DISTANCE LEARNING MATERIAL

VIDYASAGAR UNIVERSITY

DIRECTORATE OF DISTANCE EDUCATION

MIDNAPORE- 721 102

M. SC. IN APPLIED MATHEMATICS WITH OCEANOLOGY & COMPUTOR PROGRAMMING

PART - I

Paper : I

Module No. : 01, 02, 03, 04, 05, 06, 07, 08, 09, 10, 11 & 12

CONTENTS

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-I Group-A

Module No. - 01

Real Analysis

(Functions of Bounded Variations)

Module Structure :

1.1. Introduction

In real analysis we have studied many important properties of functions like continuity, differentiability, integrability, monotonicity etc. There are many interesting relations among these classes of functions. We know many basic properties of monotonic function. This module introduces a new class of functions closely related to monotonic functions. This class of functions is known as functions of bounded variation on a closed finite interval [a, b]. These functions of bounded variation are intimately connected with curves having finite arc length. They are found also to play an important role in the theory of Riemann-Stieltjes integration.

1.2. Objective

In this module a new class of functions have been introduced which plays role in the development of a number branches of analysis viz. Fourier series, Riemann-Stieltjes integration, study of rectifiable curves etc. This class of functions is closely related with the class of monotonic functions. Many interesting theorems and results have been proved and discussed. Examples are provided for explanation and clarification.

1.3. Definitions

1.3.1. Definition. Partition of [a, b]

Let [a, b] be a closed interval. A finite set P of points $x_0, x_1, x_2, ..., x_n$, where

$$a = x_0 \le x_1 \le x_2 \le ... \le x_n = b$$

is called a partition of the interval [a, b] and is denoted by $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$.

We note that for a given interval any number of partitions are possible. Collection of all partitions of [a, b] is denoted by $\wp[a,b]$ or simply by $\wp$ if there is no confusion.

The intervals $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$ are called the subintervals of the partition. The *i*th subinterval is $[x_{i-1}, x_i]$. The length of *i*th subinterval is denoted by Δx_i i.e. $\Delta x_i = x_i - x_{i-1}$.

1.3.2. Definition. Function of Bounded Variation

Let *f*(*x*) be a real valued function defined on a closed interval [a, b]. Corresponding to each partition $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ we define $\Delta f_k = f(x_k) - f(x_{k-1})$ for $k = 1, 2, ..., n$. Then the sum $\sum_{k=1}^{n} |\Delta f_k|$ is called the variation of *f* over [a, b] for the partition P. It depends on the partition P and is denoted by $V_P(f, a, b)$. If the least upper bounded of $V_P(f, a, b)$ over all possible partitions of [a, b] exists then *f*(*x*) is said to be of bounded variation over [a, b].

1.3.3. Definition. Total Variation of a function *f* on [*a*, *b*]

If a function *f* is of bounded variation on [*a*, *b*] then the least upper bound of $V_P(f,a,b)$ over all possible partitions of [*a*, *b*] exists i.e. $\sup\limits_{P\in\wp} V_P(f,a,b)$ exists. This supremum is called the total variation of *f* on [*a*, *b*] and is denoted by $V(f, a, b)$.

Thus $V(f,a,b) = \sup\limits_{P\in\wp} V_P(f,a,b)$.

1.4. Examples of functions of bounded variation and functions not of bounded variation.

1.4.1. Example. Any constant function is of bounded variation

Solution : Let $f(x)=c$ be defined on [*a*, *b*].

For any partition $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ of [*a*, *b*]

we have

$$\begin{aligned} V_P(f,a,b) &= \sum_{k=1}^{n} \left| f(x_k) - f(x_{k-1}) \right| \\ &= \sum_{k=1}^{n} |c - c| \\ &= 0 \end{aligned}$$

This is true for any partition *P*. As the RHS is independent of *P* we have $\sup\limits_{P\in\wp} V_P(f,a,b) = 0$

Hence $f(x)$ is of bounded variation on [*a*, *b*] and the total variation of *f* on [*a*, *b*] is $V(f, a, b) = 0$

1.4.2. Example. The function $f(x)$ defined by $f(x) = 3x^2 + 2x + 5$ on the interval [3, 8] is of bounded variation on [3, 8].

Solution. Here $f(x) = 3x^2 + 2x + 5$ and is defined over the interval [3, 8]. We have $f(x) = 6x + 2 > 0$ for all $x \in [3,8]$.

Hence $f(x)$ is monotonic increasing on [3, 8].

For any partition $P = \{3 = x_0, x_1, x_2, \ldots, x_n = 8\}$ of [3, 8] we have

$$V_P(f,3,8)=\sum_{k=1}^{n}\left|f(x_k)-f(x_{k-1})\right|$$

$$=\sum_{k=1}^{n}\left\{f(x_k)-f(x_{k-1})\right\}\quad\left[\because f(x)\text{ is m.i. and } x_k\geq x_{k-1}\right]$$

$$=f(x_1)-f(x_0)$$
$$+f(x_2)-f(x_1)$$
$$+f(x_3)-f(x_2)$$
....
....
$$+f(x_{n-1})-f(x_{n-2})$$
$$+f(x_n)-f(x_{n-1})$$
$$=f(x_n)-f(x_0)$$
$$=f(8)-f(3)$$
$$=(3.8^2+2.8+5)-(3.3^2+2.3+5)$$
$$=175$$

This is true for all P of [3, 8]. Since the RHS is independent of P we have

$$\sup_{P\in\wp}V_P(f,3,8)=175.$$

Hence f is of bounded variation on [3, 8] and the total variation of f on [3, 8] is $V(f,3,8)=175$.

1.4.3. Example. The function $f(x)$ defined on $[a, b]$ by

$$f(x)=\alpha \text{ for rational } x\in[a,b]$$
$$=\beta \text{ for irrational } x\in[a,b]$$

where $\alpha\neq\beta$ is not of bounded variation on $[a, b]$.

Solution. Here $f(x)=\alpha$ for rational x in $[a, b]$.

$=\beta$ for irrational x in $[a, b]$.

Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$. Then

$$V_P(f,a,b) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

$$= \sum_{k=1}^{n} |\alpha - \beta| \text{ [Each subinterval contains rational as well as irrational points]}$$

$$= n|\alpha - \beta|$$

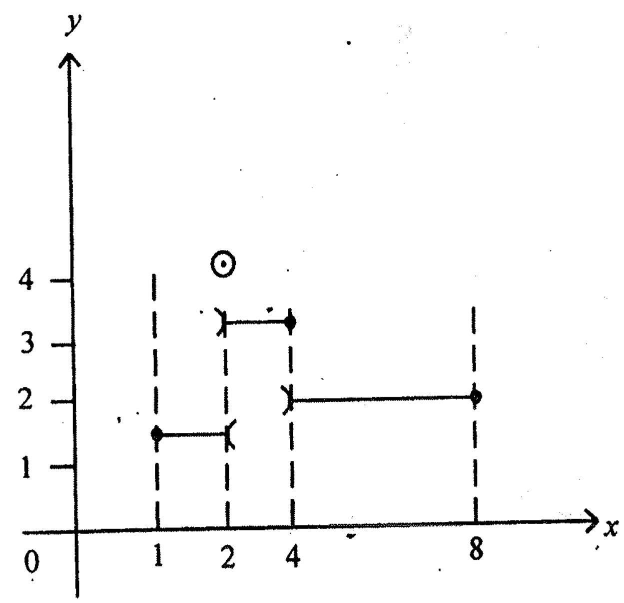
This is true for any partitiion P i.e. for any n however large.

Taking $n \to \infty$ we have $V_P(f,a,b) \to \infty$. Hence $\sup_{P \in \wp} V_P(f,a,b)$ does not exist i.e. f is not of bounded variation on $[a, b]$.

1.4.4. Example. The function $f(x)$ defined on $[1, 8]$ by

is of bounded variation over $[1, 8]$

$$f(x) = \begin{aligned} &1, \quad 1 \le x < 2 \\ &= 4, \quad x = 2 \\ &= 3, \quad 2 < x \le 4 \\ &= 2, \quad 4 < x \le 8 \end{aligned}$$



Solution.

Let $P = \{1 = x_0, x_1, x_2, ..., x_n = 8\}$ be any partition of $[1, 8]$.

Now four cases may arise.

Case 1. P does not contain 2 and 4

Case 2. P contains 2 but does not contain 4

Case 3. P contains 4 but does not contain 2

Case 4. P contains both 2 and 4

Case 1. Let $x_{i-1} < 2 < x_i$ and $x_{j-1} < 4 < x_j$

Hence $P = \{1 = x_0, x_1, x_2, ..., x_{i-1}, x_i, x_{i+1}, ..., x_{j-1}, x_j, x_{j+1}, ..., x_n = 8\}$

Now $V_P(f, 1, 8)$

$$= \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

$$= \sum_{k=1}^{n} |\Delta f_k|$$

$$= \sum_{k=1}^{i-1} |\Delta f_k| + |\Delta f_i| + \sum_{k=i+1}^{j-1} |\Delta f_k| + |\Delta f_j| + \sum_{k=j+1}^{n} |\Delta f_k|$$

$$= \sum_{k=1}^{i-1} |1-1| + |3-1| + \sum_{k=i+1}^{j-1} |3-3| + |2-3| + \sum_{k=j+1}^{n} |2-2|$$

$$= 0 + 2 + 0 + 1 + 0$$

$$= 3.$$

Case 2. Let $x_{i-1} < 2 = x_i$ and $x_{j-1} < 4 < x_j$

Here $V_P(f, 1, 8)$

$$= \sum_{k=1}^{n} |\Delta f_k|$$

$$= \sum_{k=1}^{i-1} |\Delta f_k| + |\Delta f_i| + |\Delta f_{i+1}| + \sum_{k=i+2}^{j-1} |\Delta f_k| + |\Delta f_j| + \sum_{k=j+1}^{n} |\Delta f_k|$$

$$= \sum_{k=1}^{i-1} |1-1| + |4-1| + |3-4| + \sum_{k=i+2}^{j-1} |3-3| + |2-3| + \sum_{k=j+k}^{n} |2-2|$$

$$= 0 + 3 + 1 + 0 + 1 + 0$$

$$= 5$$

Case 3. Let $x_{i-1} < 2 < x_i$ and $x_{j-1} < 4 = x_j$

Here $V_P(f,1,8)$

$$= \sum_{k=1}^{i-1} |\Delta f_k| + |\Delta f_i| + \sum_{k=i+1}^{j-1} |\Delta f_k| + |\Delta f_j| + |\Delta f_{j+1}| + \sum_{k=j+2}^{n} |\Delta f_k|$$

$$= \sum_{k=1}^{i-1} |1-1| + |3-1| + \sum_{k=i+1}^{j-1} |3-3| + |3-3| + |2-3| + \sum_{k=j+2}^{n} |2-2|$$

$$= 0 + 2 + 0 + 0 + 1 + 0$$

$$= 3$$

Case 4. Let $x_{i-1} < 2 = x_i$ and $x_{j-1} < 4 = x_j$

Here $V_P(f,1,8)$

$$= \sum_{k=1}^{i-1} |\Delta f_k| + |\Delta f_i| + |\Delta f_{i+1}| + \sum_{k=i+2}^{j-1} |\Delta f_k| + |\Delta f_j| + |\Delta f_{j+1}| + \sum_{k=j+2}^{n} |\Delta f_k|$$

$$= \sum_{k=1}^{i-1} |1-1| + |4-1| + |3-4| + \sum_{k=i+2}^{i-1} |3-3| + |3-3| + |2-3| + \sum_{k=j+2}^{n} |2-2|$$

$$= 0 + 3 + 1 + 0 + 0 + 1 + 0$$

$$= 5$$

Hence $\sup_{P \in \wp} V_P(f,1,8)$ exists and is 5.

Thus f is of bounded variation on [1, 8] and the total variation of f on [1, 8] is 5.

1.5. Some Theorems and results

1.5.1. Theorem. A function of bounded variation on $[a, b]$ is bounded on $[a, b]$.

Proof. Let $f(x)$ be of bounded variation on $[a, b]$. Then $\sup_{P \in \wp} V_P(f,a,b)$ exists.

Let thus suprëmum be M.

$\therefore V_P(f,a,b) \le M$ for all partition P of $[a, b]$ .... (1)

Let us consider the partition $P_x\{a,x,b\}$ where $a \le x \le b$.

Now $V_{P_x}(f,a,b) = |f(x)-f(a)| + |f(b)-f(x)|$

$\therefore$ Using (1) we have

$$|f(x)-f(a)| + |f(b)-f(x)| \le M$$

$$\Rightarrow |f(x)-f(a)| \le M$$

This is true for any x in $[a, b]$.

$\therefore$ For any x in $[a, b]$ we have

$$|f(x)|$$
$$\le |f(x)-f(a)| + |f(a)|$$
$$\le M + |f(a)|$$

Hence $f(x)$ is bounded over $[a, b]$.

1.5.2. Theorem. If a function $f(x)$ is unbounded over $[a, b]$, then it cannot be of bounded variation over $[a, b]$.

Proof. Here $f(x)$ is unbounded over $[a, b]$. So there exists $\beta \in (a,b)$ such that $|f(x)| \to \infty$ as $x \to \beta^-$ or as $x \to \beta^+$.

Let $|f(x)| \to \infty$ as $x \to \beta^-$.

We consider the partition P of $[a, b]$ where

$P = \{a = x_0, x_1, x_2, \ldots, x_n, b\}$ and $x_n \to \beta$ as $n \to \infty$ from left.

Now as $n \to \infty$ we have $x_n \to \beta$ from left and as $x_n \to \beta$ from left we have $|f(x_n)| \to \infty$.

We have

$$V_P(f,a,b) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| + |f(b) - f(x_n)| \quad \text{............... (1)}$$

Now $|f(x_n)| = |f(x_n) - f(b) + f(b)| \le |f(x_n) - f(b)| + |f(b)|$

or, $|f(x_n) - f(b)| \ge |f(x_n)| - |f(b)|$

or, $|f(b) - f(x_n)| \ge |f(x_n)| - |f(b)|$

Hence from (1)

$$V_P(f,a,b) \ge \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| + |f(x_n)| - |f(b)| \qquad \text{.............. (2)}$$

Since $|f(x_n)| \to \infty$ as $n \to \infty$ it follows from (2) that $V_P(f,a,b) \to \infty$ as $n \to \infty$. Hence f is not of bounded variation on $[a, b]$.

Similar is the proof when $|f(x)| \to \infty$ as $x \to \beta^{+}$.

1.5.3. Theorem. Prove that a bounded monotonic function is a function of bounded variation.

Proof. Let f be monotonic non-decreasing function over $[a, b]$.

Let $P = \{a = x_0, x_1, x_2, ..., x_n, b\}$ be any partition of $[a, b]$.

Then we have $f(x_k) \ge f(x_{k-1})$ for all $k = 1, 2, ..., n$ as $x_k \ge x_{k-1}$ for all $k = 1, 2, ..., n$.

Now

$$V_P(f,a,b) = \sum_{k=1}^{n} |\Delta f_k|$$

$$= \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

$$= \sum_{k=1}^{n} \{f(x_k) - f(x_{k-1})\} \quad [\because f(x_k) \ge f(x_{k-1}) \text{ for all } k = 1,2,...,n]$$

$$= f(x_1) - f(x_0)$$

$$+ f(x_2) - f(x_1)$$

$$+ f(x_3) - f(x_2)$$

....

....

$$+f(x_n)-f(x_{n-1})$$
$$=f(x_n)-f(x_0)$$
$$=f(b)-f(a)$$

We see that the RHS is independent of the partition P.

Hence $\sup_P V_P(f,a,b)$ exists and is $f(b)-f(a)$.

$\therefore f$ is of bounded variation on $[a, b]$ and the total variation of f on $[a, b]$ is $f(b)-f(a)$.

Similarly, when f is monotonic non increasing function we can show that f is of bounded variation on $[a, b]$. In this case the total variation of f on $[a, b]$ is $f(a)-f(b)$.

The converse of the previous theorem is not true i.e. a function of bounded variation on $[a, b]$ may or may not be monotonic on $[a, b]$ e.g. $f(x)=3x^2+2x+5, 3\le x\le 8$ is of bounded variation on [3, 8] and is monotonic increasing [Example 1.4.2.]. But the function

$$f(x)=1,\quad 1\le x<2$$
$$=4,\quad x=2$$
$$=3,\quad 2<x\le 4$$
$$=2,\quad 4<x\le 8$$

is of bounded variation on [1, 8] but is not mononic on [1, 8] [Example 1.4.4]

In theorem 1.5.3. we have seen that every monotonic function is of bounded variation. As continuous property of a function plays an important role in analysis similar question arises whether every continuous function is of bounded variation or not.

The answer to this question is not in the affirmative.

Obviously, if a function is continuous and monotonic in $[a, b]$ then it is of bounded variation on $[a, b]$. But the following example is continuous but not of bounded variation on $[a, b]$. So in general a continuous function may or may not be of bounded variation.

1.5.4. Example. The function

$$f(x)=x\sin\frac{\pi}{x}, x\neq 0$$
$$=0 \qquad , x=0$$

is continuous on [0, 1] but not of bounded variation there.

Solution. We have $|f(x)=f(0)|=\left|x\sin\frac{\pi}{x}-0\right|=|x|\left|\sin\frac{\pi}{x}\right|\le|x|$ $\qquad\left[\because\left|\sin\frac{\pi}{x}\right|\le 1\right]$

Thus $|f(x)-f(0)|<\varepsilon$ whenever $|x|<\varepsilon$

$\therefore$ For given $\varepsilon>0$ there exists $\delta=\varepsilon$ such that

$$|f(x)-f(0)|<\varepsilon \text{ whenever } |x-0|<\delta.$$

So $f(x)$ is continuous at $x=0$. Obviously $f(x)$ is continuous at every $x\neq 0$. Hence $f(x)$ is continuous on [0, 1].

We consider the partititon P_n of [0, 1] as

$$P_n=\left\{0,\frac{2}{2n+1},\frac{2}{2n-1},\frac{2}{2n-3},\ldots\ldots,\frac{2}{7},\frac{2}{5},\frac{2}{3},1\right\}.$$

Now $V_{P_n}(f,0,1)$

$$=\left|f\left(\frac{2}{2n+1}\right)-f(0)\right|+\left|f\left(\frac{2}{2n-1}\right)-f\left(\frac{2}{2n+1}\right)\right|+\left|f\left(\frac{2}{2n-3}\right)-f\left(\frac{2}{2n-1}\right)\right|+\ldots$$

$$\ldots\ldots+\left|f\left(\frac{2}{5}\right)-f\left(\frac{2}{7}\right)\right|+\left|f\left(\frac{2}{3}\right)-f\left(\frac{2}{5}\right)\right|+\left|f(1)-f\left(\frac{2}{3}\right)\right|$$

We have $f\left(\frac{2}{2n+1}\right)=\frac{2}{2n+1}\sin\frac{(2n+1)\pi}{2}=\frac{2}{2n+1}\sin\left(n\pi+\frac{\pi}{2}\right)$

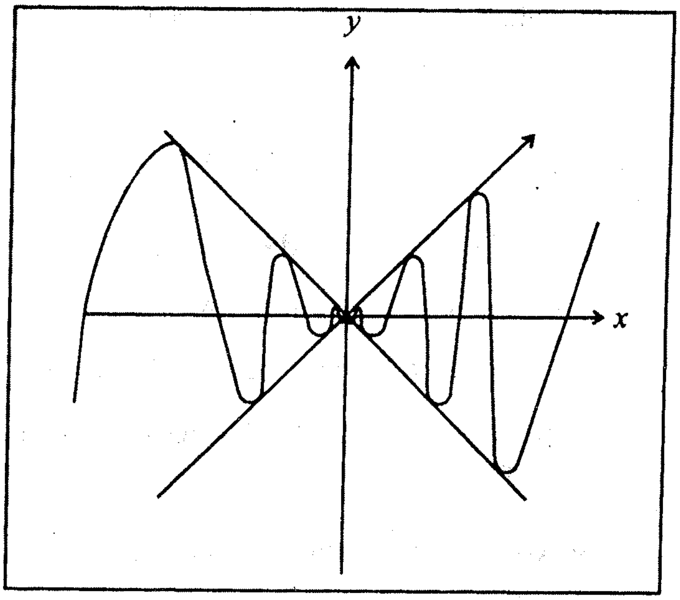
$$=\frac{2}{2n+1}(-1)^n$$

$$\therefore V_{P_n}(f,0,1)=\left|\frac{2}{2n+1}(-1)^n-0\right|+\left|\frac{2}{2n-1}(-1)^{n-1}-\frac{2}{2n+1}(-1)^n\right|+\left|\frac{2}{2n-3}(-1)^{n-2}-\frac{2}{2n-1}(-1)^{n-1}\right|+\ldots$$

$$\ldots+\left|\frac{2}{5}(-1)^2-\frac{2}{7}(-1)^3\right|+\left|\frac{2}{3}(-1)^1-\frac{2}{5}(-1)^2\right|+\left|0-\frac{2}{3}(-1)^1\right|$$

$$=\frac{2}{2n+1}+\frac{2}{2n-1}+\frac{2}{2n+1}+\frac{2}{2n-3}+\frac{2}{2n-1}+\ldots+\frac{2}{5}+\frac{2}{7}+\frac{2}{3}+\frac{2}{5}+\frac{2}{3}$$

$$=4\left(\frac{1}{2n+1}+\frac{1}{2n-1}+\frac{1}{2n-3}+....+\frac{1}{5}+\frac{1}{3}\right).$$



We note that for each positive integer n, however large, P_n is a partition of [0, 1]. As the series $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+....+\frac{1}{2n-1}+\frac{1}{2n+1}+....$ is s divergent series, it follows that $V_{P_n}(f,0,1)\to\infty$ as $n\to\infty$.

Hence $\sup_P V_P(f,0,1)$ does not exist i.e. f is not of bounded variation on [0, 1].

The above example shows that a continuous function may not be of bounded variation. The following example shows that a discontinuous function may also be not of bounded variation.

1.5.5. Example. The function

$$f(x)=\sin\frac{\pi}{x},\ x\neq 0$$
$$=0\quad,\ x=0$$

is discontinuous at $x=0$ and not of bounded variation on [0, 1].

Solution. Here $\lim_{x\to 0} f(x)$ does not exists. Hence $f(x)$ is not continuous at $x=0$.

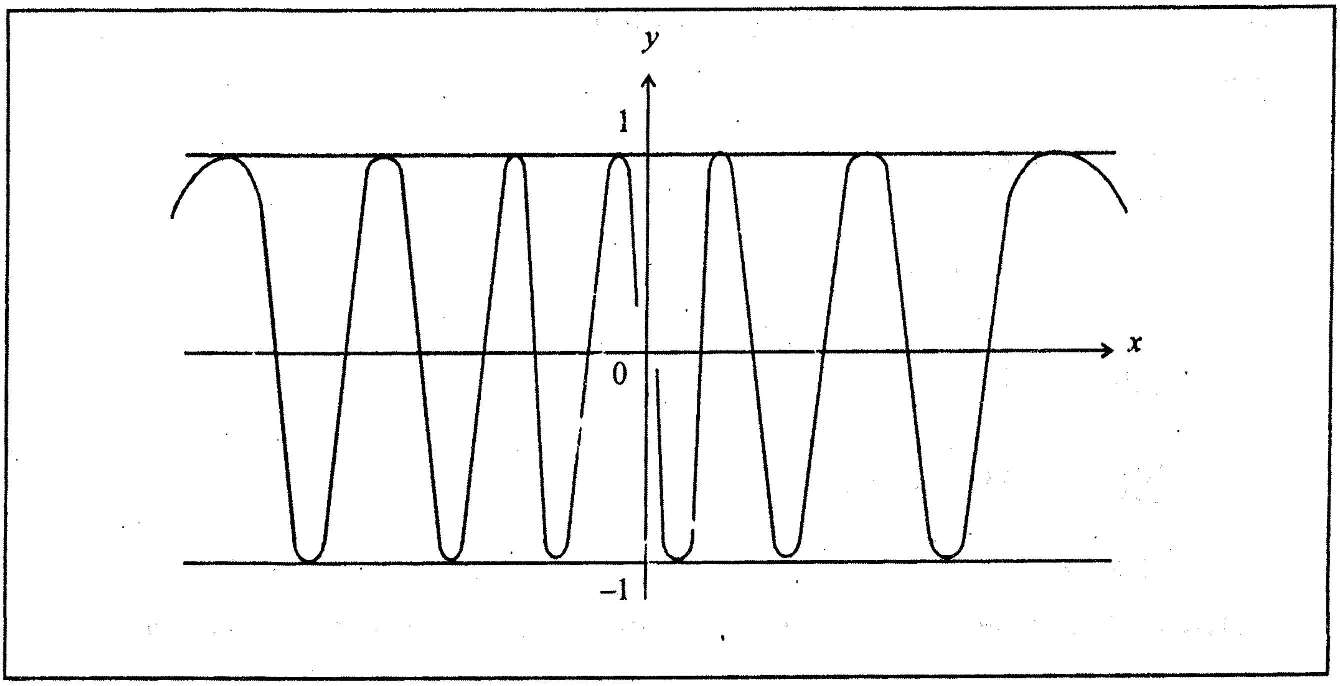
We consider the partition P_n of [0, 1] as

$$P_n=\left\{0,\frac{2}{2n+1},\frac{2}{2n-1},\frac{2}{2n-2},......,\frac{2}{7},\frac{2}{5},\frac{2}{3},1\right\}$$

Now $V_P(f,0,1)$

$$=\left|f\left(\frac{2}{2n+1}\right)-f(0)\right|+\left|f\left(\frac{2}{2n-1}\right)-f\left(\frac{2}{2n+1}\right)\right|+\left|f\left(\frac{2}{2n-3}\right)-f\left(\frac{2}{2n-1}\right)\right|+....$$

$$....+\left|f\left(\frac{2}{5}\right)-f\left(\frac{2}{7}\right)\right|+\left|f\left(\frac{2}{3}\right)-f\left(\frac{2}{5}\right)\right|+\left|f(1)-f\left(\frac{2}{3}\right)\right|$$



$$= \left|(-1)^n - 0\right| + \left|(-1)^{n-1} - (-1)^n\right| + \left|(-1)^{n-2} - (-1)^{n-1}\right| + ...$$

$$..... + \left|(-1)^2 - (-1)^3\right| + \left|(-1) - (-1)^2\right| + \left|0 - (-1)\right|$$

$= 1 + \{ 2 + 2 + ... + 2\} + 1$

$= 1 + 2(n - 1) + 1$

$= 2n$

Thus $V_{P_n}(f, 0, 1) \to \infty$ as $n \to \infty$. This shows that f is not of bounded variation on [0, 1].

The following theorem gives the sufficient condition for a function to be of bounded variation.

1.5.6. Theorem. If the deviative $\dfrac{df(x)}{dx}$ exists and is bounded on $[a, b]$, then the function $f(x)$ is of bounded variation on $[a, b]$.

Proof. Here $\dfrac{df(x)}{dx}$ exists and is bounded on the interval $[a, b]$.

Therefore, there exists M such that

$$|f'(x)| \le M \text{ for all } x \in [a, b] \qquad \text{..................(1)}$$

Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$.

Now $V_P(f, a, b)$

$$= \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

$$= \sum_{k=1}^{n} |(x_k - x_{k-1}) f'(\xi_k)| \text{ where } \xi_k$$

$$= \sum_{k=1}^{n} (x_k - x_{k-1}) |f'(\xi_k)|$$

$$\leq \sum_{k=1}^{n} (x_k - x_{k-1}) M \quad \text{[by (1)]}$$

$$= M(b-a)$$

This is true for any partition P of $[a, b]$. Since the RHS is independent of P we see that $\sup_P V_P(f, a, b)$ exists.

Hence f is of bounded variation on $[a, b]$.

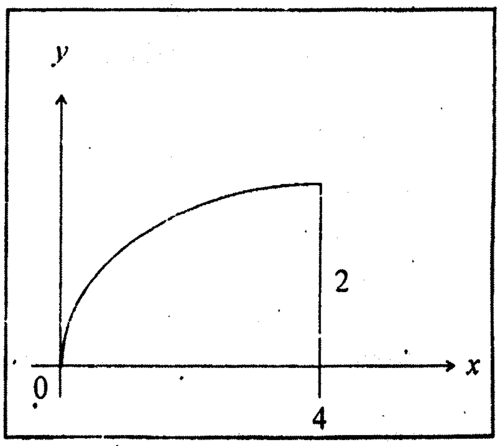
It is important to note that boundedness of $f'(x)$ is not necessary for $f(x)$ to be of bounded variation. This can be shown by the following example.

1.5.7 Example. Let $f(x) = x^{1/2}, 0 \leq x \leq 4$. Then $f(x)$ is of bounded variation on $[0, 4]$ though $f'(x)$ is unbounded on $[0,4]$.

Solution.

$f(x) = x^{1/2}$

$\therefore f'(x) = \frac{1}{2} x^{-1/2}$. So for $x > 0$ we have $f'(x) > 0$.



This shows that $f(x)$ is monotonic increasing on $[0, 4]$ and so $f(x)$ is of bounded variation on $[0, 4]$.

We note here that

$$\lim_{x \to 0+} f'(x) = \lim_{x \to 0+} \frac{1}{2} x^{-1/2} = +\infty$$

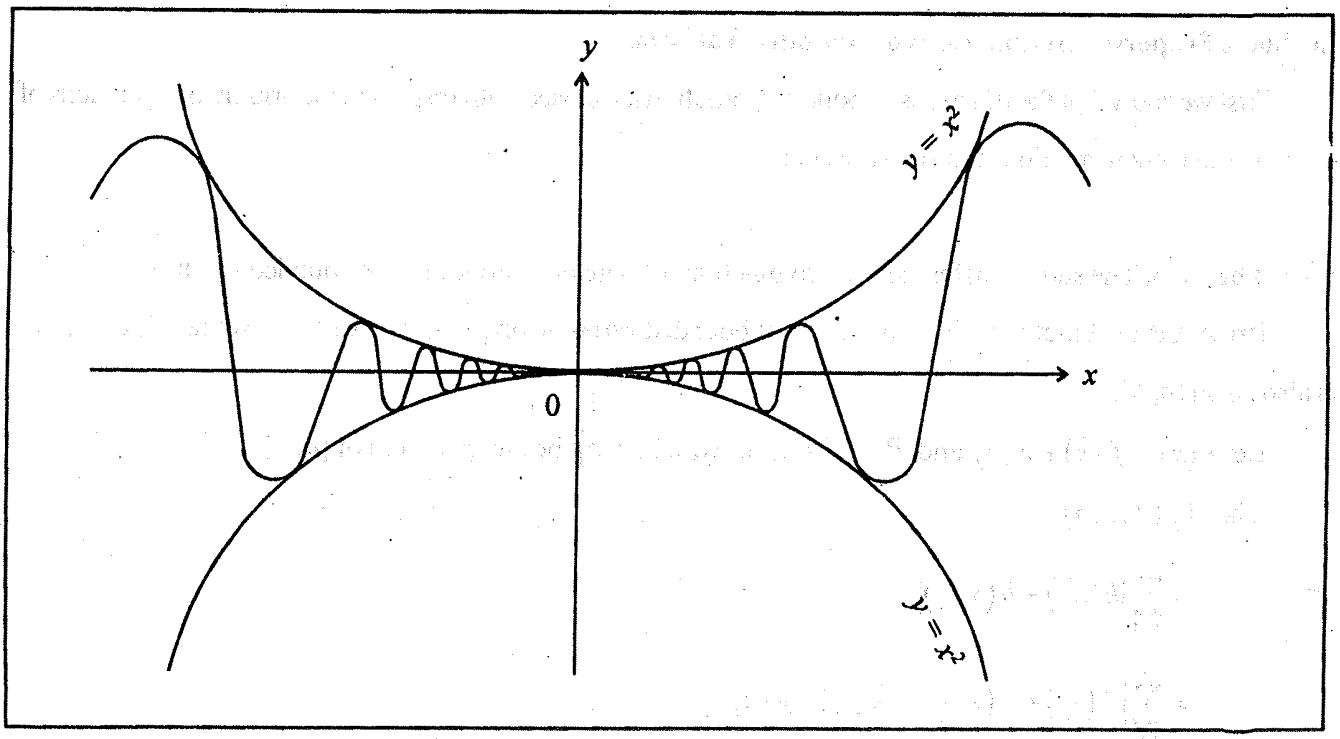
$\therefore f'(x)$ is unbounded at $x = 0$.

Note : Theorem 1.5.6. helps us to establish bounded variation nature of functions.

1.5.8. Example. Let $f(x) = x^2 \sin\frac{1}{x}, x \neq 0$ and $f(0)=0$. Then $f(x)$ is of bounded variation on $[0, 1]$.

Solution. Here $f(x) = x^2 \sin\frac{1}{x}, \quad x \neq 0$

$= 0 \quad , x = 0.$



We can easily show that $f'(0) = 0$. For $x \neq 0$ we have

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

$$\therefore |f'(x)| = \left|2x\sin\frac{1}{x} - \cos\frac{1}{x}\right|$$

$$\le \left|2x\sin\frac{1}{x}\right| + \left|\cos\frac{1}{x}\right|$$

$$\le 2|x|\left|\sin\frac{1}{x}\right| + \left|\cos\frac{1}{x}\right|$$

$$\le 2\cdot 1\cdot 1 + 1$$

$$= 3$$

Thus $|f'(x)| \le 3$ for all $x \in [0,1]$.

Hence by theorem 1.5.6., $f(x)$ is of bounded variation on [0, 1].

1.6. Some Properties of Functions of Bounded Variation.

First we show that the functions of bounded variation are closed with respect to the arithmetic operations of addition, subtraction, multiplication and division.

1.6.1. Theorem. The sum or difference of two functions of bounded variation is of bounded variation.

Proof. Let the functions $f(x)$ and $g(x)$ be of bounded variation on $[a, b]$. We are to show that $f(x)+g(x)$ is also so on $[a, b]$.

Let $h(x) = f(x)+g(x)$ and $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$.

Then $V_P(f,a,b)$

$$= \sum_{k=1}^{n} |h(x_k) - h(x_{k-1})|$$

$$= \sum_{k=1}^{n} |f(x_k) + g(x_k) - f(x_{k-1}) - g(x_{k-1})|$$

$$\le \sum_{k=1}^{n} \{|f(x_k) - f(x_{k-1})| + |g(x_k) - g(x_{k-1})|\}$$

$$\le \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| + \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})|$$

$$= V_P(f,a,b) + V_P(g,a,b)$$

$$\therefore V_P(h,a,b) \le V_P(f,a,b) + V_P(g,a,b) \qquad \text{..............} (1)$$

Now f and g are functions of bounded variation on $[a, b]$.

$\therefore \sup_P V_P(f,a,b)$ exists and is $V(f, a, b)$

and $\sup_P V_P(g,a,b)$ exists and is $V(g, a, b)$.

$\therefore V_P(f,a,b) \le V(f,a,b)$ and $V_P(g,a,b) \le V(g,a,b)$ ....(2)

Using (2) in (1) we get

$$V_P(h,a,b) \le V(f,a,b) + V(g,a,b)$$

The RHS is independent of P and so this result is true for all P.

Hence $\sup_P V_P(h,a,b)$ exists.

$\therefore$ h i.e. $f+g$ is of bounded variation on $[a, b]$.

In the same way we can show that $f-g$ is also of bounded variation on $[a, b]$.

1.6.2. Theorem. The product of two functions of bounded variation is of bounded variation

Proof. Let $f(x)$ and $g(x)$ be functions of bounded variation on $[a, b]$ and $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$.

We denote the product function $f(x)\,g(x)$ by $h(x)$.

We know that functions of bounded variation are always bounded. Hence there exists constant K such that

$|f(x)| \le K$ and $|g(x)| \le K$ for all $x \in [a,b]$. .... (1)

Now $V_P(h,a,b)$

$$= \sum_{k=1}^{n} |h(x_k) - h(x_{k-1})|$$

$$= \sum_{k=1}^{n} |f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1})|$$

$$= \sum_{k=1}^{n} |f(x_k)g(x_k) - f(x_k)g(x_{k-1}) + f(x_k)g(x_{k-1}) - f(x_{k-1})g(x_{k-1})|$$

$$= \sum_{k=1}^{n} |f(x_k)\{g(x_k) - g(x_{k-1})\} + g(x_{k-1})\{f(x_k) - f(x_{k-1})\}|$$

$$\le \sum_{k=1}^{n} \{|f(x_k)||g(x_k) - g(x_{k-1})| + |g(x_{k-1})||f(x_k) - f(x_{k-1})|\}$$

$$\leq \sum_{k=1}^{n}\{K|g(x_k)-g(x_{k-1})|+K|f(x_k)-f(x_{k-1})|\}, \text{ [by (1)]}$$

$$= K\left\{\sum_{k=1}^{n}|g(x_k)-g(x_{k-1})|+\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|\right\}$$

$$= K\{V_P(g,a,b)+V_P(f,a,b)\}$$

$$\leq K\left\{\sup_P V_P(g,a,b)+\sup_P V_P(f,a,b)\right\} \text{ [as } f \text{ and } g \text{ are functions of } bv\text{, both sup exist]}$$

$$= K\{V(g,a,b)+V(f,a,b)\}$$

This is true for all partition *P*. As the RHS is independent of *P*, it follows that $\sup_P V_P(h,a,b)$ exists. Hence *h* i.e. *fg* is of bounded variation on $[a, b]$.

1.6.3. Theorem. If *f* and *g* are functions of bounded variation on $[a, b]$ and $|g(x)| \geq \sigma > 0$ for all $x \in [a,b]$, then f/g is of bounded variation on $[a, b]$.

Proof. Let $f(x)$ and $g(x)$ be functions of bounded variation on $[a, b]$ and $P=\{a=x_0,x_1,x_2,...,x_n=b\}$ be any partition of $[a,b]$. We denote the quotient $f(x)/g(x)$ by $h(x)$.

We know that function of bounded variation are always bounded.

Hence there exists constant *M* such that

$|f(x)| \leq M$ and $|g(x)| \leq M$ for all $x \in [a,b]$ .... (1)

Also we have $|g(x)| \geq \sigma > 0$ for all $x \in [a,b]$ .... (2)

Now $V_P(h,a,b)$

$$= \sum_{k=1}^{n}|h(x_k)-h(x_{k-1})|$$

$$= \sum_{k=1}^{n}\left|\frac{f(x_k)}{g(x_k)}-\frac{f(x_{k-1})}{g(x_{k-1})}\right|$$

$$= \sum_{k=1}^{n}\left|\frac{f(x_k)g(x_{k-1})-g(x_k)f(x_{k-1})}{g(x_k)g(x_{k-1})}\right|$$

$$=\sum_{k=1}^{n}\frac{|f(x_k)g(x_{k-1})-g(x_k)f(x_{k-1})|}{|g(x_k)||g(x_{k-1})|}$$

$$\le\sum_{k=1}^{n}\frac{1}{\sigma^2}|f(x_k)g(x_{k-1})-f(x_{k-1})g(x_{k-1})+f(x_{k-1})g(x_{k-1})-g(x_k)f(x_{k-1})|, \text{[by (2)]}$$

$$=\sum_{k=1}^{n}\frac{1}{\sigma^2}|g(x_{k-1})\{f(x_k)-f(x_{k-1})\}-f(x_{k-1})\{g(x_k)-g(x_{k-1})\}|$$

$$\le\sum_{k=1}^{n}\frac{1}{\sigma^2}\{|g(x_{k-1})||f(x_k)-f(x_{k-1})|+|f(x_{k-1})||g(x_k)-g(x_{k-1})|\}$$

$$\le\sum_{k=1}^{n}\frac{1}{\sigma^2}\{M|f(x_k)-f(x_{k-1})|+M|g(x_k)-g(x_{k-1})|\}, \text{[by (1)]}$$

$$=\frac{M}{\sigma^2}\left\{\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|+\sum_{k=1}^{n}|g(x_k)-g(x_{k-1})|\right\}$$

$$=\frac{M}{\sigma^2}\{V_P(f,a,b)+V_P(g,a,b)\}$$

or, $V_P(h,a,b)\le\frac{M}{\sigma^2}\{V_P(f,a,b)+V_P(g,a,b)\}$ .... (3)

Since f and g are functions of bounded variation

$\sup_P V_P(f,a,b)$ exists and is $V(f, a, b)$

and $\sup_P V_P(g,a,b)$ exists and is $V(g, a, b)$.

From (3) we have thus $V_P(h,a,b)\le\frac{M}{\sigma^2}\{V(f,a,b)+V(g,a,b)\}$

This is true for all partitions P. Since the RHS is independent of P we see that $\sup_P(h,a,b)$ exists. Hence h

i.e. f/g is of bounded variation on $[a, b]$.

1.6.4. Theorem. If a function f is of bounded variation on $[a, b]$ then it is also of bounded variation on $[a, c]$ and $[c, b]$ where $a<c<b$ and conversely. Also $V(f,a,b)=V(f,a,c)+V(f,c,b)$.

Proof. First we assume that *f* is of bounded variation on $[a, b]$ and $a < c < b$. We are to show that *f* is of bounded variation on $[a, c]$ and on $[c, b]$.

Let $P_1 = \{a = x_0, x_1, ..., x_m = c\}$ be any partition of $[a, c]$ and $P_2 = \{c = y_0, y_1, y_2, ..., y_n = b\}$ be any partition of $[c, b]$.

Then $V_{P_1}(f, a, c) = \sum_{k=1}^{m} |f(x_k) - f(x_{k-1})|$ .... (1)

and $V_{P_2}(f, c, b) = \sum_{r=1}^{n} |f(y_r) - f(y_{r-1})|$ .... (2)

Let $P = P_1 \cup P_2$ i.e. $P = \{a = x_0, x_1, ..., x_m = c = y_0, y_2, ..., y_n = b\}$

Then P is a partition of $[a, b]$

Now $V_P(f, a, b)$

$$= \sum_{k=1}^{m} |f(x_k) - f(x_{k-1})| + \sum_{r=1}^{n} |f(y_r) - f(y_{r-1})|$$

$= V_{P_1}(f, a, c) + V_{P_2}(f, c, b)$ [by (1) and (2)] .... (3)

Since *f* is of bounded variation on $[a, b]$, $\sup_P V_P(f, a, b)$ exists and is $V(f, a, b)$. Thus from (3) we have

$V_{P_1}(f, a, c) + V_{P_2}(f, c, b) \le V(f, a, b)$ .... (4)

$\therefore V_{P_1}(f, a, c) \le V(f, a, b)$

and $V_{P_2}(f, c, b) \le V(f, a, b)$

These are true for any partition P_1 of $[a, c]$ and P_2 of $[c, b]$. Since the RHS is independent of P_1 and P_2 it follows that $\sup_{P_1} V_{P_1}(f, a, c)$ and $\sup_{P_2} V_{P_2}(f, c, b)$ exist.

This shows that *f* is of bounded variation on $[a, c]$ and on $[c, b]$.

Conversely, let *f* be of bounded variation on $[a, c]$ and on $[c, b]$ where $a < c < b$.

Let $P = \{a = z_0, z_1, z_2, ..., z_{r-1}, z_r, ..., z_n = b\}$ be any partition of $[a, b]$ and $z_{r-1} < c \le z_r$.

Now $V_P(f, a, b) = \sum_{k=1}^{r-1} |f(z_k) - f(z_{k-1})| + |f(z_r) - f(z_{r-1})| + \sum_{k=r+1}^{n} |f(z_k) - f(z_{k-1})|$

$$\leq \left\{ \sum_{k=1}^{r-1} |f(z_k) - f(z_{k-1})| + |f(c) - f(z_{r-1})| \right\} + \left\{ |f(z_r) - f(c)| + \sum_{k=r+1}^{n} |f(z_k) - f(z_{k-1})| \right\}$$

$$\left[\because |f(z_r) - f(z_{r-1})| = |f(z_r) - f(c) + f(c) - f(z_{r-1})| \leq |f(z_r) - f(c)| + |f(c) - f(z_{r-1})| \right]$$

$$= V_{P_1'}(f,a,c) + V_{P_2'}(f,c,b)$$

where $P_1' = \{a, z_0, z_1, z_2 ..., z_{r-1}, c\}$ and $P_2' = \{c, z_r, z_{r+1}, ..., z_n = b\}$.

$\therefore$ We have $V_P(f,a,b) \leq V_{P_1'}(f,a,c) + V_{P_2'}(f,c,b)$ .... (5)

Since f is of bounded variation on $[a, c]$ and on $[c, b]$ we note that $V(f, a, c)$ and $V(f, c, b)$ exists and $V_{P_1'}(f,a,c) \leq V(f,a,c)$ and $V_{P_2'}(f,c,b) \leq V(f,c,b)$. From (5) we thus have

$$V_P(f,a,b) \leq V(f,a,c) + V(f,c,b) \qquad \text{.................. (6)}$$

This is true for all P of $[a, b]$. Since the RHS is independent of P is follows that $\sup_P V_P(f,a,b)$ exists and this sup is $V(f, a, b)$.

Hence f is of bounded variation on $[a, b]$.

From (6) we have

$$V(f,a,b) \leq V(f,a,c) + V(f,c,b) \qquad \text{.................. (7)}$$

Again (4) is true for any partition P_1 of $[a, c]$ and P_2 of $[c, b]$.

Since $\sup_{P_1} V_{P_1}(f,a,c)$ and $\sup_{P_2} V_{P_2}(f,c,b)$ exist it follows from (4) that

$$\sup_{P_1} V_{P_1}(f,a,c) + V_{P_2}(f,c,b) \leq V(f,a,b)$$

or, $V(f,a,c) + V_{P_2}(f,c,b) \leq V(f,a,b)$

or, $V(f,a,c) + \sup_{P_2} V_{P_2}(f,c,b) \leq V(f,a,b)$

or, $V(f,a,c) + V(f,c,b) \leq V(f,a,b)$ .... (8)

From (7) and (8) we have finally

$$V(f,a,b) = V(f,a,c) + V(f,c,b).$$

1.7. Variation Function

1.7.1. Definition. Variation Function

Let $f(x)$ be a function of bounded variation on $[a, b]$. For any x of $[a, b]$, the total variation of f on $[a, x]$ is a function of x i.e. $V(f, a, x)$ is a function of x. This function is called the total variation function or simply the variation function of f and is denoted by $v_f(x)$ or simply by $v(x)$.

1.7.2. Theorem. The variation function $v(x)$ of the function $f(x)$ of bounded variation on $[a, b]$ is monotone increasing on $[a, b]$.

Proof. For x_1, x_2 of $[a, b]$ with $x_2 > x_1$ we have

$$\begin{aligned}
&v(x_2) - v(x_1)\\
&= V(f, a, x_2) - V(f, a, x_1)\\
&= \{V(f, a, x_1) + V(f, x_1, x_2)\} - V(f, a, x_1) \quad [\because a \le x_1 < x_2 \le b]\\
&= V(f, x_1, x_2)\\
&\ge 0
\end{aligned}$$

$\therefore v(x_2) \ge v(x_1)$ whenever $a \le x_1 < x_2 \le b$.

This shows that v is monotone increasing on $[a, b]$.

It is interesting to note that $f(x)$ may be monotone increasing or monotone decreasing or neither monotone increasing or decreasing but $v(x)$ is always monotone increasing.

This is because as x increases from x_1 to x_2 the positive term $V(f, x, x_2)$ is added with $v(x_1)$ to get $v(x_2)$.

1.7.3. Theorem. A function is of bounded variation on an interval if and only if it can be expressed as the difference of two monotone increasing functions.

Proof. Let $f(x)$ be function of bounded variation on $[a, b]$ and $v(x)$ be the variation function of $f(x)$ on $[a, b]$.

Let us define

$$p(x)=\frac{1}{2}\{v(x)+f(x)\} \qquad \text{.................. (1)}$$

and $$q(x)=\frac{1}{2}\{v(x)+f(x)\}. \qquad \text{.................. (2)}$$

We now show that $p(x)$ and $q(x)$ are monotone increasing on $[a, b]$.

Let $x_1, x_2 \in [a,b]$ and $x_2 > x_1$. Then we have

$$p(x_2)-p(x_1)$$

$$=\frac{1}{2}\{v(x_2)+f(x_2)\}-\frac{1}{2}\{v(x_1)+f(x_1)\}$$

$$=\frac{1}{2}\{v(x_2)-v(x_1)\}+\frac{1}{2}\{f(x_2)-f(x_1)\}$$

$$=\frac{1}{2}\{V(f,a,x_2)-V(f,a,x_1)\}+\frac{1}{2}\{f(x_2)-f(x_1)\}$$

$$=\frac{1}{2}\{V(f,a,x_1)+V(f,x_1,x_2)-V(f,a,x_1)\}+\frac{1}{2}\{f(x_2)-f(x_1)\}$$

$$=\frac{1}{2}V(f,x_1,x_2)-\frac{1}{2}\{f(x_1)-f(x_2)\} \qquad \text{.................. (3)}$$

Now $V(f,x,x_2)=\sup\limits_{P} V_P(f,x_1,x_2)$ where P is any partition of $[x_1,x_2]$. As $P'=\{x_1,x_2\}$ is a partition of $[x_1,x_2]$ we have

$$V(f,x_1,x_2)\geq V_{P'}(f,x_1,x_2)=|f(x_1)-f(x_2)| \qquad \text{.................. (4)}$$

$$\therefore V(f,x_1,x_2)\geq |f(x_1)-f(x_2)|\geq f(x_1)-f(x_2)$$

Using this result in (3) we have $p(x_2)-p(x_1)\geq 0$ i.e. $p(x_2)\geq p(x_1)$.

This proves that $p(x)$ is monotone increasing on $[a, b]$.

Similarly, we have

$$q(x_2)-q(x_1)$$

$$=\frac{1}{2}\{v(x_2)-f(x_2)\}-\frac{1}{2}\{v(x_1)-f(x_1)\}$$

$$=\frac{1}{2}V(f,x_1,x_2)-\frac{1}{2}\{f(x_2)-f(x_1)\}$$

From (4) we have $V(f,x_1,x_2)\geq|f(x_1)-f(x_2)|\geq f(x_2)-f(x_1)$.

Using this result it follows that $q(x_2)\geq q(x_1)$. Hence $q(x)$ is also monotone increasing on $[a, b]$.

Thus $p(x)$ and $q(x)$ defined by (1) and (2) are monotone increasing on $[a, b]$.

From (1) and (2) we have

$$f(x)=p(x)-q(x)$$

and $\quad v(x)=p(x)-q(x)$

This proves that $f(x)$ can be expressed as difference of two montone increasing functions.

Conversely, let $f(x)$ can be expressed as difference of two monotone increasing functions i.e. $f(x)=p(x)-q(x)$ where $p(x)$ and $q(x)$ are two monotone increasing functions.

We are to prove that $f(x)$ is of bounded variation.

Let $P\{a=x_0,x_1,x_2,...,x=b\}$ be any partition of $[a, b]$.

Now $V_P(f,a,b)$

$$=\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|$$

$$=\sum_{k=1}^{n}|\{p(x_k)-q(x_k)\}-\{p(x_{k-1})-q(x_{k-1})\}|$$

$$\leq\sum_{k=1}^{n}\{|p(x_k)-p(x_{k-1})|+|q(x_k)-q(x_{k-1})|\}$$

$$=\sum_{k=1}^{n}\{p(x_k)-p(x_{k-1})+q(x_k)-q(x_{k-1})\}\quad [\because p \text{ and } q \text{ are m.i}]$$

$$=p(b)-p(a)+q(b)-q(a)$$

This is true for any P of $[a, b]$. As the RHS is independent of P it follows that $\sup\limits_{P} V_P(f,a,b)$ exists. Hence f is of bounded variation on $[a, b]$. This proves the converse part of the theorem.

The following theorem gives a very interesting continuous property of a function of bounded variation and its variation function.

1.7.4. Theorem. The variation function of a function f of bounded variation is continuous at some point if and only if f is continuous at that point.

Proof. Let $f(x)$ be a function of bounded variation on $[a, b]$ and $\upsilon(x)$ be the variation function of $f(x)$.

Let $\upsilon(x)$ be continuous at $x = c \in (a,b)$. We are to prove that $f(x)$ is continuous at $x = c$.

Since $\upsilon(x)$ is continuous at $x = c$, for given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\upsilon(x)-\upsilon(c)| < \varepsilon \text{ whenever } |x-c| < \delta. \quad \text{.......... (1)}$$

For $x>c$ we have

$$|f(x)-f(c)| \le V(f,c,x) = V(f,a,x)-V(f,a,c) = \upsilon(x)-\upsilon(c)$$

$$= |\upsilon(x)-\upsilon(c)| \quad \text{.......... (2)}$$

For $x<c$ we have

$$|f(x)-f(c)| \le V(f,x,c) = V(f,a,c)-V(f,a,x) = \upsilon(c)-\upsilon(x)$$

$$= |\upsilon(x)-\upsilon(c)| \quad \text{.......... (3)}$$

From (1), (2) and (3) it follows that

$$|f(x)-f(c)| < \varepsilon \text{ whenever } |x-c| < \delta.$$

Hence $f(x)$ is continuous at $x = c$.

Conversely, let $f(x)$ be continuous at $x = c$.

Then for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x)-f(c)| < \varepsilon/2 \text{ whenever } |x-c| < \delta$$

We are to prove that $\upsilon(x)$ is continuous at $x = c$.

We shall first show that $\upsilon(x)$ is right continuous at $x = c$.

Now $V(a, c, b)$ is the total variation of f on $[c, b]$ and is $\sup_P V_P(f,c,b)$. From definition of supremum, there exists a partition $P_1 = \{c = x_0, x_1, x_2, \ldots, x_n = b\}$ of $[c, b]$ such that

$$\sum_{k=1}^{n} |f(x_k)-f(x_{k-1})| > V(f,c,b) - \varepsilon/2 \quad \text{.......... (5)}$$

In P_1 we assume that $x_1 - c < \delta$. If it is not so, then we can easily make it so by introducing additionl points in P_1 without affecting (5).

Thus we have $0 < x_1 - c < \delta$.

$\therefore$ By (4) we have $|f(x_1) - f(c)| < \varepsilon/2$ .... (6)

Using (6) we get from (5)

$$V(f,c,b) - \varepsilon/2 < \varepsilon/2 + \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \le \varepsilon/2 + V(f,x_1,b)$$

or, $V(f,c,b) - V(f,x_1,b) < \varepsilon/2 + \varepsilon/2$

or, $V(f,c,x_1) + V(f,x_1,b) - V(f,x,b) < \varepsilon$

or, $V(f,c,x_1) < \varepsilon$

or, $V(f,a,x_1) - V(f,a,c) < \varepsilon$

or, $\upsilon(x_1) - \upsilon(c) < \varepsilon$

or, $0 < \upsilon(x_1) - \upsilon(c) < \varepsilon$ [$\therefore \upsilon(x)$ is m.i. and $x_1 > c$]

or, $-\varepsilon < 0 < \upsilon(x_1) - \upsilon(c) < \varepsilon$

Hence

$$|\upsilon(x_1) - \upsilon(c)| < \varepsilon \text{ whenever } 0 < x_1 - c < \delta$$

i.e. $|\upsilon(x) - \upsilon(c)| < \varepsilon$ whenever $c < x_1 < c + \delta$

This shows that $\upsilon(x)$ is right continuous at $x = c$.

Similarly, considering $V(f, a, c)$ we can prove that $\upsilon(x)$ is left continuous at $x = c$.

Hence $\upsilon(x)$ is continuous $x = c$.

This proves the theorem.

Note. In the above theorem c is any point of $[a, b]$. Hence the above theorem may be stated as

"The variation function of a function f of bounded variation is continuous if and only if f is a continuous".

1.7.5. Definition. Positive and Negative Variation of f on $[a, b]$.

Let f be a function of bounded variation on $[a, b]$.

For any partition $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ of $[a, b]$ let

$$V_P^+(f,a,b) = \sum_{\Delta f_k > 0} \Delta f_k \text{ and } V_P^-(f,a,b) = -\sum_{\Delta f_k < 0} \Delta f_k$$

If $\sup_P V_P^+(f,a,b)$ exists then f is said to be of positive variation on $[a, b]$. If $\sup_P V_P^-(f,a,b)$ exists then f is said to be of negative variation on $[a, b]$. These supremums are called respectively the positive total variation and negative total variation of f on $[a, b]$ and are denoted respectively by $V^+(f,a,b)$ and $V^-(f,a,b)$.

1.7.6. Theorem. If f is a function of bounded variation on $[a, b]$, then

$$V(f,a,b) = V^+(f,a,b) + V^-(f,a,b)$$

and $f(b) - f(a) = V^+(f,a,b) - V^-(f,a,b)$.

Proof. Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$.

Then $V_P(f,a,b)$

$$= \sum_{k=1}^{n} |\Delta f_k|$$

$$= \sum_{\Delta f_k > 0} \Delta f_k + \sum_{\Delta f_k = 0} \Delta f_k + \sum_{\Delta f_k < 0} (-\Delta f_k)$$

$$= V_P^+(f,a,b) + 0 + V_P^-(f,a,b)$$

$$\therefore V_P(f,a,b) = V_P^+(f,a,b) + V_P^-(f,a,b) \qquad \text{.................(1)}$$

Again $V_P^+(f,a,b) - V_P^-(f,a,b)$

$$= \left(\sum_{\Delta f_k > 0} \Delta f_k\right) - \left(-\sum_{\Delta f_k < 0} \Delta f_k\right)$$

$$= \sum_{\Delta f_k > 0} \Delta f_k + \sum_{\Delta f_k < 0} \Delta f_k$$

$$= \sum_{\Delta f_k > 0} \Delta f_k + \sum_{\Delta f_k < 0} \Delta f_k + \sum_{\Delta f_k = 0} \Delta f_k$$

$$= \sum_{k=1}^{n} \Delta f_k$$

$$= \sum_{k=1}^{n} \{f(x_k) - f(x_{k-1})\}$$

$$= f(b) - f(a)$$

or, $V_P^+(f,a,b) - V_P^-(f,a,b) = f(b) - f(a)$ .... (2)

Adding (1) and (2) we have

$$V_P^+(f,a,b) = \frac{1}{2}\{V_P(f,a,b) + f(b) - f(a)\} \quad \text{................ (3)}$$

Subtracting (2) from (1) we have

$$V_P^-(f,a,b) = \frac{1}{2}\{V_P(f,a,b) - f(b) + f(a)\} \quad \text{................ (4)}$$

Since $\sup\limits_P V_P(f,a,b)$ exists it follows from (3) and (4) that $\sup\limits_P V_P^+(f,a,b)$ and $\sup\limits_P V_P^-(f,a,b)$ exist. We denote them by $V^+(f,a,b)$ and $V^-(f,a,b)$ respectively. Hence taking *sup* in (3) and (4) we get

$$V^+(f,a,b) = \frac{1}{2}\{V(f,a,b) + f(b) - f(a)\} \quad \text{................(5)}$$

and $V^-(f,a,b) = \dfrac{1}{2}\{V(f,a,b) - f(b) + f(a)\}$ .... (6)

From (5) and (6) we have

$$V(f,a,b) + V^+(f,a,b) + V^-(f,a,b)$$

and $f(b) - f(a) = V^+(f,a,b) - V^-(f,a,b)$

Hence the theorem.

1.7.7. Definition. Positive and Negative Variation Functions.

In (5) and (6) taking $b = x$ we get these definitions as follows.

For a function f of bounded variation on [a, b], the functions $V^+(x) = \dfrac{1}{2}\{v(x) + f(x) - f(a)\}$ and

$v^-(x)=\frac{1}{2}\{v(x)-f(x)+f(a)\}$ are called respectively the positive variation function and negative variation function.

1.8. Isslustrative Examples

1.8.1. Example. Prove that $f:[-1,2]\to R$ defined by $f(x)=3x^2-5x+2$ is of bounded variation on $[-1, 2]$ (Use monotone property).

Solution. Here $f(x)=3x^2-5x+2$

$\therefore f'(x)=6x-5$

Hence $f'(x)>0$ for $6x-5>0$ i.e. for $x>5/6$

and $f'(x)<0$ for $6x-5<0$ i.e. for $x<5/6$.

Thus $f(x)$ is monotone decreasing on $\left[-1, 5/6\right]$

and monotone increasing on $\left[5/6, 2\right]$.

As $f(x)$ is m.d on $\left[-1, 5/6\right]$ it follows that $f(x)$ is of b.v. on $\left[-1, 5/6\right]$. Again as $f(x)$ is m.i. on $\left[5/6, 2\right]$ it follows that $f(x)$ is of b.v. on $\left[5/6, 2\right]$.

Since $f(x)$ is of b.v. on $\left[-1, 5/6\right]$ and on $\left[5/6, 2\right]$, it follows that $f(x)$ is of b.v. on $\left[-1, 5/6\right]\cup\left[5/6, 2\right]$ i.e. on $[-1, 2]$.

1.8.2. Example. Find the total variation of the function $f(x)=3x^2-5x+2$ on $[-1, 2]$.

Solution. We have seen in the previous example that $f(x)$ is m.d. on $\left[-1, 5/6\right]$ and is m.i. on $\left[5/6, 2\right]$.

As $f(x)$ is m.d. on $\left[-1, 5/6\right]$ we have

$$V\left(f,-1, 5/6\right)=f(-1)-f\left(5/6\right)$$

Again $f(x)$ is m.i. on $\left[\frac{5}{6},2\right]$ we have

$$V\left(f,\tfrac{5}{6},2\right)=f(2)-f\left(\tfrac{5}{6}\right)$$

Now $V(f,-1,2)=V\left(f,-1,\frac{5}{6}\right)+V\left(f,\frac{5}{6},2\right)$

$$=f(-1)-f\left(\tfrac{5}{6}\right)+f(2)-f\left(\tfrac{5}{6}\right)$$

$$=f(-1)+f(2)-2f\left(\tfrac{5}{6}\right)$$

$$=10+4+\frac{1}{6}$$

$$=14\frac{1}{6}$$

1.8.3. Example. Let $f:[-1,2]\to R$ be defined by $f(x)=3x^2-5x+2$. Find the variation function, positive variation function and negative variation function. From these results find the total variation, positive variation and negative variation.

Solution :

We have $f(x)=3x^2-5x+2$

$\therefore f'(x)=6x-5$.

$\therefore f'(x)>0$ for $x>\frac{5}{6}$

and $f'(x)<0$ for $x<\frac{5}{6}$.

Thus $f(x)$ is m.d. on $\left[-1,\frac{5}{6}\right]$

and $f(x)$ is m.i. on $\left[\frac{5}{6},2\right]$.

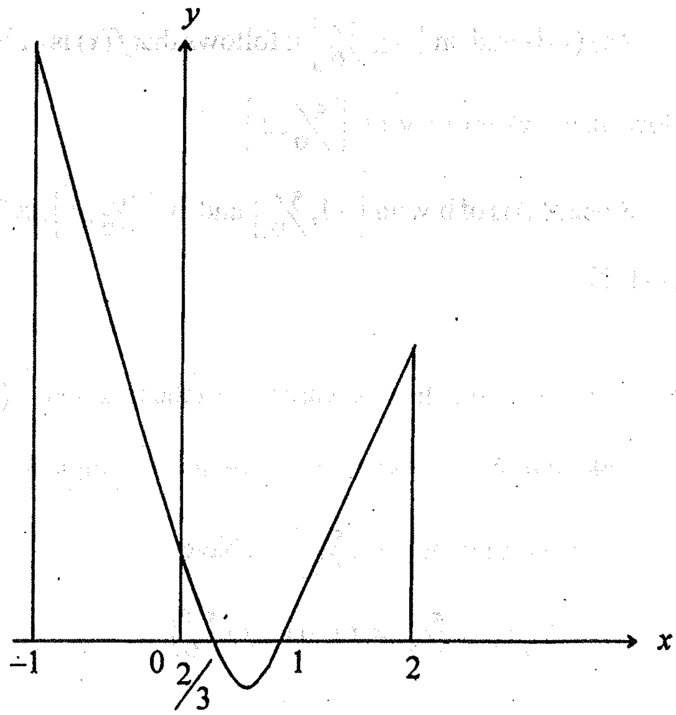
For $-1\le x\le \frac{5}{6}$ we have

$$v(x)=V(f,-1,x)$$

$$=f(-1)-f(x)$$

$$=10-3x^2+5x-2$$

$$=8+5x-3x^2.$$



$$\therefore v\left(5/6\right)=V\left(f,-1,5/6\right)=8+\frac{25}{6}-3\cdot\frac{25}{36}=\frac{121}{12}$$

For $\frac{5}{6}<x\le 2$ we have

$$\begin{aligned} v(x) &= V(f,-1,x) \\ &= V\left(f,-1,5/6\right)+V\left(f,5/6,x\right) \\ &= \frac{121}{12}+f(x)-f\left(5/6\right) \\ &= \frac{121}{12}+3x^2-5x+2+\frac{1}{12} \\ &= 3x^2-5x+73/6 \end{aligned}$$

$$\therefore v(x)=8+5x-3x^2, -1\le x\le 5/6$$

$$=3x^2-5x+\frac{73}{6}, \frac{5}{6}<x\le 2. \qquad \text{................ (1)}$$

This is the required variation function.

The positive variation function $v^+(x)$ is given by

$$v^+(x)=\frac{1}{2}\{v(x)+f(x)-f(-1)\}$$

$$\therefore v^+(x)=\frac{1}{2}\{8+5x-3x^2+3x^2-5x+2-10\} \text{ for } -1\le x\le 5/6$$

$$=\frac{1}{2}\left\{3x^2-5x+\frac{73}{6}+3x^2-5x+2-10\right\} \text{ for } 5/6<x\le 2$$

i.e. $v^+(x)=0$ for $-1\le x\le 5/6$

$$=\frac{1}{2}\left\{6x^2-10x+\frac{25}{6}\right\} \text{ for } 5/6<x\le 2$$

i.e. $v^+(x)=0$, $-1\le x\le 5/6$

$$=3x^2-5x+\frac{25}{12}, \quad 5/6<x\le 2 \qquad \text{................(2)}$$

The negative variation function $\upsilon^-(x)$ is given by

$$\upsilon^-(x)=\frac{1}{2}\{\upsilon(x)-f(x)+f(-1)\}$$

$$\therefore \upsilon^-(x)=\frac{1}{2}\{8+5x-3x^2-3x^2+5x-2+10\} \text{ for } -1\le x\le 5/6$$

$$=\frac{1}{2}\left\{3x^2-5x+\frac{73}{6}-3x^2+5x-2+10\right\} \text{ for } 5/6<x\le 2$$

i.e. $\upsilon^-(x)=\frac{1}{2}\{16+10x-6x^2\}$, $-1\le x\le 5/6$

$$=\frac{1}{2}\left\{\frac{121}{6}\right\} \quad , 5/6<x\le 2$$

i.e. $\upsilon^-(x)=8+5x-3x^2$, $-1\le x\le 5/6$

$$=121/12 \quad , 5/6<x\le 2 \qquad \text{............ (3)}$$

From (1) we get the total variation as

$$V(f,-1,2)=\upsilon(2)=3.2^2-5.2+\frac{73}{6}=\frac{85}{6}$$

From (2) we get the total positive variation as

$$V^+(f,-1,2)=\upsilon^+(2)=3.2^2-5.2+\frac{25}{12}=\frac{49}{12}$$

From (3) we get the total negative variation as

$$V^-(f,-1,2)=\upsilon^-(2)=\frac{121}{12}.$$

We have the interesting results

$$V^+(f,-1,2)+V^-(f,-1,2)=\frac{49}{12}+\frac{121}{12}=\frac{85}{6}=V(f,-1,2)$$

and $V^+(f,-1,2)-V^-(f,-1,2)=\frac{49}{12}-\frac{121}{12}=4-10=f(2)-f(-1).$

1.9. Summary. Function of bounded variation is very important property of functions. Its relation with monotone function, bounded function and continuous function have been studied here. Theorems have been proved and examples are given to illustrate them.

1.10. Self Assessment Questions

1. Show that $f(x) = x^2 - 3x + 2$ is a function of bounded variation on [–1, 4].
2. Show that $f(x) = \cos\frac{1}{x}$ if $x \neq 0$; $f(0) = 0$ is not of bounded variation on [0, 1].
3. Show that $f(x) = x\cos\frac{\pi}{2x}$ if $x \neq 0$; $f(0) = 0$ is continuous on [0, 1] but not of bounded variation there.
4. Show that $f(x) = x^2 \sin\frac{3}{2x}$ if $x \neq 0$; $f(0) = 0$ is of bounded variation on [0, 1].
5. Show that $f(x) = 3\sin^2 x + 4e^{2x} + 3x^3 + 5x^2 + \log x$ is of bounded variation on [1, 4].
6. Find the total variation of the function $f(x) = [x]$ on [1, 4].
7. Determine the variation function of $f(x) = \sin 3x$ on $[0, \pi]$
8. Compute the positive, negative and total variation functions of $f(x) = 3x^2 - 2x^3$ for $-2 \leq x \leq 2$.
9. Compute the positve, negative and the total variation function of f, where $f(x) = 2[x] - 3x$ for $-1 \leq x \leq 2$.
10. Show that $f(x) = 3\sin x + 4$ is of bounded variation over any finite interval.

1.11. Suggested Books for further reading

1. Mathematical Analysis : S.C. Malik & Savita Arora; Wiley Eastern Limited, New Age International Limited.
2. Introduction to Mathematical Analysis : Amritava Gupta; Academic Publishers, Calcutta.
3. Mathematical Analysis : Tom M. Apostol; Narosa Publishing House.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-I Group–A

Module No. - 2

Real Analysis

(Riemann-Stieltjes Integral-I)

Module Structure

2.1 Introduction :

The famous German mathematician Riemann was the first man who gave a rigorous arithmetic treatment of integration. Many generalisations of the Riemann integral have yet been done by many mathematicians. One such generalisation is Riemann-Stieltjes integral done by Thomas Joaner Stieltjes. As a special case of Riemann-Stieltjes integral we get Riemann integral. This generalised integral needs two functions know as integrand and integrator. There are several accepted definitions of Rieman-Stieltjes integral. In this module we consider the limit definition.

2.2 Objective

The problem of finding area of a region under a curve was solved by Riemann integration. With the development of science mathematicians had to face many problems which could not be handeled by Riemann integration. Riemann-Stieltjes integral is found to be a very beautiful tool to tackel many such problems. Problems in physics which involve mass distributions that are partly discrete and partly continuous can be treated by using Riemann-Stieltjes integrals. This integral is also seen to be very much useful in the mathematical theory of probability where simultaneous treatment of continuous and discrete randodm variables are essential.

2.3. Notations and Definitions

The following notations and terminology are used in this module. All functions used in this module are bounded and defined on the closed finite interval $[a, b]$.

A partition P of $[a, b]$ is a finite set of points $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ such that $x_0 < x_1 < .... < x_n$.

A partition P' of $[a, b]$ is said to be finer than the partition P if $P \subseteq P'$.

For each $i = 1, 2, ..., n$ the difference $\alpha(x_i) - \alpha(x_{i-1})$ will be denoted by $\Delta\alpha_i$ i.e. $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

We note that $\sum_{i=1}^{n} \Delta\alpha_i = \alpha(b) - \alpha(a)$.

The norm of a partition P is the length of the largest subinterval of P and is denoted by $\|P\|$.

Thus $\|P\| = \max\{\Delta x_1, \Delta x_2, ..., \Delta x_n\}$ where $\Delta x_i = x_i - x_{i-1}$ for all $i = 1, 2, ..., n$.

It may be noted that if P' is finer than P then $\|P'\| \le \|P\|$.

2.4. The Definition of the Riemann-Stieltjes integral

Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be a partition of $[a, b]$. Let ξ_i be a point in the subinterval $[x_{i-1}, x_i]$ and $\Gamma = \{\xi_1, \xi_2, ..., \xi_n\}$. A sum of the form $\sum_{i-1}^{n} f(t_i)\Delta\alpha_i = S(P, \Gamma, f, \alpha)$ is called the Riemann-Stieltjes sum of the function f with respect to the function α.

The function f is said to be Riemann-Stieltjes integrable with respect to α if there exists a real number I having the property that for every $\in > 0$, there exists a $\delta > 0$ such that for every partition P of $[a, b]$ with norm $\|P\| < \delta$ and for every choice of ξ_i in $[x_{i-1}, x_i]$, we have

$$|S(P, \Gamma, f, \alpha) - I| < \in.$$

We shall write $\lim_{\|P\|\to 0} S(P, \Gamma, f, \alpha) = I = \int_a^b f\, d\alpha.$

Thus $\int_a^b f(x)\, d\alpha(x) = \lim_{\|P\|\to 0} S(P, \Gamma, f, \alpha).$

The function f and α are called the integrand and integrator respectively and "$f \in R(\alpha)$ on $[a, b]$" means f is RS-integrable with respect to α on $[a, b]$.

2.4.1. Definition

If $a < b$, we define $\int_b^a f d\alpha = -\int_a^b f d\alpha$ whenever $\int_a^b f d\alpha$ exists. We also define $\int_a^a f d\alpha = 0$.

2.5. Linear Properties

The following theorems show that the integral operators are linear on both the integrand and the integrator.

Theorem 2.5.1. If f and g are RS-integrable on $[a, b]$ with respect to α then $f+g$ is RS-integrable on $[a, b]$. Also

$$\int_a^b (f+g)\,d\alpha = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha$$

Proof. Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$ and $\Gamma = \{\xi_1, \xi_2, ..., \xi_n\}$. where ξ_i is any point in $[x_{i-1}, x_i], i = 1, 2, ..., n$.

Now we have

$$S(P, \Gamma, f+g, \alpha)$$

$$= \sum_{i=1}^{n} (f+g)(\xi_i)\Delta\alpha_i$$

$$= \sum_{i=1}^{n} \{f(\xi_i) + g(\xi_i)\}\Delta\alpha_i$$

$$= \sum_{i=1}^{n} f(\xi_i)\Delta\alpha_i + \sum_{i=1}^{n} g(\xi_i)\Delta\alpha_i$$

$$= S(P, \Gamma, f, \alpha) + S(P, \Gamma, g, \alpha) \quad \text{.................. (1)}$$

We assume that f and g are RS-integrable on $[a, b]$ with respect to α on $[a, b]$. Then for every $\in > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ s.t.

$$= \left| S(P, \Gamma, f, \alpha) - \int_a^b f d\alpha \right| < \in/2 \quad \text{.................. (2)}$$

for every partition P of $[a, b]$ with norm $\|P\| < \delta_1$ and for every choice of ξ_i in $[x_{i-1}, x_i]$ and

$$= \left| S(P, \Gamma, f, \alpha) - \int_a^b f d\alpha \right| < \in/2 \quad \text{.................. (3)}$$

for every partition P of $[a, b]$ with norm $\|P\| < \delta_2$ and for every choice of ξ_i in $[x_{i-1}, x_i]$

We take $\delta = \min(\delta_1, \delta_2)$.

Then both (2) and (3) are true for every partition P of $[a, b]$ with norm $\|P\| < \delta$.

$\therefore$ For every partition P with norm $\|P\| < \delta$ we have

$$= \left| S(P,\Gamma,f+g,\alpha) - \left\{ \int_a^b f\,d\alpha + \int_a^b g\,d\alpha \right\} \right|$$

$$= \left| S(P,\Gamma,f,\alpha) + S(P,\Gamma,g,\alpha) - \int_a^b f\,d\alpha - \int_a^b g\,d\alpha \right|, \text{ [using (1)]}$$

$$\le \left| S(P,\Gamma,f,\alpha) - \int_a^b f\,d\alpha \right| + \left| S(P,\Gamma,g,\alpha) - \int_a^b g\,d\alpha \right|$$

$$< \epsilon/2 + \epsilon/2 \text{ [using (2) and (3)]}$$

$$= \epsilon$$

Thus for given $\epsilon > 0$, there exists $\delta > 0$ such that for every partition P with norm $\|P\| < \delta$ and for every choice of ξ_i in $[x_{i-1}, x_i]$, we have

$$\left| S(P,\Gamma,f+g,\alpha) - \left\{ \int_a^b f\,d\alpha + \int_a^b g\,d\alpha \right\} \right| < \epsilon.$$

Hence $f + g$ is RS-integrable with respect to α on $[a, b]$ and the value of the integral is $\int_a^b f\,d\alpha + \int_a^b g\,d\alpha$.

Thus $\int_a^b (f+g)\,d\alpha = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha$.

Theorem 2.5.2 If f and g are RS-integrable on $[a, b]$ w.r.t. α then $f - g$ is RS-integrable on $[a, b]$ and also

$$\int_a^b (f-g)\,d\alpha = \int_a^b f\,d\alpha - \int_a^b g\,d\alpha$$

Proof. Similar to the proof of Theorem 2.5.1.

Theorem 2.5.3. If f is RS-integrable on $[a, b]$ w.r.t. α and w.r.t. β then f is RS-integrable on $[a, b]$ w.r.t. $\alpha + \beta$.

Also $\int_a^b f\,d(\alpha \pm \beta) = \int_a^b f\,d\alpha \pm \int_a^b f\,d\beta$.

Proof. Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$ and $\Gamma = \{\xi_1, \xi_2, \ldots, \xi_n\}$ where ξ_i is is any point in $[x_{i-1}, x_i]$. for each $i = 1, 2, \ldots, n$

We have $S(P,\Gamma,f,\alpha\pm\beta)$

$$=\sum_{i=1}^{n} f(\xi_i)\Delta(\alpha\pm\beta)_i$$

$$=\sum_{i=1}^{n} f(\xi_i)\{(\alpha\pm\beta)(x_i)-(\alpha\pm\beta)(x_{i-1})\}$$

$$=\sum_{i=1}^{n} f(\xi_i)\{\alpha(x_i)\pm\beta(x_i)-\alpha(x_{i-1})\mp\beta(x_{i-1})\}$$

$$=\sum_{i=1}^{n} f(\xi_i)\{\Delta\alpha_i\pm\Delta\beta_i\}$$

$$=\sum_{i=1}^{n} f(\xi_i)\Delta\alpha_i\pm\sum_{i=1}^{n} f(\xi_i)\Delta\beta_i$$

$$=S(P,\Gamma,f,\alpha)\pm S(P,\Gamma,f,\beta). \qquad \text{.................} (1)$$

We assume that f is RS-integrable with respect to α and with respect to β on $[a, b]$.

Then for every $\in > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such taht

$$\left|S(P,\Gamma,f,\alpha)-\int_a^b f\,d\alpha\right|<\in/2 \qquad \text{.................} (2)$$

for every partition P of $[a, b]$ with norm $\|P\|<\delta_1$ and for every choice of ξ_i in $[x_{i-1},x_i]$ and

$$\left|S(P,\Gamma,f,\beta)-\int_a^b f\,d\beta\right|<\in/2 \qquad \text{.................} (3)$$

for every partition P of $[a, b]$ with norm $\|P\|<\delta_2$ and for every choice of ξ_i in $[x_{i-1},x_i]$

We take $\delta=\min\{\delta_1,\delta_2\}$. Then both (2) and (3) hold good for every partition P of $[a, b\}$ with norm $\|P\|<\delta$ and for every choice of ξ_i in $[x_{i-1},x_i]$.

Thus for every partition P of $[a, b]$ with norm $\|P\|<\delta$ and for corresponding Γ we have

$$\left|S(P,\Gamma,f,\alpha\pm\beta)-\left\{\int_a^b f\,d\alpha\pm\int_a^b f\,d\beta\right\}\right|$$

$$=\left|S(P,\Gamma,f,\alpha)\pm S(P,\Gamma,f,\beta)-\int_a^b f\,d\alpha \mp \int_a^b f\,d\beta\right|, \text{[using (1)]}$$

$$=\left|\left\{S(P,\Gamma,f,\alpha)-\int_a^b f\,d\alpha\right\}\pm\left\{S(P,\Gamma,f,\beta)-\int_a^b f\,d\beta\right\}\right|$$

$$\leq\left|\left\{S(P,\Gamma,f,\alpha)-\int_a^b f\,d\alpha\right\}+\left\{S(P,\Gamma,f,\beta)-\int_a^b f\,d\beta\right\}\right|$$

$$<\in/2+\in/2, \text{ [using (2) and (3) as } \|P\|<\delta\text{]}$$

$$=\in.$$

Hence for given $\in$, there exists $\delta>0$ such that for every partition P with norm $\|P\|<\delta$ and for every choice of ξ_i in $[x_{i-1},x_i]$ we have

$$\left|S(P,\Gamma,f,\alpha\pm\beta)-\left\{\int_a^b f\,d\alpha\pm\int_a^b f\,d\beta\right\}\right|<\in.$$

This shows that f is RS-integrable w.r.t. $\alpha\pm\beta$ on $[a, b]$ and the value of the integral is $\int_a^b f\,d\alpha\pm\int_a^b f\,d\beta$.

$$\therefore \int_a^b f\,d(\alpha\pm\beta)=\int_a^b f\,d\alpha\pm\int_a^b f\,d\beta.$$

Theorem 2.5.4. If f is RS-integrable with respect to α on $[a, b]$ then $c_1 f$ is RS-integrable with respect to $c_2\alpha$ for any real c_1 and c_2. Also $\int_a^b (c_1 f)\,d(c_2\alpha)=c_1c_2\int_a^b f\,d\alpha.$

Proof. Let $P=\{a=x_0,x_1,x_2,\ldots,x_n=b\}$ be any partition of $[a, b]$ and $\Gamma=\{\xi_1,\xi_2,\ldots,\xi_n\}$ where ξ_i is any point in $[x_{i-1},x_i]$ for each $i = 1, 2, \ldots, n$.

We have

$$S(P,\Gamma,f,c_1f,c_2\alpha)$$

$$=\sum_{i=1}^{n}(c_1f)(\xi_i)\Delta(c_2\alpha)_i$$

$$=\sum_{i=1}^{n}c_1f(\xi_i)\{(c_2\alpha)(x_i)-(c_2\alpha)(x_{i-1})\}$$

$$= \sum_{i=1}^{n} c_1 f(\xi_i)\{c_2\alpha(x_i) - c_2\alpha(x_{i-1})\}$$

$$= \sum_{i=1}^{n} c_1 f(\xi_i)\cdot c_2\alpha(x_i) - \sum_{i=1}^{n} c_1 f(\xi_i)\cdot c_2\alpha(x_{i-1})$$

$$= c_1c_2\sum_{i=1}^{n} f(\xi_i)\alpha(x_i) - c_1c_2\sum_{i=1}^{n} f(\xi_i)\alpha(x_{i-1})$$

$$= c_1c_2\sum_{i=1}^{n} f(\xi_i)\{\alpha(x_i) - \alpha(x_{i-1})\}$$

$$= c_1c_2\sum_{i=1}^{n} f(\xi_i)\Delta\alpha_i$$

$$= c_1c_2\, S(P,\Gamma,f,\alpha) \quad \text{.................... (1)}$$

We assume that f is RS-integrable with respect to α on $[a, b]$.

Then for every $\in > 0$, there exists $\delta > 0$ such that

$$\left|S(P,\Gamma,f,\alpha) - \int_a^b f\,d\alpha\right| < \frac{\in}{|c_1||c_2|} \quad \text{....................(2)}$$

for every partition P of $[a, b]$ with norm $\|P\| < \delta$ and for every choice of ξ_i in $[x_{i-1}, x_i]$.

Thus for every partition P of $[a, b]$ with norm $\|P\| < \delta$ and for corresponding Γ we have

$$\left|S(P,\Gamma,c_1f,c_2\alpha) - c_1c_2\int_a^b f\,d\alpha\right|$$

$$= \left|c_1c_2\, S(P,\Gamma,f,\alpha) - c_1c_2\int_a^b f\,d\alpha\right|, \text{ [using (1)]}$$

$$= |c_1||c_2|\left|S(P,\Gamma,f,\alpha) - \int_a^b f\,d\alpha\right|$$

$< \in$, [using (2)].

Hence c_1f is RS-integrable with respect to $c_2\alpha$ on $[a, b]$ and the value of the integral is $c_1c_2\int_a^b f\,d\alpha$.

So, we have $\int_a^b (c_1 f)\, d(c_2\alpha) = c_1 c_2 \int_a^b f\, d\alpha.$

Combining Theorems 2.5.1, 2.5.2, 2.5.3 and 2.5.4 we have the following theorem.

Theorem 2.5.5. If f and g are RS-integrable w.r.t. α as well as w.r.t. β on [a, b], then $c_1 f + c_2 f$ is RS-integrable w.r.t. $c_3\alpha + c_4\beta$ on [a, b] for any real numbers c_1, c_2, c_3 and c_4.

Also $\int_a^b (c_1 f + c_2 g)\, d(c_3\alpha + c_4\beta) = c_1 c_3 \int_a^b f\, d\alpha + c_1 c_4 \int_a^b f\, d\beta + c_2 c_3 \int_a^b g\, d\alpha + c_2 c_4 \int_a^b d\, d\beta.$

Note. This theorem shows that linear property holds good for both integrand and integrator.

2.6. Some more Theorems.

Analogous to the theorem obtained for Riemann integral the following theorem shows that the RS-integral is additive with respect to the interval of integration.

Theorem 2.6.1. Assume that $a < c < b$. If f is RS-integrable with respect to α on [a, b] then f is RS-integrable with respect to α on [a, b] and on [c, b]. Also

$$\int_a^b f\, d\alpha = \int_a^c f\, d\alpha + \int_c^b f\, d\alpha.$$

Proof. Let $P_1 = \{a = a_0, a_1, a_2, \ldots, a_m = c\}$ be the partition of [a, c] where $a_i = a + \frac{(c-a)i}{m}$ for all $i = 0, 1, 2, \ldots, m$ and $\Gamma_1 = \{a_1, a_2, \ldots, a_m\}$.

