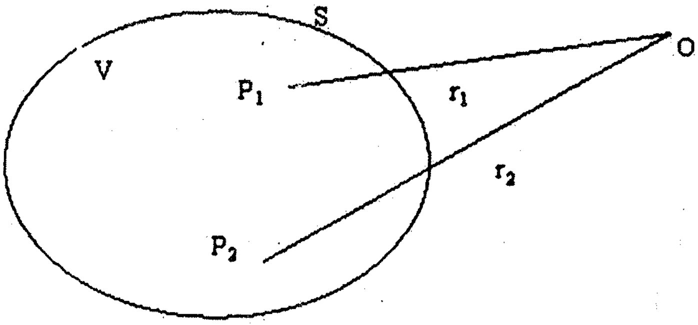


Fig - 6.3

where $H(r, r)$ is harmonic in V. Then

$\nabla^2 G = \nabla^2 \left(\frac{1}{|r-r'|}\right) = -4\pi\delta(r - r')$ $(by(3.1))$

Let r_1 and r_2 be the position vectors of P_1 and P_2 respectively and r be that of a variable point Q. Assume $u = G(r_1, r') = H(r_1, r') + \frac{1}{|r_1 - r'|}$

and $u' = G(r_2, r') = H(r_2, r') + \frac{1}{|r_2 - r'|}$,

so that $G(r_1, r') = G(r_2, r') = 0$ on S and $\nabla^2 G(r_1, r') = -4\pi\delta(r_1 - r'), \nabla^2 G(r_2, r') = -4\pi\delta(r_2 - r')$.

Then from Greens theorem

$\iiint\limits_V (u\nabla^2 u' - u'\nabla^2 u)\, dv = \iint\limits_S \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) ds$

we get

$\iiint\limits_V [G(r_1, r')\nabla^2 G(r_2, r') - G(r_2, r')\nabla^2 G(r_1, r')]\, dv = \iint\limits_S \left[G(r_1, r')\frac{\partial G}{\partial n}(r_2, r') - G(r_2, r')\frac{\partial G}{\partial n}(r_1, r')\right] ds$

or,

$-4\pi \iiint\limits_V [G(r_1, r')\delta(r_2 - r') - G(r_2, r')\delta(r_1 - r')]dv = 0$

Using the property of Dirac-delta function, we have $G(r_1, r_2) = G(r_2, r_1)$, that is Greens function is symmetric.

II. *If G be continuous and $\frac{\partial G}{\partial n}$ be discontinuous at r, then $\lim\limits_{\varepsilon \to 0} \iint\limits_\sigma \frac{\partial G}{\partial n} = -4\pi$ where σ is the surface of a small sphere Σ of radius ε.*

Proof. We have $\nabla^2 G(r, r') = -4\pi\delta(r - r')$ so that

$\iiint_{\Sigma} \nabla^2 G(r, r')\, dv = -4\pi \iiint_{\Sigma} \delta(r - r')\, dv = -4\pi$

or, $\lim_{\epsilon \to 0} \iiint_{\Sigma} \nabla^2 G(r, r')\, dv = -4\pi$

or, $\lim_{\epsilon \to 0} \iint_{s} \frac{\partial G}{\partial n} ds = -4\pi$

5. Solution of Dirichlet problem for some particular cases

I. *Half-space (semi-infinite space)*

Suppose the half space be given by $0 \le x \le \infty, -\infty < y < \infty, -\infty < z < \infty$. It is our aim

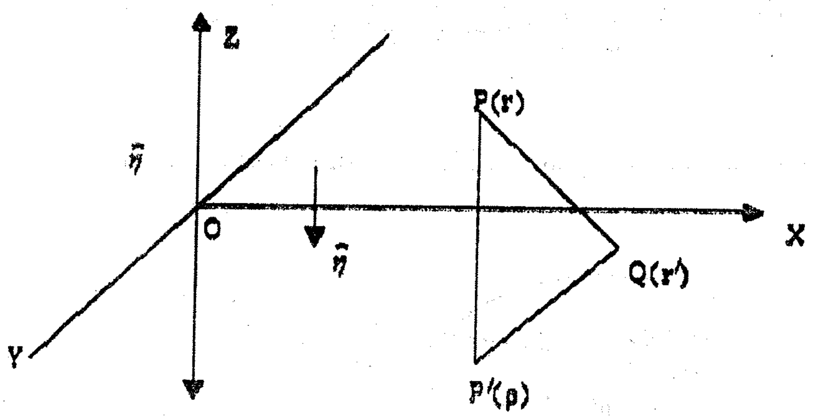


Fig - 6.1

to find the solutions of the Laplaces equation $\nabla^2 u = 0$ within $x \geq 0$ subject to u=f(y,z) on y=0 and $u \to 0$ as $r = \sqrt{x^2 + y^2 + z^2} \to \infty$, by means of Greens function $G(r, r)$ which satisfies the relations:

