Study of Soft Set Theory and Its Applications

University Grants Commission Sponsored

Minor Research Project

(Project F. No. 42-994/2013 (SR) Dated 25.03.2013)

Submitted to

University Grants Commission, New Delhi

Principal Investigator

Prof. Shyamal Kumar Mondal

Department of Applied Mathematics with Oceanology and Computer Programming

VIDYASAGAR UNIVERSITY

Midnapore, West Bengal, India, Pin- 721102

2015

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Principal Investigator

Prof. Shyamal Kumar Mondal

Department of Applied Mathematics with Oceanology and Computer Programming

Vidyasagar University

Midnapore, West Bengal, India, Pin- 721102

Research Assistant

Mrs. Tanushree Mitra Basu

Department of Applied Mathematics with Oceanology and Computer Programming

VIDYASAGAR UNIVERSITY

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DECLARATION

This is to state that the UGC Sponsored Minor Research Project, entitled Study of Soft Set Theory and Its Applications (Project F.No. 42-994/2013 (SR) Dated 25.03.2013), has been carried out by me as its Principal Investigator under the financial assistance from UGC, New Delhi. The work is based on my reading and understanding of the existing materials. The books, articles and journals which I have used for this project are acknowledged at the respective place in this report. I further declare that the project report is my own work and research which I have carried out with the financial help from UGC, New Delhi under the Minor Research Project Grant.

Prof. Shyamal Kumar Mondal
Principal Investigator of the Project
Professor of Applied Mathematics
Department of Applied Mathematics with Oceanology and Computer Programming
Vidyasagar University
West Bengal
India
PIN-721102

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Prof. Shyamal Kumar Mondal
Principal Investigator of the Project
Professor of Applied Mathematics
Department of Applied Mathematics with Oceanology and Computer Programming
Vidyasagar University
West Bengal
India
PIN-721102

1 Introduction

There exist theories viz., theory of probability; theory of fuzzy sets, interval mathematics, intuitionistic fuzzy set, rough sets which have been widely used by various researchers to solve complicated problems involving uncertainties. These theories were developed because the classical methods could not be successfully utilized to solve such problems. All these theories have their inherent difficulties. Molodtsov[3] proposed a new theory named as 'Soft Set Theory'in 1999. This theory may be regarded as another major mathematical tool to overcome inherent difficulties of the previous methods. The theory of soft set is methodologically significant to deal with problems in the field of neural networks, soft computing etc. specially in the representation of vague imprecise knowledge data analysis, machine learning etc. As there is no limitation for the description of the objects; as a result researchers can select the form of parameters they require (fuzzy or, intuitionistic fuzzy, interval-valued fuzzy etc.) which immensely simplifies the decision making process and make it more efficient in the absence of partial information.

In the thesis, we have worked on further development of the soft set theory, fuzzy soft set theory, intuitionistic fuzzy soft set theory and interval-valued fuzzy soft set theory. We have developed certain aspects in algebra and analysis in the parlance of the above theories. In addition to that, we have implemented such new aspects in real life decision making problems and also in typical problems of medical science. Furthermore we have modified the certain related theories which have been already developed in the above mentioned areas.

1.1 Some definitions

1.1.1 Soft Set

Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the set of all subsets of U. Let $A \subseteq E$. Then a pair (F, A) is called a **soft set** over U, where F is a mapping given by, $F : A \longrightarrow P(U)$.
In other words, a soft set over U is a parameterized family of subsets of the universe U.

Example 1. Let, $X = \{b_1, b_2, b_3, b_4, b_5\}$ be the set of five bags and $E = \{$*Low weight* (e_1), *good design* (e_2), *cheap* (e_3), *good chain quality* $(e_4)\}$ be the set of four parameters.
Now, assume that, F is a mapping which represents the '*quality of the bags*' as follows,

$F(e_1) = \{b_1, b_2, b_3\}$; $F(e_2) = \{b_1, b_2, b_3, b_5\}$; $F(e_3) = \{b_2, b_3, b_4, b_5\}$; $F(e_4) = \{b_1, b_3, b_5\}$.
Then, $(F, E) = \{(e_1, \{b_1, b_2, b_3\}), (e_2, \{b_1, b_2, b_3, b_5\}), (e_3, \{b_2, b_3, b_4, b_5\}), (e_4, \{b_1, b_3, b_5\})\}$ is a soft set over X, which represents '*the quality of these five bags over the four considered parameters*'.

Tabular form of the soft set (F, E) has been given in Table 1.

Table 1

Soft set (F, E)

	e_1	e_2	e_3	e_4
b_1	1	1	0	1
b_2	1	1	1	0
b_3	1	1	1	1
b_4	0	0	1	0
b_5	0	1	1	1

In the table, the entry 0 is used to represent the belongingness of a parameter to a bag and the entry 1 is used to represent the not belongingness of a parameter to a bag.

1.1.2 Fuzzy Set

If X is a collection of objects denoted generically by x then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x))/x \in X\} \tag{1}$$

$\mu_{\tilde{A}}(x)$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in $\tilde{A}$ which maps X to the membership space M. (When M contains only the two points 0 and 1, $\tilde{A}$ is nonfuzzy and $\mu_{\tilde{A}}(x)$ is identical to he characteristic function of the non-negative real numbers whose supremum is finite. Elements with a zero degree of membership are normally not listed.)

1.1.3 Fuzzy Soft Set

Let U be an initial universe set and E be a set of parameters(which are fuzzy words or sentences involving fuzzy words). Let F^U denotes the set of all fuzzy subsets of U. Let $A \subset E$. Then a pair (F, A) is called a **Fuzzy Soft Set** (FSS) over U, where F is a mapping given by, $F : A \longrightarrow F^U$.

Example 2. Assume that, $U = \{d_1, d_2, d_3, d_4\}$ be the set of four dresses and $E = \{Cheap(e_1), Modern(e_2), longevity(e_3)\}$ be the set of corresponding parameters which are in fuzzy sense. Now, assume a mapping $\tilde{F}$ which describes '*the attractiveness of the dresses*' as follows,
$\tilde{F}(e_1) = \{d_1/0.2, d_2/0.5, d_3/0.5, d_4/0.2\}$; $\tilde{F}(e_2) = \{d_1/0.6, d_2/0.1, d_3/0.8, d_4/0.3\}$;
$\tilde{F}(e_3) = \{d_1/0.8, d_2/0.5, d_3/0.6, d_4/0.3\}$.
Then, $(\tilde{F}, E) = \{(e_1, \{d_1/0.2, d_2/0.5, d_3/0.5, d_4/0.2\}), (e_2, \{d_1/0.6, d_2/0.1, d_3/0.8, d_4/0.3\}),$

$(e_3, \{d_1/0.8, d_2/0.5, d_3/0.6, d_4/0.3\})\}$ is a fuzzy soft set over U, which represents the attractiveness of the four dresses over the three parameters.
Tabular form of the fuzzy soft set $(\tilde{F}, E)$ has been provided in Table 2.

Table 2

Fuzzy soft set $(\tilde{F}, E)$

	e_1	e_2	e_3
d_1	0.2	0.6	0.8
d_2	0.5	0.1	0.5
d_3	0.5	0.8	0.6
d_4	0.2	0.3	0.3

1.1.4 Intuitionistic Fuzzy Set

If X is a collection of objects denoted generically by x then an intuitionistic fuzzy set A in X is a set of ordered triples,

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x))/x \in X\} \tag{2}$$

where $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ are functions mapping from X into $[0, 1]$. For each $x \in X, \mu_{\tilde{A}}(x)$ represents the degree of membership of the element x to the subset A of X, $\nu_{\tilde{A}}(x)$ gives the degree of non-membership. For the functions $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ mapping into $[0, 1]$ the condition $0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$ holds.

1.1.5 Intuitionistic Fuzzy Soft Set

Let U be an initial universe set and E be a set of parameters(which are intuitionistic fuzzy words or sentences involving intuitionistic fuzzy words). Let F^U denotes the set of all intuitionistic fuzzy sets of U. Let $A \subset E$. A pair (F, A) is called an **Intuitionistic Fuzzy Soft Set**(IFSS) over U, where F is a mapping given by, $F : A \longrightarrow F^U$.

Example 3. Assume that, $X = \{u_1, u_2, u_3\}$ be the universal set and $E = \{e_1, e_2, e_3, e_4\}$ be the set of four corresponding parameters which are in intuitionistic fuzzy sense. Now, assume a mapping $\tilde{F}$ such as,
$\tilde{F}(e_1) = \{u_1/(0.6, 0.2), u_2/(0.7, 0.3), u_3/(0.5, 0.1)\}$;
$\tilde{F}(e_2) = \{u_1/(0.2, 0.3), u_2/(0.1, 0.5), u_3/(0.5, 0.4)\}$;
$\tilde{F}(e_3) = \{u_1/(0.4, 0.3), u_2/(0.6, 0.4), u_3/(0.3, 0.4)\}$.
Then, $(\tilde{F}, E)$ is an intuitionistic fuzzy soft set over X, where,
$(\tilde{F}, E) = \{(e_1, \tilde{F}(e_1)), (e_2, \tilde{F}(e_2)), (e_3, \tilde{F}(e_3))\} =$
$\{(e_1, \{u_1/(0.6, 0.2), u_2/(0.7, 0.3), u_3/(0.5, 0.1)\}), (e_2, \{u_1/(0.2, 0.3), u_2/(0.1, 0.5), u_3/(0.5, 0.4)\}),$

$(e_3, \{u_1/(0.4, 0.3), u_2/(0.6, 0.4), u_3/(0.3, 0.4)\})\}$.
Tabular form of the intuitionistic fuzzy soft set $(\tilde{F}, E)$ has been provided in Table 3.

Table 3

Intuitionistic fuzzy soft set $(\tilde{F}, E)$

	e_1	e_2	e_3
u_1	(0.6,0.2)	(0.2,0.3)	(0.4,0.3)
u_2	(0.7,0.3)	(0.1,0.5)	(0.6,0.4)
u_3	(0.5,0.1)	(0.5,0.4)	(0.3,0.4)

1.1.6 Interval-Valued Fuzzy Set

An interval-valued fuzzy set X on a universe U is a mapping such that $X : U \longrightarrow \mathrm{Int}([0, 1])$, where $\mathrm{Int}([0, 1])$ stands for the set of all closed subintervals of $[0, 1]$, the set of all interval-valued fuzzy sets on U is denoted by $IVF(U)$. Suppose that $X \in IVF(U); \forall x \in U; \mu_X(x) = [\mu_X^-(x), \mu_X^+(x)]$ is called the degree of membership of an element x with respect to X. $\mu_X^-(x)$ and $\mu_X^+(x)$ are referred to as the lower and upper degrees of membership of x with respect to X where $0 \leq \mu_X^-(x) \leq \mu_X^+(x) \leq 1$.

1.1.7 Interval-Valued Fuzzy Soft Set

Let U be an initial universe, E be a set of parameters and $A \subseteq E$. Then a pair $(\bar{F}_A, E)$ is called an **interval-valued fuzzy soft set**(I-VFSS) over $P(U)$ where $\bar{F}_A$ is a mapping given by $\bar{F}_A : A \longrightarrow P(U)$.
An interval-valued fuzzy soft set is a parameterized family of interval-valued fuzzy subsets of U. An interval-valued fuzzy soft set is also a special case of a soft set because it is still a mapping from parameters to $P(U)$. $\forall e \in A, \bar{F}_A(e)$ is referred as an interval-valued fuzzy set of U. It can be written as: $\bar{F}_A(e) = \{(x, \mu_{\bar{F}_A}(x)) : x \in U\}$ where $\mu_{\bar{F}_A}(x)$ is the interval-valued fuzzy membership degree that object x holds on parameter e. If $\forall e \in E, \forall x \in U, \mu^-_{\bar{F}_A}(x) = \mu^+_{\bar{F}_A}(x)$, then $\bar{F}_A$ will degenerated to be a standard fuzzy set and then $(\bar{F}_A, E)$ will be degenerated to be a traditional fuzzy soft set.

Example 4. Assume that, $U = \{h_1, h_2, h_3\}$ be the set of three houses and $E = \{modern(e_1), cheap(e_2), ingreensurroundings(e_3)\}$ be the set of corresponding parameters which are in interval-valued fuzzy sense. Now, the availability of the parameters over these three houses have been characterized by interval-valued fuzzy soft set $(\tilde{F}, E)$ as given in Table 4.

Table 4

Interval-valued fuzzy soft set $(\tilde{F}, E)$

	e_1	e_2	e_3
h_1	[0.4,0.6]	[0.7,0.9]	[0.2,0.5]
h_2	[0.7,0.8]	[0.4,0.8]	[0.3,0.6]
h_3	[0.1,0.5]	[0.3,0.5]	[0.4,0.8]

From Table 4 we have seen that, the evaluation of the house h_1 over the parameter e_1 is $[0.4, 0.6]$ i.e., h_1 house is at least modern with degree 0.4 at most modern with degree 0.6.

Soft-Matrix and Its Application

Shyamal Kumar Mondal

Department of Applied Mathematics with Oceanology and Computer Programming,
Vidyasagar University, Midnapore -721 102,W.B., India,

Abstract: The purpose of this paper is to introduce the concept of Interval-Valued Fuzzy Soft Matrix($IVFS$ -matrix) together with some different types of matrices in interval-valued fuzzy soft set theory. We have defined here some new operations on these matrices and proposed some theorems along with few properties on these matrices. Moreover a new efficient $IVFSM$ -algorithm based on these new matrix operations has been developed to solve interval-valued fuzzy soft set based real life group decision making problems. Finally the $IVFSM$ -algorithm has been applied to a more relevant subject- **the Drastic Change of Climate** in present world scenario.

Keywords: Interval-Valued Fuzzy Soft Set; Interval-Valued Fuzzy Soft Matrix($IVFS$ -Matrix); Choice Matrix; Group Decision.

1 Introduction

In real scenario we need strategies which provide some flexible information processing capacity to deal with uncertainties. Soft set theory is generally used to solve such problems. In the year 1999 Molodtsov [1] introduced soft set as a completely generic mathematical tool for modeling uncertainties. Since there is hardly any limitation in describing the objects, researchers simplify the decision making process by choosing the form of parameters they require and subsequently makes it more efficient in the absence of partial information. Maji et al. have done further research on soft set theory [6, 7] and on fuzzy soft set theory [8]. Then Cagman et al. [4, 5] have proposed the definition of soft matrix which is the representation of a soft set and they also introduced a new soft set based decision making method.

In the year 2009 , Yang et al. [10] have combined the interval-valued fuzzy set [3] and soft set[1], from which a new soft set model: interval-valued fuzzy soft set(IVFSs)[10] is obtained. They have also given an algorithm to solve IVFSs based decision making problems. Then Feng et al[2] have shown that Yang's algorithm has some drawbacks and they have proposed another method for solving IVFSs based decision making problems.

But according to Feng's method [2], the decision maker has to form a reduct fuzzy soft set (of pessimistic or optimistic or neutral type) of the given IVFSs and then can select any level to form the level soft set. There does not exist any unique or uniform criterion for the selection of the level. So by this method the decision maker will be puzzled to decide that which type and which level is most suitable for the selection of the object. Moreover till now researchers [10, 2] have worked on finding solution of the IVFSs based decision making problems involving **only one** decision maker. There does not exist any method for solving a IVFSs-based **group** decision making problem.

Group Decision is the academic and professional field that aims to improve collective decision process by taking into account the needs and opinions of every group assisting groups or individuals within groups, as they interact and collaborate to reach a collective decision.

In this paper, we have introduced the concept of $IVFS$ -matrix. Here we have presented the concept of choice matrix which represents the choice parameters of the decision makers and then we have introduced some new operations on $IVFS$ -matrix and choice matrix. Moreover we have proposed some theorems along with few properties on these matrices. Finally we have introduced the $IVFSM$ -algorithm based on some of these new matrix operations to solve interval-valued fuzzy soft set based real life decision making problems **involving any number of decision maker**. At last to realize this newly proposed algorithm we have applied it to predict the country whose climate change [9] is mostly affected according to the opinions of scientists.

2 Preliminaries

2.1 Definition: [1]

Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U . Let $A \subseteq E$. A pair (F_A, E) is called a **soft set** over U , where F_A is a mapping given by, $F_A : A \to P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U .

2.2 Definition: [8]

Let U be an initial universe set and E be a set of parameters(which are fuzzy words or sentences involving fuzzy words). Let $P(U)$ denotes the set of all fuzzy sets of U. Let $A \subset E$. A pair (F_A, E) is called a **Fuzzy Soft Set** (FSS) over U, where F_A is a mapping given by, $F_A : E \to P(U)$ such that $F_A(e) = \widetilde{\phi}$ if $e \notin A$ where $\widetilde{\phi}$ is a null fuzzy set.

2.3 Definition: [10]

Let U be an initial universe, E be a set of parameters(which are interval-valued fuzzy words or sentences involving interval-valued fuzzy words) and $A \subseteq E$. Then a pair $(\overline{F}_A, E)$ is called an **interval-valued fuzzy soft set** over $P(U)$ where $\overline{F}_A$ is a mapping given by $\overline{F}_A : A \to P(U)$.

Example 2.2 Suppose that, U be the set of six houses $h_1, h_2, h_3, h_4, h_5, h_6$ and E be the set of parameters given by $E = \{$ *beautiful, wooden, cheap, in the green surroundings*$\} = \{e_1, e_2, e_3, e_4\}$ *and* $A = E$

The tabular representation of the interval-valued fuzzy soft set $(\overline{F}_A, E)$ is shown in Table 1. In Table 1, we can see that the precise evaluation for each object on each parameter is unknown while the lower and upper limits of such an evaluation is given. For example, we cannot present the precise degree of how beautiful the house h_1 is, however, the house h_1 is at least beautiful on the degree of 0.7 and it is at most beautiful on the degree of 0.9 .

Table-1: Tabular representation of $(\overline{F}_A, E)$

	e_1	e_2	e_3	e_4
h_1	[0.7,0.9]	[0.6, 0.7]	[0.3, 0.5]	[0.5, 0.8]
h_2	[0.6, 0.8]	[0.8,1.0]	[0.8, 0.9]	[0.9, 1.0]
h_3	[0.5, 0.6]	[0.2,0.4]	[0.5, 0.7]	[0.7, 0.9]
h_4	[0.6, 0.8]	[0.0,0.1]	[0.7, 1.0]	[0.6, 0.8]
h_5	[0.8, 0.9]	[0.1,0.3]	[0.9, 1.0]	[0.2, 0.5]
h_6	[0.8, 1.0]	[0.7, 0.8]	[0.2, 0.5]	[0.7,1.0]

2.4 Definition: [4]

Let (F_A, E) be a soft set over U. Then a subset of $U \times E$ is uniquely defined by

$$R_A = \{(u,e) : e \in A, u \in F_A(e)\}$$

which is called a relation form of (F_A, E). Now the **characteristic function** of R_A is written by,

$$\chi_{R_A} : U \times E \to \{0,1\},\ \chi_{R_A} = \begin{cases} 1, (u,e) \in R_A \\ 0, (u,e) \notin R_A \end{cases}$$

2.5 Definition: [4]

Let $U = \{u_1, u_2, u_m\}$, $E = \{e_1, e_2, \ldots\ldots\ldots, e_n\}$ and if $a_{ij} = \chi_{R_A}(u_i, e_j)$, we can define a matrix

$$[a_{ij}]_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \ldots\ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & \ldots\ldots\ldots\ldots & a_{mn} \end{pmatrix}$$

which is called a **soft matrix** of order $m \times n$ corresponding to the soft set (F_A, E) over U. A soft set (F_A, E) is uniquely characterized by the matrix $[a_{ij}]_{m \times n}$. Therefore we shall identify any soft set with its soft matrix and use these two concepts as interchangeable.