Then $S(P_1, \Gamma_1, f, \alpha)$ depends on m only. Let $S(P_1, \Gamma_1, f, \alpha) = s_m$ ....(1)

Similarly, let $P_2 = \{c = b_0, b_1, b_2, \ldots, b_n = b\}$ be the partition of [c, b] where $b_j = c + \frac{(b-c)i}{n}$ for all $j = 0, 1, 2, \ldots, n$ and $\Gamma_2 = \{b_1, b_2, \ldots, b_n\}$.

Then $S(P_2,\Gamma_2,f,\alpha)$ depends on n only. Let $S(P_2,\Gamma_2,f,\alpha)=s'_n$ .... (2)

Let $P_3=P_1\cup P_2=\{a=a_0,a_1,a_2,...,a_m=c=b_0,b_1,b_2,...,b_n=b\}$

and $\Gamma_3=\Gamma_1\cup\Gamma_2=\{a_1,a_2,...,a_m,b_1,b_2,...,b_n\}$.

Then P_3 is a partition of $[a, b]$ and

$$S(P_3,\Gamma_3,f,\alpha)=S(P_1,\Gamma_1,f,\alpha)+S(P_2,\Gamma_3,f,\alpha)=s_m+s'_n \quad \text{[using (1) and (2)]} \quad ....(3)$$

Since by hypothesis f is RS-integrable with respect to α on $[a, b]$, for every $\in>0$ there exists $\delta>0$ such that

$$|S(P,\Gamma,f,\alpha)-I|<\in/2 \quad ....(4)$$

for every partition P of $[a, b]$ and for every corresponding Γ, where $I=\int_a^b f\,d\alpha$.

We note that $\|P_3\|=\max\left\{\frac{c-a}{m},\frac{b-c}{n}\right\}$. So there exists positive integer N such that $\|P_3\|<\delta$ for all $m,n\geq N$.

Hence from (4) we have for all $m,n\geq N$

$|S(P_3,\Gamma_3,f,\alpha)-I|<\in/2$. Using (3) we have

$|s_m+s'_n-I|<\in/2$

Thus $|s_m+s'_n-I|<\in/2$ for all $m,n\geq N$ .... (5)

In particular we have from (5)

$|s_N+s'_n-I|<\in/2$ for all $n\geq N$

and $|s_N+s'_{n+p}-I|<\in/2$ for all $n\geq N$ and $\forall\, p=1,2,....$

From these we have

$|s'_{n+p}-s'_n|$

$$=\left|\left(s'_{n+1}+s_N-I\right)-\left(s'_n+s_N-I\right)\right|$$

$$\leq\left|\left(s'_{n+1}+s_N-I\right)-\left(s'_n+s_N-I\right)\right|$$

$$\leq \in/2+\in/2$$

i.e. $\left|s'_{n+P}-s'_n\right|<\in$ for all $n\geq N$ and $p=1,2,3,...$

This shows that $\lim_{n\to\infty} s'_n$ exists. Let $\lim_{n\to\infty} s'_n=s$. .... (6)

Let $P'=\{a=x_0,x_1,x_2,....,x_r=c\}$ be any partition of $[a,c]$ and $\Gamma'=\{\xi_1,\xi_2,....,\xi_r\}$ where $x_{i-1}\leq\xi_i\leq x_i$ for all $i=1,2,...,r$.

Let $P_4=P'\cup P_2$ and $\Gamma_4=\Gamma'\cup\Gamma_2$.

Then $P_4=\{a=x_0,x_1,x_2,....,x_r=c=b_0,b_1,b_2,...,b_n=b\}$ is a partition of $[a, b]$ with corresponding $\Gamma_4=\{\xi_1,\xi_2,....,\xi_r,b_1,b_2,...,b_n\}$.

We note that $\|P_4\|=\max\left\{\|P'\|,\frac{b-c}{n}\right\}$. .... (7)

Now $S(P_4,\Gamma_4,f,\alpha)=S(P',\Gamma',f,\alpha)+S(P_2,\Gamma_2,f,\alpha)$.

Using (2) we get $S(P_4,\Gamma_4,f,\alpha)=S(P',\Gamma',f,\alpha)+s'_n$

or, $S(P',\Gamma',f,\alpha)=S(P_4,\Gamma_4,f,\alpha)-s'_n$ .... (8)

Since f is RS-integrable w.r.t. α on $[a, b]$ we have

$\lim_{\|P_4\|\to 0} S(P_4,\Gamma_4,f,\alpha)=\int_a^b f\,d\alpha=I.$ Also from (6) $\lim_{n\to\infty} s'_n=s$.

$\therefore \lim_{\|P_4\|\to 0} S(P_4,\Gamma_4,f,\alpha)-\lim_{n\to\infty} s'_n$ exists and the value of this limit is $I-s$.

Thus f is RS-integrable w.r.t. α on $[a, c]$ and $\int_a^c f\,d\alpha=I-s$ .... (9)

Similarly, we can prove that f is RS-integrable w.r.t. α on $[c, b]$.

Using (2) and (6) we have

$$\therefore \lim_{\|P_2\|\to 0} S(P_2,\Gamma_2,f,\alpha)=s$$

Since f is RS integrable w.r.t. α on $[c, b]$ we have

$$\therefore \lim_{\|P_2\|\to 0} S(P_2,\Gamma_2,f,\alpha)=\int_c^b f\,d\alpha \qquad \therefore \int_c^b f\,d\alpha = s \qquad \text{.................(10)}$$

Using (10) in (9) we have

$$\int_a^c f\,d\alpha=\int_a^b f\,d\alpha-\int_c^b f\,d\alpha$$

or, $$\int_a^c f\,d\alpha+\int_c^b f\,d\alpha=\int_a^b f\,d\alpha$$

Hence the theorem.

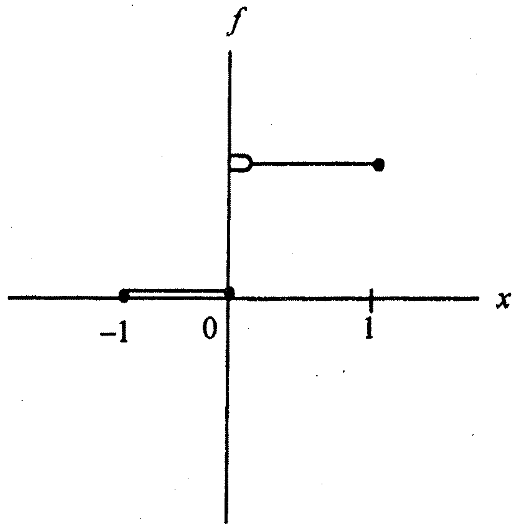
Note. In Riemann integral the Theorem 2.6.1 holds good. Also the converse of this theorem is seen to be true for Riemann integral. But for RS-integral the converse of Theorem 2.6.1 is not true i.e. if f is RS-integrable w.r.t. α over $[a, c]$ as well as over $[c, b]$ then it does not necessarily follow that f is RS-integrable w.r.t. α over $[a, b]$. The following example establishes this fact.

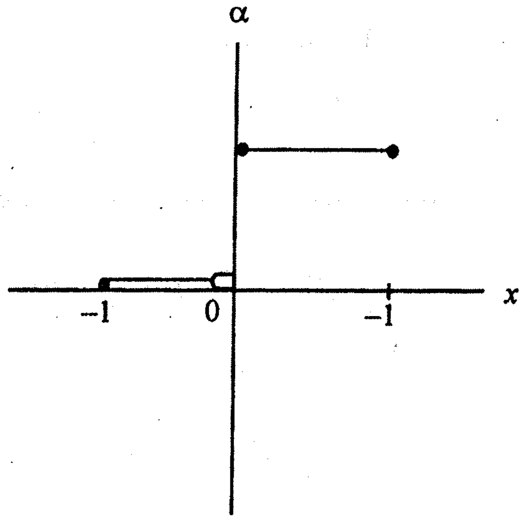
Example 2.6.1. Let $f(x)$ and $\alpha(x)$ be defined on $[-1, 1]$ as follows.

$$f(x)=0 \text{ for } -1\le x\le 0$$
$$=1 \text{ for } 0<x\le 1$$
$$\alpha(x)=0 \text{ for } -1\le x<0$$
$$=1 \text{ for } 0\le x\le 1$$

Show that f is RS-integrable w.r.t. α over $[-1, 0]$ and also over $[0, 1]$. But f is not RS-integrable w.r.t. α over $[-1, 1]$.

Solution. The graph of the functions $f(x)$ and $\alpha(x)$ are as follows.





Let $P_1 = \{-1 = x_0, x_1, x_2, ..., x_r = 0\}$ be any partition of [–1, 0] and $\Gamma_1 = \{\xi_1, \xi_2, ..., \xi_r\}$ where for each i = 1, 2, ..., r; $x_{i-1} \le \xi_i \le x_i$.

Then $S(P_1, \Gamma_1, f, \alpha)$

$$= \sum_{i=1}^{r} f(\xi_i)\{\alpha(x_i) - \alpha(x_{i-1})\}$$

$$= \sum_{i=1}^{r} 0 \cdot \{\alpha(x_i) - \alpha(x_{i-1})\} \quad [\because -1 \le f(\xi_i) \le 0 \text{ for each } i = 1, 2, ..., r]$$

$$= 0$$

This is true for all partition P_1 of [–1, 0] and corresponding Γ_1. We note that the RHS is independent of P_1 and Γ_1 and is always 0. Hence $\lim_{\|P_1\| \to 0} S(P_1, \Gamma_1, f, \alpha)$ exists and is 0.

So $\int_{-1}^{0} f\, d\alpha$ exists and is 0 i.e. $\int_{-1}^{0} f\, d\alpha = 0$.

Again let $P_2 = \{0 = z_0, z_1, z_2, ..., z_m = 1\}$ be any partition of [0, 1] and $\Gamma_2 = \{\eta_1, \eta_2, ..., \eta_m\}$ where $z_{j-1} \le \eta_j \le z_j$. for each j = 1, 2, ..., m

Now $S(P_2, \Gamma_2, f, \alpha)$

$$=\sum_{j=1}^{m} f(\eta_i)\{\alpha(z_j)-\alpha(z_{j-1})\}$$

$$=\sum_{j=1}^{m} f(\eta_j)\{1-1\} \quad [\because 0 \le z_j \le 1 \text{ for each } j=1,2,\ldots,m]$$

$= 0$. This is true for all P_2 and Γ_2.

We note that the RHS is independent P_2, Γ_2 and is always 0.

Hence $\lim_{\|P_2\| \to 0} S(P_2, \Gamma_2, f, \alpha)$ exists and is 0.

Thus $\int_0^1 f\,d\alpha$ exists and is 0 i.e. $\int_0^1 f\,d\alpha = 0$.

$\therefore$ Both $\int_{-1}^{0} f\,d\alpha$ and $\int_0^1 f\,d\alpha$ exist. Now we show that $\int_{-1}^{1} f\,d\alpha$ does not exist.

Let $P = \{-1 = x_0, x_1, \ldots, x_{k-1}, x_k, \ldots, x_n = 1\}$ be a partition of [–1, 1] where $x_{k-1} < 0 < x_k$. Also let $\Gamma = \{\xi_1, \xi_2, \ldots, \xi_n\}$ where $x_{i-1} \le \xi_i \le x_i$ for each $i = 1, 2, \ldots, n$.

Then $S(P, \Gamma, f, \alpha)$

$$=\sum_{i=1}^{n} f(\xi_i)\{\alpha(x_i)-\alpha(x_{i-1})\}$$

$$=\sum_{i=1}^{k-1} f(\xi_i)\{\alpha(x_i)-\alpha(x_{i-1})\} + f(\xi_k)\{\alpha(x_k)-\alpha(x_{k-1})\} + \sum_{i=k+1}^{n} f(\xi_i)\{\alpha(x_i)-\alpha(x_{i-1})\}.$$

$$=\sum_{i=1}^{k-1} 0\{\alpha(x_i)-\alpha(x_{i-1})\} + f(\xi_k)\{1-0\} + \sum_{i=k+1}^{n} f(\xi_i)\{1-1\}$$

$$= f(\xi_k).$$

Now $x_{k-1} < 0 < x_k$ and $x_{k-1} \le \xi_k \le x_k$.

$\therefore$ If $x_{k-1} \le \xi_k \le 0$ then $f(\xi_k) = 0$

and if $0 < \xi_k \le x_k$ then $f(\xi_k) = 1$.

Sunce ξ_k can take any value in $[x_{k-1}, x_k]$ it follows that $\lim_{\|P\| \to 0} S(P, \Gamma, f, \alpha)$ does not exit. This means $\int^1 d\alpha$

does not exist.

2.7 Integration by parts

There is a remarkable and interesting relation between the integrand and the integrator in a Rieman Stieltjes integral. In fact the existence of $\int_a^b f d\alpha$ implies the existence of $\int_a^b \alpha df$, and the converse is also true. These two integrals are related by a simple relation, which is nothing but a generalization of integration by parts in Riemann integration theory.

The following theorem provides a kind of reciprocity law for the integral and is known as the formula for integration by parts.

Theorem 2.7.1. If f is RS-integrable w.r.t. α over $[a,b]$, then α is also RS-integrable w.r.t. f over $[a,b]$. The two integrals are related as

$$\int_a^b f d\alpha + \int_a^b \alpha df = f(b)\alpha(b) - f(a)\alpha(a).$$

Proof. Let $P = \{a = x_0, x_1, x_2 ..., x_n = b\}$ be any partition of $[a,b]$ and $\Gamma = \{\xi_1, \xi_2, ..., \xi_n\}$ where ξ_i is any point in $[x_{i-1}, x_i]$ for each $i = 1,2,...,n$.

Taking $\xi_0 = a$ and $\xi_{n+1} = b$ let $P' = \{a = \xi_0, \xi_1, \xi_2 ..., \xi_n, \xi_{n+1} = b\}$. We note that $x_{i-1} \le \xi_i \le x_i$ and $x_{i-2} \le \xi_{i-1} \le x_{i-1}$. Therefore, $\xi_{i-1} \le x_{i-1} \le \xi_i$ for $i = 1,2,...,n+1$.

Let $\Gamma' = \{x_0, x_1, ..., x_n\}$. Then P' is a partition of $[a,b]$ with $\Gamma' = \{x_0, x_1, ..., x_n\}$ where $\xi_{i-1} \le x_{i-1} \le \xi_i$ for $i=1,2,...,n+1$.

Now $S(P,\Gamma,\alpha,f)$

$$= \sum_{i=1}^{n} \alpha(\xi_i)\{f(x_i) - f(x_{i-1})\}$$

$$= \sum_{i=1}^{n} \alpha(\xi_i) f(x_i) - \sum_{i=1}^{n} \alpha(\xi_i) f(x_{i-1})$$

$$= \sum_{i=2}^{n+1} \alpha(\xi_{i-1}) f(x_{i-1}) - \sum_{i=1}^{n} \alpha(\xi_i) f(x_{i-1})$$

$$=\sum_{i=1}^{n+1}\alpha(\xi_{i-1})f(x_{i-1})-\alpha(\xi_0)f(x_0)-\sum_{i=1}^{n+1}\alpha(\xi_i)f(x_{i-1})+\alpha(\xi_{n+1})f(x_n)$$

$$=-\sum_{i=1}^{n+1}f(x_{i-1})\{\alpha(\xi_i)-\alpha(\xi_{i-1})\}-\alpha(a)f(a)+\alpha(b)f(b)$$

$$=-S(P',\Gamma',f,\alpha)-\alpha(a)f(a)+\alpha(b)f(b) \qquad .....(1)$$

Since $x_{i-1}\le\xi_i\le x_i$ for all $i=1,2,\ldots,n$ and

$\xi_{i-1}\le x_{i-1}\le\xi_i$ for all $i=1,2,\ldots,n+1$ we have $\|P\|\le 2\|P'\|$ and $\|P'\|\le 2\|P\|$. $\therefore \|P\|\to 0 \Leftrightarrow \|P'\|\to 0$ ...(2)

We assume that f is RS-integrable w.r.t. α on $[a,b]$.

$\therefore \lim\limits_{\|P'\|\to 0} S(P',\Gamma',f,\alpha)$ exists and the value of the limit is $\int_a^b fd\alpha$.

$\therefore -\lim\limits_{\|P'\|\to 0} S(P',\Gamma',f,\alpha)-\alpha(a)f(a)+\alpha(b)f(b)$ exists and

the value of the limit is $\int_a^b fd\alpha-\alpha(a)f(a)+\alpha(b)f(b)$.

Hence from (1) and (2) it follows that $\lim\limits_{\|P\|\to 0} S(P,\Gamma,\alpha,f)$ exists and the value of the limit is $-\int_a^b d\alpha-\alpha(a)f(a)+\alpha(b)f(b)$.

Thus α is RS-integrable w.r.t. f on $[a,b]$ and the value of the integral is $-\int_a^b fd\alpha-\alpha(a)f(a)+\alpha(b)f(b)$.

i.e. $\int_a^b \alpha df=-\int_a^b fd\alpha-\alpha(a)f(a)+\alpha(b)f(b)$

or, $\int_a^b fd\alpha+\int_a^b \alpha df=f(b)\alpha(b)-f(a)\alpha(a)$.

2.8 Reduction of RS-integral to R-integral

The next theorem illustrates one of the situations in which Riemann Stieltjes integrals reduce to Riemann integrals. Here $d\alpha(x)$ is replaced by $\alpha'(x)dx$ in the integral $\int_a^b f(x)d\alpha(x)$ whenever $\alpha(x)$ has a continuous derivative $\alpha'(x)$.

Theorem 2.8.1. If f is RS-integrable with respect to α on $[a,b]$ and if α has a continuous derivative α' on $[a,b]$, then the Riemann integral $\int_a^b f(x)\alpha'(x)dx$ exists and

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx.$$

Proof. Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a,b]$

and $\Gamma = \{\xi_1, \xi_2, ..., \xi_n\}$ where ξ_i is any point in $[x_{i-1}, x_i]$ for each $i = 1,2,...,n$.

Let $g(x) = f(x)\alpha'(x)$.

We consider the Riemann-Stieltjes sum

$$S(P,\Gamma,f,\alpha) = \sum_{i=1}^{n} f(\xi_i)\{\alpha(x_i) - \alpha(x_{i-1})\} = \sum_{i=1}^{n} f(\xi_i)\Delta\alpha_i \qquad ....(1)$$

Also we consider the Riemann sum

$$S(P,\Gamma,g) = \sum_{i=1}^{n} g(\xi_i)\{x_i - x_{i-1}\} = \sum_{i=1}^{n} g(\xi_i)\Delta x_i \qquad ....(2)$$

Since $\alpha(x)$ has a continuous derivative $\alpha'(x)$ on $[a,b]$, using Mean Value Theorem we have for each i=1,2,..., n

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)(x_i - x_{i-1}) = \alpha'(t_i)\Delta x_i \qquad ....(3)$$

where $x_{i-1} < t_i < x_i$

From (1) and (2) we have

$$S(P,\Gamma,f,\alpha) - S(P,\Gamma,g)$$

$$= \sum_{i=1}^{n}\{f(\xi_i)\Delta\alpha_i - g(\xi_i)\Delta x_i\}$$

$$= \sum_{i=1}^{n}\{f(\xi_i)\alpha'(t_i)\Delta x_i - f(\xi_i)\alpha'(\xi_i)\Delta x_i\} \text{ [using (3)]}$$

$$= \sum_{i=1}^{n} f(\xi_i)\{\alpha'(t_i) - \alpha'(\xi_i)\}\Delta x_i$$

$\therefore \left|S(P,\Gamma,f,\alpha)-S(P,\Gamma,g)\right|$

$$\leq \sum_{i=1}^{n}\left|f(\xi_i)\right|\left|\alpha'(t_i)-\alpha'(\xi_i)\right|\Delta x_i \qquad \text{... (4)}$$

Since *f* is bounded there exists $M > 0$ such that

$$|f(x)| \leq M \text{ for all } x \text{ in } [a,b] \qquad \text{... (5)}$$

Continuity of $\alpha'(x)$ on $[a,b]$ implies uniform continuity on $[a,b]$.

Hence for given $\in > 0$ these exists a $\delta_1 > 0$, depending on $\in$ only, such that

$$0 \leq |x-y| < \delta_1 \text{ implies } \left|\alpha'(x)-\alpha'(y)\right| < \frac{\in}{2M(b-a)} \qquad \text{... (6)}$$

Since $f \in R(\alpha)$ on $[a,b]$, for given $\in$ there exists a $\delta_2 > 0$ such that for every partition *P* of $[a,b]$ with norm $\|P\| < \delta_2$ and for every corresponding Γ we have

$$\left|S(P,\Gamma,f,\alpha)-\int_a^b f d\alpha\right| < \in/2 \qquad \text{... (7)}$$

Let $\delta = \min\{\delta_1, \delta_2\}$.

$\therefore$ For $0 \leq |x-y| < \delta$, (6) holds good and for $\|P\| < \delta$, (7) holds good. If *P* be *a* partition with norm $\|P\| < \delta$ then $|t_i - \xi_i| < \delta$ for each *i* and so $\left|\alpha'(t_i)-\alpha'(\xi_i)\right| < \frac{\in}{2M(b-a)}$.

For partition *P* with $\|P\| < \delta$ we thus have from (4)

$\left|S(P,\Gamma,f,\alpha)-S(P,\Gamma,g)\right|$

$$\leq \sum_{i=1}^{n} M\left\{\frac{\in}{2M(b-a)}\right\}\Delta x_i$$

$$= \frac{M\in}{2M(b-a)}\sum_{i=1}^{n}(x_i - x_{i-1})$$

$=\frac{\in}{2(b-a)}(b-a)$

$=\in/2$

i.e. $\left|S(P,\Gamma,f,\alpha)-S(P,\Gamma,g)\right|<\in/2 \ \forall P$ with $\|P\|<\delta$ ... (8)

Now for P with $\|P\|<\delta$ we have

$\left|S(P,\Gamma,g)-\int_a^b f d\alpha\right|$

$=\left|\left\{S(P,\Gamma,f,\alpha)-\int_a^b f d\alpha\right\}-\left\{S(P,\Gamma,f,\alpha)-S(P,\Gamma,g)\right\}\right|$

$\leq\left|\left\{S(P,\Gamma,f,\alpha)-\int_a^b f d\alpha\right\}\right|+\left|S(P,\Gamma,f,\alpha)-S(P,\Gamma,g)\right|$

$<\in/2+\in/2$ [Using (7) and (8)]

$=\in$

Hence for given $\in$ there exists $\delta>0$ such that for every partition P with $\|P\|<\delta$

$$\left|S(P,\Gamma,f\alpha')-\int_a^b f d\alpha\right|<\in.$$

This shows that $f(x)\alpha'(x)$ is Riemann integrable over $[a,b]$ and the value of the integral is $\int_a^b f d\alpha$.

Hence $\int_a^b f\alpha' dx=\int_a^b f d\alpha$.

This proves the theorem.

2.9 Illustrative Examples

Example 2.9.1 If $\alpha(x)=c$ for all x, then show that $\int_a^b f(x)d\alpha(x)=0$.

Solution. Let $P=\{a=x_0,x_1,x_2,...,x_n=b\}$ be any partition of $[a,b]$ and

$\Gamma = \{\xi_1, \xi_2, ..., \xi_n\}$ where $x_{i-1} \le \xi_i \le x_i$ for each $i = 1,2,...,n$.

Then $S(P,\Gamma,f,\alpha)$

$$= \sum_{i=1}^{n} f(\xi_i)\{\alpha(x_i) - \alpha(x_{i-1})\}$$

$$= \sum_{i=1}^{n} f(\xi_i)\{c - c\}$$

$$= 0. \qquad ...(1)$$

This is true for all P and Γ. The RHS of (1) is independent of P and Γ.

Hence $\lim_{|P| \to 0} S(P,\Gamma,f,\alpha)$ exists and the value of the limit is 0.

$$\therefore \int_a^b f(x)\,d\alpha(x) = 0.$$

Example 2.9.2. If $f(x) = c$ for all x then $\int_a^b f(x)\,d\alpha(x)$

$$= c\{\alpha(b) - \alpha(a)\}$$

Solution. Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a,b]$ and $\Gamma = \{\xi_1, \xi_2, ..., \xi_n\}$ where $x_{i-1} \le \xi_i \le x_i$ for each $i = 1,2,...,n$.

Then $S(P,\Gamma,f,\alpha)$

$$= \sum_{i=1}^{n} f(\xi_i)\{\alpha(x_i) - \alpha(x_{i-1})\}$$

$$= \sum_{i=1}^{n} c\{\alpha(x_i) - \alpha(x_{i-1})\}$$

$$= c\{\alpha(x_n) - \alpha(x_0)\}$$

$$= c\{\alpha(b) - \alpha(a)\}$$

This result is true for all P and Γ. The RHS is independent of P and Γ.

$\therefore \lim_{\|P\|\to 0} S(P,\Gamma,f,\alpha)$ exists and the value of the limit is $c\{\alpha(b)-\alpha(a)\}$

Hence $\int_a^b f d\alpha = c\{\alpha(b)-\alpha(a)\}$.

Example 2.9.3. Let $f(x) \quad = 0 \quad 2 \le x < 3$

$= 1 \quad x = 3$

$= 0 \quad 3 < x \le 8$

and $\alpha(x) = 2x^2 + 5$. Show that f is RS-integrable w.r.t. α on [2,8] and $\int_2^8 f d\alpha = 0$.

Solution. Let $P = \{2 = x_0, x_1, x_2, ..., x_n = 8\}$ be any partition of [2,8] and $\Gamma = \{\xi_1, \xi_2, ..., \xi_n\}$ where $x_{i-1} \le \xi_i \le x_i$ for all $i = 1,2,...,n$.

Let $x_{k-1} \le 3 < x_k$.

Then $S(P,\Gamma,f,\alpha)$

$$= \sum_{i=1}^{n} f(\xi_i)\{\alpha(x_i) - \alpha(x_{i-1})\}$$

$$= \sum_{i=1}^{k-1} f(\xi_i)\{\alpha(x_i) - \alpha(x_{i-1})\} + f(\xi_k)\{\alpha(x_k) - \alpha(x_{k-1})\}$$

$$+ \sum_{i=k+1}^{n} f(\xi_i)\{\alpha(x_i) - \alpha(x_{i-1})\}$$

Since $f(x) = 0 \forall x \ne 3$ and $x_{k-1} \le 3 < x_k$ it follows that

$f(\xi_1) = f(\xi_2) = ... = f(\xi_{k-1}) = f(\xi_{k+1}) = ... = f(\xi_n) = 0.$

$$\therefore S(P,\Gamma,f,\alpha) = f(\xi_k)\{\alpha(x_k) - \alpha(x_{k-1})\}$$

$$= f(\xi_k)\{\alpha(x_k) - \alpha(3) + \alpha(3) - \alpha(x_{k-1})\}$$

$$= f(\xi_k)[\{\alpha(x_k) - \alpha(3)\} + \{\alpha(3) - \alpha(x_{k-1})\}] \quad ...(1)$$

Since $\alpha(x) = 2x^2 + 5$ we see $\alpha(x)$ is continuous at $x = 3$. Hence for given $\in > 0$ these exists $\delta > 0$ such that

$|\alpha(x)-\alpha(3)|<\in/2$ for $|x-3|<\delta$.

Here $\alpha(x)$ is an increasing function, so we have

$\alpha(x)-\alpha(3)<\in/2$ for $3<x<3+\delta$ ... (2)

and

$\alpha(3)-\alpha(x)<\in/2$ for $3-\delta<x<3$ ... (3)

$\therefore$ For partition P with norm $\|P\|<\delta$, using (2) and (3) we have $\alpha(x_k)-\alpha(3)<\in/2$ and $\alpha(3)-\alpha(x_{k-1})<\in/2$

$\therefore$ From (1) we have

$$S(P,\Gamma,f,\alpha)<\in f(\xi_k)$$

Now $f(\xi_k)=0$ if $\xi_k\neq 3$

$=1$ if $\xi_k=3$

$\therefore S(P,\Gamma,f,\alpha)$ <0 when $\xi_k\neq 3$

$<\in$ when $\xi_k=3$

Since $\in$ is arbitrary, in either case we have

$$\lim_{\|P\|\to 0} S(P,\Gamma,f,\alpha)=0$$

Hence f is RS-integrable w.r.t. α over [2,8]

and $\int_2^8 f d\alpha=0$.

Example 2.9.4. Let in the interval [-2,6] $f(x)$ and $g(x)$ be defined as follows

$f(x)=0$ for $-2\leq x\leq 4$

$= x^2$ for $4 < x \le 6$

and $g(x) = 3 - x^2$ for $-2 \le x < 4$

$= 5$ for $4 \le x \le 6$.

Show that *f* is not RS-integrable with respect to *g* over [-2,6].

Solution. Let $P = \{-2 = x_0, x_1, x_2, \ldots, x_{k-1}, x_k, \ldots, x_n = 6\}$ be a partition of [-2,6] where $x_{k-1} < 4 < x_k$. Also let $\Gamma = \{\xi_1, \xi_2, \ldots, \xi_n\}$ where $x_{i-1} \le \xi_i \le x_i$ for each $i = 1,2,\ldots,n$.

Then $S(P,\Gamma,f,g)$

$$= \sum_{i=1}^{n} f(\xi_i)\{g(x_i) - g(x_{i-1})\}$$

$$= \sum_{i=1}^{k-1} f(\xi_i)\{g(x_i) - g(x_{i-1})\} + f(\xi_k)\{g(x_k) - g(x_{k-1})\}$$

$$+ \sum_{i=k+1}^{n} f(\xi_i)\{g(x_i) - g(x_{i-1})\}$$

$$= \sum_{i=1}^{k-1} 0\{g(x_i) - g(x_{i-1})\} + f(\xi_k)\{5 - (3 - x_{k-1}^2)\}$$

$$+ \sum_{i=k+1}^{n} f(\xi_i)\{5 - 5\}$$

$\therefore S(P,\Gamma,f,g) = f(\xi_k)\cdot(2 + x_{k-1}^2)$ where $-2 \le x_{k-1} < 4$

Now $-2 \le x_{k-1} < 4 < x_k$ and $x_{k-1} \le \xi_k \le x_k$.

$\therefore$ If $x_{k-1} \le \xi_k \le 4$ then $f(\xi_k) = 0$ and

if $4 < \xi_k \le x_k$ then $f(\xi_k) = \xi_k^2$

$\therefore S(P,\Gamma,f,g) = 0$ if $x_{k-1} \le \xi_k \le 4$

$= \xi_k^2(2 + x_{k-1}^2)$ if $4 < \xi_k \le x_k$

As ξ_k can take any value in $[x_{k-1}, x_k]$ it follows that $\lim_{\|P\|\to 0} S(P,\Gamma,f,g)$ does not exist which means that f is not RS-integrable with respect to g over [-2,6].

Example 2.9.5. In the interval [-4,5] let $f(x)$ and $\alpha(x)$ be defined as follows

$$f(x) = e^x + 3x + 2$$

and $\alpha(x) = 0, -2 \le x < 0$

$$= \frac{1}{2}, x = 0$$

$$= 1, 0 < x \le 5.$$

Show that f is RS-integrable w.r.t. α over [-4,5]. Find $\int_{-4}^{5} f d\alpha$.

Solution. Let $P = \{-4 = x_0, x_1, x_2, \ldots, x_{k-1}, x_k, \ldots, x_n = 5\}$ be any partition of [-4,5] and $\Gamma = \{\xi_1, \xi_2, \ldots, \xi_k, \ldots, \xi_n\}$ where $x_{i-1} \le \xi_i \le x_i$ for each $i = 1,2,\ldots,n$.

Let $x_{k-1} \le 0 < x_k$.

Then $S(P,\Gamma,f,\alpha)$

$$= \sum_{i=1}^{n} f(\xi_i)\{\alpha(x_i) - \alpha(x_{i-1})\}$$

$$= \sum_{i=1}^{k-1} f(\xi_i)\{\alpha(x_i) - \alpha(x_{i-1})\} + f(\xi_k)\{\alpha(x_k) - \alpha(x_{k-1})\}$$

$$+ \sum_{i=k+1}^{n} f(\xi_i)\{\alpha(x_i) - \alpha(x_{i-1})\} \qquad \ldots (1)$$

When $x_{k-1} = 0$ then from (1) we have

$S(P,\Gamma,f,\alpha)$

$$= f(\xi_{k-1})\{\alpha(x_{k-1}) - \alpha(x_{k-2})\} + f(\xi_k)\{\alpha(x_k) - \alpha(x_{k-1})\}$$

$$+ \sum_{i=k+1}^{n} f(\xi_{k-1})\{1-1\}$$

$$= f(\xi_{k-1})\left\{\frac{1}{2}-0\right\} + f(\xi_k)\left\{1-\frac{1}{2}\right\} + 0$$

$$= \frac{1}{2}\{f(\xi_{k-1}) + f(\xi_k)\}.$$

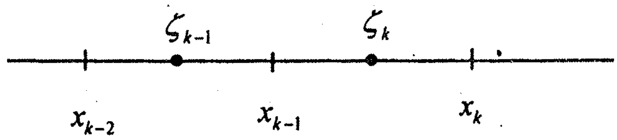
When $x_{k-1} < 0 < x_k$ then from (1) we have

$$S(P,\Gamma,f,\alpha)$$

$$= \sum_{i=1}^{k-1} f(\xi_i)\{0-0\} + f(\xi_k)\{1-0\}$$

$$+ \sum_{i=k+1}^{n} f(\xi_i)\{1-1\}$$

$$= f(\xi_k)$$



Thus $S(P,\Gamma,f,\alpha) = \begin{cases} f(\xi_k) & \text{whe. } x_{k-1} < 0 < x_k \\ \frac{1}{2}\{f(\xi_{k-1}) + f(\xi_k)\} & \text{when } x_{k-1} = 0 < x_k \end{cases}$ ... (2)

Now $f(x) = e^x + 3x + 2$ is continuous at $x = 0$.

$\therefore \lim_{x\to 0} f(x) = f(0) = e^0 + 3.0 + 2 = 3$

As either $x_{k-1} < 0 < x_k$ or $x_{k-1} = 0 < x_k$ and $x_{k-2} \le \xi_{k-1} \le x_{k-1}, x_{k-1} \le \xi_k \le x_k$, it follows that both ξ_{k-1} and $\xi_k \to 0$ as $\|P\| \to 0$.

Since $f(x)$ is continuous at $x = 0$ from (2) it follows that $\lim_{\|P\|\to 0} S(P,\Gamma,f,\alpha)$ exists and the value of the limit is $f(0)$.

Hence f is RS-integrable w.r.t. α over [-4,5] and the value of the integral is $f(0)$ i.e. $\int_{-4}^{5} f d\alpha = f(0) = 3$.

2.10 Summary. Riemann-Stieltjes integral is one kind of generisation of Riemann integral. In this module the limit definition of RS-integral has been considered. Theorems and examples have been studied for a clear conception of this integral. The bound definition of RS-integral will be studied in the next module.

2.11 Self Assessment Questions

1. If $f(x)=5$ for all x and $\alpha(x)=2x^3-3x+7$ then find $\int_2^{10} f d\alpha$.

2. If $f(x)=3e^x-\log x+\sin x+2x^2$ and $\alpha(x)=3$ for all x then find $\int_2^4 f d\alpha$.

3. If $f(x)=0, -1 \le x < 4$

   $= 4, x = 4$

   $= 0, 4 < x \le 8$

   and $\alpha(x)=5x-4$ show that f is RS-integrable w.r.t. α over $[-1,8]$ and determine the value of the integration.

4. If $f(x)=0$ for $5 \le x \le 8$

   $= 2x+1$ for $8 < x \le 10$

   and $\alpha(x)=3x^2+5$ for $5 \le x < 8$

   $= 7$ for $8 \le x \le 10$

   show that f is not RS-integrable w.r.t. α on $[5,10]$.

5. Let α be monotonically increasing function on $[a,b]$ and be continuous at x_0 where $a < x_0 < b$ and f be such that $f(x_0)=1$ and $f(x)=0$ for $x \ne x_0$. Show that f is RS-integrable with respect to α over $[a,b]$ and $\int_a^b f d\alpha = 0$.

6. If $f(x)$ is a function bounded on $[-1,1]$ and is continuous at $x=0$ and $\alpha(x)$ be defined as

   $\alpha(x)=0, x<0$

   $=\frac{1}{2}, x=0$

   $=1, x>0$

then show that f is RS-integrable w.r.t. α over $[-1,1]$. Find the value of the integral.

7. Show that $\int_0^2 x^2 d(x) = 8$

8. Lit $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & 1 \le x \le 4 \end{cases}$

and $g(x) = \begin{cases} 0, & 0 \le x \le 1 \\ 1, & 1 < x \le 4 \end{cases}$

Discuss whether f is RS-integrable w.r.t. of over [0,4]

2.12 Suggested books for further reading

1. Introduction to Mathematical Analysis: Amritava Gupta; Academic Publishers, Calcutta.
2. Mathematical Analysis: Tom. M. Apostol; Narosa Publishing House.
3. Mathematical Analysis:.S.C. Malik & Savita Arora; Wiley Eastern Limited, New Age International Limited.
4. Principle of Mathematical Analysis; Walter Rudin; International Students Edition, McGraw-Hill International Book Company.
5. Elements fo Real Analysis: Shanti Narayan & M.D. Raisinghania; S. Chand.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-I Group–A

Module No. - 3

Real Analysis

(Riemann Stieltjes Integral – II)

Module Structure

3.1 Introduction :

In module 2 we have considered the limit definition of Riemann Stieltjes integral. In this definition no conditions were imposed on the integrator. In the Riemann integral the integrator is $\alpha(x) = x$ which is monotonic increasing. So monotonic increasing integrator should give a special attention. In this module special integrators like monotonic increasing integrator, functions of bounded variation as integrators and step functions as integrators have been considered. Theorems and results for integrals with such special functions as integrators have been studied. Many remarkable and beautiful results are found for RS-integrals with these integrators.

3.2 Objective

The objective of this module is to study RS-integrals with monotonic increasing functions, functions of bounded variations and step functions as integrators. The bound definition is valied only for monotonic increasing functions as integrator. It is seen that the bound definition and limit definition for RS-integral are not equivalent though they are equivalent for Reimann integral. It is proved here that any finite sum can be expressed as RS-integral with step functions as integrator. First and second mean value theorems have been also studied in this module for RS-integral.

3.3 Monotonic increasing functions as integrators

In the Riemann integral the integrator is the function $g(x)=x$. This function is monotonic increasing, continuous and derivable. In the bound definition if Riemann integral $\Delta x_i = x_i - x_{i-1}$ plays an important role and the property that $\Delta x_i > 0$ has been used. In the RS-integral the role of Δx_i is taken by $\Delta g_i = g(x_i) - g(x_{i-1})$. To get the property $\Delta g_i > 0$ we require the monotonic increasing behaviour of the integrator $g(x)$. The upper sum-lower-sum formalism can be developed to a certain extent as follows.

Definition 3.3.1. Let f be bounded and g be monotonic increasing on $[a, b]$ and $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$. Then upper and lower Riemann-Stieltjes sums of f w.r.t. g over $[a, b]$ are defined respectively as

$$U(P, f, g) = \sum_{i=1}^{n} M_i \Delta g_i$$

and $$L(P, f, g) = \sum_{i=1}^{n} m_i \Delta g_i$$

where $\Delta g_i = g(x_i) - g(x_{i-1})$

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

Theorem 3.3.1. If f is bounded and g is monotinic increasing on $[a, b]$ then for any partition P of $[a, b]$

$$m[g(b)-g(a)] \le L(P,f,g) \le U(P,f,g) \le M[g(b)-g(a)]$$

where $M = \sup\{f(x) : x \in [a,b]\}$

$m = \inf\{f(x) : x \in [a,b]\}$.

Proof. Let $P = \{a = x_0, x_1, x_2 ..., x_n = b\}$ be any partition of $[a, b]$. Clearly we have for each $i = 1, 2, ..., n$

$$m \le m_i \le M_i \le M \qquad \text{.............(1)}$$

Now $\Delta g_i = g(x_i) - g(x_{i-1}) \ge 0$ as g is monotonic increasing.

$\therefore$ From (1) we have

$$m\Delta g_i \le m_i \, \Delta g_i \le M_i \, \Delta g_i \le M \ \Delta g_i$$

This is true for all $i = 1, 2, ..., n$. So we have

$$\sum_{i=1}^{n} m \, \Delta g_i \le \sum_{i=1}^{n} m_i \, \Delta g_i \le \sum_{i=1}^{n} M_i \, \Delta g_i \le \sum_{i=1}^{n} M \, \Delta g_i$$

or, $$m\sum_{i=1}^{n}\{g(x_i)-g(x_{i-1})\} \le L(P,f,g) \le U(P,f,g) \le M\sum_{i=1}^{n}\{g(x_i)-g(x_{i-1})\}$$

or, $$m\{g(b)-g(a)\} \le L(P,f,g) \le U(P,f,g) \le M\{g(b)-g(a)\}.$$

Let $\Pi[a,b]$ be the set of all partitions of the interval $[a, b]$.

From Theorem 3.3.1 we see that the infinite set $\{L(P,f,g) : P \in \Pi[a,b]\}$ is bounded above and the infinite set $\{U(P,f,g) : P \in \Pi[a,b]\}$ is bounded below. Hence

$\sup\{L(P,f,g) : P \in \Pi[a,b]\}$ and $\inf\{U(P,f,g) : P \in \Pi[a,b]\}$ exist.

Definition 3.3.2. If f is bounded and g is monotonic increasing on $[a, b]$ then we define

$$\sup\{L(P,f,g) : P \in \Pi[a,b]\} = \underline{\int_a^b} f dg = \underline{I}$$

and $\inf\{U(P,f,g): P \in \Pi[a,b]\} = \overline{\int_a^b} fdg = \overline{I}$

and call them respectively as lower and upper Riemann-Stieltijes integrals of f w.r.t. g over $[a, b]$.

As in Riemann-integral we can easily prove the following theorem.

Theorem 3.3.2. If f is bounded and g is monotonic increasing on $[a, b]$ and $P, P' \in \Pi[a,b]$ such that $P' \supset P$ i.e. P' is refinement of P, then

$$U(P',f,g) \leq U(P,f,g)$$

and $L(P',f,g) \geq L(P,f,g)$.

Now we prove the following theorem.

Theorem 3.3.3. If f is bounded and g is monotonic increasing on $[a, b]$ then $\underline{I} \leq \overline{I}$.

Proof. Let P and P' be any two partition of $[a, b]$. Then $P \cup P'$ is a refinement of both P and P'.

So we have

$$L(P,f,g) \leq L(P \cup P',f,g) \leq U(P \cup P',f,g) \leq U(P',f,g)$$

Keeping P fixed and taking infimum over P', we have

$$L(P,f,g) \leq \inf\{U(P',f,g): P' \in \Pi[a,b]\}$$

or, $L(P,f,g) \leq \overline{\int_a^b} f\,dg$

or, $L(P,f,g) \leq \overline{I}$

Now taking supremum over P, we get

$$\sup\{L(P,f,g): P \in \Pi[a,b]\} \leq \overline{I}$$

or, $\underline{\int_a^b} f\,dy \leq \overline{I}$

or, $\underline{I} \le \overline{I}$. This proves the theorem.

It is the time too take an attempt to show that the equality of upper and lower Riemann-Stieltjes integrals is an equivalent criterion of integrability in the sense of limit definition. But this attempt fails, as such a result essentially depends on the analogue of Darboux Theorem of Riemann integral. In this case Darboux Theorem does not hold without imposition of additionl conditions, like the continuity of the integrator g. However, more fruitful results are obtained by assuming continuity of the integrand f which at once guarantees the existence of the integral $\int_a^b f\,dg$. Thus we have the following theorem.

Theorem 3.3.4. If f is continuous and g is monotonic increasing on $[a, b]$ then the upper and lower Riemann Stieltjes integrals are equal and $\int_a^b f\,dg$ exists in the limit sense. Also for partitions P of $[a, b]$

$$\lim_{\|P\|\to 0} U(P,f,g) = \lim_{\|P\|\to 0} L(P,f,g) = \int_a^b f\,dg = \underline{I}$$

$$= \overline{I} = \lim_{\|P\|\to 0} S(P,\Gamma,f,g).$$

Proof. Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$.

Now $\underline{I} = \sup\{L(P,f,g) : P \in \Pi[a,b]\}$

and $\overline{I} = \inf\{U(P,f,g) : P \in \Pi[a,b]\}$

So, $L(P,f,g) \le \underline{I} \le \overline{I} \le U(P,f,g)$. .... (1)

Let $\xi_i \in [x_{i-1}, x_i]$ for each $i = 1, 2, ..., n$ and $\Gamma = \{\xi_1, \xi_2, ..., \xi_n\}$.

Then $L(P,f,g) \le S(P,\Gamma,f,g) \le U(P,f,g)$. .... (2)

Now f is continuous in the closed interval $[a, b]$ and so it is uniformly continuous in $[a, b]$.

$\therefore$ For given $\in > 0$ there exists $\delta > 0$, depending on $\in$ only such that

$|f(x) - f(x')| < \in / \{g(b) - g(a)\}$ .... (3)

for all $x, x' \in [a,b]$ such that $|x - x'| < \delta$.

As f is continuous in $[x_{i-1}, x_i]$ there exists ξ_i' and η_i' in it such that

$$f(\xi_i') = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = m_i$$

and $f(\eta_i') = \sup\{f(x) : x \in [x_{i-1}, x_i]\} = M_i$.

Now we choose P such that $\|P\| < \delta$. Then $|\xi_i' - \eta_i'| < \delta$ and so from (3) we have

$$|f(\xi_i') - f(\eta_i')| < \in/\{g(b) - g(a)\}$$

or, $|M_i - m_i| < \in/\{g(b) - g(a)\}$

or, $M_i - m_i < \in/\{g(b) - g(a)\}$ $[\because M_i \geq m_i]$. .... (4)

Since $g(x)$ is monotonic increasing we have

$\Delta g_i = g(x_i) - g(x_{i-1}) \geq 0$.

So (4) gives

$$M_i \Delta g_i - m_i \Delta g_i \leq \in \Delta g_i / \{g(b) - g(a)\}.$$

This is true for all $i = 1, 2, ..., n$ and if $g(x)$ is not constant function then for at least one i we have $\Delta g_i > 0$.

Hence $\sum_{i=1}^{n} M_i \Delta g_i - \sum_{i=1}^{n} m_i \Delta g_i < \sum_{i=1}^{n} \in \Delta g_i / \{g(b) - g(a)\}$

or, $U(P,f,g) - L(P,f,g) < \in$. .... (5)

From (1) and (5) we have

$$\bar{I} - \underline{I} \leq U(P,f,g) - L(P,f,g) < \in.$$

As $\in$ is arbitrary this gives $\underline{I} = \bar{I}$. Let $\underline{I} = \bar{I} = I$ .... (6)

Then from (1),

$$L(P,f,g) \leq I \leq U(P,f,g)$$ .... (7)

Using (2) and (7) we get from (5)

$$|S(P,\Gamma,f,g)-I|<\in.$$

This is true for all P with norm $\|P\|<\delta$.

Hence f is RS-integrable in the limit sense

and $\int_a^b f\,dg = I = \lim_{\|P\|\to 0} S(P,\Gamma,f,g).$

$\therefore$ From (6), $\underline{I}=\overline{I}=I=\int_a^b f\,dg.$ .... (8)

From (1) we have for P with norm $\|P\|<\delta$

$$0\le U(P,f,g)-\overline{I}\le U(P,f,g)-L(P,f,g)<\in \text{ [using (5)]}$$

and $0\le \underline{I}-L(P,f,g)\le U(P,f,g)-L(P,f,g)<\in$ [using(5)].

$$\therefore \lim_{\|P\|\to 0} U(P,f,g)=\overline{I}=I=\int_a^b f\,dg$$

and $\lim_{\|P\|\to 0} L(P,f,g)=\underline{I}=I=\int_a^b f\,dg.$

Hence the theorem is proved.

Important Note : The conditions in the Theorem 3.3.4 are sufficient. This is seen in the following example.

Example 3.3.1 Let $f(x)=0$ for $x<0$

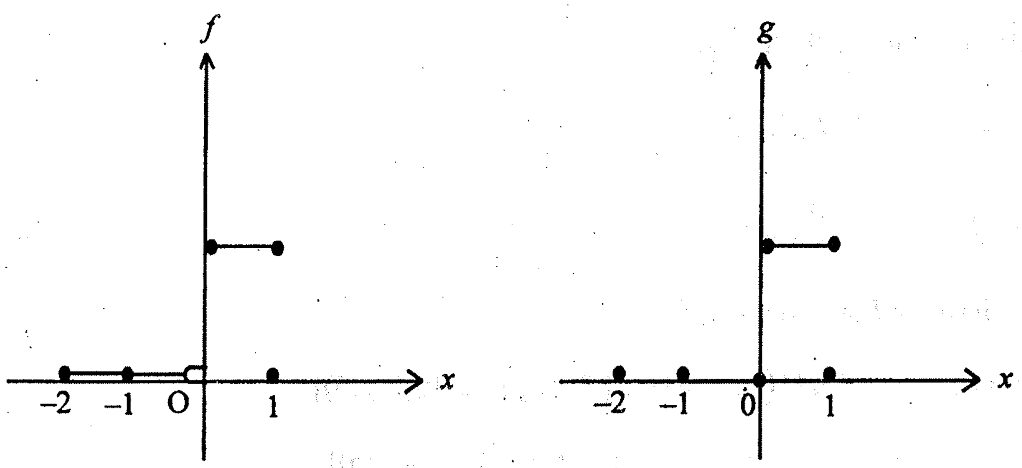
$= 1$ for $x\ge 0$

and $g(x)=0$ for $x\le 0$

$= 1$ for $x>0$.

Show that $\underline{I}=\overline{I}$ in the interval $[-2, 1]$ though f is not continuous in $[-2, 1]$.

Solution. Here $f(x)$ is not continuous at $x = 0$. So $f(x)$ is not continuous in $[-2,1]$. Let $P = [-2 = x_0, x_1, x_2, ..., x_n = 1]$ be any partition of $[-2, 1]$ where $x_{k-1} < 0 \le x_k$.



Case 1. $x_{k-1} < 0 < x_k$.

$$U(P,f,g) = \sum_{i=1}^{n} M_i \Delta g_i = \sum_{i=1}^{k-1} M_i \Delta g_i + M_k \Delta g_k + \sum_{i=k+1}^{n} M_i \Delta g_i$$

$$= \sum_{i=1}^{k-1} 0 \Delta g_i + 1 \cdot (1-0) + \sum_{i=k+1}^{n} 1 \cdot 0$$

$$= 1$$

$$L(P,f,g) = \sum_{i=1}^{n} m_i \Delta g_i = \sum_{i=1}^{k-1} m_i \Delta g_i + m_k \Delta g_k + \sum_{i=k+1}^{n} m_i \Delta g_i$$

$$= \sum_{i=1}^{k-1} 0 \Delta g_i + 0 \cdot (1-0) + \sum_{i=k+1}^{n} 1 \cdot 0$$

$$= 0$$

Case 2. $x_{k-1} < 0 < x_k$.

$$U(P,f,g) = \sum_{i=1}^{n} M_i \Delta g_i = \sum_{i=1}^{k-1} M_i \Delta g_i + M_k \Delta g_k + \sum_{i=k+1}^{n} M_i \Delta g_i$$

$$= \sum_{i=1}^{k-1} 0\,\Delta g_i + 1\cdot 0 + \sum_{i=k+1}^{n} 1\cdot 0$$

$$= 0$$

$$L(P,f,g) = \sum_{i=1}^{n} m_i\,\Delta g_i = \sum_{i=1}^{k-1} m_i\,\Delta g_i + m_k\,\Delta g_k + \sum_{i=k+1}^{n} m_i\,\Delta g_i$$

$$= \sum_{i=1}^{k-1} 0\,\Delta g_i + 0\cdot 0 + \sum_{i=k+1}^{n} 1\cdot 0$$

$$= 0$$

$\therefore \underline{I} = \sup\{L(P,f,g) : P \in \Pi[a,b]\} = \sup\{0,0\} = 0$

and $\overline{I} = \inf\{U(P,f,g) : P \in \Pi[a,b]\} = \inf\{1,0\} = 0.$

$\therefore \underline{I} = \overline{I}$ though $f(x)$ is not continuous in $[-2, 1]$.

Definition 3.3.2. Let f and α be bounded functions on $[a, b]$ and g be monotonic increasing on $[a, b]$, $b \geq a$. The function f is said to be Riemann-Stieltjes integrable with respect to g over $[a, b]$ if $\underline{I} = \overline{I}$.

This definition is called the bound definition of R.S integral.

Note : The limit definition and the bound definition of RS-integral are not equivalent though they are equivalent for Riemann integral. The following examples shows this.

Example 3.3.2. Let $f(x) = 0$ for $x < 0$

$= 1$ for $x \geq 0$

and $g(x) = 0$ for $x \leq 0$

$= 1$ for $x > 0$.

Show that f is RS-integrable w.r.t. g following bound definition over $[-2, 1]$ but not RS-integrable following limit definition.

Solution. In Example 3.3.1 we have shown that $\underline{I} = \overline{I}$.

So f is RS-integrable w.r.t. g following bound definition.

Following the arguments given in the solution of Example 2.6.1 we can easily show that f is not RS-integrable w.r.t. g over $[-2, 1]$ following limit definition.

Example 3.3.3. Let $f(x) = 4$ for all x and $g(x) = 2-x$ for all x. Show that f is RS-integrable w.r.t. g over $[1,3]$ following limit definition but not following bound definition.

Solution. Let $P = [1 = x_0, x_1, x_2, ..., x_n = 3]$ be any partition of $[1,3]$ and $\Gamma = \{\xi_1, \xi_2, ..., \xi_n\}$ where for each $i = 1, 2, ... n$, $\xi_i \in [x_{i-1}, x_i]$

Now $S(P,\Gamma,f,g)$

$$= \sum_{i=1}^{n} f(\xi_i)\{g(x_i) - g(x_{i-1})\}$$

$$= \sum_{i=1}^{n} 4\{(2-x_i)-(2-x_{i-1})\}$$

$$= \sum_{i=1}^{n} 4(x_{i-1} - x_i)$$

$$= 4(x_0 - x_n)$$

$$= 4(1-3)$$

$$= -8$$

As the RHS is independent of P & Γ, it follows that $\lim_{|P|\to 0} S(P,\Gamma,f,g)$ exists and the limit is -8.

Hence f is RS-integrable w.r.t. g over $[1, 3]$ following limit definition and $\int_1^3 f\, dg = -8$.

As here $g(x)$ is not monotonic increasing the bound definition of RS-integral is not at all applicable.

3.4. Some theorems for monotonic increasing integrators

In the Riemann integral there are a number of theorems. When the integrator $g(x)$ is monotonic increasing many theorems of Riemanns integral are found to be true for RS-integral. The proof of these theorems are exactly similar to those of Riemann integral. So without giving the proof we only state these theorems.

Theorem 3.4.1. A function f is integrable with respect to the monotonic increasing integrator g on $[a, b]$ if and only if for every $\in > 0$ there exists a partition P of $[a, b]$ such that $U(P, f, g) - L(P, f, g) < \in$.

Theorem 3.4.2. If $f_1 \in R(g)$ and $f_2 \in R(g)$ over $[a, b]$, then $f_1 + f_2 \in R(g)$ and $\int_a^b (f_1 + f_2)\,dg = \int_a^b f_1\,dg + \int_a^b f_2\,dg.$

Theorem 3.4.3. If $f \in R(g)$ and c is a constant, then $cf \in R(g)$ and $\int_a^b cf\,dg = c\int_a^b f\,dg.$

Theorem 3.4.4. If $f_1 \in R(g), f_2 \in R(g)$ and $f_1(x) \le f_2(x)$ on $[a, b]$, then $\int_a^b f_1\,dg \le \int_a^b f_2\,dg.$

Theorem 3.4.5. If $f \in R(g)$ over $[a, b]$ and if $a < c < b$, then $f \in R(g)$ on $[a, c]$ and on $[c, b]$ and

$$\int_a^b f\,dg = \int_a^c f\,dg + \int_c^b f\,dg.$$

Theorem 3.4.6. If $f \in R(g)$ over $[a, b]$, then

$$f \in R(g) \text{ and } \left|\int_a^b f\,dg\right| \le \int_a^b |f|\,dg.$$

Theorem 3.4.7. If $f \in R(g)$ on $[a, b]$, then $f^2 \in R(g)$.

The following two theorems are valid only for RS-integral.

Theorem 3.4.8. If $f \in R(g)$ and $f \in R(g_2)$ over $[a, b]$ then $f \in R(g_1 + g_2)$ over $[a, b]$ and

$$\int_a^b f\,d(g_1 + g_2) = \int_a^b f\,dg_1 + \int_a^b f\,dg_2.$$

Proof. Since $f \in R(g_1)$ and $f \in R(g_2)$ therefore for given $\in > 0$, there exist partitions P_1 and P_1 of $[a,b]$ such that

$$U(P_1, f, g_1) - L(P_1, f, g_1) < \in/2$$

$$U(P_2, f, g_2) - L(P_2, f, g_2) < \in/2.$$

Let $P = P_1 \cup P_2$.

$$\therefore U(P,f,g)-L(P,f,g)<\epsilon/2 \qquad \text{.................... (1)}$$

$$U(P,f,g_2)-L(P,f,g_2)<\epsilon/2. \qquad \text{.................... (2)}$$

Let the partition P be $\{a = x_0, x_1, x_2, ..., x_n = b\}$

and m_i, M_i be bounds of f in $[x_{i-1}, x_i]$.

Let $g = g_1 + g_2$.

$$\therefore g(x) = g_1(x) + g_2(x)$$

$$\Delta g_{1i} = g_1(x_i) - g_1(x_{i-1})$$

$$\Delta g_{2i} = g_2(x_i) - g_2(x_{i-1}).$$

$$\therefore \Delta g_i = g(x_i) - g(x_{i-1})$$

$$= \{g_1(x_i) + g_2(x_i)\} - \{g_1(x_{i-1}) + g_2(x_{i-1})\}$$

$$= \Delta g_{1i} + \Delta g_{2i}.$$

$$\therefore U(P,f,g) = \sum_{i=1}^{n} M_i \Delta g_i$$

$$= \sum_{i=1}^{n} M_i (\Delta g_{1i} + \Delta g_{2i})$$

$$= \sum_{i=1}^{n} M_i \Delta g_{1i} + \sum_{i=1}^{n} M_i \Delta g_{2i}$$

$$= U(P,f,g_1) + U(P,f,g_2)$$

Similarly, $L(P,f,g) = L(P,f,g_1) + L(P,f,g_2)$

$$\therefore U(P,f,g) - L(P,f,g)$$

$$= U(P,f,g_1) - L(P,f,g_1) + U(P,f,g_2) - L(P,f,g_2)$$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$ [using (1) and (2)]

$= \epsilon$

i.e. For given $\epsilon > 0, \exists$ partition P of $[a, b]$ such that

$$U(P, f, g) - L(P, f, g) < \epsilon.$$

This proves that $f \in R(g)$ i.e. $f \in R(g_1 + g_2)$.

Now we prove the second part.

Since $f \in R(g)$ we have

$$\begin{aligned}
\int_a^b f\, dg = \overline{\int_a^b} f\, dg &= \inf U(P, f, g) \\
&= \inf \{U(P, f, g_1) + U(P, f, g_2)\} \\
&\geq \inf U(P, f, g_1) + \inf U(P, f, g_2) \\
&= \overline{\int_a^b} f\, dg_1 + \overline{\int_a^b} f\, dg_2 \\
&= \int_a^b f\, dg_1 + \int_a^b f\, dg_2
\end{aligned}$$

i.e., $\int_a^b f\, dg \geq \int_a^b f\, dg_1 + \int_a^b f\, dg_2$. .... (3)

Similarly,

$$\begin{aligned}
\int_a^b f\, dg = \underline{\int_a^b} f\, dg &= \sup L(P, f, g) \\
&= \sup \{L(P, f, g_1) + L(P, f, g_2)\} \\
&\leq \sup L(P, f, g_1) + \sup L(P, f, g_2) \\
&= \underline{\int_a^b} f\, dg_1 + \underline{\int_a^b} f\, dg_2
\end{aligned}$$

$$= \int_a^b f\,dg_1 + \int_a^b f\,dg_2$$

i.e., $\int_a^b f\,dg \geq \int_a^b f\,dg_1 + \int_a^b f\,dg_2$. .... (4)

From (3) and (4) we have

$$\int_a^b f\,dg = \int_a^b f\,dg_1 + \int_a^b f\,dg_2.$$

Theorem 3.4.9. If $f \in R(g)$ and c is a positive constant, then $f \in R(cg)$ and $\int_a^b f\,d(cg) = c\int_a^b f\,dg$.

Proof. Let $f \in R(g)$. Therefore for given $\in > 0$, there exists partition $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ of $[a, b]$ such that

$$U(P,f,g) - L(P,f,g) < \in/c \qquad \text{..................... (1)}$$

Let $\alpha = cg$.

$\therefore \alpha(x) = (cg)(x) = cg(x)$. Since $c>0$ and g is m.i. we have α is m.i.

Now, $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

$$= cg(x_i) - cg(x_{i-1})$$

$$= c\{g(x_i) - g(x_{i-1})\}$$

$$= c\,\Delta g_i.$$

$$\therefore U(P,f,\alpha) = \sum_{i=1}^{n} M_i\,\Delta\alpha_i$$

$$= \sum_{i=1}^{n} M_i\,c\Delta g_i$$

$$= c\sum_{i=1}^{n} M_i\,\Delta g_i$$

$$= cU(P,f,g).$$

Similarly $= L(P,f,\alpha) = cL(P,f,g)$.

$$\therefore U(P,f,g) - L(P,f,g)$$

$$= cU(P,f,g) - cL(P,f,g)$$

$$= c\{U(P,f,g) - L(P,f,g)\}$$

$$< c\{\in/c\} \qquad \text{[using (1)]}$$

$$= \in.$$

i.e. For given $\in > 0, \exists$ partition P of $[a, b]$ s.t.

$$U(P,f,\alpha) - L(P,f,\alpha) < \in.$$

Hence $f \in R(\alpha)$ i.e. $f \in R(cg)$.

Since $f \in R(\alpha)$ we have

$$\int_a^b d\alpha = \overline{\int_a^b} f\, d\alpha = \inf U(P,f,\alpha)$$

$$= \inf cU(P,f,g)$$

$$= c \inf U(P,f,g) \quad (\because c > 0)$$

$$= c \overline{\int_a^b} f\, dg$$

$$= c \int_a^b f\, dg$$

i.e. $\int_a^b f\, d(cg) = c\int_a^b f\, dg.$

We now prove the Mean Value Theorems for Riemann Stieljes integral.

Theorem 3.4.10. First Mean Value Theorem.

i) If f is RS-integrable with respect to g over $[a, b]$ where g is monotonic increasing on $[a, b]$, then there exists a number μ such that $m \le \mu \le M$, where $M = \sup\{f(x) : x \in [a,b]\}$ and $m = \inf\{f(x) : x \in [a,b]\}$ and

$$\int_a^b f(x)\,dg(x) = \mu\{g(b) - g(a)\}$$

ii) If f is continuous and g is monotonic increasing on $[a, b]$, then there exists a point $\xi \in [a,b]$ such that

$$\int_a^b f(x)\,dg(x) = f(\xi)\{g(b) - g(a)\}.$$

Proof.

(i) For any $x \in [a,b]$ we have

$$m \le f(x) \le M.$$

Since $g(x)$ is monotonic increasing on $[a, b]$ and $f \in R(g)$ we have by Theorem 3.4.4. that

$$\int_a^b m\,dg \le \int_a^b f\,dg \le \int_a^b M\,dg$$

or, $m\{g(b) - g(a)\} \le \int_a^b f\,dg \le M\{g(b) - g(a)\}$

or, $m \le \left(\int_a^b f\,dg\right) \Big/ \{g(b) - g(a)\} \le M \quad [\because g(b) > g(a)]$

Let $\mu = \left(\int_a^b f\,dg\right) \Big/ \{g(b) - g(a)\}$.

Then $\int_a^b f\,dg = \mu\{g(b) - g(a)\}$ and $m \le \mu \le M$.

ii) Here f is continuous and g is monotonic increasing on $[a, b]$. Therefore, f is RS-integrable w.r.t. g on $[a,b]$. So, by the first part there exists μ such that $m \le \mu \le M$ where $m = \inf\{f(x) : x \in [a,b]\}$ and

$M = \sup\{f(x) : x \in [a,b]\}$ and

$$\int_a^b f(x)\,dg(x) = \mu\{g(b) - g(a)\}. \qquad \text{.................. (1)}$$

Since $f(x)$ is continuous over $[a, b]$, there exists at least one ξ in $[a, b]$ such that $f(\xi) = \mu$.

So from (1) we get

$$\int_a^b f(x)dg(x) = f(\xi)\{g(b)-g(a)\}.$$

Theorem 3.4.11. Second Mean Value Theorem.

If f is monotonic increasing and g is continuous on $[a, b]$, then there exists a point $\xi \in [a, b]$, such that

$$\int_a^b f(x)dg(x) = f(a)\int_a^\xi dg(x) + f(b)\int_\xi^b dg(x).$$

Proof. Here g is continuous on $[a, b]$ and f is monotonic increasing on $[a, b]$. So by First Mean Value Theorem there exists $\xi \in [a,b]$ such that

$$\int_a^b g(x)df(x) = g(\xi)\{f(b)-f(a)\} \qquad \text{..................(1)}$$

Since g is continuous and f is monotonic increasing we have $g \in R(f)$. Hence by Theorem 2.7.1 $f \in R(g)$ and

$$\int_a^b f\,dg + \int_a^b g\,df = f(b)g(b) - f(a)g(a).$$

Using (1) we thus get

$$\int_a^b f\,dg + g(\xi)\{f(b)-f(a)\} = f(b)g(b) - f(a)g(a)$$

or, $\int_a^b f\,dg = f(b)g(b) - f(a)g(a) - g(\xi)f(b) + g(\xi)f(a)$

$$= f(a)\{g(\xi)-g(a)\} + f(b)\{g(b)-g(\xi)\}$$

$$= f(a)\int_a^\xi dg + f(b)\int_\xi^b dg.$$

Theorem 3.4.12. If f is RS-integrable with respect to g over $[a, b]$ where g is monotonic increasing on $[a,b]$ and F is defined on $[a, b]$ by $F(x) = \int_a^x f(t)dg(t), x \in [a,b]$, then F is of bounded variation on $[a, b]$.

Proof. Let $P = \{a = x_0, x_1, x_2, ...x_n = b\}$ be any partition of $[a, b]$. Since f is bounded on $[a, b]$ there exists positive number M such that $|f(x)| \le M$ for all $x \in [a,b]$.

Now $|F(x_i) - F(x_{i-1})|$

$$= \left| \int_a^{x_i} f(t)\,dg(t) - \int_a^{x_{i-1}} f(t)\,dg(t) \right|.$$

$$= \left| \int_a^{x_{i-1}} f(t)\,dg(t) - \int_{x_{i-1}}^{x_i} f(t)\,dg(t) - \int_a^{x_{i-1}} f(t)\,dg(t) \right|$$

$$= \left| \int_{x_{i-1}}^{x_i} f(t)\,dg(t) \right|$$

$$\le \int_{x_{i-1}}^{x_i} |f(t)|\,dg(t), \text{ [using Theorem 3.4.6 } (\because g \text{ is } m.i.)]$$

$$\le \int_{x_{i-1}}^{x_i} M\,dg(t)$$

$$= M\{g(x_i) - g(x_{i-1})\}.$$

$$\therefore V_P(F,a,b)$$

$$= \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|$$

$$\le \sum_{i=1}^{n} M\{g(x_i) - g(x_{i-1})\}$$

$$= M\{g(b) - g(a)\}$$

Thus $V_P(F,a,b) \le M\{g(b) - g(a)\}$.

This is true for all P. Since the RHS is independent of P, $\sup_P V_P(F,a,b)$ exists. This proves that $F(x)$ is of bounded variation on $[a, b]$.

3.5. Functions of bounded variations as integrators

We know that a function of bounded variation can be written as the difference between two monotonic increasing functions. So a number of theorems for monotonic increasing functions as integrator are seen to be true here.

Theorem 3.5.1. If f is continuous and g of bounded variation on $[a, b]$, then $\int_a^b f\,dg$ exists.

Proof. Here g is of b.v. on $[a, b]$. Therefore, there exist m.i functions g_1 and g_2 on $[a,b]$ such that $g=g_1-g_2$.

As f is continuous and g_1 is m.i it follows that $f \in R(g_1)$.

Also as f is continuous and g_2 is m.i it follows that $f \in R(g_2)$.

Now $f \in R(g_1)$ and $f \in R(g_2)$ implies $f \in R(g_1 - g_2)$

i.e. $f \in R(g)$. Hence $\int_a^b f\,dg$ exists.

Theorem 3.5.2. If f is of bounded variation and g is continuous on $[a, b]$, then $\int_a^b f\,dg$ exists.

Proof. Using Theorem 3.5.1. it follows that $g \in R(f)$.

Hence by Theorem 2.7.1. we see that $f \in R(g)$.

Note. From Theorems 3.5.1 and 3.5.2 we see that if one of the two functions f and g is continuous and other is of bounded variation on $[a, b]$, then $f \in R(g)$ and $g \in R(f)$.

Theorem 3.5.3. If f is continuous and g of bounded variation on $[a, b]$ and $|f(x)| \le M$ for $x \in [a,b]$ then

$$\left|\int_a^b f\,dg\right| \le MV(g;a,b).$$

Proof. Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$ and ξ_i be any point of $[x_{i-1}, x_i]$. Let $\Gamma = \{\xi_1, \xi_2, ..., \xi_n\}$.

$$\text{Then } |S(P,\Gamma,f,g)| = \left|\sum_{i=1}^{n} f(\xi_i)\{g(x_i) - g(x_{i-1})\}\right|$$

$$\le \sum_{i=1}^{n} |f(\xi_i)||g(x_i) - g(x_{i-1})|$$

$$\leq M\sum_{i=1}^{n}|g(x_i)-g(x_{i-1})|$$

$$= MV(P,g,a,b)$$

$$\leq M\sup\{V(P,g;a,b):P\in\Pi[a,b]\}$$

$$= MV(g,a,b)$$

i.e. $|S(P,\Gamma,f,g)|\leq MV(g,a,b)$.

$\therefore \lim_{\|P\|\to 0}|S(P,\Gamma,f,g)|\leq MV(g,a,b)$, $[\because f\in R(g)$ so the limit exists$]$

or, $\left|\int_a^b f\,dg\right|\leq MV(g,a,b)$.

The following theorem is a result on change of integrator.

Theorem 3.5.4. If f and g are continuous and α is of bounded variation on $[a, b]$ and β is defined on $[a, b]$ by

$$\beta(x)=\int_a^x g(t)\,d\alpha(t),x\in[a,b] \text{ then}$$

$$\int_a^b f(x)g(x)\,d\alpha(x)=\int_a^b f(x)\,d\beta(x).$$

Proof. We first prove the existence of the two integrals and then we prove their equality.

Since $f(x)$ and $g(x)$ are continuous on $[a, b]$, $f(x)\,g(x)$ is also so in $[a, b]$. Thus fg is continuous and α is of b.v. on $[a, b]$.

Therefore $fg\in R(\alpha)$ i.e. the integral $\int_a^b fg\,d\alpha$ exists.

As α is of bounded variation on $[a, b]$, there exists m.i functinos α_1 and α_2 on $[a, b]$ such that $\alpha=\alpha_1-\alpha_2$.

Let $\beta_i=\int_a^x g(t)\,d\,\alpha(t), i=1,2.$

Now $g(x)$ is continuous and α_1 is m.i. on $[a, b]$. Theorem 3.4.12

$\beta_1(x) = \int_a^x g(t)\,d\alpha_1(t)$ is of b.v. on [a, b].

Similarly, $\beta_2^{(x)} = \int_a^x g(t)\,d\alpha_2(t)$ is of b.v. on [a, b].

Now, $\beta(x) = \int_a^x g\,d\alpha = \int_a^x g\,d(\alpha_1 - \alpha_2) = \int_a^x g\,d\alpha_1 - \int_a^x g\,d\alpha_2 = \beta_1 - \beta_2.$

$\therefore \beta$ is of b.v. on [a, b].

Thus f is continuous and β is of b.v. on [a, b].

$\therefore f \in R(\beta)$ i.e. the integral $\int_a^b f\,d\beta$ exists.

Now we prove the equality of the two integrals.

Let $P = \{a = x_0, x_1, x_2, ...x_n = b\}$ be any partition of [a, b].

Now we have

$$\beta_1(x_i) - \beta_1(x_{i-1}) = \int_a^{x_i} g(t)\,d\alpha_1(t) - \int_a^{x_{i-1}} g(t)\,d\alpha_1(t)$$

$$= \int_a^{x_{i-1}} g(t)\,d\alpha_1(t) + \int_{x_{i-1}}^{x_i} g(t)\,d\alpha_1(t) - \int_a^{x_{i-1}} g(t)\,d\alpha_1(t)$$

$$= \int_{x_{i-1}}^{x_i} g(t)\,d\alpha_1(t). \qquad \text{.................. (1)}$$

Here $g(t)$ is continuous and $\alpha_1(t)$ is m.i on $[x_{i-1}, x_i]$. So by first mean value theorem there exists $\xi_i' \in [x_{i-1}, x_i]$ s.t.

$$\int_{x_{i-1}}^{x_i} g(t)\,d\alpha_1(t) = g(\xi_i')\{\alpha_1(x_i) - \alpha_1(x_{i-1})\}. \qquad \text{.................. (2)}$$

Let $\Gamma_1 = \{\xi_1', \xi_2', ..., \xi_n'\}$. Then

$$S(P, \Gamma, fg, \alpha_1) = \sum_{i=1}^{n} (fg)(\xi_i')\{\alpha_1(x_i) - \alpha_1(x_{i-1})\}$$

$$= \sum_{i=1}^{n} f(\xi_i')\,g(\xi_i')\{\alpha_1(x_i) - \alpha_1(x_{i-1})\}$$

$$=\sum_{i=1}^{n} f(\xi_i')\{\beta_1(x_i)-\beta_1(x_{i-1})\} \text{ [using (2) and (1)]}$$

$$=S(P,\Gamma_1,f,\beta_1). \qquad \text{................ (3)}$$

Now fg is continuous and α_1 is m.i on $[a, b]$ and f is continuous and β_1 is of b.v. on $[a, b]$, so $\int_a^b fg\,d\alpha_1$ and $\int_a^b f\,d\beta_1$ exist. Hence taking limit as $\|P\|\to 0$ we get from (3)

$$\lim_{\|P\|\to 0} S(P,\Gamma,fg,\alpha_1)=\lim_{\|P\|\to 0} S(P,\Gamma_1,f,\beta_1)$$

or, $\int_a^b fg\,d\alpha_1=\int_a^b f\,d\beta_1.$

Proceeding in the same manner we can prove that

$$\int_a^b fg\,d\alpha_2=\int_a^b f\,d\beta_2.$$

$$\therefore \int_a^b fg\,d\alpha_1-\int_a^b fg\,d\alpha_2=\int_a^b fd\,\beta_1-\int_a^b fd\,\beta_2$$

or, $\int_a^b fg\,d(\alpha_1-\alpha_2)=\int_a^b fd(\beta_1-\beta_2)$

or, $\int_a^b fg\,d\alpha=\int_a^b fd\beta.$

3.6 Step functions as integrators.

Many interesting and important results are there for RS-integral with step functions as integrators. We first define step function.

Definition 3.6.1. A function defined on $[a, b]$ is said to be a step function or a piecewise constant function if there is a fixed partition $\{a=c_0,c_1,c_2,\ldots,c_m=b\}$, $(c_0<c_1<\ldots<c_m)$ such that f is constant on each open subinterval $(c_{i-1},c_i), i=1,2,\ldots,m$ so that f possibly has jump discontinuity at the points $c_0,c_1,\ldots,c_m$.

We now prove that step functions on $[a, b]$ are functions of bounded variation on $[a, b]$.

Theorem 3.6.1. If f is a step function on $[a, b]$, then f is of bounded variation on $[a, b]$.

Proof. Here f is a step function on $[a, b]$. So there exists partition $P=\{a=c_0, c_1, c_2, ..., c_m=b\}$ such that f is constant on each open subinterval $(c_{i-1}, c_i), i=1,2,...,m$. Let us choose any point $d_i \in (c_{i-1}, c_i)$ for each $i = 1, 2, ..., m$.

Now f is monotonic on $[c_{i-1}, d_i]$ and on $[d_i, c_i]$ and so f is of b.v. on $[c_{i-1}, d_i]$ and on $[d_i, c_i]$.

Thus f is of b.v. on $[c_{i-1}, d_i] U [d_i, c_i]$ i.e. on $[c_{i-1}, c_i]$

Hence f is of b.v on $\bigcup_{i=1}^{m}[c_{i-1}, c_i]$ i.e. on $[a, b]$.

The next theorem gives the formula for finding RS-integral when f is continuous and g is a step function.

Theorem 3.6.2. Let g be a step function on $[a, b]$ such that g possibly has jump discontinuity at the points $c_0, c_1, ..., c_m$ $(a=c_0<c_1<...<c_m=b)$, the height of jump at c_i being g_i $(i=0,1,...,m)$ i.e.

$$g_0 = g(c_0+0)-g(c_0)$$

$$g_i = g(c_i+0)-g(c_i-0), \quad i=1,2,...,m-1$$

$$g_m = g(c_m)-g(c_m-0).$$

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dg(x) = \sum_{i=0}^{m} f(c_i)g_i.$$

Proof. We first consider the simple case in which $f(x)$ is continuous in $[\alpha, \beta]$ and $g(x)$ is constant in $[\alpha, \beta)$.

We show that $\int_\alpha^\beta f\,dg = f(\beta)[g(\beta)-g(\beta-0)]$.

We note that $g(x)$ is monotonic in $[\alpha, \beta]$. As f is continuous in $[\alpha, \beta]$, $\int_\alpha^\beta f\,dg$ exists.

Let $A=\{\alpha=a_0, a_1, a_2, ..., a_n=\beta\}$ be the partition of $[\alpha, \beta]$ where $a_i = \alpha + i(\beta-\alpha)/n$, $i=0,1,2,...n$.

So this partition divides $[\alpha, \beta]$ into n equal parts. In the subinterval $[a_{i-1}, a_i]$ we choose the point a_i $(i = 1,2,...,n)$ and denote $\Gamma = \{a_1, a_2, ..., a_n\}$. Then

$$S(A,\Gamma,f,g) = \sum_{i=1}^{n} f(a_i)\{g(a_i) - g(a_{i-1})\}$$

$$= \sum_{i=1}^{n-1} f(a_i)\{g(a_i) - g(a_{i-1})\} + f(a_n)\{g(a_n) - g(a_{n-1})\}$$

$$= \sum_{i=1}^{n-1} f(a_i)\cdot 0 + f(\beta)\left\{g(\beta) - g\left(\beta - \frac{\beta-\alpha}{n}\right)\right\}$$

$$= f(\beta)\left\{g(\beta) - g\left(\beta - \frac{\beta-\alpha}{n}\right)\right\}$$

Since $\int_\alpha^\beta f\,dg$ exists we have from this

$$\lim_{|A|\to 0} S(A,\Gamma,f,g) = \lim_{n\to\infty} f(\beta)\left\{g(\beta) - g\left(\beta - \frac{\beta-\alpha}{n}\right)\right\}$$

or, $\int_\alpha^\beta f\,dg = f(\beta)\{g(\beta) - g(\beta - 0)\}$. .... (1)

Similarly, when f is continuous in $[\alpha, \beta]$ and g is constant in $(\alpha, \beta]$ then

$\int_\alpha^\beta f\,dg = f(\alpha)\{g(\alpha + 0) - g(\alpha)\}$. .... (2)

We now consider the general case of the theorem.

Here g is a step function on $[a, b]$ and so g is of b.v. on $[a, b]$. Since f is continuous on $[a, b]$ and g is of b.v. on $[a, b]$, it follows that f is RS-integrable w.r.t. g i.e. $\int_\alpha^\beta f\,dg$ exists.

We choose any point $d_i \in (c_{i-1}, c_i), i = 1,2,...,m$. Then g is monotonic in $(c_{i-1}, d_i]$ and also in $[d_i, c_i)$.

$\therefore$ For each $i = 1, 2, ..., m$ we have from (1) and (2)

$$\int_{c_{i-1}}^{d_i} f(x)\,dg(x) = f(c_{i-1})\{g(c_{i-1} + 0) - g(c_{i-1})\}$$

and $\int_{d_i}^{c_i} f(x)dg(x) = f(c_i)\{g(c_i) - g(c_i - 0)\}$.

$$\therefore \int_a^b f(x)dg(x) = \sum_{i=1}^{m} \int_{c_{i-1}}^{c_i} f(x)dg(x)$$

$$= \sum_{i=1}^{m} \left\{ \int_{c_{i-1}}^{d_i} f(x)dg(x) + \int_{d_i}^{c_i} f(x)dg(x) \right\}$$

$$= \sum_{i=1}^{m} f(c_{i-1})\{g(c_{i-1}+0) - g(c_{i-1})\} + \sum_{i=1}^{m} f(c_i)\{g(c_i) - g(c_i - 0)\}$$

$$= \sum_{i=0}^{m-1} f(c_i)\{g(c_i+0) - g(c_i)\} + \sum_{i=1}^{m} f(c_i)\{g(c_i) - g(c_i - 0)\}$$

$$= f(c_0)\{g(c_0+0) - g(c_0)\} + \sum_{i=1}^{m-1} f(c_i)\{g(c_i+0) - g(c_i)\}$$

$$+ \sum_{i=1}^{m-1} f(c_i)\{g(c_i+0) - g(c_i)\} + f(c_m)\{g(c_m) - g(c_m - 0)\}$$

$$= f(c_0)\{g(c_0+0) - g(c_0)\} + \sum_{i=1}^{m-1} f(c_i)\{g(c_i+0) - g(c_i - 0)\} + f(c_m)\{g(c_m) - g(c_m - 0)\}$$

$$= f(c_0)g_0 + \sum_{i=1}^{m-1} f(c_i)g_i + f(c_m)g_m$$

$$= \sum_{i=0}^{m} f(c_i)g_i.$$

Note. We note that if the integrator is a step function and integrand is continuous, then the RS-integral reduces to a finite sum.

The following theorem shows that any finite sum can be represented as RS-integral.

Theorem 3.6.3. Let $\sum_{i=1}^{n} a_i$ be a given finite sum. Prove that this sum can be represented as RS-integral.

Proof. The given sum is $\sum_{i=1}^{n} a_i$. We define a function $f(x)$ on $[0, n]$ as follows.

$$f(x) = a_i + (x-i)(a_i - a_{i-1}) \text{ for } x \in [i-1, i), \quad i = 1, 2, \dots, n$$

$$= a_n \text{ for } x = n$$

where a_0 is a constant.

$$\therefore f(x) = a_1 + (x-1)(a_1 - a_0),\ 0 \le x < 1$$

$$= a_2 + (x-2)(a_2 - a_1),\ 1 \le x < 2$$

$$= a_3 + (x-3)(a_3 - a_2),\ 2 \le x < 3$$

....

....

$$= a_n + (x-n)(a_n - a_{n-1}),\ n-1 \le x < n$$

$$= a_n,\ x = n$$

We note that $f(x)$ is continuous in $[0, n]$.

We take $g(x) = [x]$.

$$\therefore g(x) = 0, \quad 0 \le x < 1$$

$$= 1, \quad 1 \le x < 2$$

$$= 2, \quad 2 \le x < 3$$

.....

.....

$$= n-1, \quad n-1 \le x < n$$

$$= n, \quad x = n.$$

Then $g(x)$ is a step function on $[0, n]$.

$$\therefore \int_0^n f\,dg = f(0)\{g(0+0) - g(0)\}$$

$$+f(1)\{g(1+0)-g(1-0)\}$$

$$+f(2)\{g(2+0)-g(2-0)\}$$

.....

.....

$$+f(n-1)\{g(\overline{n-1}+0)-g(\overline{n-1}-0)\}$$

$$+f(n)\{g(n)-g(n-0)\}.$$

$$=a_0\{0-0\}+a_1\{1-0\}+a_2\{2-1\}+...$$

$$....+a_{n-1}\{(n-1)-(n-2)\}+a_n\{n-(n-1)\}$$

$$=a_1+a_2+a_3+...+a_{n-1}+a_n$$

$$=\sum_{i=1}^{n}a_i.$$

3.7. Illustrative Examples

Example 3.7.1. Using bound definition prove that $\int_0^3 x\,d[x]=6$. Verify the result using Theorem 3.6.2.

Solution.

Here $\alpha(x)=[x]$ is monotonic increasing function on [0, 3] and $f(x)=x$ is continuous on [0, 3]. Hence $\int_0^3 f(x)d\alpha(x)$ i.e. $\int_0^3 xd[x]$ exists.