(i) $G(r, r') = H(r, r') + \frac{1}{|r-r'|}$

(ii) $\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right) H(r, r') = 0$ in $x \geq 0$

(iii) $G(r, r) = 0 on x = 0$,

where $r = (x, y, z)$ and $r = (x, y, z)$ are the points P and Q respectively and ρ is the point P the image point of P in x=0. Then, noting that $PQ = PQ$, Q being the point on x=0, we have by using (i) and (iii) $|r - r'| = |\rho - r'|$ so that $H(\rho, r') = -\frac{1}{|\rho - r'|}$.

Hence

$$G(r, r') = \frac{1}{|r - r'|} - \frac{1}{|\rho - r'|} \tag{9}$$

Thus the required solution is obtained from (2.10) and (5.1) as

$u(r) = -\frac{1}{4\pi} \iint_S u(\rho') \frac{\partial}{\partial n} G(r, r')\, ds$

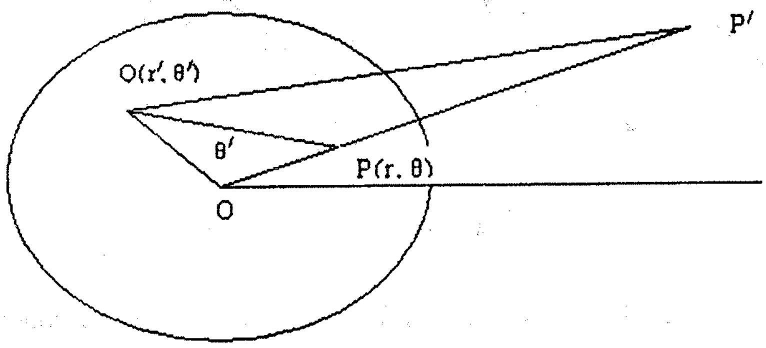
$= -\frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y', z') \left[\frac{\partial}{\partial x'} \left\{ \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x+x')^2+(y+y')^2+(z+z')^2}} \right\} \right]_{x'=0} dy'dz'$

i.e.,

$$u(r) = -\frac{x}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(y', z')}{\left\{ x^2 + (y - y')^2 + (z - z')^2 \right\}} dy'dz' \tag{10}$$

II. *Circular disc*

Let $P(r, \theta)$ and $Q(r', \theta')$ be two points within the disc having position vectors r and r



respectively and $P'(\frac{a^2}{r}, \theta)$ be the inverse point of P with respect to the disc. We construct the Greens function $G(r, r)$ defined by

$G(r, r') = H(r, r') + \ln \frac{1}{|r - r'|}$

where $H(r, r') = \ln \left(r, \frac{P'Q}{a} \right)$.

Now, $P'Q^2 = OQ^2 + OP^2 - 2OQ.OP' \cos(\theta' - \theta)$

$= r'^2 + \frac{a^4}{r^2} - 2r' \frac{a^2}{r} \cos(\theta' - \theta)$

So that $\frac{r^2 P'Q^2}{a^2} = \frac{r^2 r'^2}{a^2} + a^2 - 2rr' \cos(\theta' - \theta)$

and hence $H(r, r') = \frac{1}{2}\ln\left\{\frac{r^2r'^2}{a^2} + a^2 - 2rr'\cos(\theta' - \theta)\right\}$.

It is easily seen that $\nabla^2 H = 0$. Also $G = \ln\frac{r.P'Q}{aPQ}$ and so $G = 0$ on the circle r=a.

Since $PQ^2 = r^2 + r'^2 - 2rr'\cos(\theta' - \theta)$, we have

$G = \frac{1}{2}\ln\left\{\frac{a^2+r^2r'^2/a^2-2rr'\cos(\theta'-\theta)}{r^2+r'^2-2rr'\cos(\theta'-\theta)}\right\}$

So that on the circle $r = a$

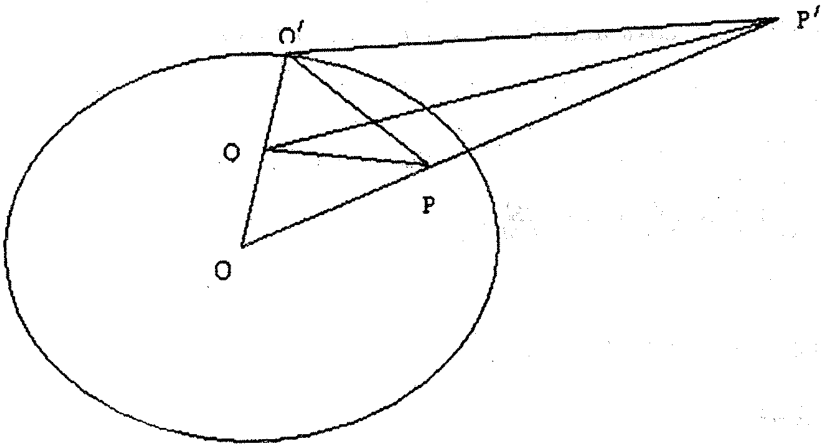
$\left(\frac{\partial G}{\partial n}\right)_{r'=a} = \left(\frac{\partial G}{\partial r}\right)_{r'=a} = \frac{r^2-a^2}{a\{a^2-2ar\cos(\theta'-\theta)+r^2\}}$.