3 Some New Concepts of Matrices in Interval-Valued Fuzzy Soft Set Theory

3.1 The Concept of $IVFS$ -Matrix:

Let $(\overline{F}_A, E)$ be an interval-valued fuzzy soft set over U. Then a subset of $U \times E$ is uniquely defined by $R_A = \{(u,e) : e \in A, u \in \overline{F}_A(e)\}$ which is called a relation form of $(\overline{F}_A, E)$. Now the relation R_A is characterized by the membership function $\mu_A : U \times E \to Int([0,1])$ such that

$$\mu_A(u,e) = \begin{cases} [\mu^-_{\overline{F}_A(e)}(u), \mu^+_{\overline{F}_A(e)}(u)],\ if\ e \in A \\ [0,0],\ if\ e \notin A \end{cases}$$

where Int($[0,1]$) stands for the set of all closed subintervals of $[0,1]$ and $[\mu^-_{\overline{F}_A(e)}(u), \mu^+_{\overline{F}_A(e)}(u)]$ denotes the interval-valued fuzzy membership degree of the object u associated with the parameter e.

Now if the set of universe $U = \{u_1, u_2, \ldots\ldots\ldots\ldots, u_m\}$ and the set of parameters $E = \{e_1, e_2, \ldots\ldots\ldots, e_n\}$, then R_A can be presented by a table in the following form

Table-2: Tabular representation of R_A

	e_1	e_2	....	e_n
u_1	$\mu_A(u_1, e_1)$	$\mu_A(u_1, e_2)$	....	$\mu_A(u_1, e_n)$

u_2	$\mu_A(u_2,e_1)$	$\mu_A(u_2,e_2)$	....	$\mu_A(u_2,e_n)$
....	....	....	...	....
u_m	$\mu_A(u_m,e_1)$	$\mu_A(u_m,e_2)$	....	$\mu_A(u_m,e_n)$

where $\mu_A(u_m,e_n)=[\mu^-_{\overline{F}_A(e_n)}(u_m),\mu^+_{\overline{F}_A(e_n)}(u_m)]$

For simplicity if we take $[\mu^-_{\overline{F}_A(e_j)}(u_i),\mu^+_{\overline{F}_A(e_j)}(u_i)]=a_{ij}$, then from Table-2 we can define a matrix

$$(\overline{a}_{ij})_{m\times n}=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

which is called an **interval-valued fuzzy soft matrix or, simply** $IVFS$ **-matrix** of order $m\times n$ corresponding to the interval-valued fuzzy soft set $(\overline{F}_A,E)$ over U. An interval-valued fuzzy soft set $(\overline{F}_A,E)$ is uniquely characterized by the matrix $(\overline{a}_{ij})_{m\times n}$. Therefore we shall identify any interval-valued fuzzy soft set with its $IVFS$ -matrix and use these two concepts as interchangeable.

Example 3.1

Let U be the set of five cities, given by, $U=\{C_1,C_2,C_3,C_4,C_5\}$.

Let E be the set of parameters (each parameter is an interval-valued fuzzy word), given by

$E=\{highly, immensely, moderately, average, less\}$
$=\{e_1,e_2,e_3,e_4,e_5\}(say)$

Let $A\subset E$, given by, $A=\{e_1,e_2,e_3,e_5\}$ (say)

Now suppose that, $\overline{F}_A:A\to P(U)$ describing the pollution of the cities and the Interval-Valued Fuzzy Soft Set $(\overline{F}_A,E)$ is given by,

$(\overline{F}_A,E)=\{$*highly polluted city* $=$
$\{C_1/[.2,.4],C_2/[.8,.9],C_3/[.4,.5],$
$C_4/[.6,.7],C_5/[.6,.8]\},$
immensely polluted city=
$\{C_1/[0,.1],C_2/[.9,1],C_3/[.3,.4],C_4/[.4,.6],C_5/[.6,.7]\},$
moderately polluted city=
$\{C_1/[.3,.5],C_2/[.4,.5],C_3/[.8,1],C_4/[.1,.2],C_5/[.3,.5]\},$
less polluted city=
$\{C_1/[.9,1],C_2/[.1,.2],C_3/[.5,.7],C_4/[.3,.5],C_5/[.1,.2]\}\}$

Hence the $IVFS$ -matrix $(\overline{a}_{ij})$ is written by,

$$(\overline{a}_{ij})=\begin{pmatrix} [.2,.4] & [0,.1] & [.3,.5] & [0,0] & [.9,1] \\ [.8,.9] & [.9,1] & [.4,.5] & [0,0] & [.1,.2] \\ [.4,.5] & [.3,.4] & [.8,1] & [0,0] & [.5,.7] \\ [.6,.7] & [.4,.6] & [.1,.2] & [0,0] & [.3,.5] \\ [.6,.8] & [.6,.7] & [.3,.5] & [0,0] & [.1,.2] \end{pmatrix}$$

3.2 Definition: Row $IVFS$ **-Matrix**

An $IVFS$ -matrix of order $1\times n$ i.e., with a single row is called a **row** $IVFS$ **-matrix**. Physically, a row $IVFS$ -matrix formally corresponds to an interval-valued fuzzy soft set whose universal set contains only one object.

3.3 Definition: Column $IVFS$ **-Matrix**

An $IVFS$ -matrix of order $m\times 1$ i.e., with a single column is called a **column** $IVFS$ **-matrix**. Physically, a column $IVFS$ -matrix formally corresponds to an interval-valued fuzzy soft set whose parameter set contains only one parameter.

3.4 Definition: Square $IVFS$ **-Matrix**

An $IVFS$ -matrix of order $m\times n$ is said to be an **square** $IVFS$ **-matrix** if $m=n$ ie., the number of rows and the number of columns are equal. That means a square $IVFS$ -matrix is formally equal to an interval-valued fuzzy soft set having the same number of objects and parameters.

3.5 Definition: Complement of an $IVFS$ **-matrix**

Let $(\overline{a}_{ij})$ be an $m\times n$ $IVFS$ -matrix, where $(\overline{a}_{ij})$ is the matrix representation of the interval-valued fuzzy soft set $(\overline{F}_A,E)$. Then the **complement** of $(\overline{a}_{ij})$ is denoted by $(\overline{a}_{ij})^c$ and is defined by,

$(\overline{a}_{ij})^c=(\overline{c}_{ij})$, where $(\overline{c}_{ij})$ is also an $IVFS$ -matrix of order $m\times n$ and it is the matrix representation of the interval-valued fuzzy soft set $(\overline{F}^c_{\neg A},E)$, ie.,

$c_{ij}=[\mu^-_{c_{ij}},\mu^+_{c_{ij}}]=[1-\mu^+_{a_{ij}},1-\mu^-_{a_{ij}}]$.

3.6 Definition: Null $IVFS$ **-Matrix**

An $IVFS$ -matrix of order $m \times n$ is said to be a **null $IVFS$ -matrix or zero $IVFS$ -matrix** if all of its elements are $[0,0]$. A null $IVFS$ -matrix is denoted by, $\overline{\Phi}$. Now the interval-valued fuzzy soft set associated with a null $IVFS$ -matrix must be a null interval-valued fuzzy soft set.

3.7 Definition: Complete $IVFS$ -Matrix or, Absolute $IVFS$ -Matrix

An $IVFS$ -matrix of order $m \times n$ is said to be a **complete $IVFS$ -matrix or, absolute $IVFS$ -matrix** if all of its elements are $[1,1]$. A complete or absolute $IVFS$ -matrix is denoted by, C_A. Now the interval-valued fuzzy soft set associated with an absolute $IVFS$ -matrix must be an absolute interval-valued fuzzy soft set.

3.8 Definition: Diagonal $IVFS$ -Matrix

A square $IVFS$ -matrix of order $n \times n$ is said to be a **diagonal $IVFS$ -matrix** if all of its non-diagonal elements are $[0,0]$.

If the diagonal elements of a diagonal $IVFS$ -matrix be all equal, then the matrix is called a **scalar $IVFS$ -matrix**.

If the diagonal elements of a diagonal $IVFS$ -matrix be all $[1,1]$, then the matrix is called a **unit or identity $IVFS$ -matrix**.

3.9 Definition: Triangular $IVFS$ -Matrix

A square $IVFS$ -matrix $\overline{a}_{ij}$ of order $n \times n$ is said to be an **upper triangular $IVFS$ -matrix** if all the elements below the leading diagonal are $[0,0]$, ie., $a_{ij} = [0,0]$ if $i > j$.

A square $IVFS$ -matrix $\overline{a}_{ij}$ of order $n \times n$ is said to be an **lower triangular $IVFS$ -matrix** if all the elements above the leading diagonal are $[0,0]$, ie., $a_{ij} = [0,0]$ if $i < j$.

3.10 Definition: Equality of $IVFS$ -Matrices

Let A and B be two $IVFS$ -matrices under the same universe U and set of parameters E. Now A and B are said to be **conformable for equality**, if they be of the same order.

Now the $IVFS$ -matrices A and B with same order are said to be **equal**, if and only if the corresponding elements of A and B be equal.

3.11 Definition: Transpose of a square $IVFS$ -Matrix

The **transpose** of a square $IVFS$ -matrix A of order $n \times n$ is another square $IVFS$ -matrix of the same order obtained from A by interchanging its rows and columns. It is denoted by A^T. Now if $A = (\overline{a}_{ij})_{n \times n}$, then its transpose A^T is defined by

$$A^T = (\overline{b}_{ij})_{n \times n} \text{ where } b_{ij} = a_{ji}$$

Therefore the interval-valued fuzzy soft set associated with A^T becomes a new interval-valued fuzzy soft set over the same universe and over the same set of parameters.

Note: Transpose of a non-square $IVFS$ -Matrix cannot be defined, as it does not carry any physical meaning.

3.12 Definition: Choice Matrix

It is a square matrix whose rows and columns both indicate parameters. If ξ is a choice matrix, then its element $\xi(i, j)$ is defined as follows:

$$\xi(i,j) = \begin{cases} [1,1] \text{ when } i^{th} \text{ and } j^{th} \text{ parameters are both choice parameters of the decision makers} \\ [0,0] \text{ when atleast one of the } i^{th} \text{ or } j^{th} \text{ parameters be not under choice of the decision maker} \end{cases}$$

Any Greek letter may be used to denote a choice matrix. There are different types of choice matrices according to the number of decision makers. We may realize this by the following example.

3.13 Definition: Symmetric $IVFS$ -Matrix

A square $IVFS$ -matrix A of order $n \times n$ is said to be a **symmetric $IVFS$ -matrix**, if its transpose be equal to it, i.e., if $A^T = A$. Hence the $IVFS$ -matrix $(\overline{a}_{ij})$ is symmetric, if $a_{ij} = a_{ji}, \forall i, j$.

Therefore if $(\overline{a}_{ij})$ be a symmetric $IVFS$ -matrix then the interval-valued fuzzy soft sets associated with $(\overline{a}_{ij})$ and $(\overline{a}_{ij})^T$ both be the same.

3.14 Definition: Addition of $IVFS$ -Matrices

Two $IVFS$ -matrices A and B are said to be **conformable for addition**, if they be of the same order and after addition the obtained sum also be an $IVFS$ -matrix of the same order. Now if $A = (\overline{a}_{ij})$ and $B = (\overline{b}_{ij})$ of the same order $m \times n$, then the **addition** of A and B is denoted by, $A \oplus B$ and is defined by,

$$(\overline{a}_{ij}) \oplus (\overline{b}_{ij}) = (\overline{c}_{ij}),$$

$$\text{where } c_{ij} = [sup\{\mu^-_{a_{ij}}, \mu^-_{b_{ij}}\}, sup\{\mu^+_{a_{ij}}, \mu^+_{b_{ij}}\}] \forall i, j$$

3.15 Definition: Subtraction of $IVFS$ -Matrices

Two $IVFS$-matrices A and B are said to be **conformable for subtraction**, if they be of the same order and after subtraction the obtained result also be an $IVFS$-matrix of the same order. Now if $A=(\overline{a}_{ij})$ and $B=(\overline{b}_{ij})$ of order $m\times n$, then **subtraction** of B from A is denoted by, $A!B$ and is defined by,

$(\overline{a}_{ij})!(\overline{b}_{ij})=(\overline{c}_{ij})$,

$where\ c_{ij}=[inf\{\mu^-_{a_{ij}},\mu^-_{b^o_{ij}}\},inf\{\mu^+_{a_{ij}},\mu^+_{b^o_{ij}}\}]\forall i,j$

where $(\overline{b}^o_{ij})$ is the complement of $(\overline{b}_{ij})$

3.16 Properties:

Let A and B be two $IVFS$-matrices of order $m\times n$. Then

i) $A\oplus B=B\oplus A$

ii) $(A\oplus B)\oplus C=A\oplus(B\oplus C)$

iii) $A!B\neq B!A$

iv) $(A!B)!C\neq A!(B!C)$

v) $A\oplus A^o\neq C_A$

vi) $A!A\neq\overline{\Phi}$

3.17 Theorems:

Theorem 1: If A be an $IVFS$-square matrix of order $n\times n$, then $(A^T)^T=A$

Theorem 2: If A and B be two $IVFS$-square matrices of order $n\times n$, then $(A\oplus B)^T=A^T\oplus B^T$

Theorem 3: If A be an $IVFS$-square matrix of order $n\times n$, then $(A\oplus A^T)$ is symmetric.

Theorem 4: If A and B be two $IVFS$-square matrices of order $n\times n$ and if A and B be symmetric, then $A\oplus B$ is symmetric.

3.18 Definition: Product of an $IVFS$-Matrix with a Choice Matrix

Let U be the set of universe and E be the set of parameters. Suppose that A be an $IVFS$-matrix and β be the choice matrix over (U,E). The product of an $IVFS$-matrix A with a choice matrix β is denoted by $A\otimes\beta$. Now A and β are said to be conformable for product $A\otimes\beta$ when the number of columns of A be equal to the number of rows of β and the product $A\otimes\beta$ becomes also another $IVFS$-matrix. If $A=(\overline{a}_{ij})_{m\times n}$ and $\beta=(\overline{\beta}_{jk})_{n\times p}$, then the product $A\otimes\beta$ is defined as

$A\otimes\beta=(\overline{c}_{ik})$

$where\ c_{ik}=[sup^n_{j=1}inf\{\mu^-_{a_{ij}},\mu^-_{\beta_{jk}}\},sup^n_{j=1}inf\{\mu^+_{a_{ij}},\mu^+_{\beta_{jk}}\}]$

3.19 Properties:

i) $\beta\otimes A$ cannot be defined here.

ii) If $A\otimes\beta=\overline{\Phi}$, then we cannot say, as in scalar algebra, that either A is a null $IVFS$-matrix or, β is a null matrix.

4 Group Decisions

Group Decision is the academic and professional field that aims to improve collective decision process by taking into account the needs and opinions of every group members. Group decision includes the development and study of methods for assisting groups or individuals within groups, as they interact and collaborate to reach a collective decision. But till now there does not exist any method to solve interval-valued fuzzy soft set(IVFSs) based group decision making problem. In the following subsection at first a general IVFSs based decision making problem is defined, then a new approach is developed to solve such types of problems.

4.1 A Generalized Interval-Valued Fuzzy Soft Set Based Group Decision Making Problem:

Let N number of decision makers want to select an object jointly from the m number of objects which have n number of features ie., parameters(E). Suppose that each decision maker has freedom to take his decision of inclusion and evaluation of parameters associated with the selected object, ie., each decision maker has his own choice parameters belonging to the parameter set E and has his own view of evaluation. Here it is assumed that parameter evaluation of the objects must be interval-valued fuzzy. Now the problem is to find out the object out of these m objects which satisfies all the choice parameters of all decision makers jointly as much as possible.

4.2 A New Approach to Solve $IVFS$-Matrix Based Group Decision Making Problems:

This new approach is specially based on choice matrices and its operations. The stepwise procedure of this new approach is presented in the following $IVFSM$-algorithm.

$IVFSM$**-Algorithm**:

Step-I: First construct the combined choice matrices with respect to the choice parameters of the decision makers.

Step-II: Compute the product $IVFS$-matrices by multiplying each given $IVFS$-matrix with the combined choice matrix as per the rule of multiplication of $IVFS$ -matrices.

Step-III: Compute the sum of these product $IVFS$ -matrices to have the resultant $IVFS$ -matrix(R_{IVFS}).

Step-IV: Then compute the weight of each object(O_i) by adding the membership values of the entries of its concerned row(i-th row) of R_{IVFS} and denote it as $W(O_i)$.

Step-V: $\forall O_i \in U$, compute the score r_i of O_i such that,

$$r_i = \sum_{O_j \in U} ((\mu_i^- - \mu_j^-) + (\mu_i^+ - \mu_j^+))$$

Step-VI: The object having the highest score becomes the jointly selected object according to all decision makers. If more than one object have the highest score then any one of these highest scorers may be chosen as the jointly selected object.

To illustrate the basic idea of the $IVFSM$ -algorithm, now we apply it to the following $IVFS$ -matrix based group decision making problem.

4.3 A Problem Regarding Change of The Climate System

There exists a controversy regarding the main cause of global climate change for which we have two school of thoughts. One group of scientists (S_A) says that **human activities** (like increasing concentration of greenhouse gases in the atmosphere, deforestation, burning of fossil fuels etc.) bears the maximum influence and another group of scientists (S_B) gives more emphasis to **natural variation** (like ocean current, increased solar activity, cosmic rays etc.). Now the role of natural variation and human activities may vary and their degree of impact on the climate change depending on the place concerned. Now considering this variety we choose three countries C_1, C_2 and C_3 which are suffering from the climate change. According to the opinions of the groups of scientists S_A and S_B the effect on climate change of these countries due to its main cause are respectively the interval-valued fuzzy soft sets $(\overline{F}_A, E)$ and $(\overline{G}_B, E)$ given by

$(\overline{F}_A, E)$ ={ effect on climate change due to human activities

$$= \left\{ \frac{C_1}{[0.8, 0.9]}, \frac{C_2}{[0.2, 0.4]}, \frac{C_3}{[0.4, 0.7]} \right\}$$

and

$(\overline{G}_B, E)$ ={ effect on climate change due to natural variation

$$= \left\{ \frac{C_1}{[0.3, 0.5]}, \frac{C_2}{[0.7, 0.8]}, \frac{C_3}{[0.5, 0.6]} \right\}$$

Now the problem is to find the country whose climate change is most affected satisfying the opinions of the both groups of scientists as much as possible.