Now $\alpha(x)=0$ for $0\le x\le 1$

$=1$ for $1\le x<2$

$=2$ for $2\le x<3$

$=3$ for $x=3$

Let us consider the partition P of [0, 3] as follows

$$P=\{0=x_0,x_1,x_2,...,x_l=1,x_{l+1},...,x_m=2,x_{m+1},...,x_n=3\}$$

Then $U(P,f,\alpha)=\sum_{i=1}^{l-1} M_i\,\Delta\alpha_i + M_l\,\Delta\alpha_l + \sum_{i=l+1}^{m-1} M_i\,\Delta\alpha_i + M_m\Delta\alpha_m + \sum_{i=m+1}^{n-1} M_i\,\Delta\alpha_i + M_n\,\Delta\alpha_n$

$$=\sum_{i=1}^{l-1} M_i(0-0)+1\times(1-0)+\sum_{i=l+1}^{m-1} M_i(1-1)+2\times(2-1)+\sum_{i=m+1}^{n-1} M_i(2-2)+3\times(3-2)$$

$= 1 + 2 + 3$

$= 6$

Similarly,

$$L(P,f,\alpha)=m_l\{\alpha(x_l)-\alpha(x_{l-1})\}+m_m\{\alpha(x_m)-\alpha(x_{m-1})\}+m_n\{\alpha(x_n)-\alpha(x_{n-1})\}$$

$$= x_{l-1}(1-0)+x_{m-1}(2-1)+x_{n-1}(3-2)$$

$$= x_{l-1}+x_{m-1}+x_{n-1}.$$

Taking x_{l-1} very close to 1, x_{m-1} very close to 2 and x_{n-1} very close to 3, we have the value of $L(P,f,\alpha)$ very close to 6. Thus for such partition we have

$$6=L(P,f,\alpha)\le \sup_P L(P,f,\alpha)=\underline{I}\le \overline{I}=\inf_P U(P,f,\alpha)\le U(P,f,\alpha)=6$$

Hence $\underline{I}=\overline{I}=6$.

This shows that $\int_0^3 f\,d\alpha=\int_0^3 x\,d[x]=6.$

Since $\alpha(x)$ is a step function and $f(x)$ is continuous function on [0, 3] we have by Theorem 3.6.2

$$\int_0^3 f\,d\alpha = f(0)[\alpha(0+0)-\alpha(0)]+f(1)[\alpha(1+0)-\alpha(1-0)]$$

$$+f(2)[\alpha(2+0)-\alpha(2-0)]+f(3)[\alpha(3)-\alpha(3-0)]$$

$= 0\,(0-0) + 1\,(1-0) + 2\,(2-1) + 3\,(3-2)$

$= 1 + 2 + 3$

$= 6.$

Example 3.7.2. Evaluate $\int_{-1}^{2} x^5 d\left(|x|^3\right)$

Solution. Let $f(x) = x^5$ and $\alpha(x) = |x|^3$. Then

$$\int_{-1}^{2} f\,d\alpha + \int_{-1}^{2} \alpha\,df = f(2)\alpha(2) - f(-1)\alpha(-1)$$

$$\therefore \int_{-1}^{2} x^5 d\left(|x|\right)^3 = 2^5 \times |2|^3 - (-1)^5 |-1|^3 - \int_{-1}^{2} |x|^3 d\left(x^5\right)$$

$$= 32 \times 8 - (-1) \times 1 - \int_{-1}^{2} |x|^3 \cdot 5x^4 dx$$

$$= 257 - 5\int_{-1}^{0} |x|^3 \cdot x^4 dx - 5\int_{0}^{2} |x|^3 \cdot x^4 dx$$

$$= 257 - 5\int_{-1}^{0} \left(-x^3\right) \cdot x^4 dx - 5\int_{0}^{2} x^3 \cdot x^4 dx$$

$$= 257 + 5\int_{-1}^{0} x^7 dx - 5\int_{0}^{2} x^7 dx$$

$$= 257 + 5\left[\frac{x^8}{8}\right]_{-1}^{0} - 5\left[\frac{x^8}{8}\right]_{0}^{2}$$

$$= 257 + 5\left(0 - \frac{1}{8}\right) - 5\left(\frac{2^8}{8} - 0\right)$$

$$= \frac{771}{8}$$

Example 3.7.3. Show that $\int_{0}^{3} x^2 d\left([x] - x\right) = 5$

Solution.

$$\int_{0}^{3} x^2 d\left([x] - x\right)$$

$$= \int_{0}^{3} x^2 d[x] - \int_{0}^{3} x^2 dx$$

$$= \int_0^3 f(x)\,dg(x) - \left[\frac{x^3}{3}\right]_0^3 \text{ where } f(x) = x^2 \text{ and } g(x) = [x]$$

$$= f(0)\{g(0+0)-g(0)\} + f(1)\{g(1+0)-g(1-0)\} + f(2)\{g(2+0)-g(2-0)\}$$
$$+ f(3)\{g(3)-g(3-0)\} - \left\{\frac{3^3}{3} - \frac{0^3}{3}\right\}$$

$= 0\ (0 - 0) + 1\ (1 - 0) + 2^2\ (2 - 1) + 3^2\ (3 - 2) - 9$

$= 1 + 4 + 9 - 9$

$= 5.$

Example 3.7.4. Evaluate $\int_{-1}^{3} f(x)\,d\alpha(x)$ where $f(x) = 2x + 3$

and $\alpha(x) = -8, -1 \le x < 0$

$= 3,\ 0 \le x < 1$

$= 4,\ x = 1$

$= -2,\ 1 < x < 2$

$= 6,\ 2 \le x < 3$

$= 5,\ x = 3$

Solution. Here $f(x)$ is continuous on $[-1, 3]$ and $\alpha(x)$ is a step function on $[-1, 3]$. Hence we have

$$\int_{-1}^{3} f\,d\alpha = f(-1)\{\alpha(-1+0) - \alpha(-1)\} + f(0)\{\alpha(0+0) - \alpha(0-0)\}$$
$$+ f(1)\{\alpha(1+0) - \alpha(1-0)\} + f(2)\{\alpha(2+0) - \alpha(2-0)\}$$
$$+ f(3)\{\alpha(3) - \alpha(3-0)\}.$$

$$= \{2(-1)+3\}(-8+8) + (2\times 0+3)\{3+8\} + (2\times 1+3)(-2-3)$$
$$+ (2\times 2+3)(6+2) + (2\times 3+3)(5-6)$$

$= 1\times 0+3\times 11+5(-5)+7\times 8+9(-1)$

$= 33 - 25 + 56 - 9$

$= 89 - 34$

$= 55.$

3.8. Summary. Riemann-Stieltijes integral with different integrators have been considered here. It is seen that any finite sum can expressed as RS-integral. Examples are considered for illustration.

3.9. Self Assessment Questions.

1. Show that $\int_0^3 x\,d([x]-x) = \frac{3}{2}$

2. Show that $\int_2^6 [5-x]\,d(\log[x]) = \log 3$

3. Show that $\int_{-1}^{4} (x^2+e^x)\,d([x]+2)$ exists. Evaluate the integral.

4. Show that $\int_0^{2\pi} \sin x\,d(\cos x) = -\frac{\pi}{2}$

5. Evaluate $\int_0^2 x\,d\alpha(x)$ where $\alpha(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2+x, & 1 < x \le 2 \end{cases}$

6. Evaluate $\int_0^3 (x-[x])\,dx^2$

7. Evaluate $\int_0^3 x(3-x)\,d\alpha(x)$ where $\alpha(x) = x$ for $0 \le x < 1$

   $= 1$ for $1 \le x < 2$

   $= 3-x$ for $2 \le x < 3.$

8. Evaluate $\int_{-2}^{4} (2x^2-3x+5)\,d\alpha(x)$ where $\alpha(x) = 3, -2 \le x < -1$

   $= 4, -1 \le x < 1$

$= 2, x = 1$

$= 5, \; 1 < x < 4$

$= 8, \; x = 4.$

3.10. Suggested books for further reading :

1. Introduction to Mathematicsl Analysis : Amritava Gupta; Academic Publishers, Calcutta.
2. Mathematical Analysis : S.C. Malik & Savita Arora; Wiley Eastern Ltd., New Age Internatinal Ltd.
3. Mathematical Analysis : Tom. M. Apostol; Narosa Publishing House.
4. Elements of Real Analysis : Shanti Narayan & M.D. Raisinghania; S. Chand.
5. Principles of Mathematical Analysis; Walter Rudin; International Students Edition; McGraw-Hill International Book Company.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-I

Group–A

Module No. - 04

(Measurable Sets and Measurable Functions)

Module Structure

4.1 Introduction

The concept of measure of a set is an outcome of the notion of finding length of a segment or finding area of a plane figure or finding the volume of a figure in three dimensional space. The measure theory is extensively used in functional analysis, probability theory, the theory of dynamical system and many other branches of mathematics.

The concept of measure of a set has been used to define a very important class of functions, known as measurable functions. The notion of measurable function will be used to develop Lebesgue integral in the next module.

4.2 Objective

To overcome the limitations of Riemann integral our aim is to develop the important integral due to Lebesgue. For this we need the notion of measurable function which again demands the idea of measurable set. It is seen that all ordinary operations of analysis when applied to measurable functions lead to measurable functions. In other words, all functions that we ordinarily meet with are seen to be measurable functions. That is why the study of measurable sets and functions have become so essential.

4.3. Measurable Sets

The concept of measure of a set is a generalization of the following concepts,

i) The length of a line segment.

ii) The area of a plane figure.

iii) The volume of a space figure.

iv) The increment of a non-decreasing function over an interval.

v) The integral of a non negative function over a set on the line or over a region in the plane or over a region in the space.

Here we consider the measure of a set of real numbers only. The concept can be easily extended in general to the case of more abstract theory.

In keeping with the concept of length familiar from geometry, we now define the measure of an interval as follows. An interval may be closed, may be open or half open.

Let $A_1 = \{x : a \le x \le b\}$

$A_2 = \{x : a \le x < b\}$

$A_3 = \{x : a < x \leq b\}$

$A_4 = \{x : a < x < b\}$.

In all these four cases we define the measure as $(b-a)$. This is because from geometry we know that the length of a point is zero.

Thus $m(A_1) = m(A_2) = m(A_3) = m(A_4) = b - a$.

Before defining the measure of any set of real numbers, we first define the measure of open sets.

For this we need the famous "Representation theorem for open sets on the real line". This theorem is based on the concept of component interval.

4.3.1. Definition : Component Interval

Let A be an open subset in R. An open interval I (which may be finite or infinite) is called a component interval of A is $I \subseteq A$ and if there is no open interval $J \neq I$ such that $I \subseteq J \subseteq A$.

In other words, a component interval of A is not a proper subset of any other open interval contained in A.

We now state the representation theorem.

4.3.2. Theorem : Every non-emply open set A in R is the union of a countable collection of disjoint component intervals of A.

Thus if A is any non-empty bounded open subset of the closed bounded interval $[a, b]$ in R, then there exist a countable disjoint open intervals whose union is the set A. If these intervals are $I_1, I_2, \ldots, I_n$ then $A = \bigcup_{i=1}^{n} I_i$ and its measure is defined as $m(A) = \sum_{i=1}^{n} m(I_i)$.

If these intervals are $I_1, I_2, I_3 \ldots$ then $A = \sum_{i=1}^{\infty} I_i$ and its measure is defined as $m(A) = \sum_{i=1}^{\infty} m(I_i)$.

4.3.3. Definition. If A is any open set of $[a, b]$ then $m(A)=\sum_{i=1}^{n} m(I_i)$ or $m(A)=\sum_{i=1}^{\infty} m(I_i)$ where I_i are the disjoint component intervals of A.

4.3.4. Definition. If B is any closed subset of the interval $[a, b]$ then the complement B' of B relative to any open subset G of $[a, b]$ containing B is $G-B=G\cap B'$ and is an open set in $[a, b]$. The measure of B is defined as

$$m(B)=m(G)-m(G-B).$$

Note. It is very important to note that the measure of the closed set B is independent of the choice of G containing B.

With the help of these notions of measure of open set and closed set we now define, the outer and the inner measure of any set of $[a, b]$ as follows.

4.3.5. Definition Outer measure

Let A be any bounded subset of $[a, b]$. The outer measure of A is denoted by $m^*(A)$ and is defined as

$$m_*(A)=\inf m(G)$$

where the infimum is taken over all open sets G which contains A.

4.3.6. Definition Inner measure

Let A be any bounded subset of $[a, b]$. The inner measure of A is denoted by $m_*(A)$ and is defined as

$$m^*(A)=\sup m(F)$$

where the supremum is taken over all the closed sets F contained in the set A.

We note that any open set G and any closed set F considered above satisfy $F\subset A\subset G$ and both sup $m(F)$ and inf $m(G)$ exists and they are related by the inequality

$$\sup m(F) \le \inf m(G)$$

i.e. $m_*(A) \le m^*(A)$.

So, in general we have either $m_*(A) < m^*(A)$ or $m_*(A) = m^*(A)$.

4.3.7. Definition Measurable Set

A set $A \subset [a,b]$ is said to be measurable if $m_*(A) = m^*(A)$.

In this case, the common value is called the measure of A and is denoted by $m(A)$. i.e. $m_*(A) = m^*(A) = m(A)$.

We now prove the following theorem.

4.3.8. Theorem. If $A \subset [a,b]$, then

$$m^*(A) + m_*(A') = b - a \text{ where } A' = [a,b] - A.$$

Proof. Let G be any open subset of $[a, b]$ containing A i.e. $A \subset G$.

Then $G' \subset A'$ where G' and A' are respectively the complements of G and A relative to $[a, b]$.

Now $m_*(A') = \sup m(G') \ge m(G')$

$$\therefore m(G) + m_*(A') \ge m(G) + m(G') = (b-a)$$

Taking infimum over all open sets $G \supset A$ we get

$$m^*(A) + m_*(A') \ge b - a \qquad \text{..............(1)}$$

Again let F be any closed set such that $F \subset A'$.

Then F' is an open set and $F' \supset A$.

From definition we have

$$m^*(A) = \inf m(F')$$

$\therefore m^*(A) \le m(F')$

or, $m(F) + m^*(A) \le m(F) + m(F') = (b-a)$

Taking supremum over all closed sets $F \subset A'$, we get

$$m_*(A') + m^*(A) \le b - a \qquad \text{............... (2)}$$

From (1) and (2), we have

$$m^*(A) + m_*(A') = b - a$$

An alternative definition of measurable set follows from the following theorem.

4.3.9. Theorem. A subset $A \subset [a,b]$ is measurable if nd only if $m^*(A) + m^*(A') \le b - a$ awhere $A' = [a,b] - A$.

Proof. Let A be measurable. Then

$$m^*(A) + m_*(A) = m(A) \qquad \text{......... (1)}$$

We have

$$m^*(A) + m_*(A') = b - a.$$

Intercharging A and A', we have

$$m^*(A') + m_*(A) = b - a.$$

Using (1), this gives

$$m^*(A') + m^*(A) = b - a.$$

Hence the condition is necessary.

Again let $m^*(A) + m^*(A') \le b - a$.

We have $m^*(A') + m_*(A) = b - a$.

$$\therefore m^*(A) + m^*(A') \le m^*(A') + m_*(A)$$

or, $m^*(A) \le m_*(A)$

or, $m_*(A) \ge m^*(A)$

But we know, $m_*(A) \le m^*(A)$

$\therefore m_*(A) = m^*(A)$ i.e. A is measurable.

Hence the condition is sufficient.

Note. We note that the above result is symmetric in A and A'. Hence we get the important result that A' is measurable whenever A is measurable.

We can easily prove the following theorems.

4.3.10. Theorem. A necessary and sufficient condition for a set A to be measurable is that for each $\in > 0$, there exist an open set $G \supset A$ and a closed set $F \subset A$ such that $m(G) - m(F) < \in$.

4.3.11. Theorem. If A_1 and A_1 are subsets of $[a, b]$, then

$$m^*(A_1) + m^*(A_2) \ge m^*(A_1 \cup A_2) + m^*(A_1 \cap A_2)$$

and $m_*(A_1) + m_*(A_2) \le m_*(A_1 \cup A_2) + m_*(A_1 \cap A_2)$.

We now prove the following theorem.

4.3.12. If A_1 and A_1 are measurable subsets of $[a, b]$, then both $A_1 \cup A_2$ and $A_1 \cap A_2$ are measurable and

$$m(A_1) + m(A_2) = m(A_1 \cup A_2) + m(A_1 \cap A_2).$$

Proof. Let A_1 and A_2 be measurable. Then

$$m^*(A_1) = m_*(A_1) \text{ and } m^*(A_2) = m_*(A_2) \qquad \text{..................(1)}$$

Thus using (1) we have

$$m(A_1)+m(A_2)=m^*(A_1)+m^*(A_2)\geq m^*(A_1\cup A_2)+m^*(A_1\cap A_2)$$

$$\geq m_*(A_1\cup A_2)+m_*(A_1\cap A_2) \qquad \text{.................. (2)}$$

and $m_*(A_1\cup A_2)+m_*(A_1\cap A_2)\geq m_*(A_1)+m_*(A_2)=m(A_1)+m(A_2)$. .... (3)

From (2) and (3) we get

$$m(A_1)+m(A_2)\leq m_*(A_1\cup A_2)+m_*(A_1\cap A_2)\leq m^*(A_1\cup A_2)+m^*(A_1\cap A_2)\leq m(A_1)+m(A_2)$$

..................(4)

$\therefore m_*(A_1\cup A_2)+m_*(A_1\cap A_2)=m^*(A_1\cup A_2)+m^*(A_1\cap A_2)$

or, $m^*(A_1\cup A_2)-m_*(A_1\cup A_2)=m_*(A_1\cup A_2)-m^*(A_1\cap A_2)$. .... (5)

Now $m^*(A_1\cup A_2)\geq m_*(A_1\cup A_2)$.

$\therefore$ From (1) we have $m_*(A_1\cap A_2)-m^*(A_1\cap A_2)\geq 0$

or, $m_*(A_1\cap A_2)\geq m^*(A_1\cap A_2)$.

But $m_*(A_1\cap A_2)\leq m^*(A_1\cap A_2)$.

Hence $m_*(A_1\cap A_2)=m^*(A_1\cap A_2)$, i.e. $A_1\cap A_2$ is measurable.

From (5) we have $m^*(A_1\cup A_2)-m_*(A_1\cup A_2)=0$

or, $m_*(A_1\cup A_2)=m^*(A_1\cup A_2)$ i.e. $A_1\cup A_2$ is measurable.

From (4) we have

$$m(A_1)+m(A_2)=m_*(A_1\cup A_2)+m_*(A_1\cap A_2)=m^*(A_1\cup A_2)+m^*(A_1\cap A_2)$$

or, $m(A_1)+m(A_2)=m(A_1\cup A_2)+m(A_1\cap A_2)$.

4.3.13. Theorem. If $A_1, A_2, A_3, ...$ are pairwise disjoint measurable subsets of $[a,b]$ then $\bigcup_{n=1}^{\infty} A_n$ is measurable and $m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$.

Proof. For each n, from definition of $m^*(A_n)$ it follows that for given $\in > 0$ there exists open subset G_n of $[a, b]$ such that $A_n \subset G_n$ and $m^*(A_n) < m(G_n) < m^*(A_n) + \in/2^n$.

Now the set $\bigcup_{n=1}^{\infty} G_n$ is an open set and $\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} G_n$.

$$\therefore m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le m\left(\bigcup_{n=1}^{\infty} G_n\right) \le \sum_{n=1}^{\infty}\left\{m^*(A_n) + \in/2^n\right\}$$

$$= \sum_{n=1}^{\infty} m^*(A_n) + \left\{\in/2 + \in/2^2 + ...\right\}$$

$$= \sum_{n=1}^{\infty} m^*(A_n) + \in$$

Since $\in$ is arbitrary, we have

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} m^*(A_n) \qquad ....(1)$$

Again for some large positive integer N we have

$$m_*\left(\bigcup_{n=1}^{\infty} A_n\right) \ge m_*\left(\bigcup_{n=1}^{N} A_n\right) \ge \sum_{n=1}^{N} m_*(A_n) - m_*\left(\bigcap_{n=1}^{N} A_n\right) \text{ [using Theorem 4.3.11]}$$

Since $A_1, A_2, A_3 ....$ are pairwise disjoint,

$$\bigcap_{n=1}^{N} A_n = \phi \quad \therefore m\left(\bigcap_{n=1}^{N} A_n\right) = m(\phi) = 0.$$

$$\therefore m_*\left(\bigcup_{n=1}^{\infty} A_n\right) \ge \sum_{n=1}^{N} m_*(A_n).$$

This is true for any large positive integer N. Hence letting $N \to \infty$ we have

$$\therefore m_*\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \sum_{n=1}^{\infty} m_*(A_n). \qquad \text{.................... (2)}$$

Since A_n is measurable for each $n = 1, 2, 3, ...$ we have

$$m_*(A_n) = m^*(A_n) = m(A_n).$$

Using this result in (1) and (2) we have

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m(A_n)$$

and $$m_*\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \sum_{n=1}^{\infty} m(A_n).$$

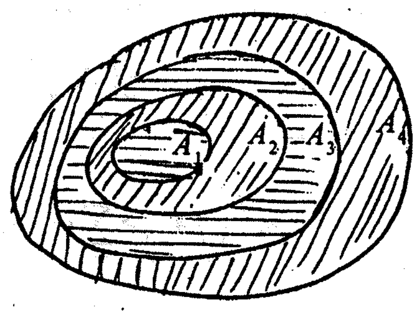
Thus $$\sum_{n=1}^{\infty} m(A_n) \leq m_*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m(A_n).$$

$$\therefore m_*\left(\bigcup_{n=1}^{\infty} A_n\right) = m^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$$

This shows that $\bigcup_{n=1}^{\infty} A_n$ is measurable and

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n).$$

4.3.14. Theorem. If $A_n, n = 1, 2, 3, ...$ are measurable subsets of $[a, b]$ and if $A_n \subset A_{n+1}$ for all n then $\bigcup_{n=1}^{\infty} A_n$ is measurable and $m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$.



Proof. We have $A_n \subset A_{n+1}$ for all $n = 1, 2, 3, ...$

$$\therefore A_1, A_2 - A_1, A_3 - A_2, A_4 - A_3, ...$$

are pairwise disjoint and measurable.

Also for any positive integer k, however large, we have

$$\bigcup_{n=1}^{k} A_n = A_1 \cup (A_2 - A_1) \cup (A_3 - A_2) \cup \ldots \cup (A_k - A_{k-1})$$

$$\therefore m\left(\bigcup_{n=1}^{k} A_n\right) = m(A_1) + \sum_{n=1}^{k-1} m(A_{n+1} - A_n)$$

$$= m(A_1) + \sum_{n=1}^{k-1} \{m(A_{n+1}) - m(A_n)\} \quad [\because A_n \subset A_{n+1} \text{ for all } n]$$

$$= m(A_1) + m(A_2) - m(A_1) + m(A_3) - m(A_2) + \ldots + m(A_k) - m(A_{k-1})$$

$$= m(A_k)$$

Letting $k \to \infty$ we have

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{k\to\infty} m(A_k)$$

i.e. $$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n).$$

4.3.15. Theorem. If $A_n, n = 1, 2, 3, \ldots$ are measurable subsets of $[a, b]$ and if $A_{n+1} \subset A_n$ for all n then $\bigcap_{n=1}^{\infty} A_n$ is measurable and $m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$.

Proof. Let A_n' be complement of A_n relative to $[a, b]$.

Since A_n is measurable we have A_n' is also measurable and $m(A_n') = b - a - m(A_n)$.

As $A_{n+1} \subset A_n$ we have i.e. $A_{n+1}' \supset A_n'$ i.e. $A_n' \subset A_{n+1}'$

Repeating the proof of Theorem 4.3.14 we get the result that

$$\bigcup_{n=1}^{\infty} A_n' \text{ is mesurable} \qquad \ldots\ldots\ldots\ldots (1)$$

and $$m\left(\bigcup_{n=1}^{\infty} A_n'\right) = \lim_{n\to\infty} m(A_n') = b - a - \lim_{n\to\infty} m(A_n) \qquad \ldots\ldots\ldots\ldots (2)$$

Now $\bigcup_{n=1}^{\infty} A_n' = \left(\bigcap_{n=1}^{\infty} A_n\right)'$ $\therefore \left(\bigcap_{n=1}^{\infty} A_n\right)'$ is measurable & so $\bigcap_{n=1}^{\infty} A_n$ is measurable.

Again from (2) we have

$$m\left\{\bigcap_{n=1}^{\infty} A_n\right\}' = b - a - \lim_{n\to\infty} m(A_n)$$

or, $$b - a - m\left\{\bigcap_{n=1}^{\infty} A_n\right\} = b - a - \lim_{n\to\infty} m(A_n)$$

or, $$m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n).$$

4.3.16. Theorem. If $A_1, A_2, A_3, \ldots$ are measurable subsets of $[a, b]$, then $\bigcup_{n=1}^{\infty} A_n$ is measurable and

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} m(A_n).$$

Moreover $\bigcap_{n=1}^{\infty} A_n$ is measurable.

Proof. Here $A_1, A_2, A_3, \ldots$ are measurable subsets of $[a, b]$. We have the important result that $\bigcup_{n=1}^{\infty} A_n$ can be expressed as the union of disjoint measurable subsets as follows.

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup (A_2 - A_1) \cup \{A_3 - (A_1 \cup A_2)\} \cup \{A_4 - (A_1 \cup A_2 \cup A_3)\} \cup \ldots$$

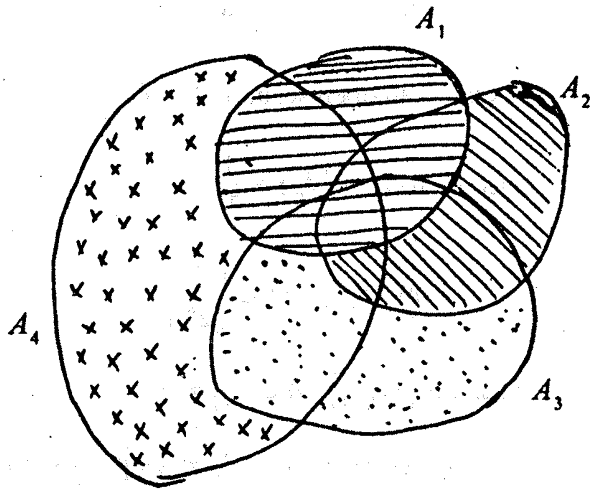
$\therefore$ By Theorem 4.3.13 we have $\bigcup_{n=1}^{\infty} A_n$ is measurable and

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = m(A_1) + m(A_2 - A_1) + m\{A_3 - (A_1 \cup A_2)\} + m\{A_4 - (A_1 \cup A_2 \cup A_3)\} + \ldots$$

$$= m(A_1)+m(A_2)-m(A_1)+m(A_3)-m(A_1\cup A_2)+m(A_4)-m(A_1\cup A_2\cup A_3)+.....$$

$$\le m(A_1)+m(A_2)+m(A_3)+m(A_4)+.... \ [\because m(A)\ge 0 \text{ for any } A]$$

$$=\sum_{n=1}^{\infty} m(A_n)$$



Thus we have $m\left(\bigcup_{n=1}^{\infty} A_n\right)\le \sum_{n=1}^{\infty} m(A_n)$.

Again we have

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n'\right)'$$

where A' is the complement of A.

Now for any $n = 1,2,3,....$ it is given that A_n is measurable. Hence for any $n = 1,2,3,....$ A_n' is measurable and so $\bigcup_{n=1}^{\infty} A_n'$ is measurable and thus $\left(\bigcup_{n=1}^{\infty} A_n'\right)'$ is measurable that is $\bigcap_{n=1}^{\infty} A_n$ is measurable.

4.4. Sets of measure zero

Sets having measure zero play an important role in measure theory.

4.4.1 Theorem. If $m^*(A)=0$ then A is measurable and $m(A)=0$.

Proof. We have the result

$$m_*(A)\le m^*(A)$$

If $m^*(A)=0$ then $m_*(A)\le 0$. But $m_*(A)\ge 0$. $\therefore m_*(A)=0$.

Hence $m_*(A)=m^*(A)=0$.

This shows that A is measurable and $m(A)=0$.

4.4.2. Theorem. A set consisting of one point is measurable and its measure is zero.

Proof. Let $A = \{x\}$ be any singleton set.

For each $n = 1, 2, 3, ...$ let $I_n = \left]x - \frac{1}{2^{n+1}}, x + \frac{1}{2^{n+1}}\right[$

Then I_n is an open set and $A \subset I_n$.

Also we have $m(I_n) = \frac{1}{2^n}$.

$$\therefore \inf_n m(I_n) = \inf_n \left\{\frac{1}{2^n}\right\} = 0$$

Thus $\inf\{m(I_n) : A \subset I_n\} = 0$

Now $\inf\{m(G) : A \subset G, G \text{ is open}\} \le \inf\{m(I_n) : A \subset I_n\}$.

$$\therefore \inf\{m(G) : A \subset G, G \text{ is open}\} \le 0$$

i.e. $\inf\{m(G) : A \subset G, G \text{ is open}\} = 0 \; [\because m(G) \not< 0]$

i.e. $m^*(A) = 0$.

We have thus $m_*(A) \le m^*(A) = 0$ i.e. $m_*(A) = 0$

Hence $m_*(A) = m^*(A) = 0$ i.e. A is measurable and $m(A) = 0$.

4.4.3. Theorem. A set consisting of a finite number of points is measurable and has measure zero.

Proof. Let $A = \{x_1, x_2, ..., x_n\}$ be any finite set.

Then $A = \bigcup_{i=1}^{n} \{x_i\}$.

$$\therefore m^*(A) = m^*\left(\bigcup_{i=1}^{n} \{x_i\}\right) \le \sum_{i=1}^{n} m^*(\{x_i\}).$$

Now $m^*(\{x_i\}) = 0$ for each $i = 1, 2, ..., n$.

Hence $m^*(A) \le 0$ i.e. $m^*(A) = 0$

We thus have $m_*(A) \le m^*(A) = 0$ i.e. $m_*(A) = 0$

$$\therefore m_*(A) = m^*(A) = 0$$

This shows that A is measurable and $m(A) = 0$

The same result is true when the set is countable. As the proof is exactly similar we only state the theorem.

4.4.4. Theorem. Any countable set is measurable and its measure is zero.

4.5. Cantor Set

Georg Cantor (1845–1918) constructed a set by a process which is very simple, but the set obtained is in fact very interesting curious and complicated. It violets many of our common senses.

4.5.1. Construction and definition of Cantor set.

To construct the Cantor set, we proceed as follows.

First we denote the closed unit interval [0, 1] by F_1. Next we delet from F_1 the open interval $G_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$ which is the middle third. The union ofo the remaining closed intervals is denoted by F_2. Thus

$$F_2 = \left[0, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, 1\right].$$

Next we delete from F_2 the open interval $G_2 = \left(\tfrac{1}{9}, \tfrac{2}{9}\right) \cup \left(\tfrac{7}{9}, \tfrac{8}{9}\right)$ which is the union middle third of its two pieces. The remaining closed set is denoted by F_3. Then

$$F_3 = \left[0, \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9}, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9}, 1\right].$$

This process is continued to obtain a sequence of closed sets F_n, each of which contains all its successors. The Cantor set P_0 is defined by

$$P_0 = \bigcap_{n=1}^{\infty} F_n.$$

We note that the Cantor set P_0 consists of those points in the closed unit interval [0, 1] which ultimately remain after the removal of all the open middle third open intervals $G_1, G_2, G_3, \ldots$

Clearly P_0 is non-empty set as it contains all end points of the closed intervals which make up the sets $F_n, n = 1,2,3\ldots$ Thus $0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}$, are members of the Cantor set P_0. It contains many other points also. e.g. $\frac{1}{4}$ is not an end point but is a member of P_0. Actually P_0 contains a multitude of points other than the above end points. The set of end points is countable but the set of Cantor set is actually uncountable. It is seen that every point of the Cantor set is in fact a limit point. In other words the Cantor set is a perfect set. Another very interesting and astonishing property of Cantor set is that its measure is zero though it is an uncountable set.

4.5.2. Theorem. Prove that the Cantor set has measure zero.

Proof. In the process of construction of the Cantor set P_0 we are to delete the middle open intervals from each of the closed intervals that remains in the previous stage.

The starting interval is $[0,1]$ and has measure 1.

The construction of Cantor set is done under the following scheme.

Stage	Number of deleted intervals	Length of each deleted interval	Union of deleted intervals
1	1	$\frac{1}{3}$	$G_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$
2	2	$\frac{1}{3^2}$	$G_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$
3	2^2	$\frac{1}{3^2}$	$G_3 = \left(\frac{1}{27}, \frac{2}{27}\right) \cup \left(\frac{7}{27}, \frac{8}{27}\right) \cup \left(\frac{19}{27}, \frac{20}{27}\right) \cup \left(\frac{25}{27}, \frac{26}{27}\right)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

If G_0 be the union of all open deleted intervals then

$$G_0 = \bigcup_{n=1}^{\infty} G_n.$$

Now $m(G_0) = m\left(\bigcup_{n=1}^{\infty} G_n\right) = \sum_{n=1}^{\infty} m(G_n)$

$$= 1 \times \tfrac{1}{3} + 2 \times \tfrac{1}{3^2} + 2^2 \times \tfrac{1}{3^2} + \ldots$$

$$= \tfrac{1}{3} + \tfrac{2}{3^2} + \tfrac{2}{3^2} + \ldots$$

$$= \frac{\tfrac{1}{3}}{1 - \tfrac{2}{3}}$$

$$= 1$$

Now $P_0 \cup G_0 = [0,1]$ and $P_0 \cap G_0 = \phi$

$$\therefore m(P_0) + m(G_0) = m([0,1])$$

or, $m(P_0) + 1 = 1$

or, $m(P_0) = 0.$

Hence the Cantor set P_0 has measure zero.

4.5.3. Theorem. Cantor set is an uncountable set.

Proof. Let P_0 be the Cantor set. If possible let P_0 be countable.

We enumerate the points of P_0 by $x_i, i = 1,2,3,\ldots$

So $P_0 = \{x_1, x_2, x_3, \ldots\}$.

Let the ternary representation of x_i in non terminating form be

$$x_i = 0 \cdot a_{i1} a_{i2} a_{i3} \ldots, \quad i = 1,2,3,\ldots$$

From the construction of P_0 it is clear that no a_{ij} is 1. i.e. a_{ij} is either 0 or 2. [See Note]

$\therefore x_1 = 0 \cdot a_{11}a_{12}a_{13}......$

$x_2 = 0 \cdot a_{21}a_{22}a_{23}......$

$x_3 = 0 \cdot a_{31}a_{32}a_{33}......$

....

....

....

We note that all points of P_0 are in this list.

Now we construct a point y as

$y = 0 \cdot b_1\, b_2\, b_3....$

where $b_i = 0$ if $a_{ii} = 2$

$= 2$ if $a_{ii} = 0.$

Then y is a point of the Cantor set P_0.

Now $y \neq x_1$ as $b_1 \neq a_{11}$

$y \neq x_2$ as $b_2 \neq a_{22}$

$y \neq x_3$ as $b_3 \neq a_{33}$

....

....

....

Thus no x_i is y i.e. $y \notin P_0$. But as noted above y is a member of the Cantor set P_0. This is a contradiction. Hence P_0 is uncountable.

Note. The ternary representation of a number x in [0, 1] is given by the series $x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + ... = 0 \cdot a_1a_2a_3.....$

If the form of this represention is terminating we can put it in non terminating form as follows. The number $\frac{1}{3}$ is 0.1

which is terminating in nature. The non-terminating form of it is

$$\frac{1}{3}=\frac{1}{3}+\frac{0}{3^2}+\frac{0}{3^3}+......=0\cdot1=\frac{0}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\frac{2}{3^4}+........=0.0222.....$$

In the ternary representation of the points of P_0 in non-terminating form always contains only 0 or 2 as in the process of construction of the Cantor set always the middle open interval is deleted. The ternary representation of $\frac{7}{9}$ is $\frac{7}{9}=\frac{2\cdot3+1}{3^2}=\frac{2}{3}+\frac{1}{3^2}=0.21=0.20222.....=\frac{2}{3}+\frac{0}{3^2}+\frac{2}{3^3}+\frac{2}{3^4}+$

Similarly,

$$\frac{8}{9}=\frac{2\cdot3+2}{3^2}=\frac{2}{3}+\frac{2}{3^2}=0.22=0.22000......$$

4.6. Measurable Functions

With the help of measurable set measurable function is defined. This class of functions plays an important role in mathematics. Almost all functions that we come accross are found to be measurable functions.

4.6.1. Definition. Let f be a function defined on $[a, b]$. We call f to be a measurable function if for each $\alpha \in R$, the set $\{x: f(x)>\alpha\}$ is a measurable set.

In other words f is said to be measurable function if for every real number α, the inverse image of $]\alpha,\infty[$ is a measurable set.

4.6.2. Theorem. The function f on $[a, b]$ is measurable if and only if one of the following conditions hold:

i) $\{x: f(x)>\alpha\}$ is a measurable set for every real α

ii) $\{x: f(x)\geq\alpha\}$ is a measurable set for every real α

iii) $\{x: f(x)<\alpha\}$ is a measurable set for every real α

iv) $\{x: f(x)\leq\alpha\}$ is a measurable set for every real α.

Proof. Let f be measurable function on $[a, b]$. Then by definition, the set $\{x : f(x) \geq \alpha\}$ is a measurable set for every real α.

Now the set $\{x : f(x) \leq \alpha\}$ is the complement of the set $\{x : f(x) > \alpha\}$ relative to $[a, b]$. Since complement of measurable set is measurable, it follows that $(i) \Leftrightarrow (iv)$.

Since f is measurable function on $[a, b]$, from definition it follows that the set $\left\{x : f(x) > \alpha - \frac{1}{n}\right\}, n = 1,2,3,.....$ is measurable. Now $\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty}\left\{x : f(x) > \alpha - \frac{1}{n}\right\}$

We know that arbitrary intersection of measurable sets is measurable. Hence $\{x : f(x) \geq \alpha\}$ is measurable set.

$\therefore (i) \Rightarrow (ii)$.

Let (iii) be true. Now $\{x : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty}\left\{x : f(x) > \alpha + \frac{1}{n}\right\}$. Since $\left\{x : f(x) < \alpha + \frac{1}{n}\right\}$ is measurable, it follows that $\{x : f(x) \leq \alpha\}$ is measurable $\therefore (iii) \Rightarrow (iv)$.

Now $\{x : f(x) < \alpha\}$ is the complement of $\{x : f(x) \geq \alpha\}$.

$\therefore (ii) \Leftrightarrow (iii)$

Hence we have finally,

$$\begin{array}{ccc} (i) & \Rightarrow & (ii) \\ \Updownarrow & & \Updownarrow \\ (iv) & \Leftarrow & (iii) \end{array}$$

This proves the theorem.

4.6.3. Theorem. If f is measurable function then $|f|$ is measurable function.

Proof. For any real α we have

$$\{x : |f(x)| < \alpha\} = \{x : -\alpha < f(x) < \alpha\}$$

$$=\{x: f(x)<\alpha\} \cap \{x: f(x)>-\alpha\}$$

If f is measurable function then both $\{x: f(x)<\alpha\}$ and $\{x: f(x)>-\alpha\}$ are measurable and so $\{x:|f(x)|<\alpha\}$ is measurable.

Hence $|f|$ is measurable function.

4.6.4. Theorem. If f is measurable function on $[a, b]$ and k is any real number then $f+k$ and kf are measurable functions.

Proof. For any real α we have

$$\{x: f(x)+k>\alpha\}=\{x: f(x)>\alpha-k\} \quad \text{.................} (1)$$

If f is measurable function then the RHS of (1) a measurable set and so LHS of (1) is also a measurable set. Hence $f(x)+k$ is measurable function.

We have for $k>0$

$$\{x: kf(x)>\alpha\}=\{x: f(x)>\alpha/k\}$$

If f is measurable then the RHS is measurable set and so the LHS is a measurable set. So kf is measurable function.

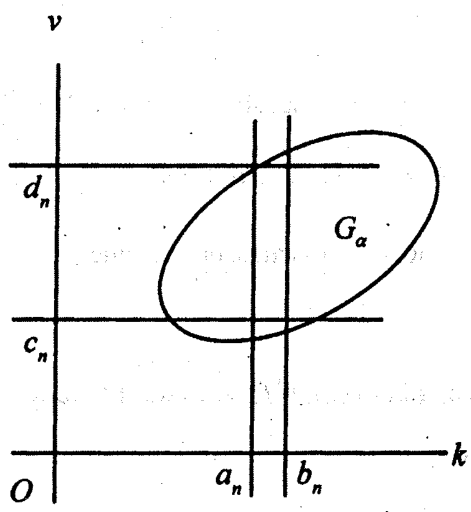
For $k<0$ we have

$$\{x: kf(x)>\alpha\}=\{x: f(x)<\alpha/k\}.$$

If f is measurable function then the RHS is measurable set and so the LHS is also so. Hence kf is measurable function.

Note. Taking $k=-1$ we get the result that if f is measurable function then $-f$ is also measurable function.

4.6.4. Theorem. Let f and g be measurable real valued functions defined on $[a, b]$ and let $F(u, v)$ be real and continuous function on R^2. Then $F(f(x), g(x))$ is measurable function on $[a, b]$.



Proof. For any real number α let

$$G_\alpha = \{(u,v) : F(u,v) > \alpha\}.$$

Then G_α is an open subset of R^2.

We can write G_α as

$$G_\alpha = \bigcup_{n=1}^{\infty} I_n$$

where $\{I_n\}$ is a sequence of open intervals.

$$I_n = \{(u,v) : a_n < u < b_n, c_n < v < d_n\}.$$

Now, $\{x : F(f(x), g(x)) > \alpha\}$

$$= \{x : (f(x), g(x)) \in G_\alpha\}$$

$$= \bigcup_{n=1}^{\infty} \{x : (f(x), g(x)) \in I_n\}$$

$$= \bigcup_{n=1}^{\infty} \{x : a_n < f(x) < b_n\} \cap \{x : c_n < g(x) < d_n\}$$

$$= \bigcup_{n=1}^{\infty} \left[\{x : f(x) > a_n\} \cap \{x : f(x) < b_n\} \cap \{x : g(x) > c_n\} \cap \{x : g(x) < d_n\}\right].$$

Since $f(x)$ and $g(x)$ are measurable functions, for each n = 1, 2, 3, ... the sets $\{x : f(x) > a_n\}$, $\{x : f(x) < b_n\}$, $\{x : g(x) > c_n\}$ and $\{x : g(x) < d_n\}$ are measurable. Hence $\{x : F(f(x), g(x)) > \alpha\}$ is a measurable set. This is true for all real α. ∴ $F(f(x), g(x))$ is a measurable function on $[a, b]$.

4.6.5. Theorem. If f and g are measurable functions on $[a, b]$ then so are $f + g, f - g, fg$ and f/g provided $g \neq 0$ on $[a,b]$.

Proof. Let $F_1(u,v) = u + v, F_2(u,v) = u - v, F_3(u,v) = uv$ and $F_4(u,v) = u/v, v \neq 0$.

Then F_1, F_2, F_3 and F_4 are all continuous functions on R^2. Hence by Theorem 4.6.4 it follows that $f+g$, $f-g, fg$ and f/g are all measurable functions on $[a, b]$.

4.6.6. Theorem. If $\{f_n\}$ is a sequence of measurable functions on $[a, b]$ such that the sequence $\{f_n(x)\}$ is bounded for every $x \in [a,b]$, then the functions

$$G(x) = \sup\{f_1(x), f_2(x), f_3(x), \ldots\}$$

$$g(x) = \inf\{f_1(x), f_2(x), f_3(x), \ldots\}$$

$$H(x) = \limsup_{n\to\infty} f_n(x)$$

$$h(x) = \liminf_{n\to\infty} f_n(x)$$

are measurable.

Proof. Let α be a real number and $G(x) > \alpha$ for all $x \in [a,b]$.

$$\therefore \sup\{f_1(x), f_2(x), f_3(x), \ldots\} > \alpha$$

i.e. $f_n(x) > \alpha$ for some n.

Now $\{x : G(x) > \alpha\}$

$$= \bigcup_n \{x : f_n(x) > \alpha\}.$$

Each f_n being measurable function, the set $\{x : f_n(x) > \alpha\}$ is measurable and hence $\bigcup_n \{x : f_n(x) > \alpha\}$ is measurable i.e. $\{x : G(x) > \alpha\}$ is measurable. This is true for each real α. Hence G is measurable function

Similarly $g(x) < \alpha$ for all $x \in [a,b]$

$$\Rightarrow \inf\{f_1(x), f_2(x), \ldots\} < \alpha$$

$$\Rightarrow f_n(x) < \alpha$$

$$\Rightarrow f_n(x) < \alpha \text{ for some } n.$$

$\therefore \{x : g(x) < \alpha\} = \bigcup_n \{x : f_n(x) < \alpha\}$. Now $\{x : f_n(x) < \alpha\}$ is measurable for all *n*.

$\therefore \{x : g(x) < \alpha\}$ is a measurable set.

This shows that $g(x)$ is a measurable function.

For each positive integer *n* we define

$$G_n(x) = \sup\{f_n(x), f_{n+1}(x), f_{n+2}(x), ...\}$$

and $g_n(x) = \inf\{f_n(x), f_{n+1}(x), f_{n+2}(x), ...\}$.

From the results proved above it follows that $G_n(x)$ and $g_n(x)$ are measurable functions.

For each $x \in [a,b]$ we have

$$G_1(x) \geq G_2(x) \geq G_3(x) \geq ...... \geq G_n(x) \geq ......$$

and $g_1(x) \leq g_2(x) \leq g_3(x) \leq ...... \leq g_n(x) \leq ......$

Now $H(x) = \limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty} G_n(x)$

and $h(x) = \liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty} g_n(x)$.

The sequences $\{G_n(x)\}$ being monotonically decreasing it follows that $H(x) < G_n(x)$ for all *n*.

Now for each real α the set

$$\{x : H(x) < \alpha\} = \bigcup_n \{x : G_n(x) < \alpha\}.$$

Since $G_n(x)$ is measurable for all *n*, it follows that $H(x)$ is measurable.

Similarly, $G_n(x)$ is monotonically increasing and so $h(x) > g_n(x)$ for all *n*.

For each real α, the set

$$\{x : h(x) > \alpha\} = \bigcup_n \{x : g_n(x) > \alpha\}.$$

As all $g_n(x)$ are measurable, $h(x)$ is measurable.

4.6.7. Definition. Almost Everywhere.

A property is said to hold almost everywhere if the property holds everywhere except on a set of measure zero. As for example if $f = g$ a.e then the set $A = \{x : f(x) \neq g(x)\}$ has measure zero.

4.6.8. Theorem. If $f = g$ almost everywhere and f is a measurable function, then g is also a measurable function.

Proof. Let $A = \{x : f(x) \neq g(x)\}$. Then $m(A) = 0$.

Let α be a positive number. Then the set

$$\{x : f(x) - g(x) > \alpha\} = \{x : f(x) \neq g(x), f(x) - g(x) > \alpha\}$$

$$\therefore \{x : f(x) - g(x) > \alpha\} \subset A$$

Since $m(A) = 0$, we have $m\{x : f(x) - g(x) > \alpha\} = 0$

i.e. for $\alpha > 0$, the set $\{x : f(x) - g(x) > \alpha\} = 0$ is measurable.

Let α a be a negative number. Then the set

$$\{x : f(x) - g(x) > \alpha\} = \{x : f(x) = g(x)\} \cup \{x : f(x) \neq g(x), f(x) - g(x) > \alpha\}.$$

The first set on the RHS is the complement of the set of measure zero and the second set on the RHS is contained in the set of measure zero. Hence both sets on the RHS are measurable.

$\therefore \{x : f(x) - g(x) > \alpha\}$ is measurable for any $\alpha < 0$.

$\therefore \{x : f(x) - g(x) > \alpha\}$ is measurable for arbitrary real α.

Hence the set $f - g$ is measurable.

Now $g = f - (f - g)$. Since f and $f - g$ are measurable we see that g is measurable.

4.6.9 Theorem. Every continuous function is measurable.

Prof. Let α be any real number and $f(x)$ be continuous function. Then the set $\{x : f(x) > \alpha\}$ is the inverse image of the open interval $]\alpha, \infty[$ under the function $f(x)$. Since $f(x)$ is toninuous function the inverse image of every open set under $f(x)$ is an open set. Hence $]\alpha,\infty[$ is an open set and so its inverse image $\{x : f(x) > \alpha\}$ is an open set. We know that every open set is measurable. Hence $\{x : f(x) > \alpha\}$ is a measurable set. This is true for any α. So $f(x)$ is measurable function.

The converse of this theorem may or may not be true, i.e. a measurable function may or may not be continuous. Still then every measurable function which is finite a.e can be approximated by some continous function as shown by N.N. Lusin. We only state this theorem.

4.6.10. Theorem. (N.N. Lusin). Let f be a measurable function which is finite a.e. on $[a, b]$. Then for every $\delta > 0$ there exists a continuous function g such that $m\{x : f \neq g\} < \delta$. Moreover if f is bounded by a constant M then g is also bounded by the same constant.

The following theorem is due to Egoroff. We only state it.

4.6.11. Theorem (Egoroff). If $\{f_n\}$ is a sequence of measurable functions which converge to a real valued function f a.e on a measurable set E of finite measure, then given $\in > 0$, there is a subset $A \subset E$ with $m(E - A) < \in$ such that $\{f_n\}$ converges to f uniformly on A.

4.7. Illustrative Examples

Example 4.7.1. If A_1 and A_2 are measurable subsets of the closed interval $[a, b]$ then $A_1 - A_2$ is measurable and if $A_2 \subset A_1$ then $m(A_1 - A_2) = m(A_1) - m(A_2)$.

Solution. We have $A_1 - A_2 = A_1 = A_1 \cap A_2'$

If A_1 and A_2 are measurable then A_1 and A_2' are measurable and so $A_1 \cap A_2'$ is measurable i.e. $A_1 - A_2$ is measurable.

If $A_2 \subset A_1$, then $A_1 = A_2 \cup (A_1 - A_2)$. .... (1)

Since A_2 and $A_1 - A_2$ are disjoint, we have

$$A_2 \cap (A_1 - A_2) = \phi \quad \therefore m[A_2 \cap A_1 - A_2] = 0.$$

From (1) we have

$$m(A_1) = m[A_2 \cup (A_1 - A_2)]$$

$$= m(A_2) + m(A_1 - A_2) - m[A_2 \cap (A_1 - A_2)]$$

$$= m(A_2) + m(A_1 - A_2) - 0$$

$$\therefore m(A_1 - A_2) = m(A_1) - m(A_2).$$

Example 4.7.2. Show that the set of even integers has measure zero.

Solution. The set of even integers is

$A = \{2, 4, 6, 8, ...]$

Let $\in$ be arbitrary small positive number.

We consider the open intervals

$$I_1 = \left(2 - \in/2^2, 2 + \in/2^2\right), I_2 = \left(4 - \in/2^3, 4 + \in/2^3\right), I_3 = \left(6 - \in/2^4, 6 + \in/2^4\right), I_4 = \left(8 - \in/2^5, 8 + \in/2^5\right), ...$$

Then $A \subset I_1 \cup I_2 \cup I_3 \cup I_4 \cup ......$

$$\therefore m^*(A) \le m(I_1) + m(I_2) + m(I_3) + m(I_4) + ...$$

$$= \in/2 + \in/2^2 + \in/2^3 + \in/2^4 + ...$$

$$= \frac{\in/2}{1 - 1/2}$$

$$= \in$$

Since $\in$ is arbitrary small $m^*(A) \le 0$.

But $m^*(A) \nless 0 \quad \therefore m^*(A) = 0.$

Now $0 \le m_*(A) \le m^*(A) = 0$

$\therefore m_*(A) = m^*(A) = 0$

i.e. A is measurable and has measure zero

i.e. $m(A) = 0$.

Example 4.7.3. Show that if f is measurable function then the set $\{x : f(x) = \alpha\}$ is measurable for each real number α.

Solution. Since f is measurable function the sets $\{x : f(x) \ge \alpha\}$ and $\{x : f(x) \le \alpha\}$ are mesurable sets. We know intersection of two measurable sets is measurable. Therefore, the set $\{x : f(x) \ge \alpha\} \cap \{x : f(x) \le \alpha\}$ is measurable and so $\{x : f(x) = \alpha\}$ is measurable.

Example 4.7.4. Show that any constant function is measurable.

Solution. Let $f(x) = c$ for all x.

For $\alpha \le c$ we have $\{x : f(x) > \alpha\}$ = Domain of definition of the function and so is a measurable set.

Again for $\alpha > c$ we have $\{x : f(x) > \alpha\} = \phi$, which is measurable set with measure zero.

Hence for each real α, the set $\{x : f(x) > \alpha\}$ is a measurable set.

$\therefore f(x)$ is a measurable function.

Example 4.7.5. Show that the function

$f(x) = 4$, for rational x in $[1, 8]$

$= -3$, for irrational x in $[1, 8]$

is a measurable function on $[1, 8]$.

Solution. For any $\alpha < -3$ we have $\{x : f(x) > \alpha\} = [1,8]$

which is measurable with measure 7.

For $-3 \le \alpha < 4, \{x : f(x) > \alpha\}$ = {all rational numbers in [1, 8]}

which is a measurable set with measure zero.

For $\alpha \ge 4$ we have $\{x : f(x) > \alpha\} = \phi$

which is measurable with measure zero.

Thus for any real α, the set $\{x : f(x) > \alpha\}$ is measurable.

Hence $f(x)$ is a measurable function.

4.8. Summary : In this module the notion of measurable set has been introduced first. Then this notion is used to define measurable function. Sets with measure zero plays an important role to define almost everywhere property. Cantor set is defined and its properties are discussed.

4.9. Self Assessment Questions

1. Find the measure of the sets

   i) $\{x : -2 < x \le 4\} \cup \{x : 5 \le x < 8\}$

   ii) $\{x : -2 < x \le 4\} \cup \{x : 1 \le x \le 8\}$

   iii) $\{x : -2 < x \le 4\} \cup \{5, 6, 8\} \cup \{x : 7 \le x \le 10\}$.

2. Show that the set of all irrational numbers in [2, 5] is measurable.

3. If A_1, A_2, A_3 are measurable sets, prove that

$$m(A_1 \cup A_2 \cup A_3) = m(A_1) + m(A_2) + m(A_3) - m(A_1 \cap A_2) - m(A_1 \cap A_3)$$
$$-m(A_2 \cap A_3) + m(A_1 \cap A_2 \cap A_2).$$

6. Show that every non-empty open set has positive measure.

7. Show that every subset of a set of measure zero is of measure zero.

8. Show that the function

$f(x) = 4$ for irrational x in $[2, 5]$

$= -3$ for rational x in $[2, 5]$

is a measurable function.

9. Show that the function

$$f(x) = 4, 1 \leq x < 3$$
$$= -2, \ x = 3$$
$$= 5, \ 3 < x < 6 \cup 8 < x < 10$$
$$= 8, \ 6 \leq x \leq 8 \cup 10 \leq x \leq 12$$

is a measurable function on $[1, 12]$.

4.10. Suggested books for further reading

1. Mathematical Analysis : S.C. Malik & Savita Arora; Wiley Eastern Limited, New Age International Limited.
2. Mathematical Analysis : Tom. M. Apostol; Narosa Publishing House.
3. Principles of Mathematical Analysis : Walter Rudin; International Students Edition, McGraw-Hill International Book Company.
4. Introductory Real Analysis : A.N. Kolmogorov & S.V. Fomin; Dover Publicatins, INC, New York.

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M.Sc. Course
in
Applied Mathematics with Occanology
and
Computer Programming

PART - I

PAPER - I **Module No - 05** GROUP -A]

Real Analysis
(Lebesgue Integral)

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Content :

5.1 Introduction

The Riemann integral $\int_a^b f(x)dx$ is simple to describe and serves all needs of elementary calculus. This integral exists only for functions which are either continuous or else do not have too many points of discontinuity. In fact if $f(x)$ has discontinuties everywhere or if $f(x)$ is defined on an abstract set then Riemann integral does not exist. For such functions we need another fully developed notion of integral which is more flexible than the notion of the Riemann integral. In 1902 Lebesgue gave a definition of measure for point sets and applied this notion of measure theory to develop a new integral more general than the Riemann integral. To honour Lebesgue, this integral is known as Lebesgue integral.

5.2 Objective

An extension of Riemann integral is introduced by Lebesgue and is known as Lebesgue integral. It permits integral for more general functions as integrands. Also it can treat both bounded as well as unbounded functions. Further it enables us to integrate functions which are defined not only on a finite interval $[a,b]$ but also over any set A of finite measure. The Lebesgue integral also gives more satisfying convergence theorems than Riemann integral. If a sequence of functions $\{f_n(x)\}$ converges pointwise to a limit function $f(x)$ on $[a,b]$, it is desirable to conclude that

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx$$

with a minimum of additional hypotheses. In Lebesgue integral we have the famous Lebesgue dominated convergence theorem, which permits term-by-term integration if each $\{f_n(x)\}$ is Lebesgue integrable and if the sequence is dominated by a Lebesgue integrable function. This theorem is true for Lebesgue integral only and is not true for Riemann integral. In fact it is seen that the class of all Riemann integrable functions is quite small in comparison with the class of all Lebesgue integrable functions.

5.3 Lebesgue Integral

5.3.1 Definition : Measurable partition

Let $f(x)$ be any bounded function defined on the closed finite interval $[a,b]$. A partition $P=\{A_1, A_2, \ldots, A_n\}$ of $[a,b]$ is called a measurable partition of $[a,b]$ if

(i) $A_1, A_2, \ldots, A_n$ are measurable subsets of $[a, b]$

(ii) $\bigcup_{i=1}^{n} A_i = [a, b]$

and (iii) $m(A_i \cap A_j) = 0$ for $i \neq j$

5.3.2 Definition Upper Lebesgue Integral, Lower Lebesgue Integral and Lebesgue Integral.

For any measurable partition $P = \{A_1, A_2, ..., A_n\}$

we define $U(P,f) = \sum_{i=1}^{n} \left(\sup_{x \in A_i} f(x) \right) m(A_i)$

and $L(P,f) = \sum_{i=1}^{n} \left(\inf_{x \in A_i} f(x) \right) m(A_i)$

as the upper and lower Lebesgue sums of the function $f(x)$ corresponding to the partition P of $[a,b]$.

Let $M = \sup_{a \le x \le b} f(x)$ and $m = \inf_{a \le x \le b} f(x)$. Then for any measurable partition $P = \{A_1, A_2, ..., A_n\}$ we have $\left(\inf_{a \le x \le b} f(x) \right) m([a, b]) \le L(P,f) \le U(P,f) \le \left(\sup_{a \le x \le b} f(x) \right) m([a, b])$

i.e. $m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$ ....(5.31)

let $\wp$ be the set of all measurable partitions of $[a,b]$. Then from (5.3.1) it is clear that $\sup\{L(P,f): P \in \wp\}$ and $\inf\{U(P,f): P \in \wp\}$ exist and they are denbted respectively by $L\underline{\int_a^b} f(x)dx$ and $L\overline{\int_a^b} f(x)dx$ and are called respectively as lower Lebesgue Integral and upper Lebesgue Integral. It can be easily shown that $L\underline{\int_a^b} f(x)dx \le L\overline{\int_a^b} f(x)dx$

A bounded function $f(x)$ defined on $[a,b]$ is said to be Lebesgue integrable if

$$L\int_a^{b} f(x)\,dx = L\int_a^{\bar{b}} f(x)\,dx$$

If $L\int_a^{b} f(x)dx \neq L\int_a^{b} f(x)dx$ then $f(x)$ is said to be not Lebesgue integrable.

The fact that f is Lebesgue integrable is expressed by writting $f \in L[a,b]$ and the Lebesgue integral is denoted by $L\int_a^{b} f(x)\,dx$ i.e.

$$L\int_a^{b} f(x)\,dx = L\int_a^{\bar{b}} f(x)\,dx = L\int_a^{b} f(x)\,dx$$

5.3.3 Lemma

Show that $L\int_a^{b} f(x)\,dx \leq L\int_a^{\bar{b}} f(x)\,dx$

Proof

Let f be a bounded function defined on $[a,b]$.

•Let $P_1 = \{A_1, A_2, \ldots\ldots, A_n\}$ and $P_2 = \{B_1, B_2, \ldots\ldots, B_m\}$ be any two measurable partitions of $[a,b]$.

Let P be the measurable partition of $[a,b]$ whose components are mn subsets $A_i \cap B_j\,(i = 1,2,\ldots\ldots,n,\ j = 1,2,\ldots\ldots,m)$ Then P is a refinement of both P_1 and P_2.

Obviously, we have

$$L(P_2,f) \leq L(P,f) \leq U(P,f) \leq U(P_1,f)$$

Taking supremum over all P_2 we have

$$\sup_{P_2} L(P_2,f) \leq U(P_1,f).$$

Taking infimum over all P_1 we have

$$\sup_{P_2} L(P_2, f) \le \inf_{P_1} U(P_1, f)$$

or, $$L\int_a^b f(x)dx \le L\int_a^{\bar{b}} f(x)dx.$$

5.3.4 Theorem

Prove that every bounded Riemann integrable function is Lebesgue integrable and the two integrals are equal.

Proof.

Let the bounded function $f(x)$ on $[a,b]$ be Riemann integrable on $[a,b]$. Then we have

$$R\int_a^b f(x)dx = R\int_a^{\bar{b}} f(x)dx = R\int_a^b f(x)dx \quad \text{..........................(5.3.4.1)}$$

Let $P' = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of the interval $[a,b]$ and $\wp'$ be the collection of all such possible partitions of $[a,b]$. We can write the partition $\wp'$ as

$$\wp' = \{[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]\}.$$

We note that $\bigcup_{i=1}^{n} [x_{i-1}, x_i] = [a, b]$ and $m([x_{i-1}, x_i] \cup [x_{j-1}, x_j]) = 0$ for all $i \ne j$. Hence $\wp'$ is also a measurable partition of $[a,b]$.

Let $P = \{A_0, A_1, ..., A_n\}$ be any measurable partition of $[a,b]$ and $\wp'$ the collection of all possible measurable partitions of $[a,b]$. Then $\wp' \subset \wp$ i.e. all partitions involved in the Riemann integral is a subset of all partitions involved in Labesgue integral.

$$\therefore \sup\{L(P':f): P' \in \wp'\} \le \sup\{L(P,f): P \in \wp\}$$

and $$\inf\{U(P,f): P \in \wp\} \le \inf\{U(P':f): P' \in \wp'\}$$

i.e. $$R\int_a^b f(x)dx \le L\int_a^b f(x)dx \quad \text{and} \quad L\int_a^{\bar{b}} f(x)dx \le R\int_a^{\bar{b}} f(x)dx$$

But we have $L\int_a^b f(x)dx \le L\int_a^{\bar{b}} f(x)dx$

Thus

$$R\int_a^b f(x)dx \le L\int_a^b f(x)dx \le L\int_a^{\bar{b}} f(x)dx \le R\int_a^{\bar{b}} f(x)dx \qquad \text{..........................(5.3.4.2)}$$

Using (5.3.4.1) we have from (5.3.4.2)

$$R\int_a^b f(x)dx = L\int_a^b f(x)dx = L\int_a^{\bar{b}} f(x)dx.$$

This shows that $f(x)$ is Lebesgue integrable on $[a, b]$ and the value of the Lebesgue integral is $R\int_a^b f(x)dx$ Hence $f(x)$ is Lebesgue integrable on $[a, b]$ if it is Riemann integral on $[a, b]$ and the two integrals are equal.

The following example shows that the converse of the above theorem is not true.

5.3.5 Example

Show that the function $f(x)$ defined by

$f(x) = 3$ when x is rational in [1, 6]

$= 4$ when x is irrational in [1, 6]

is not Riemann integrable on [1, 6] but is Lebesgue integrable on [1, 6]. Find the value of Lebesgue integral.

Ans. We first show that $f(x)$ is not Riemann integrable on [1, 6], Let $P = \{1 = x_0, x_1, x_2, ..., x_n = 6\}$ be any partition of [1, 6]. We have

$$L(P, f) = \sum_{i=1}^{n} inf\{f(x): x_{i-1} \le x \le x_i\}\Delta x_i$$

$$= \sum_{i=1}^{n} 3(x_i - x_{i-1})$$

$$= 3(x_n - x_0)$$

$$= 3(6-1)$$

$$= 15$$

$$U(P,f) = \sum_{i=1}^{n} sup\{f(x): x_{i-1} \leq x \leq x_i\}\Delta x_i$$

$$= \sum_{i=1}^{n} 4(x_i - x_{i-1})$$

$$= 4(6-1)$$

$$= 20$$

Since $L(P,f) = 15$ and $U(P,f) = 20$ for all partition P of [1, 6].

we have $\sup_P L(P,f) = 15 \neq 20 = \inf_P U(P,f)$

Hence $f(x)$ is not R-integrable on [1, 6].

Now we show that $f(x)$ is L-integrable on [1, 6].

Let $P_1 = \{A_1, A_2\}$ where

A_1 = set of all rational numbers in [1, 6]

and A_2 = set of all irrational numbers in [1, 6].

Then $A_1 \cup A_2 = [1, 6]$ and $m(A_1 \cap A_2) = 0$.

Therefore P_1 is a measurable partition of [1, 6].

Now $m(A_1) = 0$ and $m(A_2) = m([1, 6]) - m(A_1)$

$$= 6 - 1 - 0 = 5$$

$\therefore \; L(P_1, f) = inf\{f(x): x \in A_1\}\, m(A_1) + inf\{f(x): x \in A_2\}\, m(A_2)$

$= 3.0 + 4.5$

$= 20$

$$U(P_1, f) = sup\{f(x): x \in A_1\}\ m(A_1) + sup\{f(x): x \in A_2\}\ m(A_2)$$

$= 3.0 + 4.5$

$= 20$

Hence we have for any measurable partition P of [1, 6].

$$20 = L(P_1, f) \le \sup_P\{L(P, f)\} \le \inf_P\{U(P, f)\} \le U(P_1, f) = 20$$

i.e. $L\int_1^6 f(x)dx = L\int_1^{\bar{6}} f(x)dx = 20$

This shows that $f(x)$ is L-integrable and the value of the integral is 20 i.e. $L\int_1^6 f(x)dx = 20$

The following theorem gives the necessary and sufficient condition for a function to be L-integrable.

5.3.6 Theorem.

A bounded function $f(x)$ is L-integrable on $[a,b]$ if and only if for every $\in > 0$ there exists a measurable partition P of $[a,b]$ such that

$U(P, f) - L(P, f) < \in$.

Proof.

The condition is necessary :

Let $f(x)$ be L-integrable on $[a,b]$. So we have

$$\int_a^b f(x)dx = \int_a^{\bar{b}} f(x)dx = \int_a^b f(x)dx \quad \text{.............................(5.3.6.1)}$$

Let $\in$ be any pre-assigned positiove number.

Since $\int_{a}^{b} f(x)dx = \sup_{P} L(P,f)$

and $\int_{a}^{\bar{b}} f(x)dx = \inf_{P} U(P,f)$

there exist measurable partitions P_1 and P_2 of $[a,b]$ such that

$$L(P_1,f) > \int_{\underline{a}}^{b} f(x)dx - \epsilon/2 = \int_{a}^{b} f(x)dx - \epsilon/2, \ [by(5.3.6.1)]$$

and $$U(P_2,f) < \int_{a}^{\bar{b}} f(x)dx + \epsilon/2 = \int_{a}^{b} f(x)dx + \epsilon/2, \ [by(5.3.6.1)]$$

Let $P = P_1 \cup P_2$. Then

$$U(P,f) \le U(P_2,f) < \int_{a}^{b} f(x)dx + \epsilon/2 < L(P_1,f) + \epsilon/2 + \epsilon/2 \le L(P,f) + \epsilon$$

$\therefore \ U(P,f) - L(P,f) < \epsilon.$

The condition is sufficient :

Let ϵ be any pre-assigned positive number and P be a measurable partition such that

$U(P,f) - L(P,f) < \epsilon$ ....(5.3.6.2)

For this measurable partition P we have

$$L(P,f) \le \int_{a}^{b} f(x)dx \le \int_{a}^{\bar{b}} f(x)dx \le U(P,f)$$

$$\therefore \int_{a}^{b} f(x)dx - \int_{a}^{b} f(x)dx \le U(P,f) - L(P,f) < \epsilon$$

Since ϵ is arbitrary, we have

$$\int_{a}^{b} f(x)dx = \int_{a}^{\bar{b}} f(x)dx$$

This shows that $f(x)$ is L-integrable on $[a,b]$.

The following theorem shows that all bounded measurable functions are Lebesgue integrable. As almost all functions that we come accross are measurable it follows by this theorem that most of the functions that we deal with are Lebesgue integrable.

5.3.7 Theorem. Every bounded measurable function on $[a,b]$ is Lebesgue integrable on $[a,b]$.

Proof. Let $f(x)$ be bounded measurable functions on $[a,b]$.

So there exists real numbers m and M such that

$m \le f(x) < M$ for all $x \in [a,b]$.

Let $\in$ be any pre-assigned small positive number. Then there exists a finite number of points $y_0, y_1, y_2, ..., y_n$ such that

i) $y_k - y_{k-1} < \in/(b-a)$ for each $k = 1, 2, ..., n$ ....(5.3.7.1)

and ii) $m = y_0 < y_1 < ...... < y_n = M$.

For each $k = 1, 2, ..., n$ we define A_k as follows

$A_k = \{x \in [a,b] : y_{k-1} \le f(x) < y_k\}$ ....(5.3.7.2).

Then each A_k is measurable and

$\bigcup_{k=1}^{n} A_k = [a,b]$ and $m(A_i \cap A_j) = 0$ for all $i \ne j$.

$\therefore$ $P = \{A_1, A_2, ..., A_n\}$ is a measurable partition of $[a,b]$.

From (5.3.7.2) it follows that

$\therefore$ $sup\{f(x) : x \in A_k\} \le y_k$ and $inf\{f(x) : x \in A_k\} \ge y_{k-1}$

$$\therefore \; U(P,f) = \sum_{k=1}^{n}\left(\sup_{A_k} f(x)\right) m(A_k) \le \sum_{k=1}^{n} y_k \, m(A_k)$$

$$L(P,f) = \sum_{k=1}^{n}\left(\inf_{A_k} f(x)\right) m(A_k) \ge \sum_{k=1}^{n} y_{k-1} \, m(A_k)$$

Hence $U(P,f)-L(P,f)$

$$\leq \sum_{k=1}^{n}(y_k - y_{k-1})\, m(A_k)$$

$$< \sum_{k=1}^{n} \frac{\in}{b-a}.\, m(A_k),\ \text{[by (5.3.7.1)]}$$

$$= \frac{\in}{b-a}\sum_{k=1}^{n} m(A_k)$$

$$= \frac{\in}{b-a}.(b-a)$$

$$= \in$$

i.e. $U(P,f)-L(P,f)<\in$.

This shows that $f(x)$ is Lebesgue integrable on $[a,b]$.

5.4 Properties of Lebesgue Integral

Like Riemann integral the following properties can be easily proved for Lebesgue integral.

Let $f(x)$ be bounded Lebesgue integral function defined on $[a,b]$. Then the following properties hold for $f(x)$.

Property 1 . If $a<c<b$, integrable on $[a,b]$ then $f(x)$ is L-integrable on $[a,c]$ as well as on $[c,b]$ Also

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx .$$

Property 2. If $f(x)$ is L-integrable on $[a,b]$ and k is a real number then $kf(x)$ is L-integrable on $[a,b]$.

Also, $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

Property 3. If $f(x)$ and $g(x)$ is L-integrable on $[a,b]$ then $f(x)+g(x)$ is L-integrable on $[a,b]$, and

$$\int_a^b \{f(x)+g(x)\}dx = \int_a^b f(x)dx + \int_c^b g(x)dx.$$

5.4.1 Definition. Almost Everywhere

A property which holds everywhere except on a set of measure zero is said to hold almost everywhere $(a.e)$.

For example, let $f(x)=3$ for rational x and $f(x)=4$ for irrational x. Then $f(x)=4$ a.e.

The following theorems are very important and have wide applications.

5.4.2 Theorem

Let $f(x)$ be a bounded and Lebesgue integrable function on $[a,b]$ and $g(x)$ be a bounded function on $[a,b]$ such that $f(x)=g(x)$ a.e. on $[a,b]$. Then $g(x)$ is also Lebesgue integrable on $[a,b]$ and

$$\int_a^b g(x)dx = \int_a^b f(x)dx.$$

Proof. Here $f(x)$ is a bounded and Lebesgue integrable function on $[a,b]$. So $f(x)$ is a measurable function on $[a,b]$.

Also $f(x)=g(x)$ a.e. on $[a,b]$. Hence $g(x)$ is a measurable function on $[a,b]$.

Since $g(x)$ is bounded and measurable, we see that $g(x)$ is Lebesgue integrable on $[a,b]$.

We now show that $\int_a^b g(x)dx = \int_a^b f(x)dx$.

Let A be the set of all points x in $[a,b]$ for which $f(x)\neq g(x)$. Since $f(x)=g(x)$ a.e. on $[a,b]$, it follows that $m(A)=0$.

Let $B=[a,b]-A$

$\therefore\ m(B)=m([a,b])-m(A)=b-a-0=b-a$

$\therefore\ f(x) = g(x) \forall x \in B.$

Let $h(x) = g(x) - f(x)$. $\therefore\ h(x) = 0$ for $x \in B$

$\neq 0$ for $x \in A$.

Let $P' = \{A, B\}$. Then P' is a measurable partition of $[a,b]$.

Now $U(P',h) = \sup_A h(x).m(A) + \sup_B h(x).m(B)$

$= \sup_A h(x).0 + 0.m(B)$

$= 0$

and $L(P',h) = \inf_A h(x).m(A) + \inf_B h(x).m(B)$

$= \inf_A h(x).0 + 0.m(B)$

$= 0$

Thus

$0 = L(P',h) \leq \sup_P L(P,h) \leq \inf_P U(P,h) \leq U(P',h) = 0$

i.e. $\sup_P L(P,h) = \inf_P U(P,h) = 0$

This shows that $h(x)$ is Lebesgue integrable on $[a,b]$ and the value of the integral is o.

i.e. $\int_a^b h(x)dx$ exists and the value is zero.

i.e. $\int_a^b h(x)dx = 0.$

Now $\int_a^b g(x)dx = \int_a^b [h(x) + f(x)]dx$

$= \int_a^b h(x)dx + \int_a^b f(x)dx$

$$= 0 + \int_a^b f(x$$

$$= \int_a^b f(x)dx.$$

5.4.3 Theorem. If $f(x)$ is bounded and Lebesgue integrable on $[a,b]$ and if $f(x) \geq 0$ a.e on $[a,b]$ then $\int_a^b f(x)dx \geq 0.$

Proof. Let $f(x)$ be bounded and Lebesgue integrable on $[a,b]$.

Here $f(x) \geq 0$ a.e. on $[a,b]$. Let A be the set of points defined by

$$A = \{x \in [a,b] : f(x) < 0\}$$

Then we have $m(A) = 0$.. Let $B = [a,b] - A$

$$\therefore \quad m(B) = m([a,b]) - m(A)$$

$$= b - a - 0$$

$$= b - a$$

Thus we have $f(x) \geq 0 \; \forall \, x \in B$

$$< 0 \; \forall \, x \in A.$$

We now define a function $F(x)$ on $[a,b]$ such that

i) $F(x) \geq 0 \; \forall \, x \in [a,b]$

ii) $F(x) = f(x) \; \forall \, x \in B$.

and iii) $F(x) \neq f(x) \; \forall \, x \in A$

Then $F(x) = f(x)$ a.e. on $[a,b]$.

Since $f(x)$ is bounded and L-integrable on $[a,b]$, if follows that $F(x)$ is also L-integrable on $[a,b]$ and

$$\int_a^b F(x)dx = \int_a^b f(x)dx \quad \text{.........(5.4.3.1)}$$

As $F(x)$ is L-integrable on $[a,b]$, we have

$$\int_{-a}^{b} F(x)dx = \int_a^{-b} F(x)dx = \int_a^b F(x)dx \quad \text{.........(5.4.3.2)}$$

Let $P = \{A_1, A_2, \ldots\ldots, A_n\}$ be any measurable partition of $[a,b]$. Then

$$U(P,F) = \sum_{i=1}^{n} \sup_{A_i} F(x).m(A_i) \geq 0 \; [\because F(x) \geq 0 \forall x]$$

$$\therefore \inf_P U(P,F) \geq 0$$

or, $\int_a^{-b} F(x)dx \geq 0$

or, $\int_a^{b} F(x)dx \geq 0$ [by (5.4.3.2)]

or, $\int_a^{b} f(x)dx \geq 0$ [by (5.4.3.1)]

5.4.4 Theorem. If $f(x)$ and $g(x)$ are bounded and L-integrable functions on $[a,b]$ and if $f(x) \geq g(x)$ a.e. on $[a,b]$ then $\int_a^b f(x) \geq \int_a^b g(x)dx$.

Proof. Let $h(x) = f(x) - g(x)$. Then $h(x) \geq 0$ a.e. on $[a,b]$.

As theorem 5.4.3 we can prove that $\int_a^b h(x)dx \geq 0$

$$\therefore \int_a^b \{f(x) - g(x)\}dx \geq 0$$

or, $\int_a^b f(x)dx - \int_a^b g(x)dx \geq 0$

or, $\int_a^b f(x)dx \geq \int_a^b g(x)dx.$

5.4.5 Theorem. If $f(x)$ is bounded and Lebesgue integrable on $[a,b]$ then $|f(x)|$ is also Lebesgue integrable on $[a,b]$ and $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

Proof. Let $f(x)$ be bounded and Lebesgue integrable on $[a,b]$. Hence $f(x)$ is a measurable function on $[a,b]$.

Let $f^+(x) = max\{f(x), 0\}$

and $f^-(x) = -min\{f(x), 0\}$

Then $|f(x)| = f^+(x) + f^-(x)$.

Since $f(x)$ is bounded and measurable on $[a,b]$, it follows that $f^+(x)$ and $f^-(x)$ are also bounded and measurable on $[a,b]$. Hence $|f(x)|$ is also bounded and measurable on $[a,b]$. Therefore, $|f(x)|$ is Lebesgue integrable on $[a,b]$.

Now, we prove that $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

For all $x \in [a,b]$ we have $f(x) \leq |f(x)|$

and $-f(x) \leq |f(x)|$

$\therefore \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$ ....(5.4.5.1)

and $-\int_a^b f(x)dx \leq \int_a^b |f(x)|dx$ ....(5.4.5.2)

From (5.4.5.2) we have

$$\int_a^b f(x)dx \geq -\int_a^b |f(x)|dx$$

or $\quad -\int_a^b |f(x)dx| \le \int_a^b f(x)dx$ ....(5.4.5.3)

Thus from (5.4.5.1) and (5.4.5.3) we get

$$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$$

$$\therefore \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$$

5.5. Lebesgue Integral for Unbounded Functions

Now we consider Lebesgue integral for unbounded function.

5.5.1 Definition.

Let $f(x)$ be a non-negative unbounded measurable function on $[a,b]$. To get Lebesgue integral of such function we take the help of another function $F(x,n)$ of $x \in [a,b]$ and positive integer n defined as follows :

$$F(x,n) = f(x) \quad \text{if} \quad 0 \le f(x) \le n$$
$$= n \quad \text{if} \quad f(x) > n.$$

In other words $F(x,n) = min\{f(x), n\}$. This function $F(x,n)$ is called truncated function.

As $F(x,n)$ is minimum of $f(x)$ and n, it follows that $F(x,n)$ is bounded and hence is measurable.

$\therefore$ For each positive integer n, $F(x,n)$ is Lebesgue integrable.

Now if $\lim_{n\to\infty} \int_a^b F(x,n)dx$ exists finitely then we say that the unbounded function $f(x)$ is Lebesgue integrable and

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b F(x,n)dx.$$

If the limit does not exist finitely we say that $f(x)$ is not Lebesgue integrable.

We explain this by an example.

5.5.2 Example.

Let $f(x)$ be defined on $[0, 1]$ as

$$f(x)=\frac{1}{x^{3/4}},\ 0<x<1$$

$$=0,\ x=0$$

Show that $f(x)$ is Lebesgue integrable on $[0, 1]$ and find the value of the integral.

Ans. We note that $\frac{1}{x^{3/4}}\to\infty$ as $x\to 0$, so $f(x)$ is unbounded on $[0, 1]$. To examine the Lebesgue integrability of $f(x)$ on $[0, 1]$ we define the truncated function $F(x, n)$ as follows :

$$F(x,n)=\frac{1}{x^{3/4}}\ \text{if}\ 0\le\frac{1}{x^{3/4}}\le n$$

$$=n\ \text{if}\ \frac{1}{x^{3/4}}>n$$

$$=0\ \text{if}\ x=0$$

or $$F(x,n)=\frac{1}{x^{3/4}}\ \text{if}\ \frac{1}{n^{4/3}}\le x\le 1$$

$$=n\ \text{if}\ 0<x<\frac{1}{n^{4/3}}$$

$$=0\ \text{if}\ x=0$$

Now $$\int_0^1 F(x,n)dx=\int_0^{\frac{1}{n^{4/3}}} n\,dx+\int_{\frac{1}{n^{4/3}}}^{1}\frac{1}{x^{3/4}}dx$$

$$=n[x]_0^{\frac{1}{n^{4/3}}}+\left[4x^{1/4}\right]_{\frac{1}{n^{4/3}}}^{1}$$

$$=n\times\frac{1}{n^{4/3}}+4-4.\frac{1}{n^{1/3}}$$

$$= 4 - \frac{3}{n^{1/3}}$$

Thus by the definition of the Lebesgue integral of unbounded function, we have

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \int_0^1 F(x, n)dx$$

$$= \lim_{n\to\infty}\left(4 - \frac{3}{n^{1/3}}\right)$$

$$= 4$$

5.5.3 Example

Let $f(x)$ be defined on $[0, 1]$ as follows

$$f(x) = \frac{1}{2x}, \text{ if } 0 < x \le 1$$

$$= 8, \text{ if } x = 0.$$

show that $f(x)$ is not Lebesgue integrable on $[0, 1]$.

Ans. We define the truncated function $F(x, n)$ as

$$F(x,n) = \frac{1}{2x} \text{ if } 0 \le \frac{1}{2x} \le n$$

$$= n \text{ if } \frac{1}{2x} > n$$

$$= min(8, n) \text{ if } x = 0$$

or, $$F(x,n) = \frac{1}{2x} \text{ if } \frac{1}{2n} \le x \le 1$$

$$= n \text{ if } 0 < x < \frac{1}{2n}$$

$$= min(8, n) \text{ if } x = 0$$

For $n>8$ we have

$$\int_0^1 F(x,n)dx = \int_0^{\frac{1}{2n}} n\,dx + \int_{\frac{1}{2n}}^1 \frac{1}{2x}dx$$

$$= n[x]_0^{\frac{1}{2n}} + \left[\frac{1}{2}\log x\right]_{\frac{1}{2n}}^1$$

$$= \frac{1}{2} - \frac{1}{2}\log\frac{1}{2n}$$

$$= \frac{1}{2} + \frac{1}{2}\log 2n$$

$$\therefore \lim_{n\to\infty}\int_0^1 F(x,n)dx = \lim_{n\to\infty}\left[\frac{1}{2}+\frac{1}{2}\log(2n)\right]$$

$$= \infty$$

Thus by definition of the Lebesgue integral for unbounded function we see that $f(x)$ is not Lebesgue integrable.

5.6 Some Important Theorems.

The following theorems are very important and have wide applications in many fields. We only state these theorems.

5.6.1 Theorem. Lebesgue Theorem on Bounded Convergence.

Let $\{f_n(x)\}$ be a sequence of functions measurable on $[a,b]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$. If there exists a constant M such that $|f_n(x)| \le M$ for all positive integer n and for all x then

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx$$

5.6.2 Theorem. Monotone Convergence Theorem.

Let $\{f_n(x)\}$ be sequence of measurable functions on $[a,b]$ such that for $x \in [a,\ b]$

$$0 \le f_1(x) \le f_2(x) \le \ldots \le f_n(x) \le \ldots$$

If $\lim_{n\to\infty} f_n(x) = f(x)$, Then $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$

5.6.3 Theorem. Classical Lebesgue Dominated Convergence Theorem

Let $\{f_n(x)\}$ be a sequence of measurable functions on $[a,b]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e on $[a,b]$..

If there exists a Lebesgue integrable functions $g(x)$ on $[a,b]$

such that for each positive integer n

$|f_n(x)| \le g(x)$ a.e on $[a,b]$

then $f(x)$ is Lebesgue integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

5.6.4 Lemma. Fatou's Lemma.

If $\{f_n(x)\}$ be a sequence of non-negative measurable functions on $[a,b]$ such that

$\lim_{n\to\infty} f_n(x) = f(x)$ a.e on $[a,b]$,

then $\liminf_{n\to\infty} \int_a^b f_n(x)dx \ge \int_a^b f(x)dx$ if f is Lebesgue integrable on $[a,b]$.

Otherwise, $\liminf_{n\to\infty} \int_a^b f_n(x)dx = \infty$

5.6.5 Example. Verify Bounded Convergence Theorem for the sequence of functions

$$f_n(x) = \frac{2}{\left(1+\frac{3x}{n}\right)^n}, \quad 0 \le x \le 1, \quad n = 1,2,3,....$$

Ans . Here $|f_n(x)| = \left|\dfrac{2}{\left(1+\dfrac{3x}{n}\right)^n}\right| \le 2 \forall n \text{ and } \forall x.$

$\therefore$ Each $f_n(x)$ is bounded and measurable.