Thus from (2.10), the required solution is

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi a}\int_0^{2\pi}\frac{f(\theta')\,d\theta'}{a^2 - 2ar\cos(\theta' - \theta) + r^2} \qquad (11)$$

III. *Sphere*

In this case our problem is to determine the function $u(r,\theta,\phi)$ satisfying $\nabla^2 u = 0, 0 \leq r \leq$



$a, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$ subject to $u(a,\theta,\phi) = f(\theta,\phi)$.

Let us define Greens function $G(r, r)$ by

$G(r, r') = H(r, r') + \frac{1}{|r - r'|}$

where $H(r, r)$ satisfies the conditions

$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right) H(r, r') = 0$

and $G(r, r') = H(r, r') + \frac{1}{|r-r'|} = 0$ on the surface r = a of the sphere.

Let $P(r,\theta,\phi)$ be a point within the sphere and P be the inverse point of P with respect to the sphere so that if OP=r then $r = OP = \frac{a^2}{r}$. Now if Q be any variable point on the surface of the sphere, then from the similar triangles OQP and OQP we have

$\frac{PQ'}{P'Q'} = \frac{r}{a} = \frac{a}{\rho}$, i.e., $PQ' = \frac{r}{a}P'Q'$, where $OQ' = \rho$.

Noting that this relation is valid for all points on the surface r = a, we get

$H(r, r') = -\frac{a}{r}\frac{1}{\left|\overrightarrow{P'Q'}\right|} = -\frac{a}{r|\rho - r'|} = -\frac{a}{r\left|\frac{a^2}{r^2}r - r'\right|}$

It may be easily verified that this form of $H(r, r)$ satisfies Laplaces equation. Let $Q(r', \theta', \phi')$ be a variable point inside the sphere. If Q lies on the surface of the sphere, say at Q, then $\frac{PQ}{P'Q} = \frac{r}{a}$.

Thus Greens function for the present

$G(r, r') = \frac{1}{|r - r'|} - \frac{a/r}{\left|\frac{a^2}{r^2}r - r'\right|} = \frac{1}{R} - \frac{/r}{R'}$

where PQ=R and $PQ = R$. it is seen that G=0 on the surface $r = a$ of the sphere.

Now $R^2 = r^2 + r'^2 - 2rr' \cos\theta$ and $R'^2 = \frac{a^4}{r^2} + r'^2 - 2\frac{a^2}{r}r' \cos\theta$

where $\cos\theta = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi')$. Thus

$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r'} = -\frac{1}{R^2}\frac{\partial R}{\partial r'} + \frac{a/r}{R'^2}\frac{\partial R'}{\partial r'} = -\frac{1}{R^3}\left[R\frac{\partial R}{\partial r'} - \left(\frac{a}{r}\right)\frac{R^3}{R'^3}R'\frac{\partial R'}{\partial r'}\right]$

$= -\frac{1}{R^3}\left[R\frac{\partial R}{\partial r'} - \left(\frac{a}{r}\right)\frac{R^3}{a'^3}R'\frac{\partial R'}{\partial r'}\right]\left(\frac{R}{R'} = \frac{PQ}{P'Q} = \frac{r}{a}\right)$

So that

$\left(\frac{\partial G}{\partial n}\right)_{r'=a} = -\left[\frac{1}{R^3}\left\{(a - r\cos\theta) - \frac{r^2}{a^2}\left(a - \frac{a^2}{r}\cos\theta\right)\right\}\right]_{r'=a}$

$= \frac{r^2 - a^2}{a(r^2 + a^2 - 2ar\cos\theta)^{3/2}}$.