This problem may be solved by our proposed $IVFSM$ -algorithm. Here the set of universe $U = \{C_1, C_2, C_3\}$ and the set of parameters $E = \{ \text{human activities}, \text{natural variation} \} = \{e_1, e_2\}$ (say). The choice parameters of the groups S_A and S_B are respectively, $A = \{e_1\}, B = \{e_2\}$. Therefore the $IVFS$ -matrices associated with $(\overline{F}_A, E)$ and $(\overline{G}_B, E)$ describing the effect on climate change of the countries due to its main cause according to the groups of scientists S_A and S_B are respectively,

$$(\overline{a}_{ij}) = \begin{pmatrix} [0.8,0.9] & [0,0] \\ [0.2,0.4] & [0,0] \\ [0.4,0.7] & [0,0] \end{pmatrix}$$

and

$$(\overline{b}_{ij}) = \begin{pmatrix} [0,0] & [0.3,0.5] \\ [0,0] & [0.7,0.8] \\ [0,0] & [0.5,0.6] \end{pmatrix}$$

1) The combined choice matrices of S_A and S_B in different forms are,

$$e_A \overset{e_B}{\begin{pmatrix} [0,0] & [1,1] \\ [0,0] & [0,0] \end{pmatrix}},$$

$$e_B \overset{e_A}{\begin{pmatrix} [0,0] & [0,0] \\ [1,1] & [0,0] \end{pmatrix}}$$

2) Corresponding product $IVFS$ -matrices are,

$$\begin{pmatrix} [0.8,0.9] & [0,0] \\ [0.2,0.4] & [0,0] \\ [0.4,0.7] & [0,0] \end{pmatrix} \otimes e_A \overset{\quad e_B}{\begin{pmatrix} [0,0] & [1,1] \\ [0,0] & [0,0] \end{pmatrix}} = \begin{pmatrix} [0,0] & [0.8,0.9] \\ [0,0] & [0.2,0.4] \\ [0,0] & [0.4,0.7] \end{pmatrix}$$

$$\begin{pmatrix} [0,0] & [0.3,0.5] \\ [0,0] & [0.7,0.8] \\ [0,0] & [0.5,0.6] \end{pmatrix} \otimes e_B \overset{\quad e_A}{\begin{pmatrix} [0,0] & [0,0] \\ [1,1] & [0,0] \end{pmatrix}} = \begin{pmatrix} [0.3,0.5] & [0,0] \\ [0.7,0.8] & [0,0] \\ [0.5,0.6] & [0,0] \end{pmatrix}$$

3) The sum of these product $IVFS$ -matrices is,

$$\begin{pmatrix} [0,0] & [0.8,0.9] \\ [0,0] & [0.2,0.4] \\ [0,0] & [0.4,0.7] \end{pmatrix} \oplus \begin{pmatrix} [0.3,0.5] & [0,0] \\ [0.7,0.8] & [0,0] \\ [0.5,0.6] & [0,0] \end{pmatrix} = \begin{pmatrix} [0.3,0.5] & [0.8,0.9] \\ [0.7,0.8] & [0.2,0.4] \\ [0.5,0.6] & [0.4,0.7] \end{pmatrix} = \overline{R}_{IVFS}$$

4) Now the weights of the countries are respectively,

$W(C_1) = [0.3 + 0.8, 0.5 + 0.9] = [1.1, 1.4]$, $W(C_2) = [0.9, 1.2]$ and $W(C_3) = [0.9, 1.3]$

5) Now the scores for the countries are, $r_1 = (0.2 + 0.2) + (0.2 + 0.1) = 0.7$, $r_2 = -0.5$ and $r_3 = -0.2$

6) Since the score r_1 is maximum (0.7), then the climate change of the corresponding country C_1 is most affected fulfilling the opinions of the both groups of scientists as much as possible.

7 Conclusion

In this paper first we have proposed the concept of $IVFS$ -matrix and defined different types of matrices in interval-valued fuzzy soft set theory. Then we have introduced some new operations, proposed some theorems along with few properties on these matrices. At last the new efficient $IVFSM$ -algorithm(§4) has been developed to solve interval-valued fuzzy soft set based real life decision making problems which may contain more than one decision maker. Finally this has been applied to a problem of the Drastic Change in The Climate System.

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A New Solution Approach for Solving Intuitionistic Fuzzy Soft Set Based Multi-Criteria Ranking Problems And Its Application in Medical Science

Tanushree Mitra Basu*and Shyamal Kumar Mondal

Department of Applied Mathematics with Oceanology and Computer Programming,

Vidyasagar University, Midnapore -721 102,W.B., India,

E-mail : tanushreemitra13@gmail.com, shyamal_260180@yahoo.com

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Abstract: In this paper we have introduced the concept of intuitionistic fuzzy soft set based multi-criteria ranking problem. Then an algorithm has been developed for solving intuitionistic fuzzy multi-criteria ranking problems and finally we have applied this new algorithm to solve an intuitionistic fuzzy soft set based multicriteria ranking problem in medical science.

Keywords: Ranking Problem, Intuitionistic Fuzzy Multi-Criteria, Algorithm, Medical Science.

1 Introduction

Most of the recent mathematical methods meant for formal modeling,reasoning and computing are crisp, accurate and deterministic in nature. But in ground reality, crisp data is not always the part and parcel of the problems encountered in different fields like economics, engineering, social science, medical science, environment etc. As a consequence various theories viz. theory of probability, theory of fuzzy sets introduced by Zadeh [4], theory of intuitionistic fuzzy sets

*To whom all correspondence should be addressed

by Atanassov[3], theory of vague sets by Gau[12], theory of interval mathematics by Gorzalczany[5], theory of rough sets by Pawlak[14] have been evolved in process. But difficulties present in all these theories have been shown by Molodtsov [1]. The cause of these problems is possibly related to the inadequacy of the parametrization tool of the theories. As a result Molodtsov proposed the concept of soft theory as a new mathematical tool for solving the uncertainties which is free from the above difficulties. Maji et al. [7, 9] have further done various research works on soft set theory. For presence of vagueness Maji et al.[6, 8] have introduced the concept of Fuzzy Soft Set. Then Mitra Basu et al. [11] proposed the mean potentiality approach to get a balanced solution of a fuzzy soft set based decision making problem.

But satisfactory evaluation of membership values is not always possible because of the insufficiency in the available information (besides the presence of vagueness) situation. Evaluation of non-membership values is also not always possible for the same reason and as a result there exists an indeterministic part upon which hesitation survives. Certainly fuzzy soft set theory is not suitable to solve such problems. In those situations Intuitionistic Fuzzy Soft Set theory (IFSS)[10] may be more applicable. In the year 2001, Maji et al.[10] have introduced the concept of **intuitionistic fuzzy soft set**. Moreover they have proposed an algorithm for solving intuitionistic fuzzy soft set based decision making problem. Recently, Yuncheng Jiang et al.[13] present an adjustable approach to intuitionistic fuzzy soft sets based decision making by using level soft sets of intuitionistic fuzzy soft sets. They mainly extend the decision making approach presented by Feng et al.[2] to the intuitionistic fuzzy case.

But it seems that there is little investigation on multi-criteria decision making using intuitionistic fuzzy soft sets with multiple criteria being explicitly taken into account. On the other hand, most of them mainly solve how select an optimal object from the entire candidate rather than completely rank or sort the objects. However, many practical problem in economics, engineering, environment, social science, medical science, etc., that involve completely ranking, such as it need to rank the supplier in supply chain according to their service level.

So in this paper we have introduced the concept of intuitionistic fuzzy soft set based multi-criteria ranking problem. Then we have developed an algorithm to solve intuitionistic fuzzy soft set based multi-criteria ranking problems. At last we have applied this new algorithm for solving such type of problems in medical science.

2 Preliminaries

2.1 Definition: Soft Set [1]

Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the set of all subsets of U. Let $A \subseteq E$. Then a pair (F, A) is called a **soft set** over U, where F is a mapping given by, $F : A \longrightarrow P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U.

2.2 Definition: Fuzzy Soft Set (FSS) [6]

Let U be an initial universe set and E be a set of parameters which are fuzzy words or sentences involving fuzzy words. Let F^U denotes the set of all fuzzy subsets of U. Let $A \subset E$. Then a pair $(\tilde{F}, A)$ is called a **Fuzzy Soft Set** (FSS) over U, where $\tilde{F}$ is a mapping given by, $\tilde{F} : A \longrightarrow F^U$

2.3 Definition: Intuitionistic Fuzzy Soft Set(IFSS) [10]

Let U be an initial universe set and E be a set of parameters which are intuitionistic fuzzy words or sentences involving intuitionistic fuzzy words. Let IF^U denotes the set of all intuitionistic fuzzy sets of U. Let $A \subset E$. A pair $(\hat{F}, A)$ is called an **Intuitionistic Fuzzy Soft Set**(IFSS) over U, where $\hat{F}$ is a mapping given by, $\hat{F} : A \longrightarrow IF^U$

2.4 Definition: Level Soft Set of an Intuitionistic Fuzzy Soft Set [13]

Let $\varpi = (\hat{F}, A)$ be an intuitionistic fuzzy soft set over U and $A \subseteq E$. A (s,t)**-level soft set** of the intuitionistic fuzzy soft set ϖ is a crisp soft set $L(\varpi; (s,t))$ which is defined as,

$$L(\varpi; (s,t)) = (F_{(s,t)}, A) = \{x \in U : \mu_a(x) \geq s \text{and} \nu_a(x) \leq t\} \forall a \in A$$

In this definition, $s, t \in [0,1]$ also are called threshold values, in which $s \in [0,1]$ can be regarded as a given least threshold on membership values and $t \in [0,1]$ can be regarded as a given greatest threshold on non-membership values.

3 A Generalized Intuitionistic Fuzzy Soft Set Based Ranking Problem

Let a decision maker wants to rank m number of objects which have n number of features i.e., parameters(E). Suppose that the decision maker has freedom to take his decision of inclusion and evaluation of parameters associated with the selected object, i.e., the decision maker has his own choice parameters belonging to the parameter set E and has his own view of evaluation. Here it is assumed that **the parameter evaluation of the objects by the decision maker must be intuitionistic fuzzy** and may be presented in linguistic form or intuitionistic fuzzy soft set format. **Now the problem is to rank these m objects in respect to the maximum degree of satisfaction of all the choice parameters of the decision maker.**

4 An Approach to Solve Intuitionistic Fuzzy Soft Set Based Ranking Problem

To solve an intuitionistic fuzzy soft set based ranking problem, we introduce a new approach to rank the solutions of an intuitionistic fuzzy soft set based decision making problem with equally weighted choice parameters which comprises of some new notions as follows.

Definition: Potentiality

The **potentiality of an intuitionistic fuzzy soft set** (p_{ifs}) is defined as the sum of all membership values as well as non-membership values of all objects with respect to all parameters i.e., mathematically it is defined as,

$$p_{ifs} = (\sum_{i=1}^{m}\sum_{j=1}^{n}\mu_{ij}, \sum_{i=1}^{m}\sum_{j=1}^{n}\nu_{ij})$$

where μ_{ij} and ν_{ij} are respectively the membership and non-membership value of the i-th object with respect to the j-th parameter, m is the number of objects and n is the number of parameters.

Definition: Mean Potentiality

The **mean potentiality** (m_p) of an intuitionistic fuzzy soft set is defined as it's average weight among the total potentiality i.e., mathematically it is defined as,

$$m_p = \frac{p_{ifs}}{m \times n}$$

Definition: Hesitation Function

Let $(\hat{F}, A)$ be an intuitionistic fuzzy soft set over the universal set of objects U then it can be viewed as,

$$(\hat{F}, A) = \{(x_i, \mu_{a_j}(x_i), \nu_{a_j}(x_i))/x_i \in U, a_j \in A, i = 1, 2, ..., n; j = 1, 2, ..., m\}$$

Then the **hesitation function** $\pi_A(x_i)$ of the object $x_i \in U$ in all attributes A of $(\hat{F}, A)$ is defined as,

$$\pi_A(x_i) = \sum_{j=1}^{m} \pi_{a_j}(x_i) = \sum_{j=1}^{m}(1 - \mu_{a_j}(x_i) - \nu_{a_j}(x_i))$$

Definition: Score Function

Let $(\hat{F}, A)$ be an intuitionistic fuzzy soft set over the universal set of objects U then it can be viewed as,

$$(\hat{F}, A) = \{(x_i, \mu_{a_j}(x_i), \nu_{a_j}(x_i))/x_i \in U, a_j \in A, i = 1, 2, ..., n; j = 1, 2, ..., m\}$$

Then the **score function** $\zeta_A(x_i)$ of the object $x_i \in U$ in all attributes A of $(\hat{F}, A)$ is defined as,

$$\zeta_A(x_i) = \sum_{j=1}^{m} \zeta_{a_j}(x_i) = \sum_{j=1}^{m}(\mu_{a_j}(x_i) - \nu_{a_j}(x_i))$$

Definition: Accuracy Function

Let $(\hat{F}, A)$ be an intuitionistic fuzzy soft set over the universal set of objects U then it can be viewed as,

$$(\hat{F}, A) = \{(x_i, \mu_{a_j}(x_i), \nu_{a_j}(x_i))/x_i \in U, a_j \in A, i = 1, 2, ..., n; j = 1, 2, ..., m\}$$

Then the **accuracy function** $\theta_A(x_i)$ of the object $x_i \in U$ in all attributes A of $(\hat{F}, A)$ is defined as,

$$\theta_A(x_i) = \sum_{j=1}^{m} \theta_{a_j}(x_i) = \sum_{j=1}^{m}(\mu_{a_j}(x_i) + \nu_{a_j}(x_i))$$

This is an approach by which we can find the appropriate level to get the level soft set of an intuitionistic fuzzy soft set and then using hesitation, score and accuracy function we can solve an intuitionistic fuzzy soft set based multi-criteria ranking problem. The detailed step-wise procedure as an algorithm is given in the following subsection.

4.1 The Methodology For Solving Intuitionistic Fuzzy Soft Set Based Ranking Problem

In many real life problems of economics, engineering, environment, social science, medical science etc., that involves completely ranking rather than select an optimal one. For example, evaluation of supply chain partners is an important decision involving multiple criteria and risk factors and it needs to rank the supplier completely in the supply chain of engineering project according to their service level. Because in practical business of the economics and management, ranking the cooperative supplier according to their service level is more significant in most of the time to the decision maker for using different cooperation mode. The stepwise procedure to solve such type of problems is given below.

Algorithm:

Step 1: Compute the potentiality $(p_{ifs}) = (\sum \mu_i, \sum \nu_i)$ of the intuitionistic fuzzy soft set $(\hat{F}, A)$

Step 2: Find Mean Potentiality $(m_p) = (\frac{\sum \mu_i}{m \times n}, \frac{\sum \nu_i}{m \times n})$

Step 3: Form a m_p-level soft set of the IFSS.

Step 4: Present the level soft set in tabular form and compute the choice value C_i of objects $\forall i$

Step 5: Ranking the objects according to the choice value C_i from the largest to the smallest. If all the alternatives can be sorted by the strict order then stop otherwise go to the next step.

Step 6: Compute the degree of hesitation $\zeta_a(x)$ with the membership of element $x \in U$ to $a \in A$ in $(\hat{F}, A)$ and then compute the sum degree-hesitation $\zeta_A(x)$

Step 7: Ranking the objects which cannot be sorted by the choice value C_i, according to $\zeta_A(x)$ from the smallest to the largest. If all the alternatives can be sorted by the strict order, then stop otherwise go to the next step.

Step 8: Compute the degree of accuracy $\theta_a(x)$ with the membership of element$x \in U$ to $a \in A$ in $(\hat{F}, A)$ and furthermore compute the sum degree-hesitation $\theta_A(x)$ with the element $x \in U$.

Step 9: Ranking the objects which cannot be sorted by the degree of hesitation $\zeta_a(x)$ and the choice value C_i, according to $\theta_A(x)$ from the largest to the smallest.

5 Application of this Algorithm For Solving Intuitionistic Fuzzy Soft Set Based Ranking Problem in Medical Science

In medical science there also exist intuitionistic fuzzy soft set based ranking problem and we may apply the above new Algorithm for solving those problems. Now we will discuss a such type of problem with its solution.

Problem: There are different types of diseases and various modalities of treatments in respect to them. On the basis of different aspects of the treatment procedure we may measure the degree of effectiveness of the treatment for the disease. Here we consider three common diseases of oral cavity such as dental caries, gum disease and oral ulcer. Now medicinal treatment, extraction and scaling that are commonly executed, have more or less impacts on the treatment of these three diseases. According to the statistics, the degree of effectiveness in case of medicinal treatment for dental caries, gum disease, oral ulcer are $(0.5, 0.2), (0.3, 0.5)$ and $(0.8, 0.1)$ respectively; by extraction the degrees of effectiveness for dental caries, gum disease and oral ulcer are $(0.5, 0.3)$,$(0.4, 0.5)$ and $(0.2, 0.7)$ respectively and by scaling the degrees of effectiveness for dental caries, gum disease and oral ulcer are $(0.2, 0.7)$, $(0.6, 0.3)$ and $(0.5, 0.4)$ respectively. Suppose a patient simultaneously having these three diseases. **Now the problem is to rank the effectiveness of these three treatment procedures in respect to these three diseases.**

Solution:

The set of universe $U = \{\text{medicinal treatment, extraction, scaling}\} = \{t_1, t_2, t_3\}$ and the set of parameters $E = \{\text{dental caries, gum disease, oral ulcer}\} = \{e_1, e_2, e_3\}$(say). Now from the given data we have the intuitionistic fuzzy soft set $(\hat{F}, E)$ describing "the effectiveness of the treatments for the diseases" in tabular form as

Table-1

	e_1	e_2	e_3
t_1	(0.5,0.2)	(0.3,0.5)	(0.8,0.1)
t_2	(0.5,0.3)	(0.4,0.5)	(0.2,0.7)
t_3	(0.2,0.7)	(0.6,0.3)	(0.5,0.4)

Now let us apply our algorithm to solve this problem.

1) Potentiality of $(\hat{F}, E)$ is $p_{ifs} = (4, 3.7)$

2) Mean Potentiality of $(\hat{F}, E)$ is $m_p = (0.44, 0.41)$

3) and 4) The tabular representation of the m_p level soft set of $(\hat{F}, E)$ with choice values is

Table-2

	e_1	e_2	e_3	choice value
t_1	1	0	1	2
t_2	1	0	0	1
t_3	0	1	1	2

5) Using maximum choice value principle, the ranking result is $t_1 = t_3 \succ t_2$

6) Now the table representing the degree of hesitation is,

Table-3

	ζ_{e_1}	ζ_{e_2}	ζ_{e_3}	ζ_E
t_1	0.3	0.2	0.1	0.6
t_2	0.2	0.1	0.1	0.4
t_3	0.1	0.1	0.1	0.3

Using the minimum degree-hesitation principle on the basis of the result from the first principle, we will get the ranking result as,

$t_3 \succ t_1 \succ t_2$

i.e., scaling$\succ$ medicinal treatment$\succ$ extraction

Hence if a patient is simultaneously suffering from the three diseases dental caries, gum disease and oral ulcer then the effectiveness of medicinal treatment is better than extraction and the effectiveness of scaling is better than both medicinal treatment and extraction.

6 Conclusion:

In this paper we have introduced the concept of intuitionistic fuzzy soft set based multi-criteria ranking problem. Then we have developed an algorithm to solve intuitionistic fuzzy soft set based multi-criteria ranking problems. At last we have applied this new algorithm for solving such type of problems in medical science.

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A New Solution Approach Using Intuitionistic Fuzzy Cognitive Map to Solve Intuitionistic Fuzzy Multi-Criteria Ranking Problems in Medical Science

Tanushree Mitra Basu and Shyamal Kumar Mondal *

Department of Applied Mathematics with Oceanology and Computer Programming,

Vidyasagar University, Midnapore -721 102,W.B., India,

E-mail : tanushreemitra13@gmail.com, shyamal_260180@yahoo.com

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Abstract: In this paper we have introduced the concept of a generalized intuitionistic fuzzy multi-criteria ranking problem. We have also reformulated the equation evaluating the value of each node of an intuitionistic fuzzy cognitive map (IFCM). Moreover we have introduced some new notions such as potentiality, mean potentiality, accuracy function, hesitation function of an intuitionistic fuzzy soft set(IFSS) and by utilizing these new notions together with the concept of IFCM we have developed a new algorithm, named as $IFCM$-Algorithm for solving intuitionistic fuzzy multi-criteria ranking problems. Finally we have applied the $IFCM$-Algorithm to rank the effectiveness of the treatment procedures in respect to the diseases from medical science.