Now

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{2}{\left(1+\frac{3n}{n}\right)^n} = \lim_{n\to\infty} \frac{2}{\left[\left(1+\frac{3n}{n}\right)^{\frac{n}{3x}}\right]^{3x}} = \frac{2}{e^{3x}} = f(x), \text{ say.}$$

Then $|f(x)| = \left|\dfrac{2}{e^{3x}}\right| \le 2$. Thus the limit function $f(x)$ is bounded and measurable.

We have

$$\int_0^1 f_n(x)dx = \int_0^1 \frac{2}{\left(1+\frac{3x}{n}\right)^n} dx = \left[\frac{2\left(1+\frac{3x}{n}\right)^{-n+1}}{(-n+1)} \times \frac{n}{3}\right]_0^1$$

$$= \frac{2n}{3(1-n)}\left[\left(1+\frac{3}{n}\right)^{-n+1} - 1\right]$$

$$= \frac{2n}{3(n-1)}\left[1 - \frac{1}{\left(1+\frac{3}{n}\right)^{n-1}}\right]$$

$$= \frac{2n}{3(n-1)}\left[1 - \frac{\left(1+\frac{3}{n}\right)}{\left(1+\frac{3}{n}\right)^n}\right]$$

$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)dx$

$$= \lim_{n\to\infty}\frac{2}{3\left(1-\frac{1}{n}\right)}\left[1-\frac{\left(1+\frac{3}{n}\right)}{\left\{\left(1+\frac{3}{n}\right)^{\frac{n}{3}}\right\}^3}\right]$$

$$=\frac{2}{3}\left[1-\frac{1}{e^3}\right]$$

$$=\frac{2}{3}-\frac{2}{3e^3}$$

Also $\int_0^1 f(x)dx$

$$=\int_0^1 \frac{2}{e^{3x}}dx=\int_0^1 2e^{-3x}dx=\left[2.\frac{e^{-3x}}{-3}\right]_0^1=\left\{-\frac{2}{3}e^{-3}+\frac{2}{3}\right\}$$

$$=\frac{2}{3}-\frac{2}{3e^3}$$

Hence $\lim_{n\to\infty}\int_0^1 f_n(x)dx=\int_0^1 f(x)dx.$

5.6.6 Example. Define the sequence functions $f_n(x)$ on $[0, 1]$ as follows.

$$f_n(x)=-2xn^2+2n \text{ for } 0\le x\le\frac{1}{n}$$

$$=0 \text{ for } \frac{1}{n}<x\le 1$$

Show that $\lim_{n\to\infty}\int_0^1 f_n(x)dx \ne \int_0^1\left\{\lim_{n\to\infty} f_n(x)\right\}dx$

Ans. We have

$$\int_0^1 f_n(x)dx = \int_0^{1/n}\left(-2xn^2+2n\right)dx + \int_{1/n}^1 0\,dx$$

$$= \left[-x^2n^2+2nx\right]_0^{1/n}$$

$$= -\frac{1}{n^2}.n^2 + 2n.\frac{1}{n} - 0$$

$$= -1+2$$

$$= 1$$

Thus $\int_0^1 f_n(x)dx = 1$ for each $n = 1, 2, 3, \ldots\ldots$

$$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)dx = \lim_{n\to\infty} = 1 = 1$$

Also $\lim_{n\to\infty} f_n(x) = 0$ for every x.

$$\therefore \int_0^1\left[\lim_{n\to\infty} f_n(x)\right]dx = \int_0^1 0\,dx = 0$$

Hence $\therefore \lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1\left\{\lim_{n\to\infty} f_n(x)\right\}dx$

5.7 Illustrative Examples

5.7.1 Example. Using definition of Lebesgue integral find the integral $\int_a^b f(x)dx$ when

$$f(x) = \alpha,\ x \in [a, b] \text{ and rational}$$
$$= \beta, x \in [a, b] \text{ and irrational } (\beta > \alpha).$$

Ans. Let A_1 = set of all rational numbers in $[a,b]$.

and A_2 = set of all irrational numbers in $[a,b]$.

Then $A_1 \cup A_2 = [a,b]$ and $A_1 \cup A_2 = \phi \quad \therefore m(A_1 \cup A_2) = 0$

$\therefore P_1 = \{A_1, A_2\}$ is a measurable partition of $[a,b]$.

Now $m(A_1) = 0$ & $m(A_2) = m([a,b]) - m(A_1) = b - a - 0 = b - a$.

$$\therefore L(P_1, f) = inf\{f(x): x \in A_1\}.m(A_1) + inf\{f(x): x \in A_2\}.m(A_2)$$

$$= \alpha.0 + \beta.(b-a)$$

$$= \beta(b-a)$$

$$U(P_1, f) = sup\{f(x): x \in A_1\}.m(A_1) + sup\{f(x): x \in A_2\}.m(A_2)$$

$$= \alpha.0 + \beta.(b-a)$$

$$= \beta(b-a)$$

Hence for any measurable partition P of $[a,b]$ we have

$$\beta(b-a) = L(P_1, f) \le \sup_P\{L(P,f)\} \le \inf_P\{U(P,f)\} \le U(P_1, f) = \beta(b-a)$$

$$\therefore \int_{-a}^{b} f(x)dx = \int_{a}^{-b} f(x)dx = \beta(b-a)$$

i.e. f is L-integrable on $[a,b]$ and $\int_a^b f(x)dx = \beta(b-a)$

5.7.2 Example.

Show that any constant function is lebesgue integrable on $[a,b]$

Ans. Let $f(x) = k$ for $x \in [a, b]$.

Let $P = \{A_1, A_2, \ldots\ldots, A_n\}$ be any measurable partition of $[a,b]$.

Then $A_1 \cup A_2 \cup .... \cup A_n = [a, b]$ and $m(A_i \cap A_j) = 0 \forall \quad i \neq j$..

$$\therefore \sum_{i=1}^{n} m(A_i) = m([a,b]) = b-a.$$

Now, $L(P,f) = \sum_{i=1}^{n} \left\{ \inf_{Ai} f(x) \right\} m(A_i)$

$$= \sum_{i=1}^{n} k\, m(A_i)$$

$$= k(b-a)$$

and $U(P,f) = \sum_{i=1}^{n} \left\{ \sup_{A_i} f(x) \right\} m(A_i)$

$$= \sum_{i=1}^{n} k\, m(A_i)$$

$$= k(b-a)$$

Since P is any measurable partition of $[a,b]$ & $k(b-a)$ is independent of P, it follows that

$$\sup_P L(P,f) = k(b-a) = \inf_P U(P,f)$$

$\therefore f(x)$ is L-inlegrable on $[a,b]$ and $\int_a^b f(x)\, dx = k(b-a)$

5.7.3 Example.

Evaluate the Lebesgue integral $\int_0^5 f(x)\, dx$ where

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2, \quad 3 \leq x < 4 \\ 2, & 2 \leq x < 3, \quad 4 \leq x \leq 5 \end{cases}$$

Ans. Let $P_1 = \{A_1, A_2, A_3\}$ where

$A_1 = \{x : 0 \le x < 1\}$, $A_2 = \{x : 1 \le x < 2 \text{ or } 3 \le x < 4\}$ and $A_3 = \{x : 2 \le x < 3 \text{ or } 4 \le x \le 5\}$

$\therefore m(A_1) = 1$, $m(A_2) = 1 + 1 = 2$ and $m(A_3) = 1 + 1 = 2$

Also $A_1 \cup A_2 \cup A_3 = [0, 5]$ & $m(A_i \cap A_j) = 0$.

Thus P_1 is a measurable partition of $[0, 5]$.

$$\text{Now } L(P_1, f) = \left\{\inf_{A_1} f(x)\right\} m(A_1) + \left\{\inf_{A_2} f(x)\right\} m(A_2) + \left\{\inf_{A_3} f(x)\right\} m(A_3)$$

$$= 0.1 + 1.2 + 2.2$$

$$= 6$$

$$U(P_1, f) = \left\{\sup_{A_1} f(x)\right\} m(A_1) + \left\{\sup_{A_2} f(x)\right\} m(A_2) + \left\{\sup_{A_3} f(x)\right\} m(A_3)$$

$$= 0.1 + 1.2 + 2.2$$

$$= 6$$

Hence

$$6 = L(P_1, f) \le \sup_P L(P, f) \le \inf_P U(P, f) \le U(P_1, f) = 6$$

i.e. $$\int_{-0}^{5} f(x)dx = \int_{0}^{-5} f(x) = 6$$

$\therefore f(x)$ is L-integrable on $[0, 5]$ and $\int_0^5 f(x)dx = 6$.

5.7.4 Example.

If $f(x) = 0$ for every x in the Cantor set P_0 and $f(x) = k$ for x in each of the deleted intervals of length $\frac{1}{3^k}$ in $P_0^c = [a, b] - P_0$ during the formation of Cantor set then prove that $f(x)$ is Lebesgue integrable on $[0, 1]$ and $\int_0^1 f(x)dx = 3$.

Ans. In the formation of Cantor set we have the following in different stages.

Stage	Number of intervals deleted	Length of each deleted interval	Set of deleted intervals	Measure of deleted intervals
1	1	$1/3$	$A_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$	$m(A_1) = 1/3$
2	2	$1/3^2$	$A_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$	$m(A_2) = 2/3^2$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
K	2^{k-1}	$1/3^k$	A_k	$m(A_k) = 2^{k-1}/3^k$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

Then we have

$$f(x) = 0 \text{ for } x \in P_0$$

$$= k \text{ for } x \in A_k \quad k = 1, 2, \ldots..$$

Let $P_1 = \{P_0 A_1, A_2, \ldots.., A_k, \ldots..\}$

Then $\bigcup_{i=0}^{\infty} A_i = [0, 1]$ where $P_0 = A_0$ and $m(A_i \cap A_j) = 0$.

$\therefore$ P_1 is a measurable partition of $[0, 1]$.

Now $L(P_1, f) = \left\{\inf_{P_0} f(x)\right\} m(P_0) + \sum_{K=1}^{\infty}\left\{\inf_{A_k} f(x)\right\}.m(A_k)$

$$= 0 + \sum_{k=1}^{\infty} k.\frac{2^{k-1}}{3^k}$$

$$= \frac{1}{3}\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^{k-1}$$

$$=\frac{1}{3}\left\{1+2.\left(\frac{2}{3}\right)+3.\left(\frac{2}{3}\right)^2+.........\right\}$$

$$=\frac{1}{3}\left(1-\frac{2}{3}\right)^{-2}$$

$$=\frac{1}{3}\left(\frac{1}{3}\right)^{-2}$$

$$=\frac{1}{3}\times 3^2$$

$$=3$$

Similarly we have $U(P_1, f)=3$.

$$\therefore 3=L(P_1,f)\leq \sup_P L(P,f)\leq \inf_P U(P,f)\leq U(P_1,f)=3$$

Thus $\int_{-0}^{1} f(x)dx=3=\int_{0}^{-1} f(x)dx$

i.e. $f(x)$ is L-integrable on $[0, 1]$ and $\int_0^1 f(x)dx=3.$

5.8. Summary

To enlarge the class of Riemann integrable functions Lebesgue integrable functions are considered here. If a functions is Riemann integrable then it must be Lebesgue integrable but the converse is not true. It is seen that Riemann integral is inadequate from a more general point of view. The class of Riemann integrable functions is quite small. The limiting operations for Riemann integral is often lead to great difficulties. On the other hand it is seen that the class of all Lebesgue integrable functions is very large and the limiting operations for this integrals are found easy to perform in comparison with Riemann integral. Some important theorems have stated here. Examples are given for illustrations.

5.9. Self Assessment Questions

1. Give an example which is Lebesque integrable but not Riemann integrable in $[a, b]$.

2. Show that the functions defined on $[-1, 5]$ as

$$f(x) = 3 \text{ for } x \text{ rational in } [-1, 5]$$
$$= 3 \text{ for } x \text{ irrational in } [-1, 5]$$

is Lebesgue integrable and find the value of the integral.

3. Find the Lebesgue integral of $f(x)$ on $[2, 8]$ where.

$$f(x) = 2,\ 2 \le x < 3,\ 4.5 \le x \le 5,\ 7.5 \le x < 8$$
$$= 3,\ 4 \le x < 4.5,\ 6 \le x < 6.5$$
$$= -1,\ 5 < x \le 6,\ 6.5 \le x < 7$$
$$= 4,\ 3 \le x < 4,\ x = 8$$
$$= 8,\ 7 \le x < 7.5$$

4. Evaluate the Lebesgue integral $\int_{-1}^{5} f(x)\,dx$ where

$$f(x) = 5,\ -1 \le x \le 0,\ 2 < x < 3$$
$$= 6,\ 0 < x \le 2,\ 3 \le x < 4$$
$$= 3,\ x \text{ rational in } [4, 5]$$
$$= 4,\ x \text{ irrational in } [4, 5].$$

5. Show that the unbounded function $f(x)$ defined by

$$f(x) = \frac{1}{x^{2/3}},\ 0 < x \le 1$$
$$= 0,\ x = 0$$

is Lebesgue integrable on $[0, 1]$. Find $\int_{0}^{1} f(x)\,dx$.

6. Determine whether the unbounded function $f(x)$ defined by

$$f(x) = 2 + \frac{3}{x},\ 0 < x \le 1$$
$$= 8,\ x = 0$$

is Lebesgue integrable on $[0, 1]$.

7. Verity Bounded Convergence Theorem for the sequence of functions

$$f_n(x) = \frac{5}{\left(2 + \frac{3x}{n}\right)^n},\ 0 \le x \le 1,\ n = 1, 2, 3, \ldots$$

8. For $n \ge 2$ define $f_n(x)$ on $[0, 1]$ as

$$f_n(x) = n^2 x,\ 0 \le x \le \frac{1}{n}$$
$$= -n^2(x - 2/n),\ \frac{1}{n} < x \le 2/n$$
$$= 0,\ \frac{2}{n} < x \le 1.$$

Show that $f_n(x) \to f(x)$ pointwise on $[0, 1]$ where $f(x)$ is zero function and

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1 \ne 0 = \int_0^1 f(x)\,dx.$$

5.10 Suggested Further Readings

1. S. C. Malik & Savita Arora; Mathematical Analysis; Wiley Eastern Limited

2. T M. Apostol; Mathematical Analysis; Narosa Publishing House

3. A. N. Kolmogorou & S. V. Fomin; Introductory Real Analysis; Dover Publications.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-I Group–B

Module No. - 6
FUNCTIONS OF A COMPLEX VARIABLE

INTRODUCTION :

In solving algebraic equations we are often forced to land outside the domain of real numbers. This urges to extend the domain of real numbers to perform the basic algebraic operations within the domain itself. Complex numbers have this property, which tempted to consider functions that assumes when the argument is real or complex. In general we study functions of a complex variable; at the elementary stage they are considered to be single-valued. Studying limit, continuity and differentiability of such functions we introduce the notion of analytic function. Actually venture the properties of such functions / study the nature of such functions and its consequences are the basis of the theory of functions of a complex variable. The theory of analytic functions play a key role in solving numerous mathematical problems in applied sciences.

An analytic function whose derivative is non-zero in a domain deserves a special property called 'angle preserving property' and discussed under conformal representation.

Module Content

1.1 Complex numbers
1.2 Complex plane
1.3 Functions of a complex variable
1.4 Limit, continuity and differentiability
1.5 Analytic function
1.6 Conformal representation

Objectives

- Complex number system forms a field
- Operations on complex numbers, the limit of a sequence, point at infinity
- Equivalency of the limit (or continuity) of a function to that of its real and imaginary parts
- Necessary and sufficient conditions for differentiability
- Construction of an analytic function from its real part .
- Bilinear transformation and some of its properties

Keywords

Field, Sequence, Point at infinity, limit, and continuity of a real function, Bilinear.

1.1 **Complex numbers** In solving algebraic equations like $ax^2+bx+c=0, a,b,c\in\mathbb{R}$ we find solutions of the form

(1) $\alpha\pm\sqrt{-1}\,\beta$ if $b^2<4ac$ where $\alpha=-\dfrac{b}{2a}$ and $\beta=\dfrac{\sqrt{4ac-b^2}}{2a}$ are real numbers. Such form of expression (1) which contains un extra symbol $\sqrt{-1}$ is termed as complex numbers.

Definition A complex number is an ordered pair of real numbers (x,y) which obey the following binary operations of addition and multiplication :

I. The sum of two complex numbers (x_1,y_1) and (x_2,y_2) is the complex number (x_1+x_2,y_1+y_2)

II. The product of two complex numbers (x_1,y_1) and (x_2,y_2) is the complex number Moreover, they are the same if and only if

III. $x_1=y_1$ and $x_2=y_2$.

In particular if $y_1=y_2=0$ it follows from I and II that $(x_1,0)+(x_2,0)=(x_1+x_2,0)$ and $(x_1,0)+(x_2,0)=(x_1x_2,0)$, which show that complex number of the form $(x,0)$ obey the addition and multiplication rules of real numbers and we are tempted to consider complex numbers of the form $(x,0)$ as the real number x.

The imaginary number i

The complex number (0, 1) is named as the unit imaginary number and is denoted by i. We find that $i^i=i\times i=(0,1)(0,1)=(-1,0)=-1$. So that (x,y) can be expressed as

$(x, y) = (x, 0) + (0, y)$

$= (x, 0) + (0, 1)(y, 0)$

$= x + iy$

Field. A set F of elements, at least two in number, is called a field if there are two binary laws of composition known as addition and multiplication such that the properties are satisfied.

If $a, b, c \in F$, then

(*i*) $a + b = b + a$ and $ab = ba$

(*ii*) $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$

(*iii*) there is a unique element θ in F such that $\theta + a = a = a + \theta$.

(*iv*) for each element a in F there is an element $\tilde{a}$ in F such that $a + \tilde{a} = \theta = \tilde{a} + a$

(*v*) for any a in F there is a unique element $e \neq \theta$ in F such that $ea = ae = a$

(*vi*) for each element a $\neq \theta$ in F there is an element a' in F such that $aa' = a'a = e$.

(*vii*) $a(b + c) = ab + bc$

Note. We know that the set of real numbers forms a field.

Complex number system forms a field

Given any complex numbers $\alpha_j = (x_j, y_j), j = 1,2,3 x_j, y_j \in \mathbb{R}$.

Addition is commutative

$$\alpha_1 + \alpha_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$= (x_2 + x_1, y_2 + y_1) = (x_2, y_2) + (x_1, y_1) = \alpha_2 + \alpha_1$$

Multiplication is commulative

$$\alpha_1\alpha_2 = (x_1, y_1)(x_2, y_2) = (x_1 y_1 - y_1 y_2, x_1 y_2 + y_1 x_2)$$

$$= (x_2 x_1 - y_2 y_1, x_2 y_1 + y_2 x_1) = (x_2, y_2)(x_1, y_1) = \alpha_2\alpha_1$$

Additive and Multiplicative identity

$$\alpha + \theta = (x, y) + (0,0) = (x + 0, y + 0) = (x, y)$$

$$\alpha e = (x, y)(1,0) = (x - 0, 0 + y) = (x, y)$$

Here (0, 0) plays the role of ero element and (1, 0) serves as the role of unit element.

Additive inverse

$$\alpha+(-\alpha)=(x,y)+(-x,-y)=(x-x,y-y)=(0,0)$$

Multiplicative inverse

The complex number (p, q) is the inverse of the complex number (x, y) if $(x, y)(p, q) = (1, 0)$. Applying the property II of complex numbers we find that

$$(xp - yq, xq + yp) = (1, 0)$$

On solving $xp - yq = 1$ and $xq + p = 0$, we obtain

$$(p,q)=(x,y)\rightarrow -1=\left(\frac{x}{x^2+y^2},-\frac{y}{x^2+y^2}\right)$$

and it is well defined since $x^2 + y^2 \neq 0$.

Addition and Multiplication we are associative

$$\begin{aligned}\alpha_1+(\alpha_2+\alpha_3) &=(x_1,y_1)+(x_2+x_3,y_2+y_3)=(x_1+x_2+x_3,y_1+y_2+y_3)\\ &=(x_1+x_2,y_1+y_2)+(x_3,y_3)=(\alpha_1+\alpha_2)+\alpha_3\end{aligned}$$

$$\begin{aligned}\alpha_1(\alpha_2\alpha_3) &=(x_1,y_1)(x_2x_3-y_2y_3,x_2y_3+y_2x_3)\\ &=(x_1x_2x_3-x_1y_2y_3-y_1x_2y_3-y_1y_2x_3,x_1x_2y_3+x_1y_2x_3+y_1x_2x_3-y_1y_2y_3)\end{aligned}$$

Multiplication is distributive with respect to addition

$$\begin{aligned}\alpha_1(\alpha_2+\alpha_3) &=(x_1,y_1)(x_2+x_3,+y_2+y_3)\\ &=(x_1x_2+x_1x_3-y_1y_2-y_1y_3,x_1y_2+x_1y_3+y_1x_2+y_1x_3)\\ &=(x_1x_2-y_1y_2,x_1y_2+y_1x_2)+(x_1x_3-y_1y_3,x_1y_3+y_1x_3)\\ &=\alpha_1\alpha_2+\alpha_1\alpha_3\end{aligned}$$

Thus $\mathbb{C}$ satisfies all the properties of a field.

Note. Real number system is a special case of the complex number system. For the complex numbers of the form $(x, 0)$ have the same arithmetic properties as the real numbers. The complex numbers $(x, 0)$ and $(0, 0)$ are actually the real numbers x and 0 respectively.

Ordering is not possible in complex number system

If not then either $(0, 1)$ or $-(0, 1)$ is positive. In either case their square must be positive, but $(0,1)(0, 1) =$

$[-(0, 1)] = [-(0, 1)] = (-1, 0) = -(1, 0) = - < 0$ which is a contradiction.

1.2 Complex Plane

We take a plane with two mutually perpendicular axes, the x-axis and the y-axis intersecting at the origin 0. Then the complex number $z = x + iy$ is associated with the coordinate (x, y) of R^2 plane. So there exists a one-to-one correspondence between complex numbers and points in a plane. The real numbers are represented by the points on the horizontal axis, which is for this reason called the real axis, while the pure imaginary numbers are denoted by the points on the vertical axis and in complex terminology this axis is called imaginary axis. The plane as a whole which represent complex numbers is called the complex plane or Argand plane in the name of the French mathematician d'Argand (1768 – 1822).

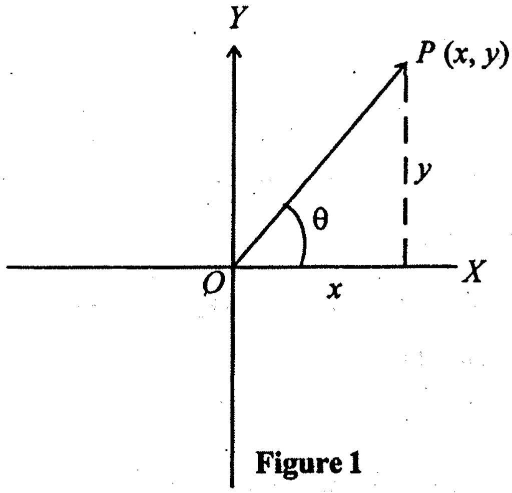


Figure 1

The complex number z can equivalently be represented by the position vector OP.

Also the point $z\,(z \neq 0)$ is uniquely defined by polar coordinates (r, ϕ) where $r = |z|$, the absolute value or modules of the complex number $z = x + iy$ is given by

$$|z| = \left(x^2 + y^2\right)^{\frac{1}{2}},$$

ϕ is the angle between the real axis and the vector OP and is called the argument or the amplitude of the number. Angle is taken to be positive if it is ma anticlockwise otherwise negative.

For the number $z = 0$ the argument is undefined.

From Figure 1 it follows that

$$x = r\cos\phi,\ \ y = r\sin\phi$$

Any complexd number $z \neq 0$ can be expressed as

$$z = r(\cos\phi + i\sin\phi)$$

where argument of z, simply $\arg z$ is one of the values

$$\arg z = \phi + 2k\pi, \quad k = 0, \pm1, \pm2, ...$$

The value of ϕ his in the interval

$$-\pi < \phi \leq \pi$$

The complex conjugate of a complex number $z = x + iy$ is defined by $x - iy$ and is denoted by $\bar{z}$.

Some simple properties of absolute values and conjugates:

(*i*) $\operatorname{Re} z = \frac{1}{2}(z + \bar{z}),\ \operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$

(*ii*) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(*iii*) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(*iv*) $|z| = (\bar{z})$

(*v*) $|z|^2 = z\bar{z}$

(*vi*) $\left(\overline{\frac{z_1}{z_2}}\right) = \frac{\bar{z}_1}{\bar{z}_2}, |z_2| > 0$

(*vii*) $|z| > 0; |z| = 0$ if and only if $z = 0$.

(*ix*) $|z_1 z_2| = |z_1||z_2|$

(*x*) $|z_1 + z_2| \leq |z_1| + |z_2|$

(*xi*) $|z_1 - z_2| \geq |z_1| - |z_2|$

(*xii*) $|\operatorname{Re} z| \leq |z|, |\operatorname{Im} z| \leq |z|$

(*xiii*) $z^n = r^n(\cos n\phi + i\sin n\phi); z = r(\cos\phi + i\sin\phi),\ n =$ integer

To prove (*xiii*), we calculate z^2

$$z^2 = r^2(\cos^2\phi - \sin^2\phi + i2\sin\phi\cos\phi) = r^2(\cos 2\phi + i\sin 2\phi)$$

Again,

$$z^3 = r^3\left(\cos^3\phi + 3i\cos^2\phi\sin\phi - 3\cos\phi\sin^2\phi - i\sin^3\phi\right)$$

$$= r^3\left\{\left(4\cos^3\theta - 3\cos\theta\right) + i\left(3\sin\theta - 4\sin^3\theta\right)\right\}$$

$$= r^3\left(\cos 3\theta + i\sin 3\theta\right)$$

Proceeding in this way it can be proved that

$$z^n = r\left(\cos n\theta + i\sin n\theta\right)$$

for integral values of n. The result in (*xiii*) is known as de Moivre's formula.

Example – 1.

Find the real and imaginary parts of $\dfrac{z-1}{z+1}$ where $z = x + iya, z \neq -1$.

Here,

$$\frac{z-1}{z+1} = \frac{x-1+iy}{x+1+iy} = \frac{(x+1)-iy}{(x+1)-iy}\cdot\frac{(x-1)+iy}{(x+1)+iy}$$

$$= \frac{\left(x^2-1\right)+y^2+i\left\{y(x+1)-y(x-1)\right\}}{(x+1)^2+y^2} = \frac{x^2+y^2-1+2iy}{(x+1)^2+y^2}$$

So that

$$Re\left(\frac{z-1}{z+1}\right) = \frac{x^2+y^2-1}{(x+1)^2+y^2} \text{ and } \operatorname{Im}\left(\frac{z-1}{z+1}\right) = \frac{2y}{(x+1)^2+y^2}$$

Example – 2.

Find all the values of $\sqrt[4]{1+i\sqrt{3}}$.

We first express $1+i\sqrt{3}$ in polar form :

$$1+i\sqrt{3} = 2\left(\frac{1}{2}+i\sqrt{\frac{3}{2}}\right) = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

So, $$\sqrt[4]{1+i\sqrt{3}} = 2^{1/4}\left(\cos\frac{\frac{\pi}{3}+2k\pi}{4}+i\sin\frac{\frac{\pi}{3}+2k\pi}{4}\right)$$

Putting $k = 0, 1, 2, 3$ produces

$$2^{1/4}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right), 2^{1/4}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right),$$

$$2^{1/4}\left(\cos\frac{13\pi}{12}+i\sin\frac{13\pi}{12}\right) \text{and } 2^{1/4}\left(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right).$$

Example – 3.

Evaluate $\left(\frac{1+i\sqrt{3}}{1+i}\right)^{30}$

Let us transform the expression is polar form :

$$\left(\frac{1+i\sqrt{3}}{1+i}\right)^{30} = \left[\frac{(1-i)(1+i\sqrt{3})}{(1-i)(1+i)}\right]^{30} = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}}-i\cdot\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right]^{30}$$

$$= 2^{15}\left[\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}\left\{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right\}\right]^{30}$$

$$= 2^{15}\left(\cos 15\cdot\frac{\pi}{2}-i\sin 15\cdot\frac{\pi}{2}\right)(\cos 10\pi+i\sin 10\pi)$$

$$= 2^{15}\left(0-i(-1)\right)\left(1+i(0)\right) = 2^{15}i.$$

Sequence of complex numbers

Formal definition of the limit of a sequence of complex numbers $z_1, z_2, \ldots, z_n, \ldots$ is the same as that of the limit of a sequence of real numbers.

Definition. A sequence of complex numbers $\{z_n\}$ is said to converge to a limit ζ if for energy positive ε there exists a positive integer N, depending on ε such that

$$|z_n - \zeta| < \varepsilon$$

for all $n > N$. Equivalently we write

$$\lim_{n\to\infty} z_n = \zeta.$$

Now if $z_n = x_n + iy_n$ and $\zeta = \alpha = \alpha + i\beta$, then $|x_n - \alpha| \le |z_n - \zeta| < \varepsilon$ and $|y_n - \beta| \le |z_n - \zeta| < \varepsilon$ for $n>N$

Therefore,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(x_n + i\,y_n) = \alpha + i\beta$$

implies that $\lim_{n\to\infty} x_n = \alpha$ and $\lim_{n\to\infty} y_n = \beta$. Consequently we can apply the theory of sequences of real numbers onto the sequences of complex numbers and have the following Cauchy criterion for convergence:

A necessary and sufficient condition for convergence of a sequence $\{z_n\}$, of complex numbers is that for given $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that $|z_{n+p} - z_n| < \varepsilon$ whenever $n > N(\varepsilon)$ and $p \equiv +ve$ integer.

Moreover if, $\lim_{n\to\infty} z_n$ and $\lim_{n\to\infty} w_n$ exist then

(*i*) $\lim_{n\to\infty}\{z_n \pm w_n\} = \lim_{n\to\infty} z_n \pm \lim_{n\to\infty} w_n$

(*ii*) $\lim_{n\to\infty}\{z_n w_n\} = \lim_{n\to\infty} z_n \cdot \lim_{n\to\infty} w_n$

(*iii*) $\lim_{n\to\infty}\left\{\frac{z_n}{w_n}\right\} = \frac{\lim_{n\to\infty} z_n}{\lim_{n\to\infty} w_n}$ provided $\lim_{n\to\infty} w_n \neq 0$ hold.

A sequence $\{z_n\}$ of complex numbers is said to be founded if there is a number R such that $|z_n| < R$ for all positive integer n.

A sequence $\{z_n\}$ converges to ∞ i.e. $\lim_{n\to\infty} z_n = \infty$ if for every $R > 0$ there exists a positive integer $N(R)$ such that $|z_n| > R$ for $n > N$. The complex plane $\mathbb{C}U\{\infty\}$ is termed as extended complex plane and denoted simply as $\mathbb{C}_\infty$. With proper complex α numbers $\beta(\beta \neq 0)$ we have the following properties :

(*i*) $\infty \pm \alpha = \alpha \pm \infty = \infty$

(*ii*) $\infty \cdot \beta = \beta \cdot \infty = \infty \cdot \infty = \infty$

(*iii*) $\frac{\alpha}{\infty} = 0$

(*iv*) $\frac{\infty}{a} = \infty$

(*v*) $\frac{\beta}{0} = \infty$

The notions of real and imaginary parts or the argument do not for improper complex number ∞.

The Point at Infinity

It is true that the points of the complex plane have one-to-one correspondence with the points of the Euclidean plane, but they differ in one point-that is point at infinity. In the later there may be so many points at infinity, but in the complex plane it is one and is defined to be the point corresponding to the origin ($z = 0$) in the transformation $\zeta = 1/z$. This is interpreted geometrically by use of Reimann's spherical transformation.

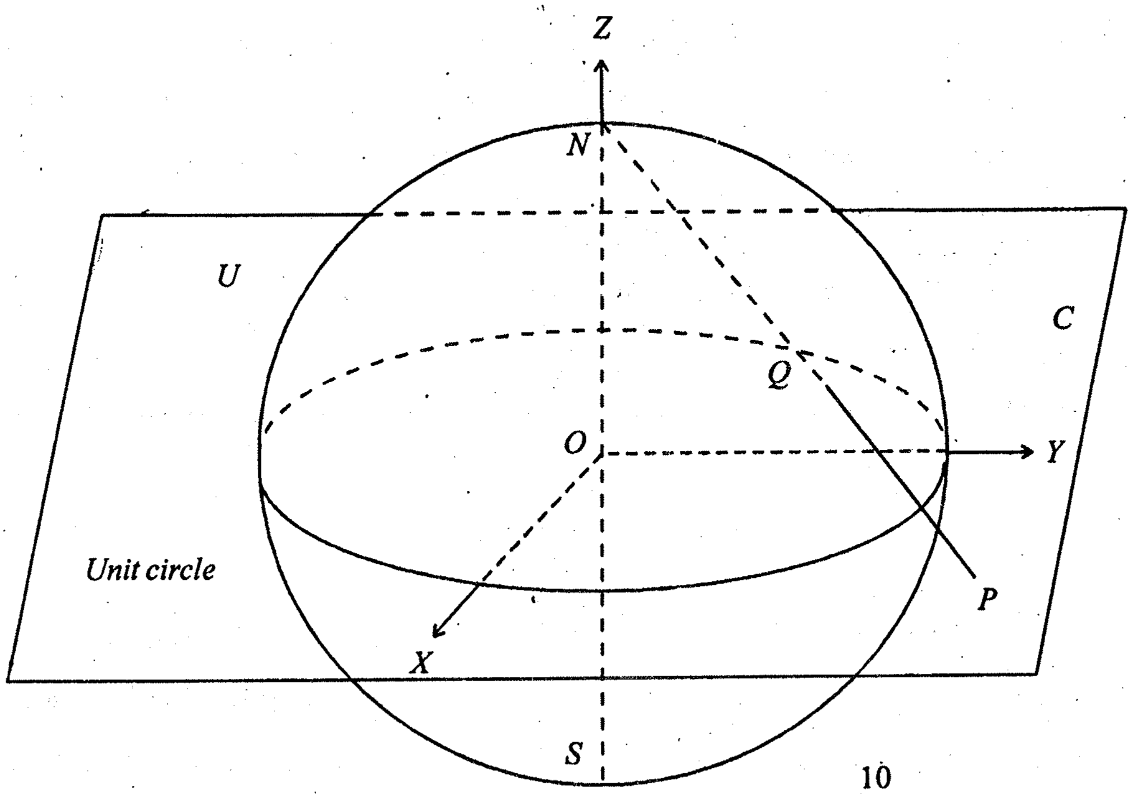


Figure 2

To this end, let U be the unit sphere and the complex plane C cuts the sphere U and passes through its origin. The straight line that passes through 0 and perpendicular to the complex plane is the axis of the sphere and the points N and S at which the axis intersects the sphere termed as North and South pole respectively. Let P be any point lying on C and the line NP intersects the sphere U at Q. Thus points outside the unit circle are mapped onto the upper hemisphere whereas the points inside the unit circle are mapped onto the lower hemisphere. The origin $z = 0$ is mapped onto the south pole S of the sphere U and the north pole N corresponds to the point at infinity of the complex plane. This creates a one-to-one correspondence between the points of the sphere U and the points in the extended complex plane. This kind of mapping-plane to sphere is called the <u>Stereographic Projection</u> adn the sphere is called the <u>Riemann sphere</u>.

1.3 Functions of a Complex Variable

Let S be a set of points in the complex z-plane and to every point if we can assign one or more complex numbers w then a function of a complex variable z is defined on S. Symbolically, we write $w = f(z)$. If to each value of z there corresponds only one value of w, we say that the function $f(z)$ is single-valued and in any other case it is called multivalued or many-valued. For example, $f(z)=z^2+1$, $g(z)=2z^2-3z+1$ are single-valued functions, whereas $\phi(z) = z^{1/2}$ and $\psi(z) = \log z$ are multi-valued functions.

If $z = x + iy$, x, $\psi \equiv$ real, then $f(z)$ can be expressed as

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

where the functions $u(x, y)$ and $v(x, y)$ of two real variables x, y are respectively the real and imaginary parts of $f(z)$.

Curves :

A continuous curve in the complex plane is represented parametrically by $x = x(t), y - y(t),\ a \le t \le b$; $x(t)$, $y(t)$ are continuous functions of t. In complex notation it is expressed as $z = \lambda(t), a \le t \le b$, it is a continuous mapping of the real line segment $[a, b]$. A curve $\gamma : z = \lambda(t)$ in the complex plane is said to be <u>smooth</u> if the derivative $\lambda'(t)$ is continous and non zero on $[a, b]$. A continous curve γ is said to be <u>piecewise smooth</u> or <u>sectionally smooth</u> on $[a, b]$ if there exists a division of the segment $[a, b]$ by points $a = t_0 < t_1 < t_2 < ... < t_n = b$ so that γ is smooth on each $[t_{j-1}, t_j], j = 1, 2, ..., n$ but tangents on γ at the points corresponding $t = t_j, j = 1, 2, ...,$ $n - 1$ do not exist.

A curve $\gamma : z = \lambda(t) a \le t \le b$ is <u>closed</u> if $\lambda(a) = \lambda(b)$ and is a <u>simple closed</u> if $\lambda(\alpha) = \lambda(\beta)$ implies $\alpha = a$ and $\beta = b$ or $\alpha = b$ and $\beta = a$, that the curve does not cross itself.

A set D is called a <u>domain</u> (or <u>region</u>) if the following conditions are satisfied :

(*i*) All the points of the set D are interior points.

(*ii*) Any two points of the set D can be connected by a polygonal line that lies entirely within D.

Example – 4

(*i*) The set of points $|z| \leq 1$ is not α domain since the point $|z| = 1$ are not interior points.

(*ii*) The set of points satisfying $|z| \neq 2$ do not form a domain since the points $|z| < 2$ cannot be linked with the points satisfying $|z| > 2$ by any polygonal path consisting the given set of points.

1.4 Limit, Continuity and Differencitability :

Let a function $f(x)$ be defined in some neighbourhood $r(z_0)$ of a point $z = z_0$ except possibly for z_0 itself. We say that $f(z)$ tends to a limit ℓ as z approaches z_0 along any path lying in $r(z_0)$, if given any positive number ε no matter how small there exists a positive number $\delta(\varepsilon)$ such that

$$|f(z) - \ell| < \varepsilon \text{ whenever } |z - z_0| < \delta(\varepsilon).$$

Example – 5. Show that

$$\lim_{z \to i} \frac{z^3 - iz^2 + 2iz + 2}{z - i} = 2i - 1$$

Here we show that given ε there exists a $\delta > 0$ such that

$$\left| \frac{z^3 - iz^2 + 2iz + 2}{z - i} - (2i - 1) \right| < \varepsilon \text{ whenever } |z - i| < \delta$$

But the first inequality implies that

$$\left| \frac{(z^2 - 2i)}{z - i} - (2i - 1) \right| < \varepsilon$$

i.e. $|z + i||z - i| < \varepsilon$

This holds whenever $|z - i| < \sqrt{1+\varepsilon} - 1$. Thus $\delta = \sqrt{1+\varepsilon} - 1$.

Example – 6. $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist.

To verify this we evaluate the limit $z \to 0$ on two differents paths.

First Case : $z \to 0$ along the real axis

$$\lim_{z\to 0}\frac{\overline{z}}{z}=\lim_{x\to 0}\frac{\overline{x}}{x}=\lim_{x\to 0}\frac{x}{x}=1$$

Second case : $z \to 0$ along the imaginary axis

$$\lim_{z\to 0}\frac{\overline{z}}{z}=\lim_{y\to 0}\frac{\overline{iy}}{iy}=\lim_{y\to 0}\frac{-iy}{iy}=-1$$

So we get a different limit depending on the path approaching the origin.

It is easy check that $\lim_{z\to z_0} f(z)=L$ holds if and only if $\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y)=a$ and $\lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y)=b$, where $z_0=x_0+iy_0$, $f(z)=u(x,y)+iv(x,y)$ and $L=a+ib$.

On the other hand, the following properties are the usual limit rules for complex functions.

Lemma – 1.

Let *f* and *g* be complex functions and $z_0 \in D$

(*i*) $\lim_{z\to z_0}\{f(z)\pm g(z)\}=\lim_{z\to z_0} f(z)\pm \lim_{z\to z_0} g(z)$

(*ii*) $\lim_{z\to z_0}\{f(z)\cdot g(z)\}=\lim_{z\to z_0} f(z)\cdot \lim_{z\to z_0} g(z)$

(*iii*) $\lim_{z\to z_0}\frac{f(z)}{g(z)}=\lim_{z\to z_0} f(z)\Big/\lim_{z\to z_0} g(z)$, provided $\lim_{z\to z_0} g(z)\neq 0$

Cotinuity

Definition

A function *f*(*z*) defined in a domain *D* is said to be continuous at a point ζ belonging to *D* if

$$\lim_{z\to\zeta} g(z)=f(\zeta)$$

In otherwords, *f*(*z*) is continuous at a point ζ if for every $\varepsilon>0$ there exists a $\delta>0$ such that $|f(z)-f(\zeta)|$ holds for all *z* satisfying $|z-\zeta|<\delta$.

A function *f*(*z*) is continuous in a domian *D* if it is continuous at each point of *D*.

The results in Lemma 1 on limits of sums and products of functions also be used to establish that sums and products of continuous functions are continuous.

Lemma – 2.

A function $f(z)$ is continuous at a point $z_0 = (x_0 + iy_0)$ if and only if its real and imaginary parts which are functions of two real variables are continuous at the point $(x_0 + y_0)$.

Let $f(z) = u(x,y) + iv(x,y)$. Then

$\lim_{z \to z_0} f(z) = \lim_{\substack{x \to x_0 \\ y \to y_0}} \{u(x,y) + iv(x,y)\} = u(x_0, y_0) + iv(x_0, y_0) = f(z_0)$ conversely, given $f(z)$ is continuous at z_0, for every positive z we can find $\delta > 0$ such that

$$|f(z) - f(z_0)| < \delta \text{ for all } z \text{ satisfying } |z - z_0| < \delta$$

Thus,

$$|u(x,y) - u(x_0, y_0)| \leq |f(z) - f(z_0)| < \varepsilon$$

$$|v(x,y) - v(x_0, y_0)| \leq |f(z) - f(z_0)| < \epsilon$$

for all z, $|x - x_0|, |y - y_0| \leq |z - z_0| < \delta$.

Example – 7.

The function $f(z) = z^3$ is continous for all values of z.

Let z_0 be any point in the entire plane and $\varepsilon > 0$ be arbitrary, we have to prove that there exists a $\delta > 0$ such that $|z^3 - z_0^3| < \varepsilon$ for $|z - z_0| < \delta$. We allow z tending to z_0, then there is a positive r such that $|z| < R$ and $|z_0| < R$. Then

$$|z^3 - z_0^3| = |z - z_0||z^2 + zz_0 + 1| \leq (M^2 + M^2 + 1)|z - z_0|$$

Let us take $\delta = \varepsilon / 2M^2 + 1$ then $|z^3 - z_0^3| < (2M^2 + 1)\delta = \varepsilon$

Uniform Continuity

A function $f(z)$ is said to be uniformly continuous in a domain D if for given $\varepsilon > 0$ there exists a positive δ, depending on ε only such that

$$|f(z_1) - f(z_2)| < \varepsilon \text{ for } |z_1 - z_2| < \delta$$

for all $z_1, z_2 \in D$.

Exmaple – 8.

The function $f(z)=\frac{1}{z}$ is not uniformly continous in $D=\{|z| 0<|z|<1\}$.

The given function is continuous in *D*. To prove the problem let z_0 be any point lying in *D*. Then

$$|f(z)-f(z_0)|=\left|\frac{1}{z}-\frac{1}{z_0}\right|=\frac{|z-z_0|}{|z||z_0|}$$

If $|z_0|>\delta, \delta<1$ and $|z-z_0|<\delta$, so $|z|=|z-z_0+z_0|\geq|z_0|-|z-z_0|>|z_0|-\delta$

Thus, $|f(z)-f(z_0)|<\dfrac{\delta}{|z_0|^2-\delta|z_0|}$ so $\varepsilon=\dfrac{\delta}{|z_0|^2-\delta|z_0|}$ and that $\delta=\dfrac{\varepsilon|z_0|^2}{1+\varepsilon|z_0|}$ which depends on z_0 and ε both and hence *f*(*z*) is not uniformly continuous in *D*.

Remark :

A function continuous in a closed founded domain *D* is uniformly continous in *D*.

Complex Differentiation.

Definition :

Let *f*(*z*) be a function defined in a domain *D* and z_0 be an interior point of *D*. The derivative of *f*(*z*) at z_0 denoted by $f'(z_0)$ or $\frac{df}{dz}(z_0)$ is defined by

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} \qquad \text{.....................(1)}$$

provided this limit exists. In this case *f* is called <u>differentiable</u> at z_0. If *f* is differentiable at every point of a neighbourhood of z_0 then *f* is said to be <u>analytic</u> at z_0.

Example – 9.

Given $f(z)=z^2$. Find $f'(z_0)$.

Let $\Delta z=z-z_0$. Then

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{(z_0+\Delta z)^2-z_0^2}{\Delta z}=\lim_{\Delta z\to 0}\frac{2z_0+\Delta z+(\Delta z)^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z_0 + \Delta z) = 2z_0$$

Example – 10.

The function $f(z) = z\bar{z}$ is nowhere differentiable except at the origin.

Here we consider the given function's differentiability at z_0.

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\overline{z_0 + \Delta z}) - z_0\bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0 + \overline{\Delta z} + \bar{z}_0 \Delta z + \Delta z \overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(z_0 + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z} \right) \qquad \text{.................... (2)}$$

The limit in (2) attains the value $\bar{z}_0 + z_0$ when the limit approaches 0 along the real axis and it equals to $\bar{z}_0 + z_0$ while the limit towards the origin along the imaginary axis.

Clearly the two values of (2) coincide when $z_0 = 0$.

The basic properties for derivatives are similar to the differentiation formulae of elementary calculus.

Lemma – 3.

Let f and g are differentiable at $z \in \mathbb{C}$ and that $c \in \mathbb{C}$, then the following relations hold :

(*i*) $\{f(z) + c\,g(z)\}' = f'(z) + c\,g'(z)$,

(*ii*) $\{f(z) \cdot g(z)\}' = f'(z) + g(z) + f(z)g'(z)$

(*iii*) $\{f(z)/g(z)\}' = \dfrac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}$ provided $g(z) \neq 0$.

Remark

A function differentiable at z_0 is continuous at z_0 but the converse is not always true.

Chain Rule :

Let f is differentiable and g is differentiable at $f(z)$, then $g_0 f$ is differentiable and

$$(g_0 f)'(z) = g'(f(z))f'(z)$$

Necessary conditions for Differentiability.

Therem. If a function $f(z)=u(x,y)+iv(x,y)$ is differentiable at $z_0=x_0+iy_0$ then the partial derivatives u_x,u_y,v_x,v_y exist at (x_0,y_0) and satisfy

$$u_x(x_0,y_0)=v_y(x_0,y_0) \text{ and } u_y(x_0,y_0)=-v_x(x_0,y_0) \qquad \text{..................(3)}$$

The equations in (3) are called Cauchy-Riemann equations.

Proof. Following definition of differentiability

$$f'(z_0)=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

Here Δz approaching zero along any path. Let us two paths :

(*i*) $\Delta z\to 0$ along the real axis

$$f'(z_0)=\lim_{\Delta x\to 0}\left\{\frac{u(x_0+\Delta x,y_0)-u(x_0,y_0)}{\Delta x}+i\frac{v(x_0+\Delta x,y)-v(x_0,y_0)}{\Delta x}\right\}$$

$$=u_x(x_0,y_0)+i\,v_x(x_0,y_0)$$

(*ii*) $\Delta z\to 0$ along the imaginary axis

$$f'(z_0)=\lim_{\Delta y\to 0}\left\{\frac{u(x_0+y_0+\Delta y)-u(x_0,y_0)}{i\,\Delta y}+i\frac{v(x_0,y_0+\Delta y)-v(x_0,y_0)}{i\,\Delta y}\right\}$$

$$=-i\,u_y(x_0,y_0)+v_y(x_0,y_0)$$

Comparing the two derivatives, we obtain the relations in (3).

Sufficient conditions for Differentiability.

Theorem suppose the partial derivatives of $u(x,y)$ and $v(x,y)$ are continous at (x_0,y_0) and satisfy the *C–R* equations (3) then $f(z)=u(x,y)+i\,v(x,y)$ is differentiable at $z_0=x_0+i\,y_0$.

Proof. Let us first start with quotient

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\frac{\Delta u+i\,\Delta v}{\Delta x+i\Delta y}$$

$$=\frac{\{u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0)\}+i\{v(x_0+\Delta x,y_0+\Delta y)-v(x_0,y_0)\}}{\Delta x+i\,\Delta y}$$

But we observe that

$$u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0)=$$
$$\{u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0+\Delta y)\}+\{u(x_0,y_0+\Delta y)-u(x_0,y_0)\}$$

Again,

$\{u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0+\Delta y)\}=\Delta x\, u_x(x_0+\theta_1\Delta x, y_0+\Delta y)$ using Mean-Value theorem for one real variable and $z_0=x_0+i\,y_0$.

Likewise,

$$u(x_0,y_0+\Delta y)-u(x_0,y_0)=\Delta y\, u_y(x_0,y_0+\theta_2\Delta y), 0<\theta_2<1.$$

Also, we have

$u_x(x_0+\theta_1\Delta x, y_0+\Delta y)=u_x(x_0,y_0)+\varepsilon_1$ and

$u_y(x_0,y_0+\theta_2+\Delta y)=u_y(x_0,y_0)+\varepsilon_2$, using continuity of u_x, u_y at (x_0,y_0)

where $\varepsilon_1,\varepsilon_2\to 0$ as $\Delta x+i\,\Delta y\to 0$

So,

$$\Delta u=\Delta x\{u_x(x_0,y_0)+\varepsilon_1\}+\Delta y\{u_y(x_0,y_0)+\varepsilon_2\}$$

Siimilarly using continuity of v_x and v_y at (x_0,y_0)

$$\Delta v=\Delta x\{v_x(x_0,y_0)+\varepsilon_3\}+\Delta y\{v_y(x_0,y_0)+\varepsilon_4\}$$

where $\varepsilon_3,\varepsilon_4\to 0$ as $\Delta x+i\,\Delta y\to 0$.

Thus

$$\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\lim_{(\Delta x+i\Delta y)\to 0}\frac{\Delta u+i\,\Delta v}{\Delta x+i\,\Delta y}$$

$$=\lim_{(\Delta x+i\Delta y)\to 0}\Delta x\{u_x(x_0,y_0)+\varepsilon_1+i\,v_x(x_0,y_0)+\varepsilon_3\}+\Delta y$$
$$+\Delta y\{u_y(x_0,y_0)+\varepsilon_2+i\,v_y(x_0,y_0)+\varepsilon_4\}/\Delta x+i\,\Delta y$$

$$=\lim_{(\Delta x+i\Delta y)\to 0}\left[\frac{\Delta x(u_x+i\,v_x)+i\Delta y(u_x+i\,v_x)}{\Delta x+i\,\Delta y}+\frac{\Delta x(\varepsilon_1+i\,\varepsilon_3)+\Delta y(\varepsilon_2+i\,\varepsilon_4)}{\Delta x+i\,\Delta y}\right]$$

[using $C-R$ equations]

$$=u_x(x_0,y_0)+i\,v_x(x_0,y_0)$$

Since

$$\left|\frac{\Delta x(\varepsilon_1 + i\varepsilon_3) + \Delta y(\varepsilon_2 + i\varepsilon_4)}{\Delta x + i\,\Delta y}\right| \le |\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| + |\varepsilon_4| \to 0 \text{ as } \Delta x,$$

$\Delta y \to 0.$

Thus $f(z)$ is differentiable at z_0.

Example – 11.

Consider the function

$$f(z) = \begin{cases} xy(x+iy)/x^2 + y^2 & if\ z \neq 0 \\ 0 & if\ z \neq 0 \end{cases}.$$

Show that f satisfies the Cauchy-Riemann equations at the origin though it is not differentiable there.

First we evaluate u_x, u_y, v_x, v_y at the point (0, 0). Here

$$u(x,y) = \frac{x^2 y}{x^2 + y^2} \quad \text{and} \quad v(x,y) = \frac{xy^2}{x^2 + y^2}$$

$$u_x = \frac{2xy^3}{(x^2+y^2)^2}\left(\frac{0}{0}\right) \qquad u_y = \frac{x^4 - x^2y^2}{(x^2+y^2)^2}\left(\frac{0}{0}\right)$$

$$= \frac{y^3}{2x(x^2+y^2)}\left(\frac{0}{0}\right) \qquad = \frac{-x^2}{2y(x^2+y^2)}\left(\frac{0}{0}\right)$$

$$= 0 \qquad = 0$$

$$v_x = \frac{y^4 - x^2y^2}{(x^2+y^2)^2}\left(\frac{0}{0}\right) \qquad v_y = \frac{2x^3y}{(x^2+y^2)^2}\left(\frac{0}{0}\right)$$

$$= \frac{-y^3}{2(x^2+y^2)}\left(\frac{0}{0}\right) \qquad = \frac{x^2}{2y(x^2+y^2)}\left(\frac{0}{0}\right)$$

$$= 0 \qquad = 0$$

These show that $C-R$ equations are satisfied. Now to check the differentiability we evaluate the limit

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z}$$

along any radius vector. For this sake, we allow the limit $z \to 0$ along the line $y = mx$,

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{\substack{x\to 0\\ y\to 0}}\frac{\frac{xy(x+iy)}{x^2+y^2}}{x+iy} = \lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x^2+y^2} = \lim_{z\to 0}\frac{mx^2}{x^2\left(1+m^2\right)}$$

$$= \frac{m}{1+m^2}$$

This limit is not unique as it depends on m. Hence $f(z)$ is not differentiable at the origin.

Example – 12.

Where are the functions

$$(a)\, f(z) = 2xy - i(x+y)^2 \quad (b)\, f(z) = e^x e^{-iy}$$

differentiable?

(a) Here, $u(x, y) = 2xy$ and $v(x, y) = -(x + y)^2$

$u_x = 2y,\ u_y = 2x,\ v_x = -2(x + y) = v_y$

We see that the $C-R$ equations

$$u_x = v_y \Rightarrow 2y + x = 0$$

$$u_y = -v_x \Rightarrow y = 0$$

are satisfied only at the origin, also the partial derivatives are continuous there. Hence the given function is differentiable only at the origin.

(b) Here $u(x,y) = e^x \cos y$ and $v(x,y) = -e^x \sin y$

$$u_x = e^x \cos y, u_y = e^x \sin y, v_x = -e^x \sin y, v_y = -e^x \cos y$$

We observe that the $C-R$ equations $e^x \cos y = 0$ and $e^x \sin y = 0$ are nowhere satisfied. Hence the given function is not where differentiable.

1.5 Analytic functions

We knew that a function $f(z)$ analytic at $z = z_0$ means that in some neighbourhood of the point z_0 the function is throughout differentiable. We shall see later that if f is differentiable at every point of an open set in $\mathbb{C}$ it is analytic in $\mathbb{C}$; in fact, it is unfinitely differentiable in $\mathbb{C}$ which is in contrast with the real case. The functions e^z, $\sin z$, $\cos z$

are all analytic in the finite complex plane. The function $f(z) = 1/z$ is analytic for all finite $z \neq 0$.

Example – 13.

There are no points at which the function $f(z) = |z^2|$ is analytic.

Definition :

If f is not analytic at $z = z_0$ but is analytic some point in every neighbourhood of z_0, then z_0 is termed as a singular point of f

Example – 14.

$z = 0$ is the singular point of

$$f(z) = \frac{1}{z}.$$

Harmonic functions

Suppose f is analytic in a domain D,

$$f(x+iy) = u(x,y) + iv(x,y)$$

and that u and v have continous partial derivatives of all orders. From the Cauchy-Riemann equations we have

$$u_x(x,y) = v_y(x,y) \text{ and } u_y(x,y) = -v_x(x,y) \qquad \text{.................. (4)}$$

Differentiating with respect to x, we get

$$u_{xx}(x,y) = v_{yx}(x,y) \text{ and } u_{yx}(x,y) = -v_{xx}(x,y) \qquad \text{.................. (5)}$$

Differentiating with respect to y, we find

$$u_{xy}(x,y) = v_{yy}(x,y) \text{ and } u_{yy}(x,y) = -v_{xy}(x,y) \qquad \text{.................. (6)}$$

From euations (5), (6) we obtain

$$u_{xx}(x,y) = u_{yy}(x,y) = 0$$

and $\quad v_{xx}(x,y) = v_{yy}(x,y) = 0$

for all $x + iy \in D$, whence it follows that the functions $u(x, y)$ and $v(x, y)$ satisfy the Laplace equation $\Delta^2 w = 0$. The solutions of various problems in Physics and Mechanics satisfy boundary value problems for the Laplace equation. The function w which satisfy the Laplace equation in a domain D is called a harmonic function in

D. The two functions $u(x,y)$ and $v(x,y)$, respectively the real and imaginary parts of $f(z)$, are harmonic functions in D and we say, v is a harmonic conjugate of u.

Example – 16.

Given $u(x,y)$, find a harmonic conjugate of $u(x,y)$, hence construct an analytic function

$f(z) = u(x,y) + i\,v(x,y)$:

(a) $x^3 - 3xy^2$ (b) $x^3 + 3x^2y - 3xy^2 - y^3$

(a) Given $u(x,y) = x^3 - 3xy^2$

$u_x(x,y) = 3x^2 - 3y^2$ and $u_{xx} = 6x$

$u_y(x,y) = -6xy$ and $u_{yy} = -6x$

So, $u(x,y)$ satisfies Laplace equation. To find a harmonic conjugate v of u, we have following C–R equations

$u_x(x,y) = v_y(x,y)$ and $u_y(x,y) = -v_x(x,y)$

From the first we have

$v_y(x,y) = 3x^2 - 3y^2$

which on integration w.r. to y yields

$$v(x,y) = 3x^2y - y^3 + \phi(x)$$

It now follows from the second equation

$$-6xy = -v_x(x,y) = -6xy - \phi'(x),$$

which shows that $\phi'(x) = 0$. Hence $\phi(x) = k$, a real constant and $v(x,y) = 3x^2y - y^3 + k$.

Thus,

$$f(z) = x^3 - 3xy^2 + i(3x^2y - y^3) = (x+iy)^3 = z^3$$

(b) $u(x,y) = x^3 + 3x^2y - 3xy^2 - y^3$

$u_{xx}(x,y) = 6x + 6y$ and $u_{yy} = 6x - 6y$

$u(x,y)$ satisfies the Laplace equation $\nabla^2 w = 0$.

Using $C-R$ equations,

$u_x = v_y$ and $u_y = -v_x$

We find that

$3x^2 - 3y^2 + 6xy = v_y$ and $3x^2 - 6xy - 3y^2 = -v_x$

Integrating the first we find,

$$v(x,y) = 3x^2y - y^3 + 3xy^2 + \psi(x)$$

so, $v_x(x,y) = 6xy + 3y^2 + \psi'(x) = -3x^2 + 6xy + 3y^2$, using the second equations. So $\psi'(x) = -3x^2$ and $\psi(x) = -x^3$ and $v(x,y) = 3x^2y - y^3 + 3xy^2 - x^3$.

Hence $f(z)$ is of the form

$$f(z) = x^3 + 3x^2y - 3xy^2 - y^3 + i(3x^2y - y^3 + 3xy^2 - x^3)$$

$$= (1-i)x^3 + (1-i)3x^2y - (1-i)3xy^2 - (1+i)y^3$$

$$= (1-i)(x+iy)^3$$

$$= (1-i)z^3.$$

Example – 17.

Show that the function $u(x,y) = y^3 - 3x^2y - 2x - 1$ is harmonic in $\mathbb{C}$ and find it's harmonic conjugate. Hence construct an analytic function with $u(x,y)$ as its real part.

$u(x,y)$ satisfies the Laplace equation :

$$u_{xx} = -6y \text{ and } u_{yy} = 6y \text{ so, } u_{xx} + u_{yy} = 0$$

Let $v = (x,y)$ be its harmonic conjugate, then

$$v_x = v_y \text{ and } u_y = -v_x$$

From the first

$$v_y = -6xy - 2 \text{ so, } v(x,y) = 3xy^2 - 2y + \phi(x)$$

using the second

$$-3y^2 + \phi'(x) = -u_y = -3y^2 + 3x^2$$

i.e., $\phi(x) = x^3$ and hence $v(x,y) = -3xy^2 - 2y + x^3$

The desired analytic function $f(z) = u + iv$ takes the form

$$f(z) = y^3 - 3x^2y - 2x - 1 + i(-3xy^2 - 2y + x^3)$$

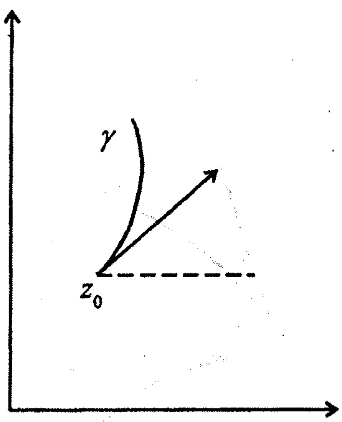
$$= i(x^3 + i^3y^3 + 3i^2xy^2 + 3ix^2y) - 2(x+iy) - 1$$

$$= i(x+iy)^3 - 2(x+iy) - 1$$

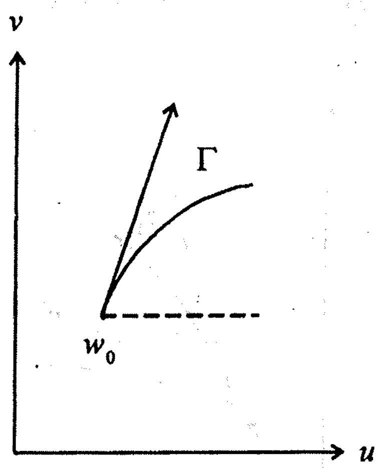
$$= iz^3 - 2z - 1.$$

1.6 Conformal Representation

Given a function of a complex variable $w = f(z)$ analytic in a domain D. Let z_0 be any point lying within D; $\gamma : z = \sigma(t), a \le t \le b, \sigma \le (t_0) = z_0$ be a curve passing through z_0 (and lying within D). The function $\sigma(t)$ has a non-zero derivative $\sigma'(t_0)$ at the point z_0 and the curve γ has a tangent at this point with a slope equal to Arg $\sigma'(t_0)$.



z-plane



w-plane

Under the mapping $w = f(z)$ the curve γ is transformed into a curve $\Gamma : w = f(\sigma(t)) = \mu(t)$, $a \le b \le b, \mu(t_0) = f(z_0) = w_0$ in the w-plane. $\mu(t)$ is differentiable at $t = t_0$ and the curve Γ has a tangent at $w_0 = f(z_0)$. Then following the chain rule for differentiation of compositve functions, assuming $f'(z_0) \neq 0$

$$\mu'(t_0) = f'((t_0))\sigma'(t_0)$$

It follows that

$$Arg\, \mu'(t_0) = Arg\, f'(z_0) + Arg\, \sigma'(t_0)$$

i.e. $$Arg\, \mu'(t_0) - Arg\, \sigma'(t_0) = Arg\, f'(z_0) \qquad \text{..............(7)}$$

This shows that change in slope of a curve at a point under a transformation depends only on the point and not on the particular curve through that point.

Definition :

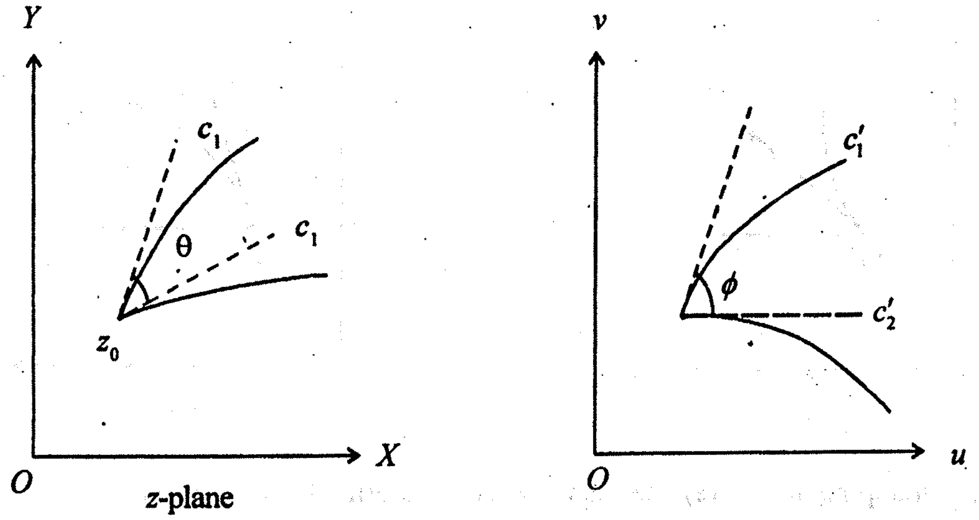
A mapping $w = f(z)$ is said to be conformal at a point $z = z_0$, if it preserves angles between oriented curves, passing the rought z_0, in magnitude and in sense of rotation.

Theorem :

Let $f(z)$ be an analytic function defined in a domain D and z_0 be any point lying within D.

If $f'(z_0) \neq 0$, then $f(z)$ is conformal at z_0.

Proof. Let $c_1 : z = \lambda_1(t)$ and $c_2 : z = \lambda_2(t), t \equiv$ parameter be two curves which intersect at some $t = t_0$ where $\lambda_1(t_0) = \lambda_2(t_0) = z_0; c_1', c_2'$ be their images under the mapping $w = f(z)$.



tangent lines are

$$z' = \lambda_1'(t_0), z'\lambda_2'(t_0) \, at \, t = t_0$$

tangent lines are

$$w_1'(t_0) = f'(\lambda_1(t_0)) \cdot \lambda_1'(t_0)$$
$$w_2'(t_0) = f'(\lambda_2(t_0)) \cdot \lambda_2'(t_0)$$

Following the result given in equation (7),

$$Arg(w_1'(t_0)) - Arg(\lambda_1'(t_0)) = Arg(f'(\lambda_1(t_0))) = Arg(f'(z_0))$$

and

$$Arg(w_2'(t_0)) - Arg(\lambda_2'(t_0)) = Arg(f'(\lambda_2(t_0))) = Arg(f'(z_0))$$

Subtaracting,

$$\{Arg(w_1'(t_0)) - Arg(w_2'(t_0))\} - \{Arg(\lambda_1'(t_0)) - Arg(\lambda_2'(t_0))\} = 0$$

i.e. $\theta = \phi$.

We now consider a simple type of conformal mapping viz. Möbius transformation named after August Ferdinand Möbius (1790-1878). This transformation is also known as Linear fractional transformation / Bilinear transformation

Definition :

A Möbius transformation is a function of the form

$$w = f(z) \equiv \frac{az+b}{cz+d}, \qquad ad - bc \neq 0 \qquad \text{.................. (8)}$$

where $a,b,c,d \in \mathbb{C}$. It is analytic in $\mathbb{C}/\left\{-\frac{d}{c}\right\}$ (unless $c = 0$, in which case f is entire). The inverse transformation f^{-1} is defined by

$$z = f^{-1}(w) \equiv \frac{-dw+b}{cw-a}, \qquad \text{.................. (9)}$$

which is another Möbius transformation. If $c \neq 0$, (*i*) the point $z = -\frac{d}{c}$ is mapped onto the point at infinity in the w-plane by f and ∞ to a/c (*ii*) the points $w = \frac{a}{c}$ and ∞ are mapped respectively to the points ∞ and $-\frac{d}{c}$ by f^{-1}. Some of the remarkable properties of Möbius transformation are following:

A. Möbius transfomratins are bijections on $\mathbb{C}_\infty$.

Suppose $f(z_1) = f(z_2)$ in (8) that is

$$\frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d}$$

This is equivalent to (provided denominators are not zero)

$$(az_1+b)(cz_2+d) = (az_2+b)(cz_1+d)$$

i.e. $(ad-bc)(z_1-z_2) = 0$

Since $ad - bc \neq 0$, this implies $z_1 = z_2$. So f is one-to-one and likewise f^{-1}, which implies f is onto.

B. Suppose $f(z) = \dfrac{az+b}{cz+d}$ is a Möbius transformation. If

(*i*) If $c = 0$, $f(z) = \frac{a}{d}z + \frac{b}{d}$

(*ii*) If $c \neq 0$, $f(z) = \frac{bc - ad}{c^2} + \frac{1}{z + \frac{d}{c}} \frac{a}{c}$

In particular, every Möbius transformation is a composition of translation, dilation and inversion. It can be established simply rearranging the term on the right.

C. Möbius transformation maps circles and lines into circles and lines.

Clearly, translation and dilation map circles and lines into circles and lines. So following the last property *B* we only prove the property for the inversion $f(z) = \frac{1}{z}$. Let us start with the equation of a straight line –

$$ax + by = c \quad , a, b, c \text{ are real constants}, z = -x + iy$$

Let $\alpha = a + bi$, then the equation changes to

$$\frac{\alpha + \bar{\alpha}}{2} \frac{z + \bar{z}}{2} + \frac{\alpha - \bar{\alpha}}{2i} \frac{z - \bar{z}}{2i} = c$$

i.e., $\bar{\alpha} z + \alpha \bar{z} = 2c$ or $\text{Re}(\bar{\alpha} z) = c$ .... (10)

First Case. Möbius transformation maps circles to circles or lines :

Consider the circle $|z - z_0| = r$, which modify as follows:

$$|z - z_0|^2 = r^2$$

or, $|z - z_0|\left(\overline{z - z_0}\right) = r^2$

or, $z\bar{z} - z_0\bar{z} - z\bar{z}_0 + z_0\bar{z}_0 = r^2$

i.e. $|z|^2 - z_0\bar{z} - z\bar{z}_0 + |z_0|^2 - r^2 = 0$

Transforming the equation by $w = \frac{1}{z}$, we find

$$\left|\frac{1}{w}\right|^2 - z_0 \frac{1}{w} - \bar{z}_0 \frac{1}{w} + |z_0|^2 - r^2 = 0$$

Multiplying by $|w|^2 = w\bar{w}$, we get

$$\left|-z_0 w - \overline{z_0 w} + |w|^2 \left(|z_0|^2 - r^2\right)\right| = 0$$ .... (11)

If $r = |z_0|$, this equation reduces to an equation straight line (1) with $\alpha = \bar{z}_0$ and $c = \frac{1}{2}$.

If $r \neq |z_0|$, we divide the equation (11) by $|z_0|^2 = r^2$ and obtain

$$|w|^2 - \frac{z_0}{|z_0|^2 - r^2} w - \frac{\overline{z}_0}{|z_0|^2 - r^2} \overline{w} + \frac{1}{|z_0|^2 - r^2} = 0$$

Take $w_0 = \frac{\overline{z}_0}{|z_0|^2 - r^2}, k^2 = |w_0|^2 - \frac{1}{|z_0|^2 - r^2} = \frac{|z_0|^2}{\left(|z_0|^2 - r^2\right)} - \frac{1}{|z_0|^2 - r^2} = \frac{r^2}{\left(|z_0|^2 - r^2\right)^2}$

and rewrite the equation as

$$|w|^2 - \overline{w}_0 w - w_0 \overline{w} + |w_0|^2 - k^2 = 0$$

or, $\quad (w - w_0)\left(\overline{w - w_0}\right) = k^2$

i.e. $\quad |w - w_0|^2 = k^2$

This is an equation of circle in terms of w withcentre at w_0 and radius k.

Second Case. Möbius transformation maps lines to lines or circles. Now we trasform the equation of straight line (10) by $w = \frac{1}{z}$ and get

$$\overline{z_0 w} + z_0 w = 2c\, w\overline{w}$$

If $c = 0$, this describes a line in the form (10) in terms of w.

If $c \neq 0$, we divide by $2c$ and find

$$w\overline{w} - \frac{\overline{z}_0}{2c}\overline{w} - \frac{z_0}{2c} w = 0$$

or, $\quad \left(w - \frac{\overline{z}_0}{2c}\right)\left(\overline{w} - \frac{\overline{z}_0}{2c}\right) - \frac{|z_0|^2}{4c^2} = 0$

i.e. $\quad \left|w - \frac{\overline{z}_0}{2c}\right|^2 = \left(\frac{|z_0|}{2c}\right)^2$

This is an equation of a circle with centre at $\frac{\overline{z}_0}{2c}$ and radius $|z_0|/2$.

D. Composition of two Möbius tranformations is Möbius.

Let $f_j(z) = \frac{a_j z + b_j}{c_j z + d_j}, j = 1, 2$ be the two Möbius transformation. Then we find that

$$(f_2 o f_1)(z) = f_2(f_1(z)) = \frac{(a_1a_2 + b_2c_1)z + (a_2b_1 + b_2d_1)}{(a_1c_2 + c_1d_2)z + (b_1c_2 + d_1d_2)}$$

is Möbius, since

$$(a_1a_2 + b_2c_1)(b_1c_2 + d_1d_2) - (a_1c_2 + c_1d_2)(a_2b_1 + b_2d_1)$$
$$= (a_1d_1 - b_1c_1)(a_2d_2 - b_2c_2) \neq 0$$

Similarly, $(f_1 o f_2)(z)$ is Möbius.

Definition. If z, z_1, z_3 and z_3 are any four points in $\mathbb{C}$ with z_1, z_2 and z_3 distinct, their cross ratio is defined by

$$\{z, z_1, z_2, z_3\} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

Here if $z = z_3$, the result is infinity. Moreover if one of z, z_1, z_3 or z_3 is infinity, then one term each in the numerator and denominator cancel each other.

E. A Möbius transformation, other than the identity map, has atmost two fixed points.

Let $f(z) = \frac{az + b}{cz + d}$ be a Möbius transformation, not an identity map and z be a fixed point, then $f(z) = z$

i.e. $\frac{az + b}{cz + d}$ or $cz^2 + (d - a)z - b = 0$

This is a quadratic equation, hence there exist at most two distinct points satisfying $f(z) = z$.

F. f be a Möbius transformation and $f(z_j) = w_j, j = 1,2,3$ where all the z_j's are distinct then f is unique.

Let there exists another Möbius transformation $g(z)$ satisfying $g(z_j) = w_j, j = 1,2,3$. Then

$$f(z_j) = w_1 = g(z_1)$$

so, $z_1 = f^{-1}(g(z_1))$. Similarly, $z_i = f^{-1}(g(z_i)), i = 2,3.$

f is Möbius, we have seen f^{-1} is also Möbius. Again g is Möbius, by property D $f^{-1} o g$ is Möbius. Thus a Möbius transformation $f^{-1} o g$ satisfies

$$Tz = z$$

at three distinct points z_1, z_2, z_3, but by property E it can only happen when $f^{-1} o g$ is an identity map and hence $f = g$ as asserted.

G A Möbius transformation leaves a cross-ratio in variant.

In particular, if f be a Möbius transformation and z_1, z_2, z_3 and z_4 are all distinct points in $\mathbb{C}$ then

$$\{z_1,z_2,z_3,z_4\}=\{f(z_1),f(z_2),f(z_3),f(z_4)\}$$

Let T be a Möbius transformation defined by

$$T(z)=\{z,z_2,z_3,z_4\} \text{ for all } z$$

We can easily check that T is Möbius. Now since f^{-1} is Möbius, by property D, $T\,o\,f^{-1}$ is Möbius. Again we can have

$$\left(T\,o\,f^{-1}\right)\left(f(z_2)\right)=T(z_2)=0$$

$$\left(T\,o\,f^{-1}\right)\left(f(z_3)\right)=T(z_3)=1$$

$$\left(T\,o\,f^{-1}\right)\left(f(z_4)\right)=T(z_4)=\infty$$

But we observe that the Möbius transformation defined by

$$\int(z)\{z,f(z_2),f(z_3),f(z_4)\}$$

satisfies the same properties as $T\,o\,f^{-1}$. Following property F,

$$\left(T\,o\,f^{-1}\right)=\int$$

So, at $f(z_1)$,

$$\left(T\,o\,f^{-1}\right)\left(f(z_1)\right)=\int\left(f(z_1)\right)$$

or, $T(z_1)=\{f(z_1),f(z_2),f(z_3),f(z_4)\}$

i.e., $\{z_1,z_2,z_3,z_4\}=\{f(z_1),f(z_2),f(z_3),f(z_4)\}$

Example – 18.

Construct the Möbius transformation that maps the point $z_1=-1, z_2=i, z_3=1$ into the points $w_1=1$, $w_2=0$ and $w_3=-1$ respectively.

Let f be the required transformation and as Möbius transformation leaves a cross-ratio invariant, we have

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}=\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

or, $$\frac{(z+1)(i-1)}{(z-1)(i+1)}=\frac{(w-1)(0+1)}{(w+1)(0-1)} \text{ or } \frac{(z+1)(i-1)}{(z-1)(i+1)}=\frac{w-1}{w+1}$$

i.e. $\frac{z-i}{iz-1} = w$

Example – 19.

Find the Möbius transformation that maps the points

$z_1 = 1$ to $w_1 = i$

$z_2 = -1$ to $w_1 = \infty$

$z_3 = -i$ to $w_3 = 1$

The Möbius transformation that maps z_j to $w_j, j = 1, 2, 3$ can be expressed as

$$\frac{(z-1)(-1+i)}{(z+i)(-1-1)} = \frac{(w-i)\cdot 1}{(w-1)\cdot 1} \text{ or } \frac{2w-(1+i)}{i-1} = \frac{z(i-3)+1-3i}{-(z+1)(1+i)}$$

i.e., $$w = \frac{i+1}{2} - \frac{i-1}{2}\left[\frac{z(i-3)+(1-3i)}{(z+1)(1+i)}\right]$$

$$= \frac{i+1}{2} - \frac{z(1-2i)+(1+2i)}{(z+1)(1+i)}$$

Module Summary

In this module we introduced the theory of functions of a complex variable and explained in what way it differs from real analysis. Next limit and continuity; necessary and sufficient conditions for a function to be differentiable at a point were defined. It may happen that a function is differentiable only at a point were defined. It may happen that a function is differentiable only at a point not in any neighbourhood of that point.

Our object of study is of such functions, which are differentiable not only at the point itself but throughout in neighbourhood of that point, called 'analytic functions'.

We conclude our discussion with an important class of conformal mapping, namely Möbius transformation. The general theory will be studied at an advanced stage.

Self assessment questions :

1. Find the real and imaginary parts of the following :

(a) $\frac{z-2}{z+2}$ (b) $\frac{2+7i}{5i+1}$ (c) $\left(\frac{-1+i\sqrt{3}}{5}\right)^3$

2. Sketch the follwoing sets in the complex plane :

(a) $\{z \in \mathbb{C} : |z+1-i| = 2\}$ (b) $\{z \in \mathbb{C} : |z+1-i| = 2\}$

(c) $\{z \in \mathbb{C} : \mathrm{Re}|z+3-i| = 2\}$ (d) $\{z \in \mathbb{C} : |z-2|-|z+2| = 5\}$

3. Evaluate the following limits :

(a) $\lim\limits_{z\to 3i} \dfrac{z^2+2}{z^3+3oi}$ (b) $\lim\limits_{z\to\infty} \dfrac{z+4i}{4z+i}$ (c) $\lim\limits_{z\to\infty} \dfrac{2z^2-3}{5z+7}$

4. Show that the function

$$f(z) = (x^2 - 2xy) + i\cos(x+y)$$

is continuous on $\mathbb{C}$.

5. Whose are the following functions diffrentiable?

(a) $f(z) = z\,\mathrm{Im}\,z$ (b) $f(z) = \dfrac{ix+1}{y}$ (c) $f(z) = z^2 - \bar{z}^2$

6. Show that the function defined by

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, z \neq 0$$
$$= 0 \quad , \quad z = 0$$

is continous and satisfies the Cauchy-Riemann equations at the origin but $f'(0)$ does not exist.

7. Find the analytic function whose real part is $u(x,y) = \frac{1}{2}(e^y - e^{-y})(\sin x - \cos x)$

8. Show that the function $u(x, y) = ax^2 + bxy + cy^2$ is harmonic if and only if $a = -c$.

9. Show that the derivative of a Möbius transformation is no-zero in the entire plane.

10. Find the Möbius transformation which maps 1, 1+i and 2 to 0, 1 and ∞ respectively.

Suggested further readings :

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-I Group–B

Module No. - 7
FUNCTIONS OF A COMPLEX VARIABLE

INTRODUCTION

At a glance, complex integration seems nothing different from real integration. For an indefinite complex integral is a function whose derivative equals a given analytic function defined in some domain D and the definite integrals are performed over rectifiable arcs and are restricted to continuous functions.

As usual complex integral is defined by a limit process which on simplification reduces to an expression of line integrals of real-valued functions. Moreover if it is complex definite integral, replacement of the path of integration by a parametric form $\gamma(t.)$ $a \le t \le b$ leads to calculation of real definite integrals. So in considering the existence of complex integrals we are to familiar with the theory of existence of real integrals of continuous functions.

The theory of complex integral is utmost essential to establish many important and interesting properties of analytic functions in a very elegant way.

Module content

1.1 Rectifiable arcs

1.2 Complex integration

1.3 Properties of complex integrals

1.4 Cauchy's integral theorem

1.5 Cauchy's integral formula and its consequences

Objectives

- Measurement of length of an arc.
- Integration of a complex-valued function over a curve lying on the complex plane
- Some useful results on complex integrals
- Integration of complex function over various closed contours
- Evaluation of complex function at a point lying in its region of analyticity. Some important results of analytic functions.
- Detection of the maximum value of the modulus of a complex function on a closed bounded domain
- Explansion of a complex function when its region of analyticity is a disk or an annuulus.

Key words

Rectifiable arc. Line integral. Simply and multiply connected domains. Power series.

1.1 Rectifiable arcs

Definition. Jordan arc (or, contour)

A simple curve which is piecewise smooth is called a Jordan arc or, simply a contour.

Definition. Jordan contour (or, closed contour).

A simple closed curve if it is piecewise smooth is called a Jordan contour or simple closed contour.

Let $z = \gamma(t), a \le t \le b$ be a given curve, we can have the following possibilities

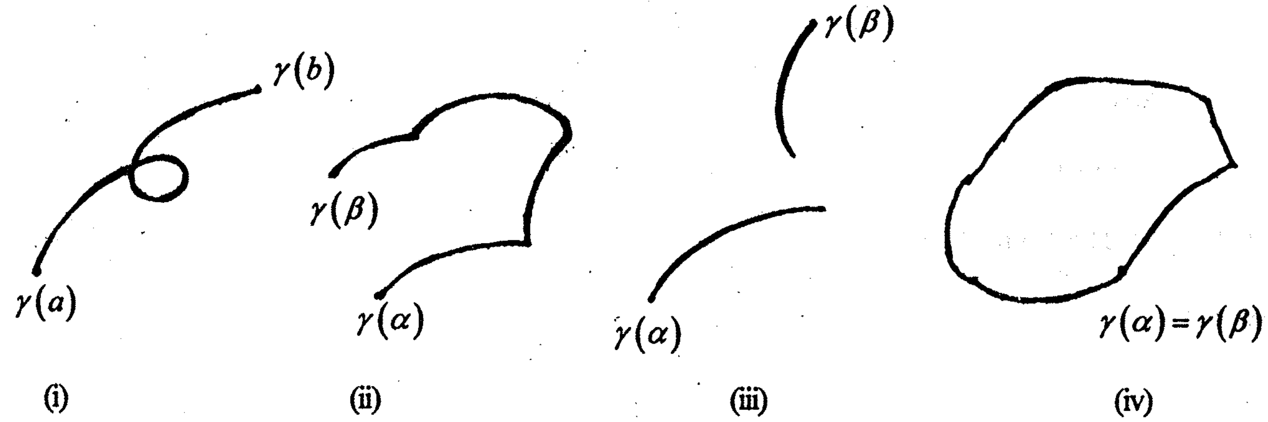


Figure 1

(i) smooth curve but not a Jordan arc.

(ii) it is a Jordan arc

(iii) discontinuous curve

(iv) Jordan contour

Rectifiable arc

A curve is said to be rectifiable if it is of finite length. A piecewise smooth curve $z = \gamma(t), a \le t \le b$ is rectifiable and its length can be calculated by the integral

$$L = \int_a^b |\gamma'(t)|\,dt \tag{1}$$

1.2 Complex Integral

Definition, complex integral (or, contour integral).

Let us suppose $z = \gamma(t), a \le t \le b$ be a piecewise smooth curve and *f* is a complex function continuous on γ. Then we define the integral of *f* on γ as

$$\int_\gamma f = \int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt \tag{2}$$

Example 1 Compute the integral of the function $f(z) = z^2$ over any curve from the point $z = -1$ to $z = 1$.