Thus the solution of the *interior Dirichlet problem for a sphere is* obtained from (2.10) as

$u(r, \theta, \phi) = \frac{r^2 - a^2}{4\pi a}\iint \frac{f(\theta', \phi')}{(r^2 + a^2 - 2ar\cos\theta)^{3/2}}ds'$

i.e.,

$$u(r, \theta, \phi) = \frac{a(r^2 - a^2)}{4\pi}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\frac{f(\theta'.\phi')\sin\theta'}{(r^2 + a^2 - 2ar\cos\theta)^{3/2}}d\theta' d\phi' \quad (12)$$

$ds' = a^2 \sin\theta' d\theta' d\phi'$. The equation (5.3) is known as *Poisson integral formula.* In a similar way, the *solution of exterior Dirichlet problem* can be obtained in the form

$$u(r,\theta,\phi) = \frac{a(r^2-a^2)}{4\pi} \int_{\theta'=0}^{\pi} \int_{\phi'=0}^{2\pi} \frac{f(\theta',\phi')\sin\theta'}{(r^2+a^2-2ar\cos\theta)^{3/2}} d\theta' d\phi' \quad (13)$$

where the function u is harmonic outside the sphere (thus implying regularity at infinity) and assumes the boundary values $f(\theta,\phi)$.

6. Greens function for Neumann problem

Let us now proceed to see whether there exists a function similar to Greens function which can solve Laplaces equation for Neumann problem. We consider the case of a bounded region and follow the analogy of the work of section-2. We wish to eliminate $u(r)$ from under the integral sign in

$u(r) = \frac{1}{4\pi} \iint\limits_S \left[\frac{1}{|r-r'|}\frac{\partial u(r')}{\partial n} - u(r')\frac{\partial}{\partial n}\left(\frac{1}{|r-r'|}\right)\right] ds$

by means of

$0 = \frac{1}{4\pi} \iint\limits_S \left[H(r,r')\frac{\partial u(r')}{\partial n} - u(r')\frac{\partial}{\partial n}H(r,r')\right] ds$

in order that u(r) may be expressed in terms of the boundary values of its normal derivative alone. This can be done if it is possible to find $H(r,r)$, harmonic in V, and having a normal derivative which is the negative of that of $\frac{1}{|r-r'|}$. But this is impossible, because by Gausss theorem on the integral of the normal derivative,

$\iint\limits_S \frac{\partial}{\partial n}\left(\frac{1}{|r-r'|}\right) ds = -4\pi;$

On the other hand, if $H(r,r)$ is harmonic in V, then

$\iint\limits_S \frac{\partial}{\partial n} H(r,r') ds = 0.$

We, therefore, demand that the normal derivative of $H(r,r)$ shall differ from that of $-\frac{1}{|r-r'|}$ by a constant, and this will serve our perpose.

The function defined by

$G(r,r') = H(r,r') + \frac{1}{|r-r'|}$

if it exists is known as the *Greens function of the second kind* for V. in terms of this function, we obtain the following expression for u(r) by adding the last two equations :

$$u(r) = \frac{1}{4\pi} \iint_S \frac{\partial}{\partial n} u(r') G(r, r') \, ds + \frac{c}{4\pi} \iint_S u(r') \, ds \tag{14}$$

Thus u(r) is obtained in terms of its normal derivatives except for an additive constant, which is all that could be expected, since u(r) is determined by its normal derivatives onlyto within an additive constant.

Exercise

1. Show that the solution of the Laplace's equation in the upper half-plane defined by $y \geq 0, -\infty < x < \infty$ by using Green's function method, subject to the condition u=f(x) on y=0 is

$u(x,y) = \frac{y}{\pi} \int\limits_{-\infty}^{\infty} \frac{f(x')dx'}{(x-x')^2+y'^2}.$

2. Use the method of images to show that the harmonic Green's function for the half-space $z \geq 0$ is

$G(r,r') = \frac{1}{4\pi}\left(\frac{1}{r} - \frac{1}{r'}\right),$

where $r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$ and $r'^2 = (x-x')^2 + (y-y')^2 + (z+z')^2$.

3. Show that $G(r,r')$=$G(r,r')$ for Green's function of the second kind.

4. Show the formula

$u(r,\theta,\phi) = \frac{1}{4\pi} \iint_S f(\theta',\phi') \left[\frac{2}{r} + \frac{1}{a}\log\left(\frac{2a}{a - r\cos\theta + r}\right)\right] ds$

where $f(\theta',\phi')$ is any continuous function such that $\iint_S f(\theta',\phi')ds = 0$ and $\cos\Theta = \cos\theta.\cos\theta' + \sin\theta.\sin\theta'\cos(\phi-\phi')$ solves the Neumann problem for the sphere.

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....
....

3. Write the number of Modules / Units which according to you should be re-written :
....
....
....

4. Which portion / page is not understandable to you? (mention the page no. and portion)
....
....
....
....

5. Write a short comment about the study material as a learner.
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....

....
(Full Signature of the Learner)

Date : ....

Enrolment No. ....

Phone / Mobile No. ....


BADAL DINESH BHAVAN
DIRECTORATE OF DISTANCE EDUCATION