Keywords: Ranking Problem, Intuitionistic Fuzzy Multi-Criteria, Intuitionistic Fuzzy Cognitive Map (IFCM), $IFCM$-Algorithm, Medical Science.

*To whom all correspondence should be addressed: E-mail : tanushreemitra13@gmail.com, shyamal_260180@yahoo.com

1 Introduction

Most of the recent mathematical methods meant for formal modeling,reasoning and computing are crisp, accurate and deterministic in nature. But in ground reality, crisp data is not always the part and parcel of the problems encountered in different fields like economics, engineering, social science, medical science, environment etc. As a consequence various theories viz. theory of probability, theory of fuzzy sets introduced by Zadeh [7], theory of intuitionistic fuzzy sets by Atanassov[6], theory of vague sets by Gau[15], theory of interval mathematics by Gorzalczany[8], theory of rough sets by Pawlak[17] have been evolved in process. But difficulties present in all these theories have been shown by Molodtsov [3]. The cause of these problems is possibly related to the inadequacy of the parametrization tool of the theories. As a result Molodtsov proposed the concept of soft theory as a new mathematical tool for solving the uncertainties which is free from the above difficulties. Maji et al. [10, 12] have further done various research works on soft set theory. For presence of vagueness Maji et al.[9, 11] have introduced the concept of Fuzzy Soft Set. Then Mitra Basu et al. [14] proposed the mean potentiality approach to get a balanced solution of a fuzzy soft set based decision making problem.

But satisfactory evaluation of membership values is not always possible because of the insufficiency in the available information (besides the presence of vagueness) situation. Evaluation of non-membership values is also not always possible for the same reason and as a result there exists an indeterministic part upon which hesitation survives. Certainly fuzzy soft set theory is not suitable to solve such problems. In those situations Intuitionistic Fuzzy Soft Set theory (IFSS)[13] may be more applicable. In the year 2001, Maji et al.[13] have introduced the concept of **intuitionistic fuzzy soft set**. Moreover they have proposed an algorithm for solving intuitionistic fuzzy soft set based decision making problem. Recently, Yuncheng Jiang et al.[16] present an adjustable approach to intuitionistic fuzzy soft sets based decision making by using level soft sets of intuitionistic fuzzy soft sets. They mainly extend the decision making approach presented by Feng et al.[5] to the intuitionistic fuzzy case.

More recently, significant results have been obtained in modeling medical decision making by an alternative fuzzy logic-based approach known as fuzzy cognitive map (FCM) [4], whereas the application of generalizations of the conventional fuzzy sets, such as the intuitionistic fuzzy sets (IFSS) [13], have already provided indications for their applicability in the medical domain.

In the year 2011, Dimitris K. Iakovidis[2] have proposed the **intuitionistic fuzzy cognitive map(IFCM)** as an extension of the original FCM model, aiming to exploit the advantages of both FCM and IFS.

But it seems that there is little investigation on multi-criteria decision making using intuitionistic fuzzy soft sets with multiple criteria being explicitly taken into account. On the other hand, most of them mainly solve how select an optimal object from the entire candidate rather than completely rank or sort the objects. However, many practical problem in economics, engineering, environment, social science, medical science, etc., that involve completely ranking, such as it need to rank the supplier in supply chain according to their service level.

So in this paper we have introduced the concept of a generalized intuitionistic fuzzy multi-criteria ranking problem. Then we have developed an $IFCM$-Algorithm based on intuitionistic fuzzy cognitive map (IFCM) and few basic properties of intuitionistic fuzzy soft set to solve intuitionistic fuzzy multi-criteria ranking problems. At last we have applied this $IFCM$-Algorithm for solving such type of problem in medical science[1].

2 Preliminaries

2.1 Definition: Soft Set [3]

Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the set of all subsets of U. Let $A \subseteq E$. Then a pair (F, A) is called a **soft set** over U, where F is a mapping given by, $F : A \longrightarrow P(U)$.
In other words, a soft set over U is a parameterized family of subsets of the universe U.

2.2 Definition: Fuzzy Soft Set (FSS) [9]

Let U be an initial universe set and E be a set of parameters which are fuzzy words or sentences involving fuzzy words. Let F^U denotes the set of all fuzzy subsets of U. Let $A \subset E$. Then a pair $(\tilde{F}, A)$ is called a **fuzzy soft set** (FSS) over U, where $\tilde{F}$ is a mapping given by, $\tilde{F} : A \longrightarrow F^U$

2.3 Definition: Intuitionistic Fuzzy Soft Set(IFSS) [13]

Let U be an initial universe set and E be a set of parameters which are intuitionistic fuzzy words or sentences involving intuitionistic fuzzy words. Let IF^U denotes the set of all intuitionistic fuzzy sets of U. Let $A \subset E$. A pair $(\hat{F}, A)$ is called an **intuitionistic fuzzy soft set**(IFSS) over U, where $\hat{F}$ is a mapping given by, $\hat{F} : A \longrightarrow IF^U$

2.4 Definition: Level Soft Set of an Intuitionistic Fuzzy Soft Set [16]

Let $\varpi = (\hat{F}, A)$ be an intuitionistic fuzzy soft set over U and $A \subseteq E$. A (s,t)**-level soft set** of the intuitionistic fuzzy soft set ϖ is a crisp soft set $L(\varpi; (s,t))$ which is defined as,

$$L(\varpi; (s,t)) = (F_{(s,t)}, A) = \{x \in U : \mu_a(x) \geq s \text{ and } \nu_a(x) \leq t\} \forall a \in A$$

In this definition, $s, t \in [0,1]$ also are called threshold values, in which $s \in [0,1]$ can be regarded as a given least threshold on membership values and $t \in [0,1]$ can be regarded as a given greatest threshold on non-membership values.

2.5 Definition: Intuitionistic Fuzzy Cognitive Map (IFCM) [2]

Intuitionistic fuzzy cognitive map (IFCM) is a decision-making model, enhanced, so that it captures the degree of hesitancy in the relations defined by the experts between its concepts. Once the IFCM is constructed, it can receive data from its input concepts, perform reasoning and infer medical decisions as values of its output concepts. During reasoning, the IFCM iteratively calculates its state until convergence. The state is represented by a *state vector* S^k, which consists of real node values $s_i^k \in [0,1], i = 1, 2, ..., N$, at an iteration k. The value of each node may be calculated by the following equation:

$$s_i^{k+1} = f(s_i^k + \sum_{j=1, j\neq i}^{N} s_j^k.(w_{ji}^{\mu} - w_{ji}^{\pi}))$$

where f is a threshold function such as its functional value lies within $[0,1]$, $w_{ji}^{\mu} \in [-1,1]$ and $w_{ji}^{\pi} \in [0,1]$ represent the *influence weight* and the *hesitancy weight* corresponding to the edge directed from node j to node i. However, the weight factor $(w_{ji}^{\mu} - w_{ji}^{\pi})$ does not always preserve the sign of the influence, whereas for unrelated concepts it could generate a non-negative causal relation between them if the hesitancy weight is nonzero. To overcome these limitations, Dimitris K. Iakovidis [2] has proposed the following equation to find the value of each node,

$$s_i^{k+1} = f(s_i^k + \sum_{j=1, j \neq i}^{N} s_j^k . w_{ji}^{\mu} . (1 - w_{ji}^{\pi}))$$

The weight factor $w_{ji}^{\mu}.(1 - w_{ji}^{\pi})$ preserves the sign of the influence and takes a zero value if two concepts are unrelated or if the hesitancy weight becomes equal to unity.

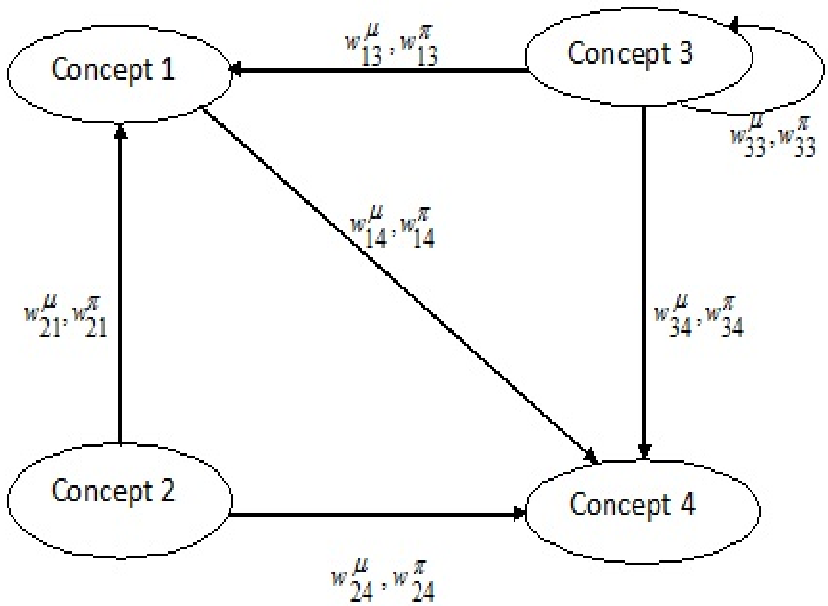

Figure 1: Example of an Intuitionistic Fuzzy Cognitive Map(IFCM) With Four Nodes

3 A Generalized Intuitionistic Fuzzy Multi-Criteria Ranking Problem

Let a decision maker wants to rank m number of objects which have n number of features i.e., criteria or, parameters(E). Suppose that the decision maker has freedom to take his decision of inclusion and evaluation of parameters associated with the selected object, i.e., the decision maker has his own choice parameters belonging to the parameter set E and has his own view of evaluation. Here it is assumed that **the parameter evaluation of the objects by the decision maker must be intuitionistic fuzzy** and may be presented in linguistic form or intuitionistic fuzzy soft set format. **One parameter may be influenced by other parameters as well as by itself also.**

The parameters are assumed to be intuitionistic fuzzy as it can be considered the most generalized form (crisp or fuzzy sets are intuitionistic fuzzy sets but the converse is not true). This is the most appropriate way to mathematically express the human perception. Now an object x having an intuitionistic fuzzy parameter (or criterion) e implies that x satisfies the criterion e with a grade of membership value $\mu_e(x)$ (evaluated by the decision maker), with a grade of non-membership value $\nu_e(x)$ (evaluated by the same decision maker) and the amount $\pi_e(x) = 1 - \mu_e(x) - \nu_e(x)$ which is termed as hesitation, the degree of indeterminacy concerning the membership of x with respect to e.

Now the problem is to rank these m objects in respect to the maximum degree of satisfaction of all the choice parameters of the decision maker.

4 An Approach to Solve Intuitionistic Fuzzy Multi-Criteria Ranking Problem

To solve an intuitionistic fuzzy multi-criteria ranking problem, we introduce a new approach to rank the solutions of an intuitionistic fuzzy soft set based decision making problem with inter related choice parameters which comprises of some new notions and an equation as follows.

Definition: Potentiality

The **potentiality of an intuitionistic fuzzy soft set** (p_{ifs}) is defined as the sum of all membership values as well as non-membership values of all objects with respect to all parameters i.e., mathematically it is defined as,

$$p_{ifs} = (\sum_{i=1}^{m}\sum_{j=1}^{n}\mu_{ij}, \sum_{i=1}^{m}\sum_{j=1}^{n}\nu_{ij})$$

where μ_{ij} and ν_{ij} are respectively the membership and non-membership value of the i-th object with respect to the j-th parameter, m is the number of objects and n is the number of parameters.

Definition: Mean Potentiality

The **mean potentiality** (m_p) of an intuitionistic fuzzy soft set is defined as it's average weight among the total potentiality i.e., mathematically it is defined as,

$$m_p = \frac{p_{ifs}}{m \times n}$$

Definition: Hesitation Function

Let $(\hat{F}, A)$ be an intuitionistic fuzzy soft set over the universal set of objects U then it can be viewed as,

$$(\hat{F}, A) = \{(x_i, \mu_{a_j}(x_i), \nu_{a_j}(x_i))/x_i \in U, a_j \in A, i = 1, 2, ..., m; j = 1, 2, ..., n\}$$

Let $w : A \longrightarrow [0, 1]$ is a weight function specifying the weight $w_j = w(a_j)$ for each attribute $a_j \in A$. Then the weighted **hesitation function** $\pi_A(x_i)$ of the object $x_i \in U$ in all attributes A of $(\hat{F}, A)$ is defined as,

$$\pi_A(x_i) = \sum_{j=1}^{n} w(a_j)\pi_{a_j}(x_i) = \sum_{j=1}^{n} w(a_j)(1 - \mu_{a_j}(x_i) - \nu_{a_j}(x_i)), \qquad i = 1, 2, .., m$$

Definition: Accuracy Function

Let $(\hat{F}, A)$ be an intuitionistic fuzzy soft set over the universal set of objects U then it can be viewed as,

$$(\hat{F}, A) = \{(x_i, \mu_{a_j}(x_i), \nu_{a_j}(x_i))/x_i \in U, a_j \in A, i = 1, 2, ..., m; j = 1, 2, ..., n\}$$

Let $w : A \longrightarrow [0, 1]$ is a weight function specifying the weight $w_j = w(a_j)$ for each attribute $a_j \in A$. Then the weighted **accuracy function** $\theta_A(x_i)$ of the object $x_i \in U$ in all attributes A of $(\hat{F}, A)$ is defined as,

$$\theta_A(x_i) = \sum_{j=1}^{n} w(a_j)\theta_{a_j}(x_i) = \sum_{j=1}^{n} w(a_j)(\mu_{a_j}(x_i) + \nu_{a_j}(x_i)), \qquad i = 1, 2, .., m$$

Equation: Evaluating The Value of Each Node of An Intuitionistic Fuzzy Cognitive Map

Dimitris K. Iakovidis [2] has proposed the following equation to find the value(after reaching convergence, the fixed point attractor) of each node of an intuitionistic fuzzy cognitive map,

$$s_i^{k+1} = f(s_i^k + \sum_{j=1, j\neq i}^{N} s_j^k.w_{ji}^{\mu}.(1 - w_{ji}^{\pi})) \tag{1}$$

But in an intuitionistic fuzzy cognitive map any node may be influenced by itself also, so we have reformulated the equation (1) introduced by Dimitris K. Iakovidis [2] and proposed the following equation (2) to find the value of each node as:

$$s_i^{k+1} = f(s_i^k + \sum_{j=1}^{N} s_j^k.w_{ji}^{\mu}.(1 - w_{ji}^{\pi})) \tag{2}$$

In the equation (2) we are also considering the case when $j = i$, i.e., when the i-th node is influenced by itself.

At first from the given influences of all criteria using the properties of intuitionistic fuzzy cognitive map we will calculate the individual weight of each criterion. Then by utilizing potentiality as well as mean potentiality we can find the appropriate level to get the level soft set of an intuitionistic fuzzy soft set and then involving hesitation, accuracy function we can solve an intuitionistic fuzzy multi-criteria ranking problem. The detailed step-wise procedure as an algorithm is given in the following subsection.

4.1 The Methodology For Solving Intuitionistic Fuzzy Multi-Criteria Ranking Problem

In many real life problems of economics, engineering, environment, social science, medical science etc., that involves completely ranking rather than select an optimal one. For example, evaluation of supply chain partners is an important decision involving multiple criteria and risk factors which are influenced by each other and it needs to rank the supplier completely in the supply chain of engineering project according to their service level. Because in practical business of the economics and management, ranking the cooperative supplier according to their service level is more significant in most of the time to the decision maker for using different cooperation mode. The stepwise procedure to solve such type of problems is given below.

$IFCM$-**Algorithm:**

Step 1: Draw the intuitionistic fuzzy cognitive map(IFCM) associated with the criteria.

Step 2: From the given information, set the initial value of each node(i.e., the initial weight of each criterion) to construct the initial state vector(S^0) which consists of initial real node values $s_i^0 \in [0,1]; i = 1,2,...,N; N$ be the number of nodes.

Step 3: Till attaining convergence calculate the value of each node by the following equation:

$$s_i^{k+1} = f(s_i^k + \sum_{j=1}^{N} s_j^k.w_{ji}^{\mu}.(1 - w_{ji}^{\pi}))$$

Where the state is represented by a *state vector* S^k, which consists of real node values $s_i^k \in [0,1], i = 1,2,...,N,$ at an iteration k; f is a threshold function such as its functional value lies within $[0,1]$, $w_{ji}^{\mu} \in [-1,1]$ and $w_{ji}^{\pi} \in [0,1]$ respectively represent the *influence weight* and the *hesitancy weight* corresponding to the edge directed from node j to node i.

Step 4: After getting convergence collect the final value $w(e_i)$ of each node presenting the criterion $e_i; i = 1, 2, .., N$.

Step 5: Compute the potentiality of the intuitionistic fuzzy soft set $(\hat{F}, A)$,

$$p_{ifs} = (\sum_{i=1}^{m}\sum_{j=1}^{n}\mu_{ij}, \sum_{i=1}^{m}\sum_{j=1}^{n}\nu_{ij})$$

where μ_{ij} and ν_{ij} respectively be the membership and non-membership value of the i-th object with respect to the j-th parameter, m be the number of objects and n be the number of parameters.

Step 6: Find the mean potentiality (m_p) of $(\hat{F}, A)$ as, $m_p = \frac{p_{ifs}}{m \times n}$

Step 7: Form the m_p-level soft set of the IFSS $(\hat{F}, A)$.

Step 8: Present the level soft set in tabular form with the individual weights of each criterion and compute the choice value C_i of i^{th} object for $i = 1, 2, ..., m$.

Step 9: Rank the objects according to the choice value C_i from the largest to the smallest. If all alternatives can be sorted by the strict order then stop otherwise go to the next step.

Step 10: Compute the degree of hesitation $\pi_a(x)$ of each object $x \in U$ with respect to each parameter $a \in A$ in $(\hat{F}, A)$ and then compute the functional values of weighted hesitation function $\pi_A(x) \forall x \in U$.

Step 11: Rank the objects which cannot be sorted by the choice value C_i, according to $\pi_A(x)$ from the smallest to the largest. If all alternatives can be sorted by the strict order, then stop otherwise go to the next step.

Step 12: Compute the degree of accuracy $\theta_a(x)$ of each object $x \in U$ with respect to each parameter $a \in A$ in $(\hat{F}, A)$ and furthermore compute the functional values of weighted accuracy function $\theta_A(x) \forall x \in U$.

Step 13: Rank the objects which cannot be sorted by the degree of hesitation $\pi_A(x)$ and the choice value C_i, according to $\theta_A(x)$ from the largest to the smallest.