We perform the integral on the three curves from $z = -1$ to $z = 1$ as follows:

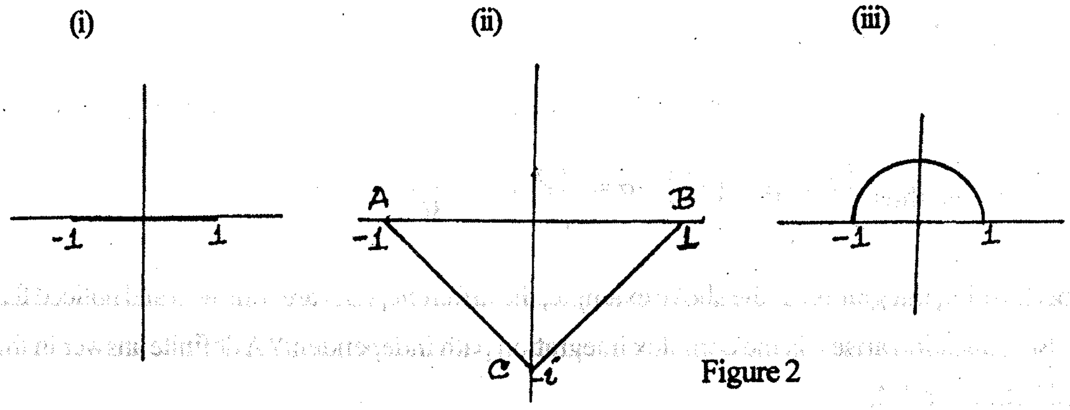


Figure 2

(i) Let γ be the straight line segment from $z = x = -1$ to $z = x = 1$. A parametrization of this curve is $\gamma(t) = t$, $-1 \le t \le 1$. Then following (2),

$$\int_\gamma f = \int_{-1}^{1} t^2 dt = \frac{2}{3}$$

(ii) In this the integration is taken over the curve ACB which consists of the straight line segments AC and CB. A parametric form of these straight line segments is expressed as:

$$\gamma(t) = \begin{cases} (t-1) - it \\ (1-t) - it \end{cases} ; 0 \le t \le 1$$

So, $$\int_\gamma f = \int_0^1 \{(t-1) - it\}^2 \{(t-1) - it\}' dt$$

$$+ \int_1^0 \{(1-t) - it\}^2 \{(1-t) - it\}' dt$$

$$= \int_0^1 \{1 + 2(i-1)t - 2it^2\}(1-i)dt + \int_1^0 \{1 - 2(i+1)t + 2it^2\}(-1-i)dt$$

$$=(1-i)\left(1+i-1-\frac{2i}{3}\right)-(1+i)\left(-1+i+1-\frac{2i}{3}\right)$$

$$=(1-i)\frac{i}{3}-(1+i)\frac{i}{3}=\frac{2}{3}$$

(iii) Here the path of integration is the upper half of the unit circle with centre at the origin, so we take a parametric form $\gamma(\theta)=e^{i\theta},\pi\le\theta\le 0$ and thus $\int_\gamma f=\int_\pi^0 e^{2i\theta}\left(e^{i\theta}\right)'d\theta=i\int_\pi^0 e^{3i\theta}d\theta=\frac{i}{3i}e^{3i\theta}\Big|=\frac{2}{3}$

Observation While performing integration in the above example, the different paths are followed and noticed that the result is invariant. So a question arises, is the complex integration path independent? A definite answer in this regard will be attained in Section 1.4.

1.3 Properties of complex integrals

Let γ be a smooth curve and f, g are complex functions continuous on γ, then

(a) $\int_\gamma(\alpha f+\beta g)\,dz=\alpha\int_\gamma fdz+\beta\int_\gamma gdz;\alpha,\beta\equiv$ constants

(b) $\int_{-\gamma} fdz=-\int_\gamma fdz,-\gamma$ denotes orientation of γ in opposite direction.

(c) If $\gamma=[\gamma_1\gamma_2]$, where γ_1 and γ_2 are smooth curves,

$$\int_\gamma fdz=\int_{\gamma_1} fdz+\int_{\gamma_2} fdz$$

(d) $\left|\int_\gamma fdz\right|\le\int_\gamma |f|d\sigma$

where σ is the arc length from the initial point of γ to any arbitrary point there.

(e) $\left|\int_\gamma fdz\right|=\left|\int_a^b f(\lambda(t))\lambda'(t)\,dt\right|\le\int_a^b\left|f(\lambda(t))\right|\left|\lambda'(t)\right|dt\le ML$

where $\lambda(t), a \le t \le b$ is a parametric form of the curve γ, $m = \max|f(z)|, z\varepsilon\gamma$ and L = length of the curve γ.

Theorem 1 Let *f* be continuous on a domain D, F is a primitive on D and γ be any curve lying in D with α, β as its end points, then

$$\int_\gamma f(z)dz = F(\beta) - F(\alpha)$$

Proof. We consider a parametric form of the given curve $\gamma : z = \lambda(t), a \le t \le b$ with $\lambda(a) = \alpha$ and $\lambda(b) = \beta$. Thus following (2)

$$\int_\gamma f(z)dz = \int_a^b F'(\lambda(t))\lambda'(t)dt$$

$$= \int_a^b \frac{d}{dt}F(\lambda(t))dt = F(\lambda(b)) - F(\lambda(a)) = F(\beta) - F(\alpha)$$

Note The above theorem shows that integral is path independent, but mind that the integrand *f* must possess antiderivative in a region enclosed by the paths with initial point α and terminal point β. The converse of the above theorem is also true.

Theorem 2 Let *f* be continuous on a domain D and $\int fdz$ is path independent on D. Then *f* has an antiderivative on D.

Proof. Let $F(z) = \int_{z_0}^{z} f(z)dz$ and by given hypothesis it is path independent. Here

$F(z + \Delta z) - F(z) = \int_z^{z+\Delta z} f(\varsigma)d\varsigma$ and so we can write

$$-f(z) + \frac{F(z + \Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\int_z^{z+\Delta z}\{f(\varsigma) - f(z)\}d\varsigma$$

Applying property (e) of complex integral on the r.h.s. of (3) we find

$$\left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z}\{f(\varsigma)-f(z)\}\,d\varsigma\right| \le \frac{1}{|\Delta z|}|\Delta z|\max_{\varsigma\in\gamma_{\Delta z}}|f(\varsigma)-f(z)| \to 0, \text{ as } \Delta z \to 0 \quad (4)$$

where $\gamma_{\Delta z}$ is a curve connecting z_0 to $z+\Delta z$.

Thus, taking modulus in (3) we obtain on using (4) as $\Delta z \to 0$

$$F'(z) = f(z), z \in D.$$

Example 2 Evaluate

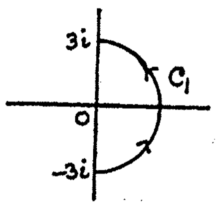
(i) $\int_{C_1}\frac{dz}{z}, C_1$ the right half of the circle $|z| = 3$, starting from $-3i$ to $3i$.

(ii) $\int_{C_2}\frac{dz}{z}, C_2$ the left half of the circle $|z| = 3$, starting from $3i$ to $-3i$.

(iii) $\int_{|z|=3}\frac{dz}{z}$ the circle $|z| = 3$ is positively oriented.

Here we put $z = 3e^{i\theta}$ and find $\int\frac{dz}{z} = i\theta$

In (i) we take the branch $-\pi/2 \le \theta \le \frac{\pi}{2}$ and obtain



$$\int_{C_1}\frac{dz}{z} = i\theta\Big|_{-\pi/2}^{\pi/2} = i\pi$$

Figure 3

In evaluating (ii), the branch we choose is $\pi/2 \le \theta \le \frac{3\pi}{2}$

and hence $\int_{C_2}\frac{dz}{z} = i\theta\Big|_{\pi/2}^{3\pi/2} = i\pi$

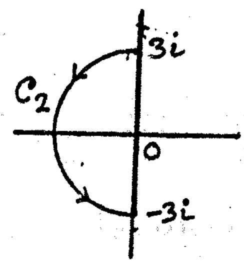


Figure 4

Finally in evaluating (iii) we choose the branch $-\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

Then we can express

$$\int_{|z|=3} \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} = i\pi + i\pi = 2i\pi$$

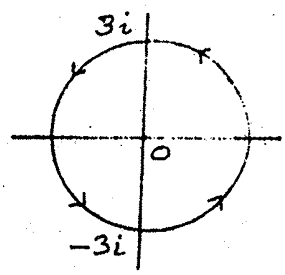


Figure 5

Note: We say a simple closed contour is positively oriented, if at the time of traversing the curve it's interior always lies to the left. For example, a circle oriented in the anti-clockwise direction is positively oriented. Hence forth a simple closed contour, until otherwise stated, is assumed to be positively oriented.

1.4 Cauchy's Integral Theorem

In this section we draw attention to a theorem which is a matter of central importance in complex analysis. It is concerned with the result

$$\int_{\gamma} f(z)\,dz = 0$$

where γ is a closed contour. Such an interesting result was first initiated by A.L. cauchy in 1825. He proved the theorem under the following conditions:

(i) f' is continuous on γ.

(ii) f is analytic in the interior of γ.

Later in 1831, F.G.B. Goursat modified the condition (i) of Cauchy's version and showed that continuity of f' on γ is not necessary, only f is to be taken continuous on γ.

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Theorem 3 [Cauchy's integral theorem] (modified version)

Let γ be a simple closed contour and D be its interior, $f(z)$ is continuous in the closed domain $\bar{D}$ and analytic in D. Then

$$\int_{\gamma} f(z)\,dz = 0 \tag{5}$$

Proof. We first prove the theorem for the case where γ is the contour Δ of a triangle.

Let be γ the triangle ABC and we denote $I = \int_{\gamma} f(z)\,dz$, supposed to be not equal to zero. We join the mid points of the sides of the triangle ABC and obtain four congruent triangles $\gamma_1, \gamma_2, \gamma_3$ and γ_4, say.

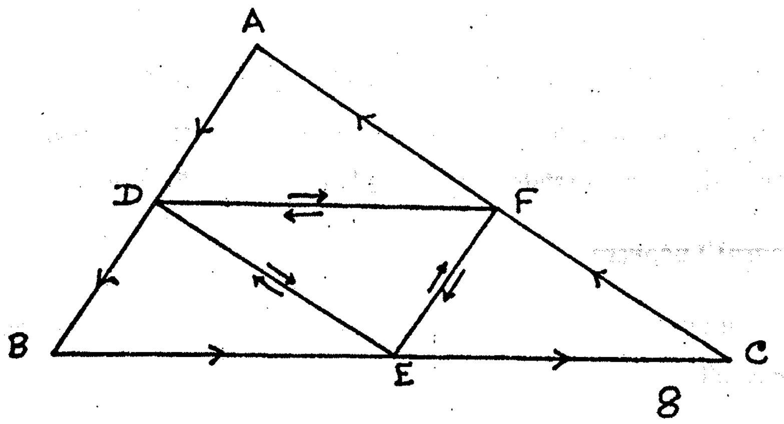

Figure 6

Then calculating we find

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz$$

Taking moduli

$$\left|\int_{\gamma} f(z)\,dz\right| \le \sum_{j=1}^{4} \left|\int_{\gamma_j} f(z)\,dz\right| \le 4\left|\int_{\gamma_1^*} f(z)\,dz\right|$$

where

$$\left|\int_{\gamma_1^*} f(z)\,dz\right| = \max\left|\int_{\gamma_j} f(z)\,dz\right|, j = 1,2,3,4$$

Dividing the triangle ABC likewise n times we obtain

$$\left|\int_{\gamma} f(z)\,dz\right| \le 4^n \left|\int_{\gamma_n^*} f(z)\,dz\right| \qquad (6)$$

Let us denote

γ and it's interior by $\bar{\gamma}_0$.

γ_j^* and it's interior by $\bar{\gamma}_j, j = 1,2,\dots,n$

Proceeding in this way we obtain a collection of closed nested sets $\bar{\gamma}_k$ satisfying

$$\bar{\gamma}_0 \supset \bar{\gamma}_1 \supset \bar{\gamma}_2 \supset \dots \supset \bar{\gamma}_n \supset \dots$$

If l be the perimeter of the triangle γ then the length between any two points belonging to $\bar{\gamma}_0$ will always be less than $\frac{l}{2}$ and in general the distance between any two points of $\bar{\gamma}_n$ will be less than $l/2^{n+1}$, which tends to zero as n tending to infinity. So applying Bolzano-Weier-strass theorem we conclude that there exists a unique point ς common to all the $\bar{\gamma}_k$'s. Since $f(z)$ is analytic at this point, we have

$$f(z) = f(\varsigma) + f'(\varsigma)(z-\varsigma) + \eta(z,\varsigma)(z-\varsigma) \quad (7)$$

where given $\varepsilon > 0$ there exists a $\delta(\varepsilon)$ such that $|\eta| < \varepsilon$ whenever $|z-\varsigma| < \delta$.

From (7) on integration, utilizing the fact that

$$\int_{\gamma_n} dz = 0 \text{ and } \int_{\bar{\gamma}_n} z\,dz = 0$$

We finally achieve

$$\left|\int_{\gamma_n^*} f(z)\,dz\right| = \left|\int_{\gamma_n^*} \eta(z,\varsigma)(z-\varsigma)\,d\varsigma\right|$$

$$< \varepsilon \sup_{z\in\dot{\gamma}_n} |z-\varsigma| \text{ length of } \dot{\gamma}_n < \varepsilon.\frac{l}{z^{n+1}}.\frac{l}{z^n} = \frac{\varepsilon}{2}\frac{l^2}{4^n}$$

Thus from (6),

$$|I| \le 4^n \left| \int_{\dot{\gamma}_n} f(z)\,dz \right| < \frac{\varepsilon l^2}{2}$$

But ε is arbitrarily small, so $I = 0$ and the theorem is proved.

Next we consider γ to be a closed polygon. If it happens to be simple, it can always be triangulated by joining all its vertices to a common point lying inside the polygon, see figure 7. Here the integration over the polygon ABCDEFGA is equivalent to integration over the seven triangles, as integrations on the common sides of the adjacent triangles are opposite in direction, which clearly reduce to zero by virtue of the first case proved earlier.

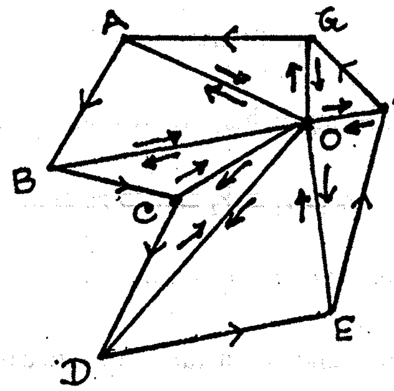

Figure 7

In case the polygon is not simple integration over γ may lead to integration over finite number of triangles and may be over finite number of straight line segments twice but in opposite direction and hence truth of the theorem follows.

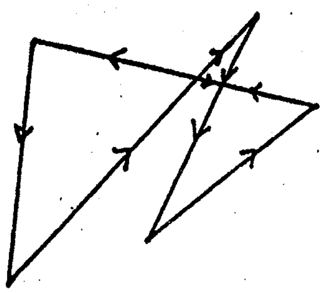

Figure 8

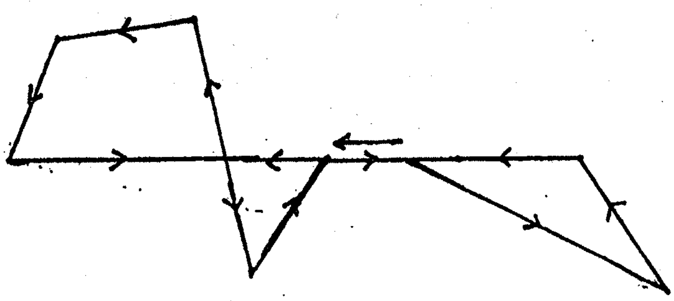

Figure 9

Finally, when γ is an arbitrary simple closed curve, we take a simple polygon P_ε lying within γ and each point of which is at a distance less than ε from γ. Then it is clear from earlier conclusions,

$$\int_{P_\varepsilon} f(z)\,dz = 0 \tag{8}$$

Now it can be shown following some sophisticated theories that

$$\lim_{\varepsilon \to 0} \int_{P_\varepsilon} f(z)\,dz = \int_{\gamma} f(z)\,dz \tag{9}$$

combining (8) and (9), we get the result.

Simply and Multiply connected domains

Definition A domain D is said to be simply connected for any simple closed contour γ belonging in D, the interior of γ also belongs to D. The followings are a few examples of simply connected domains.

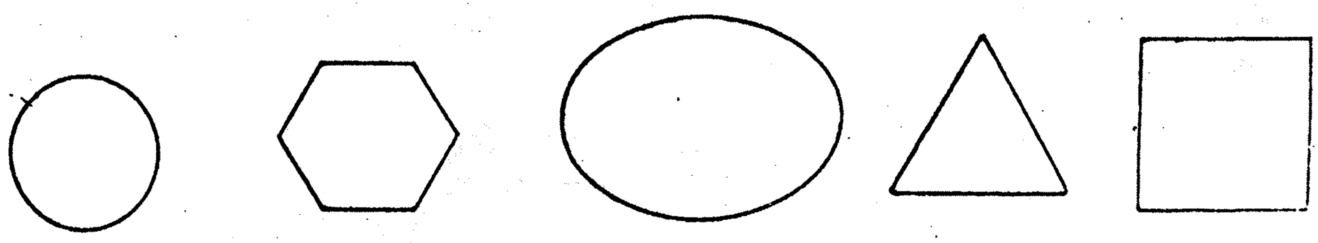

Figure 10

Definition The domains which are not simply connected are called multiply connected. Some of these are as follows:

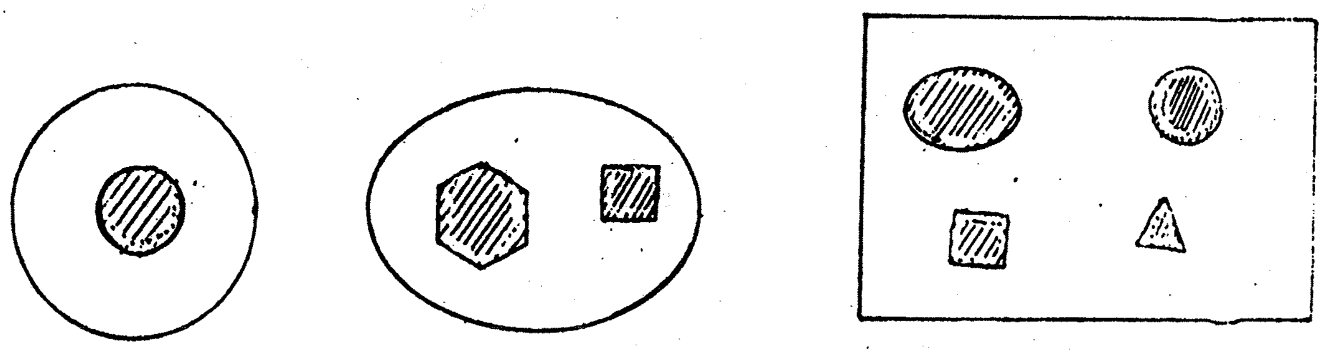

Figure 11

Theorem 4

[Cauchy's integral theorem for a multiply connected domain]

Let γ be a simple closed curve and a function $f(z)$ is continuous on γ and analytic inside γ except the regions enclosed by the simple closed curves $\gamma_1,...,\gamma_k$, then

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + ... + \int_{\gamma_k} f(z)\,dz \tag{10}$$

Proof. Let us join the curve γ with the closed curves γ_j by the curves $l_j, j=1,...,k$. We consider the region enclosed by the closed curve C formed by traversing $l_1, -\gamma_1, -l_1, P_1P_2, l_2, -\gamma_2, -l_2, P_2P_3, l_3, -\gamma_3, -l_3, ..., l_{k-1}, -\gamma_{k-1}, -l_{k-1}, P_{k-1}P_k, l_k, -\gamma_k, -l_k, P_kP_1$ and by the given hypothesis $f(z)$ is analytic there, hence

$$\int_{C} f(z)\,dz = 0$$

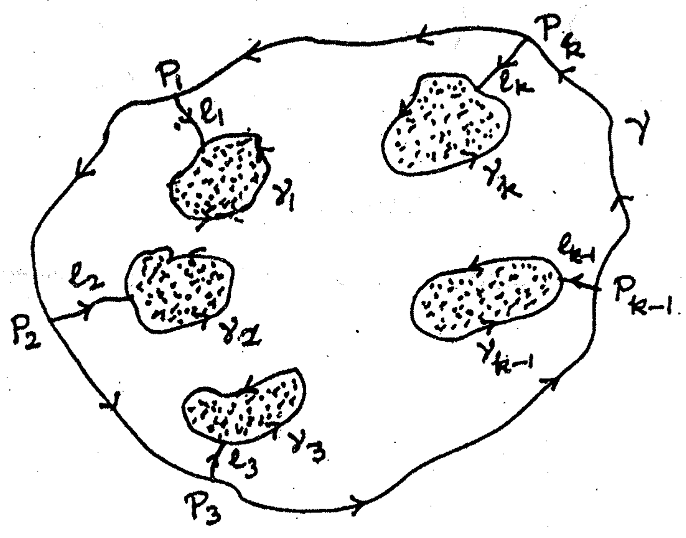

Figure 12

following Cauchy's integral theorem and written explicitly it becomes,

$$\int_{P_1P_2+P_2P_3+...+P_{k-1}P_k+P_kP_1} f(z)\,dz + \int_{-\gamma_1-\gamma_2-\gamma_3-...-\gamma_{k-1}-\gamma_k} f(z)\,dz = 0$$

i.e. $$\int_{\gamma} f(z)\,dz = \sum_{j=1}^{k} \int_{\gamma_j} f(z)\,dz$$

1.5 Cauchy's Integral formula and It's consequences

This formula illutrates how the value of a function at a point can be determined in terms of its values on the closed curve enclosing that point.

Theorem 5 [Cauchy's Integral Formula] Let γ be a simple closed contour and let $f(z)$ be

(i) continuous in the interior and on γ

(ii) regular in the interior of γ

then, for any point z_0 lying in the interior of γ

$$f(z_0)=\frac{1}{2\pi i}\int_{\gamma}\frac{f(\varsigma)}{\varsigma-z_0}d\varsigma \tag{11}$$

[The integral on the right is called Cauchy's integral]

Proof. Let us denote Interior of γ by D. Clearly the function

$$\psi(z)=\frac{f(z)}{z-z_0}$$

is regular on the domain $\tilde{D}=D-\{z_0\}$.

Let us choose ρ such that $C_\rho:|z-z_0|=\rho$ lies completely inside γ.

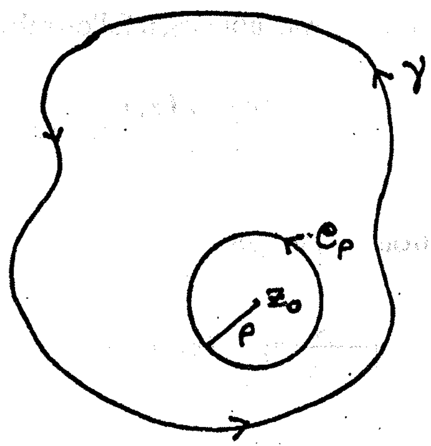

Figure 13

Now using Theorem 4, we find that

$$\int_{\gamma}\psi(\varsigma)d\varsigma=\int_{C_\rho}\psi(\varsigma)d\varsigma \tag{12}$$

To establish the theorem, we consider

$$\left|\int_{C_\rho}\frac{f(\varsigma)}{\varsigma-z_0}d\varsigma-2\pi i f(z_0)\right|=\left|\int_{C_\rho}\frac{f(\varsigma)}{\varsigma-z_0}d\varsigma-f(z_0)\int_{C_\rho}\frac{d\varsigma}{\varsigma-z_0}\right|$$

$$= \left| \int_{C_\rho} \frac{f(\varsigma) - f(z_0)}{\varsigma - z_0} d\varsigma \right|$$

Since $f(z)$ is continuous at z_0; given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. If we choose $\rho < \delta$, then

$$\left| \frac{f(\varsigma) - f(z_0)}{\varsigma - z_0} \right| = \frac{|f(\varsigma) - f(z_0)|}{|\varsigma - z_0|} < \frac{\varepsilon}{\rho}, \varsigma \in C_\rho$$

Hence

$$\left| \int_{C_\rho} \frac{f(\varsigma) - f(z_0)}{\varsigma - z_0} d\varsigma \right| < \frac{\varepsilon}{\rho} 2\pi\rho = 2\pi\varepsilon$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\int_{C_\rho} \frac{f(\varsigma) - f(z_0)}{\varsigma - z_0} d\varsigma = 0$$

So from (13) we get

$$\int_{C_\rho} \frac{f(\varsigma)}{\varsigma - z_0} d\varsigma = 2\pi i f(z_0) \qquad (14)$$

combining (14) with (12) the theorem follows.

Example 3. Evaluate $\int_\gamma \frac{dz}{z^2 + 4}$ where γ is the circle $|z| = 3$ oriented positively.

Here we can express $\quad I = \int_{|z|=3} \frac{1}{(z-2i)(z+2i)} dz = \int_{\gamma_1} \frac{\frac{1}{z+2i}}{z-2i} dz + \int_{\gamma_2} \frac{\frac{1}{z-2i}}{z+2i} dz$

[Using Theorem 4]

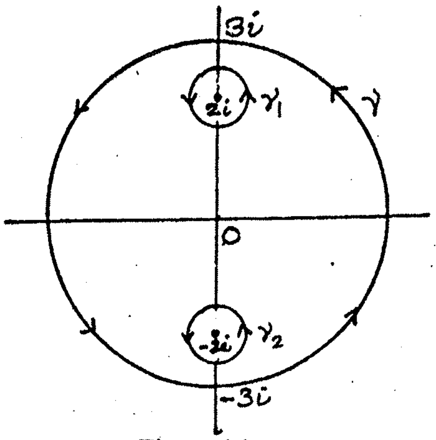

Figure 14

where γ_1 is the circle $|z-2i| = \frac{1}{2}$ and γ_2 is the circle $|z+2i| = \frac{1}{2}$

Hence applying the result of Theorem 5, we obtain

$$I = \frac{1}{4i} - \frac{1}{4i} = 0$$

Example 4. Evaluate the integral

$$\int_{|z|=2} \frac{\sin(3\Pi z)}{(6z-1)(z-3)} dz$$

where the circle $|z| = 2$ is positively oriented.

Here the integrand is not analytic at the two points $z = \frac{1}{6}$ and $z = 3$ of which $z = \frac{1}{6}$ lies in the circle $|z| = 2$.

So, in order to apply the Cauchy's integral formula we express the given integral as

$$\int_{|z|=2} \frac{\frac{1}{6}\frac{\sin(3\Pi z)}{z-3}}{z-\frac{1}{6}} dz = \frac{1}{6} 2\Pi i \frac{\sin\left(3\Pi \cdot \frac{1}{6}\right)}{\frac{1}{6}-3}$$

$$= -\frac{2}{17}\Pi i.$$

The derivative of an analytic function

Theorem 6 Let C be a simple closed contour and R be the region consisting of C and all it's interior points. If $f(z)$ is analytic in R, then

$$f'(z) = \frac{1}{2\Pi i}\int_C \frac{f(\varsigma)}{(\varsigma - z)^2} d\varsigma \tag{15}$$

for all z lying in the interior of C.

Proof. We consider the difference quotient

$$\frac{f(z+h)-f(z)}{h} = \frac{1}{h}\left[\frac{1}{2\Pi i}\int_C \frac{f(\varsigma)}{\varsigma-(z+h)}d\varsigma - \frac{1}{2\Pi i}\int_C \frac{f(\varsigma)}{\varsigma - z}d\varsigma\right]$$

$$= \frac{1}{2\Pi i}\int_C \frac{f(\varsigma)}{(\varsigma - z - h)(\varsigma - z)}d\varsigma$$

Hence we shall have to show that the following expression tends to zero as $h \to 0$,

$$\frac{f(z+h)-f(z)}{h} - \frac{1}{2\Pi i}\int_C \frac{f(\varsigma)}{(\varsigma - z)^2}d\varsigma$$

$$= \frac{1}{2\Pi i}\int_C \left\{\frac{f(\varsigma)}{(\varsigma - z - h)(\varsigma - z)} - \frac{f(\varsigma)}{(\varsigma - z)^2}\right\}d\varsigma$$

$$= h \cdot \frac{1}{2\Pi i}\int_C \frac{f(\varsigma)}{(\varsigma - z - h)(\varsigma - z)^2}d\varsigma$$

This can be made arbitrarily small if we can show that the integral remains bounded as h tending to zero. For this sake, let $M = \max_{\varsigma \in c}|f(\varsigma)|$ and since the region enclosed by C is closed and bounded there exists a positive d such that the open disk of radius d with centre at z does not intersect C i.e., d is the minimum distance from z to the boundary C of the region in which z lies.

So that

$$|\varsigma - z| \geq d, \varsigma \in C$$

and $|\varsigma - z - h| \geq \big||\varsigma - z| - |h|\big| \geq d - |h|$

Then $\left|\dfrac{f(\varsigma)}{(\varsigma-z-h)(\varsigma-z)^2}\right| \le \dfrac{|f(\varsigma)|}{(|\varsigma-z|-|h|)|\varsigma-z|^2} \le \dfrac{M}{(d-|h|)d^2}$

which is clearly bounded as $h \to 0$, that is

$$f'(z)=\frac{1}{2\Pi i}\int_C \frac{f(\varsigma)}{(\varsigma-z)^2}d\varsigma$$

Now we show that there is a similar formula for $f''(z)$.

We consider

$$\frac{f'(z+h)-f'(z)}{h}=\frac{1}{h}\cdot\frac{1}{2\Pi i}\left[\int_C \frac{f(\varsigma)d\varsigma}{(\varsigma-z-h)^2}-\int_C \frac{f(\varsigma)}{(\varsigma-z)^2}d\varsigma\right]$$

$$=\frac{1}{2\Pi i}\int_C \frac{2(\varsigma-z)-h}{(\varsigma-z-h)^2(\varsigma-z)^2}f(\varsigma)d\varsigma$$

and hence,

$$\frac{f'(z+h)-f'(z)}{h}-\frac{1}{\Pi i}\int_C \frac{f(\varsigma)}{(\varsigma-z)^3}d\varsigma=\frac{1}{2\Pi i}\int_C\left\{\frac{2(\varsigma-z)-h}{(\varsigma-z-h)^2(\varsigma-z)}-\frac{2}{(\varsigma-z)^3}\right\}f(\varsigma)d\varsigma$$

$$=\frac{1}{2\Pi i}\int_C \frac{3(\varsigma-z)h-2h^2}{(\varsigma-z-h)^2(\varsigma-z)^3}f(\varsigma)d\varsigma$$

and that

$$\left|\frac{f'(z+h)-f'(z)}{h}-\frac{1}{\Pi i}\int_C \frac{f(\varsigma)}{(\varsigma-z)^3}d\varsigma\right|\le\frac{1}{2\Pi}\frac{(3M+2|h|)|h|}{(d-|h|)^2d^3}$$

Then making h tending to zero, we obtain

$$f''(z)=\frac{2!}{2\Pi i}\int_C \frac{f(\varsigma)}{(\varsigma-z)^3}d\varsigma \qquad (16)$$

and in general

$$f^{(n)}(z)=\frac{n!}{2\Pi i}\int_C \frac{f(\varsigma)}{(\varsigma-z)^{n+1}}d\varsigma, n=1,2,... \qquad (17)$$

where $f(\varsigma)$ is regular in the interior of a simple closed contour C and continuous in the interior and on C and z is any point lying in the interior of C. Applying the formula in (17), we observe that;

Example 5. Suppose γ is the circle $|z|=1$ with positive orientation, then

$$\int_\gamma \frac{e^{4z}}{\left(z-\frac{1}{2}\right)^3}dz=\frac{2\Pi i}{2!}4^2e^2=16e^2\Pi i$$

Example 6. Suppose γ be a simple closed contour enclosing a point ς, then

$$\int_\gamma \frac{dz}{z-\varsigma}=2\Pi i$$

and $n=2,3,...$

$$\int_\gamma \frac{dz}{(z-\varsigma)^n}=0$$

Example 7. If γ be a circle $|z-2i|=1$ with positive orientation, then

$$\int_\gamma \frac{dz}{(z^2+4)^3}=\int_\gamma \frac{\left(\frac{1}{z+2i}\right)^3}{(z-2i)^3}dz=\frac{2\Pi i}{2!}\left(-\frac{3i}{256}\right)=\frac{3\Pi}{256}$$

Cauchy's Inequalities If $f(z)$ is regular in $D:|z-\varsigma|<R$ and continuous in $\bar{D}$ and suppose $|f(z)|\le M$ for all $z\in D$, then

$$\left|f^{(n)}(\varsigma)\right|\le\frac{n!M}{R^n}, \qquad (18)$$

$n = 1, 2, \ldots$

Proof. Taking modulus in (17), we find that

$$\left|f^{(n)}(\varsigma)\right| = \frac{n!}{2\Pi}\left|\int_{|z-\varsigma|=r} \frac{f(z)}{(z-\varsigma)^{n+1}}dz\right|, r < R$$

$$\leq \frac{n!}{2\Pi}\int_0^{2\Pi} \frac{\left|f(\varsigma + re^{i\theta})\right|}{r^n}d\theta \leq \frac{n!M}{r^n},$$

since r is arbitrary, the result follows on $r \to R-0$.

Theorem 7. [Liouville's Theorem] If $f(z)$ is an entire function and bounded for all z, then $f(z)$ is constant throughout the complex plane.

Alternate statement: Bounded entire function is constant.

Proof. Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then following Cauchy's inequality, for any $\zeta \in \mathbb{C}$

$$|f'(\varsigma)| \leq \frac{M}{R}$$

This holds for any $R > 0$ since f is entire. Now letter $R \to \infty$, we have $|f'(\varsigma)| = 0$ and hence $f'(\varsigma) = 0$ for any $\zeta \in \mathbb{C}$. Thus $f(z) = k$ for some constant k and all $z \in \mathbb{C}$.

Usefulness of Liouville's theorem is realized in solving a difficult problem in algebra stated below.

Theorem 8 [Fundamental Theorem of Algebra]

Every nonconstant polynomial has a root in $\mathbb{C}$.

Proof. Let us consider a polynomial

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0,$$

with $a_n \neq 0$. suppose there is no such $z_0 \in \mathbb{C}$ for which $P(z_0) = 0$. Then

$$f(z)=\frac{1}{P(z)}$$

is entire. Now we show that $f(z)$ is bounded.

We choose R large enough so that

$$\left|\frac{a_{n-j}}{z^j}\right|<\frac{|a_n|}{2n} \text{ whenever } |z|>R$$

for $j = 1,2,...n$.

Since

$$|P(z)|\geq|z|^n\left||a_n|-\left|\frac{a_{n-1}}{z}+\frac{a_{n-2}}{z^2}+...+\frac{a_0}{z^n}\right|\right|$$

$$\geq|z|^n\left||a_n|-\left|\frac{a_{n-1}}{z}\right|-\left|\frac{a_{n-2}}{z^2}\right|-....-\left|\frac{a_0}{z^n}\right|\right|$$

$$>|z|^n\left||a_n|-\frac{|a_n|}{2n}-\frac{|a_n|}{2n}-...-\frac{|a_n|}{2n}\right|$$

$$=\frac{|a_n|}{2}|z|^n,$$

$$\frac{1}{|P(z)|}<\frac{2}{|a_n||z|^n}<\frac{2}{|a_n|R}$$

i.e., if $|z|>R, \frac{1}{P(z)}$ is bounded. But $\frac{1}{P(z)}$ is continuous, so $\frac{1}{P(z)}$ is bounded in $|z|\leq R$ also.

It follows that $\frac{1}{P(z)}$ is a bounded entire function on $\mathbb{C}$. By Liouville's theorem, $f(z)$ is a constant function, which is true only if $n = 0$. That is, the polynomial is constant if it has no zero on $\mathbb{C}$.

Theorem 8. [Morera's Theorem] If $f(z)$ is continuous on a domain D and

$$\int_{\gamma} f(z)\,dz = 0$$

for every simple closed contour γ in D, then $f(z)$ is analytic in D.

Proof. Let γ be any simple closed contour lying in D and z_0 be any fixed point, by given hypothesis

$$\int_{\gamma} f(z)\,dz = 0$$

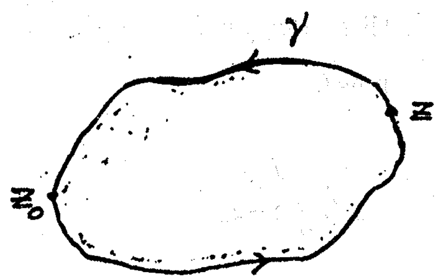

Figure 15

which can written equivalently as,

$$\int_{\gamma_1} f(z)\,dz + \int_{-\gamma_2} f(z)\,dz = 0 \text{ i.e., } \int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

where γ_1 and γ_2 are the simple curves obtained on traversing from z_0 to z (any arbitrary point on γ) clockwise and anti-clockwise respectively.

So that $\int_{z_0}^{z} f(z)\,dz$

is path-independent. Thus by theorem 2, there exists an analytic function $F(z)$ with $F'(z) = f(z)$ for all z in D and since derivative of an analytic function is analytic, $f(z)$ is analytic in D.

This theorem is also known as "Converse of Cauchy's Integral Theorem". Next article deals with a fundamental property of analytic functions.

1.6 Maximum Modulus Principle

Theorem 9. Let $f(z)$ be analytic in a domain D and continuous on $\bar{D}$, then maximum of $|f(z)|$ is attained on the boundary of *D* unless it is a constant.

Proof. Given $f(z)$ is continuous on the closed domain $\bar{D}$ and hence $|f(z)|$ being continuous must attain its maximum value some where on $\bar{D}$. Suppose $|f(z)| \le M$ for $z \varepsilon \bar{D}$ and suppose on the contrary, let $|f(z)| \le M$ holds for some z_0 lying in *D*, not on the foundation of *D*. We consider a disk $K : |z - z_0| \le R$ with centre at z_0 and radius so that it lies entirely in *D*; let γ_ρ be the circle $|z - z_0| = \rho \le R$, oriented positively. From Cauchy's integral formula, we know that

$$f(z_0) = \frac{1}{2\Pi i} \int_{\gamma_\rho} \frac{f(z)}{z - z_0} dz$$

converting γ_ρ parametrically by $z = z_0 + \rho e^{i\theta}, 0 \le \theta \le 2\Pi$, we find

$$f(z_0) = \frac{1}{2\Pi i} \int_0^{2\Pi} f\left(z_0 + \rho e^{i\theta}\right) d\theta$$

and so

$$M = |f(z_0)| \le \frac{1}{2\Pi} \int_0^{2\Pi} \left|f\left(z_0 + \rho e^{i\theta}\right)\right| d\theta \le M$$

since $\left|f\left(z_0 + \rho e^{i\theta}\right)\right| \le M$. This implies that

$$M = \frac{1}{2\Pi} \int_0^{2\Pi} \left|f\left(z_0 + \rho e^{i\theta}\right)\right| d\theta$$

i.e., $$0 = M - \frac{1}{2\Pi} \int_0^{2\Pi} \left|f\left(z_0 + \rho e^{i\theta}\right)\right| d\theta$$

$$= \frac{1}{2\Pi} \int_0^{2\Pi} \left\{M - \left|f\left(z_0 + \rho e^{i\theta}\right)\right|\right\} d\theta \qquad (19)$$

Since $M - \left|f\left(z_0 + \rho e^{i\theta}\right)\right|$ is a continuous function of θ and

$$M - \left|f\left(z_0 + \rho e^{i\theta}\right)\right| \geq 0$$

for all $\theta \varepsilon [0, 2\Pi]$, it follows from (19) that

$$M = \left|f\left(z_0 + \rho e^{i\theta}\right)\right|$$

for all $\theta \varepsilon [0, 2\Pi]$ i.e. $|f(z)| = M$ for all $z \varepsilon \gamma_\rho$. But $\rho \leq R$ is arbitrary, it follows that $|f(z)| = M$ for all $z \varepsilon D$. So $f(z)$ is constant for all $z \varepsilon D$.

1.7 Power Series: Taylor's Series and Laurent's series

Power series Inifinite series of the form

$$a_0 + a_1(z - z_0) + ... + a_n(z - z_0)^n + ... \quad (20)$$

is called power series about the point z_0, where $a_0, a_1, a_2, ...$ are complex numbers. Many interesting properties of analytic function can easily be established by the use of power series.

The region of convergence $|z - z_0| < R$ is determined by the formulae

(i) $R = \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$: Cauchy-Hadamard formula

(ii) $R = \lim\limits_{n \to \infty} \left|\dfrac{a_n}{a_{n+1}}\right|$: so called Ratio Test

if the respective limit exists. The number R is called the radius of convergence and the set $|z - z_0| = R$ is known as the circle of convergence of the power series. Some useful criteria in this connection are the following:

(i) If $R = 0$, the series diverges for all $z \neq z_0$.

(ii) If $0 < R < \infty$, the series converges absolutely for all z satisfying $|z - z_0| < R$ and diverges for all z satisfying $|z - z_0| > R$.

(iii) If $R = \infty$, the series converges absolutely in the entire complex plane.

So far, the points on the circufetence of the circle i.e., for all z satisfying $|z-z_0| = R$ are not yet considered. The series

(i) may converge at all these points

(ii) may diverge at all these points

(iii) may converge at some of these points and diverge at the remaining points.

Uniformly Convergent Series of Complex Functions

Definition. A series $\sum_{k=1}^{\infty} f_k(z)$ of complex functions defined in a bounded closed domain D, converges uniformly in D if the sequence of partial sums

$$s_n(z) = \sum_{k=1}^{n} f_k(z)$$

converges uniformly in D.

We state without proof some useful theorems in this regard.

Theorem 10 A series $\sum_{k=1}^{\infty} f_k(z)$ of complex functions defined in a bounded closed domain D, converges uniformly in D if and only if given $\varepsilon > 0$ there exists a positive real number $N(\varepsilon)$ such that

$$\left|\sum_{k=n+1}^{n+p} f_k(z)\right| < \varepsilon, z \in D$$

for all $n > N, p \equiv +ve$ integer.

Theorem 11. Let $\{f_k(z)\}$ be a sequence of continuous functions defined in a bounded closed domain D. Suppose the series $\sum f_k(z)$ converges uniformly to a function $f(z)$ in D. Then $f(z)$ is continuous in D.

Theorem 12 Let $f_1(z), f_2(z), \ldots$ are all continuous on a simple contour γ and moreover if the series $\sum f_k(z)$ converges uniformly to $f(z)$ on γ, then

$$\sum_{k=1}^{\infty} \int_{\gamma} f_k(z)\,dz = \int_{\gamma} f(z)\,dz$$

Theorem 13. If $\{f_k(z)\}$ be a sequence of analytic functions in a bounded closed domain *D* and moreover if the series $\sum f_k(z)$ converges uniformly to $f(z)$ in *D*, then $f(z)$ is analytic in *D* and

$$\sum_{k=1}^{\infty} f_k^{(m)}(z) = f^{(m)}(z), (m) \text{ denotes mth derivative}$$

Weierstrass M-test Given a sequence $\{f_k\}$ of complex-valued functions defined in a bounded closed domain *D* and let $\sum_{k=1}^{\infty} M_k$ be a convergent series of non-negative real terms. Then the series $\sum_{k=1}^{\infty} f_k(z)$ converges uniformly and absolutely in *D* if $|f_k(z)| \le M_k$ hold.

Taylor's Series

Denefition. If $f(z)$ is analytic at a point $z = z_0$, the power series

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

is known as Taylor's series in the neighbourhood of the point $z = z_0$.

Theorem 14 [Taylor's theorem] Suppose $f(z)$ be analytic in a domain *D*. Then $f(z)$ can be represented in the neighbourhood of z_0 as a power series (with a radius of convergence at least *R*)

$$f(z) = \sum_{k=0}^{\infty} C_k (z - z_0)^k \quad \text{with } C_k = \frac{1}{2\Pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{k+1}} dz$$

where γ is the circle $|z - z_0| = R, R = Inf\{|z - z_0| : z\varepsilon\partial D\}$

Proof. Let γ_r be any circle $|z - z_0| = r, 0 < r < R$. Oriented positively and let ς be any point lying interior of γ_r. Then by Cauchy's integral formula (Th.5),

$$f(\varsigma)=\frac{1}{2\Pi i}\int_{\gamma_r}\frac{f(z)}{z-\varsigma}dz \tag{21}$$

The factor $1/(z-\varsigma)$ in the integrand can expanded into a geometric series

$$\frac{1}{z-\varsigma}=\frac{1}{(z-z_0)-(\varsigma-z_0)}$$

$$=\frac{1}{z-z_0}\left\{1-\frac{\varsigma-z_0}{z-z_0}\right\}^{-1}$$

$$=\frac{1}{z-z_0}\left\{1+\frac{\varsigma-z_0}{z-z_0}+\left(\frac{\varsigma-z_0}{z-z_0}\right)^2+...\right\}, \text{clearly } |\varsigma-z_0|<|z-z_0|$$

Hence

$$f(\varsigma)=\frac{1}{2\Pi i}\int_{\gamma_r}\frac{1}{z-z_0}\left\{f(z)+\frac{\varsigma-z_0}{z-z_0}f(z)+\left(\frac{\varsigma-z_0}{z-z_0}\right)^2 f(z)+...\right\}dz$$

$$=\frac{1}{2\Pi i}\int_{\gamma_r}\sum_{k=0}^{\infty}\frac{f(z)}{(z-z_0)^{k+1}}(\varsigma-z_0)^k dz \tag{22}$$

Here

$\left|\frac{f(z)}{(z-z_0)^{k+1}}(\varsigma-z_0)^k\right|\le\frac{M(\gamma_r)}{r}\left\{\frac{|\varsigma-z_0|}{r}\right\}^k$, where $M(\gamma_r)=\max_{z\in\gamma_r}|f(z)|$ and the geometric series $\sum_{k=0}^{\infty}\left\{\frac{|\varsigma-z_0|}{r}\right\}^k$ converges since $|\varsigma-z_0|/r<1$. Hence the series in the integrand of (22) converges uniformly by Weiestrass M-test. Moreover each term of the series is continuous, term-wise integration is permissible by Th.12. Therefore

$$f(\varsigma)=\sum_{k=0}^{\infty}\frac{(\varsigma-z_0)^k}{2\Pi i}\int_{\gamma_r}\frac{f(z)}{(z-z_0)^{k+1}}dz$$

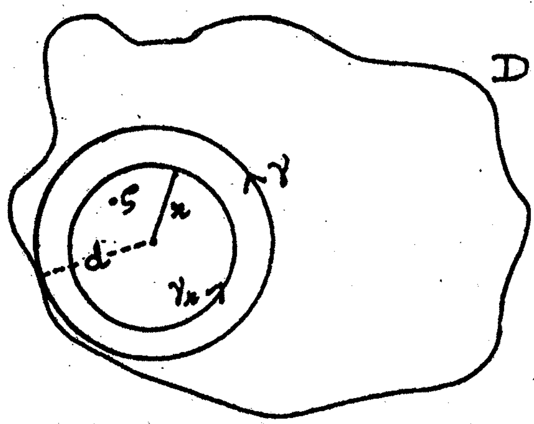

Figure 16

Since ς is rbitrary, $|\varsigma-z_0|=r, 0<r<R$

$$f(z)=\sum_{k=0}^{\infty}C_k(z-z_0)^k$$

where $C_k=\frac{1}{2\Pi i}\int_{\gamma}\frac{f(z)}{(z-z_0)^{k+1}}dz$

$$=\frac{f^{(k)}(z_0)}{k!}$$

following Cauchy's integral formula.

Example 8. Expand the following functions in form of a Taylor's series:

(i) e^z

(ii) e^{iz} in the neighbourhood of origin.

(iii) $\cosh z$ in the neighbourhood of origin.

(iv) $\frac{1}{1-z}$ in the neighbourhood of $z=2$.

(i) $f(z)=e^z$ is an entire function and hence it can expressed in the form a Taylor's series in the neighbourhood of any finite point. Now $f^{(n)}(0)=e^0=1$ for $n=0, 1, 2, \ldots$ Therefore,

$$e^z=1+z+\frac{z^2}{z!}+\frac{z^3}{z!}+\ldots=\sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

(ii) $g(z)=e^{iz}$. It is also an entire function. Similar arguments like (i) hold here too. We find that $f^{(n)}(0)=i^n$,

$n = 0, 1, 2, \ldots$ and

$$e^{iz} = \sum_{n=0}^{\infty} \frac{i^n}{n!} z^n$$

(iii) $\cosh z$ is an entire function. We have

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

$$= 1 + \frac{z^2}{z!} + \frac{z^4}{z!} + \ldots = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \text{ using (i)}$$

(iv) $\phi(z) = \dfrac{1}{1-z}$ is analytic in the whole complex plane except the point $z = 1$.

We now express

$$\frac{1}{1-z} = -\frac{1}{1+(z-2)}$$

$$= -1 + (z-2)^2 + \ldots, |z-2| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} (z-2)^n$$

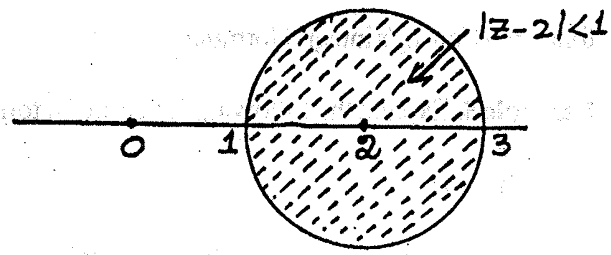

Figure 17

Example 9. Expand $f(z) = \dfrac{1}{(z-1)(z^2+4)}$ in the neighbourhood of $z = -\dfrac{3}{2}$.

We express $f(z)$ as

$$\frac{1}{(z-1)(z^2+4)} = \frac{1}{5}\left(\frac{1}{z-1} - \frac{z+1}{z^2+4}\right)$$

$$= \frac{1}{5}\left\{\frac{1}{z-1} - \left(\frac{1}{2} + \frac{1}{4i}\right)\frac{1}{z-20} - \left(\frac{1}{2} - \frac{1}{4i}\right)\frac{1}{z+2i}\right\} \quad (23)$$

Now

$$\frac{1}{z-1}=\frac{1}{z+\frac{3}{2}-\frac{5}{2}}=-\frac{2}{5}\left\{1-\left(z+\frac{3}{2}\right)\frac{2}{5}\right\}^{-1}$$

$$=-\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^{n+1}\left(z+\frac{3}{2}\right)^{n},\left|z+\frac{3}{2}\right|<\frac{5}{2}$$

Again

$$\frac{1}{z-2i}=\frac{1}{\left(z+\frac{3}{2}\right)-\left(2i+\frac{3}{2}\right)}$$

$$=-\left(2i+\frac{3}{2}\right)^{-1}\left\{1-\left(z+\frac{3}{2}\right)\left(2i+\frac{3}{2}\right)^{-1}\right\}^{-1}$$

$$=-\left(2i+\frac{3}{2}\right)^{-1}\sum_{n=0}^{\infty}\left(\frac{z+\frac{3}{2}}{2i+\frac{3}{2}}\right)^{n},\left|z+\frac{3}{2}\right|<\left|2i+\frac{3}{2}\right|=\frac{5}{2}$$

Likewise

$$\frac{1}{z+2i}=-\left(\frac{3}{2}-2i\right)^{-1}\sum_{n=0}^{\infty}\left(\frac{z+\frac{3}{2}}{\frac{3}{2}-2i}\right)^{n},\left|z+\frac{3}{2}\right|<\left|\frac{3}{2}-zi\right|=\frac{5}{2}$$

substitution all these expressions in (23), we find

$$f(z)=-\frac{1}{5}\left[-\frac{8}{25}+\frac{176}{(25)^{2}}\left(z+\frac{3}{2}\right)+\frac{2112}{(25)^{3}}\left(z+\frac{3}{2}\right)^{2}+\ldots\right],\left|z+\frac{3}{2}\right|<\frac{5}{2}$$

Laurent Series

Definition A Laurent series in the neighbourhood of α point z_0 is of the form

$$\sum_{k=-\infty}^{\infty} C_k (z-z_0)^k = \sum_{k=0}^{\infty} C_k (z-z_0)^k + \sum_{k=1}^{\infty} C_{-k} (z-z_0)^{-k} \quad (24)$$

i.e., it consists of two power series - one with non-negative powers and the other with negative powers. Let the radius of convergence of the first of the series on the right-hand side be R_2, that is, it converges for $|z-z_0|<R_2$ and the second series be convergent for $|z-z_0|>R_1$. For the convergence of Laurent series in (24) it is essential to have common region of convergence of the above two series.

If $R_1 > R_2$, there is a common region of convergence - the annulus $R_1 < |z-z_0| < R_2$, that is, the series (24) converges to an analytic funtion

$$f(z) = f_1(z) + f_2(z) = \sum_{k=-\infty}^{\infty} C_k (z-z_0)^k$$

for, $R_1 < |z-z_0| < R_2$

If $R_1 < R_2$, there is no such common region of convergence and hence the series in (24) converges nowhere to any function. Conversely, it can be seen in the next theorem that any function analytic in an annulus can represented by a Laurent series.

Laurent's Theorem

Theorem 15. Given $f(z)$ be analytic in the annular region $D: R_1 < |z-z_0| < R_2$. Then $f(z)$ can be represented uniquely in D by a Laurent series in the neighbourhood of z_0:

$$f(z) = \sum_{k=-\infty}^{\infty} C_k (z-z_0)^k \text{ with } C_k = \frac{1}{2\Pi i}\int_\gamma \frac{f(\varsigma)}{(\varsigma-z_0)^{k+1}} d\varsigma$$

where γ is an arbitrary positively oriented closed contour lying in D and enclosing the point z_0.

Proof. We choose real numbers R_1' and R_2' such that $R_1 < R_1' < R_2' < R_2$. Let γ_1 and γ_2 be two positively oriented closed contours defined by

$$\gamma_1 : |\varsigma - z_0| = R_1' \text{ and } \gamma_2 : |\varsigma - z_0| = R_2'$$

respectively. A straight forward application of Cauchy's integral formula for multiplying connected domain shows that

$$f(z) = \frac{1}{2\Pi i}\int_{\gamma_2}\frac{f(\varsigma)}{\varsigma - z}d\varsigma - \frac{1}{2\Pi i}\int_{\gamma_1}\frac{f(\varsigma)}{\varsigma - z}d\varsigma \qquad (26)$$

for all $z \in \mathbb{C}$ satisfying $R_1' < |z - z_0| < R_2'$.

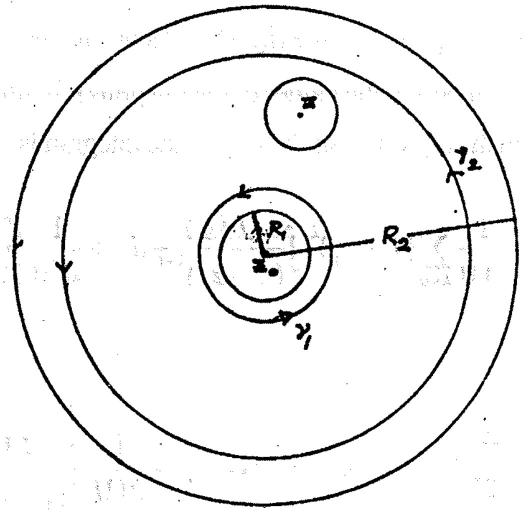

Figure 18.

But in the integrand of the first integral we write

$$\frac{1}{\varsigma - z} = \frac{1}{\varsigma - z_0}\left[1 - \frac{z - z_0}{\varsigma - z_0}\right]^{-1}$$

$$= \frac{1}{\varsigma - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\varsigma - z_0}\right)^n$$

since $\left|\frac{z - z_0}{\varsigma - z_0}\right| < 1$ and moreover the infinite series converges uniformly in ς, for values of ς that lie on the circle γ_2.

Also for the second integral we express

$$-\frac{1}{\varsigma - z} = \frac{1}{z - z_0}\left[1 - \frac{\varsigma - z_0}{z - z_0}\right]^{-1}$$

$$= \frac{1}{z - z_0}\sum_{n=0}^{\infty}\left(\frac{\varsigma - z_0}{z - z_0}\right)^n$$

as $\left|\frac{\varsigma - z_0}{z - z_0}\right| < 1$ and the series converges uniformly in ς for all ς satisfying $|\varsigma| = R_1'$ i.e., when $\varsigma \varepsilon \gamma_1$. Now from (26) it follows on integrating term-wise that

$$f(z)=\frac{1}{2\Pi i}\sum_{k=0}^{\infty}(z-z_0)^k\int_{\gamma_2}\frac{f(\varsigma)}{(\varsigma-z_0)^{k+1}}d\varsigma+\frac{1}{2\Pi i}\sum_{k=0}^{\infty}(z-z_0)^{-k-1}\int_{\gamma_1}\frac{f(\varsigma)}{(\varsigma-z_0)^{-k}}d\varsigma$$

$$=\frac{1}{2\Pi i}\sum_{k=0}^{\infty}(z-z_0)^k\int_{\gamma_2}\frac{f(\varsigma)}{(\varsigma-z_0)^{k+1}}d\varsigma+\frac{1}{2\Pi i}\sum_{k=1}^{\infty}(z-z_0)^{-k}\int_{\gamma_1}\frac{f(\varsigma)}{(\varsigma-z_0)^{-k+1}}d\varsigma$$

We note that the integrands in the right hand side are analytic in the annulus $R_1' < |z-z_0| < R_2'$ and so by virtue of Cauchy's integral theorem the values of the integrals remain unaltered under an arbitrary deformation of the contours of integration in the region of analyticity of the integrands. This helps us in writing

$$f(z)=\frac{1}{2\Pi i}\sum_{k=0}^{\infty}(z-z_0)^k\int_{\gamma}\frac{f(\varsigma)}{(\varsigma-z_0)^{k+1}}d\varsigma+\frac{1}{2\Pi i}\sum_{k=1}^{\infty}(z-z_0)^{-k}\int_{\gamma}\frac{f(\varsigma)}{(\varsigma-z_0)^{-k+1}}d\varsigma$$

and that

$$f(z)=\sum_{k=-\infty}^{\infty}C_k(z-z_0)^k \text{ where } C_k=\frac{1}{2\Pi i}\int_{\gamma}\frac{f(\varsigma)}{(\varsigma-z_0)^{k+1}}d\varsigma$$

where γ is an arbitrary closed contour lying in the annulus $R_1 < |z-z_0| < R_2$ and enclosing the point z_0.

Uniqueness

Let us suppose that $f(z)$, analytic in the annulus $R_1 < |z-z_0| < R_2$, possesses two expansions in the neighbour of z_0

$$f(z)=\sum_{k=-\infty}^{\infty}c_k(z-z_0)^k=\sum_{k=-\infty}^{\infty}d_k(z-z_0)^k$$

which converge uniformly in the annulus. Multiplying both sides by $(z-z_0)^{-m-1}$, where m is a fixed integer, we get

$$\sum_{k=-\infty}^{\infty}c_k(z-z_0)^{k-m-1}=\sum_{k=-\infty}^{\infty}d_k(z-z_0)^{k-m-1}$$

Integrating term-wise over the closed contour γ taking into consideration

$$\int_{\gamma}(z-z_0)^j\,dz=\begin{cases}0, j\neq -1\\ 2\Pi i, j=-1\end{cases}, j\equiv \text{integer}$$

we find that $c_m = d_m$ for each integer m. Hence the uniqueness of the two series follows.

Example 10. Find the Taylor or, Laurent series expansion for the function

$$f(z)=\frac{z}{z^2-3z+z},$$

in the following region of analyticity:

(i) $|z|<1$ (ii) $1<|z|<2$ and (iii) $|z|>2$

We express $f(z)$ as

$$f(z)=\frac{z}{z^2-3z+2}$$

$$=\frac{z}{(z-1)(z-2)}$$

$$=\frac{2}{z-2}-\frac{1}{z-1}$$

(i) If $|z|<1$,

$$f(z)=\frac{1}{1-z}-\frac{2}{2-z}=\frac{1}{1-z}-\frac{1}{1-\frac{z}{2}}$$

Now, $\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n, |z|<1$ and $\frac{1}{1-\frac{z}{2}}=\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n, \left|\frac{z}{2}\right|<1$

Thus,

$$f(z)=\sum_{n=0}^{\infty} z^n-\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n, |z|<1$$

$$=\sum_{n=0}^{\infty}\left(1-\frac{1}{2^n}\right)z^n$$

(ii) $1<|z|<2$

$$f(z)=-\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}-\frac{1}{1-\frac{z}{2}}$$

But, $\frac{1}{1-\frac{1}{z}}=\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n, \left|\frac{1}{z}\right|<1$ and $\frac{1}{1-\frac{z}{2}}=\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n, \left|\frac{z}{2}\right|<1$

and hence, $f(z)=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}-\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n, 1<|z|<2$

$$=-\sum_{n=0}^{\infty}\frac{z^n}{2^n}-\sum_{n=1}^{\infty}\frac{1}{z^n}$$

(iii) $|z|>2$

Here we rewrite $f(z)$ as

$$f(z)=-\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}+\frac{2}{z}\cdot\frac{1}{1-\frac{2}{z}}$$

Clearly, $\frac{1}{1-\frac{1}{z}}=\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n, 1<|z|$ and $\frac{1}{1-\frac{2}{z}}=\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n, 2<|z|$

Therefore, $f(z) = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^{n+1}, |z| > 2$

$$= \sum_{n=0}^{\infty} \frac{2^{n+1} - 1}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{2^n - 1}{z^n}$$

Example 11. Find the Laurent series expansion for $\frac{z-2}{z+1}$ in the neighbourhood of $z = -1$ and specify the region in which it converges.

Here we can express the given function as

$$\frac{z-2}{z+1} = \frac{z+1-3}{z+1}$$

$$= 1 - \frac{3}{z+1}$$

it converges everywhere for $z \neq -1$.

Example 12. Find the Taylor series expansion of sin*z* in the neighbourhood of $z = 0$.

$$f(z) = \sin z$$

$$f^{I}(z) = \cos z, f^{II}(z) = -\sin z, f^{III}(z) = -\cos z, f^{IV}(z) = \sin z$$

which indicate that

$$f^{2n}(0) = 0 \text{ and } f^{(2n+1)}(0) = (-1)^n, n = 0, 1, 2, \ldots$$

So, using the formula $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$

We find that

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, |z| < \infty$$

Example 13. Derive the Laurent series expansion for the function $f(z) = z^3 \sinh \frac{1}{z}$ in the neighbourhood of the origin.

The given function is analytic throughout the complex except at the point $z = 0$.

Using the result of example 12, we have

$$\sinh z = -i \sin iz = -i^2 \sum_{n=0}^{\infty} (-1)^{2n} \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

So,

$$z^3 \sinh \frac{1}{z} = z^3 \sum_{n=0}^{\infty} \frac{z^{-(2n+1)}}{(2n+1)!}, 0 < |z| < \infty$$

$$= \sum_{n=0}^{\infty} \frac{z^{2-2n}}{(2n+1)!}$$

$$= z^2 + \frac{1}{3!} + \sum_{n=2}^{\infty} \frac{1}{(2n+1)! z^{2n-2}}$$

$$= \frac{1}{6} + z^2 + \sum_{n=1}^{\infty} \frac{1}{(2n+3)! z^{2n}}$$

1.8 Module Summary

In this module complex integration is introduced and its related properties are studied. Later, by virtue of complex integration theory some important properties complex functions, which are analytic throughout a region (regular) are established.

At the end, it is found that a complex function analytic in a simply connected domain can be expanded in the form of a Taylor series where as the expansion will be in the form of a Laurent series if the region of analyticity of the function is a multiply connected domain. The converse in both the cases is also true.

1.9 Self Assessment Questions

1. Integrate the following functions over the circle $|z| = 3$ oriented positively;

   (i) $z^2 - 5z + 7$ (ii) $1/z^3$

2. Let γ be the ellipse $25x^2 + 9y^2 = 225$ traversed in the counter-clockwise direction. Define the function $f(z)$ by

$$f(z) = \int_\gamma \frac{\varsigma^2 + 2\varsigma + 3}{\varsigma - z} d\varsigma$$

Find (i) $f(2)$ (ii) $f(6i)$.

3. Evaluate $\int_\gamma \frac{e^{3z}}{z^2 - 9} dz$

Where γ is the closed contour given as follows:

(i)

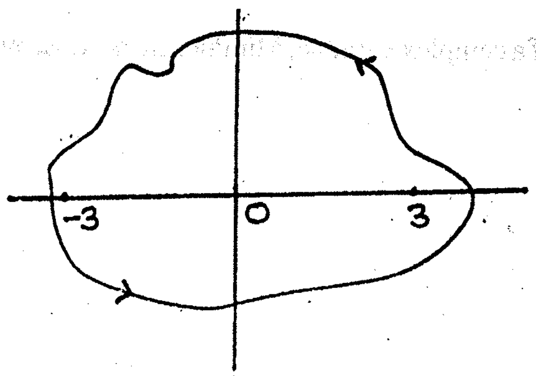

Figure 19

(ii)

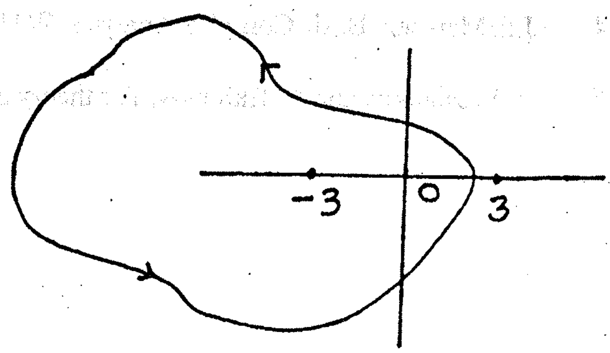

Figure 20

4. Prove that $\int_\gamma z e^{z^2} dz = 0$ for any closed curve γ.

5. Suppose $f(z)$ be continuous on $\bar{D}$ and satisfies the mean-value property, then show that $|f(z)|$ attains its maximum on the boundary of D.

6. Express log (1+z) in the form of a Taylor's series about $z = 0$ and specify the region in which it converges.

7. Find an entire function $f(z)$ such that $\text{Re}\{f(z)\} = e^x \sin y$.

8. Find the maximum and minimum of $|f(z)|$ on the unit disk $|z| \leq 1$, where $f(z) = z^2 - 3$.

9. Find the Taylor series expansion $e^z \cos z$ in the neighbourhood of $z = 0$.

10. Find the Laurent series expansion for $\dfrac{1}{(z-1)(z+2i)}$ about $z = 1$ and mention the region of convergence.

1.10 Suggested Further Readings

1. R.V. Churchill and J.W. Brown. Complex variables and Applications. McGraw-Hill, New York.
2. J.W. Dettman, Applied complex variables, MacMillan, New York.
3. A.I. Markushevich, The theory of analytic functions: A brief course, Mir Publishers, Moscow.
4. J.E. Marsden, Basic Complex Analysis, W.H. Freeman & Co., Sanfrancisco.
5. A. Sveshnikov and A. Tikhonov, The theory of functions of a complex variable, Mir Publishers, Moscow.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-I

Group–B

Module No. - 8
FUNCTIONS OF COMPLEX VARIABLE

Introduction

In our earlier discussion we have seen a function analytic throughout a region can be expanded in the form of a Taylor series, whereas it can be represented by a Laurent series if the region of analyticity is an annulus. From now on the region of analyticity to be considered will be a multiply connected domain i.e., the function is now analytic in a region except for a finite number of points. As in annular region, the function in this case too can expressed in the neighbourhood of each of these points in the form of a Laurent series. Such points are taken to be isolated in nature and require a classification whose consequences are far reaching. As a result two important nations arises:

One is 'Residue Calculus' and the other is 'Argument Principle'. By virtue of these we can

(i) solve some real integrals which are extremely tedius in nature to handle with in real analysis

(ii) determine the number of zeros of a function and detect their position in the complex plane.

Module content

1.1 Classification of Isolated singularities.

1.2 Residue and its applications.

1.3 Logarithmic residue.

Objectives

- Singular point of an analytic function and its classification
- Residue calculus and evaluation of definite and improper integrals.
- Determination of number of zeros of an analytic function.

Key words

Singular point, Principal part, Residue, Variation of argument of a function.

1.1 Classification of Isolated Singularities

In the end of preceding module we had taken function $f(z)$ analytic in a region with exception of a single point z_0 (say) i.e. the function taken there was analytic in a domain D satisfying $0 < |z - z_0| < R$ for some $R > 0$.

Regarding behaviour of the function $f(z)$ near z_0 there exist two possibilities:

(i) We can find a finite complex number C_0 such that $f(z_0) = C_0$ and get a function $f(z)$ analytic throughout the disc $|z - z_0| < R$, say

(ii) We can not find any such number.

These situations tempted us to classify such points.

Isolated Singularity

If a function $f(z)$ is analytic in the punctured disc $\{z \in \mathbb{C}; 0 < |z - z_0| < R\}$ but not analytic at $z = z_0$, then the point z_0 is called an isolated singularity of $f(z)$.

Example 1

Consider the function

$$f(z)=\frac{1}{z(z-i)}$$

The isolated singularities are $z=0$ and i.

Example 2

$$f(z)=\frac{\sin z}{z}$$

Here $z=0$ is the isolated singularity.

Example 3

The function

$$f(z)=e^{1/z}$$

possesses an isolated singularity at $z=0$.

Depending on the behaviour of $f(z)$ near an isolated singularity z_0, classifications occur as follows. These are

I(i) removable singularity if $\lim_{z\to z_0} f(z)$ exists and is finite.

I(ii) pole if $\lim_{z\to z_0} f(z)=\infty$

I(iii) essential singularity if $\lim_{z\to z_0} f(z)$ does not exist.

Again if z_0 is an isolated singularity of $f(z)$, following Th. 15 (Laurent theorem) of module 7 $f(z)$ can be expressed in the form of Laurent series.

$$f(z)=\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

This can be decomposed as

$$f(z)=\sum_{n=0}^{\infty}c_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{c_{-n}}{(z-z_0)^n} \tag{1}$$

where the first term is referred to as the regular part and the second term as the principal part. The three possible cases are of much importance to us. These are:

II(i) The principal part is identically zero.

II(ii) The principal part contains only a finite number of terms.

II(iii) The principal part possesses an infinite number of terms.

Note Actually there exists a one-to-one correspondence between two sets of observations I and II and it will be proved in theorems to come.

Behaviour of analytic functions - Removable singularity

Theorem 1 Suppose z_0 is an isolated singularity of $f(z)$. It is removable if and only if the principal part of the Laurent series of this function in the neighbourhood of z_0 is identically zero.

Proof. Assume z_0 be an removable singularity of $f(z)$. Following the definition of removable singularity [see I(i)] we have

$$\lim_{z\to z_0} f(z)=\ell,$$

$\ell \equiv a$ finite number. Then $f(z)$ is analytic and bounded in the punctured disc $\{z\in\mathbb{C}:0<|z-z_0|<r\}$ i.e., there exists a $M>0$ such that

$$|f(z)<M| \text{ for } 0<|z-z_0|<r \tag{2}$$

Now let us consider the Laurent series expansion (1) of $f(z)$ in the neighbourhood of z_0 where the coefficients are given by

$$c_n=\frac{1}{2\pi i}\int\frac{f(\varsigma)}{(\varsigma-z_0)^{n+1}}d\varsigma, n=0,\pm1,... \tag{3}$$

and the contour of integrating γ is the circle $|\varsigma - z_0| = \rho < r$. Using (2) in (3) we find that

$$|C_n| \leq \frac{1}{2\pi}\int_{\gamma}\frac{|f(\varsigma)|}{|\varsigma - z_0|^{n+1}}|d\varsigma|$$

$$< \frac{1}{2\pi}\frac{M}{\rho^n}\cdot 2\pi = \frac{M}{\rho^n}$$

Since the coefficients c_n do not depend on ρ and moreover it is permissible to consider ρ as small as we please. We can have $c_n = 0$ for $n < 0$ i.e. the principal part of the Laurent series vanishes.

Conversely, suppose that principal part of the Laurent series is identically zero, then $f(z)$ has the expansion

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + ..., 0 < |z - z_0| < r$$

But this is the regular part of the Laurent series and hence converges in the entire disc $|z - z_0| < \rho$. Therefore $\lim_{z \to z_0} f(z) = c_0$, i.e., z_0 is a removable singularity of $f(z)$.

Conclusion: Observations I(i) and II(i) are equivalent.

Example 4. Determine the type of the isolated singularity $z_0 = 0$ of the function

$$f(z) = \frac{1 - e^{-z}}{z}$$

We know the Taylor series expansion of e^{-z} as

$$e^{-z} = 1 - \frac{z}{1!} + \frac{z^2}{2!} - ... + (-1)^n \frac{z^n}{n!} + ..., |z| < \infty$$

Substituting this in the given function, we obtain the Laurent series expansion in the neighbourhood of origin

$$f(z) = \frac{1 - \left(1 - \frac{z}{1!} + \frac{z^2}{2!} - ...\right)}{z}$$

$$= 1 - \frac{z}{2!} + \frac{z^2}{3!} - \ldots$$

This expansion possess no principal part. So the point $z_0 = 0$ will a removable singularity of $f(z)$ if we define $f(z)$ as

$$f(z) = \begin{cases} \dfrac{1 - e^{-z}}{z} & \text{for } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

– Pole

As stated in II(ii), the Laurent series expansion (1) contains only a finite number of terms with negative exponent of $(z - z_0)$ and is of the form

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n + C_{-1}(z - z_0)^{-1} + C_{-2}(z - z_0)^{-2} + \ldots + C_{-k}(z - z_0)^{-k} \quad \ldots (4)$$

for some $k > 0$ and $c_{-k} \neq 0$. In this case the function $f(z)$ is said to possess a pole of order k at $z = z_0$.

Theorem 2 The following statements are equivalent:

(i) The coefficients in the Laurent series (1) satisfy, $c_{-k} \neq 0$, but $c_{-k-1} = c_{-k-2} = \ldots = 0$.

(ii) $\lim_{z \to z_0} (z - z_0)^k f(z) = \ell$ a non-zero finite constant.

(iii) $\lim_{z \to z_0} f(z) = \infty$.

(iii) $\Rightarrow (ii)$.

Suppose z_0 is a pole of $f(z)$. By definition $f(z) \to \infty$ as $z \to z_0$. Then there exist some $R > 0$ so that $|f(z)| > 1$ in the punctured neighbourhood of $z_0, 0 < |z - z_0| < R$. We consider the function

$$g(z) = \frac{1}{f(z)}$$

Then $g(z)$ is analytic in the annulus $0<|z-z_0|<R$ and $g(z) \to 0$ as $z \to z_0$. So, by definition of removable singularity, z_0 is a removable singularity of $g(z)$ and hence $g(z)$ can be defined as

$$g(z)=\begin{cases} 1/f(z), & 0<|z-z_0|<R \\ 0, & z=z_0 \end{cases}$$

Now $g(z)$ is analytic in the disc $|z-z_0|<R$ and vanishes at $z=z_0$. Considering it's zeros,

$$g(z)=(z-z_0)^k h(z)$$

where $h(z)$ is analytic in $|z-z_0|<R$ and $h(z_0)\neq 0$. In fact $h(z)\neq 0$ for all z in $|z-z_0|<R$ since $g(z)\neq 0$ for z satisfying $0<|z-z_0|<R$. Hence $\phi=\frac{1}{h}$ is analytic in $|z-z_0|<R$ and

$$f(z)=\frac{1}{g(z)}=\frac{1}{(z-z_0)^k h(z)}=\frac{\phi(z)}{(z-z_0)^k}$$

Again, since $\phi(z)$ is continuous at z_0

$$\lim_{z\to z_0}(z-z_0)^k f(z)=\lim_{z\to z_0}\phi(z)=\phi(z_0)\equiv \textit{finite}$$

(ii) $\Rightarrow (i)$

By the given hypothesis, $(z-z_0)^k$ has a removable singularity at z_0 and using Theorem 1, we have the expansion

$$(z-z_0)^k f(z)=c_0+c_1(z-z_0)+c_2(z-z_0)^2+\ldots$$

i.e., $$f(z)=\frac{c_0}{(z-z_0)^k}+\frac{c_1}{(z-z_0)^{k-1}}+\ldots+\frac{c_{k-1}}{z-z_0}+\sum_{j=k}^{\infty}c_j(z-z_0)^{j-k}$$

which proves the statement given in (i)

(i) $\Rightarrow$ (iii)

Suppose $f(z)$ has the expansion in the neighbourhood of $z_0, 0<|z-z_0|<R$

$$f(z)=\sum_{k=0}^{\infty}c_k(z-z_0)^k+\frac{c_{-1}}{z-z_0}+\frac{c_{-2}}{(z-z_0)^2}+...+\frac{c_{-k}}{(z-z_0)^k}, c_{-k}\neq 0$$

$$=(z-z_0)^{-k}\sum_{j\geq -k}c_j(z-z_0)^{j+k}$$

$$=(z-z_0)^{-k}\sum_{j\geq 0}c_{j-k}(z-z_0)^j$$

$$=(z-z_0)^{-k}\phi(z)$$

where $\phi(z)$ is analytic in the disc $|z-z_0|<R$ and $\phi(z_0)=c_{-k}\neq 0$ (by the given hypothesis).

Following the continuity of $\phi(z)$ we can have a region $|z-z_0|<\delta$ where

$$|\phi(z)-\phi(z_0)|<\frac{|c_{-k}|}{2}$$

So that

$$|\phi(z)|\geq \big||\phi(z_0)|-|\phi(z)-\phi(z_0)|\big|>\frac{|c_{-k}|}{2}$$

when $|z-z_0|<\delta$.. Therefore, in a neighbourhood $0<|z-z_0|<\delta$ of isolated singularity z_0 we find

$$|f(z)|\geq \frac{1}{2}|c_{-k}||z-z_0|^{-k}$$

Hence $|f(z)|\to\infty$ as $z\to z_0$ in any manner. So, the point $z=z_0$ is a pole of $f(z)$.

Example 5. Determine the nature of isolated singularities of the function

$$|f(z)|=\frac{z^2+1}{z^5-2iz^4-z^3}$$

at $z = 0$ and $z = i$.

Let us rewrite the function as

$$f(z) = \frac{(z+i)(z-i)}{z^3(z-i)^2}$$

$$= \frac{z+i}{z^3(z-i)}$$

Isolated singularity $z = 0$

$$f(z) = \frac{z+i}{-iz^3\left(1-\frac{z}{i}\right)} = \frac{z+i}{-iz^3}\left(1-\frac{z}{i}\right)^{-1}$$

$$= \frac{z+i}{-iz^3}\left(1+\frac{z}{i}+\frac{z^2}{i^2}+\ldots\right), 0<|z|<1$$

$$= -\frac{1}{z^3}+\frac{2i}{z^2}\left\{1+\frac{z}{i}+\frac{z^2}{i^2}+\ldots\right\}, 0<|z|<1$$

So, $z = 0$ is a pole of order three of the given function.

Now for the isolated singularity at $z = i$

$$f(z) = \frac{z-i+zi}{i^3(z-i)}\left\{1+\frac{z-i}{i}\right\}^{-3}$$

$$= \left(\frac{1}{i^3}-\frac{2}{z-i}\right)\left(1-\frac{3}{1!}\frac{(z-i)}{i}+\frac{12}{2!}\frac{(z-i)^2}{i^2}-\ldots\right)$$

$$= -\frac{2}{z-i}-5i+9(z-i)+14i(z-i)^2+\ldots$$

which shows that $z = i$ is a simple pole of $f(z)$.

Theorem 3. If $f(z)$ has a pole at $z = z_0$ of order *m*, then $1/f(z)$ has a zero of order *m* there and vice versa.

Proof. For $z = z_0$ is a pole of order *m*.

$$f(z) = (z - z_0)^{-m} \phi(z_0), \phi(z_0) \neq 0$$

The function $\phi(z)$ is redefined and make $\phi(z)$ analytic in the disc $|z - z_0| < R$ by assigning $\phi(z_0) = C_{-m}$, say $(\neq 0)$, then $(z - z_0)^m f(z)$ is analytic and not equal to zero at $z = z_0$. Thus

$$g(z) = \frac{1}{(z - z_0)^m f(z)}$$

is analytic and not zero at $z = z_0$. From this we see that

$$\frac{1}{f(z)} = (z - z_0)^m g(z)$$

and the assertion follows.

Conversely, suppose $\frac{1}{f} = (z - z_0)^m g(z)$ is a single-valued analytic function defined in a neighbourhood of z_0 and at this point it possesses a zero of order *m*. Then we can find a $\delta > 0$ small enough such that $h(z)$ has no other zero except z_0 in the disc $|z - z_0| < \delta$.

Now we consider the function

$$f(z) = \frac{1}{h(z)}$$

It is single-valued and analytic in the annulus $0 < |z - z_0| < \delta$ and approaches infinity as $z - z_0$, we express

$$f(z) = \frac{1}{h(z)}$$

$$= \frac{1}{(z-z_0)^m \psi(z)},\ \psi(z_0) \neq 0 \text{ and } \psi(z) \text{ is analytic in } |z-z_0| < R$$

$$= \frac{g(z)}{(z-z_0)^m \psi(z)},\ g(z) \text{ is analytic and not zero in } |z-z_0| < R$$

whence it follows that the point z_0 is a pole of order *m* of $f(z)$.

Remark. The above theorem establishes a relation between the zeros and poles of an analytic function.

Example 6 Classify the isolated singularity of the function $f(z) = 1/z^3 + 6z - 6\sinh z$ at $z = 0$

Let $\phi(z) = z^3 + 6z - 6\sinh z$. The point $z = 0$ is a zero of $\phi(z)$, so the given function $f(z)$ has a pole at $z = 0$. Now to test for its order. We observe that

$$\phi^{I}(z) = 3z^2 + 6 - 6\cosh z, \phi^{I}(0) = 0$$

$$\phi^{II}(z) = 6z - 6\sinh z, \phi^{II}(0) = 0$$

$$\phi^{III}(z) = 6 - 6\cosh z, \phi^{III}(0) = 0$$

$$\phi^{IV}(z) = -6\sinh z, \phi^{IV}(0) = 0$$

but, $\phi^{V}(z) = -6\cosh z$ and $\phi^{V} = -6 \neq 0$

Therefore, the point $z = 0$ is a pole of order five for $f(z)$.

Example 7. Test for isolated singularity of the function

$$f(z) = \frac{\cos z}{z^3 + z^2 - 9z - 9}$$

The function can be expressed as

$$f(z) = \frac{\cos z}{(z+1)(z^2-9)}$$

which shows that $z=-1, z=\pm 3$ are the simple poles of $f(z)$.

Essential Singularity

If an isolated singularity of a function $f(z)$ is neither a removable singularity nor a pole then it will be an essential singularity. The behaviour of a function in the neighbourhood of an essential singularity is strange enough. The following theorem illustrates this fact in a nice way.

Theorem 4. If z_0 is an essential singularity of $f(z)$ and $D=\{z \in \mathbb{C}: 0<|z-z_0|<R\}$ for some $R>0$, then for any number L, there exists a sequence of points $\{z_n\}$ in D converging to z_0 such that $\lim_{n\to\infty} f(z_n)=L$.

Proof. Let us suppose on the contrary that there is no such sequence $\{z_n\}$, then there exists a neighbourhood of z_0, $0<|z-z_0|<\rho$ and a positive number $\in$ such that

$$|f(z)-L| \geq \in \text{ whenever } 0<|z-z_0|<\rho$$

It then follows that the function

$$h(z)=\frac{1}{f(z)-L}$$

is analytic in the punctured disc $0<|z-z_0|<\rho$ and remains bounded as $z \to z_0$ and so by definition $[I(i)]$ z_0 is a removable singularity. Defining $h(z)$ at z_0, the function turns out to be analytic in the disc $|z-z_0|<\rho$. Let us express

$$f(z)=L+\frac{1}{h(z)}$$

Now if (i) $h(z_0)\neq 0$, then $f(z)$ is analytic at z_0 or, if (ii) $h(z_0)=0$, then $f(z)$ has a pole at z_0. So whatever be the case, we arrive at a contradiction that z_0 is an essential singularity of $f(z)$ and hence the theorem follows.

Example 8. Show that $z=-\pi$ is an isolated essential singularity of $\cos^{1/z+\pi}$. Find a sequence $\{z_n\}$ converging to $-\pi$ such that $\lim_{n\to\infty} \cos\frac{1}{z_n+\pi}=\sqrt{7}$.

We can write express $\cos^{1/z+\pi}$ as

$$\cos\frac{1}{z+\pi}=\frac{e^{i\frac{1}{z+\pi}}+e^{-i\frac{1}{z+\pi}}}{2}$$

Now, using the expansion

$$e^{s}=1+\frac{s}{1!}+\frac{s^{2}}{2!}+\ldots$$

We find that

$$\cos\frac{1}{z+\pi}=1-\frac{1}{2!(z+\pi)^{2}}+\frac{1}{4!(z+\pi)^{4}}-\ldots$$

This expansion contains an infinite number of terms with negative power of $(z+\pi)$. Hence $z=-\pi$ is an isolated essential singularity of $f(z)$.

For the second part of the question, we solve the equation

$$\cos\frac{1}{z+\pi}=\sqrt{7}$$

i.e. $\dfrac{e^{i/z+\pi}+e^{-i/z+\pi}}{2}=\sqrt{7}$ or, $\alpha^{2}-2\sqrt{7}\alpha+1=0$, taking $e^{i/z+\pi}=\alpha$

so that, $e^{i/z+\pi}\equiv\alpha=\dfrac{2\sqrt{7}\pm\sqrt{24}}{2}=\sqrt{7}\pm\sqrt{6}$

and $\quad z=-\pi+\dfrac{1}{Log\left(\sqrt{7}\pm\sqrt{6}\right)}=-\pi+\dfrac{1}{\log\left(\sqrt{7}\pm\sqrt{6}\right)+2k\pi i}$

Thus we get a sequence $\{z_n\}$, where $z_n=-\pi+\left[\log\left(\sqrt{7}\pm\sqrt{6}\right)+2n\pi i\right]^{-1}$, as asked for.

Example 9. Construct an analytic function having a removable singularity at $z=i\pi/2$, a zero of two at $z=2$ and a pole of order three at $z = 2i$.

We note that

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

Equating right hand side to zero we find that $\cosh z$ has a zero at $z = \frac{i\pi}{2}$. We can now present a function

$$f(z) = \frac{(z-2)^2 \cosh z}{(z-2i)^3\left(z - \frac{i\pi}{2}\right)}$$

having the desired property.

Isolated singularities at infinity.

The behaviour of a function $f(z)$ at $z = \infty$ can equivalently be judged by studying the behaviour of the function $f(1/\varsigma)$ at $\varsigma = 0$. The punctured neighbourhood $\{z : R < |z| < \infty\}$ of ∞ takes the form $\{\varsigma : 0 < |\varsigma| < R^{-1}\}$, the punctured neighbourhood of $\varsigma = 0$.

Let us suppose that a function $f(z)$ is analytic in the domain $\{z : R < |z| < \infty\}$. Then by transforming z by $z = 1/\varsigma$ and studying the nature of $\varsigma = 0$ we confirm that $f(z)$ has an isolated singularity at $z = \infty$. This may be classified as a removable singularity, a pole or an essential singularity.

1.2 Residue and its applications

We are concerned with single-valued functions analytic in a region enclosed by a closed rectifiable curve Γ, say except for a finite number of singularities which are isolated in nature. In the course of performing integration of such functions $f(z)$ over the curve Γ a new idea emerges known as 'Theory of Residues'. As an application of this idea the process of evaluating contour integrals turns out to be much simple and consequently values of some real integrals (proper or improper), which were difficult in earlier consideration are determined easily.

Definition Residue

Let z_0 be an isolated singularity of $f(z)$ and in the neighbourhood of z_0, it has the Laurent series expansion

$$f(z)=\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n, 0<|z-z_0|<R$$

where C_n is given by

$$c_n=\frac{1}{2\pi i}\int_\gamma \frac{f(\varsigma)}{(\varsigma-z_0)^{n+1}}d\varsigma,$$

γ is the circle $|z-z_0|=\rho(0<\rho<R)$ oriented positively. The coefficient c_{-1} of $(z-z_0)^{-1}$ in the Laurent series expansion is termed as 'Residue' of $f(z)$ at $z=z_0$ and is denoted by

$$\text{Res}\left[f(z),z_0\right]=c_{-1}=\frac{1}{2\pi i}\int_\gamma \frac{f(\varsigma)}{(\varsigma-z_0)^{n+1}}d\varsigma \quad (5)$$

Example 10. Find the residue of the function $e^{1/z}$ at $z=0$.

The Laurent series expansion of $e^{1/z}$ in the neighbourhood $z=0$ is is

$$e^{1/z}=1+\frac{1}{z}+\frac{1}{2z^2}+...$$

Here $c_{-1}=1$ and Res $\left[e^{1/z},0\right]=1$.

Example 11. Find the residue of the function

$$f(z)=z^2\cos\frac{1}{z+1}$$

at its singularity.

The point $z=-1$ is the singularity of $f(z)$. Let us expand the function in the neighbourhood of $z=-1$.

$$z^2\cos\frac{1}{z+1}=\{(z+1)-1\}^2\left(1-\frac{1}{2!(z+1)^2}+\frac{1}{4!(z+1)^4}-\ldots\right)$$

$$=\left\{(z+1)^2-2(z+1)+1\right\}\left\{1-\frac{1}{2!(z+1)^2}+\frac{1}{4!(z+1)^4}-\ldots\right\}$$

$$=(z+1)^2-2(z+1)+\left(1-\frac{1}{2!}\right)+\frac{2}{2!}\frac{1}{z+1}-\left(\frac{1}{2}-\frac{1}{4!}\right)\frac{1}{(z+1)^2}+\ldots$$

The point $z=-1$ is the essential singularity of $f(z)$, since the Laurent series expansion contains infinite number of terms with negative powers of $(z-z_0)$. The residue of $f(z)$ at $z=-1$ is 1.

Theorem 5 [Cauchy's Residue Theorem]

Let $f(z)$ be continuous within and on a closed contour Γ and regular, except for the isolated singularities $z_1, z_2, \ldots, z_n$, within Γ. Then

$$\int_\Gamma f(z)\,dz=2\pi i\sum_{k=1}^{n}\text{Res}\left[f(z),z_k\right] \tag{6}$$

Proof. We draw circle γ_k (oriented in positive sense), around each isolated singularity $z_k, k=1,\ldots,n$ with sufficiently small radius, lying entirely within Γ. Then applying Cauchy's Integral Theorem for multiply connected domain [Th.4, Module 7] we find that

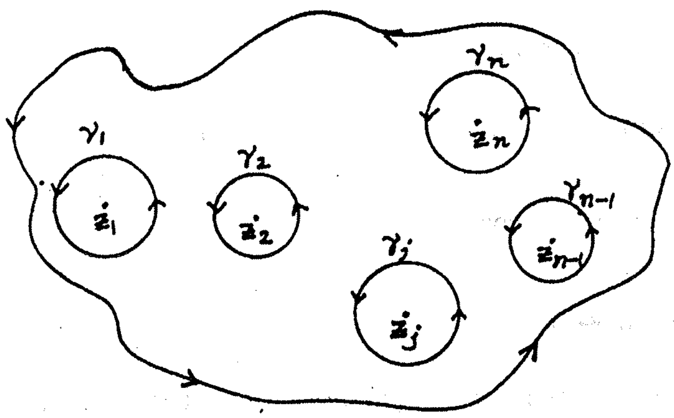

Figure 1

$$\int_{\Gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + ... + \int_{\gamma_n} f(z)\,dz$$

$$= 2\pi i \operatorname{Res}\left[f(z), z_1\right] + 2\pi i \operatorname{Res}\left[f(z), z_2\right] + ...$$

$$+2\pi i \operatorname{Res}\left[f(z), z_n\right], \text{[using (5)]}$$

and the theorem follows.

Calculation of Residues at Poles

Lemma 1 Given $f(z)$ and $g(z)$ are analytic in a region containing z_0 and $g(z)$ has a simple zero at $z = z_0$ but $f(z_0) \neq 0$. Then $f(z)/g(z)$ has a simple pole at $z = z_0$ and

$$\operatorname{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)} \tag{7}$$

Proof. By the given hypothesis, $g(z)$ can be expressed in the neighbourhood of $z = z_0$ as

$g(z) = (z - z_0)\{c_1 + c_2(z - z_0) + ...\} = (z - z_0)h(z)$, say with $h(z_0) = c_1 \neq 0$. Then we can write

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z - z_0)h(z)}$$

and $f(z)/h(z)$ is analytic at $z = z_0$, so $f(z)/g(z)$ possesses a simple at $z = z_0$.

To find the residue of $f(z)/g(z)$ at the simple pole $z = z_0$, we observe that

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z - z_0)h(z)} = \frac{1}{z - z_0}\{a_0 + a_1(z - z_0) + ...\}$$

Clearly, a_0 is the residue of $\frac{f(z)}{g(z)}$ at $z = z_0$. We calculate this as

$$\lim_{z\to z_0}(z-z_0)\frac{f(z)}{g(z)} = \lim_{z\to z_0}\{a_0 + a_1(z-z_0)+...\}$$

i.e. $$a_0 = \lim_{z\to z_0}\frac{f(z)}{\frac{g(z)}{z-z_0}} = \frac{f(z_0)}{g'(z_0)}$$

Lemma 2 Suppose z_0 is a pole of $f(z)$ of order *k*.

Then $$\text{Res}\left[f(z), z_0\right] = \frac{1}{(k-1)!}\lim_{z\to z_0}\frac{d^{k-1}}{dz^{k-1}}\left\{(z-z_0)^k f(z)\right\} \quad (8)$$

Proof. $f(z)$ can be expressed in the neighbourhood of z_0 as

$$f(z) = \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + ... + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0)+...$$

whence

$$(z-z_0)^k f(z) = a_{-k} + a_{-k+1}(z-z_0)+...+a_{-1}(z-z_0)^{k-1}+...$$

Differentiating $k-1$ times, we get

$$\frac{d^{k-1}}{dz^{k-1}}\left\{(z-z_0)^k f(z)\right\} = (k-1)!a_{-1} + k(k-1)...2a_0(z-z_0)+...$$

and finally allowing $z \to z_0$ we find that

$$(k-1)!a_{-1} = \lim_{z\to z_0}\frac{d^{k-1}}{dz^{k-1}}\left\{(z-z_0)^k f(z)\right\}$$

i.e., $$a_{-1} = \text{Res}\left[f(z), z_0\right] = \frac{1}{(k-1)!}\lim_{z\to z_0}\frac{d^{k-1}}{dz^{k-1}}\left\{(z-z_0)^k f(z)\right\}.$$

Example 12. Find the residues of the following functions at their singularities:

(a) $\frac{1-\cos z}{z^3(z-1)^3}$ (b) $\frac{e^{i\pi z}}{z(z^2-1)}$ (c) $\frac{\sin 1/z}{1-z}$

Let us consider the function

(a) $f(z)=\frac{1-\cos z}{z^3(z-1)^3}$

Here the numerator of the function has a double zero at $z=0$ and as a result the function possesses simple pole at $z=0$ and triple pole at $z=1$.

Following formula (8)

$$\text{Res } [f(z),0] = \text{Res}\left[\frac{1-\cos z}{z^3(z-1)^3},0\right]$$

$$=\lim_{z\to 0} z\cdot\frac{1-\cos z}{z^3(z-1)^3}=\frac{\lim\limits_{z\to 0}\frac{1-\cos z}{z^2}}{-1}=-\frac{1}{2!}=-\frac{1}{2}$$

$$\text{Res } [f(z),1] = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left\{(z-1)^3\frac{1-\cos z}{z^3(z-1)^3}\right\}$$

$$=\frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left(\frac{1-\cos z}{z^3}\right)=\frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left\{z^{-3}(1-\cos z)\right\}$$

$$=\frac{1}{2!}\lim_{z\to 1}\left(12z^{-5}(1-\cos z)+(-3)z^{-4}.2\sin z+z^{-3}\cos z\right)$$

$$=\frac{1}{2!}\left(12(1-\cos 1)-6\sin 1+\cos 1\right)$$

$$=\frac{1}{2!}(12-6\sin 1-11\cos 1)$$

(b) $f(z) = \frac{e^{i\pi z}}{z(z^2-1)}$

The given function possesses simple poles at $z = 0, z = \pm 1$ in the finite complex plane. Thus

$$\text{Res } [f(z),0] = \lim_{z\to 0} z\frac{e^{i\pi z}}{z(z^2-1)} = -1$$

$$\text{Res } [f(z),1] = \lim_{z\to 0}(z-1)\frac{e^{i\pi z}}{z(z^2-1)} = \lim_{z\to 1}\frac{e^{i\pi z}}{z(z+1)} = -\frac{1}{2}$$

$$\text{Res } [f(z),-1] = \lim_{z\to -1}(z+1)\frac{e^{i\pi z}}{z(z^2-1)} = \lim_{z\to -1}\frac{e^{i\pi z}}{z(z-1)} = -\frac{1}{2}$$

(c) $f(z) = \frac{\sin(1/z)}{1-z}$

This function has an essential singularity at $z = 0$.

Let us consider the Laurent series expansion of $f(z)$ in the neighbourhood $0 < |z| < 1$,

$$f(z) = \frac{\sin 1/z}{1-z}$$

$$= \left(\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \ldots\right)\left(1 + z + z^2 + \ldots\right)$$

$$= \sum_{n=0}^{\infty} z_n + \left(1 - \frac{1}{3!} + \frac{1}{5!} - \ldots\right)\frac{1}{z} + \left(-\frac{1}{3!} + \frac{1}{5!} - \ldots\right)\frac{1}{z} + \ldots$$

So, $\text{Res } [f(z),0] = 1 - \frac{1}{3!} - + \frac{1}{5!} - \ldots = \sin 1$

For the residue at the pole $z = 1$, we find

$$\text{Res}\ \left[f(z),1\right] = \lim_{z\to 1}(z-1)\frac{\sin 1/z}{1-z}$$

$$= -\sin 1$$

Evaluation of Integrals

As an application of Cauchy's Residue Theorem we now show how various types of real integrals, some of which proved to be complicated earlier, can be evaluated by means of complex integrals.

I. Integrals of the form $I = \int_0^{2\pi} R(\sin\theta, \cos\theta)\, d\theta$, where $R\left(f(x), g(x)\right)$ is a rational function of $f(x)$ and $g(x)$.

In this case we change the variable by a substitution $z = e^{i\theta}$ and the given integral reduces to an integral of an analytic function of a complex variable over closed contour.

$$I = \frac{1}{i}\int_{|z|=1} R\left(z - \frac{1}{z}, z + \frac{1}{z}\right)\frac{dz}{z}$$

clearly the integrand is a rational function and can expressed as

$$R(z) = R\left(z - \frac{1}{z}, z + \frac{1}{z}\right) = \frac{\alpha_0 + \alpha_1 z + \ldots + \alpha_n z^m}{\beta_0 + \beta_1 z + \ldots \beta_n z^n},$$

which is analytic inside the circle $|z| = 1$ except at the zeros $z_1, z_2, \ldots, z_N, N \le n$ of the denominator. Thus, using Theorem 5, we obtain

$$I = 2\pi \sum_{k=1}^{N} \text{Res}\left[R(z), z_k\right]$$

[The points z_k are the poles of the function $R(z)$].

Example 13. Evaluate $\int_0^{2\pi} \frac{\cos\theta}{5 + 4\cos\theta} d\theta$

We substitute $z = e^{i\theta}$ and the given integral reduces to

$$\frac{1}{2i}\int_{|z|=1}\frac{z^2+1}{(z+2)(2z+1)z}dz = \frac{1}{4i}\int_{|z|=1}\frac{z^2+1}{z(z+2)\left(z+\frac{1}{2}\right)}dz$$

The integrand has three simple poles of which two $z = 0$ and $z = -\frac{1}{2}$ lie inside $|z| = 1$ and the residues are :

$$\text{at } z = 0 : \lim_{z\to 0}\frac{z(z^2+1)}{z(z+2)\left(z+\frac{1}{2}\right)} = 1$$

$$\text{at } z = -\frac{1}{2} : \lim_{z\to -\frac{1}{2}}\frac{\left(z+\frac{1}{2}\right)(z^2+1)}{z(z+2)\left(z+\frac{1}{2}\right)} = \frac{\frac{5}{4}}{-\frac{1}{2}\cdot\frac{3}{2}} = -\frac{5}{3}.$$

Therefore,

$$\int_0^{2\pi}\frac{\cos\theta}{5+4\cos\theta}d\theta = 2\pi i\cdot\frac{1}{4i}\left(1-\frac{5}{3}\right) = -\frac{\pi}{3}$$

II. Integrals of the form $\int_{-\infty}^{\infty} f(x)dx$

Here $f(x)$ is a rational function of the form $p(x)/q(x)$, say, where the degree of the denominator $q(x)$ is greater than the degree of $p(x)$ by atleast two units and that $q(x)$ never vanishes for real x.

Example 14. Evalute the integral $\int_{-\infty}^{\infty}\frac{x}{(x^2+1)(x^2+2x+2)}dx$

We consider the complex integral

$$\int_{\Gamma}\frac{z}{(z^2+1)(z^2+2z+2)}dz$$

where Γ is the semi-circular contour as shown in the figure 2.

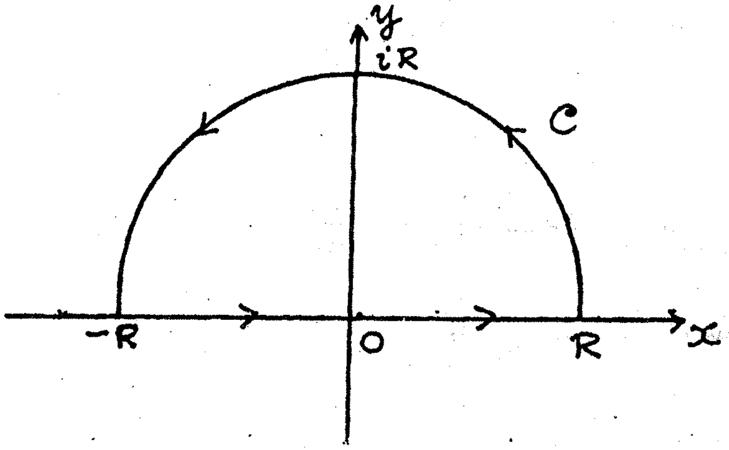

Figure 2

Let us perform the contour integral

$$\int_{\Gamma}\frac{z}{(z^2+1)(z^2+2z+2)}dz=\int_{-R}^{R}\frac{x}{(x^2+1)(x^2+2x+2)}dx+\int_{c}\frac{zdz}{(z^2+1)(z^2+2z+2)} \qquad (9)$$

since Γ is the contour consisting of the line segment $-R$ to $R(R>0)$ and the semi-circle c with radius R and centre at the origin 0 in the upper half plane oriented positively.

The second integral on the right hand side of (9) approaches zero as R increases without bound, for

$$\left|\int_{c}\frac{zdz}{(z^2+1)(z^2+2z+2)}\right|\le\frac{\pi R^2}{(R^2-1)(R-\sqrt{2})^2}\to 0 \text{ as } R\to\infty .$$

Now the integrand in the integral on the left hand side of (9) possesses simple poles at $z=\pm i$ and $z=-1\pm i$ of which the points i and $-1+i$ lie inside the closed contour Γ.

Residue at $z=i$

$$\lim_{z\to i}\frac{(z-i)z}{(z^2+1)(z^2+2z+2)}=\lim_{z\to i}\frac{z}{(z+i)(z^2+2z+2)}=\frac{i}{2i(2i+1)}=\frac{1-2i}{10}$$

Residue at $z=-1+i$

$$\lim_{z \to -1+i} \frac{(z+1-i)z}{(z^2+1)(z+1-i)(z+1+i)} = \frac{i-1}{\{(i-1)^2+1\} \cdot 2i} = \frac{-1+3i}{10}$$

Hence,

$$\int_{\Gamma} \frac{z}{(z^2+1)(z^2+2z+2)} dz = 2\pi i = 2\pi i \left\{ \frac{1-2i}{10} + \frac{-1+3i}{10} \right\} = -\frac{\pi}{5} \qquad (11)$$

Allowing $R \to \infty$ in (9), we obtain on using (10)

$$\int_{-\infty}^{\infty} \frac{xdx}{(x^2+1)(x^2+2x+2)} = -\frac{\pi}{5}$$

Jordan's Inequality

We know that $\cos\theta$ decreases steadily as θ increases from 0 to $\pi/2$. In the graph of the function $y = \cos\theta$ (see figure 3) the ordinate PQ decreases as θ increases. Obviously it is true for its mean ordinate.

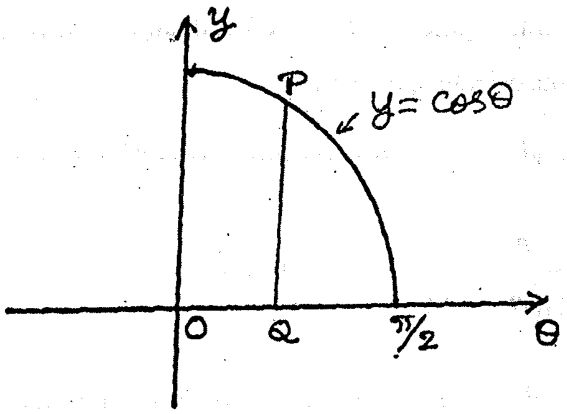

Figure 3

i.e. $\frac{1}{\theta}\int_0^{\theta} \cos\theta d\theta = \frac{\sin\theta}{\theta}$

which proved that

$$\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1 \text{ whenever } 0 \le \theta \le \frac{\pi}{2}$$

equivalently,

$$\frac{2\theta}{\pi} \le \sin\theta \le \theta \tag{10}$$

III. Integrals of the form $\int_{-\infty}^{\infty} e^{iax} f(x)\,dx, \alpha > 0$

The function $f(x)$ is defined on the entire real axis $-\infty < x < \infty$ and that it tends to zero uniformly as $|x| \to \infty$.

Before going to evaluate the above integrals we establish the so called Jordan's Lemma which has various applications in the calculation of a wide class of improper integrals.

Jordan's Lemma

Let c_R be the semicircle $|z| = R$ in the upper half plane $\operatorname{Im} z \ge 0$ and if $a > 0$, then

$$\lim_{R\to\infty} \int_{c_R} e^{iaz} f(z)\,dz = 0 \tag{11}$$

subject to the condition that $f(\mathrm{Re}^{i\theta}) \to 0$, uniformly with respect to θ when $0 \le \theta \le \pi$, as $R \to \infty$.

Proof. Suppose $z \in C_R$ and we choose R so large that $|f(z)| < \epsilon$. Then taking $z = \mathrm{Re}^{i\theta}, 0 \le \theta \le \pi$

$$\left|e^{iaz}\right| = \left|\exp\left\{ia\left(R\cos\theta + i\sin\theta\right)\right\}\right| = \exp\left(-aR\sin\theta\right)\Big|$$

Let us estimate the integral given in (11)

$$\left|\int_{C_R} e^{iaz} f(z)\,dz\right| = \left|\int_0^{\pi} \exp\left(ia\mathrm{Re}^{i\theta}\right) f\left(\mathrm{Re}^{i\theta}\right) \mathrm{Re}^{i\theta}\,d\theta\right|$$

$$< \varepsilon \int_0^{\pi} e^{-aR\sin\theta} R\,d\theta$$

$$= 2R\varepsilon \int_0^{\pi/2} e^{-mR\sin\theta} d\theta$$

$$\le 2R\varepsilon \int_0^{\pi/2} e^{-2R\frac{a}{\pi}\theta} d\theta \text{ [using (10]}$$

$$= \frac{\pi\varepsilon}{a}\left[1 - e^{-aR}\right] \to 0 \text{ as } R \to \infty$$

which proves the lemma.

Remark. If $a < 0$ and the condition on $f(z)$ holds in the lower half plane $\operatorname{Im} z \le 0$, when $0 \ge \theta \ge -\pi$ the result in (11) is also valid.

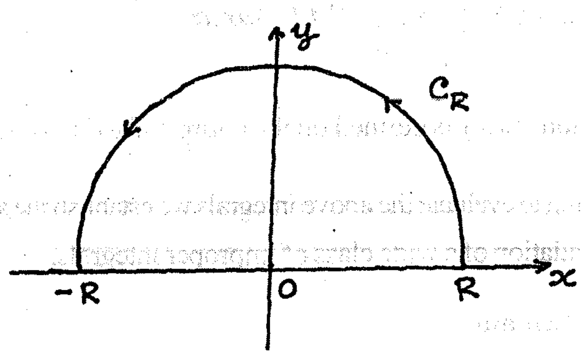

Figure 4

Example 15 Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{(x-1)\sin 4x}{x^2 - 2x + 10} dx$$

Let us consider the complex integral

$$\int_\Gamma \frac{(z-1)e^{i4z}}{z^2 - 2z + 10} dz$$

where Γ is closed contour consisting of the straight line segmei $[-R, R]$ and the semi-circular arc $c_R : |z| = R$ in the upper halı plane $\operatorname{Im} z \ge 0$, $0 \le \arg z \le \pi$

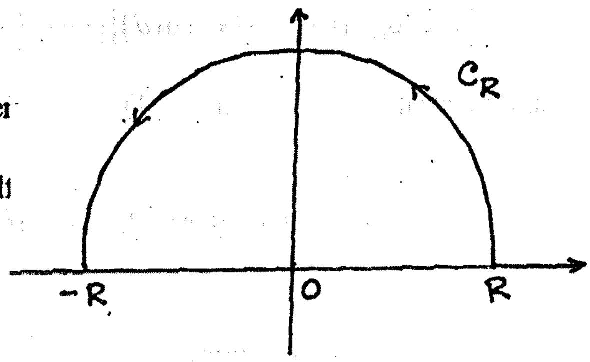

Figure 5

The integrand $\frac{(z-1)e^{i4z}}{z^2-2z+10} = e^{i4z}f(z)$, say possesses simple poles at $z = 1 \pm 3i$ of which $z = 1+3i$ lies in the upper half plane. Therefore,

$$\int_{\Gamma} e^{i4z} f(z)\,dz = \int_{-R}^{R} e^{i4z} f(x)\,dx + \int_{c_R} e^{i4z} f(z)\,dz$$

Allowing $R \to \infty$,

$$2\pi i \text{ Res } \left[e^{i4z} f(z), 1+3i\right] = \int_{-\infty}^{\infty} e^{i4z} f(x)\,dx + \lim_{R\to\infty} \int_{c_R} e^{i4z} f(z)\,dz$$

The second term on the r.h.s. approaches zero following Jordan's lemma and hence

$$\int_{-\infty}^{\infty} \frac{(x-1)e^{i4x}}{x^2-2x+10}\,dx = \lim_{z\to 1+3i} \frac{(z-1-3i)(z-1)e^{i4z}}{(z-1-3i)(z-1+3i)} \cdot 2\pi i$$

$$= \frac{3ie^{i4(1+3i)}}{6i} \cdot 2\pi i$$

$$= \pi i e^{-12}(\cos 4 + i \sin 4)$$

Equating the imaginary parts, we obtain

$$\int_{-\infty}^{\infty} \frac{(x-1)\sin 4x}{x^2-2x+10}\,dx = \pi e^{-12} \cos 4$$

Lemma 3. Let $f(z)$ be analytic in a region D except at the point ξ which is its simple pole and let c_R be a portion of a circular arc of radius r subtends an angle θ at its centre ξ. Then

$$\lim_{r\to 0} \int_{c_r} f(z)\,dz = i\theta \text{ Res } \left[f(z), \xi\right] \tag{12}$$

Proof. Given the point $z = \xi$ is a simple pole of $f(z)$ and hence $f(z)$ can be expressed in the neighbourhood $z = \xi$ as

$$f(z) = \frac{\alpha}{z - \varsigma} + \psi(z)$$

where $\psi(z)$ is analytic at $z = \xi$ and $\alpha =$ Res $[f(z), \varsigma]$.

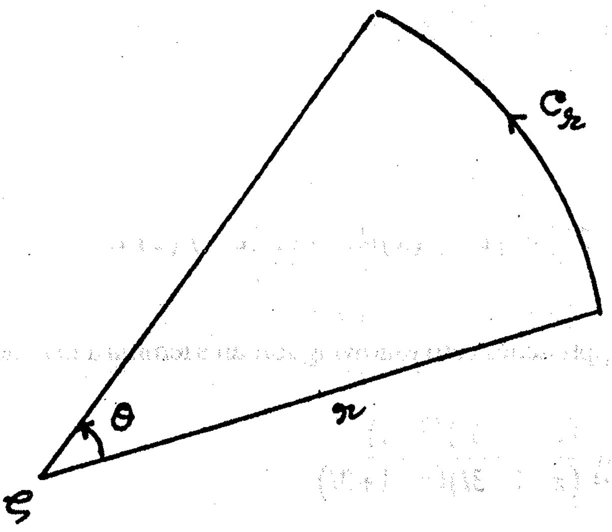

Figure 6

Thus,

$$\int_{c_r} f(z)\,dz = \int_{C_r} \frac{\alpha}{z-\varsigma}\,dz + \int_{C_r} \psi(z)\,dz$$

but, $\int_{c_r} \frac{\alpha}{z-\varsigma}\,dz = \alpha \int_{C_r} \frac{ire^{i\phi}}{re^{i\phi}}\,d\phi$, taking $z - \varsigma = re^{i\phi}$

$$= i\theta\alpha$$

Again $\psi(z)$ is analytic at $z = \varsigma$, so it is bounded in the neighbourhood of $z = \varsigma$, say by M. Thus $\left|\int_{c_r} \psi(z)\,dz\right| \leq M$

length of the arc $c_r = M\theta r \to 0$ as $r \to 0$

Therefore, $\int_{c_r} f(z)\,dz = i\theta\alpha = i\theta$ Res $[f(z), \varsigma]$

In the course of evaluating improper real integrals so far we have considered complex integral of those functions which do not possess any singularities on the real axis. To evaluate a broad class of improper integrals we need to modify the contour of integration of the functions which have singularities on the real axis. The following problems will highlighten the matter.

Example 16. Evaluate the integral

$$\int_0^\infty \frac{\sin 4x}{x\left(x^2+9\right)}dx$$

Let us consider the function

$$f(z)=\frac{e^{i4z}}{z\left(z^2+9\right)}$$

it has simple poles at $z=0$ and $z=\pm 3i$. We choose the contour of integration as shown in Fig. 7.

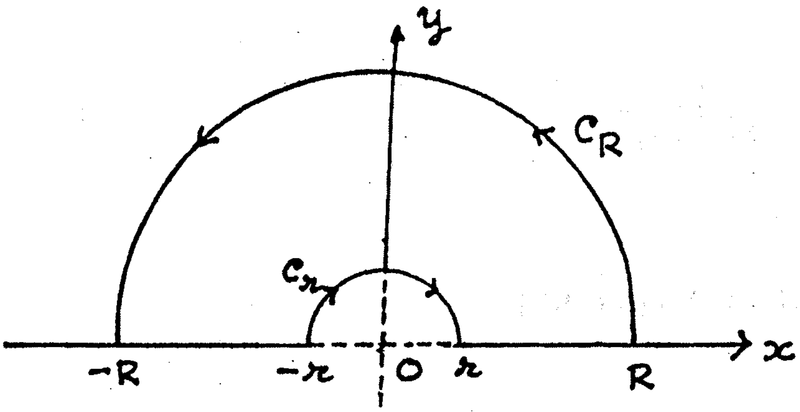

Figure 7

c_r is the small semi-circular arc $|z|=r, +\pi \geq \arg z \geq 0$ oriented clockwise with $r<3$ and c_r is another semi-circular arc $|z|=R, R>3$ with $0\leq \arg z\leq \pi$, oriented positively. Thus we construct a closed contour Γ in the upper half-plane consisting of segments of the real axis $[-R,-r],[r,R]$ and semicircular arcs c_r and c_R.

The function $f(z)$ is continuous on Γ and analytic within Γ except at the simple pole z=3i. Following the Cauchy's residue theorem,

$$\int_{\Gamma} f(z)\,dz = \int_{-R}^{-r} f(z)\,dz + \int_{c_r} f(z)\,dz + \int_{r}^{R} f(z)\,dz + \int_{c_R} f(z)\,dz \tag{13}$$

Now,

$$\int f(z)\,dz = 2\pi i \ \text{Res}\ \left[f(z),3i\right] = 2\pi i \lim_{z\to 3i} \frac{(z-3i)e^{i4z}}{z(z-3i)(z+3i)}$$

$$= 2\pi i \frac{e^{-12}}{18i^2} = -\frac{i\pi e^{-12}}{9}$$

$$\int_{-R}^{-r} f(z)\,dz + \int_{r}^{R} f(z)\,dz = \int_{-R}^{-r} \frac{e^{i4x}}{x(x^2+9)}\,dx + \int_{r}^{R} \frac{e^{i4x}}{x(x^2+9)}\,dx$$

$$= \int_{r}^{R} \frac{e^{i4x} - e^{i4x}}{x(x^2+9)}\,dx$$ [Replacing x by $-x$ in the first of R.H.S.]

$$= 2i \int_{r}^{R} \frac{\sin 4x}{x(x^2+9)}\,dx \tag{14}$$

Again following Lemma 3,

$$\lim_{r\to 0} \int_{c_r} f(z)\,dz = i(-r)\ \ \text{Res}\ \left[f(z),0\right]$$

$$= -ir \lim_{z\to 0} \frac{ze^{i4z}}{z(z^2+9)} = -\frac{i\pi}{9} \tag{15}$$

Finally, applying Jordan's Lemma we find that the fourth integral in the r.h.s. of (13) tends to zero as $R \to \infty$, since the function $\frac{1}{z(z^2+9)} \to 0$ uniformly as $|z| \to \infty$ i.e.

$$\lim_{R\to\infty} \int_{c_R} f(z)\,dz = 0 \tag{16}$$

Thus, as $R \to \infty$ and $r \to 0$ we get from (13) using (14) - (16) that

$$2i\int_0^\infty \frac{\sin 4x}{x\left(x^2+9\right)}dx - \frac{i\pi}{9} = -\frac{i\pi e^{-12}}{9}$$

whence,

$$\int_0^\infty \frac{\sin 4x}{x\left(x^2+9\right)}dx = \frac{\pi}{18}\left(1-e^{-12}\right)$$

Example 17 Evaluate the integral

$$\int_0^\infty \frac{\sin \pi x}{x\left(1-x^2\right)}dx$$

To evaluate the integral we consider the complex integral

$$\int_\Gamma \frac{e^{\pi z i}}{z\left(1-z^2\right)}dz$$

where Γ is the closed contour consisting of the line segments $[-R,-1-P],[-1+P,-E],[\in,1-r],[1+r,R]$ and the semicircular arcs $c_P, c_\in, c_r$ in the upper half-plane with radii ρ, ε and r respectively.

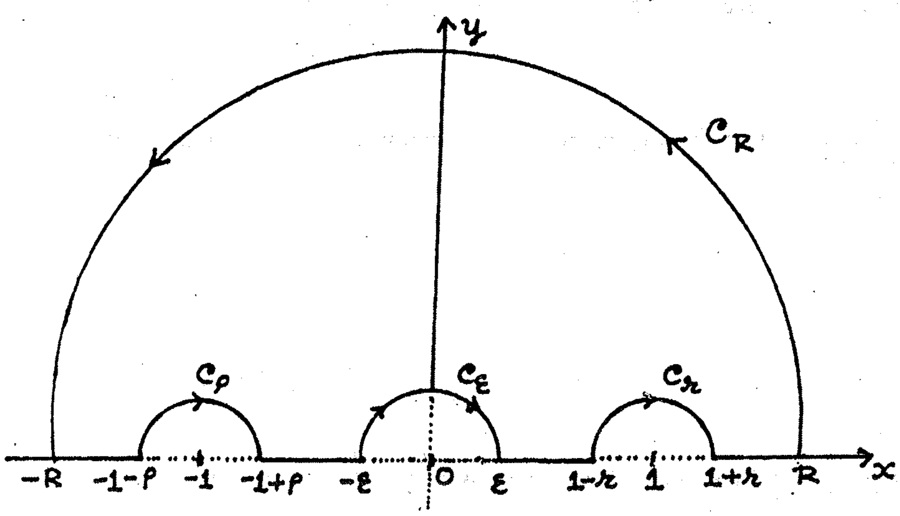

Figure 8

We see that the function $f(z)=\dfrac{e^{\pi zi}}{z(1-z^2)}$ is analytic within and on the contour Γ. Then using the Cauchy's integral theorem, we find that $\int_{\Gamma} f(z)\,dz=0$. Again we can express

$$\int_{\Gamma} f(z)dz = \int_{-R}^{-1-\rho} f(z)d + \int_{c_\rho} f(z)dz + \int_{-1+\rho}^{-\varepsilon} f(z)dz + \int_{c_\varepsilon} f(z)dz +$$

$$+\int_{\epsilon}^{1-r} f(z)dz + \int_{c_r} f(z)dz + + \int_{l+r}^{R} f(z)dz + \int_{c_R} f(z)dz \tag{17}$$

Applying lemma 3 in the second, fourth and sixth integral on the r.h.s. of (17), we obtain

$$\lim_{\rho\to 0}\int_{c_\rho} f(z)dz = -i\pi \operatorname{Res}\left[f(z),-1\right] = -i\pi\cdot\frac{e^{-i\pi}}{2} = -\frac{i\pi}{2} \tag{18}$$

$$\lim_{\epsilon\to 0}\int_{c_\epsilon} f(z)dz = -i\pi \operatorname{Res}\left[f(z),0\right] = -i\pi\cdot 1 = -i\pi \tag{19}$$

and

$$\lim_{r\to 0}\int_{c_r} f(z)dz = -i\pi \operatorname{Res}\left[f(z),1\right] = -i\pi\cdot\frac{-e^{i\pi}}{2} = -\frac{i\pi}{2} \tag{20}$$

Again $\dfrac{1}{z(1-z^2)} \to 0$ uniformly as $|z|\to\infty$, so using Jordan's lemma, we find

$$\lim_{R\to\infty}\int_{c_R} f(z)dz = 0$$

allowing ρ,ε,r tend to zero and $R\to\infty$ in (17) we finally get on utilizing (18) – (20).

$$\int^{-1} f(z)dx - \frac{i\pi}{2} + \int_{-1}^{0} f(x)dx - i\pi + \int_{0}^{1} f(x)dx - \frac{i\pi}{2} + \int_{1}^{\infty} f(x)dx = 0$$

i.e., $\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{x(1-x^2)} dx = 2i\pi$

Equating imaginary parts, we arrive at

$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{x(1-x^2)} dx = 2\pi$$

Now since the integrand is an even function of x,

$$\int_{0}^{\infty} \frac{\sin \pi x}{x(1-x^2)} dx = \pi$$

1.3 Logarithmic Residues

The concept of logarithmic residue If $f(z)$ is analytic and not equal to zero at any point of the closed contour Γ and regular inside Γ except at a finite number of points which are all poles, then the quantity

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} dz$$

is called the logarithmic residue of $f(z)$ with respect to the closed contour Γ and the integrand is known as logarithmic derivative of $f(z)$.

Theorem 6 Let $f(z)$ be a meromorphic functino defined within and on a simple closed contour Γ except at a finite number of poles lying inside Γ. If $f(z) \neq 0$ on Γ, then the difference between the total number of zeros (N) and the total number of poles (P) lying inside Γ is expressed by the relation.

$$N - P = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} dz \tag{21}$$

[The total number of zeros (poles) are counted according to their multiplicities]

Proof. Let $n_1, n_2, ..., n_k$ be the zeros and $p_1, p_2, ..., p_j$ be the poles of $f(z)$ lying inside Γ. In order to perform the integration in (21), before hand we should know the behaviour of the integrand $\left(f'(z)/f(z)\right)$ within and on Γ.

If $f(z)$ has a zero of order m at z_0. Then we can write $f(z)$ as

$$f(z)=(z-z_0)^m g(z)$$

where $g(z_0)\neq 0$ and $g(z)$ is analytic in the neighbourhood of $z=z_0$. Then

$$\frac{f'(z)}{f(z)}=\frac{m(z-z_0)^{m-1}g(z)+(z-z_0)^m g'(z)}{(z-z_0)^m g(z)}=\frac{m}{z-z_0}+\frac{g'(z)}{g(z)} \tag{22}$$

The second term on r.h.s. of (22) is analytic at z_0 by virtue of the behaviour of $g(z)$ at $z=z_0$. So the function $f'(z)/f(z)$ possesses simple pole at $z=z_0$ with residue m.

Again if $f(z)$ has a pole of order s, then $f(z)$ will be of the form

$$f(z)=\frac{h(z)}{(z-z_0)^s},$$

where $h(z)$ is analytic and not equal to zero at $z=z_0$.

In this case, we find

$$\frac{f'(z)}{f(z)}=\frac{-s(z-z_0)^{-s-1}h(z)+(z-z_0)^{-s}h'(z)}{(z-z_0)^{-s}h(z)}=\frac{-s}{z-z_0}+\frac{h'(z)}{h(z)} \tag{23}$$

The behaviour of $h(z)$ at $z=z_0$ confirms that the function $f'(z)/f(z)$ has a simple at $z=z_0$ with residue $-s$.

Suppose now that the order of the given zeros at $z=n_i, i=1,2,\ldots,k$ be α_i and the order of the given poles at $z=p_i$ be $\beta_i, i=1,2,\ldots,j$. Application of the Cauchy's Residue Theorem yields on using the results in (22) and (23),

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f'(z)}{f(z)}dz=(\alpha_1+\alpha_2+\ldots+\alpha_k)-(\beta_1+\beta_2+\ldots+\beta_j)$$

$$=N-P$$

counting the number of zeros and the number of poles according to their multiplicities.

Geometric Interpretation

We transform the integral on the right of (21):

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f'(z)}{f(z)}dz=\frac{1}{2\pi i}\int_{\Gamma}d\log f(z)=\frac{1}{2\pi i}\int_{\Gamma}d\{\log|f(z)|+i\arg f(z)\}$$

$$=\frac{1}{2\pi i}\int_{\Gamma}d\log|f(z)|+\frac{1}{2\pi}\int_{\Gamma}d\arg f(z)$$

where we have taken the principal branch of the logarithm. Here $\log|f(z)|$ is real single-valued function and hence its variation as the point z traverses the closed contour Γ is zero. Thus the first term in the above equation is zero, while the second term is the total variation of the argument of the function $f(z)$ as the point z traverses Γ multiplied by $1/2\pi$. Therefore

$$N-P=\frac{1}{2\pi}[\arg f(z),\Gamma] \tag{24}$$

We interpret the equation (24) geometrically as follows:

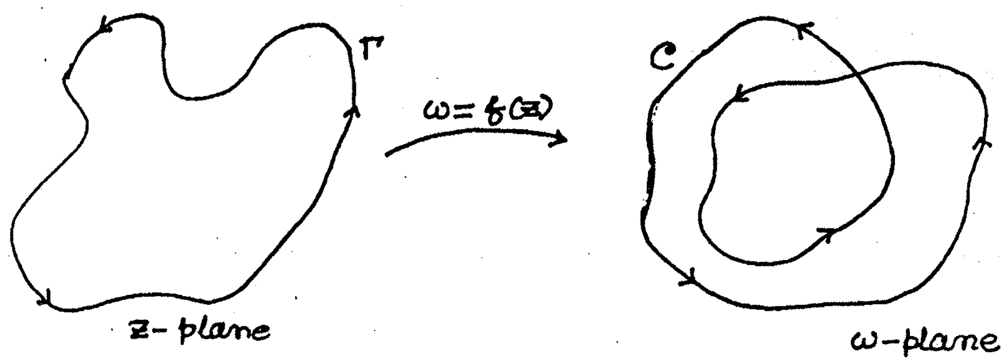

Figure 9

Let w be be the image of the point z under the mapping $w = f(z)$. Now since the function $f(z)$ is continuous within and on Γ, a complete traversal of the contour Γ in z-plane given rise to a certain closed contour C in the w-plane. The contour c may or may not enclose the point $w = 0$ within it. In the

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{c}\frac{dw}{w} = n(c,0)$$

case, the variation of the argument w in a complete traversal of c is clearly zero, while in the latter case, the variation of the argument w is determined by the number of times the point w winds around the origin $w = 0$ in its motion along the contour c. The motion of w while forming the closed contour c may be clockwise or anti-clockwise as z traverse the contour Γ in a positive sense.

Thus we see that the difference between the number of zeros and poles of $f(z)$ that lie inside a closed contour Γ is equal to the variation of argument $f(z)$ when z traverses Γ in the positive sense. This is known as "Argument Principle".

Example 19. Find the logarithmic residue of the function

$$f(z) = \frac{z^2 - z - 6}{(z^2 + z - 6)(z^2 + 1)}$$

with respect to the contour $\Gamma : |z| = 4$.

Here the given function $f(z)$ has zeros at $z = -2, 3$ and poles at $z = 2, -3, \pm i$ since it can be expressed as

$$f(z) = \frac{(z+2)(z-3)}{(z-2)(z+3)(z-i)(z+i)}$$

and hence the logarithmic residue with respect to $\Gamma : |z| = 4$ (all the zeros and poles lying inside Γ) is

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f'(z)}{f(z)}dz = N - P = 2 - 4 = -2$$

Example 20. Find the number of zeros of $f(z) \equiv z^5 - 3z^3 + 2z + 5 = 0$ in the right half-plane.

Let Γ be the closed contour consisting of the semicircle $c_R : |z| = R, \operatorname{Re} z > 0$ and the diameter of c_R along the imaginary axis. We shall take R so large that all the zeros of $f(z)$ that lie in the right half-plane will stay inside the semi-circle $|z| < R, \operatorname{Re} z > 0$.

Now, $Var\left[\arg f(z), c_R\right]$

$$= \underset{C_R}{Var} \arg f(z)$$

$$= \underset{C_R}{Var} \arg\left[z^5 - 3z^3 + 2z + 5\right]$$

$$= \underset{C_R}{Var} \arg\left[z^5\left(1 - \frac{3}{z^2} + \frac{2}{z^4} + \frac{5}{z^5}\right)\right]$$

$$= \underset{C_R}{Var} \arg z^5 + \underset{C_R}{Var} \arg\left(1 - \frac{3}{z^2} + \frac{2}{z^4} + \frac{5}{z^5}\right)$$

$$= 5\underset{C_R}{Var} \arg z + \underset{C_R}{Var} \arg\left(1 - \frac{3}{z^2} + \frac{2}{z^4} + \frac{5}{z^5}\right)$$

$\rightarrow 5\pi$ as $R \rightarrow \infty$

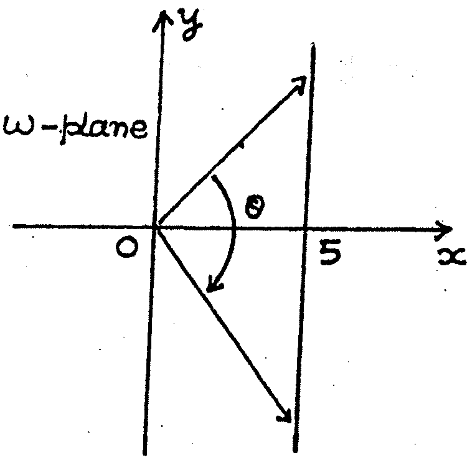

Figure 10

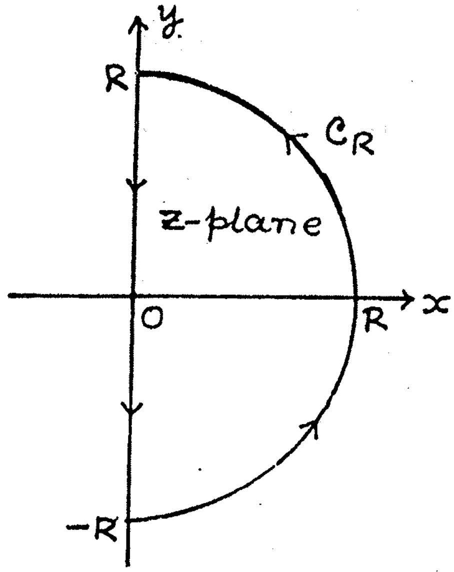

Figure 11

Now when z traverses the imaginary axis, $z = iy, -R \le y \le R$, then

$$w = f(iy) = (iy)^5 - 3(iy)^3 + 2iy + 5 = 5 + i(y^5 + 3y^3 + 2y) \quad (25)$$

So, $f(z) \ne 0$ for $\text{Re}z = 0$. Hence

$$Var[\arg f(z), \text{Re}\, z = 0] = -\pi$$

since the image of the imaginary axis in the z-plane is the straight line $\text{Re}\, w = 5$ in the w-plane and when z traverses the imaginary axis from upper half to lower half-plane it appears from eq. (25) that $\text{Im}\, w$ also traverses the line $\text{Re}\, w = 5$ from upper half-plane to lower half-plane.

Hence applying the argument principle (Theorem 6)

we get $N = \frac{1}{2\pi} Var[\arg f(z), \Gamma] = \frac{1}{2\pi}(5\pi - \pi) = 2$

So the given equation possesses two zeros in the right half-plane.

The computations of total number of zeros of an analytic function in a given domain can be made appreciably simpler in some cases by the following theorem which is due to Eugene Rouche (1832-1910).

Theorem 7 [Rouche's Theorem] Let $f(z)$ and $g(z)$ be analytic within and on a closed contour Γ and $|f(z)| > |g(z)|$ on Γ. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros within Γ.

Proof. By the given hypothesis $f(z)$ and $\psi(z) = f(z) + g(z)$ are analytic within and on Γ. Moreover, since on Γ, we find that

$$|\psi(z)||f(z) + g(z)| \ge |f(z)| - g(z) > 0 \;\; \forall z \in \Gamma$$

Thus, $f(z)$ and and $\psi(z)$ satisfy all the conditions of theorem 6 and we have (note that $\psi(z)$ and $f(z)$ have no poles within Γ,

$$N_\psi = \frac{1}{2\pi} Var[\arg \psi(z), \Gamma]$$

$$N_f = \frac{1}{2\pi} Var\left[\arg f(z), \Gamma\right]$$

where N_ψ and N_f are the total number of zeros of $\psi(z)$ and $f(z)$ respectively lying within Γ.

Let us consider their difference

$$N_\psi - N_f = \frac{1}{2\pi} Var\left[\arg \psi(z) - \arg f(z), \Gamma\right]$$

$$= \frac{1}{2\pi} Var\left[\arg\{f(z) + g(z)\} - \arg f(z)\Gamma\right]$$

$$= \frac{1}{2\pi} Var\left[\arg\left\{\frac{f(z)+g(z)}{f(z)}\right\}, \Gamma\right]$$

In order to ascertain the variation in argument of $\frac{f(z)+g(z)}{f(z)}$ on Γ we consider the function $w(z) = 1 + \frac{g(z)}{f(z)}$

and find that $|w-1| < \left|\frac{g(z)}{f(z)}\right| < 1$ on Γ, following the given condition. This interprets geometrically that as z traverses Γ, w traverses a circle with centre at $w = 1$ and radius $r < 1$, which does not contain origin within it. Consequently $Var\left[\arg w, \Gamma\right] = Var\left[\arg \frac{f(z)+g(z)}{f(z)}, \Gamma\right] = 0$. Hence the theorem follows.

Example 21. Find the number of zeros of the function $f(z) = z^7 - 5z + 2$ lying in the annulus $1 < |z| < 2$.

Solution. To detect the number of zeros as desired, we find the number of zeros inside the circles $|z| = 1$ and $|z| = 2$.

First we take the circle $|z| = 1$. Let $F(z) = -5z$ and $G(z) = z^7 + 2$ on $|z| = 1$.

$$|f(z)| = |z^7 - 5z + 2| \geq 5|z| - |z|^7 - 2 = 2 > 0$$

$|F(z)|=|-5z|=5$ and $|G(z)|=|z^7+2|\leq|z|^7+2=3$

So, $|F(z)|>|G(z)|$ on $|z|=1$. There $F(z)$ and $F(z)+G(z)$ i.e. $f(z)$ have same number of zeroz inside $|z|=1$, which means that $f(z)=z^7-5z+2$ has one zero inside $|z|=1$.

Now let us consider the circle $|z|=2$. We take $F(z)=z^7$ and $G(z)=-5z+2$.

On $|z|=2$,

$|f(z)|=|z^7-5z+2|\geq|z|^7-5|z|-2=116>0$

$|F(z)|=|z^7|=123$ and $|G(z)|=|1-5z+2|\leq 5|z|+2=12$

Consequently, $|F(z)|>|G(z)|$ on $|z|=2$.

So by Rouche's theorem $F(z)=z^7$ and $F(z)+G(z)=z^7-5z+2$ have the same number of zeros inside $|z|=2$. The function $F(z)=z^7$ has seven zeros inside $|z|=2$, the same is true for the given function. Finally we conclude that there lie six zeros of the function $f(z)=z^7-5z+2$ in the annulus $1<|z|<2$.

Example 22. Prove that there is one point belonging to $|z|<1$ for which $e^z=2z+1$ holds.

Solution. We consider the function

$$F(z)=2e^z-5z-1 \quad .$$

Let $f(z)=-5z$ and $g(z)+2e^z-1$. Then $|f(z)|=5|z|$ and

$$|g(z)|=|2e^z-1|\leq 2|e^z|-1\leq 2\left(1+\frac{|z|}{1!}+\frac{|z|^2}{2!}+\ldots\right)-1$$

Now consider the values of these functions on $c:|z|=1$. $|f(z)|_c=5$ and $|g(z)|_c\leq 2e-1<5$.

So $|f(z)|_c>|g(z)|_c |F(z)|_c=|f(z)+g(z)|_c\geq|f(z)|_c-|g(z)|_c>0$

Then following Rouche's theorem, $f(z)$ and $F(z)$ have the same number of zeros within c. Thus the given equation is satisfied by only one point lying in $|z|=1$ since $f(z)$ vanishes once within c.

1.4 Module Summary

We have studied to some extent what is an analytic function and how it behaves near a point where it is not analytic. These points are known as 'singularities' of the function. Our investigation is limited to singularities which are isolated in nature. Consequent to the expansion of a function in form of a Laurent series we are made familiar to two important theories : one is "Residue Theory" and the other is "Argument Principle". Applicating these theories we can evaluate real and complex integrals in a much more easier way than we did earlier and can determine the number of zeros of a function lying in its region of analyticity.

1.5 Self Assessment Questions

1. Determine and classify the singularities of the following functions :

   (i) $\dfrac{\sin \pi z}{6\varepsilon^{z-3}+z^2+3}$ (ii) $\dfrac{1-\cos z}{z^3}$ (iii) $z\sinh\dfrac{1}{z}$

2. If $z=\zeta$ is a pole of $f(z)$ then $\lim\limits_{z\to\zeta}(z-\zeta)^{n+1} f(z)=0$ for some positive integer n.

3. Show that if $f(z)$ has an essential singularity at $z=\zeta$, the same is true for $1/f(z)$.

4. Find the residue of the function $f(z)=\sin(z+1).\cos\dfrac{1}{z+1}$ at its singularity $z=-1$.

5. Find the residues of the following functions at their singularities :

   (i) $\dfrac{e^{2z}}{2z(z+2)}$ (ii) $\dfrac{e^{\frac{1}{z}}}{1+\frac{1}{z}}$ (iii) $\dfrac{e^{3z}-1}{\sin^2 3z}$

6. Using residue theory evaluate the following integrals:

   (i) $\displaystyle\int_{|z+i|=2}\frac{dz}{z(z^2+3z+2)}$ (ii) $\displaystyle\int_{|z|=4}\frac{dz}{z(z+3)(z^2+1)}$ (iii) $\displaystyle\int_{|z|=2} z^2\cos\frac{2}{z}\,dz$

7. Evaluate the following real integrals:

(i) $\int_{-\infty}^{\infty} \frac{dx}{\left(x^2+1\right)^2}$ (ii) $\int \frac{x \sin x}{x^2+1} dx$ (iii) $\int_0^{\infty} \frac{\log\left(x^2+1\right)}{x^2+1} dx$

8. Find the logarithmic residue of the function

$$f(z) = \frac{z^2+1}{\cos 2\pi z - 1}$$

with respect to the circle $|z| = 4$.

9. Find the number of zeros of the function $f(z) = z^5 - z^3 + 2z + 6$ in the right half-plane $\operatorname{Re} z > 0$.

10. Using Rouche's theorem, determine the number of zeros of the function

$$f(z) = z^4 - 2z^3 + 2z^2 + 2z + 21$$

that lie in the annulus $2 < |z| < 3$.

1.6 Suggested Further Readings

1. H. Cartan, Elementary theory of analytic functions of one or several variables, Addison-Wesley, Reading, Masachusetts, 1963.
2. E.T. Copson, An introduction to the theory of functions of a complex variable, Clarendon Press, Oxford, 1935.
3. J.W. Dettman, Applied Complex Variables, Macmillan, New York, 1965.
4. Stephen D. Fisher, Complex Variables, 2nd edition. New York. Dover Publications. 1990.
5. A.I. Markushevich, The theory of analytic functions : A Brief Course, Mir Publishers, Moscow, 1983.
6. J.E. Marsden, Basic Complex Variables, W.H. Freeman and Company, Sanfrancisco, 1973.
7. A. Sveshnikov and A. Tikhonov, The theory of functions of a complex variable, Mir Publishers, Moscow, 1973.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-I Group-C

Module No. - 9

Ordinary Differential Equations
(SERIES SOLUTION)

Content :

The equations containing one or more unknown functions or one or more variables and their ordinary or partial derivatives are called differential equations. If the differential equation involving only one independent and one dependent variables then the equation is called ordinary differential equation (ODE). If the differential equation contains more than one independent variables the equation is called partial differential equation (PDE). In this unit, we consider only ordinary differential equation, in particular, we restrict only to second order linear differential equations as these are used frequently in many physical applications.

Objectives

* Singular point of differential equation.
* Solution of ODE at the neighbourhood of ordinary point.
* Solution of ODE at the neighbourhood of singular point (Frobenius method).
* Exercise.

1.1 Second Order Linear Differential Equations

The general second order linear differential equation is of the form

$$a(z)w''(z)+b(z)w'(z)+c(z)w(z)=f(z) \tag{1.1}$$

with $a(z)\neq 0$. If $f(z)=0$ then the equation (1.1) is called *homogeneous* differential equation otherwise it is called *non-homogeneous* differential equation. The homogeneous differential equation

$$a(z)w''(z)+b(z)w'(z)+c(z)w(z)=0 \tag{1.2}$$

or equivalently

$$w''(z)+p_1(z)w'(z)+p_2(z)w(z)=0 \tag{1.3}$$

where $p_1(z)=b(z)/a(z)$ and $q(z)=c(z)/a(z)$ has two linearly independent solutions. If one independent solution is known then other independent solution may be obtained by using the method of variation of parameter.

The independent variable *z*, in general, is taken as complex variable in *z*-plane. This is convenient for theoretical study even though *z* is used as a real variable in many real life applications.

A point z_0 in the complex *z*-plane is called an *ordinary point* of the differential equation (1.3) if $p_1(z)$ and $p_2(z)$ are analytic at z_0; otherwise, the point is called a *singular point*. The singular points are of two types - *regular singular point* and *irregular singular point*. A singular point z_0 is called a *regular singular point* at z_0, if $p_1(z)$ has a pole of order not greater than one and $p_2(z)$ has a pole of order not greater than two. In other words, a point z_0 is called a regular singular point if

$$\lim_{z\to z_0}(z-z_0)p_1(z) \text{ is finite} \quad \text{and} \quad \lim_{z\to z_0}(z-z_0)^2 p_2(z) \text{ is finite.}$$

If a singular point is not a regular singular point is called *irregular singular* point.

EXAMPLE 1.1 *Examine the singular points of the (Legendre) equation*

$$\left(1-z^2\right)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+n(n+1)w=0.$$

SOLUTION : Here $p_1(z)=-\dfrac{2z}{\left(1-z^2\right)}$ and $p_2(z)=\dfrac{n(n+1)}{1-z^2}$. The singular point of this equation are $z=\pm 1$.

Now,

$$\lim_{z\to 1}(z-1)p_1(z)=\lim_{z\to 1}(z-1)\frac{-2z}{1-z^2}=\lim_{z\to 1}\frac{2z}{z+1}=1$$

and

$$\lim_{z\to 1}(z-1)^2 p_2(z) = \lim_{z\to 1}(z-1)^2 \frac{n(n+1)}{1-z^2} = \lim_{z\to 1}\frac{(z-1)(-n)(n+1)}{z+1} = 0.$$

Since these limits are finite, the point $z = 1$ is a regular singular point. It can be shown in a similar way $z = -1$ is also a regular singular point.

Example 1.2 *Determine the singular points of the differential equation*

$$2(z-2)^2 zw'' + 3zw' + (z-2)w = 0.$$

and classify them as regular or irregular.

Solution : Here $p_1(z) = \dfrac{3z}{2(z-2)^2 z}$ and $p_2(z) = \dfrac{z-2}{2(z-2)^2 z}$ The singular points are given by $2(z-2)^2 z = 0$,

i.e., $z = 0, 2$.

<u>The point $z = 0$.</u>

$$\lim_{z\to 0} zp_1(z) = \lim_{z\to 0} z\frac{3}{2(z-2)^2} = 0 \qquad \text{and}$$

$$\lim_{z\to 0} z^2 p_2(z) = \lim_{z\to 0} z^2 \frac{1}{2z(z-2)} = 0.$$

Since these limits are finite, $z = 0$ is a regular singular point.

<u>The point $z = 2$.</u>

$$\lim_{z\to 2}(z-2)p_1(z) = \lim_{z\to 2}(z-2)\frac{3}{2(z-2)^2} = \lim_{z\to 2}\frac{3}{2(z-2)}$$

does not exists

Hence $z = 2$ is an irregular singular point.

1.2 Power Series Method

Some times it may happened that the solution of a differential equation can not be obtained as a compact form, i.e., in the form of known functions. In this case, the solution of differential equation may be expressed in terms of infinite series, called *power series.*

We now state a basic theorem relating to power series solutions at the neighbourhood of an ordinary point of the equation

$$w''(z) + p_1(z)w' + p_2(z)w = 0. \tag{1.4}$$

Theorem 1.1 *If z_0 is an ordinary point of the differential equation* (1.4) *then it has two linear independent power series solutions of the form*

$$w(z)=\sum_{k=0}^{\infty}c_k(z-z_0)^k \tag{1.5}$$

at the neighbourhood of z_0.

We consider the differential equation (known as Hermite's differential equation)

$$\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+2\gamma w=0. \tag{1.6}$$

The point $z=0$ is an ordinary point of this equation. To find a series solution at the neighbourhood of $z=0$, let

$$w=\sum_{k=0}^{\infty}a_kz^k.$$

Then $\frac{dw}{dz}=\sum_{k=0}^{\infty}ka_kz^{k-1}$ and $\frac{d^2w}{dz^2}=\sum_{k=0}^{\infty}k(k-1)a_kz^{k-2}$.

Substituting these values to (1.6), we get

$$\sum_{k=0}^{\infty}k(k-1)a_kz^{k-2}-2z\sum_{k=0}^{\infty}ka_kz^{k-1}+2\gamma\sum_{k=0}^{\infty}a_kz^k=0$$

i.e.,

$$\sum_{k=0}^{\infty}k(k-1)a_kz^{k-2}-2\sum_{k=0}^{\infty}ka_kz^k+2\gamma\sum_{k=0}^{\infty}a_kz^k=0.$$

Equating the coefficient of z^k to zero, we obtain

$$(k+2)(k+1)a_{k+2}-2ka_k+2\gamma a_k=0$$

so that

$$a_{k+2}=-\frac{2(\gamma-k)}{(k+1)(k+2)}a_k.$$

Substituting $k=0,1,2,\ldots\ldots$ we have

$$a_2=-\frac{2\gamma}{1.2}a_0=-\frac{2\gamma}{2!}a_0$$

$$a_3=-\frac{2(\gamma-1)}{2.3}a_1=-\frac{2(\gamma-1)}{3!}a_1$$

$$a_4 = -\frac{2(\gamma-2)}{3.4}a_2 = (-1)^2\frac{2^2\gamma(\gamma-2)}{4!}a_0$$

$$a_5 = -\frac{2(\gamma-3)}{4.5}a_3 = (-1)^2\frac{2^2(\gamma-1)(\gamma-3)}{5!}a_1$$

and so on.

Hence

$$w(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + ...$$

$$= (a_0 + a_2 z^2 + a_4 z^4 + ...) + (a_1 z + a_3 z^3 + a_5 z^5 + ...)$$

$$= a_0\left[1 - \frac{2\gamma}{2!}z^2 + \frac{2^2\gamma(\gamma-2)}{4!}z^4 - .....\right]$$

$$+ a_1\left[z - \frac{2(\gamma-1)}{3!}z^3 + \frac{2^2(\gamma-1)(\gamma-3)}{5!}z^5 - .....\right]$$

$$= a_0 w_0(z) + a_1 w_1(z)$$

where a_0 and a_1 are arbitrary constants and

$$w_0(z) = 1 - \frac{2\gamma}{2!}z^2 + \frac{2^2\gamma(\gamma-2)}{4!}z^4 - ....$$

$$w_1(z) = z - \frac{2(\gamma-1)}{3!}z^3 + \frac{2^2(\gamma-1)(\gamma-3)}{5!}z^5 - ....$$

This is the required series solution of the differential equation (1.6). It can be shown that $w_0(z)$ and $w_1(z)$ are two linearly independent solutions of (1.6).

1.3 Singularity at Infinity

To determine whether the point at infinite is a singular point or not, we transform the equation by $z = 1/t$. Then

$$\frac{dw}{dz} = -t^2\frac{dw}{dt},$$

$$\frac{d^2w}{dz^2} = \frac{d}{dt}\left(-t^2\frac{dw}{dt}\right)\frac{dt}{dz} = t^4\frac{d^2w}{dt^2} + 2t^3\frac{dw}{dt}$$

Therefore from (1.4),

$$\frac{d^2w}{dt^2}+\left(\frac{2}{t}-\frac{1}{t^2}p_1(1/t)\right)\frac{dw}{dt}+\frac{1}{t^4}p_2(1/t)w=0 \tag{1.7}$$

If $t=0$ is a singular point of (1.7) then the original equation (1.4) has a singularity at $z=\infty$.

The condition that $z=\infty$ should be an ordinary point of (1.4) is stated below:

If $t=0$ is an ordinary point of the equation (1.7) then $2t-p_1(1/t)$ or $2/z-p_1(z)$ should have a zero of second order at $t=0$ or $z=\infty$; $p_2(1/t)$ or, $p_2(z)$ should have a zero of forth order at $t=0$ or, $z=\infty$.

Example 1.3 *Show that the equation*

$$\frac{d^2w}{dz^2}-\frac{2z}{1-z^2}\frac{dw}{dz}+\frac{n(n+1)}{1-z^2}w=0 \tag{1.8}$$

has a singularity at $z=\infty$.

Solution : Substituting $z=1/t$ to the given equation. Then

$$\frac{dw}{dz}=-t^2\frac{dw}{dt},$$

$$\frac{d^2w}{dz^2}=\frac{d}{dt}\left(-t^2\frac{dw}{dt}\right)\frac{dt}{dz}=t^4\frac{d^2w}{dt^2}+2t^3\frac{dw}{dt}.$$

Using these substitution the given equation reduces to

$$t^4\frac{d^2w}{dt^2}+2t^3\frac{dw}{dt}-\frac{2(1/t)}{1-1/t^2}\left(-t^2\frac{dw}{dt}\right)+\frac{n(n+1)}{1-1/t^2}w=0$$

or, $$t^4\frac{d^2w}{dt^2}+\frac{2t^3}{t^2-1}\frac{dw}{dt}+\frac{n(n+1)t^2}{t^2-1}w=0.$$

Since $t=0$ is a singularity of $p_2(1/t), z=\infty$ is a singularity of the given equation.

The method to obtain the solution of the equation (1.9) near a singular point is known as *Frobenius* method which is described below.

1.4 Frobenius Method

Now we discuss about the solution of a differential equation at the neighbourhood of a singular point. The method to obtain the solution at the neighbourhood of a singular point is not straight forward like ordinary point. Due to the presence of singularity, the form of the solution is also changed, which is stated below:

Theorem 1.2 *The necessary and sufficient condition that the integrals in the neighbourhood of a singularity $z = z_0$ of the equation*

$$\frac{d^2w}{dz^2} + p_1(z)\frac{dw}{dz} + p_2(z)w = 0 \tag{1.9}$$

are regular is that $p_1(z)$ has a pole at most of order one at the point z_0 and $p_2(z)$ has a pole at most of order two at z_0.

Proof. The condition is necessary :

Let the integrals of equation (1.9) be

$$w_1 = (z-z_0)^{\rho_1}\sum_{k=0}^{\infty} a_k (z-z_0)^k \text{ and} \tag{1.10}$$

$$w_2 = (z-z_0)^{\rho_2}\sum_{k=0}^{\infty} b_k (z-z_0)^k$$

or

$$w_1 = (z-z_0)^{\rho_1}\sum_{k=0}^{\infty} a_k (z-z_0)^k \text{ and} \tag{1.11}$$

$$w_2 = w_1\left[A\log(z-z_0) + \sum_{k=0}^{\infty} b_k (z-z_0)^k\right].$$

The differential equation (1.9) can be written as

$$w'' - \frac{w_1''w_2 - w_1w_2''}{w_1'w_2 - w_1w_2'}w' + \frac{w_1''w_2' - w_1'w_2''}{w_1'w_2 - w_1w_2'}w = 0 \tag{1.12}$$

where w_1 and w_2 are two fundamental solutions of (1.9).

[This equation is obtain from the Wronskian $\begin{vmatrix} w & w' & w'' \\ w_1 & w_1' & w_1'' \\ w_2 & w_2' & w_2'' \end{vmatrix} = 0.$]

Therefore,

$$p_1(z) = -\frac{w_1''w_2 - w_1w_2''}{w_1'w_2 - w_1w_2'} = -\frac{d}{dz}\left[\log\left\{w_1^2\frac{d}{dz}\left(\frac{w_2}{w_1}\right)\right\}\right].$$

Now, $\frac{w_2}{w_1} = A\log(z-z_0) + (z-z_0)^{\rho_2-\rho_1} P_3$, where P_3 is the power series. Then

$$\frac{d}{dz}\left(\frac{w_2}{w_1}\right)=\frac{A}{(z-z_0)}+(\rho_2-\rho_1)(z-z_0)^{\rho_2-\rho_1-1}P_3+(z-z_0)^{\rho_2-\rho_1}P_3'.$$

Therefore, $w_1^2\frac{d}{dz}\left(\frac{w_2}{w_1}\right)=A(z-z_0)^k P_4+P_5$, where P_3', P_4, P_5 are power series in $(z-z_0)$.

This gives

$$\log\left\{w_1^2\frac{d}{dz}\left(\frac{w_2}{w_1}\right)\right\}=\log A+k\log(z-z_0)+P_6,$$

where P_6 is the power series of $(z-z_0)$. Finally,

$$\frac{d}{dz}\left[\log\left\{w_1^2\frac{d}{dz}\left(\frac{w_2}{w_1}\right)\right\}\right]=\frac{k}{z-z_0}+P_7,$$

where P_7 is the power series of $(z-z_0)$.

Thus $p_1(z)=-\left(\frac{k}{z-z_0}+P_7\right)$.

Hence $p_1(z)$ has a pole of order one at most $z=z_0$.

Since w_1 is a solution of the given differential equation (1.9), therefore, we must have

$$w_1''+p_1(z)w_1'+p_2(z)w_1=0$$

i.e.,

$$p_2(z)=-\frac{w_1''}{w_1}-p_1(z)\frac{w_1'}{w_1}.$$

Since $\frac{w_1''}{w_1}$ has a pole at most of order two. Also, $p_1(z)$ and $\frac{w_1''}{w_1}$ both have pole of order one at most at $z=z_0$. Hence there product has a pole at most of order two. Therefore, $p_2(z)$ has a pole of order two at most at $z=z_0$. Hence the form of the equation near its regular singularity $z=z_0$ is

$$w''+\frac{p_1(z)}{z-z_0}w'+\frac{p_2(z)}{(z-z_0)^2}w=0$$

where $p_1(z)$ and $p_2(z)$ are power series near the point $z=z_0$.

<u>The condition is sufficient.</u>

The conditions (i) $p_1(z)$ should have a pole at most of order one and (ii) $p_2(z)$ have a pole at most of order two are the sufficient condition for the existence of regular integral of the differential equation

$$w'' + \frac{p_1(z)}{z-z_0}w' + \frac{p_2(z)}{(z-z_0)^2}w = 0$$

$$w'' + \frac{p_1(z)}{z}w' + \frac{p_2(z)}{z^2}w = 0 \tag{1.13}$$

(for simplicity we assume that z_0=0) where $p_1(z)$ and $p_2(z)$ are analytic at $z = 0$, i.e., they are of the form

$$p_1(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + ... = \sum_{n=0}^{\infty} \alpha_n z^n \text{ and}$$

$$p_2(z) = \beta_0 + \beta_1 z + \beta_2 z^2 + ... = \sum_{n=0}^{\infty} \beta_n z^n.$$

We have to show that (1.13) have a solution of the form

$$w = \sum_{k=0}^{\infty} c_k z^{\rho+k} \tag{1.14}$$

where c_k are the constants. Substituting the values of w, p_1 and p_2 in (1.13) we obtain

$c_0(\rho-1)\rho z^{\rho} + c_1(\rho+1)\rho z^{\rho+1} + ...+ c_k(\rho+k)(\rho+k-1)z^{\rho+k} + (\alpha_0 + \alpha_1 z + ...)\ (c_0 \rho z^{\rho} +$
$c_1(\rho+1)z^{\rho+1} + ...) + (\beta_0 + \beta_1 z + ...)(c_0 z^{\rho} + c_1 z^{\rho+1} + ...) = 0.$

Equating the coefficients of the different powers of z starting from the lowest power namely z^{ρ}, we have

$$\begin{aligned}
&c_0\{\rho(\rho-1) + \alpha_0\rho + \beta_0\} = 0\\
&c_1\{\rho(\rho+1) + \alpha_0(\rho+1) + \beta_0\} + c_0(\rho\alpha_1 + \beta_1) = 0\\
&c_2\{(\rho+1)(\rho+2) + \alpha_0(\rho+2) + \beta_0\} + c_1\{(\rho+1)\alpha_1 + \beta_1\} + c_0(\rho\alpha_2 + \beta_2) = 0 \qquad (1.15)\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&c_k\{(\rho+k)(\rho+k-1) + \alpha_0(\rho+k) + \beta_0\} + c_{k-1}\{(\rho+k-1)\alpha_1 + \beta_1\} + ...\\
&+ c_0(\rho\alpha_k + \beta_k) = 0.
\end{aligned}$$

These equations can be written in the simplest form as

$$\begin{aligned}
&c_0 f_0(\rho) = 0 \text{ where } f_0(\rho) = \rho(\rho-1) + \alpha_0\rho + \beta_0\\
&c_1 f_0(p+1) + c_0 f_1(\rho) = 0\\
&c_2 f_0(p+2) + c_1 f_1(\rho+1) + c_0 f_2(\rho) = 0 \qquad (1.16)
\end{aligned}$$

$$c_3 f_0(\rho+3)+c_2 f_1(\rho+2)+c_1 f_2(\rho+1)+c_0 f_3(\rho)=0$$

....

$$c_k f_0(\rho+k)+c_{k-1} f_1(\rho+k-1)+\ldots+c_0 f_k(\rho)=0,$$

where $f_k(\rho)=\rho\alpha_k+\beta_k, k\geq 1$.

Equations (1.15) or (1.16) give the coefficients c_1, c_2, ..., c_k in terms of c_0. The equation $f_0(\rho)=\rho(\rho-1)+\alpha_0\rho+\beta_0=0$ is called *indicial equation* corresponding to the singularity at $z=0$. The values of ρ are called the *exponents* at the singularity; they determine the qualitative nature of the solution in the neighbourhood of the singular point. Depending on the values of ρ, i.e., the roots of the indicial equation, the solutions are written into two different forms; these are discussed in the following.

Case I. *Roots are distinct and do not differ by an integer*, i.e., $\rho_1\neq\rho_2$ *and* $\rho_1-\rho_2\neq$ *an integer.*

Since $\rho_1-\rho_2\neq$ an integer or zero therefore, $f_k(\rho_1+k)\neq 0$ unless $k=0$. Thus c_k, $k=0, 1, \ldots$ can always be calculated in terms of previous coefficients. Taking $\rho=\rho_1$ and $\rho=\rho_2$ we get two sets of values of $c_0, c_1, \ldots$ which give the two fundamental solutions

$$w_1=z^{\rho_1}\sum_{k=0}^{\infty}c_k z^k, \quad \text{and} \quad w_2=z^{\rho_2}\sum_{k=0}^{\infty}c_k' z^k$$

where c_k and c_k' are two sets of constants.

Case II. *Roots are equal or differ by an integer, i.e.,* $\rho_1=\rho_2$ *and* $\rho_1-\rho_2=$ *an integer.* When $\rho_1-\rho_2=0$, i.e., $\rho_1=\rho_2$ we get only one solution of the form $w_1=z^{\rho_1}\sum_{k=0}^{\infty}c_k z^k$.

Again when $\rho_1=\rho_2+n$ (say) taking $\rho_1>\rho_2$. Starting with ρ_1 the equations (1.15) or (1.16) give all the c_k's. But for ρ_2 we have to determine c_k's to get the second independent solution. Since one solution is known, the second solution is to be determined by the method of variation of parameter.

Let w_1 be the solution obtain for ρ_1 the larger value of ρ, obtain from the indicial equation. Let the other solution be

$$w=w_1\int u\,dz \quad \text{(where } u \text{ is unknown).} \tag{1.17}$$

Therefore,

$$w'=w_1'\int u\,dz+w_1 u \tag{1.18}$$

and

$$w'' = w_1'' \int u\,dz + 2uw_1' + w_1u'. \tag{1.19}$$

Multiplying (1.19) by z^2, (1.18) by zp_1 and (1.17) by p_2 and adding we get

$$z^2w'' + zp_1w' + p_2w = \left[z^2w_1'' + zp_1w_1' + p_2w_1\right]\int u\,dz + z^2u\left[2w_1' + w_1p_1/z\right] + z^2u'w_1 = 0.$$

Remembering, w_1 is a solution of (1.13), we have

$$z^2u\left[2w_1' + w_1p_1/z\right] + z^2u'w_1 = 0.$$

Dividing both sides by z^2 we have

$$u\left[w_1p_1/z + 2w_1'\right] + u'w_1 = 0.$$

i.e.

$$u' = -u\left[p_1/z + 2w_1'/w_1\right].$$

Since $w = z^\rho \sum_0^\infty c_kz^k$, then

$$w' = \rho z^{\rho-1}\sum_0^\infty c_kz^k + z^\rho \sum_0^\infty kc_kz^{k-1}.$$

Therefore, the ratio $2w_1'/w_1$ is given by

$$\frac{2w_1'}{w_1} = \frac{2\rho_1}{z}\left(\frac{c_0' + c_1'z + c_2'z^2 + \ldots}{c_0 + c_1z + c_2z^2 + \ldots}\right)$$

$$= \frac{2\rho_1}{z} + h_1(z) \quad \text{and} \quad \frac{p_1}{z} = \frac{\alpha_0}{z} + h_2(z).$$

where $h_1(z)$ is an analytical function in the neighbourhood of $z = 0$. Therefore,

$$u' = -u\left[\frac{\alpha_0 + 2\rho_1}{z} + h_3(z)\right].$$

If $\rho_2 - \rho_1 = n$ (say) then we have from the indicial equation $\rho_1 + \rho_2 = 1 - \alpha_0$. This gives $2\rho_1 + \alpha_0 = 1 + n$. Hence

$$\frac{u'}{u} = -\left[\frac{1+n}{z} + h_3(z)\right].$$

Integrating,

$$\log u = -\int\left[\frac{1+n}{z} + h_3(z)\right]dz$$

i.e., $u=-e^{-(n+1)\log z-\int h_3 dz}$

$$=-\frac{1}{z^{n+1}}e^{-\int h_3 dz}$$

$$=\frac{1}{z^{n+1}}\left[A_0+A_1z+A_2z^2+....\right]$$

Therefore,

$$\int u\,dz=\int\left\{\frac{A_n}{z}+\frac{1}{z^{n+1}}\left[A_0+A_1z+...+A_{n-1}z^{n-1}+....+A_{n+1}z^{n+1}+...\right]\right\}dz$$

$$=A_n\log z+h_4(z)$$

$$=A_n\log z+\sum_0^{\infty}b_kz^k.$$

Hence,

$$w_2=w_1\int u\,dz=z^{\rho_1}\sum_0^{\infty}c_kz^k\left[A\log z+\sum_0^{\infty}b_kz^k\right].\qquad(1.20)$$

Note. If $\rho_1=\rho_2$ then the second solution can be obtain by another simple method discussed below:

Let us consider the relation

$$w''+p_1(z)w'+p_2(z)w=(\rho-\rho_1)^2z^{\rho-2}.$$

Differentiating with respect to ρ,

$$\frac{d^2}{dz^2}\left(\frac{dw}{d\rho}\right)+p_1(z)\frac{d}{dz}\left(\frac{dw}{d\rho}\right)+p_2(z)\left(\frac{dw}{d\rho}\right)=(\rho-\rho_1)^2\frac{d}{d\rho}\left(z^{\rho-2}\right)+2(\rho-\rho_1)z^{\rho-2}.$$

When $\rho=\rho_1$ then the right hand side is zero, i.e. $\left.\frac{dw}{d\rho}\right)_{\rho=\rho_1}$ satisfies the differential equation and hence it also a solution of (1.9). Hence the other independent solution can be determined by

$$w_2=\left.\frac{dw}{d\rho}\right)_{\rho_1}\qquad(1.21)$$

when the roots of indicial equation are equal.

EXAMPLE 1.4 *Solve the equation*

$$z(1-z)w'' + (1/2 - z)w' + w = 0 \tag{1.22}$$

at the neighbourhood of z = 0.

SOLUTION : The point $z = 0$ is a singular point. Let

$$w = \sum_{k=0}^{\infty} a_k z^{\rho+k} \tag{1.23}$$

be the solution of (1.22).

Substituting (1.23) in (1.22) we have

$$z(1-z)\sum_{k=0}^{\infty}\{a_k(\rho+k)(\rho+k-1)z^{\rho+k-2}\} + (1/2 - z)\sum_{k=0}^{\infty} a_k(\rho+k)z^{\rho+k-1} + \sum_{k=0}^{\infty} a_k z^{\rho+k} = 0.$$

Equating the coefficient of $z^{\rho-1}$ to zero we obtain the indicial equation as

$$\rho(\rho-1) + \rho/2 = 0 \qquad \rho = 0, 1/2$$

Again, equating the coefficient of $z^{\rho+k}$ to zero, we have

$$a_{k+1}\left[(\rho+k+1)(\rho+k) + \frac{1}{2}(\rho+k+1)\right] - a_k\left[\{(\rho+k)(\rho+k-1)\} + (\rho+k) - 1\right] = 0$$

$$\text{or, } a_{k+1}(\rho+k+1)(\rho+k+1/2) = a_k\{(\rho+k)^2 - 1\}$$

$$\text{or, } \frac{a_{k+1}}{a_k} = \frac{(\rho+k)^2 - 1}{(\rho+k+1)(\rho+k+1/2)}, \qquad k = 0, 1, 2, \ldots\ldots.$$

Therefore,

$$\frac{a_1}{a_0} = \frac{\rho^2 - 1}{(\rho+1)(\rho+1/2)}, \; \frac{a_2}{a_1} = \frac{(\rho+1)^2 - 1}{(\rho+2)(\rho+3/2)}, \ldots..$$

Thus

$$w = a_0 z^{\rho}\left[1 + \frac{a_1}{a_0}z + \frac{a_2}{a_1}\frac{a_1}{a_0}z^2 + \frac{a_3}{a_2}\frac{a_2}{a_1}\frac{a_1}{a_0}z^3 + \ldots\right]$$

$$= a_0 z^{\rho}\left[1 + \frac{(\rho^2 - 1)}{(\rho+1)(\rho+1/2)}z + \frac{(\rho+1)^2 - 1}{(\rho+2)(\rho+3/2)}\frac{(\rho^2 - 1)}{(\rho+1)(\rho+1/2)}z^2 + \ldots\right]$$

One fundamental solution for $\rho = 0$ is given by

$$w_0 = a_0(1-2z)$$

and the other fundamental solution for $\rho = 1/2$ is

$$w_1 = a_0 z^{1/2}\left[1 + \frac{(1/2)^2 - 1}{(3/2)\cdot 1} z + \frac{\{(1/2)^2 - 1\}\{(3/2)^2 - 1\}}{1\cdot 3/2\cdot 2\cdot 5/2} z^2 \right.$$

$$\left. + \frac{\{(1/2)^2 - 1\}\{(3/2)^2 - 1\}\{(5/2)^2 - 1\}}{1\cdot 3/2\cdot 2\cdot 5/2\cdot 3\cdot 7/2} z^3 + \ldots\right]$$

$$= a_0\sqrt{z}\left[1 + \frac{(1^2-4)}{3!} z + \frac{(1^2-4)(3^2-4)}{5!} z^2 + \frac{(1^2-4)(3^2-4)(5^2-4)}{7!} z^3 + \ldots\right]$$

Therefore the general solution is

$$w(z) = c_0 w_0(z) + c_1 w_1(z).$$

where c_0 and c_1 are arbitrary constants.

Example 1.5 *Obtain the solution in the neighbourhood of the origin of the following equation*

$$z\frac{d^2w}{dz^2} + \frac{dw}{dz} - w = 0. \qquad (1.24)$$

Solution : Let the solution be

$$w = \sum_0^\infty c_k z^{\rho+k}.$$

Then

$$\frac{dw}{dz} = \sum_0^\infty c_k(\rho+k) z^{\rho+k-1} \quad \text{and} \quad \frac{d^2w}{dz^2} = \sum_0^\infty c_k(\rho+k)(\rho+k-1) z^{\rho+k-2}$$

Substituting these values to (1.24) we have

$$\sum_0^\infty c_k(\rho+k)(\rho+k-1) z^{\rho+k-1} + \sum_0^\infty c_k(\rho+k) z^{\rho+k-1} - \sum_0^\infty c_k z^{\rho+k} = 0.$$

Equating lowest power of z, i.e., $z^{\rho-1}$ to zero we get the indicial equation as

$$(\rho-1)\rho + \rho = 0 \qquad \text{or} \qquad \rho = 0, 0.$$

Equating the coefficient of $z^{\rho+k}$ to zero we obtain

$$(\rho+k)(\rho+k+1)c_{k+1}+(\rho+k+1)c_{k+1}-c_k=0.$$

Therefore,

$$\frac{c_{k+1}}{c_k}=\frac{1}{(\rho+k+1)^2}.$$

Substituting $k=0, 1, \ldots\ldots$ successively to find the values of $c_1, c_2, \ldots.$ as follows :

$$c_1=\frac{1}{(\rho+1)^2}c_0$$

$$c_2=\frac{1}{(\rho+2)^2}c_1=\frac{1}{(\rho+1)^2(\rho+2)^2}c_0$$

and so on.