Figure - 2,3,4: Flow Chart of $IFCM-$Algorithm

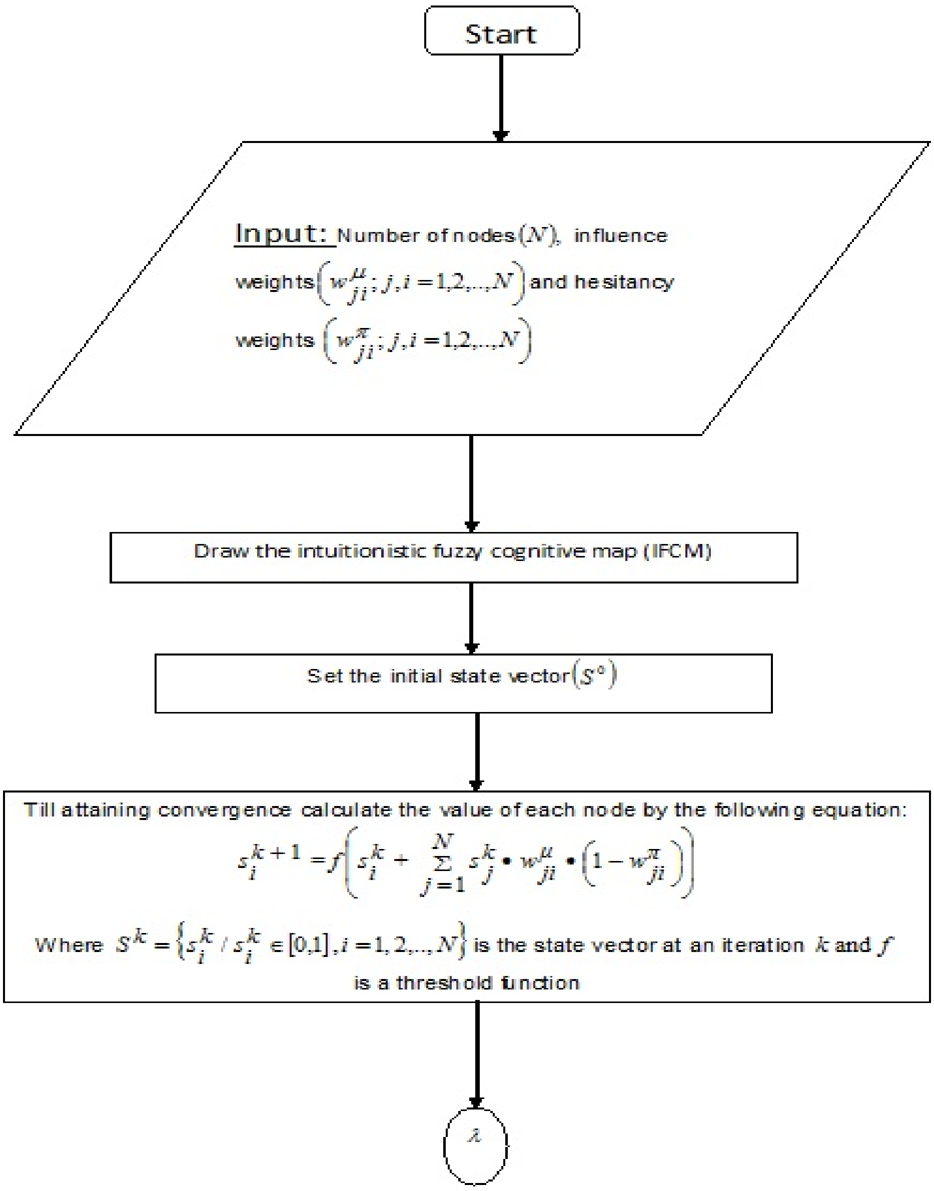

Figure 2: Flow Chart of $IFCM-$Algorithm(Part-1)

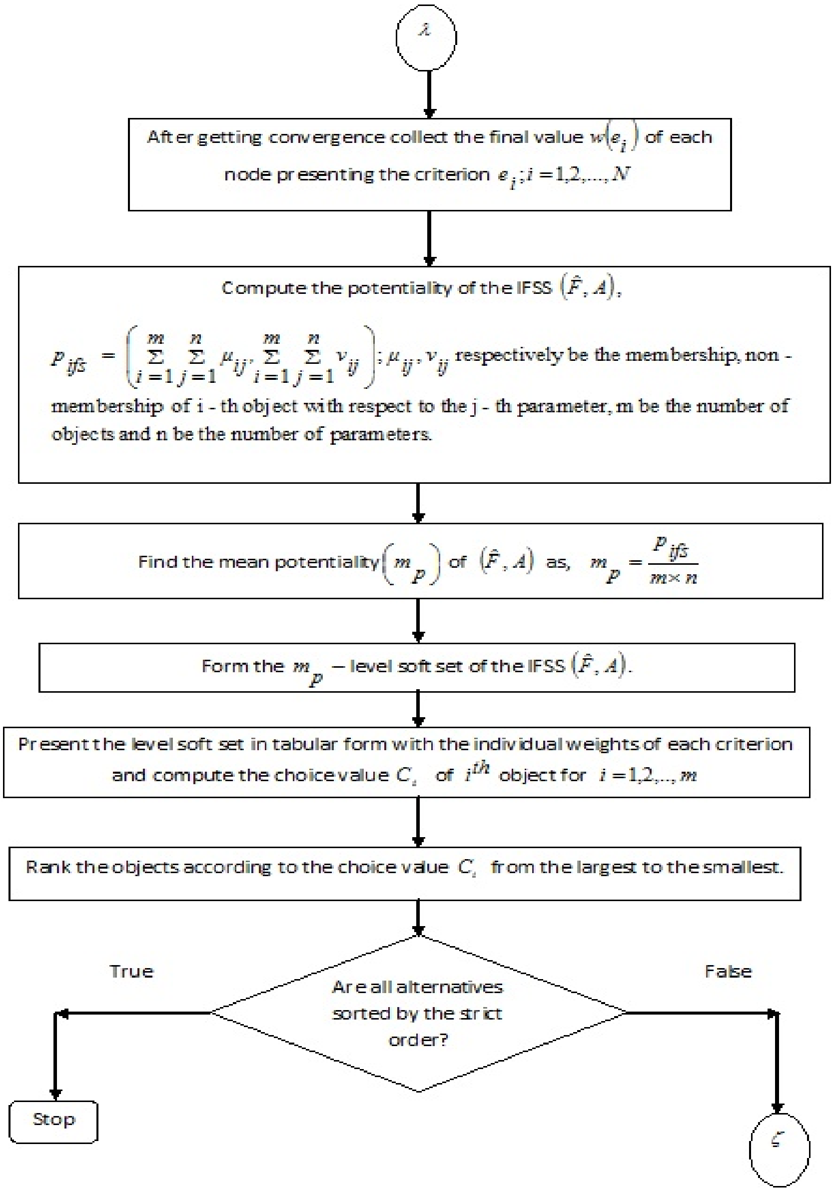

Figure 3: Flow Chart of $IFCM-$Algorithm(Part-2)

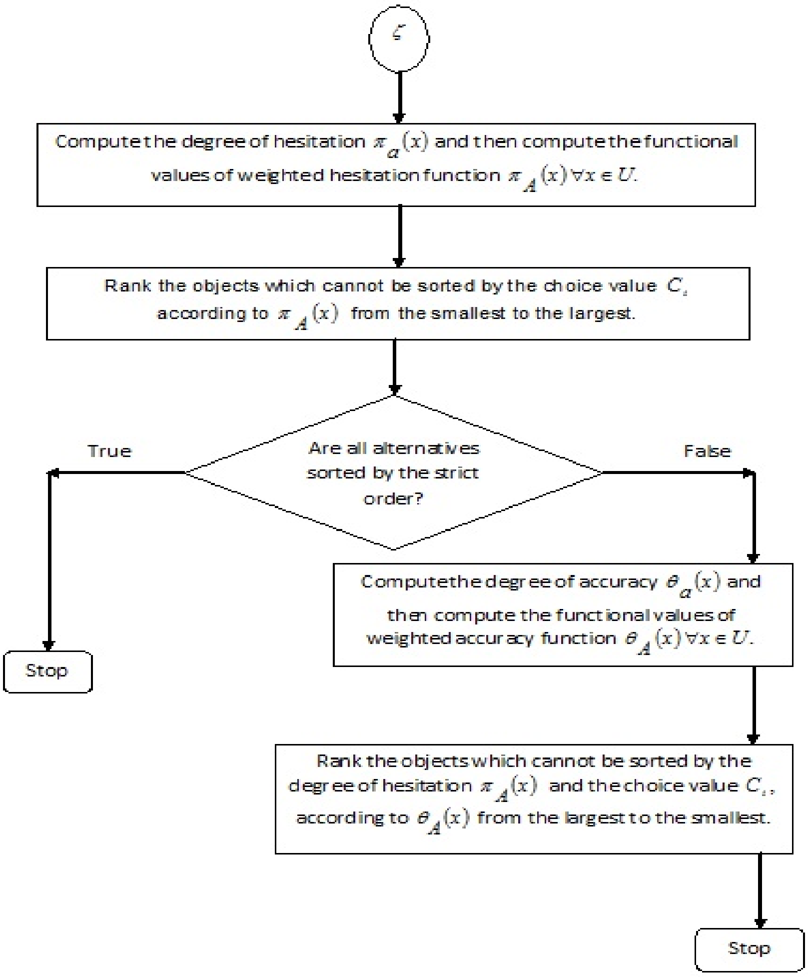

Figure 4: Flow Chart of $IFCM-$Algorithm(Part-3)

5 Application in Medical Science

In medical science there also exist intuitionistic fuzzy soft set based ranking problem and we may apply the $IFCM$-Algorithm for solving those problems. Now we will discuss a such type of problem with its solution.

Problem: There are different types of diseases and various modalities of treatments in respect to them. On the basis of different aspects of the treatment procedure we may measure the degree of effectiveness of the treatment for the disease. Here we consider three common diseases of oral cavity such as dental caries, gum disease and oral ulcer. Now medicinal treatment, extraction and scaling that are commonly executed, have more or less impacts on the treatment of these three diseases. According to the statistics[1], the degree of effectiveness in case of medicinal treatment for dental caries, gum disease, oral ulcer are $(0.5, 0.2), (0.3, 0.5)$ and $(0.8, 0.1)$ respectively; by extraction the degrees of effectiveness for dental caries, gum disease and oral ulcer are $(0.5, 0.3)$,$(0.4, 0.5)$ and $(0.2, 0.7)$ respectively and by scaling the degrees of effectiveness for dental caries, gum disease and oral ulcer are $(0.2, 0.7)$, $(0.6, 0.3)$ and $(0.5, 0.4)$ respectively. Suppose a patient simultaneously having these three diseases. Moreover these three diseases are more or less influenced by each other. Now the degrees that gum disease is influenced by itself is $(0.15, 0.65)$, by dental caries is $(0.4, 0.5)$ and by oral ulcer is $(0.3, 0.6)$. Similarly oral ulcer is influenced by dental caries $(0.2, 0.7)$ and by gum disease $(0.1, 0.8)$. But dental caries is only influenced by gum disease with a degree $(0.4, 0.6)$. **Now the problem is to rank the effectiveness of these three treatment procedures in respect to these three diseases.**

Solution:

The set of universe U = {medicinal treatment, extraction, scaling} $= \{t_1, t_2, t_3\}$ and the set of parameters E = {dental caries, gum disease, oral ulcer} $= \{e_1, e_2, e_3\}$(say). Now from the given data we have the intuitionistic fuzzy soft set $(\hat{F}, E)$ describing "the effectiveness of the treatments for the diseases" in tabular form as

Table-1: Tabular representation of $(\hat{F}, E)$

	e_1	e_2	e_3
t_1	(0.5,0.2)	(0.3,0.5)	(0.8,0.1)
t_2	(0.5,0.3)	(0.4,0.5)	(0.2,0.7)
t_3	(0.2,0.7)	(0.6,0.3)	(0.5,0.4)

Now let us apply $IFCM-$algorithm to solve this problem.

1) The intuitionistic fuzzy cognitive map(IFCM) associated with the criteria e_1, e_2, e_3 is depicted in Figure - 2.

2) Since according to the problem each of these three diseases of oral cavity such as dental

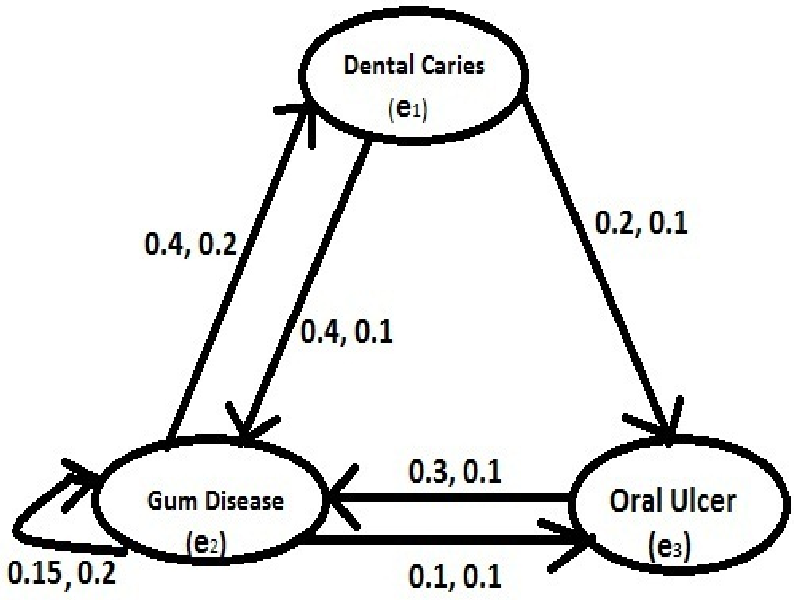

Figure 5: Intuitionistic Fuzzy Cognitive Map(IFCM) Associated With The Criteria

caries, gum disease and oral ulcer are equally common to be seen, so initially each disease seems to be of same important to be treated. Therefore initially every node value of this IFCM are same and let it be 1.

Thus the initial state vector,

$S^0 = \{1, 1, 1\}$

3) After eight iteration cycles of the IFCM model,

$$s_i^{k+1} = f(s_i^k + \sum_{j=1}^{3} s_j^k.w_{ji}^{\mu}.(1 - w_{ji}^{\pi})); i = 1, 2, 3; k = 0, 1, .., 7$$

(where the state is represented by a *state vector* S^k, which consists of real node values $s_i^k \in [0, 1], i = 1, 2, 3,$ at an iteration k; let here f be the sigmoid threshold function which is calculated as $f(x) = \frac{1}{(1+e^{-\lambda x})}$, λ be a positive constant, $f(x)$ lies between $[0, 1]$, $w_{ji}^{\mu} \in [-1, 1]$ and $w_{ji}^{\pi} \in [0, 1]$ respectively represent the *influence weight* and the *hesitancy weight* corresponding to the edge directed from node j to node i.) a fixed point attractor is obtained.

4) The final value of node e_1 is $w(e_1) = 0.727 \simeq 0.7$, node e_2 is $w(e_2) = 0.792 \simeq 0.8$ and node e_3 is $w(e_3) = 0.714 \simeq 0.7$

5) Potentiality of $(\hat{F}, E)$ is $p_{ifs} = (4, 3.7)$

6) Mean Potentiality of $(\hat{F}, E)$ is $m_p = (0.44, 0.41)$

7) and 8) The tabular representation of the m_p level soft set of $(\hat{F}, E)$ with the individual weight of each criterion together with the choice value of each treatment is

Table-2

	$e_1(w(e_1) = 0.7)$	$e_2(w(e_2) = 0.8)$	$e_3(w(e_3) = 0.7)$	choice value
t_1	1	0	1	1.4
t_2	1	0	0	0.7
t_3	0	1	1	1.5

9) Using maximum choice value principle, the ranking result is $t_3 \succ t_1 \succ t_2$
i.e., scaling $\succ$ medicinal treatment $\succ$ extraction

Hence if a patient is simultaneously suffering from the three diseases dental caries, gum disease and oral ulcer then the effectiveness of medicinal treatment is better than extraction and the effectiveness of scaling is better than both medicinal treatment and extraction.

6 Conclusion:

In this paper we have introduced the concept of a generalized intuitionistic fuzzy multi-criteria ranking problem. We have also reformulated the equation introduced by Dimitris K. Iakovidis [2] evaluating the value of each node of an intuitionistic fuzzy cognitive map (IFCM). Moreover we have introduced some new notions and by utilizing these new notions together with the concept of IFCM we have developed a new algorithm, named as $IFCM$-Algorithm for solving intuitionistic fuzzy multi-criteria ranking problems. Finally we have applied the $IFCM$-Algorithm to rank the effectiveness of the treatment procedures in respect to the diseases from medical science.

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Matrices in interval-valued fuzzy soft set theory and their application

T. Mitra Basu ①*, N. Kumar Mahapatra ②, S. Kumar Mondal ①

① Department of Applied Mathematics, Vidyasagar University, Midnapore -721 102,W.B., India
② Department of Mathematics, Panskumar Banamali College, Panskura RS-721152, WB, India
E-mail: tanushreemitra13@gmail.com

Received: Dec-9-2013; Accepted: Jan-5-2014 *Corresponding author

Abstract The purpose of this paper is to introduce the concept of Interval-Valued Fuzzy Soft Matrix ($IVFS$-matrix) together with some different types of matrices in interval-valued fuzzy soft set theory. We have defined here some new operations on these matrices and discussed all these definitions and operations by appropriate examples. In addition we have proven some theorems along with few properties on these matrices. Moreover a new efficient $IVFSM$-algorithm based on these new matrix operations has been developed to solve interval-valued fuzzy soft set based group decision making problems. After that the $IVFSM$-algorithm has been applied to a real life group decision making problem and then we have described the feasibility of this proposed method.

Key Words Interval-valued Fuzzy Soft Set; Interval-Valued Fuzzy Soft Matrix($IVFS$-Matrix); Choice Matrix;
MSC 2010 03E72, 65F30

1 Introduction

In real scenario we need strategies which provide some flexible information processing capacity to deal with uncertainties. Soft set theory is generally used to solve such problems. In the year 1999 Molodtsov [2] introduced soft set as a completely generic mathematical tool for modeling uncertainties. Since there is hardly any limitation in describing the objects, researchers simplify the decision making process by choosing the form of parameters they require and subsequently makes it more efficient in the absence of partial information. Maji et al. have done further research on soft set theory [9, 10] and on fuzzy soft set theory [11]. Then the different researchers Chen et al.[1] in 2005, Zou et al. [16] and Kong et al. [18] in 2008 have worked on parameter reduction of soft set and fuzzy soft set theory.

All of these works are based on the classical soft set theory. The soft set model, however, can also be combined with other mathematical models. For example, by amalgamating the soft set and algebra, Akta et al.[5] proposed the definition of soft groups, Feng et al. [3] proposed the concept of soft semirings, soft subsemirings, soft ideals, idealistic soft semirings and soft semiring homomorphisms. Then Y.B. Jun [17] applied the notion of soft sets to the theory of BCK/BCI-algebras. Maji et al. presented the concept of the fuzzy soft set [11, 12] which is based on a combination of the fuzzy set and soft set models. Yang et al. [13] defined the operations on fuzzy soft sets, which are based on three fuzzy logic operators: negation, triangular norm and triangular conorm. Zou et al.[15] introduced the soft set and fuzzy soft set into the

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incomplete environment. Cagman et al. [7, 8] have proposed the definition of soft matrix which is the representation of a soft set and they also introduced a new soft set based decision making method. Then in the year 2009, Yang et al. [14] have combined the interval-valued fuzzy set [6] and soft set[2], from which a new soft set model: interval-valued fuzzy soft set(IVFSs)[14] is obtained. They have also given an algorithm to solve IVFSs based decision making problems. Then Feng et al[4] have shown that Yang's algorithm has some drawbacks and they have proposed another method for solving IVFSs based decision making problems.

But according to Feng's method [4], the decision maker has to form a reduct fuzzy soft set (of pessimistic or optimistic or neutral type) of the given IVFSs and then can select any level to form the level soft set. There does not exist any unique or uniform criterion for the selection of the level. So by this method the decision maker will be puzzled to decide that which type and which level is most suitable for the selection of the object. Moreover till now researchers [14, 4] have worked on finding solution of the IVFSs based decision making problems involving **only one** decision maker. There does not exist any method for solving a IVFSs-based **group** decision making problem.