Thus

$$w=c_0z^\rho\left[1+\frac{1}{(\rho+1)^2}z+\frac{1}{(\rho+1)^2(\rho+2)^2}z^2+\ldots\right]. \qquad (1.25)$$

Let w be

$$w=c_0z^\rho F(\rho), \qquad (1.26)$$

where

$$F(\rho)=1+\frac{1}{(\rho+1)^2}z+\frac{1}{(\rho+1)^2(\rho+2)^2}z^2+\ldots.$$

On substituting $\rho=0$ in (1.25) we have

$$w_1=c_0\left(1+z+\frac{1}{1^2.2^2}z^2+\frac{1}{1^2.2^2.3^2}z^3+\ldots.\right). \qquad (1.27)$$

The other solution is given by the relation

$$w_2=\left.\frac{\partial w}{\partial \rho}\right)_{\rho=0}=\left[c_0z^\rho\log z\cdot F(\rho)+c_0z^\rho\frac{\partial F(\rho)}{\partial \rho}\right]_{\rho=0}$$

$$=c_0\left(\frac{\partial F}{\partial \rho}\right)_{\rho=0}+c_0\log z\cdot F(0).$$

The general term of $F(\rho)$ is

$$\frac{z^n}{(\rho+1)^2(\rho+2)^2...(\rho+n)^2} = y \text{ (say)}.$$

Taking logarithm on both sides and differentiating we obtain

$$\frac{1}{y}\frac{\partial y}{\partial \rho} = -2\left[\frac{1}{\rho+1}+\frac{1}{\rho+2}+....+\frac{1}{\rho+n}\right].$$

This gives

$$\frac{\partial y}{\partial \rho} = -2\frac{z^n}{(\rho+1)^2(\rho+2)^2...(\rho+n)^2}\left[\frac{1}{\rho+1}+\frac{1}{\rho+2}.....+\frac{1}{\rho+n}\right].$$

At $\rho = 0$, the above relation finally gives

$$\left.\frac{\partial y}{\partial \rho}\right)_{\rho=0} = -2\frac{z^n}{1^2\cdot 2^2...n^2}\left[1+\frac{1}{2}+...+\frac{1}{n}\right].$$

Hence

$$w_2 = -2c_0\left[\frac{z}{1^2}\cdot 1+\frac{z^2}{1^2\cdot 2^2}\left(1+\frac{1}{2}\right)+\frac{z^3}{1^2\cdot 2^2\cdot 3^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)\right.$$
$$\left.+\frac{z^4}{1^2\cdot 2^2\cdot 3^2\cdot 4^2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+...\right]+c_0\log z\cdot F(0)$$
$$= -2c_0\left[z+\frac{z^2}{1^2\cdot 2^2}\cdot\frac{3}{2}+\frac{z^3}{1^2\cdot 2^2\cdot 3^2}\cdot\frac{11}{6}+\frac{z^4}{1^2\cdot 2^2\cdot 3^2\cdot 4^2}\cdot\frac{50}{24}+....\right]$$
$$+c_0\log z\left[1+z+\frac{1}{1^2\cdot 2^2}z^2+\frac{1}{1^2\cdot 2^2\cdot 3^2}z^3+....\right]. \quad (1.28)$$

Hence the two independent solutions w_1 and w_2 of the equation (1.24) are given by (1.27) and (1.28).

Example 1.6 *Solve the differential equation*

$$z^2w'' + 4zw' + (z^2+2)w = 0 \quad (1.29)$$

by finding two independent solutions.

Solution : It is easy to note that $z = 0$ is a regular singular point. The given equation should have at least one solution of the form

$$w=\sum_{k=0}^{\infty}a_k z^{\rho+k}$$

Substituting this in the given equation, we get

$$\sum_{k=0}^{\infty}a_k(\rho+k)(\rho+k-1)z^{\rho+k}+\sum_{k=0}^{\infty}4a_k(\rho+k)z^{\rho+k}$$

$$+\sum_{k=0}^{\infty}a_k z^{\rho+k+2}+\sum_{k=0}^{\infty}2a_k z^{\rho+k}=0. \qquad (1.30)$$

Equating the coefficient of lowest power of z i.e., the coefficient of z^{ρ} to zero, we obtain the indicial equation

$$\{\rho(\rho-1)+4\rho+2\}a_0=0$$

this gives $\rho^2+3\rho+2=0$ or, $\rho^2=-1,-2$. Equating the coefficient of $z^{\rho+1}$ to zero in equation (1.30), we have $a_1(\rho^2+5\rho+6)=0$. For $\rho=-1$ as well as for $\rho=-2$ this implies that $a_1=0$.

By equating the coefficient of $z^{\rho+k+2}$ to zero in equation (1.30), we get

$$a_{k+2}\left[(\rho+k+2)(\rho+k+5)+2\right]=-a_k. \qquad (1.31)$$

For $\rho=-1$, we have

$$a_{k+2}=-\frac{a_k}{(k+2)(k+3)}$$

and this gives

$$a_k=\frac{(-1)^{k/2}}{(k+1)!}a_0, \quad k=2,4,6,.....$$

Since $a_1=0$, it follows that a_k with odd k vanish.

Hence, corresponding to the root $\rho=-1$, we have the solution

$$w_1(z)=a_0z^{-1}\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-...\right)=a_0\frac{\sin z}{z^2}.$$

Since the difference between the roots is an integer, the second solution is given by the relation

$$w_2(z)=Aw_1(z)\log z+z^{-2}\sum_{k=0}^{\infty}b_k z^k. \qquad (1.32)$$

Substituting this and the derivatives into the differential equation, we get

$$A\left(-w_1+2zw_1'+z^2w_1''\log z\right)+4A\left(w_1+zw_1'\log z\right)+A\left(z^2+2\right)w_1\log z$$

$$+\sum_{k=0}^{\infty} b_k(k-2)(k-3)z^{k-2} + \sum_{k=0}^{\infty} 4b_k(k-2)z^{k-2} + \sum_{k=0}^{\infty} b_k z^k + 2\sum_{k=0}^{\infty} b_k z^{k-2} = 0.$$

Equating the coefficient of z^{-1} to zero, we get $Aa_0 = 0$ which gives $A = 0$. Again, equating the coefficient of z^k to zero we obtain

$$b_{k+2}k(k-1) + 4b_{k+2}k + b_k + 2b_{k+2} = 0$$

which gives

$$b_{k+2} = -\frac{b_k}{(k+1)(k+2)}$$

We take $b_1 = 0$, because the odd powers just give a constant multiple of w_1. Then

$$b_k = (-1)^{k/2}\frac{1}{k!}b_0, \quad k = 2,4,6,......$$

Therefore, the other solution is

$$w_2(z) = b_0 z^{-2}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ...\right] = b_0\frac{\cos z}{z^2}.$$

Example 1.7 *Find the series solution of the following equation at the neighbour of $z = 0$.*

$$2z^2w'' - zw' + (1 - z^2)w = 0.$$

Solution : Since $z = 0$ is a singular point, let

$$w = \sum_{k=0}^{\infty} a_k z^{\rho+k}$$

be the series solution of the equation

$$2z^2w'' - zw' + (1 - z^2)w = 0. \tag{1.33}$$

Therefore,

$$w' = \sum_{k=0}^{\infty} (\rho + k)a_k z^{\rho+k-1}$$

and

$$w'' = \sum_{k=0}^{\infty} (\rho + k)(\rho + k - 1)a_k z^{\rho+k-2}$$

Substituting these values in (1.33) we obtain

$$2z^2\sum_{k=0}^{\infty}(\rho+k)(\rho+k-1)a_k z^{\rho+k-2}-z\sum_{k=0}^{\infty}(\rho+k)a_k z^{\rho+k-1}$$

$$+(1-z^2)\sum_{k=0}^{\infty}a_k z^{\rho+k}=0. \qquad (1.34)$$

Equating the coefficient of the lowest power of z, i.e. of z^p to zero we get the indicial equation

$$2a_0(\rho+0)(\rho-1)-a_0(p+0)+a_0=0$$

or, $2\rho(\rho-1)-\rho+1=0$

or, $2\rho^2-3\rho+1=0$

or, $(\rho-1)(2\rho-1)=0.$

Therefore, $\rho=1,1/2$

Euqating the cofficient of $z^{\rho+k+2}$ to zero we get

$$2a_{k+2}(\rho+k+2)(\rho+k+1)-a_{k+2}(\rho+k+2)+a_{k+2}-a_k=0$$

or, $a_{k+2}(\rho+k+2)(2\rho+2k+1)+a_{k+2}=a_k$

or, $a_{k+2}\{2(\rho^2+k^2)+5(\rho+k)+4\rho k+3\}=a_k$

or, $a_{k+2}=\dfrac{a_k}{2(\rho^2+k^2)+5(\rho+k)+4\rho k+3}.$

Equating the coefficient of $z^{\rho+1}$ in (1.34) to zero we get,

$$2a_1(\rho+1)\rho-a_1(\rho+1)+a_1=0$$

or, $a_1=0.$

Putting $k=0, 1, 2, \ldots\ldots$ we obtain

$$a_2=\frac{1}{2\rho^2+5\rho+3}a_0$$

$$a_3=\frac{1}{2(\rho^2+1^2)+5(\rho+1)+4\rho\cdot1+3}a_1=0$$

$$a_4=\frac{1}{2(\rho^2+2^2)+5(\rho+2)+4\rho\cdot2+3}a_2$$

$$=\frac{1}{2\{(\rho^2+2^2)+5(\rho+2)+4\rho\cdot2+3\}\{2\rho^2+5\rho+3\}}a_0$$

$$a_5 = 0$$

$$a_6 = \frac{1}{\{2(\rho^2+4^2)+5(\rho+4)+4\rho\cdot 4+3\}\{2(\rho^2+2^2)+5(\rho+2)+4\rho\cdot 2+3\}\{2\rho^2+5\rho+3\}} a_0$$

and so on.

Therefore,

$$w(z) = \sum_{k=0}^{\infty} a_k z^{\rho+k} = a_0 z^\rho + a_1 z^{\rho+1} + a_2 z^{\rho+2} + \ldots$$

$$= a_0 z^\rho \left\{ 1 + \frac{z^2}{2\rho^2+5\rho+3} + \frac{z^4}{\{2(\rho^2+2^2)+5(\rho+2)+4\cdot\rho\cdot 2+3\}(2\rho^2+5\rho+3)} + \ldots \right\}.$$

When $\rho = 1$,

$$w_1(z) = a_0 z \left\{ 1 + \frac{z^2}{2.1^2+5.1+3} \right.$$

$$\left. + \frac{z^4}{\{2(1^2+2^2)+5(1+2)+4\cdot 1\cdot 2+3\}(2\cdot 1^2+5\cdot 1+3)} + \ldots \right\}$$

$$= a_0 z \left\{ 1 + \frac{z^2}{10} + \frac{z^4}{360} + \ldots \right\}$$

When $\rho = \frac{1}{2}$

$$w_2(z) = a_0 z^{1/2} \left\{ 1 + \frac{z^2}{2\cdot(\frac{1}{2})^2 + 5\cdot\frac{1}{2} + 3} \right.$$

$$\left. + \frac{z^4}{\{2((\frac{1}{2})^2+2^2)+5(\frac{1}{2}+2)+4\cdot\frac{1}{2}\cdot 2+3\}(2(\frac{1}{2})^2+5\cdot\frac{1}{2}+3)} + \ldots \right\}.$$

$$= a_0 \sqrt{z} \left\{ 1 + \frac{z^2}{6} + \frac{z^4}{168} + \ldots \right\}$$

Therefore, the general solution of the given equation is $w(z) = Aw_1(z) + Bw_2(z)$ where A, B are arbitrary constants.

Example 1.8 *Find the solution of the differential equation*

$$z\frac{d^2w}{dz^2}+\frac{dw}{dz}+zw=0$$

at the neighbourhood of $z = 0$.

Solution : Clearly, $z = 0$ is a singular point of the given equation.

Let

$$w(z)=\sum_{k=0}^{\infty}a_k z^{\rho+k}$$

be a series solution of the given equation

Then

$$w'(z)=\sum_{k=0}^{\infty}a_k(\rho+k)z^{\rho+k-1}$$

$$w''(z)=\sum_{k=0}^{\infty}a_k(\rho+k)(\rho+k-1)z^{\rho+k-2}.$$

Substituting these values to the given equation we obtain

$$z\sum_{k=0}^{\infty}(\rho+k)(\rho+k-1)a_k z^{\rho+k-2}+\sum_{k=0}^{\infty}(\rho+k)a_k z^{\rho+k-1}$$

$$+z\sum_{k=0}^{\infty}a_k z^{\rho+k}=0. \quad (1.35)$$

Equating the coefficient of the lowest power of z, i.e., of $z^{\rho-1}$ to zero we get the indicial equation

$$a_0\rho(\rho-1)+a_0\rho=0$$

or, $\rho^2-\rho+\rho=0$

or, $\rho=0,0.$

Again, equating the coefficient of z^{ρ} in (1.35) to zero we get

$$a_1(\rho+1)\rho+a_1(\rho+1)=0$$

or,

$$a_1=0.$$

Equating the coefficient of $z^{\rho+k}$ to zero we get

$$a_{k+1}(\rho+k+1)(\rho+k)+a_{k+1}(\rho+k+1)+a_{k-1}=0$$

or, $a_{k+1}(\rho+k+1)^2=-a_{k-1}$

or, $$a_{k+1}=-\frac{1}{(\rho+k+1)^2}a_{k-1}.$$

Putting $k=1,2,\ldots$ we get

$$a_2=-\frac{1}{(\rho+1+1)^2}a_0=-\frac{1}{(\rho+2)^2}a_0$$

$$a_3=-\frac{1}{(\rho+2+1)^2}a_1=0$$

$$a_4=-\frac{1}{(\rho+3+1)^2}a_2=\frac{1}{(\rho+2)^2(\rho+4)^2}a_0$$

$$a_5=0$$

$$a_6=-\frac{1}{(\rho+5+1)^2}a_4=-\frac{1}{(\rho+2)^2(\rho+4)^2(\rho+6)^2}a_0$$

and so on.

Therefore,

$$a_k=\begin{cases}(-1)^{k/2}\dfrac{1}{\{(\rho+2)(\rho+4)\ldots(\rho+k)\}^2}a_0, & k \text{ is even}\\ 0, & k \text{ is odd.}\end{cases}$$

Thus,

$$w(z)=\sum_{k=0}^{\infty}a_k z^{\rho+k}$$

$$=z^{\rho}\left(a_0+a_2z+a_4z^4+\ldots\right)$$

$$=z^{\rho}a_0\left\{1-\frac{z^2}{(\rho+2)^2}+\frac{z^4}{\{(\rho+2)(\rho+4)\}^2}-\ldots\right\}$$

$$=a_0z^{\rho}\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{\{(\rho+2)(\rho+4)\ldots(\rho+2k)\}^2}.$$

Now, for $\rho=0$,

$$w_0(z)=a_0\left\{1-\frac{z^2}{2^2}+\frac{z^4}{(2.4)^2}-\frac{z^6}{(2.4.6)^2}+\ldots\right\}.$$

Another solution for $\rho = 0$ is determine from the relation

$$w_1(z) = \left(\frac{dw}{d\rho}\right)_{\rho=0}.$$

Let $w(z) = a_0 z^{\rho} F(\rho)$ where

$$F(\rho) = 1 - \frac{z^2}{2^2} + \frac{z^4}{(2.4)^2} - \frac{z^6}{(2.4.6)^2} + \ldots.$$

Therefore,

$$\left(\frac{dw}{d\rho}\right)_{\rho=0} = a_0\left[z^{\rho}\log z . F(\rho) + z^{\rho} . F'(\rho)\right]_{\rho=0}$$

$$= a_0\left[\log z . F(0) + F'(0)\right]$$

Now,

$$F'(\rho) = \sum_{k=0}^{\infty}(-1)^k z^{2k} \frac{d}{d\rho}\left\{\frac{1}{(\rho+2)^2(\rho+4)^2 \ldots .(\rho+2k)^2}\right\}.$$

Let

$$Y = \frac{1}{(\rho+2)^2(\rho+4)^2 \ldots ..(\rho+2k)^2}.$$

Then

$$\log Y = -2\log(\rho+2) - 2\log(\rho+4) - \ldots -2\log(\rho+2k).$$

Differentiating both sides with respect to ρ we get

$$\frac{dY}{d\rho} = -Y.2\left[\frac{1}{\rho+2} + \frac{1}{\rho+4} + \ldots .. + \frac{1}{\rho+2k}\right].$$

Therefore,

$$\left(\frac{dY}{d\rho}\right)_{\rho=0} = -\frac{2}{(2.4 \ldots .2k)^2}\left\{\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2k}\right\}.$$

$$F'(0) = -\sum_{k=0}^{\infty}(-1)^k z^{2k} \frac{2}{(2.4 \ldots .2k)^2}\left\{\frac{1}{2} + \frac{1}{4} + \ldots \ldots + \frac{1}{2k}\right\}.$$

Hence the other solution is

$$w_1(z) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k . z^{2k}}{(2.4....2k)^2}\left[\log z - 2\left(\frac{1}{2}+\frac{1}{4}+....+\frac{1}{2k}\right)\right]$$

$$= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k.(z/2)^{2k}}{(k!)^2}\left[\log z - \left(1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{k}\right)\right].$$

Thus the general solution is

$$w(z) = c_0 w_0(z) + c_1 w_1(z)$$

where c_0, c_1 are arbitrary constants.

1.5 Unit Summary

In this unit regular and irregular singular points of an ODE are defined. The need of series solution of an ODE is described. The solution methods to find the series solution near an ordinary point as well as a singular point are discussed. The necessary theorems are established. The examples for different types of cases are worked out. The unit ends with an exercise.

1.6 Self Assessment Questions

1. Determine whether each of the points −1, 0, 1 is an ordinary point, a regular singular point or an irregular singular point for the following differential equations :

   (i) $zw'' + (1-z)w' + zw = 0$

   (ii) $z^2(1-z^2)w'' + 2zw' + 4w = 0$

   (iii) $2z^4(1-z^2)w'' + 2zw' + 2z^2 w = 0$

   (iv) $(1-z^2)^2 w'' + z(1-z)w' + (1+z)w = 0.$

2. Obtain the integral in the neighbourhood of $z = 1$ of the differential equation

$$z(z-1)\frac{d^2w}{dz^2} - 3z\frac{dw}{dz} - w = 0.$$

3. Find the general solutions of the following differential equations

   (i) $z^2 w'' - 2w = 0$

   (ii) $z^2 w'' + zw' + z^2 w = 0$

   (iii) $w'' - 2zw' + 2nw = 0$ for $n = 0$ and $n = 1$.

4. Solve the following equations in power series

   (i) $2z^2w'' - zw' + (1-z^2)w = 0$

   (ii) $zw'' + w' + zw = 0.$

5. The differential equation $zw'' + (1-z)w' + \lambda w = 0$, where λ a constant, is known as Laguerre differential equation.

   (i) Show that the indicial equation of the Laguerre equation is $\rho^2 = 0$.

   (ii) Find a series solution of this equation of the form

$$w(z) = \sum_{k=0}^{\infty} c_k z^k.$$

   (iii) Show that this solution reduces to a polynomial of degree n if $\lambda = n$, where n is an integer.

6. Find two linearly independent series solution when $|z| < 1$, for Chebyshev's equation

$$(1-z^2)w'' - zw' + n^2w = 0$$

where n is a constant.

7. Show that the Legendre's differential equation

$$(1-z^2)w'' - 2zw' + n(n+1)w = 0$$

has a regular solution $P_n(z)$ and an irregular solution $Q_n(z)$. Show that the Wronskian of $P_n(z)$ and $Q_n(z)$ is given by

$$P_n(z)Q_n'(z) - P_n'(z)Q_n(z) = \frac{A}{1-z^2},$$

where A is independent of z.

8. Obtain a series solution of the hypergeometric equation

$$z(z-1)w'' + [(1+\alpha+\beta)z - \gamma]w' - \alpha\beta w = 0.$$

9. Obtain two series solutions of the confluent hypergeometric equatin

$$zw'' + (\gamma - z)w' - \alpha w = 0.$$

10. Find a series solution in powers of z of Airy's equation $w'' - zw = 0.$

11. Solve

$$z^2w'' + 5zw' + 4w = 0.$$

12. Find a series solution of the Bessel equation of order one-half

$$z^2 w'' + zw' + \left(z^2 - 1/4\right)w = 0.$$

1.7 Suggested Further Readings

1. I.N. Sneddon, *Special Functions of Mathematical Physics and Chemistry.*
2. N.N. Lebeder, *Special Functions and their Applications.*
3. D. Rainville, *Special Functions.*
4. M. Birkhoff and G.C. Rota, *Ordinary Differential Equations.*
5. E.A. Coddington and N.Levinson, *Theory of Ordinary Differential Equations.*
6. G.B. Arfken and H.J. Weber, *Mathematical Methods for Physicists.*

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-I | Group-C

Module No. - 10

Ordinary Differential Equations
(HYPERGEOMETRIC EQUATION)

Content :

Hypergeometric function have many applications in different discipline in science and engineering. The main objective of this unit is given below.

Objectives:

- Hypergeometric equation and its series solution.
- Hypergeometric function and its properties.
- Integral representation.

- Confluent Hypergeometric function and its properties.
- Integral representation of confluent hypergeometric function.
- Error function in terms of confluent hypergeometric function.
- Exercise.

2.1 Hypergeometric Equation

The differential equation

$$z(1-z)\frac{d^2w}{dz^2}+\{c-(a+b+1)z\}\frac{dw}{dz}-abw=0 \tag{2.1}$$

is known as hypergeometric equation or Gaussian differential equation. The equation (2.1) has three regular singularities at 0, 1, ∞.

Case I: *Solution at the neighbour of* $z=0$

As $z=0$ is a regular singular point, let

$$w=\sum_{k=0}^{\infty}d_k z^{p+k} \tag{2.2}$$

be the solution of the equation (2.1).

Then $w'=\sum_{k=0}^{\infty}(p+k)d_k z^{p+k-1}$ and $w''=\sum_{k=0}^{\infty}(p+k)(p+k-1)d_k z^{p+k-2}$.

Using above relations, equation (2.1) becomes

$$z(1-z)\sum_{k=0}^{\infty}\{d_k(p+k)(p+k-1)z^{p+k-2}\}+\{c-(a+b+1)z\}\sum_{k=0}^{\infty}d_k(p+k)z^{p+k-1}$$

$$-ab\sum_{k=0}^{\infty}d_k z^{p+k}=0.$$

The indicial equation is

$$\rho(\rho-1)+c\rho=0 \text{ or } \rho=0,1-c$$

By equating the coefficient of z^{p+k} we obtain

$$d_{k+1}[(\rho+k+1)(\rho+k)+c(\rho+k+1)]-d_k[(\rho+k)(\rho+k-1)+(\rho+k)(a+b+1)]-abd_k=0$$

or, $$d_{k+1}[(\rho+k+1)(\rho+k)+c(\rho+k+1)]=d_k[(\rho+k)(\rho+k-1)+(\rho+k)(a+b+1)+ab]$$

i.e., $\dfrac{d_{k+1}}{d_k} = \dfrac{(\rho+k+a)(\rho+k+b)}{(\rho+k+1)(\rho+k+c)}, \quad k = 0,1,2,...$

Then

$$\frac{d_1}{d_0} = \frac{(\rho+a)(\rho+b)}{(\rho+1)(\rho+c)}, \quad \frac{d_2}{d_1} = \frac{(\rho+a+1)(\rho+b+1)}{(\rho+2)(\rho+c+1)}...$$

Thus

$$w = d_0 z^{\rho}\left[1 + \frac{d_1}{d_0}z + \frac{d_2 d_1}{d_1 d_0}z^2 + \frac{d_3 d_2 d_1}{d_2 d_1 d_0}z^3 + ...\right]$$

$$= d_0 z^{\rho}\left[1 + \frac{(\rho+a)(\rho+b)}{(\rho+1)(\rho+c)}z + \frac{(\rho+a)(\rho+a+1)(\rho+b)(\rho+b+1)}{(\rho+1)(\rho+2)(\rho+c)(\rho+c+1)}z^2 + ...\right]$$

and the fundamental solution for $\rho = 0$ is given by

$$w_1^0 = d_0\left[1 + \frac{a.b}{1.c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + ...\right]$$

This is denoted by

$$w_1^0 = d_0 F(a,b,c;z), \tag{2.3}$$

where

$$F(a,b,c;z) = 1 + \frac{a.b}{1.c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + ... \tag{2.4}$$

and this function is called *hypergeometric function.*

For $\rho = 1 - c$, if c is not an integer, then

$$w_2^0 = d_0 z^{1-c}\left[1 + \frac{(1-c+a)(1-c+b)}{1.(2-c)}z + \frac{(1-c+a)(2-c+a)(1-c+b)(2-c+b)}{2!(2-c)(3-c)}z^2 + ...\right].$$

Therefore,

$$w_2^0 = d_0 z^{1-c} F(1-c+a,\ 1-c+b,\ 2-c; z) \tag{2.5}$$

Hence the general solution at the neighbour of $z = 0$ is $w = Aw_1^0 + Bw_2^0$, where A and B are arbitrary constants.

Case II: *Solution at the neighbour of z = 1.*

To obtain the solution of (2.1), applying the transformation $z = 1 - t$. Then the solution near $t = 0$ of the transform equation is the solution of the original equation near $z = 1$.

Under this transformation equation (2.1) becomes

$$t(1-t)\frac{d^2w}{dt^2}+[(a+b+1-c)-(a+b+1)t]\frac{dw}{dt}-abw=0. \tag{2.6}$$

Thus the solution of this equation near $t = 0$ are as in the previous Case I. Substituting again $1 - z$ and $a+b+1-c$ in places of z and c; a, b remain unchanged in position. Then the solutions are

$$w_1^1 = d_0F(a,b,a+b+1-c,1-z) \tag{2.7}$$

$$w_2^1 = d_0(1-z)^{c-a-b}F(c-a,c-b,1-a-b+c,1-z). \tag{2.8}$$

It's general solution is

$$w = A_2w_1^1 + B_2w_2^1 \tag{2.9}$$

Case III: *Solution near z = ∞.*

To obtain the solution near $z = \infty$, applying the transformation $z = 1/t$, i.e.,

$$\frac{dw}{dz} = -t^2\frac{dw}{dz} \text{ and } \frac{d^2w}{dz^2} = t^4\frac{d^2w}{dz^2} + 2t^3\frac{dw}{dt}$$

Then the equation (2.1) becomes

$$\frac{d^2w}{dt^2}+\left[\frac{2}{t}+\frac{-ct+(a+b+1)}{t(t-1)}\right]\frac{dw}{dt}-\frac{ab}{t^2(t-1)}w=0$$

$$t(1-t)\frac{d^2w}{dt^2}+[2(1-t)+tc-(a+b+1)]\frac{dw}{dt}+\frac{ab}{t}w=0. \tag{2.10}$$

The indicial equation near $t = 0$ of the above equation is $\rho(\rho-1)+(1-a-b)\rho+ab=0$ i.e., $(\rho-a)(\rho-b)=0$. This gives $\rho = a, b$.

The indicial equation do not give a zero root. To get the zero solution using the transformation $w = w_1t^a$.

Then $w' = w_1't^a + at^{a-1}w_1$ and $w'' = w_1''t^a + at^{a-1}w_1' + a(a-1)t^{a-2}w_1 + at^{a-1}w_1'$.

The transform equation (2.10) under these substitution, becomes

$$\{w_1''t^a + 2at^{a-1}w_1' + a(a-1)t^{a-2}w_1\}t^2(t-1) + t^{a+1}[2(t-1)-ct+(a+b+1)]w_1' -$$

$$abt^aw_1 + at^a[2(t-1)-ct+(a+b+1)]w_1 = 0$$

or, $$w_1''+\frac{(1+a-b)-(1+a+\overline{1+a-c})t}{t(1-t)}w_1'-\frac{a(1+a-c)}{t(1-t)}w_1=0$$

The indicial equation near $t=0$ is

$\rho(\rho-1)+(1+a-b)\rho=0$ or, $\rho=0, b-a$

Thus the solution w_1 is obtain as in Case I by changing a by a, b by $1+a-c$, c by $1+a-b$ and z by t.

Therefore, $w_1=F(a,1+a-c;1+a-b,t)$.

Hence one solution at the neighbourhood of $z=\infty$ is

$$w_1^\infty=\frac{1}{z^a}F(a,1+a-c,1+a-b,1/z). \tag{2.11}$$

Similarly, for the root $t=b$, the other solution near $z=\infty$ is

$$w_2^\infty=\frac{1}{z^b}F(b,1+b-c,1+b-a,1/z). \tag{2.12}$$

2.2 Some Properties of Hypergeometric Function

The hypergeometric function is

$$F(a,b,c;z)=1+\frac{a.b}{1.c}z+\frac{a(a+1)b(b+1)}{2!c(c+1)}z^2+\ldots.$$

If $z=0$ then we have

$$F(a,b,c;0)=\lim_{z\to0}\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}z^k$$

$$=\lim_{z\to0}\left[1+\frac{ab}{c}z+\frac{a(a+1)b(b+1)}{2!c(c+1)}z^2+\ldots\right]$$

$$=1.$$

If a or $b=-n$ (n is a positive integer), then the series $F(-n,b,c;z)$ reduces to a polynomial known as Jacobi's polynomial, i.e.,

$$F(-n,b,c;z)=1-\frac{nb}{c}z-\frac{n(1-n)b(b+1)}{2!c(c+1)}z^2+\ldots. \tag{2.13}$$

If $b=c$ and $a=-n$, n is a positive integer, then

$$F(-n,b,b;z)=1-nz+\frac{n(n-1)}{2!}z^2+...+(-1)^n z^n=(1-z)^n. \quad (2.14)$$

Replacing z by $-z$ to the above relation we obtain

$$F(-n,b,b;-z)=(1+z)^n. \quad (2.15)$$

$$-zF(1,1,2;z)=-z\left[1+\frac{1}{2}z+\frac{1}{3}z^2+...\right]=\log(1-z). \quad (2.16)$$

Similarly,

$$zF(1,1,2;-z)=z\left[1-\frac{1}{2}z+\frac{1}{3}z^2-...\right]=\log(1+z). \quad (2.17)$$

$$2z^2F(1/2,1,3/2;z)=2z^2\left(1+\frac{1.1}{1.3}z+\frac{1.3.1.2}{1.2.3.5}z^2+...\right)$$

$$=2z^2\left(1+\frac{1}{3}z+\frac{1}{5}z^2+...\right)=\log\left|\frac{1+z}{1-z}\right|. \quad (2.18)$$

Also,

$$F(1,b,b;z)=1+z+z^2+...=(1-z)^{-1}. \quad (2.19)$$

The derivative of hypergeometric function is also a hypergeometric function.

$$\frac{d}{dz}F(a,b,c;z)=\frac{ab}{c}+\frac{a(a+1)b(b+1)}{c(c+1)}z+...$$

$$=\frac{ab}{c}\left[1+\frac{(a+1)(b+1)}{c+1}z+\frac{(a+1)(a+2)(b+1)(b+2)}{2!(c+1)(c+2)}z^2+...\right]$$

$$=\frac{ab}{c}F(a+1,b+1,c+1;z). \quad (2.20)$$

Similarly,

$$\frac{d^2}{dz^2}F(a,b,c;z)=\frac{a(a+1)b(b+1)}{c(c+1)}F(a+2,b+2,c+2;z). \quad (2.21)$$

By repeating the process m times we get

$$\frac{d^m}{dz^m}F(a,b,c;z)=\frac{a(a+1)...(a+m-1)b(b+1)...(b+m-1)}{c(c+1)...(c+m-1)}$$

$$\times F(a+m,b+m,c+m;z). \quad (2.22)$$

Since $F(a,b,c;0)=1$ for all a, b, c, so $\lim\limits_{z\to 0} F(a+1,b+1,c+1;z)=1$ and hence

$$\left[\frac{d}{dz}\{F(a,b,c;z)\}\right]_{z=0}=\frac{ab}{c}. \tag{2.23}$$

In general,

$$\left[\frac{d^m}{dz^m}\{F(a,b,c;z)\}\right]_{z=0}=\frac{a(a+1)...(a+m-1)b(b+1)...(b+m-1)}{c(c+1)...(c+m-1)}. \tag{2.24}$$

2.3 Integral Representation

Let

$$F(a,b,c;zc)=1+\frac{ab}{c}z+\frac{a(a+1)b(b+1)}{2!c(c+1)}z^2+...=\sum_0^\infty \frac{(a)_n(b)_n}{n!(c)_n}z^n,$$

where

$$(a)_n=a(a+1)(a+2)...(a+n-1), n>1 \quad \text{and} \quad (a)_0=1$$

The expression of $(a)_n$ can be written as

$$(a)_n=a(a+1)(a+2)...(a+n-1)=\frac{1\cdot 2...(a-1)a(a+1)...(a+n-1)}{1\cdot 2...(a-1)}=\frac{\Gamma(a+n)}{\Gamma(a)}.$$

Now,

$$\frac{(b)_n}{(c)_n}=\frac{\Gamma(c)}{\Gamma(c+n)}\cdot\frac{\Gamma(b+n)}{\Gamma(b)}=\frac{\Gamma(c)}{\Gamma(b)}\frac{\Gamma(b+n)}{\Gamma(c+n)}\frac{\Gamma(c-b)}{\Gamma(c-b)}$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(\overline{b+n}+\overline{c-b})}$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}B(b+n,c-b)$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b+n-1}(1-t)^{c-b-1}dt.$$

Thus

$$F(a,b,c;z) = \sum_0^\infty \frac{\Gamma(a+n)}{n!\Gamma(a)}z^n\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b+n-1}(1-t)^{c-b-1}dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{0}^{\infty}\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\frac{(a)_n}{n!}z^n dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\sum_{0}^{\infty}\frac{(a)_n}{n!}(zt)^n dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\left[1+a(zt)+\frac{a(a+1)}{2!}(zt)^2+\ldots\right]dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt.$$

Thus the integral representation of the hypergeometric function is

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt. \quad (2.25)$$

In particular,

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)}.$$

2.4 Analytic Continuation of *F* (*a, b, c; z*)

The value of *F* (*a, b, c; z*) having known, we can obtain the solution outside $|z|=1$, i.e., we can determine the constants c_1, c_2 in the equation

$$F(a,b,c;z) = c_1 w_1^1 + c_2 w_2^1 \quad (2.26)$$

in the common region $|z|<1$, $|z-1|<1$. The region in which w_1^1 and w_2^1 are convergent assume that the real part $R(c-a-b)$ is positive.

Equation (2.26) can be written as

$$F(a,b,c;z) = c_1 F(a,b,a+b-c+1,1-z)$$
$$+c_2(1-z)^{c-a-b}F(c-a,c-b,1+c-a-b,1-z). \quad (2.27)$$

We suppose that $R(a+b)<R(c)<1$ so that all the function are convergent.

Let $z \to 1$ along the real axis, so we have from (2.27),

$$F(a,b,c;1) = c_1 F(a,b,a+b-c+1;0)+c_2.0$$

Therefore, $c_1 = F(a,b,c;1)$. [since $F(a,b,a+b-c+1;0)=1$]

Again, let $z \to 0$, then we get from (2.27)

$$F(a,b,c;0) = c_1 F(1,b,a+b-c+1;1) + c_2 F(c-a,c-b,1+c-a-b;1)$$

or,

$$1 - F(a,b,c;1)F(a,b,a+b-c+1;1) = c_2 F(c-a,c-b,1+c-a-b;1)$$

or,

$$c_2 F(c-a,c-b,1+c-a-b;1)$$

$$= 1 - \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b,a+b-c+1;1)$$

$$= 1 - \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \frac{\Gamma(1-\overline{c-a-b})\Gamma(1-c)}{\Gamma(1-\overline{c-a})\Gamma(1-\overline{c-b})}$$

$$= 1 - \frac{\sin\pi(c-a)\sin\pi(c-b)}{\sin\pi c \sin\pi(c-a-b)} \quad [\text{since } \Gamma(n)\Gamma(1-n) = \pi/\sin n\pi]$$

$$= \frac{1}{2} \frac{[\cos\pi(a+b) - \cos\pi(2c-a-b)] - [\cos\pi(a-b) - \cos\pi(2c-a-b)]}{\sin\pi c \sin\pi(c-a-b)}$$

$$= -\frac{\sin\pi a \sin\pi b}{\sin\pi c \sin\pi(c-a-b)}.$$

That is,

$$c_2 \frac{\Gamma(1+c-a-b)\Gamma(1-c)}{\Gamma(1-b)\Gamma(1-c)} = -\frac{\sin\pi a \sin\pi b}{\sin\pi c \sin\pi(c-a-b)}$$

or,

$$c_2 \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(a+b-c)} \frac{\Gamma(a+b-c)\Gamma(1-a+b-c)}{\Gamma(b)\Gamma(1-b)} \frac{\Gamma(c)\Gamma(1-c)}{\Gamma(a)\Gamma(1-a)} = -\frac{\sin\pi a \sin\pi b}{\sin\pi c \sin\pi(c-a-b)}$$

or,

$$c_2 \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(a+b-c)} \frac{\sin\pi a \sin\pi b}{\sin\pi c \sin\pi(a+b-c)} = -\frac{\sin\pi a \sin\pi b}{\sin\pi c \sin\pi(c-a-b)}$$

$[\sin(c-a-b)\pi = -\sin\pi(a+b-c)]$

or,

$$c_2 \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(a+b-c)} = 1$$

or,

$$c_2 = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$

and hence we have

$$F(a,b,c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b,1+b+a-c,1-z)$$

$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-b-a} F(c-a,c-b,1+c-a-b;1-z)$$

holds when $|z|<1$ and $|1-z|<1$ provided that $R(a+b)<R(c)<1$.

EXAMPLE 2.1 *Use integral representation of hypergeometric function, prove that*
$F(a,b,c;z) = (1-z)^{-a} F(a,c-b,c;z/(z-1))$, *where* $-\pi < amp(1-z) < \pi$.

SOLUTION: $F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt$. Then

$$F(a,c-b,c;z/(z-1)) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{c-b-1}(1-t)^{b-1}\left(1-\frac{z}{z-1}t\right)^{-a}dt.$$

$$(1-z)^{-a} F(a,c-b,c;z/(z-1)) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{c-b-1}(1-t)^{b-1}(1-z+zt)^{-a}dt.$$

Substituting, $1-z+zt = 1-zx$. Then $dt=-dx$ and $(t-1)=-x$.
Hence,

$$(1-z)^{-a} F(a,c-b,c;z/(z-1)) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_1^0 (1-x)^{c-b-1}x^{b-1}(1-zx)^{-a}(-dx)$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 (1-x)^{c-b-1}x^{b-1}(1-zx)^{-a}(-dx)$$

$$= F(a,b,c;z)$$

2.5 Confluent Hypergeometric Equation

The hypergeometric equation is

$$z(1-z)w'' + [c-(1+a+b)z]w' - abw = 0. \tag{2.28}$$

Substitute $z = t/b$. Then $\frac{dw}{dz} = \frac{dw}{dt}\frac{dt}{dz} = b\frac{dw}{dt}$ and $\frac{d^2w}{dz^2} = b^2\frac{d^2w}{dt^2}$.

Using these relations, (2.28) becomes

$$\frac{t}{b}(1-t/b)b^2\frac{d^2w}{dt^2} + \left[c-(1+a+b)\frac{t}{b}\right]b\frac{dw}{dt} - abw = 0.$$

That is,

$$t(1-t/b)\frac{d^2w}{dt^2} + \left[c-(1+a+b)\frac{t}{b}\right]\frac{dw}{dt} - aw = 0.$$

The limit $b \to \infty$ reduces the above equation as

$$t\frac{d^2w}{dt^2} + (c-t)\frac{dw}{dt} - aw = 0. \qquad (2.29)$$

This equation is known as confluent hypergeometric equation.

The hypergeometric function is given by

$$F(a,b,c;z) = 1 + \frac{ab}{1!c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + \ldots$$

Substituting $z = t/b$ in the above equation we obtain

$$F(a,b,c;t/b) = 1 + \frac{ab}{1!c}\frac{t}{b} + \frac{a(a+1)b(b+1)}{2!c(c+1)}\frac{t^2}{b^2} + \ldots$$

$$= 1 + \frac{a}{1!c}t + \frac{a(a+1)(1+1/b)}{2!c(c+1)}t^2 + \ldots$$

Then

$$\lim_{b\to\infty} F(a,b,c;t/b) = 1 + \frac{a}{1!c}t + \frac{a(a+1)}{2!c(c+1)}t^2 + \ldots$$

Let

$$G(a,c;t) = \lim_{b\to\infty} F(a,b,c;t/b).$$

Then the function

$$G(a,c;t) = 1 + \frac{a}{1!c}t + \frac{a(a+1)}{2!c(c+1)}t^2 + \ldots \qquad (2.30)$$

is called the confluent hypergeometric function.

2.6 Solution of Confluent Hypergeometric Equation

The confluent hypergeometric equation (2.29) is

$$z\frac{d^2w}{dz^2}+(c-z)\frac{dw}{dz}-aw=0. \tag{2.31}$$

The point $z=0$ is a singular point. So the solution of this equation at $z=0$ is determine by Frobenius method.

Let $w=\sum_0^\infty c_k z^{p+k}$ be the solution of (2.31). Then

$$\frac{dw}{dz}=\sum_0^\infty c_k(p+k)z^{p+k-1} \text{ and } \frac{d^2w}{dz^2}=\sum_0^\infty c_k(p+k)(p+k-1)z^{p+k-2}.$$

Substituting these values of w, w' and w'' in (2.31) we obtain

$$z\sum_0^\infty c_k(\rho+k)(\rho+k-1)z^{p+k-2}+(c-z)\sum_0^\infty c_k(\rho+k)z^{p+k-1}-a\sum_0^\infty c_k z^{p+k}=0$$

After simplification, the above equation becomes

$$\sum_0^\infty\{c_k(\rho+k)(\rho+k-1)+c(\rho+k)c_k\}z^{p+k-1}-\sum_0^\infty\{c_k(\rho+k)+ac_k\}z^{p+k}=0. \tag{2.32}$$

Equating the coefficient of $z^{\rho-1}$ we have the indicial equation as

$$\rho(\rho-1)+c\rho=0 \quad \text{or} \quad \rho=0,\ 1-c.$$

Equating the coefficient of $z^{\rho+k}$ to zero we obtain

$$c_{k+1}[(\rho+k+1)(\rho+k)+c(\rho+k+1)]=c_k(\rho+k+a).$$

Hence

$$\frac{c_{k+1}}{c_k}=\frac{\rho+k+a}{(\rho+k+1)(\rho+k+c)}. \tag{2.33}$$

Case I: $\rho=0$

In this case, the relation (2.33) reduces to

$$\frac{c_{k+1}}{c_k}=\frac{k+a}{(k+1)(k+c)} \qquad k=0,1,2\ldots$$

Thus

$$c_1=\frac{a}{c}c_0,\ c_2=\frac{1+a}{2(1+c)}c_1=\frac{a(1+a)}{2!c(1+c)}c_0,\ldots$$

Therefore,

$$w = c_0\left[1+\frac{a}{c}z+\frac{a(a+1)}{2!c(c+1)}z^2+\ldots\right] = c_0G(a,c;z).$$

Case I: $\rho=1-c$

In this case, the relation (2.33) becomes

$$\frac{c_{k+1}}{c_k} = \frac{1-c+k+a}{(k+2-c)(k+1)} = \frac{a'+k}{(c'+k)(k+1)}$$

where $a' = 1+a-c,\ c' = 2-c.$ (2.34)

Substituting $k=0, 1, 2, \ldots$ in (2.34) we have

$$c_1 = \frac{a'}{c'}c_0,\ c_2 = \frac{a'+1}{2(c'+1)}c_1 = \frac{a'(a'+1)}{2!c'(c'+1)}c_0\ldots$$

Therefore,

$$w = c_0z^{1-c}\left[1+\frac{a'}{c'}z+\frac{a'(a'+1)}{2!c'(c'+1)}z^2+\ldots\right]$$

$$= c_0z^{1-C}G(a',c';z) = c_0z^{1-C}G(1+a-c,2-c;z). \quad (2.35)$$

Hence the two fundamental integrals of the confluent hypergeometric equation are

$w_1^0 = G(a,c;z)$ and $w_2^0 = z^{1-c}G(1+a-c;2-c;z)$.

Finally the general solution of the confluent hypergeometric equation is $w = Aw_1^0 + Bw_2^0$.

2.7 Integral Representation of Confluent Hypergeometric Function

The Confluent hypergeometric function is

$$G(a,b;z) = 1+\frac{a}{b}z+\frac{a(a+1)}{2!b(b+1)}z^2+\ldots = \sum_0^\infty \frac{(a)_n}{(b)_n}\frac{z^n}{n!}.$$

Now,

$$(a)_n = \frac{a(a+1)(a+2)\ldots(a+n+1)1\cdot2\ldots(a-1)}{1\cdot2\ldots(a-1)} = \frac{\Gamma(n+a)}{\Gamma(a)}.$$

Therefore,

$$\frac{(a)_n}{(b)_n} = \frac{\Gamma(a+n)}{\Gamma(a)}\frac{\Gamma(b)}{\Gamma(b+n)}$$

$$= \frac{\Gamma(a+n)\Gamma(b-a)}{\Gamma(\overline{a+n}+\overline{b-a})}\cdot\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}$$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}B(a+n,b-a).$$

Hence,

$$G(a,b;z) = \sum_0^\infty \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\frac{z^b}{n!}\int_0^1 t^{a+n-1}(1-t)^{b-a-1}dt$$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\sum_0^\infty \int_0^1 t^{a-1}(1-t)^{b-a-1}\frac{(zt)^n}{n!}dt$$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}\left(1+\frac{zt}{1!}+\frac{(zt)^2}{2!}+\ldots\right)dt$$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}e^{zt}\,dt.$$

Finally, the integral representation of confluent hypergeometric function is

$$G(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}e^{zt}dt. \tag{2.36}$$

EXAMPLE 2.2 *Shows that* $G(a,b;z) = e^z G(b-a,b;-z)$.

SOLUTION : By integral representation of confluent hypergeometric function.

$$e^z G(b-a,b;-z) = e^z \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{b-a-1}(1-t)^{a-1}e^{-zt}dt$$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{b-a-1}(1-t)^{a-1}e^{z(1-t)}dt$$

Substituting, $1-t=x$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 (1-x)^{b-a-1} x^{a-1} e^{zx} dx$$

$$= G(a,b;z).$$

The error function $erf(x)$ is widely used function in mathematics and engineering and it is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2} du. \tag{2.37}$$

EXAMPLE 2.3 *Prove that*

$$\frac{\sqrt{\pi}}{2x} erf(x) = \frac{1}{x}\int_0^x e^{-u^2} du = G(1/2, 3/2, -x^2).$$

SOLUTION : By integral representation of confluent hypergeometric function (2.36), we have

$$G(1/2, 3/2, -x^2) = \frac{\Gamma(3/2)}{\Gamma(1/2)\Gamma(1)}\int_0^1 t^{-1/2}(1-t)^0 e^{-x^2 t} dt$$

$$= \frac{1}{2}\int_0^1 t^{-1/2} e^{-x^2 t} dt$$

Substituting, $x^2 t = u^2$. Then $x^2 dt = 2u\,du$

$$= \frac{1}{2}\int_0^x \left(\frac{u^2}{x^2}\right)^{-1/2} e^{-u^2} . \frac{2u\,du}{x^2}$$

$$= \frac{1}{x}\int_0^x e^{-u^2} du$$

$$= \frac{\sqrt{\pi}}{2x} erf(x).$$

The confluent hypergeometric function satisfy the following relations.

(i) When $a = b$ then (2.36) is the exponential series, i.e.,

$$G(a,a;z) = e^z$$

(ii) When $a = -n$, where n is a non-negative integer, then $G(a,b;z)$ is a polynomial of degree n.

(ii) The function $G(a,b;z)$ is related to the Bessel function (discussed in Unit 4) by

$$J_n(z) = \frac{1}{\Gamma(n+1)}\left(\frac{z}{2}\right)^n e^{iz} G(n+1/2, 2n+1; -2iz).$$

2.8 Worked out Examples

Example 2.4 *Shows that*

$$\tan^{-1} z = zF(1/2,1,3/2;-z^2).$$

Solution : We have

$$\tan^{-1} z = z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}+\ldots$$

$$= z\left[1-\frac{z^2}{3}+\frac{z^4}{5}-\frac{z^6}{7}+\ldots\right]$$

$$= z\left[1+\frac{\frac{1}{2}\cdot 1}{1\cdot\frac{3}{2}}(-z^2)+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\cdot 1\cdot(1+1)}{1\cdot 2\cdot\frac{3}{2}\left(\frac{3}{2}+1\right)}(-z^2)^2\right.$$

$$\left.+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\cdot 1\cdot(1+1)(1+2)}{1\cdot 2\cdot 3\cdot\frac{3}{2}\left(\frac{3}{2}+1\right)\left(\frac{3}{2}+2\right)}(-z^2)^3+\ldots\right]$$

$$= zF(1/2,1,3/2;-z^2).$$

Example 2.5 *Prove that*

$$F(a,b,c;z)-F(a,b,c-1;z)=-\frac{ab}{c(c-1)}zF(a+1,b+1,c+1;z).$$

Solution : We have

$$F(a,b,c;z)=1+\frac{a.b}{1.c}z+\frac{a(a+1)\cdot b(b+1)}{2!c(c+1)}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{3!(c+1)(c+2)}z^3+\ldots$$

and

$$F(a,b,c-1;z)=1+\frac{a.b}{1.(c-1)}z+\frac{a(a+1)\cdot b(b+1)}{2!(c-1)c}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{3!(c-1)c(c+1)}z^3+\ldots$$

Subtracting we get

$$F(a.b,c;z)-F(a,b,c-1;z)$$

$$= \frac{a.b}{1}\left\{\frac{1}{c}-\frac{1}{c-1}\right\}z+\frac{a(a+1)b(b+1)}{2!c}\left\{\frac{1}{c+1}-\frac{1}{c-1}\right\}z^2$$

$$+\frac{a(a+1)(a+2)b(b+1)(b+2)}{2!c(c+1)}\left\{\frac{1}{c+2}-\frac{1}{c-1}\right\}z^3+\ldots$$

$$= -\frac{ab}{(c-1)c}z-\frac{a(a+1)b(b+1)}{1!(c-1)c(c+1)}z^2-\frac{a(a+1)(a+2)b(b+1)(b+2)}{2!(c-1)c(c+1)(c+2)}z^3-\ldots$$

$$= -\frac{ab}{(c-1)}z\left\{1+\frac{(a+1)(b+1)}{1!(c+1)}z+\frac{(a+1)(a+2)(b+1)(b+2)}{2!(c+1)(c+2)}z^2+\ldots\right\}$$

$$= -\frac{ab}{c(c-1)}zF(a+1,b+1,c+1;z)$$

EXAMPLE 2.6 *Prove that*

$$(a-b)F(a,b,c;z)=aF(a+1,b,c;z)-bF(a,b+1,c;z)$$

SOLUTION :

$$(a-b)F(a,b,c;z)$$

$$= (a-b)\left[1+\frac{a.b}{1.c}z+\frac{a(a+1)b(b+1)}{2!c(c+1)}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}z^3+\ldots\right]$$

$$= (a-b)+\frac{a.b(a-b)}{1.c}z+\frac{a(a+1)b(b+1)(a-b)}{2!c(c+1)}z^2$$

$$+\frac{a(a+1)(a+2)b(b+1)(b+2)(a-b)}{3!c(c+1)(c+2)}z^3+\ldots$$

$$= (a-b)+\frac{a.b}{1.c}\{(a+1)-(b+1)\}z+\frac{a(a+1)b(b+1)}{2!c(c+1)}\{(a+2)-(b+2)\}z^2$$

$$+\frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}\{(a+3)-(b+3)\}z^3+\ldots\Big]$$

$$= \left\{a+\frac{a(a+1)b}{1.c}z+\frac{a(a+1)(a+2)b(b+1)}{2!c(c+1)}z^2+\ldots\right\}$$

$$-\left\{b+\frac{ab(b+1)}{1.c}z+\frac{a(a+1)b(b+1)(b+2)}{2!.c(c+1)}z^2+\ldots\right\}$$

$$= a\left\{1+\frac{(a+1)b}{1.c}z+\frac{(a+1)(a+2)b(b+1)}{2!c(c+1)}z^2+\ldots\right\}$$

$$-b\left\{1+\frac{a(b+1)}{1.c}z+\frac{a(a+1)(b+1)(b+2)}{2!c(c+1)}z^2+\ldots\right\}$$

$$= aF(a+1,b,c;z)-bF(a,b+1,c;z).$$

EXAMPLE 2.7 *Prove that*

$$F\left(a/2,a/2+1/2,1/2;z^2\right)=\frac{1}{2}\left[(1-z)^{-a}+(1+z)^{-a}\right]$$

SOLUTION : From definition of hypergeometric function.

$$F\left(a/2,a/2+1/2,1/2;z^2\right)$$

$$= 1+\frac{\frac{1}{2}\left(\frac{a}{2}+\frac{1}{2}\right)}{1\cdot\frac{1}{2}}z^2+\frac{\frac{a}{2}\left(\frac{a}{2}+1\right)\left(\frac{a}{2}+\frac{1}{2}\right)\left(\frac{a}{2}+\frac{3}{2}\right)}{2!\frac{1}{2}\left(\frac{1}{2}+1\right)}z^4+\ldots$$

$$= 1+\frac{a(a+1)}{2}z^2+\frac{a(a+1)(a+2)(a+3)}{4!}z^4+\ldots$$

$$= \frac{1}{2}\left\{2+a(a+1)z^2+2\frac{a(a+1)(a+2)(a+3)}{4!}z^4+\ldots\right\}$$

$$= \frac{1}{2}\left\{1+az+\frac{a(a+1)}{2!}z^2+\frac{a(a+1)(a+2)}{3!}z^3+\frac{a(a+1)(a+2)(a+3)}{4!}z^4+\ldots\right\}$$

$$+\frac{1}{2}\left\{1-az+\frac{a(a+1)}{2!}z^2-\frac{a(a+1)(a+2)}{3!}z^3+\frac{a(a+1)(a+2)(a+3)}{4!}z^4+\ldots\right\}$$

$$= \frac{1}{2}\left[(1-z)^{-a}+(1+z)^{-a}\right].$$

EXAMPLE 2.8 *Prove that*

$$F(1/2,1,3/2;z^2)=\frac{1}{2z}\log\frac{1+z}{1-z}.$$

SOLUTION : From definition we have

$$F(1/2,1,3/2;z^2)$$

$$= 1+\frac{\frac{1}{2}\cdot 1}{1\cdot\frac{3}{2}}z^2+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)1(1+1)}{2!\frac{3}{2}\left(\frac{3}{2}+1\right)}z^4+\ldots$$

$$= 1+\frac{z^2}{3}+\frac{1.3.2}{2.3.5}z^4+\ldots$$

$$= 1+\frac{z^2}{3}+\frac{z^4}{5}+\ldots$$

$$= \frac{1}{2z}\left\{2z+\frac{2z^3}{3}+\frac{2z^5}{5}+\ldots\right\}$$

$$= \frac{1}{2z}\left[\left\{z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\frac{z^5}{5}-\ldots\right\}-\left\{-z-\frac{z^2}{2}-\frac{z^3}{3}-\frac{z^4}{4}-\frac{z^5}{5}-\ldots\right\}\right]$$

$$= \frac{1}{2z}\left[\log(1+z)-\log(1-z)\right]$$

$$= \frac{1}{2z}\log\frac{1+z}{1-z}.$$

EXAMPLE 2.9 *Prove that*

$$(c-a)G(a-1,c;z)+(2a-c+z)G(a,c;z)-aG(a+1,c;z)=0.$$

SOLUTION :

$$(c-a)G(a-1,c;z)+(2a-c+z)G(a,c;z)-aG(a+1,c;z)$$

$$= (c-a)G(a-1,c;z)-(c-a)G(a,c;z)+aG(a,c;z)+zG(a,c;z)-aG(a+1,c;z)$$

$$= (c-a)\left\{1+\frac{a.1}{1.c}z+\frac{(a-1)a}{2!c(c+1)}z^2+\frac{(a-1)a(a+1)}{3!c(c+1)(c+2)}z^3+\ldots\right\}$$

$$-(c-a)\left\{1+\frac{a}{1.c}z+\frac{(a+1)}{2!c(c+1)}z^2+\frac{a(a+1)(a+2)}{3!c(c+1)(c+2)}z^3+\ldots\right\}$$

$$+a\left\{1+\frac{a}{1.c}z+\frac{a(a+1)}{2!c(c+1)}z^2+\frac{a(a+1)(a+2)}{3!c(c+1)(c+2)}z^3+\ldots\right\}$$

$$-a\left\{1+\frac{(a+1)}{1.c}z+\frac{(a+1)(a+2)}{2!c(c+1)}z^2+\frac{(a+1)(a+2)(a+3)}{3!c(c+1)(c+2)}z^3+\ldots\right\}$$

$$+z\left\{1+\frac{a}{1.c}z+\frac{a(a+1)}{2!c(c+1)}z^2+\frac{a(a+1)(a+2)}{3!c(c+1)(c+2)}z^3+\ldots\right\}$$

$$=\left\{\frac{c-a}{c}(a-1-a)+\frac{a}{c}(a-a-1)+1\right\}z$$

$$+\left\{\frac{(c-a)a}{2!c(c+1)}(a-1-a-1)+\frac{a(a+1)}{2!c(c+1)}(a-a-2)+\frac{a}{c}\right\}z^2$$

$$+\left\{\frac{(c-a)a(a+1)}{3!c(c+1)(c+2)}(a-1-a-2)\right.$$

$$\left.+\frac{a(a+1)(a+2)}{3!c(c+1)(c+2)}(a-a-3)+\frac{a(a+1)}{2!c(c+1)}\right\}z^3+\ldots$$

$$= \quad (-1+a/c-a/c+1)z+\frac{a}{2!c(c+1)}\{-2(c-a)-2(a+1)+2(c+1)\}z^2$$

$$+\frac{a(a+1)}{3!c(c+1)(c+2)}\{-3(c-a)-3(a+2)+3(c+2)\}z^3+\ldots$$

$$= \quad 0.$$

2.9 Unit Summary

The Hypergeometric differential equation is introduced in this unit. The series solution for different cases are obtained. The Hypergeometric function is also defined here. Some useful properties of this function are established. This function is represented in terms of integration. The confluent Hypergeometric differential equation is deduced from Hypergeometric equation. Some properties including integral representation are also established. An exercise is included at the end of this unit.

2.10 Self Assessment Questions

1. Prove that

$$F(a,b,c;z) - F(a,b,c-1;z) = -\frac{ab}{c(c-1)} zF(a+1,b+1,c+1;z).$$

2. Prove that

$$(a-b)F(a,b,c;z) = aF(a+1,b,c;z) - bF(a,b+1,c;z).$$

3. Prove that

(i) $F(a/2, a/2+1/2, 1/2; z^2) = \frac{1}{2}\left[(1-z)^{-a} + (1+z)^{-a}\right].$

(ii) $F(a/2+1/2, a/2+1, 3/2; z^2) = \frac{1}{2az}\left[(1-z)^{-a} - (1+z)^{-a}\right].$

(iii) $F(1/2, 1, 3/2; z^2) = \frac{1}{2z}\log\frac{1+z}{1-z}.$

(iv) $F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$

(v) $F(a,b,c;z) = (1-z)^{-a} F(a, c-b, c; -z/(1-z)).$

(vi) $F(a, 1-b, a+1; z) = az^{-a}\int_0^z t^{a-1}(1-t)^{b-1}\,dt.$

(vii) $\cos(wz) = F(w/2, -w/2, 1/2; \sin^2 z), \quad -\pi/2 \le z \le \pi/2.$

4. Prove the following relations:

(i) $(c-a)G(a-1,c;z) + (2a-c+z)G(a,c;z) - aG(a+1,c;z) = 0.$

(ii) $\int_0^\infty e^{-tz} G(a,c;z)\,dz = \frac{1}{t} F(a,1,c;1/t).$

5. Prove that the Wronskian of $G(a,b;z)$ and $z^{1-b}G(1+a-b, 2-b; z)$ is given by

$$-\left[(b-1)!\,e^z/(a-1)z^b\right].$$

6. Prove that

$$\frac{d^n}{dx^n} G(a,b;z) = \frac{(a)_n}{(b)_n} G(a+n, b+n; z),$$

where $(a)_n = a(a+1)(a+2)\ldots(a+n-1).$

2.11 Suggested Further Readings

1. I.N. Sneddon, *Special Functions of Mathematical Physics and Chemistry.*
2. N.N. Lebeder, *Special Functions and their Applications.*
3. D. Rainville, *Special Functions.*
4. M. Birkhoff and G.C. Rota, *Ordinary Differential Equations.*
5. E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations.*
6. G.B. Arfken and H.J. Weber, *Mathematical Methods for Physicists.*

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming
PART-I

Paper-I Group-C

Module No. - 11

Ordinary Differential Equations
(LEGENDRE EQUATION)

CONTENT :

Legendre polynomial is used in many different mathematical and physical problems. These polynomials may be constructed as a consequence of demanding a complete, orthogonal set of functions over the interval $[-1, 1]$. In quantum mechanics they represent angular momentum eigenfunctions.

Objectives:

- Legendre differntial equation and its series solution.
- Legendre polynomial.

- Rodrigues Formula.
- Properties of Legendre polynomial including orthogonal property.
- Expansion of function using Legendre polynomials.
- Generating function.
- Recurrence relations for Legendre polynomial.
- Integral Representation and Laplace's Integral.
- Exercise.

3.1 Legendre Differential Equation

The euqation

$$\left(1-z^2\right)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+n(n+1)w=0, \tag{3.1}$$

where n is a real parameter, is known as Legendre equation of order n. This equation has regular singularities at $z=\pm 1$. The physically interesting range for $|z|$ is generally $|z|\le 1$ (usually z is the cosine of some angle). The equation (3.1) can be written as

$$\frac{d}{dz}\left[\left(1-z^2\right)\frac{dw}{dz}\right]+n(n+1)w=0. \tag{3.2}$$

To study the point of infinite we transform the equation by substituting $z=1/t$. The transforms equation is

$$\left(t^2-1\right)\frac{d^2w}{dt^2}+2t\frac{dw}{dt}+\frac{1}{t^2}n(n+1)w=0, \tag{3.3}$$

which shows that the point at ∞ is a regular singularity. Further, this equation is a particular type of hypergeometric equation as in seen by substituting $z_1=(1-z)/2$, which transforms $z=1$ to $z_1=0$ and $z=-1$ to $z_1=1$. Also, $2dz_1=-dz$.

Under this transformation the Legendre equation becomes

$$\left(\frac{1-z}{2}\right)\left(\frac{1+z}{2}\right)\frac{d^2w}{dz_1^2}+\left[1-\left(\overline{-n+n+1}+1\right)z_1\right]\frac{dw}{dz_1}+n(n+1)w=0$$

i.e.,

$$z_1(1-z_1)\frac{d^2w}{dz_1^2}+\left[1-\left(\overline{-n+n+1}+1\right)z_1\right]\frac{dw}{dz_1}-(-n)(n+1)w=0. \tag{3.4}$$

Here $a=-n$, $b=n+1$, $c=1$ and $z=z_1$. Therefore, the solution of this equation is

$$w=F(-n,n+1,1;z_1)=F(-n,n+1,1;(1-z)/2) \tag{3.5}$$

which shows that when n is integer it reduces to a polynomial of degree n and this is known as *Legendre function of first kind.*

3.2 Solution of Legendre Equation at $z = 0$

It is obvious that $z = 0$ is the ordinary point of the Legendre equation. Therefore,

$$w = \sum_{0}^{\infty} c_k z^k, \tag{3.6}$$

for suitable values of c_k, is the solution (3.1). Using this substitution (3.1) becomes

$$\left(1-z^2\right)\sum_{0}^{\infty} c_k k(k-1)z^{k-2} - 2z\sum_{0}^{\infty} c_k kz^{k-1} + n(n+1)\sum_{0}^{\infty} c_k z^k = 0.$$

The coefficient of z^k leads to the following equation

$$c_{k+2}(k+1)(k+2) - c_k(k-1)k - 2c_k k + n(n+1)c_k = 0.$$

and this equation can be written as

$$c_{k+2} = \frac{n(n+1)-k(k+1)}{(k+1)(k+2)}c_k = -\frac{(n-k)(n+k+1)}{(k+1)(k+2)}c_k.$$

Thus

$$c_2 = -\frac{n(n+1)}{2!}c_0, \qquad c_3 = -\frac{(n-1)(n+2)}{2.3}c_1$$

$$c_4 = -\frac{(n-1)(n+3)}{3.4}c_2, \qquad c_5 = -\frac{(n-3)(n+4)}{4.5}c_3$$

and so on

Thus the solution of (3.1) at $z = 0$ is

$$w = c_0\left[1 - \frac{n(n+1)}{2!}z^2 + \frac{n(n+1)(n-2)(n+3)}{4!}z^4 - \ldots\right]$$

$$+c_1\left[z - \frac{(n-1)(n+2)}{3!}z^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}z^5 - \ldots\right] \tag{3.7}$$

From this solution it may be noted that whether n be even or odd one of the series terminates. When n is even the first one terminate while the second gives an infinite series and otherwise when n is odd first one is infinite and the second expression is a polynomial. The series which terminates is called Legendre polynomial of degree n and it is denoted by $P_n(z)$.

The interval of convergence of the infinite series is $|z| < 1$. Moreover, $|z| \geq 1$ gives divergent series. Hence the general solution of Legendre equation is

$$w = AP_n(z) + BQ_n(z),$$

where $Q_n(z)$ is an infinite series and satisfied (3.1).

3.2.1 General expression of $P_n(z)$

Irrespective of the value of n (odd or even) the polynomial $P_n(z)$ of degree n can be written as

$$P_n(z) = c_n\left[z^n + \frac{c_{n-2}}{c_n}z^{n-2} + \frac{c_{n-2}}{c_n}\frac{c_{n-4}}{c_{n-2}}z^{n-4} + ...\right]$$

where

$$c_{k+2} = -\frac{(n-k)(n+k+1)}{(k+1)(k+2)}c_k,$$

$k = n-2, n-4, ...$

Thus

$$P_n(z) = c_n\left[z^n - \frac{n(n-1)}{2(2n-1)}z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}z^{n-4} + ... p \text{ terms}\right]$$

where $p = n/2+1$, when n is even and $p = (n-1)/2+1$ when n is odd.

Since the constant c_n is arbitrary, we can choose $c_n = \frac{a.3.5...(2n-1)}{n!}$ and hence the Legendre polynomial becomes

$$P_n(z) = \frac{1.3.5.....(2n-1)}{n!}\left[z^n - \frac{n(n-1)}{2(2n-1)}z^{n-2}\right.$$

$$\left. + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}z^{n-4} + ...p \textit{ terms}\right]$$

$$= \frac{(2n)!}{2^n(n!)^2}\left[z^n - \frac{n(n-1)}{2(2n-1)}z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}z^{n-4}\right. \quad (3.8)$$

$$\left. + ...p \text{ terms}\right].$$

The last term is constant when n is even or contain z when n is odd.

Some lower order Legendre polynomials :

$$P_0(z) = 1$$
$$P_1(z) = z$$
$$P_2(z) = \frac{1}{2}(3z^2 - 1)$$
$$P_3(z) = \frac{1}{2}(5z^3 - 3z)$$
$$P_4(z) = \frac{1}{8}(35z^4 - 30z^2 + 3)$$
$$P_5(z) = \frac{1}{8}(63z^5 - 70z^3 + 15z)$$
$$P_6(z) = \frac{1}{16}(231z^6 - 315z^4 + 105z^2 - 5)$$
$$P_7(z) = \frac{1}{16}(429z^7 - 693z^4 + 315z^3 - 35z)$$
$$P_8(z) = \frac{1}{128}(6435z^8 - 1201z^6 + 6930z^4 - 1260z^2 + 35).$$

Conversely the terms $1, z, z^2$ etc. can be expressed using Legendre polynomial. Thus every algebraic function is expressible in terms of Legendre polynomial.

$$1 = P_0(z)$$
$$z = P_1(z)$$
$$z^2 = \frac{1}{3}[2P_2(z) + 1]$$
$$z^3 = \frac{1}{5}[2P_3(z) + 3P_1(z)]$$
$$z^4 = \frac{1}{35}[8P_4(z) + 20P_2(z) + 7]$$
$$z^5 = \frac{1}{63}[8P_5(z) + 28P_3(z) + 27P_1(z)]$$

and so on.

Example 3.1 *Show that*

$$4z^3 - 5z^2 - 3z = \frac{8}{5}P_3(z) - \frac{10}{3}P_2(z) - \frac{3}{5}P_1(z) - \frac{5}{3}P_0(z).$$

SOLUTION :

$$4z^3-5z^2-3z=\frac{4}{5}[2P_3(z)+3P_1(z)]-\frac{5}{3}[2P_2(z)+1]-3P_1(z)$$

$$=\frac{8}{5}P_3(z)-\frac{10}{3}P_2(z)-\frac{3}{5}P_1(z)-\frac{5}{3}P_0(z).$$

3.3 Rodrigues Formula

The Legendre polynomial (3.9) is

$$P_n(z)=\frac{(2n)!}{2^n(n!)^2}\left[z^n-\frac{n(n-1)}{2(2n-1)}z^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}z^{n-4}+....p \text{ terms}\right].$$

The last term being a constant or multiple of z according as ni is even or odd. When n is even the number of term is $n/2+1$ and when n is odd then number of term is $(n-1)/2+1$. So the number of term is denoted by p.

$$P_n(z)=\frac{1.3.5....(2n-1)}{n!}$$

$$\times\frac{d}{dz}\left[\frac{z^{n+1}}{n+1}-\frac{n(n-1)}{2(2n-1)(n-1)}z^{n-1}\right.$$

$$\left.+\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)(n-3)}z^{n-3}+.....p \text{ terms}\right]$$

$$=\frac{1.3.5....(2n-1)}{(n+1)!}$$

$$\times\frac{d}{dz}\left[z^{n+1}-\frac{(n+1)n}{2(2n-1)}z^{n-1}+\frac{(n+1)n(n-1)(n-2)}{2.4(2n-1)(2n-3)}z^{n-3}+....p \text{ terms}\right]$$

$$=\frac{1.3.5....(2n-1)}{(n+2)!}$$

$$\times\frac{d^2}{dz^2}\left[z^{n+2}-\frac{(n+2)(n+1)}{2(2n-1)}z^{n}+\frac{(n+2)(n+1)n(n-1)}{2.4(2n-1)(2n-3)}z^{n-2}+....p \text{ terms}\right]$$

$$=\frac{1.3.5...(2n-1)}{(2n)!}\times\frac{d^n}{dz^n}\left[z^{2n}-\frac{2n(2n-1)}{2(2n-1)}z^{2n-2}\right.$$

$$+\frac{2n(2n-1)(2n-2)(2n-3)}{2.4(2n-1)(2n-3)}z^{2n-4}+.....p\text{ terms}\Big].$$

The last term being a multiple of z^n or z^{n+1} according as n is even or odd.

Therefore,

$$P_n(z)=\frac{1.3.5....(2n-1)}{(2n)!}\times\frac{d^n}{dz^n}\left[z^{2n}-\frac{2n(2n-1)}{2(2n-1)}z^{2n-2}\right.$$

$$\left.+\frac{2n(2n-1)(2n-2)(2n-3)}{2.4(2n-1)(2n-3)}z^{2n-4}+....(n+1)\text{ terms}\right].$$

The terms that are added within the third bracket to make $(n+1)$ terms are at most of degree z^{n-2} when n is even and z^{n-1} when n is odd. So that these terms when differentiated n times with respect to z become zero. Therefore,

$$P_n(z)=\frac{1.3.5....(2n-1)}{(2n)!}\frac{d^n}{dz^n}\left[\left(z^2-1\right)^n\right]$$

$$=\frac{1}{2.4.6....(2n)}\frac{d^n}{dz^n}\left[\left(z^2-1\right)^n\right]$$

$$=\frac{1}{2^b(n!)}\frac{d^n}{dz^n}\left[\left(z^2-1\right)^n\right]. \qquad (3.9)$$

The equation (3.9) is called *Rodrigues formula* for Legendre plynomial.

The Legendre polynomial has many interesting properties, some of them can be deduced from Rodrigues formula.

Recall that $P_n(z)$ is a solution of the Legendre equation

$$\left(1-z^2\right)w''-2zw'+n(n+1)w=0$$

and since the equation is unchanged on changing n to $-n$ to $-n-1$, therefore,

$$P_n(z)=P_{-n-1}(z). \qquad (3.10)$$

The Legendre polynomial is even or odd function according as n is even or odd, i.e.,

$$P_n(-z)=\frac{(-1)^n}{2^n.n!}\frac{d^n}{dz^n}\left[\left(z^2-1\right)^n\right]=(-1)^nP_n(z). \qquad (3.11)$$

The lower order polynomials can be deduced from Rodrigues formula by substituting $n=0,1,2,....$

The value of the Legendre polynomial at $z=1$ is unity, i.e.,

$$P_n(1) = 1. \tag{3.12}$$

This proposition is justified in the following.

$$P_n(1) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\left[(z+1)^n (z-1)^n\right]_{z=1}$$

Differentiating $\left[(z+1)^n (z-1)^n\right]$, n times using Leibnitz theorem we obtain

$$P_n(1) = \frac{1}{2^n n!}\left[n!(z+1)^n + n(n-1)!(z+1)^{n-1}(z-1) + \ldots + n!(z-1)^n\right]_{z=1}$$

$$= \frac{1}{2^n n!}\left[n!2^n + 0\right] = 1.$$

Example 3.2 *Use Rodrigues formula to show that*

$$P'_{n+1}(z) - P'_{n-1}(z) = (2n+1)P_n(z), \tag{3.13}$$

where prime represents the derivative with respect to z.

Solution :

$$P'_{n+1}(z) - P'_{n-1}(z)$$

$$= \frac{d}{dz}\left[\frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}}{dz^{n+1}}\left[(z^2-1)^{n+1}\right]\right] + \frac{d}{dz}\left[\frac{1}{2^{n-1}\cdot(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z^2-1)^{n-1}\right]\right]$$

$$= \frac{1}{2^{n+1}(n+1)!}\frac{d^{n+2}}{dz^{n+2}}\left[(z^2-1)^{n+1}\right] - \frac{1}{2^{n-1}(n-1)!}\frac{d^n}{dz^n}\left[(z^2-1)^{n-1}\right]$$

$$= \frac{1}{2^n\cdot n!}\frac{d^n}{dz^n}\left[\frac{1}{2(n+1)}\frac{d^2}{dz^2}\left[(z^2-1)^{n+1}\right] - 2n(z^2-1)^{n-1}\right]$$

$$= \frac{1}{2^n\cdot n!}\frac{d^n}{dz^n}\left[\frac{d}{dz}\left[z(z^2-1)^n\right] - 2n(z^2-1)^{n-1}\right]$$

$$= \frac{1}{2^n\cdot n!}\frac{d^n}{dz^n}\left[(z^2-1)^n + 2nz^2(z^2-1)^{n-1} - 2n(z^2-1)^{n-1}\right]$$

$$= \frac{1}{2^n\cdot n!}\frac{d^n}{dz^n}\left[(z^2-1)^n + 2n(z^2-1)(z^2-1)^{n-1}\right]$$

$$= \frac{1}{2^n \cdot n!} \frac{d^n}{dz^n}\left[(2n+1)(z^2-1)^n\right]$$

$$= \frac{2n+1}{2^n \cdot n!} \frac{d^n}{dz^n}\left[(z^2-1)^n\right]$$

$$= (2n+1) P_n(z).$$

The Legendre polynomial of order n contains n zeros and these zeros lie between -1 and 1. This proposition is proved in the following theorem.

Theorem 3.1 *The zeros of Legendre polynomial are all real, distinct and lie between* -1 *and* 1.

Proof. Let us consider the Rodrigues formula

$$P_n(z) = \frac{1}{2^n \cdot n!} \frac{d^n}{dz^n}\left[(z^2-1)^n\right].$$

Rewriting this equation is

$$P_n(z) = \frac{1}{2^n \cdot n!} \frac{d^n}{dz^n}\left[(z+1)^n (z-1)^n\right]$$

and applying Leibnitz theorem n times we obtain

$$P_n(z) = \frac{1}{2^n \cdot n!}\left[n!(z+1)^n + n.(n-1)!(z-1)(z+1)^{n-1} + \ldots + n!(z-1)^n\right].$$

If z be real and $|z| > 1$ all the terms in the above are positive, i.e., $P_n(z)$ is always positive. Hence there cannot be any zero for all z with $|z| > 1$. Hence the zeros lie between -1 and 1.

Again, the equation $(z^2-1)^n = 0$ has $2n$ roots of which n roots are positive and are equal to 1 and n roots are negative, i.e., -1.

The equation $\frac{d}{dz}\left[(z^2-1)^n\right] = 0$ must have at least one roots between -1 and 1 also this is the same as

$$(z+1)^{n-1}.n(z-1)^n + n(z+1)^n (z-1)^{n-1} = 0$$

$$\text{or, } (z+1)^{n-1}(z-1)^{n-1}.2z = 0.$$

Hence the equation $\frac{d}{dz}\left[(z^2-1)^n\right] = 0$ has $(2n-1)$ roots, $n-1$ roots equal to 1 and $n-1$ roots equal to -1 and one root say, $\alpha_0 (= 0)$ between -1 and 1.

Again from the above result it follows that the equation $\frac{d^2}{dz^2}\left[\left(z^2-1\right)^n\right]=0$ must have at least one root, α_1 (say) between -1 and α_0 and at least one root α_2 (say) between α_0 and 1. Hence $\frac{d^2}{dz^2}\left[\left(z^2-1\right)^n\right]=0$ has two roots between -1 and 1.

But,

$$\frac{d^2}{dz^2}\left[\left(z^2-1\right)^n\right]=(z-1)^{n-2}(z+1)^{n+2}f_2(z)=0$$

where $f_2(z)$ is polynomial of degree 2 of z. Thus the equation has $n-2$ roots equal to 1, $n-2$ roots equal to -1 and two roots α_1 and α_2, between -1 and 1.

Similarly, $\frac{d^3}{dz^3}\left[\left(z^2-1\right)^n\right]=0$ has three roots $\alpha_1',\alpha_2',\alpha_3'$ say, between -1 and $\alpha_1;\alpha_1$ and α_2 and α_2, 1.

Proceeding this way, $\frac{d^n}{dz^n}\left[\left(z^2-1\right)^n\right]=0$ has n roots all real, distinct and lie between -1 and 1. Hence the theorem.

3.4 Orthogonality

Let z be real and $P_m(z)$ and $P_n(z)$ are the Legendre polynomials of degree m and n respectively. Since the Legendre polynomial is the solution of Legendre equation (3.1), then

$$\frac{d}{dz}\left[\left(1-z^2\right)\frac{dP_m}{dz}\right]+m(m+1)P_m=0 \text{ and} \tag{3.14}$$

$$\frac{d}{dz}\left[\left(1-z^2\right)\frac{dP_n}{dz}\right]+n(n+1)P_n=0. \tag{3.15}$$

Multiplying (3.14) by P_n and (3.15) by P_m and subtracting, we have

$$\frac{d}{dz}\left[\left(1-z^2\right)\left(P_m'P_n-P_n'P_m\right)\right]+\left[m(m+1)-n(n+1)\right]P_nP_m=0.$$

Integrating between the limits -1 and 1 with respect z,

$$\left[\left(1-z^2\right)\left(P_m'P_n-P_n'P_m\right)\right]_{-1}^{1}+\int_{-1}^{1}\left[m(m+1)-n(n+1)\right]P_m\,P_n\,dz=0$$

or, $$\left[m(m+1)-n(n+1)\right]\int_{-1}^{1}P_m\,P_n\,dz=0$$

or, $\int_{-1}^{1} P_m P_n \, dz = 0$ since $m \neq n$. (3.16)

It can be shown that

$$\int_{-1}^{1} P_n(z)^2 \, dz = \frac{2}{2n+1} \qquad (3.17)$$

This fact is justified by considering the Rodrigues formula in the following.

$$\int_{-1}^{1} P_n(z)^2 \, dz = A\int_{-1}^{1} u_n^{(n)} u_n^{(n)} \, dz$$

where $A = \frac{1}{(2^n.n!)^2}$ and $u_n = (z^2-1)^n$,

$u_n^{(n)}$ denotes *nth* derivative of u_n.

$$= \left[A u_n^{(n-1)} u_n^{(n)}\right]_{-1}^{1} - A\int_{-1}^{1} u_n^{(n-1)} u_u^{(n+1)} \, dz$$

$$= 0 + (-1)A\int_{-1}^{1} u_n^{(n-1)} u_n^{(n+1)} \, dz$$

$$= (-1)^n A\int_{-1}^{1} u_n u_n^{(2n)} \, dz$$

$$= (-1)^n (2n)!\, A\int_{-1}^{1} u_n \, dz \quad \text{since } u_n^{(2n)} = (2n)!.$$

Now,

$$\int_{-1}^{1} (z^2-1)^n \, dz = \int_{-1}^{1} (z+1)^n (z-1)^n \, dz$$

Putting $1 - z = 2y$, then $-dz = 2\,dy$

$$= (-1)^n \int_{1}^{0} 2^{2n+1} y^n (1-y)^n (-dz)$$

$$= (-1)^n 2^{2n+1} B(n+1, n+1) = (-1)^n 2^{2n+1} \frac{(n!)^2}{(2n+1)!}.$$

Hence

$$\int_{-1}^{1} P_n^2(z) \, dz = A(-1)^n (2n)! (-1)^n 2^{2n+1} \frac{(n!)^2}{(2n+1)!} = \frac{2}{2n+1}.$$

Note. The orthogonality of Legendre polynomial in terms of trigonometric form is

$$\int_{0}^{\pi} P_m(\cos\theta) P_n(\cos\theta) \sin\theta \, d\theta = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n. \end{cases} \qquad (3.18)$$

3.4.1 Expansion of Functions : Legendre Series

If $f(z)$ be a piecewise continuous function then it can be expanded in Legendre series. Let us assume that the series

$$\sum_{n=0}^{\infty} a_n P_n(z) = f(z), \tag{3.19}$$

in the sense of convergence in the interval [–1, 1]. This means that the functions $f(z)$ and $f'(z)$ be at least spice wise continuous in this interval. To find a_n, multiplying (3.19) by $P_m(z)$ and integrating term by term. Using orthogonal property, we obtain

$$\frac{2}{2m+1} a_m = \int_{-1}^{1} f(z) P_m(z)\, dz. \tag{3.20}$$

Changing the variable of integration z by t and the index m by n. Then we have

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(t) P_n(t)\, dt. \tag{3.21}$$

The constants a_n are called Legendre constants.

Using (3.21), (3.19) becomes

$$f(z) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \int_{-1}^{1} \left(f(t) P_n(t)\, dt \right) P_n(z). \tag{3.22}$$

This expansion in a series of Legendre polynomials is usually referred to as a Legendre series.

Legendre polynomial is also satisfied the following properties.

Property 3.1 *Prove that*

$$\int_{-1}^{1} P_n(z)\, dz = \begin{cases} 0, n \neq 0 \\ 2, n = 0. \end{cases} \tag{3.23}$$

Solution : Since $P_n(z)$ is a solution of Legendre equation, then

$$\frac{d}{dz}\left[(z^2 - 1) \frac{dP_n(z)}{dz} \right] + n(n+1) P_n(z) = 0.$$

Integrating between the limits –1 and 1 obtain

$$\int_{-1}^{1} \frac{d}{dz}\left[(z^2 - 1) \frac{dP_n(z)}{dz} \right] dz + \int_{-1}^{1} n(n+1) P_n(z)\, dz = 0 .$$

or, $\left[(z^2-1)\dfrac{dP_n(z)}{dz}\right]_{-1}^{1}+n(n+1)\int_{-1}^{1}P_n(z)\,dz=0$

or, $0+n(n+1)\int_{-1}^{1}P_n(z)\,dz=0$

or, $\int_{-1}^{1}P_n(z)\,dz=0,\ n\neq 0.$

When $n=0$,

$$\int_{-1}^{1}P_n(z)\,dz=\int_{-1}^{1}P_0(z)\,dz=\int_{-1}^{1}dz=2.$$

Property 3.2 *For integrer m, n,*

$$\int_{-1}^{1}z^m P_n(z)\,dz=\begin{cases}0, & \text{if } m<n.\\ \frac{2^{n+1}(n!)^2}{(2n+1)!}, & \text{if } m=n.\end{cases} \tag{3.24}$$

Proof. Let $A=\dfrac{1}{2^n.n!}$ and $u_n=(z^2-1)^n$.

Then

$$\int_{-1}^{1}z^m P_n(z)\,dz = A\int_{-1}^{1}z^m u_n^{(n)}\,dz$$

$$=A\left[u_n^{(n-1)}z^m\right]_{-1}^{1}-A\int_{-1}^{1}mz^{m-1}u_n^{(n-1)}\,dz$$

$$=(-1)A\int_{-1}^{1}mz^{m-1}u_n^{(n-1)}\,dz$$

$$=(-1)^2Am\int_{-1}^{1}(m-1)z^{m-2}u_n^{(n-2)}dz$$

$$=(-1)^m Am(m-1)(m-2)\ldots 2.1\int_{-1}^{1}u_n^{(n-m)}\,dz \tag{3.25}$$

$$=(-1)^m Am!\left[u_n^{(n-m-1)}\right]_{-1}^{1}dz$$

since $n>m, n-m-1\geq 0$, so $u_n^{(n-m-1)}$ contains at least one z^2-1

$$=(-1)^m Am!.0=0.$$

From equation (3.25), for $n=m$ we have

$$\int_{-1}^{1}z^n P_n(z)\,dz=(-1)^n An!\int_{-1}^{1}u_n\,dz$$

$$=(-1)^n\frac{1}{2^n.n!}n!(-1)^n\frac{(n!)^2}{(2n)!}\frac{2^{2n+1}}{2n+1}=\frac{(n!)^2}{(2n)!}\frac{2^{n+1}}{2n+1}$$

3.5 Generating Function

The generating function for Legendre polynomial is

$$g(z,t) = (1-2zt+t^2)^{-1/2}, |z| \le 1, |t| < 1. \tag{3.26}$$

That is, the coefficient of t^n is the Legendre polynomial of order n. This generating function generates Legendre polynomial of all orders.

For $|z| \le 1$ and $|t| < 1$, the expansion of $\left(1-2zt+t^2\right)^{-1/2}$ is shown below.

Since $|z| \le 1$ and $|t| < 1$, then $\left(1-2zt+t^2\right)^{-1/2}$ can be expanded in power series.

$$\left(1-2zt+t^2\right)^{-1/2} = \left[1-t(2z-t)\right]^{-1/2}$$

$$= 1+\frac{1}{2}t(2z-t)+\frac{1.3}{2.4}t^2(2z-t)^2+.....$$

$$+\frac{1.3....(2n-1)}{2.4....(2n)}t^n(2z-t)^n+...$$

Now, the coefficient of t^n in the term

$$\frac{1.3....(2n-1)}{2.4...(2n)}t^n(2z-t)^n$$

$$= \frac{1.3....(2n-1)}{2.4...(2n)}(2z)^n = \frac{1.3...(2n-1)}{2^n.(n!)}2^n z^n$$

$$= \frac{1.3....(2n-1)}{n!}z^n.$$

Again the coefficient of t^n in the term

$$\frac{1.3...(2n-3)}{2.4...(2n-2)}t^{n-1}(2z-t)^{n-1}$$

$$= \frac{1.3...(2n-3)}{2.4...(2n-2)}\left\{-(n-1)(2z)^{n-2}\right\}$$

$$= -\frac{1.3...(2n-3)}{2^{n-1}.1.2...(n-1)}\frac{2n-1}{n}\frac{n}{2n-1}\left\{(n-1)2^{n-2}z^{n-2}\right\}$$

$$= -\frac{1.3...(2n-2)}{n!}\frac{n(n-1)}{2(2n-1)}z^{n-2}$$

and so on.

Hence the coefficient of t^n in the expansion of $\left(1-2zt+t^2\right)^{-1/2}$ is

$$\frac{1.3.5....(2n-1)}{n!}\left[z^n-\frac{n(n-1)}{2(2n-1)}z^{n-2}+.....\right]$$

i.e., the coefficient of t^n in the expansion of $\left(1-2zt+t^2\right)^{-1/2}$ is $P_n(z)$. Thus

$$\left(1-2zt+t^2\right)^{-1/2}=\sum_{n=0}^{\infty}t^nP_n(z).$$

3.5.1 Special values

The generating function gives more information about Legendre polynomials. For $z=1$ the equation (3.26) becomes

$$\left(1-2t+t^2\right)^{-1/2}=\sum_{n=0}^{\infty}P_n(1)t^n$$

or, $$\frac{1}{1-t}=\sum_{n=0}^{\infty}P_n(1)t^n$$

or, $$\sum_{n=0}^{\infty}t^n=\sum_{n=0}^{\infty}P_n(1)t^n$$

Hence

$$P_n(1)=1 \tag{3.27}$$

for all values of n.

For $z=0$ the equation (3.26) becomes

$$\left(1+t^2\right)^{1/2}=\sum_{n=0}^{\infty}P_n(z)t^n. \tag{3.28}$$

Also, by binomial expansion

$$\left(1+t^2\right)^{-1/2}=1-\frac{1}{2}t^2+\frac{3}{8}t^4+....+(-1)^n\frac{1.3....(2n-1)}{2^n.n!}t^{2n}+......$$

Hence

$$P_{2n}(0)=\text{coefficient of } t^{2n} \text{ in the expansion of } \left(1+t^2\right)^{-1/2}$$

$$= (-1)^n \frac{1.3...(2n-1)}{2^n.n!} = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}. \qquad (3.29)$$

$P_{2n+1}(0) =$ coefficient of t^{2n+1} in the expansion of $(1+t^2)^{-1/2}$

$$= 0. \qquad (3.30)$$

The generating function remains unchamged if we replace z by $-z$ and t by $-t$. That is,

$$g(t,z) = g(-t,-z)$$

$$= [1-2(-t)(-z)+(-z)^2]^{-1/2} = \sum_{n=0}^{\infty} P_n(-z)(-t)^n$$

i.e. $\sum_{n=0}^{\infty} P_n(z)t^n = \sum_{n=0}^{\infty} (-1)^n P_n(-z)t^n.$

Hence

$$P_n(-z) = (-1)^n P_n(z). \qquad (3.31)$$

Thas is, the Legendre polynomials are odd or even with respect to $z = 0$ according to whether the index n is odd or even. This is called the *parity* or *reflection* property of Legendre polynomial.

From the relation (3.31), it is easy to observe that

$$P_{2n+1}(0) = 0.$$

The Legendre polynomial can be expressed in terms of cosine function. To find the expression for $P_n(\cos\theta)$, recall the generating function (3.26). Substituting $z = \cos\theta = (e^{i\theta} + e^{-i\theta})/2$ to

$$(1-2zt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(z)t^n$$

we get

$$\sum_{n=0}^{\infty} P_n(\cos\theta)t^n = \left(1-t\left(e^{i\theta}+e^{-i\theta}\right)+t^2\right)^{-1/2} = \left(1-te^{i\theta}\right)^{-1/2}\left(1-te^{-i\theta}\right)^{-1/2}.$$

Using binomial expansion, we have

$$\left(1-te^{i\theta}\right)^{-1/2} = 1+\frac{1}{2}te^{i\theta}+\frac{1.3}{2.4}t^2e^{2i\theta}+.....+\frac{1.2...(2n-1)}{2.4...(2n)}t^ne^{in\theta}+....$$

$$\left(1-te^{-i\theta}\right)^{-1/2} = 1+\frac{1}{2}te^{-i\theta}+\frac{1.3}{2.4}t^2e^{-2i\theta}+.....+\frac{1.2...(2n-1)}{2.4...(2n)}t^ne^{-in\theta}+....$$

Multiplying the above two expressions and equating the coefficient of t^n we obtain

$$P_n(\cos\theta)=\frac{1.2...(2n-1)}{2.4...(2n)}2\cos n\theta+\frac{1}{2}\frac{1.2....(2n-3)}{2.4...(2n-2)}2\cos(n-2)\theta+... \qquad (3.32)$$

Example 3.3 *If* (R,ϕ,z) *and* (r,θ,ϕ) *are cylindrical polar and spherical polar coordinates of the same point, show that*

$$P_n(\cos\theta)=(-1)^n\frac{r^{n+1}}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right). \qquad (3.33)$$

Solution : We have $r^2=x^2+y^2+z^2=R^2+z^2$. So $1/r=\left(R^2+z^2\right)^{-1/2}=f(R,z)$. Then

$$f(R,z+h)=\left[R^2+(z+h)^2\right]^{-1/2}=\left[R^2+z^2+2hz+h^2\right]^{-1/2}$$

$$=\left[r^2+2hr\cos\theta+h^2\right]^{-1/2}=\frac{1}{r}\left[1+\frac{2h\cos\theta}{r}+\frac{h^2}{r^2}\right]^{-1/2}$$

$$=\frac{1}{r}\sum_{n=0}^{\infty}(-1)^n\left(\frac{h}{r}\right)^n P_n(\cos\theta). \qquad (3.34)$$

Again, by Taylor's series, for several variables

$$f(R,z+h)=\sum_{n=0}^{\infty}\frac{h^n}{n!}\frac{\partial^n}{\partial z^n}f(R,z)=\sum_{n=0}^{\infty}\frac{d^n}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right). \qquad (3.35)$$

From the equation (3.34) and (3.35) we have

$$\frac{1}{r}\sum_{n=0}^{\infty}(-1)^n\left(\frac{h}{r}\right)^n P_n(\cos\theta)=\sum_{n=0}^{\infty}\frac{h^n}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right).$$

Equating the coefficients of h^n on both sides of the above equation, we get

$$\frac{(-1)^n}{r^{n+1}}P_n(\cos\theta)=\frac{1}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right).$$

That is,

$$P_n(\cos\theta)=\frac{(-1)^n}{n!}r^{n+1}\frac{1}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right). \qquad (3.36)$$

3.5.2 Bounds of $P_n(\cos\theta)$

The generating function may be used to find the bounds of $P_n(\cos\theta)$. We have for $z = \cos\theta$.

$$\left(1-2t\cos\theta+t^2\right)^{-1/2} = \left(1-te^{i\theta}\right)^{-1/2}\left(1-te^{-i\theta}\right)^{-1/2}$$

$$= \left(1+\frac{1}{2}te^{i\theta}+\frac{3}{8}t^2e^{2i\theta}+....\right)\times\left(1+\frac{1}{2}te^{-i\theta}+\frac{3}{8}t^2e^{-2i\theta}+....\right). \quad (3.37)$$

It may be noted that all the coefficient of above expression are positive. So the each term of right hand side of (3.37) is positive. Each term of the (3.37) is of the form

$$a_n\left(e^{im\theta}+e^{-im\theta}\right)/2 = a_m \cos m\theta,$$

where each a_m is positive. Then (3.37) reduces to

$$P_n(\cos\theta) = \sum_{m=0 or 1}^{n} a_m \cos m\theta. \quad (3.38)$$

The right hand side of (3.38) is clearly a maximum when $\theta = 0$ and $m\theta = 1$. That is, the maximum value of $P_n(\cos\theta)$ is $P_n(1)$. But, for $z = \cos\theta = 1, P_n(1) = 1$ (see (3.27)).

Therefore,

$$\left|P_n(\cos\theta)\right| \le P_n(1) = 1 \qquad \text{i.e., } -1 \le P_n(\cos\theta) \le 1. \quad (3.39)$$

From equations (3.32) and (3.38) we may conclude that the Legendre polynomial is a linear combination of $\cos m\theta$. This means that the Legendre polynomials form a complete set for any functions that may be expanded by a Fourier cosine series over the interval $(0, \pi)$.

3.6 Recurrence Relations

The generating function for Legendre polynomials provides a better way to deduce the recurrence relations and some special properties. Differentiating the relation

$$\left(1-2zt+t^2\right)^{-1/2} = \sum_0^{\infty} t^n P_n(z) \quad (3.40)$$

with respect to t, obtain

$$-\frac{1}{2}\left(1-2zt+t^2\right)^{-3/2}(-2z+2t) = \sum_0^{\infty} nt^{n-1}P_n(z)$$

$$(z-t)\left(1-2zt+t^2\right)^{-1/2} = \left(1-2zt+t^2\right)\sum_0^{\infty} nt^{n-1}P_n(z)$$

$$(z-t)\sum_{0}^{\infty} t^n P_n(z) = (1-2zt+t^2)\sum_{0}^{\infty} nt^{n-1} P_n(z).$$

Setting the coefficient of t^n equal to zero, we get

$$zP_n(z) - P_{n-1}(z) = (n+1)P_{n+1}(z) - 2znP_n(z) + (n-1)P_{n-1}(z).$$

On simplification,

$$(2n+1)zP_n(z) = (n+1)P_{n+1}(z) + nP_{n-1}(z). \qquad (3.41)$$

Again, differentiating (3.40) with respect to t, we obtain

$$-\frac{1}{2}(1-2zt+t^2)^{-3/2}(-2z+2t) = \sum_{0}^{\infty} nt^{n-1} P_n(z). \qquad (3.42)$$

Differentiation of (3.40) with respect to z yields

$$-\frac{1}{2}(1-2zt+t^2)^{-3/2}(-2t) = \sum_{0}^{\infty} t^{n-1} P_n'(z). \qquad (3.43)$$

Dividing (3.42) by (3.43),

$$\frac{t-z}{-t} = \frac{\sum_{0}^{\infty} nt^{n-1} P_n(z)}{\sum_{0}^{\infty} t^n P_n'(z)}.$$

That is,

$$\sum_{0}^{\infty} t^{n+1} P_n'(z) - z\sum_{0}^{\infty} t^n P_n'(z) + \sum_{0}^{\infty} nt^n P_n(z) = 0.$$

Equating t^n to zero,

$$P_{n-1}'(z) - zP_n'(z) + nP_n(z) = 0.$$

After simplification we get

$$nP_n(z) = zP_n'(z) - P_{n-1}'(z). \qquad (3.44)$$

Another recurrence relation can be obtained by differentiating (3.40) with respect to z. Differentiating (3.40) with respect to z,

$$-\frac{1}{2}(1-2zt+t^2)^{-3/2}(-2t) = \sum_{0}^{\infty} t^n P_n'(z)$$

$$t\sum_{0}^{\infty} t^n P_n(z) = (1-2zt+t^2)\sum_{0}^{\infty} t^n P_n'(z).$$

Equating the coefficient of t^{n+1} in both sides we obtain

$$P_n(z) = P'_{n+1}(z) - 2zP'_n(z) + P'_{n-1}(z). \tag{3.45}$$

From (3.44),

$$zP'_n(z) = nP_n(z) + P'_{n-1}(z).$$

Substituting this value in (3.45), we have

$$P_n(z) = P'_{n+1}(z) - 2\left[nP_n(z) + P'_{n-1}(z)\right] + P'_{n-1}(z)$$

i.e.,

$$(2n+1)P_n(z) = P'_{n+1}(z) - P'_{n-1}(z). \tag{3.46}$$

Combination of the results (3.45) and (3.44) yields

$$P_n(z) = P'_{n+1}(z) - 2zP'_n(z) + \left[zP'_n(z) - nP_n(z)\right]$$

i.e.,

$$(n+1)P_n(z) = P'_{n+1}(z) - zP'_n(z). \tag{3.47}$$

Replacing n by $n-1$ of (3.47), we have

$$nP_{n-1}(z) = P'_n(z) - zP'_{n-1}(z). \tag{3.48}$$

Multiplying (3.47) by z and adding with (3.48) we obtain

$$\begin{aligned} nP_{n-1}(z) + z(n+1)P_n(z) &= zP'_{n+1}(z) - z^2P'_n(z) + P'_n(z) - zP'_{n-1}(z) \\ &= \left(1-z^2\right)P'_n(z) + z\left(P'_{n+1}(z) - P'_{n-1}(z)\right) \\ &= \left(1-z^2\right)P'_n(z) + z\cdot(2n+1)P_n(z) \text{ [using (3.46)]} \end{aligned}$$

This relation can be written as

$$\left(1-z^2\right)P'_n(z) = n\left[P_{n-1}(z) - zP_n(z)\right]. \tag{3.49}$$

Differentiating (3.49) we have

$$\begin{aligned} \left(1-z^2\right)P''_n(z) - 2zP'_n(z) &= nP'_{n-1}(z) - nP_n(z) - nzP'_n(z) \\ &= n\left[zP'_n(z) - nP_n(z)\right] - nP_n(z) - nzP'_n(z) \text{ [using (3.44)]} \end{aligned}$$

Thus

$$\left(1-z^2\right)P''_n(z) - 2zP'_n(z) + n(n+1)P_n(z) = 0. \tag{3.50}$$

The equations (3.41), (3.44), (3.46), (3.47) and (3.49) are the recurrence relations for Legendre polynomials.

The equations (3.44), (3.46), (3.47) and (3.49) are all first order differential equations, but with two different indices. The indices of (3.50) are equal but of second order differential equation, and this equation is Legendre equation.

It may be noted that the recurrence relations are deduced from the generating function of the Legendre polynomial without considering the Legendre differential equation, and finally we obtain the Legendre differential equation (3.50).

The equation (3.50) can be written as

$$\frac{d}{dz}\left[\left(1-z^2\right)\frac{dP_n(z)}{dz}\right]+n(n+1)P_n(z)=0.$$

Substituting $z=\cos\theta$. Then $\frac{dz}{d\theta}=-\sin\theta$.

$$\frac{d}{d\theta}\left[\left(1-z^2\right)\frac{dP_n(z)}{d\theta}\frac{d\theta}{dz}\right]\frac{d\theta}{dz}+n(n+1)P_n(z)=0.$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dP_n(\theta)}{d\theta}\right]+n(n+1)P_n(\cos\theta)=0. \quad (.3.51)$$

3.7 Integral Representation

Schläfli's Integral

The Rodrigue's formula for integral n is

$$P_n(z)=\frac{1}{2^n.n!}\frac{d^n}{dz^n}\left\{\left(z^2-1\right)^n\right\}. \quad (3.52)$$

From the theory of analytic function, it is known that if $f(z)$ be analytic then its nth derivative can be represented in the integral form

$$\frac{d^n}{dz^n}\{f(z)\}=\frac{n!}{2\pi i}\int_C\frac{f(t)}{(t-z)^{n+1}}dt, \quad (3.53)$$

where C is a closed contour within and on which $f(z)$ is analytic.

The function $\left(z^2-1\right)^n$ is analytic for all values of z. Therefore,

$$\frac{d^n}{dz^n}\left\{\left(z^2-1\right)^n\right\}=\frac{n!}{2\pi i}\int_C\frac{\left(t^2-1\right)^n}{(t-z)^{n+1}}dt, \quad (3.54)$$

where C_0 is a closed contour in t-plane enclosing $t = z$.

Therefore,

$$P_n(z) = \frac{1}{2^n . 2\pi i} \int_{C_0} \frac{(t^2 - 1)^n}{(t - z)^{n+1}} dt. \tag{3.55}$$

This is *Schiläfli's integral* for Legendre's polynomial when n is an integer.

For non-integral values of n, the solution of Legendre equation can also be represent in terms of integration. Assume

$$w = \frac{1}{2^n . 2\pi i} \int_C \frac{(t^2 - 1)^n}{(t - z)^{n+1}} dt \tag{3.56}$$

where the contour C is to be so chosen that this satisfies the Legendre differential equation

$$\frac{d}{dz}\left\{(1 - z^2)\frac{dw}{dz}\right\} + n(n+1)w = 0. \tag{3.57}$$

The left hand side of (3.57) by substituting (3.56) is

$$\frac{1}{2\pi i} \int_C \frac{(t^2 - 1)^n}{2^n} \left[\frac{d}{dz}\left\{\frac{(n+1)(1-z^2)}{(t-z)^{n+2}}\right\} + \frac{n(n+1)}{(t-z)^{n+1}}\right] dt$$

$$= \frac{(n+1)}{2^n . 2\pi i} \int_C \frac{(t^2 - 1)^n}{(t - z)^{n+3}} \left[(n+2)(1-z^2) - 2z(1-z) + n(t - z^2)\right] dt$$

$$= \frac{(n+1)}{2^n . 2\pi i} \int_C \frac{(t^2 - 1)^n}{(t - z)^{n+3}} \left[2t(n+1)(1-z) - (n+2)(t^2 - 1)\right] dt$$

$$= \frac{(n+1)}{2^n . 2\pi i} \int_C \frac{d}{dt}\left[\frac{(t^2 - 1)^{n+1}}{(t - z)^{n+2}}\right] dt$$

$$= \frac{(n+1)}{2^n . 2\pi i} \left[\frac{(t^2 - 1)^{n+1}}{(t - z)^{n+2}}\right]_C \tag{3.58}$$

If n be an integer then (3.58) vanishes when C is any closed curve surrounding the point $t = z$ and thus (3.56) satisfies the differential equation (3.57) for such a contour.

If $n = \lambda$ (say) where λ is not an integer then the contour C is to be such that

$$\frac{(t^2-1)^{\lambda+1}}{(t-z)^{\lambda+2}}$$

returns to its original value after describing the contour C. The singularities of the above expression are at $t=1$, $t=-1$ and at $t = z$.

If we consider a circle or any closed curve C_1 surrounding the singularities $t = z$ and $t=1$ in the clockwise sense the expression

$$\frac{(t^2-1)^{\lambda+1}}{(t-z)^{\lambda+2}}$$

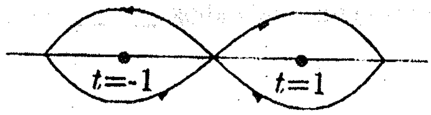

Figure 3.1 : The contour C_2

is multiplied by

$$\frac{e^{-2\pi i(\lambda+1)}}{e^{-2\pi i(\lambda+2)}} = 1$$

when we go round the contour once.

Apart from C_1, there is another contour C_2 which is shown in Figure 3.1. C_2 consits of two loops one surrounding $t=-1$ in anticlockwise sense and the other surrounding $t = 1$ in clockwise sense. On moving round this contour the expression is multiplied by $e^{-2\pi i(\lambda+1)}e^{2\pi i(\lambda+1)} = 1$ so that the contour C_2 also serves our purpose. The function with this contour integral form is Legendre formula of second kind.

3.7.1 Laplace's Integral

Recall the Schläfli's integral representation (3.55) for Legendre polynomials

$$P_n(z) = \frac{1}{2^n.2\pi i}\int_C \frac{(t^2-1)^n}{(t-z)^{n+1}}dt \tag{3.59}$$

where C is the closed contour enclosing the point $t = 1$ and $t = z$ but not $t = -1$.

Let us take for the contour C, the circle with the point z as centre and of radius $|z^2-1|^{1/2}$. In order that this circle may enclosed the point $t=1$. We must have $|z-1|<|z^2-1|^{1/2}$ i.e., $|z-1|<|z+1|$, also in order that tis circle may not enclosed the point $t=-1$ i.e., $t=-1$ be outside the circle. We must have $|z^2-1|^{1/2}<|z+1|$.

In order that this circle may enclosed of point $z=1$ and may not $t=-1$ the point z not lie to the right of the imaginary axis. Therefore,

$$|\arg z|<\frac{\pi}{2}.$$

If we take

$$t=z+\left(z^2-1\right)^{1/2}e^{i\phi}$$

be any point on the above circle with z as centre and radius $\sqrt{z^2-1}$. Then t describes C once in the positive sense ϕ changes from $-\pi$ to π. Now,

$$\begin{aligned} t^2-1 &= z^2+\left(z^2-1\right)e^{2i\phi}+2z\left(z^2-1\right)^{1/2}e^{i\phi}-1 \\ &= \left(z^2-1\right)^{1/2}e^{2i\phi}\left[2z+\left(z^2-1\right)^{1/2}\left(e^{i\phi}+e^{-i\phi}\right)\right] \\ &= 2\left(z^2-1\right)^{1/2}e^{2i\phi}\left[z+\left(z^2-1\right)^{1/2}\cos\phi\right] \end{aligned}$$

$$dt=\left(z^2-1\right)^{1/2}e^{i\phi}i\,d\phi.$$

Therefore,

$$\begin{aligned} P_n(z) &= \frac{1}{2^n.2\pi i}\int_{-\pi}^{\pi}\frac{2^n\left(z^2-1\right)^{n/2}e^{in\phi}\left\{z+\left(z^2-1\right)^{1/2}\cos\phi\right\}^n}{\left(z^2-1\right)^{(n+1)}e^{(n+1)i\phi}}\left(z^2-1\right)^{1/2}ie^{i\phi}d\phi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[z+\left(z^2-1\right)^{1/2}\cos\phi\right]^n d\phi \\ &= \frac{1}{\pi}\int_{0}^{\pi}\left[z+\left(z^2-1\right)^{1/2}\cos\phi\right]^n d\phi. \end{aligned} \tag{3.60}$$

This is known as *Laplace first integral.*

3.8 Unit Summary

Legendre polynomial has many applications in different disciplines. In this unit the series solution of Legendre differential equation is deduced and hence obtained the Legendre polynomial. In this unit it is shown that the *n*th order Legendre polynomial can be expressed in terms of *n*th order derivative, called Rodrigues formula. Some properties of this polynomial are established using Rodrigues formula. The generating function for this polynomial is deduced and several properties are proved using it. The integral representation including Laplace's integral are also established. An exercise is also given at the end of this unit.

3.9 Self Assessment Questions

1. Express the following expressions terms of Legendre polynomials (i) $3x-2x^2+x^3$, (ii) x^4-8x^2-1, (iii) $1+x+x^2$, (iv) $\sin x$.

2. Expand the following function in terms of Legendre polynomials

$$f(x)=\begin{cases}-1, & if\ -1\le x<0\\ 1, & if\ \ 0<x\le 1.\end{cases}$$

3. Show that

$$1+3P_1(z)+5P_2(z)+7P_3(z)+\ldots+(2n+1)P_n(z)=\frac{d}{dz}[P_{n+1}(z)+P_n(z)].$$

4. Prove the following recurrence relations

(i) $P'_{n+1}(z)-zP'_n(z)=(n+1)P_n(z)$

(ii) $(n+1)P_{n+1}(z)-(2n+1)zP_n(z)+nP_{n-1}(z)=0$

(iii) $zP'_n(z)-P'_{n-1}(z)=nP_n(z)$

(iv) $P'_{n+1}(z)-P'_{n-1}(z)=(2n+1)P_n(z)$

(v) $(1-z^2)P'_n(z)=nP_{n-1}(z)-nzP_n(z)$.

5. Show that $(1-2zt+t^2)^{-1/2}$ is the generating function of Legendre polynomials.

6. Show that $P_n(z)$ is the coefficient of t^n in the expansion of $(1-2zt+t^2)^{-1/2}$.

7. Show that the Legendre differential equation can be written in the form

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dP_n(\theta)}{d\theta}\right]+n(n+1)P_n(\cos\theta)=0.$$

8. Show that

$$\int_0^{\pi} P_m(\cos\theta)P_n(\cos\theta)\sin\theta d\theta = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n. \end{cases}$$

9. Show that

$$\frac{1+t}{t\sqrt{1-2tz+t^2}} - \frac{1}{t} = \sum_{n=0}^{\infty}[P_n(z)+P_{n+1}(z)]t^n.$$

10. Prove that

$$\frac{1-t^2}{\left(1-2tz+t^2\right)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)P_n(z)t^n.$$

11. Show that

$$1+\sum_{n=0}^{\infty}\frac{1}{n+1}P_n(\cos\theta) = \log\left[\frac{1+\sin\theta/2}{\sin\theta/2}\right].$$

12. Show that

$$P_n(z) = \frac{(-1)^n}{n!}\frac{d^n}{dz^n}\left[(1-z)^n\left\{1-\frac{1-z}{2}\right\}^n\right].$$

13: Show that

$$\int_0^1 P_{2n}(z)P_{2n+1}(z)dz = \int_0^1 P_{2n}(z)P_{2n-1}(z)dz.$$

14. Prove that

(i) $\int_{-1}^{1}[P_n(z)]^2 dz = \frac{2}{2n+1}$

(ii) $\int_{-1}^{1}[P_n'(z)]^2 dz = n(n+1)$

(iii) $\int_{-1}^{1} zP_n(z)P_n'(n)dz = \frac{2n}{2n-1}$

(iv) $\int_{-1}^{1} zP_n(z)P_{n-1}(z)dz = \frac{2n}{4n^2-1}$.

15. Show that

(i) $\int_{-1}^{1} z^2P_n^2(z)dz = \frac{1}{8}\left[\frac{1}{2n-1}+\frac{6}{2n+1}+\frac{1}{2n+3}\right]$

(ii) $\int_{-1}^{1}(z^2-1)P_{n+1}(z)P_n'dz = \frac{2n(n+1)}{(2n+1)(2n+3)}$

16. Prove that

$$P_n(\cos\theta) = \sum_{k=0}^{n}(-1)^k\left({}^nC_k\right)^2(\cos\theta)^{2n-2}(\sin\theta)^{2k}.$$

17. Show that

$$\int_0^{\pi} P_{2n}(\cos\theta)\,d\theta = \left[\frac{\Gamma(n+1/2)}{\mathrm{T}(n+1)}\right]^2.$$

18. Prove that

$$\int_{-1}^{1} z(1-z^2)P_n'(z)P_m'(z)dz = \begin{cases} 0, & unless\ m = n\pm 1 \\ \frac{2n(n^2-1)}{4n^2-1}\delta_{m,n-1}, & if\ m<n \\ \frac{2n(n+2)(n+1)}{(2n+1)(2n+3)}\delta_{m,n+1}, & if\ m>n. \end{cases}$$

19. Operating in spherical polar coordinates, show that

$$\frac{\partial}{\partial z}\left[\frac{P_n(\cos\theta)}{r^{n+1}}\right] = -(n+1)\frac{P_{n+1}(\cos\theta)}{r^{n+2}}.$$

20. Prove that

$$P_n'(1) = \frac{1}{2}n(n+1).$$

21. Prove that, for $n \geq 1$,

$$\int_{-1}^{1}\log(1-z)P_n(z)dz = -\frac{2}{n(n+1)}.$$

22. Prove that

$$\int_z^1 P_n(z)dz = \frac{1}{2n+1}\left[P_{n-1}(z) - P_{n+1}(z)\right].$$

23. Prove that (i) $P_n(0) = 0$ for odd n and (ii) $P_n(0) = \frac{(-1)^{n/2}.n!}{2^n.\{(n/2)!\}^2}$ for even n.

3.10 Suggested Further Readings

1. I.N. Sneddon, *Special Functions of Mathematical Physics and Chemistry.*
2. N.N. Lebeder, *Special Functions and their Applications.*
3. D. Rainville, *special Functions.*
4. M. Birkhoff and G.C. Rota, *Ordinary Differential Equations.*
5. E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations.*
6. G.B. Arfken and H.J. Weber, *Mathematical Methods for Physicists.*

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-I **Group-C**

Module No. - 12

Ordinary Differential Equations

BESSEL EQUATION

Content :

Bessel functions appear in a large class of physical problems. Bessel functions and closely related functions form a rich area of applied mathematics with many representations, many interesting and useful properties and many interrelations.

Objectives

* Bessel differential equation and its series solutions.
* Generating functions
* Recurrent relations and some properties of Bessel function.
* Integral representation of Bessel functions.
* Orthogonality.
* Hankel functions and its contour integral representation.
* Exercise.

4.1 Bessel Differential Equation and its Solutions

The Bessel differential equation of order λ is

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+\left(z^2-\lambda^2\right)w=0. \qquad (4.1)$$

It is obvious that $z=0$ is of regular singular point. To investigate the point at ∞ transfer the independent variable z by $z=1/t$, the equation becomes

$$t^4\frac{d^2w}{dt^2}+t^3\frac{dw}{dt}+\left(1-\lambda^2t^2\right)w=0. \qquad (4.2)$$

This shows that the equation (4.2) has a regular singular point at $t=0$ and hence the equation (4.1) has a singular point at $z=\infty$.

To obtain a series solution in the neighbourhood of $z=0$ by Frobenius method, let us put

$$w=z^{\rho}\sum_{k=0}^{\infty}c_kz^k \qquad (4.3)$$

in the equation (4.1)

Then (4.1) becomes

$$\sum_{k=0}^{\infty}c_k(\rho+k)(\rho+k-1)z^{\rho+k}+\sum_{k=0}^{\infty}c_k(\rho+k)z^{\rho+k}+\sum_{k=0}^{\infty}c_kz^{\rho+k+2}-\lambda^2\sum_{k=0}^{\infty}c_kz^{\rho+k}=0.$$

The indicial equation is obtained by equating the coefficient of z^{ρ} to zero as

$$\rho(\rho-1)+\rho-\lambda^2=0$$

That is, $\rho=\pm\lambda$.

Now the coefficient of $z^{\rho+k}$ is

$$c_k\left[(\rho+k)^2-\lambda^2\right]+c_{k-2}=0$$

i.e. $$c_k=-\frac{c_{k-2}}{(\rho+\lambda+k)(\rho-\lambda+k)}.$$

Coefficient of $z^{\rho+1}$ is $c_1\left[(\rho+1)^2-\lambda^2\right]=0.$

But $\left[(\rho+1)^2-\lambda^2\right]\neq 0$ for $\rho=\pm\lambda$. Thus $c_1=0$. This leads to the relation $c_3=c_5=c_7=...=0$, i.e., the coefficient of all odd indices are zero and

$$c_{2k}=-\frac{c_{2k-2}}{(\rho+\lambda+2k)(\rho-\lambda+2k)},\quad k=1,2,...$$

Case I : $\rho=\lambda$. Without loss of generality we assume that $\lambda\geq 0$.

Therefore,

$$c_{2k}=-\frac{c_{2k-2}}{2k(2\lambda+2k)}=-\frac{1}{2^2}\frac{c_{2k-2}}{k(\lambda+k)}.$$

Hence $$c_2=-\frac{1}{2^2 1!(\lambda+1)}c_0$$

$$c_4=(-1)^2\frac{1}{2^4 2!(\lambda+1)(\lambda+2)}c_0$$

$$c_6=(-1)^3\frac{1}{2^6 3!(\lambda+1)(\lambda+2)(\lambda+3)}c_0$$

and so on.

Hence

$$\begin{aligned}w&=z^\lambda\left[c_0+c_2z^2+c_4z^4+c_6z^6+...\right]\\&=c_0z^\lambda\left[1-\frac{z^2}{2^2(\lambda+1)}+\frac{z^4}{2^4 2!(\lambda+1)(\lambda+2)}-\frac{z^6}{2^6 3!(\lambda+1)(\lambda+2)(\lambda+3)}+...\right]\\&=c_0z^\lambda\left[1-\frac{(z/2)^2}{1!(\lambda+1)}+\frac{(z/2)^4}{2!(\lambda+1)(\lambda+2)}-\frac{(z/2)^6}{3!(\lambda+1)(\lambda+2)(\lambda+3)}+...\right]\end{aligned}\qquad(4.4)$$

With a special choice

$$c_0 = \frac{1}{2^{\lambda}\Gamma(\lambda+1)}$$

w is given by

$$w \quad = (z/2)^{\lambda}\frac{1}{\Gamma(\lambda+1)}\left[1 - \frac{(z/2)^2}{1^2(\lambda+1)} + \frac{(z/2)^4}{2!(\lambda+1)(\lambda+2)} - \frac{(z/2)^6}{3!(\lambda+1)(\lambda+2)(\lambda+3)} + \ldots\right]$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k (z/2)^{\lambda+2k}}{\Gamma(k+1)\Gamma(\lambda+k+1)}. \tag{4.5}$$

This is called Bessel function (of first kind) of order λ and is denoted by $J_{\lambda}(z)$.

Thus

$$J_{\lambda}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k (z/2)^{\lambda+2k}}{\Gamma(k+1)\Gamma(\lambda+k+1)}. \tag{4.6}$$

This is a convergent series for all values of $\lambda(\lambda \geq 0)$ and for all values of z.

Case II: $\lambda - (-\lambda) = 2\lambda$ is not an integer.

In this case the other independent solution is obtained by putting $-\lambda$ for λ. Hence

$$J_{-\lambda}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k (z/2)^{2k-\lambda}}{\Gamma(k+1)\Gamma(-\lambda+k+1)}. \tag{4.7}$$

This series is also convergent for all values of z.

It can be verified that both $J_{\lambda}(z)$ and $J_{-\lambda}(z)$ are solution of (4.1). If 2λ is not an integer then $J_{\lambda}(z)$ and $J_{-\lambda}(z)$ are two independent solutions of (4.1) and the general solution of Bessel equation is

$$w = AJ_{\lambda}(z) + BJ_{-\lambda}(z) \tag{4.8}$$

Case III: $\lambda = 0$

In this case $J_{\lambda}(z)$ and $J_{-\lambda}(z)$ are not linearly independent. One solution is $J_0(z)$, where

$$J_0(z) = \sum_{k=0}^{\infty}\frac{(-1)^k (z/2)^{2k}}{\Gamma(k+1)\Gamma(k+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k (z/2)^{2k}}{(k!)^2}. \tag{4.9}$$

To find the second solution let

$$w=\sum_{k=0}^{\infty} c_k z^{\rho+k}.$$

Proceed in same way as above, $c_{2k+1}=0, k=0,1,2,\ldots$ and

$$c_{2k}=-\frac{c_{2k-2}}{(\rho+\lambda+2k)(\rho-\lambda+2k)}=-\frac{c_{2k-2}}{(\rho+2k)^2},(as\,\lambda=0)k=1,2,\ldots$$

Therefore,

$$w=\sum_{k=0}^{\infty} c_k z^{k+\rho}=c_0 z^{\rho}\left(1+\frac{c_2}{c_0}z^2+\frac{c_4}{c_2}\frac{c_2}{c_0}z^4+\ldots\right)$$

$$=c_0 z^{\rho}\left[1-\frac{z^2}{(\rho+2)^2}+\frac{z^4}{\{(\rho+2)(\rho+4)\}^2}-\frac{z^6}{\{(\rho+2)(\rho+4)(\rho+6)\}^2}+\ldots\right]$$

$$=c_0 z^{\rho}\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{\{(\rho+2)(\rho+4)\ldots(\rho+2k)\}^2}$$

$$=c_0 z^{\rho}\sum_{k=0}^{\infty}(-1)^k z^{2k}F(\rho)$$

where

$$F(\rho)=\frac{1}{\{(\rho+2)(\rho+4)\ldots(\rho+2k)\}^2}.$$

Now, $\log F(\rho)=-2\{\log(\rho+2)+\log(\rho+4)+\ldots+\log(\rho+2k)\}$.

Differentiating w.r.t. ρ gives

$$\frac{1}{F(\rho)}F'(\rho)=-2\left[\frac{1}{\rho+2}+\frac{1}{\rho+4}+\ldots+\frac{1}{\rho+2k}\right].$$

That is,

$$F'(\rho)=-2F(\rho)\sum_{s=1}^{k}\frac{1}{\rho+2s}.$$

This gives

$$F'(0) = -F(0)\sum_{s=1}^{k}\frac{1}{s} = -\frac{1}{(2.4...2k)^2}\sum_{s=1}^{k}\frac{1}{s}$$

$$= -\frac{1}{2^{2k}(k!)^2}\sum_{s=1}^{k}\frac{1}{s}.$$

Hence

$$\frac{\partial w}{\partial \rho} = c_0 z^{\rho}\log z\sum_{k=0}^{\infty}(-1)^k z^{2k}F(\rho) + c_0 z^{\rho}\sum_{k=0}^{\infty}(-1)^k z^{2k}F'(\rho).$$

Therefore, the second solution denoted by $Y_0(z)$ is given by

$$Y_0(z) = \left.\frac{\partial w}{\partial \rho}\right)_{\rho=0} = c_0\log z\sum_{k=0}^{\infty}(-1)^k z^{2k}F(0) + c_0\sum_{k=0}^{\infty}(-1)^k z^{2k}F'(0)$$

$$= \log z\left(c_0\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{2^{2k}(k!)^2}\right) - c_0\sum_{k=0}^{\infty}(-1)^k z^{2k}\frac{1}{2^{2k}(k!)^2}\sum_{s=1}^{k}\frac{1}{s}.$$

As $c_0 = 1$ because $\lambda = 0$. Therefore

$$Y_0(z) = J_0(z)\log z - \sum_{k=1}^{\infty}\frac{(-1)^k (z/2)^{2k}}{(k!)^2}\sum_{s=1}^{k}\frac{1}{s}. \quad (4.10)$$

The function $Y_0(z)$ is called *Neumann's function of the second kind of order zero.*

The function

$$\overline{Y_0(z)} = \frac{2}{\pi}\left[Y_0(z) - (\log 2 - \gamma)J_0(z)\right] \quad (4.11)$$

where γ is Euler's constant, being obtained by adding to $Y_0(z)$ a function which is a constant multiple of $J_0(z)$, will also satisfy the Bessel equation. It is called *Weber's Bessel function of the second kind of order zero.*

Therefore,

$$\overline{Y_0(z)} = \frac{2}{\pi}\left\{\log(z/2) + \gamma\right\}J_0(z) - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k (z/2)^{2k}}{(k!)^2}\sum_{s=1}^{k}\frac{1}{s}. \quad (4.12)$$

The general solution of Bessel equation when $\lambda = 0$ is

$$w = AJ_0(z) + BY_0(z) \text{ or } w = AJ_0(z) + B\overline{Y_0(z)}. \quad (4.13)$$

Case IV: 2λ is an integer.

In this case the general solution of Bessel equation can be written as

$$w = AJ_\lambda(z) + B\overline{Y_\lambda(z)}$$

where

$$\overline{Y_\lambda(z)} = \frac{1}{\pi}\{\gamma + \log(z/2)\}J_\lambda(z) - \frac{1}{\pi}\sum_{k=0}^{\lambda-1}\frac{(\lambda-k-1)!}{k!}\left(\frac{2}{z}\right)^{\lambda-2k}$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\lambda+2k}}{k!(\lambda+k)!}\left\{\sum_{s=1}^{\lambda+k}\frac{1}{s}+\sum_{s=1}^{k}\frac{1}{s}\right\}. \qquad (4.14)$$

The function $\overline{Y_\lambda(z)}$ is called *Weber's Bessel function of the second kind of order* λ.

The Neumann Bessel function of second kind of order λ *is*

$$Y_\lambda(z) = J_0(z)\log z - \frac{1}{\pi}\sum_{k=0}^{\lambda-1}\frac{(\lambda-k-1)!}{k!}\left(\frac{2}{z}\right)^{\lambda-2k}$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\lambda+2k}}{k!(k+\lambda)!}\left\{\sum_{s=1}^{k+\lambda}\frac{1}{s}+\sum_{s=1}^{k}\frac{1}{s}\right\}. \qquad (4.15)$$

The equation (4.7) is also valid for integer values of λ. Let $\lambda = n$ where n is a positive integer. That is

$$J_{-n}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k-n)!}\left(\frac{z}{2}\right)^{2k-n}.$$

Since n is an integer, $1/(k-n)! = 1/\Gamma(k-n+1) \to 0$ for $k = 0,1,\ldots,n-1$. Hence the series may be considered to start with $k = n$. Replacing k by $k + n$, we obtain

$$J_{-n}(z) = \sum_{k=0}^{\infty}\frac{(-1)^{k+n}}{k!(k+n)!}\left(\frac{z}{2}\right)^{2k+n} \qquad (4.16)$$

Theorem 4.1 *The Bessel functions* $J_n(z)$ *and* $J_{-n}(z)$ *are not independent, but, they are related by*

$$J_n(z) = (-1)^n J_{-n}(z), \qquad (4.17)$$

for integral n,

Proof.

$$J_{-n}(z) = \sum_{k=0}^{\infty}\frac{(-1)^{k+n}}{k!(k+n)!}\left(\frac{z}{2}\right)^{2k+n} = (-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{z}{2}\right)^{2k+n} = (-1)^n J_n(z).$$

4.2 Generating Function

The generating function has the advantage of focusing on the functions themselves rather than on the differential equations they satisfy. The generating function of Bessel differential equation is $g(z,t) = e^{(z/2)(t-1/t)}$. Thus when λ is a positive integer $J_\lambda(z)$ is equal to the coefficient of t^λ in the expansion of $e^{(z/2)(t-1/t)}$, i.e.,

$$e^{(z/2)(t-1/t)} = \sum_{\lambda=-\infty}^{\infty} t^\lambda J_\lambda(z). \tag{4.18}$$

Proof.

$$\begin{aligned} e^{(z/2)(t-1/t)} &= e^{zt/2}e^{-z/(2t)} \\ &= \left(1+\frac{zt}{2}+\frac{1}{2!}\frac{z^2t^2}{2^2}+\ldots+\frac{1}{\lambda!}\frac{z^\lambda t^\lambda}{2^\lambda}+\frac{1}{(\lambda+1)!}\frac{z^{\lambda+1}t^{\lambda+1}}{2^{\lambda+1}}+\ldots\right) \\ &\times\left(1-\frac{z}{2t}+\frac{1}{2!}\frac{z^2}{2^2t^2}+\ldots+\frac{(-1)^\lambda}{\lambda!}\frac{z^\lambda}{2^\lambda t^\lambda}+\frac{1}{(\lambda+1)!}\frac{(-1)^{\lambda+1}z^{\lambda+1}}{2^{\lambda+1}t^{\lambda+1}}+\ldots\right) \end{aligned}$$

Coefficient of t^λ in the product is

$$\begin{aligned} &\frac{z^\lambda}{2^\lambda\cdot\lambda!}-\frac{z^{\lambda+2}}{2^{\lambda+2}\cdot(\lambda+1)!}+\frac{z^{\lambda+4}}{2!2^{\lambda+4}\cdot(\lambda+2)!}-\ldots \\ &= \frac{z^\lambda}{2^\lambda\cdot\lambda!}\left[1-\frac{1}{(\lambda+1)}\left(\frac{z}{2}\right)^2+\frac{1}{2!(\lambda+1)(\lambda+2)}\left(\frac{z}{2}\right)^4-\ldots\right]=J_\lambda(z). \end{aligned}$$

Coefficient of $t^{-\lambda}$ in the product is

$$\begin{aligned} &\frac{(-1)^\lambda z^\lambda}{2^\lambda\cdot\lambda!}+\frac{(-1)^{\lambda+1}z^{\lambda+2}}{2^{\lambda+2}\cdot(\lambda+1)!}+\frac{(-1)^{\lambda+2}z^{\lambda+4}}{2!2^{\lambda+4}\cdot(\lambda+2)!}+\ldots \\ &= \frac{(-1)^\lambda z^\lambda}{2^\lambda\cdot\lambda!}\left[1-\frac{1}{(\lambda+1)}\left(\frac{z}{2}\right)^2+\frac{1}{2!(\lambda+1)(\lambda+2)}\left(\frac{z}{2}\right)^4-\ldots\right] \\ &= (-1)^\lambda J_\lambda(z) = J_{-\lambda}(z). \end{aligned}$$

The following result can easily be deduced using the concept of generating function rather than using the differential equation.

EXAMPLE 4.1 *Shows that*

$$\cos(z\sin\theta) = J_0(z)+2\sum_{n=1}^{\infty} J_{2n}(z)\cos 2n\theta \tag{4.19}$$

$$\sin(z\sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin(2n+1)\theta. \tag{4.20}$$

What will be the cases where $\theta = \pi/2$.

Solution: From generating function

$$e^{(z/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z)t^n$$

$$= \left(J_0 + J_1 t + J_2 t^2 + \ldots\right) + \left\{(-1)^1 J_1 t^{-1} + (-1)^2 J_2 t^{-2} + (-1)^3 J_3 t^{-3} + \ldots\right\}$$

$$= J_0 + J_1\left(t - t^{-1}\right) + J_2\left(t^2 + t^{-2}\right) + J_3\left(t^3 + t^{-3}\right) + \ldots$$

Let $t = \cos\theta + i\sin\theta$.

Then $t^n = \cos n\theta + i\sin n\theta$ and $t^{-n} = \cos n\theta - i\sin n\theta$.

Therefore, $t^n + t^{-n} = 2\cos n\theta$ and $t^n - t^{-n} = 2i\sin n\theta$.

Substituting these values to the above relation we obtain

$e^{iz\sin\theta} = J_0 + 2iJ_1\sin\theta + 2J_2\cos 2\theta + 2iJ_3\sin 3\theta + 2J_4\sin 4\theta + \ldots$

This can be written as

$$\cos(z\sin\theta) + i\sin(z\sin\theta) = \left(J_0 + 2J_2\cos 2\theta + 2J_4\sin 4\theta + \ldots\right) + i2\left(J_1\sin\theta + J_3\sin 3\theta \ldots\right).$$

Separation of real and imaginary parts give

$$\cos(z\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\sin 4\theta + \ldots = J_0 + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos 2n\theta \text{ and}$$

$$\sin(z\sin\theta) = 2(J_1\sin\theta + J_3\sin 3\theta + \ldots) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin(2n+1)\theta.$$

<u>*Second Part:*</u> When $\theta = \pi/2$ then above relations become

$$\cos z = J_0 + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(z) \text{ and}$$

$$\sin z = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(z)$$

This example suggest that $\sin z$ and $\cos z$ can be expressed in terms of Bessel functions.

4.3 Recurrence Relation of $J_\lambda(z)$

Different recurrence relations are available for Bessel functions which are most important to many applications. These are presented below.

Differentiating partially the relation

$$e^{(z/2)(t-t^{-1})}=\sum_{-\infty}^{\infty}J_{\lambda}(z)t^{\lambda}$$

with respect to z we obtain

$$\frac{1}{2}\left(t-\frac{1}{t}\right)e^{(z/2)(t-t^{-1})}=\sum_{-\infty}^{\infty}J_{\lambda}'(z)t^{\lambda}\quad\text{or}\quad\left(t-\frac{1}{t}\right)\sum_{-\infty}^{\infty}t^{\lambda}J_{\lambda}(z)=2\sum_{-\infty}^{\infty}J_{\lambda}'(z)t^{\lambda}.$$

Equating the coefficient of t^{λ} we get

$$2J_{\lambda}'(z)=J_{\lambda-1}(z)-J_{\lambda+1}(z). \tag{4.21}$$

Again differentiating (4.18) partially with respect t we obtain

$$\frac{z}{2}\left(1+\frac{1}{t^2}\right)e^{(z/2)(t-1/t)}=\sum_{-\infty}^{\infty}\lambda J_{\lambda}(z)t^{\lambda-1}.$$

This gives,

$$z\left(1+\frac{1}{t^2}\right)\sum_{-\infty}^{\infty}J_{\lambda}(z)t^{\lambda}=\sum_{-\infty}^{\infty}2\lambda J_{\lambda}(z)t^{\lambda-1}.$$

Equating the coefficient the coefficient of $t^{\lambda-1}$ from both sides we obtain

$$\frac{2\lambda}{z}J_{\lambda}(z)=J_{\lambda-1}(z)+J_{\lambda+1}(z). \tag{4.22}$$

Adding (4.21) and (4.22) we get

$$2J_{\lambda}'(z)+\frac{2\lambda}{z}J_{\lambda}(z)=2J_{\lambda-1}(z).$$

This relation can be written as

$$zJ_{\lambda}'(z)=zJ_{\lambda-1}(z)-\lambda J_{\lambda}(z). \tag{4.23}$$

Subtracting (4.21) from (4.22) we have

$$\frac{2\lambda}{z}J_{\lambda}(z)-2J_{\lambda}'(z)=2J_{\lambda+1}(z).$$

That is,

$$zJ_{\lambda}'(z)=\lambda J_{\lambda}(z)-zJ_{\lambda+1}(z). \tag{4.24}$$

Adding (4.23) and (4.24) we obtain another recurrence relation as

$$J_{\lambda-1}(z)-J_{\lambda+1}(z)=2J_{\lambda}'(z). \tag{4.25}$$

Some more recurrence relation in terms of derivatives are presented below.

Example 4.2 *Prove that*

$$\frac{d}{dz}\left[z^{\lambda}J_{\lambda}(z)\right]=z^{\lambda}J_{\lambda-1}(z) \tag{4.26}$$

$$\frac{d}{dz}\left[z^{-\lambda}J_{\lambda}(z)\right]=-z^{-\lambda}J_{\lambda+1}(z). \tag{4.27}$$

Solution : Multiplying both sides of (4.23) by $z^{\lambda-1}$ we obtain

$z^{\lambda}J_{\lambda}'(z)=z^{\lambda}J_{\lambda-1}(z)-\lambda z^{\lambda-1}J_{\lambda}(z)$

or, $z^{\lambda}J_{\lambda}'(z)+\lambda z^{\lambda-1}J_{\lambda}(z)=z^{\lambda}J_{\lambda-1}(z)$

i.e., $\frac{d}{dz}\left[z^{\lambda}J_{\lambda}(z)\right]=z^{\lambda}J_{\lambda-1}(z)$.

Multiplying both sides of (4.24) by $z^{-\lambda-1}$ we obtain

$z^{-\lambda}J_{\lambda}'(z)=\lambda z^{-\lambda-1}J_{\lambda}(z)-z^{-\lambda}J_{\lambda+1}(z)$

or, $z^{-\lambda}J_{\lambda}'(z)-\lambda z^{-\lambda-1}J_{\lambda}(z)=-z^{-\lambda}J_{\lambda+1}(z)$

i.e. $\frac{d}{dz}\left[z^{-\lambda}J_{\lambda}(z)\right]=-z^{-\lambda}J_{\lambda+1}(z)$.

Example 4.3 *Prove that*

(i) $J_0'(z)=-J_1(z)$ (4.26)

(ii) $J_2(z)=J_0''(z)-\frac{1}{z}J_0'(z)$ (4.27)

Solution : The recurrence relation (4.24) is

$$zJ_n'(z)=nJ_n(z)-zJ_{n+1}(z). \tag{4.28}$$

$n=0$ gives $zJ_0'(z)=-zJ_1(z)$ i.e., $J_0'(z)=-J_1(z)$.

Differentiating with respect to z we obtain

$$J_1'(z)=-J_0''(z). \tag{4.29}$$

Putting $n=1$ in (4.28) we have

$zJ_1'(z)=J_1(z)-zJ_2(z)$

or, $z\left[-J_0''(z)\right]=-J_0'(z)-zJ_2(z)$ [using (4.29)]

i.e. $J_2(z)=J_0''(z)-\frac{1}{z}J_0'(z)$.

4.4 Bessel Differential Equation from Recurrence Relations

Suppose we consider a set of functions $w_\lambda(z)$ which satisfies the recurrence relation (4.23), but $w_\lambda(z)$ not necessarily given by the series (4.6). The equation (4.23) can be written as

$$zw'_\lambda(z) = zw_{\lambda-1}(z) - \lambda w_\lambda(z). \tag{4.30}$$

Differentiating (4.30) with respective to z, we have

$$zw''_\lambda(z) + (\lambda+1)w'_\lambda(z) - zw'_{\lambda-1} - w_{\lambda-1} = 0. \tag{4.31}$$

Multiplying this equation by z and then subtracting (4.30) multiplied by λ gives

$$z^2w''_\lambda(z) + zw'_\lambda(z) - \lambda^2 w_\lambda + (\lambda-1)zw_{\lambda-1} - z^2w'_{\lambda-1} = 0. \tag{4.32}$$

We rewrite the equation (4.24) and replacing λ by $\lambda-1$.

$$zw'_{\lambda-1} = (\lambda-1)w_{\lambda-1} - zw_\lambda.$$

Substituting the value of $zw'_{\lambda-1}$ to the last term of (4.32) we obtain

$$z^2w''_\lambda(z) + zw'_\lambda(z) - \lambda^2 w_\lambda + (\lambda-1)zw_{\lambda-1} - z\{(\lambda-1)w_{\lambda-1} - zw_\lambda\} = 0.$$

After simplification this equation becomes

$$z^2w''_\lambda(z) + zw'_\lambda + (z^2 - \lambda^2)w_\lambda(z) = 0.$$

This is the well known Bessel equation. Hence any functions, $w_\lambda(z)$ that satisfy the recurrence relations (4.23) and (4.24) satisfy Bessel differential equation; that is, the unknown function $w_\lambda(z)$ is Bessel function.

4.5 Representation of Bessel Function as Continued Fraction

The Bessel functions can be expressed as continued fraction. The equation (4.22) is rewrite as

$$J_{\lambda-1} + J_{\lambda+1} = \frac{2\lambda}{z}J_\lambda.$$

That is,

$$\frac{J_{\lambda-1}}{J_\lambda} = \frac{2\lambda}{z} - \frac{J_{\lambda+1}}{J_\lambda} = \frac{2\lambda}{z} - \cfrac{1}{\cfrac{J_\lambda}{J_{\lambda+1}}}$$

$$= \frac{2\lambda}{z} - \cfrac{1}{\cfrac{2(\lambda+1)}{z} - \cfrac{1}{\cfrac{2(\lambda+2)}{z} - \cfrac{1}{\cfrac{2(\lambda+3)}{z} - \cdots}}} \tag{4.33}$$

Multiplying (4.33) by z we obtain

$$z\frac{J_{\lambda-1}}{J_\lambda}=2\lambda-\cfrac{z^2}{2(\lambda+2)-\cfrac{z^2}{2(\lambda+4)-\cfrac{z^2}{(2\lambda+6)-\ldots}}} \qquad (4.34)$$

In particular when $\lambda = 0, 1/2$ we have

$$z\frac{J_{-1}}{J_0}=-\cfrac{z^2}{2-\cfrac{z^2}{4-\cfrac{z^2}{6-\ldots}}} \qquad (4.35)$$

$$z\frac{J_{-1/2}}{J_{1/2}}=1-\cfrac{z^2}{3-\cfrac{z^2}{5-\cfrac{z^2}{7-\ldots}}} \qquad (4.36)$$

EXAMPLE 4.4 *Show that*

$$J_{1/2}(z)=\sqrt{\frac{2}{\pi z}}\sin z \qquad (4.37)$$

$$J_{-1/2}(z)=\sqrt{\frac{2}{\pi z}}\cos z \qquad (4.38)$$

$$z\cot z=1-\cfrac{z^2}{3-\cfrac{z^2}{5-\cfrac{z^2}{7-\ldots}}} \qquad (4.39)$$

$$\{J_{1/2}(z)\}^2+\{J_{-1/2}(z)\}^2=\frac{2}{\pi^2}. \qquad (4.40)$$

SOLUTION : Substitution $\lambda = 1/2$ to the relation (4.6) we get

$$J_{1/2}(z) = \sum_0^\infty \frac{(-1)^k (z/2)^{1/2+2k}}{\Gamma(k+1)\Gamma(1/2+k+1)}$$

$$= \sqrt{\frac{2}{z}} \sum_{0}^{\infty} \frac{(-1)^k (z/2)^{1+2k}}{\Gamma(k+1)\Gamma(k+3/2)}$$

$$= \sqrt{\frac{2}{z}} \frac{1}{\Gamma(1/2)} \left[\frac{z/2}{\frac{1}{2}} - \frac{(z/2)^3}{1! \cdot \frac{3}{2} \cdot \frac{1}{2}} + \frac{(z/2)^5}{2! \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}} - \ldots \right]$$

$$= \sqrt{\frac{2}{\pi z}} \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \right] = \sqrt{\frac{2}{\pi z}} \sin z. \tag{4.41}$$

Similarly,

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z. \tag{4.42}$$

Dividing (4.42) by (4.41) we obtain

$$\frac{J_{-1/2}(z)}{J_{1/2}(z)} = \cot z. \tag{4.43}$$

Combination of the results (4.36) and (4.43) give

$$z \cot z = 1 - \cfrac{z^2}{3 - \cfrac{z^2}{5 - \cfrac{z^2}{7 - \ldots}}} \tag{4.44}$$

Squaring and adding (4.41) and (4.42) we obtain

$$[J_{1/2}(z)]^2 + [J_{-1/2}(z)]^2 = \frac{2}{\pi z}.$$

4.6 Integral Representation

Recall the equations (4.19) and (4.20) as

$$\cos(z \sin\theta) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z) \cos 2n\theta \tag{4.45}$$

$$\sin(z \sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(z) \sin(2n+1)\theta \tag{4.46}$$

By employing the orthogonality properties of cosine and sine functions

$$\int_0^{\pi} \cos n\theta \cos m\theta d\theta = \frac{\pi}{2}\delta_n^m \tag{4.47}$$

$$\int_0^{\pi} \sin n\theta \sin m\theta d\theta = \frac{\pi}{2}\delta_n^m \tag{4.48}$$

in which n and m are positive integers (zero is excluded).

Now,

$$\frac{1}{\pi}\int_0^{\pi} \cos(z\sin\theta)\cos n\theta d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[J_0(z)+2\sum_{m=1}^{\infty} J_{2m}(z)\cos(2m\theta)\right]\cos n\theta d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} J_0(z)\cos n\theta d\theta + \frac{2}{\pi}\sum_{m=1}^{\infty} J_{2m}(z)\int_0^{\pi}\cos(2m\theta)\cos n\theta d\theta$$

$$= \frac{1}{\pi}J_0(z)\left[\frac{\sin n\theta}{n}\right]_0^{\pi} + \frac{2}{\pi}\sum_{m=1}^{\infty} J_{2m}(z)\cdot\frac{\pi}{2}\cdot\delta_n^{2m}$$

$$= 0+\frac{2}{\pi}\cdot\frac{\pi}{2}J_n(z)$$

(since $\delta_n^{2m} = 1$ when $m = n/2$ i.e., when is even otherwise $\delta_n^{2m} = 0$).

$$= \begin{cases} J_n(z), \text{ when } n \text{ is even} \\ 0, \text{ when } n \text{ is odd.} \end{cases}$$

Similarly,

$$\frac{1}{\pi}\int_0^{\pi} \sin(z\sin\theta)\sin n\theta d\theta = \begin{cases} 0, & \text{when } n \text{ is even} \\ J_n(z), & \text{when } n \text{ is odd.} \end{cases}$$

Addition of these two relatives gives

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi}\left[\cos(z\sin\theta)\cos n\theta + \sin(z\sin\theta)\sin n\theta\right]d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - z\sin\theta)d\theta. \quad n = 0, 1, \ldots.. \tag{4.49}$$

As a special case,

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\theta)\,d\theta. \tag{4.50}$$

Nothing that $\cos(z\sin\theta)$ repeats itself in all four quadrants $(\theta_1 = \theta, \pi-\theta, \pi+\theta, -\theta)$, the equation (4.50) can be written as

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos(z\sin\theta)\,d\theta. \tag{4.51}$$

On the other hand, $\sin(z\sin\theta)$ reverses its sign in the third and fourth quadrants so that

$$\frac{1}{2\pi}\int_0^{2\pi} \sin(z\sin\theta)\,d\theta = 0. \tag{4.52}$$

Adding (4.51) and i times (4.52), we obtain

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\cos\theta}\,d\theta. \tag{4.53}$$

The Bessel function can be expressed in terms of contour integration as follows.

Theorem 4.2 *For a suitable contour C enclosing $t = 0$. the Bessel function of order λ is given by*

$$J_\lambda(z) = \frac{1}{2\pi i}\int_C \frac{e^{(z/2)(t-1/t)}}{t^{\lambda+1}}\,dt. \tag{4.54}$$

Proof. The Laurent series for an analytic function $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} a_n t^n \text{ where } a_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{t^{n+1}}\,dt. \tag{4.55}$$

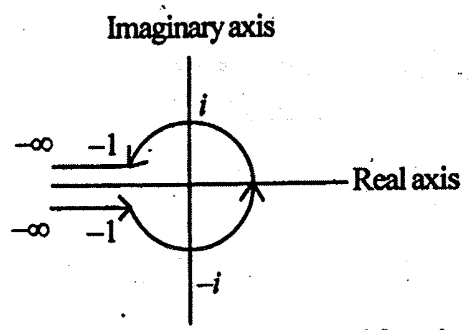

Figure 4.1 : Contour of Bessel function.

Let $f(t) = e^{(z/2)(t-1/t)}$. Again

$$e^{(z/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} t^{\lambda} J_{\lambda}(z). \tag{4.56}$$

For this $f(t)$, (4.54) becomes

$$e^{(z/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} a_{\lambda} t^{n} \quad \text{where } a_{\lambda} = \frac{1}{2\pi i}\int_C \frac{e^{(z/2)(t-1/t)}}{t^{\lambda+1}} dt. \tag{4.57}$$

Comparing (4.56) and (4.57) we have

$$J_{\lambda}(z) = a_{\lambda} = \frac{1}{2\pi i}\int_C \frac{e^{(z/2)(t-1/t)}}{t^{\lambda+1}} dt. \tag{4.58}$$

The contour C is shown in Figure 4.1.

The contour C consists of

1. $-\infty$ to -1 along the real axis,
2. the unit circle in the anticlockwise sense,
3. the real axis from -1 to $-\infty$ in the opposite direction of (i).

<u>*When λ is not an integer :*</u>

In this case, $t^{-(\lambda+1)}$ has branch points at $t = 0$ and $\pm\infty$ and a cut is necessary along the negative real axis from 0 to $-\infty$ and thus we get

$$J_{\lambda}(z) = \frac{1}{2\pi i}\int_{C_1} \frac{e^{(z/2)(t-1/t)}}{t^{\lambda+1}} dt. \tag{4.59}$$

<u>*When λ is not an integer :*</u>

The contour C becomes a unit circle. Since the two integrals along the negative real axis cancel each other and final

$$J_{\lambda}(z) = \frac{1}{2\pi i}\int_{C_0} \frac{e^{(z/2)(t-1/t)}}{t^{\lambda+1}} dt, \tag{4.60}$$

where C_0 is a unit circle containing $t = 0$.

EXAMPLE 4.5 *For an integral valve of n, show that*

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(z\sin\theta - n\theta)} d\theta. \tag{4.61}$$

Solution : Recall the equation (4.60),

$$J_n(z) = \frac{1}{2\pi i}\int_C \frac{e^{(z/2)(t-1/t)}}{t^{n+1}}\,dt$$

where C is the contour represents a unit circle with centre at the origin.

Substitute $t = \cos\theta + i\sin\theta = e^{i\theta}$.

Then $t-1/t=2i\sin\theta$ and θ varies from 0 to 2π.

Therefore,

$$J_n(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{(z/2)(2i\sin\theta)}}{\left(e^{i\theta}\right)^{n+1}}\cdot ie^{i\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{i(z\sin\theta-n\theta)}d\theta.$$

4.7 Orthogonality

It can easily be verified that $u = J_n(\alpha z)$ and $v = J_n(\beta z)$ are the solutions of the equations

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + \left(\alpha^2 z^2 - n^2\right)u = 0 \qquad (4.62)$$

$$z^2\frac{d^2v}{dz^2} + z\frac{dv}{dz} + \left(\beta^2 z^2 - n^2\right)v = 0 \qquad (4.63)$$

respectively.

Multiplying (4.62) by v/z and (4.63) by u/z and subtracting, we obtain

$$z\left(\frac{d^2u}{dz^2}\cdot v - u\frac{d^2v}{dz^2}\right) + \left(\frac{du}{dz}\cdot v - u\frac{dv}{dz}\right) + \left(\alpha^2-\beta^2\right)zuv = 0.$$

On simplification

$$\frac{d}{dz}\left\{z\left(v\frac{du}{dz} - u\frac{dv}{dz}\right)\right\} + \left(\alpha^2-\beta^2\right)zuv = 0.$$

Integration with respect to z between 0 and 1,

$$\left(\alpha^2-\beta^2\right)\int_0^1 zuv\,dz = -\left[z\left(v\frac{du}{dz} - u\frac{dv}{dz}\right)\right]_0^1 = \left[\left(v\frac{du}{dz} - u\frac{dv}{dz}\right)\right]_{z=1} \qquad (4.64)$$

Differentiating $u = J_n(\alpha z)$ and $v = J_n(\beta z)$ we get

$$\frac{du}{dz} = \alpha J_n'(\alpha z) \text{ and } \frac{dv}{dz} = \beta J_n'(\beta z).$$

Using these results (4.64) becomes

$$(\alpha^2-\beta^2)\int_0^1 zJ_n(\alpha z)J_n(\beta z)dz = \beta J_n(\alpha)J_n'(\beta)-\alpha J_n(\beta)J_n'(\beta)J_n'(\alpha). \tag{4.65}$$

Suppose α and β are two distinct roots of the equation $J_n(z)=0$, then $J_n(\alpha)=0$ and $J_n(\beta)=0$. Therefore, equation (4.65) becomes

$$\int_0^1 zJ_n(\alpha z)J_n(\alpha z)J_n(\beta z)dz = 0 \qquad (\alpha\neq\beta). \tag{4.66}$$

Hence the functions $J_n(\alpha z)$ and $J_n(\beta z)$ are orthogonal with respect to the weight function z over the interval in (0, 1).

If $\beta=\alpha$, then the right hand side of (4.65) is not zero. Its value may be obtained as follows. From (4.65),

$$\lim_{\beta\to\alpha}\int_0^1 zJ_n(\alpha z)J_n(\beta z)dz$$

$$= \lim_{\beta\to\alpha}\frac{\beta J_n(\alpha)J_n'(\beta)-\alpha J_n'(\alpha)J_n(\beta)}{\alpha^2-\beta^2}$$

$$= \lim_{\beta\to\alpha}\frac{-\alpha J_n'(\alpha)J_n(\beta)}{\alpha^2-\beta^2}\left[J_n(\alpha)=0 \text{ as } \alpha \text{ is a root of } J_n(z)\right].$$

$$= \lim_{\beta\to\alpha}\frac{-\alpha J_n'(\alpha)J_n'(\beta)}{-2\beta}=\frac{1}{2}\left[J_n'(\alpha)\right]^2.$$

Thus

$$\int_0^1 z\left[J_n(\alpha z)\right]^2 dz=\frac{1}{2}\left[J_n'(\alpha)\right]^2. \tag{4.67}$$

4.8 Hankel Functions

We have already introduce the Neumann functions in Seetion 4.1. Now, we define Hankel functions $H_\lambda^{(1)}(z)$ and $H_\lambda^{(2)}(z)$ using Neumann and Bessel functions as follows:

$$H_\lambda^{(1)}(z)=J_\lambda(z)+iY_\lambda(z) \tag{4.68}$$

and

$$H_\lambda^{(2)}(z)=J_\lambda(z)-iY_\lambda(z). \tag{4.69}$$

It may be noted that if the argument is real then the Hankel functions are complex conjugate. Series expansion of $H_\lambda^{(1)}(z)$ and $H_\lambda^{(2)}(z)$ may be obtained by combining the expansion of $J_\lambda(z)$ and $Y_\lambda(z)$. Often only the first term is of interest; it is given by

$$H_0^{(1)}(z) \approx i\frac{2}{\pi}\log z + 1 + i\frac{2}{\pi}(\gamma - \log 2) + \ldots \tag{4.70}$$

$$H_0^{(1)}(z) \approx -i\frac{(\lambda-1)!}{\pi}\left(\frac{2}{z}\right)^\lambda + \ldots, \quad \lambda > 0 \tag{4.71}$$

$$H_0^{(2)}(z) \approx -i\frac{2}{\pi}\log z + 1 - i\frac{2}{\pi}(\gamma - \log 2) + \ldots \tag{4.72}$$

$$H_0^{(2)}(z) \approx i\frac{(\lambda-1)!}{\pi}\left(\frac{2}{z}\right)^\lambda + \ldots, \quad \lambda > 0. \tag{4.73}$$

Since the Hankel functions are linear combinations of J_λ and Y_λ, they satisfy the same recurrence relations (4.22) and (4.25).

$$H_{\lambda-1}^{(1)}(z) + H_{\lambda+1}^{(1)}(z) = \frac{2\lambda}{z} H_\lambda^{(1)}(z) \tag{4.74}$$

$$H_{\lambda-1}^{(2)}(z) + H_{\lambda+1}^{(2)}(z) = \frac{2\lambda}{z} H_\lambda^{(2)}(z) \tag{4.75}$$

$$H_{\lambda-1}^{(1)}(z) - H_{\lambda+1}^{(1)}(z) = 2H_\lambda'^{(1)}(z) \tag{4.76}$$

$$H_{\lambda-1}^{(2)}(z) - H_{\lambda+1}^{(2)}(z) = 2H_\lambda'^{(2)}(z). \tag{4.77}$$

A number of Wronskian formulas can be developed:

$$H_\lambda^{(2)}(z)H_{\lambda+1}^{(1)}(z) - H_\lambda^{(1)}(z)H_{\lambda+1}^{(2)}(z) = \frac{4}{i\pi z} \tag{4.78}$$

$$J_{\lambda-1}(z)H_\lambda^{(1)}(z) - J_\lambda(z)H_{\lambda-1}^{(1)}(z) = \frac{2}{i\pi z} \tag{4.79}$$

$$J_\lambda(z)H_{\lambda-1}^{(2)}(z) - J_{\lambda-1}(z)H_\lambda^{(2)}(z) = \frac{2}{i\pi z}. \tag{4.80}$$

4.8.1 Contour Integral Representation of Hankel Functions

The integral representation for Bessel functions is

$$J_\lambda(z) = \frac{1}{2\pi i}\int_C e^{(z/2)(t-1/t)} \frac{dt}{t^{\lambda+1}}, \tag{4.81}$$

where the contour C is shown in Figure 4.1.

We now deform the contour so that it approaches the origin along the positive real axis shown in Figure 4.2. This particular approach guarantees that the exact differential mentioned will vanish as $t \to 0$ because of the presence of the term $e^{-z/2t}$. Hence each of the separate portions $-\infty(\infty e^{-i\pi})$ to 0 and 0 to $-\infty(\infty e^{i\pi})$ is a solution of Bessel equation. We define

$$H_\lambda^{(1)}(z) = \frac{1}{\pi i}\int_{C_1} e^{(z/2)(t-1/t)}\frac{dt}{t^{\lambda+1}} = \frac{1}{\pi i}\int_0^{\infty e^{i\pi}} e^{(z/2)(t-1/t)}\frac{dt}{t^{\lambda+1}}. \tag{4.82}$$

$$H_\lambda^{(2)}(z) = \frac{1}{\pi i}\int_{C_2} e^{(z/2)(t-1/t)}\frac{dt}{t^{\lambda+1}} = \frac{1}{\pi i}\int_{\infty e^{-i\pi}}^{\infty e^{i\pi}} e^{(z/2)(t-1/t)}\frac{dt}{t^{\lambda+1}}. \tag{4.83}$$

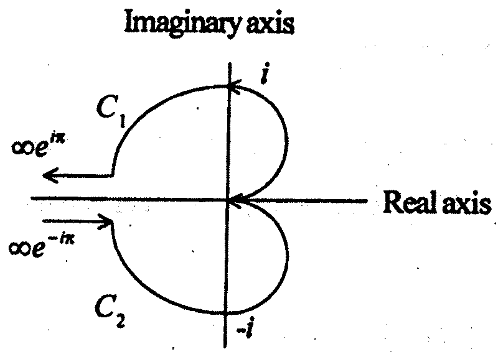

FIGURE 4.2: Contour of Hankel functions.

The Hankel functions $H_\lambda^{(1)}(z)$ has a saddle point at $t = i$, whereas $H_\lambda^{(2)}(z)$ has a saddle point at $t = -i$. Substituting $t = e^{i\pi}/s$ to (4.82) and $t = e^{-i\pi}/s$ to (4.83), we obtain

$$H_\lambda^{(1)}(z) = e^{-i\lambda\pi} H_{-\lambda}^{(1)}(z) \tag{4.84}$$

$$H_\lambda^{(2)}(z) = e^{i\lambda\pi} H_{-\lambda}^{(2)}(z). \tag{4.85}$$

By adding and subtracting (4.68) and (4.69) we have

$$J_\lambda(z) = \frac{1}{2}\left[H_\lambda^{(1)}(z) + H_\lambda^{(2)}(z)\right] \tag{4.86}$$

$$Y_\lambda(z) = \frac{1}{2i}\left[H_\lambda^{(1)}(z) - H_\lambda^{(2)}(z)\right]. \tag{4.87}$$

Replacing λ by $-\lambda$ in (4.86) and using the facts of (4.84) and (4.85) we get

$$J_{-\lambda}(z) = \frac{1}{2}\left[e^{i\lambda\pi} H_\lambda^{(1)}(z) + e^{-i\lambda\pi} H_\lambda^{(2)}(z)\right]. \tag{4.88}$$

4.9 Unit Summary

Bessel differential equation of order λ is introduced here. The series solution for different values of λ are determined. Also Neumann and Weber functions are deduced. The generating function for Bessel function is defined and several recurrence relations are established from it. Here it is shown that the Bessel function can also be represented in terms of integration. In general, this function does not obey the orthogonal property but, under certain condition it follow this property. A brief introduction is made for Hankel functions. The unit completed with an exercise.

4.10 Self Assessment Questions

1. Show that the Bessel equation of order one-half,

$$x^2y'' + xy' + (x^2 - 1/4)y = 0, \qquad x > 0$$

can be reduced to the equation

$$u'' + u = 0$$

by change of dependent variable $y = x^{-1/2}u(x)$. From this conclude that $y_1(x) = x^{-1/2}\cos x$ and $y_2(x) = x^{-1/2}\sin x$ are solutions of the Bessel equation of order one-half.

2. Show that

$$J_0^2(z) + 2\sum_{n=1}^{\infty} J_n^2(z) = 1$$

and prove that for real $z, |J_0(z)| \le 1$ and $|J_0(z)| \le 1/\sqrt{2}$ for $n \ge 1$.

3. Prove that

$$\int_0^t J_0\left(\left\{\sqrt{x(t-x)}\right\}\right)dx = 2\sin t/2.$$

4. Prove that

$$\sum_{n=0}^{\infty} J_{2n+1}(z) = \frac{1}{2}\int_0^z J_0(t)dt.$$

5. Prove that

$$\frac{d}{dz}\left[zJ_n(z)J_{n+1}(z)\right] = z\left[J_n^2(z) - J_{n+1}^2(z)\right].$$

6. Using the generating function

$$g(x,t) = g(u+v,t) = g(u,t)\cdot g(v,t)$$

show that

(i) $$J_n(u+v)=\sum_{k=-\infty}^{\infty}J_k(u)J_{n-k}(v)$$

$$=\sum_{k=0}^{n}J_n(u)J_{n-k}(v)+\sum_{k=1}^{\infty}\left[J_{-k}(u)J_{n+k}(v)+J_{-k}(v)J_{n+k}(u)\right]$$

(ii) $$J_0(u+v)=J_0(u)J_0(v)+\sum_{k=1}^{\infty}\left[J_k(u)J_{-k}(v)+J_{-k}(u)J_k(v)\right]$$

$$=J_0(u)J_0(v)+2\sum_{k=1}^{\infty}(-1)^k J_k(u)J_k(v).$$

7. Prove that

$$=\int_0^{\infty}e^{-ax}J_0(bx)dx=\frac{1}{\sqrt{a^2+b^2}},\ a>0.$$

8. Prove that $J_n(z)=0$ has no repeated roots except at $z=0$.

9. Prove that

$$\frac{d}{dx}\left[J_n^2+J_{n+1}^2\right]=2\left(\frac{n}{x}J_n^2-\frac{n+1}{x}J_{n+1}^2\right).$$

10. Prove that

$$\sqrt{\frac{\pi z}{2}}J_{3/2}(z)=\frac{1}{z}\sin z-\cos z.$$

11. Prove that

$$J_{-3/2}(z)=-\sqrt{\frac{2}{\pi z}}\left[\sin z+\frac{1}{z}\cos z\right].$$

12. Prove that

$$J_{5/2}(z)=\sqrt{\frac{2}{\pi z}}\left[\frac{3-z^2}{z^2}\sin z-\frac{3}{z}\cos z\right].$$

13. Prove that (i) $\int_0^z z^n J_{n-1}(z)dz=z^n J_n(z)$

(ii) $\int_0^z z^{n+1}J_n(z)dz=z^{n+1}J_{n+1}(z)$

(iii) $\int_0^{\infty} \frac{J_n(z)}{z} dz = \frac{1}{n}$

(iv) $\int_0^{\infty} J_o(z)\, dz = 1.$

14. Prove that (i) $4\int J_{n+1}(z)dz = \int J_{n-1}(z)dz - 2J_n(z)$

(ii) $4\frac{d^2}{dz^2}[J_n(z)] = J_{n-2}(z) - 2J_n(z) + J_{n+2}(z)$

(iii) $\int_0^{\infty} e^{-z} J_0(z)dz = \frac{1}{\sqrt{2}}.$

15. Show by the use of recurrence formulae that

(i) $J_0'' = (J_2 - J_0)/2,$

(ii) $\frac{J_2(z)}{J_1(z)} = \frac{1}{z} - \frac{J_0''(z)}{J_0'(z)}$

(iii) $J_0'''(z) + 3J_0'(z) + J_3(z) = 0.$

16. Deduce the Jacobi series

(i) $\cos(z\cos\theta) = J_0 - 2J_2 \cos 2\theta + 2J_4 \cos 4\theta - ....$

(ii) $\sin(z\cos\theta) = 2[J_1 \cos\theta - J_3 \cos 3\theta + J_5 \cos 5\theta - ....]$

Hence show that

(i) $\sin z = 2[J_1 - J_3 + J_5 - ...]$

(ii) $\cos z = J_0 - 2[J_2 - J_4 + J_6 - ....].$

17. Using only the generating function

$$e^{(z/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z) t^n$$

and not explicit series form of $J_n(z)$, show that

$$J_n(z) = (-1)^n J_n(-z).$$

18. Show, by direct differentiation, that

$$J_\lambda(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+\lambda)!} \left(\frac{z}{2}\right)^{\lambda+2k}$$

satisfies the two recurrence relations

$$J_{\lambda-1}(z)+J_{\lambda+1}(z)=\frac{2\lambda}{z}J_{\lambda}(z),$$

$$J_{\lambda-1}(z)-J_{\lambda+1}(z)=2J_{\lambda}'(z),$$

and

$$z^2J_{\lambda}''(z)+zJ_{\lambda}'(z)+\left(z^2-\lambda^2\right)J_{\lambda}(z)=0.$$

19. Prove that

(i) $$\frac{\sin z}{z}=\int_0^{\pi/2}J_0(z\cos\theta)\cos\theta d\theta$$

(ii) $$\frac{1-\cos z}{z}=\int_0^{\pi/2}J_1(z\cos\theta)d\theta$$

20. Show that

$$J_0(z)=\frac{2}{\pi}\int_0^1\frac{\cos zt}{\sqrt{1-t^2}}dt.$$

21. Derive the relation

$$J_n(z)=(-1)^n z^n\left(\frac{d}{zdz}\right)^n J_0(z).$$

22. Show that between any two consecutive zeros of $J_n(z)$ there is one and only one zero of $J_{n+1}(z)$.

23. Deduce the recurrence relation

$$J_n'(z)=\frac{1}{2}\left[J_{n-1}(z)-J_{n+1}(z)\right]$$

directly from differentiation of

$$J_n(z)=\frac{1}{\pi}\int_0^{\pi}\cos(n\theta-z\sin\theta)d\theta$$

24. Verify the Wronskian formulas

(i) $$J_{\lambda}(z)H_{\lambda}^{(1)'}(z)-J_{\lambda}'(z)H_{\lambda}^{(1)}(z)=\frac{2i}{\pi z}$$

(ii) $$J_{\lambda}(z)H_{\lambda}^{(2)'}(z)-J_{\lambda}'(z)H_{\lambda}^{(2)}(z)=\frac{2i}{\pi z}.$$

25. Show that the integral forms

$$H_\lambda^{(1)}(z)=\frac{1}{i\pi}\int_0^{-\infty} e^{(z/2)(t-1/t)}\frac{dt}{t^{\lambda+1}} \text{ and } H_\lambda^{(2)}(z)=\frac{1}{i\pi}\int_{-\infty}^{0} e^{(z/2)(t-1/t)}\frac{dt}{t^{\lambda+1}}$$

satisfy Bessel differential equation.

26. From

$$H_0^{(1)}(z)=\frac{2}{i\pi}\int_0^\infty e^{iz\cosh t}\,dt$$

show that

(i) $J_0(z)=\frac{2}{\pi}\int_0^\infty \sin(z\cosh t)\,dt$

(ii) $J_0(z)=\frac{2}{\pi}\int_1^\infty \frac{\sin(zt)}{\sqrt{t^2-1}}\,dt.$

27. A set of functions $C_n(z)$ satisfies the recurrence relations

$$C_{n-1}(z)-C_{n+1}(z)=\frac{2n}{z}C_n(z).$$

$$C_{n-1}(z)+C_{n+1}(z)=2C_n'(z).$$

What linear second order differential equation does the $C_n(z)$ satisfy?

4.11 Suggested Further Readings

1. I.N. Sneddon, *Special Functions of Mathematical Physics and Chemistry.*
2. N.N. Lebeder, *Special Functions and their Applications.*
3. D. Rainville, *Special Functions.*
4. M. Birkhoff and G.C. Rota, *Ordinary Differential Equations.*
5. E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations.*
6. G.B. Arfken and H.J. Weber, *Mathematical Methods for Physicists.*

"Learner's Feed-back"

After going through the Modules / Units please answer the following questionnaire.
Cut the portion and send the same to the Directorate.

To
The Director
Directorate of Distance Education,
Vidyasagar University
Midnapore - 721 102

1. The modules are : (give ✓ in appropriate box)
   ☐ easily understandable; ☐ very hard; ☐ partially understandable.

2. Write the number of the Modules/Units which are very difficult to understand :

   ....

   ....

   ....

3. Write the number of Modules / Units which according to you should be re-written :

   ....

   ....

   ....

4. Which portion / page is not understandable to you? (mention the page no. and portion)

   ....

   ....

   ....

   ....

5. Write a short comment about the study material as a learner.

   ....

   ....

   ....

   ....

   ....

....
(Full Signature of the Learner)

Date : ....

Enrolment No. ....

Phone / Mobile No. ....


BADAL DINESH BHAVAN