Group Decision is the academic and professional field that aims to improve collective decision process by taking into account the needs and opinions of every group assisting groups or individuals within groups, as they interact and collaborate to reach a collective decision.

In this paper, we have introduced the concept of $IVFS$-matrix. Here we have presented the concept of choice matrix which represents the choice parameters of the decision makers and then we have introduced some new operations on $IVFS$-matrix and choice matrix. Moreover we have proven some theorems along with few properties on these matrices. Finally we have proposed the new $IVFSM$-algorithm based on some of these new matrix operations to solve interval-valued fuzzy soft set based decision making problems **involving any number of decision maker.** To realize this newly proposed algorithm we have applied it to a real life group decision making problem and also described the purpose of its introduction.

2 Preliminaries

Definition 2.1([2]) Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U. Let $A \subseteq E$. A pair (F_A, E) is called a **soft set** over U, where F_A is a mapping given by, $F_A : A \longrightarrow P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U.

Definition 2.2([11]) Let U be an initial universe set and E be a set of parameters(which are fuzzy words or sentences involving fuzzy words). Let $P(U)$ denotes the set of all fuzzy sets of U. Let $A \subset E$. A pair (F_A, E) is called a **Fuzzy Soft Set** (FSS) over U, where F_A is a mapping given by, $F_A : E \longrightarrow P(U)$ such that $F_A(e) = \tilde{\phi}$ if $e \notin A$ where $\tilde{\phi}$ is a null fuzzy set.

Example 2.1. Let U be the set of four cities, given by, $U = \{C_1, C_2, C_3, C_4\}$. Let E be the set of parameters (each parameter is a fuzzy word), given by, $E = \{$ highly, immensely, moderately, average, less$\} = \{e_1, e_2, e_3, e_4, e_5\}$, where e_1 stands for the parameter 'highly', e_2 stands for the parameter 'immensely', e_3 stands for the parameter 'moderately', e_4 stands for the parameter 'average'. e_5 stands for the parameter 'less'. Let $A \subset E$, given by, $A = \{e_1, e_2, e_3, e_5\}$, Now suppose that, $F_A(e_1) = \{C_1/.2, C_2/.9, C_3/.4, C_4/.6\}$, $F_A(e_2) = \{C_2/1, C_3/.3, C_4/.4\}$, $F_A(e_3) = \{C_1/.3, C_2/.4, C_3/.8\}$, $F_A(e_5) = \{C_1/.9, C_2/.1, C_3/.5, C_4/.3\}$.

Then the fuzzy soft set is given by,

$$\begin{aligned}(F_A, E) \quad = \quad & \{\text{highly polluted city } = \{C_1/.2, C_2/.9, C_3/.4, C_4/.6\},\\ & \text{immensely polluted city} = \{C_2/1, C_3/.3, C_4/.4\},\\ & \text{moderately polluted city} = \{C_1/.3, C_2/.4, C_3/.8\},\\ & \text{average polluted city} = \tilde{\phi},\\ & \text{less polluted city} = \{C_1/.9, C_2/.1, C_3/.5, C_4/.3\}\}\end{aligned}$$

Definition 2.3([14]) Let U be an initial universe, E be a set of parameters and $A \subseteq E$. Then a pair $(\bar{F}_A, E)$ is called an **interval-valued fuzzy soft set** over $P(U)$ where $\bar{F}_A$ is a mapping given by $\bar{F}_A : A \longrightarrow P(U)$.

An interval-valued fuzzy soft set is a parameterized family of interval-valued fuzzy subsets of U. An interval-valued fuzzy soft set is also a special case of a soft set because it is still a mapping from parameters to $P(U)$. $\forall e \in A, \bar{F}_A(e)$ is referred as an interval-valued fuzzy set of U. It can be written as: $\bar{F}_A(e) = \{(x, \mu_{\bar{F}_A}(x)) : x \in U\}$ where $\mu_{\bar{F}_A}(x)$ is the interval-valued fuzzy membership degree that object x holds on parameter e. If $\forall e \in E, \forall x \in U, \mu^-_{\bar{F}_A}(x) = \mu^+_{\bar{F}_A}(x)$, then $\bar{F}_A$ will degenerated to be a standard fuzzy set and then $(\bar{F}_A, E)$ will be degenerated to be a traditional fuzzy soft set.

Example 2.2. Suppose that, U be the set of six houses $h_1, h_2, h_3, h_4, h_5, h_6$ and E be the set of parameters given by $E = \{$ beautiful, wooden, cheap, in the green surroundings$\} = \{e_1, e_2, e_3, e_4\}$ and $A = E$. The tabular representation of the interval-valued fuzzy soft set $(\bar{F}_A, E)$ is shown in Table 1. In Table 1, we can see that the precise evaluation for each object on each parameter is unknown while the lower and upper limits of such an evaluation is given. For example, we cannot present the precise degree of how beautiful the house h_1 is, however, the house h_1 is at least beautiful on the degree of 0.7 and it is at most beautiful on the degree of 0.9.

Table-1: Tabular representation of $(\bar{F}_A, E)$

	e_1	e_2	e_3	e_4
h_1	[0.7,0.9]	[0.6, 0.7]	[0.3, 0.5]	[0.5, 0.8]
h_2	[0.6, 0.8]	[0.8,1.0]	[0.8, 0.9]	[0.9, 1.0]
h_3	[0.5, 0.6]	[0.2,0.4]	[0.5, 0.7]	[0.7, 0.9]
h_4	[0.6, 0.8]	[0.0,0.1]	[0.7, 1.0]	[0.6, 0.8]
h_5	[0.8, 0.9]	[0.1,0.3]	[0.9, 1.0]	[0.2, 0.5]
h_6	[0.8, 1.0]	[0.7, 0.8]	[0.2, 0.5]	[0.7,1.0]

Definition 2.4([7]) Let (F_A, E) be a soft set over U. Then a subset of $U \times E$ is uniquely defined by $R_A = \{(u, e) : e \in A, u \in F_A(e)\}$ which is called a relation form of (F_A, E). Now the **characteristic function** of R_A is written by,

$$\chi_{R_A} : U \times E \longrightarrow \{0, 1\}, \ \chi_{R_A} \quad = \quad \begin{cases} 1, (u, e) \in R_A \\ 0, (u, e) \notin R_A \end{cases}$$

Definition 2.5([7]) Let $U = \{u_1, u_2, .u_m\}$, $E = \{e_1, e_2, \cdots, e_n\}$, then R_A can be presented by a table as in the following form

Table-2

	e_1	e_2	$\cdots$	e_n
u_1	$\chi_{R_A}(u_1, e_1)$	$\chi_{R_A}(u_1, e_2)$	$\cdots$	$\chi_{R_A}(u_1, e_n)$
u_2	$\chi_{R_A}(u_2, e_1)$	$\chi_{R_A}(u_2, e_2)$	$\cdots$	$\chi_{R_A}(u_2, e_n)$
$\cdots$	v	$\cdots$	$\cdots$	$\cdots$
u_m	$\chi_{R_A}(u_m, e_1)$	$\chi_{R_A}(u_m, e_2)$	$\cdots$	$\chi_{R_A}(u_m, e_n)$

If $a_{ij} = \chi_{R_A}(u_i, e_j)$, we can define a matrix

$$[a_{ij}]_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

which is called a **soft matrix** of order $m \times n$ corresponding to the soft set (F_A, E) over U. A soft set (F_A, E) is uniquely characterized by the matrix $[a_{ij}]_{m\times n}$. Therefore we shall identify any soft set with its soft matrix and use these two concepts as interchangeable.

3 Some New Concepts of Matrices in Interval-Valued Fuzzy Soft Set Theory

Definition 3.1 (The Concept of $IVFS$-Matrix) Let $(\bar{F}_A, E)$ be an interval-valued fuzzy soft set over U. Then a subset of $U \times E$ is uniquely defined by

$$R_A = \{(u, e) : e \in A, u \in \bar{F}_A(e)\}$$

which is called a relation form of $(\bar{F}_A, E)$. Now the relation R_A is characterized by the membership function $\mu_A : U \times E \longrightarrow \text{Int}([0, 1])$ such that

$$\mu_A(u, e) = \begin{cases} [\mu^-_{\bar{F}_A(e)}(u), \mu^+_{\bar{F}_A(e)}(u)], & if \ \ e \in A \\ [0, 0], & if \ \ e \notin A \end{cases}$$

where Int([0, 1]) stands for the set of all closed subintervals of $[0, 1]$ and $[\mu^-_{\bar{F}_A(e)}(u), \mu^+_{\bar{F}_A(e)}(u)]$ denotes the interval-valued fuzzy membership degree of the object u associated with the parameter e.

Now if the set of universe $U = \{u_1, u_2, \cdots, u_m\}$ and the set of parameters $E = \{e_1, e_2, \cdots, e_n\}$, then R_A can be presented by a table in the following form

Table-3: Tabular representation of R_A

	e_1	e_2	$\cdots$	e_n
u_1	$\mu_A(u_1, e_1)$	$\mu_A(u_1, e_2)$	$\cdots$	$\mu_A(u_1, e_n)$
u_2	$\mu_A(u_2, e_1)$	$\mu_A(u_2, e_2)$	$\cdots$	$\mu_A(u_2, e_n)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
u_m	$\mu_A(u_m, e_1)$	$\mu_A(u_m, e_2)$	$\cdots$	$\mu_A(u_m, e_n)$

where $\mu_A(u_m, e_n) = [\mu^-_{\bar{F}_A(e_n)}(u_m), \mu^+_{\bar{F}_A(e_n)}(u_m)]$. If $a_{ij} = [\mu^-_{\bar{F}_A(e_j)}(u_i), \mu^+_{\bar{F}_A(e_j)}(u_i)]$, then from table-4 we can define a matrix

$$(\bar{a}_{ij})_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

which is called an **interval-valued fuzzy soft matrix or, simply $IVFS$-matrix** of order $m \times n$ corresponding to the interval-valued fuzzy soft set $(\bar{F}_A, E)$ over U. An interval-valued fuzzy soft set $(\bar{F}_A, E)$ is uniquely characterized by the matrix $(\bar{a}_{ij})_{m\times n}$. Therefore we shall identify any interval-valued fuzzy soft set with its $IVFS$-matrix and use these two concepts as interchangeable.

Example 3.1. Let U be the set of five cities, given by, $U = \{C_1, C_2, C_3, C_4, C_5\}$. Let E be the set of parameters (each parameter is an interval-valued fuzzy word), given by,

$E = \{$ highly, immensely, moderately, average, less $\} = \{e_1, e_2, e_3, e_4, e_5\}$(say)

Let $A \subset E$, given by, $A = \{e_1, e_2, e_3, e_5\}$ (say) Now suppose that, $\bar{F}_A : A \longrightarrow P(U)$ describing "the pollution of the cities"is given by,

$\bar{F}_A(e_1) = \{C_1/[.2, .4], C_2/[.8, .9], C_3/[.4, .5], C_4/[.6, .7], C_5/[.7, .8]\}$,
$\bar{F}_A(e_2) = \{C_1/[0, .1], C_2/[.9, 1], C_3/[.3, .4], C_4/[.4, .6], C_5/[.6, .7]\}$,
$\bar{F}_A(e_3) = \{C_1/[.3, .5], C_2/[.4, .5], C_3/[.8, .9], C_4/[.1, .2], C_5/[.3, .5]\}$,
$\bar{F}_A(e_5) = \{C_1/[.9, 1], C_2/[.1, .2], C_3/[.5, .6], C_4/[.3, .5], C_5/[.1, .2]\}$

Therefore the Interval-Valued Fuzzy Soft Set is,

$$\begin{aligned}(\bar{F}_A, E) \;=\; &\{\text{highly polluted city} = \{C_1/[.2, .4], C_2/[.8, .9], C_3/[.4, .5], C_4/[.6, .7], C_5/[.6, .8]\},\\ &\text{immensely polluted city} = \{C_1/[0, .1], C_2/[.9, 1], C_3/[.3, .4], C_4/[.4, .6], C_5/[.6, .7]\},\\ &\text{moderately polluted city} = \{C_1/[.3, .5], C_2/[.4, .5], C_3/[.8, 1], C_4/[.1, .2], C_5/[.3, .5]\},\\ &\text{less polluted city} = \{C_1/[.9, 1], C_2/[.1, .2], C_3/[.5, .7], C_4/[.3, .5], C_5/[.1, .2]\}\}\end{aligned}$$

Then the relation form of $(\bar{F}_A, E)$ is written by,

$$\begin{aligned}R_A \;=\; &\{(\{C_1/[.2, .4], C_2/[.8, .9], C_3/[.4, .5], C_4/[.6, .7], C_5/[.6, .8]\}, e_1),\\ &(\{C_1/[0, .1], C_2/[.9, 1], C_3/[.3, .4], C_4/[.4, .6], C_5/[.6, .7]\}, e_2),\\ &(\{C_1/[.3, .5], C_2/[.4, .5], C_3/[.8, 1], C_4/[.1, .2], C_5/[.3, .5]\}, e_3),\\ &(\{C_1/[0, 0], C_2/[0, 0], C_3/[0, 0], C_4/[0, 0], C_5/[0, 0]\}, e_4),\\ &(\{C_1/[.9, 1], C_2/[.1, .2], C_3/[.5, .7], C_4/[.3, .5], C_5/[.1, .2]\}, e_5)\}\end{aligned}$$

Hence the $IVFS$-matrix $(\bar{a}_{ij})$ is written by,

$$(\bar{a}_{ij}) = \begin{pmatrix} [.2, .4] & [0, .1] & [.3, .5] & [0,0] & [.9, 1] \\ [.8, .9] & [.9, 1] & [.4, .5] & [0,0] & [.1, .2] \\ [.4, .5] & [.3, .4] & [.8, 1] & [0,0] & [.5, .7] \\ [.6, .7] & [.4, .6] & [.1, .2] & [0,0] & [.3, .5] \\ [.6, .8] & [.6, .7] & [.3, .5] & [0,0] & [.1, .2] \end{pmatrix}.$$

Definition 3.2(Row $IVFS$-Matrix) An $IVFS$-matrix of order $1 \times n$ i.e., with a single row is called a **row $IVFS$-matrix**. Physically, a row $IVFS$-matrix formally corresponds to an interval-valued fuzzy soft set whose universal set contains only one object.

Example 3.2. Suppose the universe set U contains only one dress d_1 and parameter set $E = \{\text{costly, beautiful, cheap, comfortable}\} = \{e_1, e_2, e_3, e_4\}$. Let $A = \{e_2, e_3, e_4\} \subset E$ and $\bar{F}_A : A \longrightarrow P(U)$ s.t., $\bar{F}_A(e_2) = \{d_1/[.8, .9]\}, \bar{F}_A(e_3) = \{d_1/[.2, .4]\}, \bar{F}_A(e_4) = \{d_1/[.5, .6]\}$. Then the interval-valued fuzzy soft set $(\bar{F}_A, E) = \{(e_2, \{d_1/[.8, .9]\}), (e_3, \{d_1/[.2, .4]\}), (e_4, \{d_1/[.5, .6]\})\}$ and then the relation form of $(\bar{F}_A, E)$ is written by, $R_A = \{(\{d_1/[0, 0]\}, e_1), (\{d_1/[.8, .9]\}, e_2), (\{d_1/[.2, .4]\}, e_3), (\{d_1/[.5, .6]\}, e_4)\}$ Hence the $IVFS$-matrix $(\bar{a}_{ij})$ is written by,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0,\ 0] & [.8,\ .9] & [.2,.4] & [.5,\ .6] \end{pmatrix}$$

which contains a single row and so it is a row $IVFS$-matrix.

Definition 3.3(Column $IVFS$-Matrix) An $IVFS$-matrix of order $m \times 1$ i.e., with a single column is called a **column** $IVFS$**-matrix**. Physically, a column $IVFS$-matrix formally corresponds to an interval-valued fuzzy soft set whose parameter set contains only one parameter.

Example 3.3. Suppose the initial universe set U contains four dresses d_1, d_2, d_3, d_4 and the parameter set E contains only one parameter given by, $E = \{\text{ beautiful }\} = \{e_1\}$. $\bar{F} : E \longrightarrow P(U)$ s.t, $\bar{F}(e_1) = \{d_1/[0.6, 0.8], d_2/[0.1, 0.3], d_3/[0.7, 0.9], d_4/[0.3, 0.5]\}$. Then the interval-valued fuzzy soft set $(\bar{F}, E) = \{(e_1, \{d_1/[0.6, 0.8], d_2/[0.1, 0.3], d_3/[0.7, 0.9], d_4/[0.3, 0.5]\})\}$ and then the relation form of $(\bar{F}, E)$ is written by, $R_E = \{(\{d_1/[0.6, 0.8], d_2/[0.1, 0.3], d_3/[0.7, 0.9], d_4/[0.3, 0.5]\}, e_1)\}$. Hence the $IVFS$-matrix $(\bar{a}_{ij})$ is written by,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0.6,\ 0.8] \\ [0.1,\ 0.3] \\ [0.7,\ 0.9] \\ [0.3,\ 0.5] \end{pmatrix}$$

which contains a single column and so it is an example of column $IVFS$-matrix.

Definition 3.4(Square $IVFS$-Matrix) An $IVFS$-matrix of order $m \times n$ is said to be an **square** $IVFS$**-matrix** if $m = n$ ie., the number of rows and the number of columns are equal. That means a square $IVFS$-matrix is formally equal to an interval-valued fuzzy soft set having the same number of objects and parameters.

Example 3.4. Consider the example 3.1. Here since the $IVFS$-matrix $(\bar{a}_{ij})$ contains five rows and five columns, so it is a square $IVFS$-matrix.

Definition 3.5 (Complement of an $IVFS$-matrix) Let $(\bar{a}_{ij})$ be an $m \times n$ $IVFS$-matrix, where $(\bar{a}_{ij})$ is the matrix representation of the interval-valued fuzzy soft set $(\bar{F}_A, E)$. Then the **complement** of $(\bar{a}_{ij})$ is denoted by $(\bar{a}_{ij})^c$ and is defined by, $(\bar{a}_{ij})^c = (\bar{c}_{ij})$, where $(\bar{c}_{ij})$ is also an $IVFS$-matrix of order $m \times n$ and it is the matrix representation of the interval-valued fuzzy soft set $(\bar{F}^c_{\neg A}, E)$, ie., $c_{ij} = [\mu^-_{c_{ij}}, \mu^+_{c_{ij}}] = [1 - \mu^+_{a_{ij}}, 1 - \mu^-_{a_{ij}}]$.

Example 3.5. Consider the example 3.1. Then the complement of $(\bar{a}_{ij})$ is,

$$(\bar{a}_{ij})^c = \begin{pmatrix} [\ .6,\ .8\ ] & [\ .9,\ 1\ ] & [\ .5,\ .7\ ] & [1,1] & [\ 0,\ .1\ ] \\ [\ .1,\ .2\ ] & [\ 0,\ .1\ ] & [\ .5,\ .6\ ] & [1,1] & [\ .8,\ .9] \\ [\ .5,\ .6] & [\ .6,\ .7] & [\ 0,\ .2\ ] & [1,1] & [\ .3,\ .5\ ] \\ [\ .3,\ .4\ ] & [\ .4,\ .6\ ] & [\ .8,\ .9\ ] & [1,1] & [\ .5,\ .7\ ] \\ [.2,\ .4\ ] & [.3,\ .4] & [.5,\ .7] & [1,1] & [.8,\ .9] \end{pmatrix}.$$

Definition 3.6 (Null $IVFS$-Matrix) An $IVFS$-matrix of order $m \times n$ is said to be a **null $IVFS$-matrix or zero $IVFS$-matrix** if all of its elements are $[0, 0]$. A null $IVFS$-matrix is denoted by, $\bar{\Phi}$. Now the interval-valued fuzzy soft set associated with a null $IVFS$-matrix must be a null interval-valued fuzzy soft set.

Example 3.6. Let there are four dresses in the universe set U given by, $U = \{d_1, d_2, d_3, d_4\}$ and the parameter set $E = \{\text{beautiful, cheap, comfortable, gorgeous}\} = \{e_1, e_2, e_3, e_4\}$.
Let $A = \{e_1, e_2, e_3\} \subset E$. Now let $\bar{F}_A : E \longrightarrow P(U)$ s.t, $\bar{F}_A(e_1) = \{d_1/[0, 0], d_2/[0, 0], d_3/[0, 0], d_4/[0, 0]\} = \bar{\phi}, \bar{F}_A(e_2) = \bar{\phi}, \bar{F}_A(e_3) = \bar{\phi}$ (where $\bar{\phi}$ is a null interval-valued fuzzy set). Then the interval-valued fuzzy soft set $(\bar{F}_A, E) = \{(e_1, \bar{\phi}), (e_2, \bar{\phi}), (e_3, \bar{\phi})\}$ and then the relation form of $(\bar{F}_A, E)$ is written by, $R_A = \{(\bar{\phi}, e_1), (\bar{\phi}, e_2), (\bar{\phi}, e_3)\}$ Hence the $IVFS$-matrix $(\bar{a}_{ij})$ is written by,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix} = \bar{\Phi}.$$

Definition 3.7 (Complete $IVFS$-Matrix or, Absolute $IVFS$-Matrix) An $IVFS$-matrix of order $m \times n$ is said to be a **complete $IVFS$-matrix or, absolute $IVFS$-matrix** if all of its elements are $[1, 1]$. A complete or absolute $IVFS$-matrix is denoted by, $\bar{C}_A$. Now the interval-valued fuzzy soft set associated with an absolute $IVFS$-matrix must be an absolute interval-valued fuzzy soft set.

Example 3.7. Let there are four dresses in the universe set U given by, $U = \{d_1, d_2, d_3, d_4\}$ and the parameter set $E = \{\text{beautiful, cheap, comfortable, gorgeous}\} = \{e_1, e_2, e_3, e_4\}$. Let $A = \{e_1, e_2, e_3, e_4\} \subseteq E$ and $\bar{F}_A : E \longrightarrow P(U)$ s.t,

$$\bar{F}_A(e_1) = \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\},$$
$$\bar{F}_A(e_2) = \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\},$$
$$\bar{F}_A(e_3) = \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\},$$
$$\bar{F}_A(e_4) = \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}.$$

Then the interval-valued fuzzy soft set

$$\begin{aligned}(\bar{F}_A, E) = \{&(e_1, \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}),\\ &(e_2, \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}),\\ &(e_3, \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}),\\ &(e_4, \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\})\}\end{aligned}$$

and then the relation form of $(\bar{F}_A, E)$ is written by,

$$\begin{aligned}R_A = \{&(\{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}, e_1), (\{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}, e_2),\\ &(\{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}, e_3), (\{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}, e_4)\}.\end{aligned}$$

Hence the $IVFS$-matrix $(\bar{a}_{ij})$ is written by,

$$(\bar{a}_{ij}) = \begin{pmatrix} [1,1] & [1,1] & [1,1] & [1,1] \\ [1,1] & [1,1] & [1,1] & [1,1] \\ [1,1] & [1,1] & [1,1] & [1,1] \\ [1,1] & [1,1] & [1,1] & [1,1] \end{pmatrix} = \bar{C}_A.$$

Definition 3.8 (Diagonal $IVFS$-Matrix) A square $IVFS$-matrix of order $n \times n$ is said to be a **diagonal**

$IVFS$**-matrix** if all of its non-diagonal elements are $[0,0]$. If the diagonal elements of a diagonal $IVFS$-matrix be all equal, then the matrix is called a **scalar** $IVFS$**-matrix**. If the diagonal elements of a diagonal $IVFS$-matrix be all $[1,1]$, then the matrix is called a **unit or identity** $IVFS$**-matrix**.

Example 3.8. Suppose the initial universe set $U = \{d_1, d_2, d_3, d_4, d_5\}$ and the parameter set $E = \{e_1, e_2, e_3, e_4, e_5\}$. Let $\bar{F} : E \longrightarrow P(U)$ s.t,

$$\bar{F}(e_1) = \{d_1/[0.8,1], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0,0]\},$$
$$\bar{F}(e_2) = \{d_1/[0,0], d_2/[0.3,0.5], d_3/[0,0], d_4/[0,0], d_5/[0,0]\},$$
$$\bar{F}(e_3) = \{d_1/[0,0], d_2/[0,0], d_3/[0.8,1], d_4/[0,0], d_5/[0,0]\},$$
$$\bar{F}(e_4) = \{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0.7,0.9], d_5/[0,0]\},$$
$$\bar{F}(e_5) = \{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0.6,0.8]\}.$$

Then the interval-valued fuzzy soft set

$$\begin{aligned}(\bar{F}, E) \quad = \quad & \{(e_1, \{d_1/[0.8,1], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}),\\ & (e_2, \{d_1/[0,0], d_2/[0.3,0.5], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}),\\ & (e_3, \{d_1/[0,0], d_2/[0,0], d_3/[0.8,1], d_4/[0,0], d_5/[0,0]\}),\\ & (e_4, \{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0.7,0.9], d_5/[0,0]\}),\\ & (e_5, \{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0.6,0.8]\})\}\end{aligned}$$

and then the relation form of $(\bar{F}, E)$ is written by,

$$\begin{aligned}R_E \quad = \quad & \{(\{(\{d_1/[0.8,1], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}, e_1),\\ & (\{d_1/[0,0], d_2/[0.3,0.5], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}, e_2),\\ & (\{d_1/[0,0], d_2/[0,0], d_3/[0.8,1], d_4/[0,0], d_5/[0,0]\}, e_3),\\ & (\{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0.7,0.9], d_5/[0,0]\}, e_4),\\ & (\{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0.6,0.8]\}, e_5)\}\end{aligned}$$

Now the $IVFS$-matrix $(\bar{a}_{ij})$ is written by,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0.8,1] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0.3,0.5] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0.8,1] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0.7,0.9] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0.6,0.8] \end{pmatrix}$$

Hence by definition $(\bar{a}_{ij})$ is a diagonal $IVFS$-matrix.
Similarly,

$$(\bar{s}_{ij}) = \begin{pmatrix} [a,b] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [a,b] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [a,b] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [a,b] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [a,b] \end{pmatrix}$$

is a scalar $IVFS$-matrix (where $a, b \in [0,1]$) which may be denoted as $([a,b])_5$ and

$$(\bar{d}_{ij}) = \begin{pmatrix} [1,1] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [1,1] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [1,1] \end{pmatrix}$$

is a unit or identity $IVFS$-matrix which may be simply denoted as $\bar{I}_5$.

Definition 3.9 (Triangular $IVFS$-Matrix) A square $IVFS$-matrix $\bar{a}_{ij}$ of order $n \times n$ is said to be an **upper triangular $IVFS$-matrix** if all the elements below the leading diagonal are $[0,0]$, ie., $a_{ij} = [0,0]$ if $i > j$. A square $IVFS$-matrix $\bar{a}_{ij}$ of order $n \times n$ is said to be an **lower triangular $IVFS$-matrix** if all the elements above the leading diagonal are $[0,0]$, ie., $a_{ij} = [0,0]$ if $i < j$.

Example 3.9. Suppose the initial universe set $U = \{d_1, d_2, d_3, d_4, d_5\}$ and the parameter set $E = \{e_1, e_2, e_3, e_4, e_5\}$. Let $\bar{F} : E \longrightarrow P(U)$ s.t,

$$\bar{F}(e_1) = \{d_1/[0.8,1], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0,0]\},$$
$$\bar{F}(e_2) = \{d_1/[0.2,0.5], d_2/[0.3,0.5], d_3/[0,0], d_4/[0,0], d_5/[0,0]\},$$
$$\bar{F}(e_3) = \{d_1/[0.4,0.6], d_2/[0.9,1], d_3/[0.8,1], d_4/[0,0], d_5/[0,0]\},$$
$$\bar{F}(e_4) = \{d_1/[0.1,0.3], d_2/[0.7,0.9], d_3/[0.2,0.4], d_4/[0.7,0.9], d_5/[0,0]\},$$
$$\bar{F}(e_5) = \{d_1/[0.5,0.7], d_2/[0.1,0.4], d_3/[0.6,0.8], d_4/[0.3,0.5], d_5/[0.6,0.8]\}.$$

Then the interval-valued fuzzy soft set

$$\begin{aligned} (\bar{F}, E) \quad = \quad & \{(e_1, \{d_1/[0.8,1], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}), \\ & (e_2, \{d_1/[0.2,0.5], d_2/[0.3,0.5], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}), \\ & (e_3, \{d_1/[0.4,0.6], d_2/[0.9,1], d_3/[0.8,1], d_4/[0,0], d_5/[0,0]\}), \\ & (e_4, \{d_1/[0.1,0.3], d_2/[0.7,0.9], d_3/[0.2,0.4], d_4/[0.7,0.9], d_5/[0,0]\}), \\ & (e_5, \{d_1/[0.5,0.7], d_2/[0.1,0.4], d_3/[0.6,0.8], d_4/[0.3,0.5], d_5/[0.6,0.8]\})\} \end{aligned}$$

and so the relation form of $(\bar{F}, E)$ is,

$$\begin{aligned} R_E \quad = \quad & \{(\{(\{d_1/[0.8,1], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}, e_1), \\ & (\{d_1/[0.2,0.5], d_2/[0.3,0.5], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}, e_2), \\ & (\{d_1/[0.4,0.6], d_2/[0.9,1], d_3/[0.8,1], d_4/[0,0], d_5/[0,0]\}, e_3), \\ & (\{d_1/[0.1,0.3], d_2/[0.7,0.9], d_3/[0.2,0.4], d_4/[0.7,0.9], d_5/[0,0]\}, e_4), \\ & (\{d_1/[0.5,0.7], d_2/[0.1,0.4], d_3/[0.6,0.8], d_4/[0.3,0.5], d_5/[0.6,0.8]\}, e_5)\} \end{aligned}$$

Therefore the $IVFS$-matrix $(\bar{a}_{ij})$ is written by,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0.8,1] & [0.2,0.5] & [0.4,0.6] & [0.1,0.3] & [0.5,0.7] \\ [0,0] & [0.3,0.5] & [0.9,1] & [0.7,0.9] & [0.1,0.4] \\ [0,0] & [0,0] & [0.8,1] & [0.2,0.4] & [0.6,0.8] \\ [0,0] & [0,0] & [0,0] & [0.7,0.9] & [0.3,0.5] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0.6,0.8] \end{pmatrix}$$

Here $a_{ij} = [0,0]$ if $i > j$, hence $(\bar{a}_{ij})$ is an upper triangular $IVFS$-matrix. Now let $\bar{G} : E \longrightarrow P(U)$ s.t,

$$\begin{aligned}(\bar{G}, E) \quad = \quad & \{(e_1, \{d_1/[0.8,1], d_2/[0.2,0.5], d_3/[0.4,0.6], d_4/[0.1,0.3], d_5/[0.5,0.7]\}),\\ & (e_2, \{d_1/[0,0], d_2/[0.3,0.5], d_3/[0.8,1], d_4/[0.6,0.8], d_5/[0.4,0.6]\}),\\ & (e_3, \{d_1/[0,0], d_2/[0,0], d_3/[0.8,0.9], d_4/[0.2,0.5], d_5/[0.7,0.9]\}),\\ & (e_4, \{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0.6,0.9], d_5/[0.2,0.5]\}),\\ & (e_5, \{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0.6,0.8]\})\}\end{aligned}$$

Then similarly the associated $IVFS$-matrix will be,

$$(\bar{b}_{ij}) = \begin{pmatrix} [0.8,1] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0.2,0.5] & [0.3,0.5] & [0,0] & [0,0] & [0,0] \\ [0.4,0.6] & [0.8,1] & [0.8,0.9] & [0,0] & [0,0] \\ [0.1,0.3] & [0.6,0.8] & [0.2,0.5] & [0.6,0.9] & [0,0] \\ [0.5,0.7] & [0.4,0.6] & [0.7,0.9] & [0.2,0.5] & [0.6,0.8] \end{pmatrix}$$

Here $b_{ij} = [0,0]$ if $i < j$, therefore $(\bar{b}_{ij})$ is a lower triangular $IVFS$-matrix.

Definition 3.10 (Equality of $IVFS$-Matrices) Let A and B be two $IVFS$-matrices under the same universe U and set of parameters E. Now A and B are said to be **conformable for equality**, if they be of the same order. Now the $IVFS$-matrices A and B with same order are said to be **equal**, if and only if the corresponding elements of A and B be equal.

Definition 3.11 (Transpose of a square $IVFS$-Matrix) The **transpose** of a square $IVFS$-matrix A of order $n \times n$ is another square $IVFS$-matrix of the same order obtained from A by interchanging its rows and columns. It is denoted by A^T. Now if $A = (\bar{a}_{ij})_{n\times n}$, then its transpose A^T is defined by $A^T = (\bar{b}_{ij})_{n\times n}$, where $b_{ij} = a_{ji}$. Therefore the interval-valued fuzzy soft set associated with A^T becomes a new interval-valued fuzzy soft set over the same universe and over the same set of parameters.

Note: Transpose of a non-square $IVFS$-Matrix cannot be defined, as it does not carry any physical meaning.

Example 3.10. Consider the example 3.1. Here $(\bar{F}_A, E)$ be an interval-valued fuzzy soft set over the universe U and over the set of parameters E, given by,

$$\begin{aligned}(\bar{F}_A, E) \quad = \quad & \{\text{highly polluted city } = \{C_1/[.2,.4], C_2/[.8,.9], C_3/[.4,.5], C_4/[.6,.7], C_5/[.6,.8]\},\\ & \text{immensely polluted city} = \{C_1/[0,.1], C_2/[.9,1], C_3/[.3,.4], C_4/[.4,.6], C_5/[.6,.7]\},\\ & \text{moderately polluted city} = \{C_1/[.3,.5], C_2/[.4,.5], C_3/[.8,1], C_4/[.1,.2], C_5/[.3,.5]\},\\ & \text{less polluted city} = \{C_1/[.9,1], C_2/[.1,.2], C_3/[.5,.7], C_4/[.3,.5], C_5/[.1,.2]\}\}\end{aligned}$$

whose associated $IVFS$-matrix is,

$$(\bar{a}_{ij}) = \begin{pmatrix} [\,.2,\,.4\,] & [\,0,\,.1\,] & [\,.3,\,.5\,] & [0,0] & [\,.9,\,1\,] \\ [\,.8,\,.9\,] & [\,.9,\,1\,] & [\,.4,\,.5\,] & [0,0] & [\,.1,\,.2] \\ [\,.4,\,.5] & [\,.3,\,.4] & [\,.8,\,1\,] & [0,0] & [\,.5,\,.7\,] \\ [\,.6,\,.7\,] & [\,.4,\,.6\,] & [\,.1,\,.2\,] & [0,0] & [\,.3,\,.5\,] \\ [.6,\,.8\,] & [.6,\,.7] & [.3,\,.5] & [0,0] & [.1,\,.2] \end{pmatrix}$$

Now its transpose $IVFS$-matrix is,

$$(\bar{a}_{ij})^T = \begin{pmatrix} [\ .2,\ .4\ ] & [\ .8,\ .9\ ] & [\ .4,\ .5\ ] & [.6,.7] & [\ .6,\ .8\ ] \\ [\ 0,\ .1\ ] & [\ .9,\ 1\ ] & [\ .3,\ .4\ ] & [.4,.6] & [\ .6,\ .7] \\ [\ .3,\ .5] & [\ .4,\ .5] & [\ .8,\ 1\ ] & [.1,.2] & [\ .3,\ .5\ ] \\ [\ 0,\ 0\ ] & [\ 0,\ 0\ ] & [\ 0,\ 0\ ] & [0,0] & [\ 0,\ 0\ ] \\ [.9,\ 1\ ] & [.1,\ .2] & [.5,\ .7] & [.3,.5] & [.1,\ .2] \end{pmatrix}$$

Therefore the interval-valued fuzzy soft set associated with $(\bar{a}_{ij})^T$ is,

$$\begin{aligned} (\bar{G}_B, E) \;=\; & \{\text{highly polluted city } = \{C_1/[.2, .4], C_2/[0, .1], C_3/[.3, .5], C_4/[0, 0], C_5/[.9, 1]\}, \\ & \text{immensely polluted city} = \{C_1/[.8, .9], C_2/[.9, 1], C_3/[.4, .5], C_4/[0, 0], C_5/[.1, .2]\}, \\ & \text{moderately polluted city} = \{C_1/[.4, .5], C_2/[.3, .4], C_3/[.8, 1], C_4/[0, 0], C_5/[.3, .5]\}, \\ & \text{average polluted city} = \{C_1/[.6, .7], C_2/[.4, .6], C_3/[.1, .2], C_4/[0, 0], C_5/[.3, .5]\}, \\ & \text{less polluted city} = \{C_1/[.6, .8], C_2/[.6, .7], C_3/[.3, .5], C_4/[0, 0], C_5/[.1, .2]\}\} \end{aligned}$$

Definition 3.12 (Choice Matrix) It is a square matrix whose rows and columns both indicate parameters. If ξ is a choice matrix, then its element $\xi(i, j)$ is defined as follows:

$$\xi(i,j) = \begin{cases} [1,1] \text{ when } i^{th} \text{ and } j^{th} \text{ parameters are both } \textbf{choice} \text{ parameters of the decision makers} \\ [0,0] \text{ when atleast one of the } i^{th} \text{ or } j^{th} \text{ parameters be } \textbf{not under choice} \text{ of the decision maker} \end{cases}$$

Any Greek letter may be used to denote a choice matrix. There are different types of choice matrices according to the number of decision makers. We may realize this by the following example.

Example 3.11. Suppose that U be a set of four factories, say, $U = \{f_1, f_2, f_3, f_4\}$ Let E be a set of parameters, given by, $E = \{$ costly, excellent work culture, assured production, good location, cheap$\} = \{e_1, e_2, e_3, e_4, e_5\}$(say). Now let the interval-valued fuzzy soft set $(\bar{F}, A)$ describing "the quality of the factories", is given by,

$$\begin{aligned} (\bar{F}, E) \;=\; & \{ \text{ costly factories } = \{f_1/[0.9, 1], f_2/[0.2, 0.5], f_3/[0.4, 0.5], f_4/[0.8, 1]\}, \\ & \text{factories with excellent work culture } = \{f_1/[0.8, 1], f_2/[0.3, 0.5], f_3/[0.5, 0.7], f_4/[0.4, 0.5]\}, \\ & \text{factories with assured production } = \{f_1/[0.9, 1], f_2/[0.2, 0.5], f_3/[0.4, 0.6], f_4/[0.8, 1]\}, \\ & \text{factories with good location} = \{f_1/[0.7, 0.9], f_2/[0.9, 1], f_3/[0.4, 0.7], f_4/[0.8, 0.9]\}, \\ & \text{cheap factories} = \{f_1/[0.1, 0.4], f_2/[0.7, 0.9], f_3/[0.5, 0.7], f_4/[0.2, 0.5]\}\} \end{aligned}$$

Suppose Mr.X wants to buy a factory on the basis of his choice parameters excellent work culture, assured production and cheap which form a subset P of the parameter set E. Therefore $P = \{e_2, e_3, e_5\}$. Now **the choice matrix of Mr.X** is,

$$(\xi_{ij})_P = e_P \begin{pmatrix} & & e_P & & \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [1,1] \end{pmatrix}$$

Now suppose Mr.X and Mr.Y together wants to buy a factory according to their choice parameters. Let the choice parameter set of Mr.Y be, $Q = \{e_1, e_2, e_3, e_4\}$. Then **the combined choice matrix of Mr.X and Mr.Y** is

$$(\xi_{ij})_{(P,Q)} = e_P \begin{pmatrix} & & e_Q & & \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [1,1] & [1,1] & [1,1] & [1,1] & [0,0] \\ [1,1] & [1,1] & [1,1] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [1,1] & [1,1] & [1,1] & [1,1] & [0,0] \end{pmatrix}$$

Here the entries $e_{ij} = [1,1]$ indicates that e_i is a choice parameter of Mr.X and e_j is a choice parameter of Mr.Y. Now $e_{ij} = [0,0]$ indicates either e_i fails to be a choice parameter of Mr.X or e_j fails to be a choice parameter of Mr.Y.] Again the above combined choice matrix of Mr.X and Mr.Y may be also presented in its transpose form as,

$$(\xi_{ij})_{(Q,P)} = e_Q \begin{pmatrix} & & e_P & & \\ [0,0] & [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix}$$

Now let us see the form of the combined choice matrix associated with three decision makers. Suppose that Mr.Z is willing to buy a factory together with Mr.X and Mr.Y on the basis of his choice parameters excellent work culture, assured production and good location which form a subset R of the parameter set E. Therefore $R = \{e_2, e_3, e_4\}$. Then **the combined choice matrix of Mr.X, Mr.Y and Mr.Z** will be of three different types which are as follows,

$$\text{i) } (\xi_{ij})_{(R,P\wedge Q)} = e_R \begin{pmatrix} & & e_{(P\wedge Q)} & & \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix}$$

Since the set of common choice parameters of Mr.X and Mr.Y is, $P \wedge Q = \{e_2, e_3\}$. Here the entries $e_{ij} = [1,1]$ indicates that e_i is a choice parameter of Mr.Z and e_j is a common choice parameter of Mr.X and Mr.Y. Now $e_{ij} = [0,0]$ indicates either e_i fails to be a choice parameter of Mr.Z or e_j fails to be a common choice parameter of Mr.X and Mr.Y.

$$\text{ii) } (\xi_{ij})_{(P,Q\wedge R)} = e_P \begin{pmatrix} & & e_{(Q\wedge R)} & & \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [1,1] & [0,0] \\ [0,0] & [1,1] & [1,1] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [1,1] & [0,0] \end{pmatrix} \text{[Since } Q \wedge R = \{e_2, e_3, e_4\}]$$

$$\text{iii) } (\xi_{ij})_{(Q,R\wedge P)} = e_Q \begin{pmatrix} & & e_{(R\wedge P)} & & \\ [0,0] & [1,1] & [1,1] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix} [\text{ Since } R \wedge P = \{e_2, e_3\}]$$

Definition 3.13 (Symmetric $IVFS$-Matrix A square $IVFS$-matrix A of order $n \times n$ is said to be a **symmetric** $IVFS$**-matrix**, if its transpose be equal to it, i.e., if $A^T = A$. Hence the $IVFS$-matrix $(\bar{a}_{ij})$ is symmetric, if $a_{ij} = a_{ji}, \forall i, j$. Therefore if $(\bar{a}_{ij})$ be a symmetric $IVFS$-matrix then the interval-valued fuzzy soft sets associated with $(\bar{a}_{ij})$ and $(\bar{a}_{ij})^T$ both be the same.

Example 3.12. Let the set of universe $U = \{u_1, u_2, u_3, u_4\}$ and the set of parameters $E = \{e_1, e_2, e_3, e_4\}$. Now suppose that, $A \subseteq E$ and $\bar{F}_A : E \longrightarrow P(U)$ s.t, $(\bar{F}_A, E)$ forms an interval-valued fuzzy soft set given by,

$$\begin{aligned} (\bar{F}_A, E) \quad = \quad & \{(e_1, \{u_1/[0.2, 0.4], u_2/[0.3, 0.5], u_3/[0.8, 1], u_4/[0.5, 0.7]\}), \\ & (e_2, \{u_1/[0.3, 0.5], u_2/[0.6, 0.8], u_3/[0.1, 0.3], u_4/[0.7, 1]\}), \\ & (e_3, \{u_1/[0.8, 1], u_2/[0.1, 0.3], u_3/[0.7, 0.9], u_4/[0.2, 0.5]\}), \\ & (e_4, \{u_1/[0.5, 0.7], u_2/[0.7, 1], u_3/[0.2, 0.5], u_4/[0.4, 0.6]\})\} \end{aligned}$$

The $IVFS$-matrix associated with this interval-valued fuzzy soft set $(\bar{F}_A, E)$ is,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0.2,0.4] & [0.3,0.5] & [0.8,1] & [0.5,0.7] \\ [0.3,0.5] & [0.6,0.8] & [0.1,0.3] & [0.7,1] \\ [0.8,1] & [0.1,0.3] & [0.7,0.9] & [0.2,0.5] \\ [0.5,0.7] & [0.7,1] & [0.2,0.5] & [0.4,0.6] \end{pmatrix}.$$

Definition 3.14 (Addition of $IVFS$-Matrices) Two $IVFS$-matrices A and B are said to be **conformable for addition**, if they be of the same order and after addition the obtained sum also be an $IVFS$-matrix of the same order. Now if $A = (\bar{a}_{ij})$ and $B = (\bar{b}_{ij})$ of the same order $m \times n$, then the **addition** of A and B is denoted by, $A \oplus B$ and is defined by,

$$(\bar{a}_{ij}) \oplus (\bar{b}_{ij}) \quad = \quad (\bar{c}_{ij}), \text{ where } c_{ij} = [\sup\{\mu^-_{a_{ij}}, \mu^-_{b_{ij}}\}, \sup\{\mu^+_{a_{ij}}, \mu^+_{b_{ij}}\}] \forall i, j.$$

Example 3.13. Consider the $IVFS$-matrix of example 3.1,

$$(\bar{a}_{ij}) = \begin{pmatrix} [.2, .4] & [0, .1] & [.3, .5] & [0,0] & [.9, 1] \\ [.8, .9] & [.9, 1] & [.4, .5] & [0,0] & [.1, .2] \\ [.4, .5] & [.3, .4] & [.8, 1] & [0,0] & [.5, .7] \\ [.6, .7] & [.4, .6] & [.1, .2] & [0,0] & [.3, .5] \\ [.6, .8] & [.6, .7] & [.3, .5] & [0,0] & [.1, .2] \end{pmatrix}$$

Now consider another $IVFS$-matrix $(\bar{b}_{ij})$ associated with the interval-valued fuzzy soft set $(\bar{G}_B, E)$. (also describing "the pollution of the cities") over the same universe U. Let $B = \{e_1, e_4, e_5\} \subset E$ and

$$(\bar{G}, B) \quad = \quad \{\text{highly polluted city } = \{C_1/[.3, .5], C_2/[.9, 1], C_3/[.4, .5], C_4/[.7, .9], C_5/[.6, .9]\},$$

average polluted city $= \{C_1/[.2,.5], C_2/[.3,.5], C_3/[.7,.9], C_4/[.2,.4], C_5/[.3,.6]\}$,
less polluted city $= \{C_1/[.8,1], C_2/[.2,.3], C_3/[.6,.8], C_4/[.3,.5], C_5/[.2,.4]\}\}$

Hence the $IVFS$-matrix $(\bar{b}_{ij})$ is written by,

$$(\bar{b}_{ij}) = \begin{pmatrix} [\,.3,\,.5\,] & [0,0] & [0,0] & [.2,.5] & [\,.8,\,1\,] \\ [\,.9,\,1\,] & [0,0] & [0,0] & [.3,.5] & [\,.2,\,.3] \\ [\,.4,\,.5] & [0,0] & [0,0] & [.7,.9] & [\,.6,\,.8\,] \\ [\,.7,\,.9\,] & [0,0] & [0,0] & [.2,.4] & [\,.3,\,.5\,] \\ [.6,\,.9\,] & [0,0] & [0,0] & [.3,.6] & [.2,\,.4] \end{pmatrix}.$$

Therefore the sum of the $IVFS$-matrices $(\bar{a}_{ij})$ and $(\bar{b}_{ij})$ is,

$$(\bar{a}_{ij}) \oplus (\bar{b}_{ij}) = \begin{pmatrix} [\,.3,\,.5\,] & [\,0,\,.1\,] & [\,.3,\,.5\,] & [.2,.5] & [\,.9,\,1\,] \\ [\,.9,\,1\,] & [\,.9,\,1\,] & [\,.4,\,.5\,] & [.3,.5] & [\,.2,\,.3] \\ [\,.4,\,.5] & [\,.3,\,.4] & [\,.8,\,1\,] & [.7,.9] & [\,.6,\,.8\,] \\ [\,.7,\,.9\,] & [\,.4,\,.6\,] & [\,.1,\,.2\,] & [.2,.4] & [\,.3,\,.5\,] \\ [.6,\,.9\,] & [.6,\,.7] & [.3,\,.5] & [.3,.6] & [.2,\,.4] \end{pmatrix}.$$

Definition 3.15 (Subtraction of $IVFS$-Matrices) Two $IVFS$-matrices A and B are said to be **conformable for subtraction**, if they be of the same order and after subtraction the obtained result also be an $IVFS$-matrix of the same order. Now if $A = (\bar{a}_{ij})$ and $B = (\bar{b}_{ij})$ of order $m \times n$, then **subtraction** of B from A is denoted by, $A \ominus B$ and is defined by,

$$(\bar{a}_{ij}) \ominus (\bar{b}_{ij}) \quad = \quad (\bar{c}_{ij}), \text{ where } c_{ij} = [\inf\{\mu^-_{a_{ij}}, \mu^-_{b^o_{ij}}\}, \inf\{\mu^+_{a_{ij}}, \mu^+_{b^o_{ij}}\}] \forall i, j$$

where $(\bar{b}^o_{ij})$ is the complement of $(\bar{b}_{ij})$

Example 3.14. Consider the $IVFS$-matrices $(\bar{a}_{ij})$ and $(\bar{b}_{ij})$ of example 3.13. Now

$$(\bar{a}_{ij}) = \begin{pmatrix} [\,.2,\,.4\,] & [\,0,\,.1\,] & [\,.3,\,.5\,] & [0,0] & [\,.9,\,1\,] \\ [\,.8,\,.9\,] & [\,.9,\,1\,] & [\,.4,\,.5\,] & [0,0] & [\,.1,\,.2] \\ [\,.4,\,.5] & [\,.3,\,.4] & [\,.8,\,1\,] & [0,0] & [\,.5,\,.7\,] \\ [\,.6,\,.7\,] & [\,.4,\,.6\,] & [\,.1,\,.2\,] & [0,0] & [\,.3,\,.5\,] \\ [.6,\,.8\,] & [.6,\,.7] & [.3,\,.5] & [0,0] & [.1,\,.2] \end{pmatrix}$$

and

$$(\bar{b}_{ij})^o = \begin{pmatrix} [\,.5,\,.7\,] & [1,1] & [1,1] & [.5,.8] & [\,0,\,.2\,] \\ [\,0,\,.1\,] & [1,1] & [1,1] & [.5,.7] & [\,.7,\,.8] \\ [\,.5,\,.6] & [1,1] & [1,1] & [.1,.3] & [\,.2,\,.4\,] \\ [\,.1,\,.3\,] & [1,1] & [1,1] & [.6,.8] & [\,.5,\,.7\,] \\ [.1,\,.4\,] & [1,1] & [1,1] & [.4,.7] & [.6,\,.8] \end{pmatrix}.$$

Therefore the subtraction of the $IVFS$-matrix $(\bar{b}_{ij})$ from the $IVFS$-matrix $(\bar{a}_{ij})$ is,

$$(\bar{a}_{ij}) \ominus (\bar{b}_{ij}) = \begin{pmatrix} [\ .2,\ .4\ ] & [\ 0,\ .1\ ] & [\ .3,\ .5\ ] & [0,0] & [\ 0,\ .2\ ] \\ [\ 0,\ .1\ ] & [\ .9,\ 1\ ] & [\ .4,\ .5\ ] & [0,0] & [\ .1,\ .2] \\ [\ .4,\ .5] & [\ .3,\ .4] & [\ .8,\ 1\ ] & [0,0] & [\ .2,\ .4\ ] \\ [\ .1,\ .3\ ] & [\ .4,\ .6\ ] & [\ .1,\ .2\ ] & [0,0] & [\ .3,\ .5\ ] \\ [.1,\ .4\ ] & [.6,\ .7] & [.3,\ .5] & [0,0] & [.1,\ .2] \end{pmatrix}.$$

Properties. *Let A and B be two $IVFS$-matrices of order $m \times n$. Then*

i) $A \oplus B = B \oplus A$

ii) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

iii) $A \ominus B \neq B \ominus A$

iv) $(A \ominus B) \ominus C \neq A \ominus (B \ominus C)$

v) $A \oplus A^o \neq \bar{C}_A$

vi) $A \ominus A \neq \bar{\Phi}$.

Proof: The proofs of (i)-(vi) are directly obtained from the definitions of addition, subtraction and complement. □

Theorem 1. *If A be an $IVFS$-square matrix of order $n \times n$, then $(A^T)^T = A$.*

Proof: Let $A = (\bar{a}_{ij})_{n\times n}$. Then by definition, $A^T = (\bar{b}_{ij})_{n\times n}$ where $b_{ij} = a_{ji} \forall i, j$. ie., $A^T = (\bar{a}_{ji})_{n\times n}$. Therefore $(A^T)^T = (\bar{c}_{ij})_{n\times n}$ where $c_{ij} = a_{ij}$ ie., $(A^T)^T = (\bar{a}_{ij})_{n\times n} = A$.

Theorem 2. *If A and B be two $IVFS$-square matrices of order $n \times n$, then $(A \oplus B)^T = A^T \oplus B^T$.*

Proof: Let $A = (\bar{a}_{ij})_{n\times n}$ and $B = (\bar{b}_{ij})_{n\times n}$. Then

$$\begin{aligned} L.H.S &= (A \oplus B)^T = C^T \text{ where } C = (\bar{c}_{ij})_{n\times n} \\ &= (\bar{c}_{ji})_{n\times n} \text{ where } c_{ji} = [\sup\{\mu^-_{a_{ji}}, \mu^-_{b_{ji}}\}, \sup\{\mu^+_{a_{ji}}, \mu^+_{b_{ji}}\}] \forall i, j \end{aligned}$$

and

$$\begin{aligned} R.H.S &= A^T \oplus B^T = (\bar{a}_{ji})_{n\times n} \oplus (\bar{b}_{ji})_{n\times n} \\ &= (\bar{d}_{ji})_{n\times n} \text{ where } d_{ji} = [\sup\{\mu^-_{a_{ji}}, \mu^-_{b_{ji}}\}, \sup\{\mu^+_{a_{ji}}, \mu^+_{b_{ji}}\}] \forall i, j \\ &= C^T = L.H.S \end{aligned}$$

Hence $(A \oplus B)^T = A^T \oplus B^T$. □

Theorem 3. *If A be an $IVFS$-square matrix of order $n \times n$, then $(A \oplus A^T)$ is symmetric.*

Proof: Let $A = (\bar{a}_{ij})_{n\times n}$. Then by definition, $A^T = (\bar{a}_{ji})_{n\times n}$. Now

$$\begin{aligned} A \oplus A^T &= (\bar{a}_{ij})_{n\times n} \oplus (\bar{a}_{ji})_{n\times n} \\ &= (\bar{c}_{ij})_{n\times n} \text{ where } c_{ij} = [\sup\{\mu^-_{a_{ij}}, \mu^-_{a_{ji}}\}, \sup\{\mu^+_{a_{ij}}, \mu^+_{a_{ji}}\}] \forall i, j. \end{aligned}$$

Now $c_{ji} = [\sup\{\mu^-_{a_{ji}}, \mu^-_{a_{ij}}\}, \sup\{\mu^+_{a_{ji}}, \mu^+_{a_{ij}}\}] = c_{ij} \forall i, j$. Therefore $(\bar{c}_{ij})_{n\times n}$ ie., $(A \oplus A^T)$ is symmetric. □

Theorem 4. *If A and B be two $IVFS$-square matrices of order $n \times n$ and if A and B be symmetric, then $A \oplus B$ is symmetric.*

Proof: Since A and B be symmetric, $A^T = A$ and $B^T = B$. Therefore $A^T \oplus B^T = A \oplus B$. Thus from theorem-2 we have, $(A \oplus B)^T = A^T \oplus B^T = A \oplus B$. Hence $A \oplus B$ is symmetric. □

Definition 3.16 (Product of an $IVFS$-Matrix with a Choice Matrix) Let U be the set of universe and E be the set of parameters. Suppose that A be an $IVFS$-matrix and β be the choice matrix over (U, E). The product of an $IVFS$-matrix A with a choice matrix β is denoted by $A \otimes \beta$. Now A and β are said to be conformable for product $A \otimes \beta$ when the number of columns of A be equal to the number of rows of β and the product $A \otimes \beta$ becomes also another $IVFS$-matrix. If $A = (\bar{a}_{ij})_{m\times n}$ and $\beta = (\bar{\beta}_{jk})_{n\times p}$, then the product $A \otimes \beta$ is defined as

$$A \otimes \beta \quad = \quad (\bar{c}_{ik}) \text{ where } c_{ik} = [\sup_{j=1}^{n} \inf\{\mu^-_{a_{ij}}, \mu^-_{\beta_{jk}}\}, \sup_{j=1}^{n} \inf\{\mu^+_{a_{ij}}, \mu^+_{\beta_{jk}}\}]$$

Properties. *i) $\beta \otimes A$ cannot be defined here. ii) If $A \otimes \beta = \bar{\Phi}$, then we cannot say, as in scalar algebra, that either A is a null $IVFS$-matrix or, β is a null matrix.*

Example 3.15. Let U be the set of four dresses, given by, U = $\{d_1, d_2, d_3, d_4\}$. Let E be the set of parameters, given by, E = { cheap, beautiful, comfortable, gorgeous } = $\{e_1, e_2, e_3, e_4\}$(say). Suppose that the set of choice parameters of Mr.X be, $A = \{e_1, e_3\}$. Now let according to the choice parameters of Mr.X, we have the interval-valued fuzzy soft set $(\bar{F}, A)$ which describes "the attractiveness of the dresses" and the $IVFS$-matrix of the interval-valued fuzzy soft set $(\bar{F}, A)$ be,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0.8,1] & [0.2,0.3] & [0.7,0.9] & [0.3,0.5] \\ [0.3,0.4] & [0.7,0.9] & [0.4,0.6] & [0.8,1] \\ [0.7,1] & [0.4,0.5] & [0.5,0.8] & [0.6,0.7] \\ [0.5,0.7] & [0.1,0.3] & [0.9,1] & [0.2,0.5] \end{pmatrix}$$

Again the choice matrix of Mr.X is,

$$\text{`}\ (\xi_{ij})_A = e_A \begin{pmatrix} & e_A & & \\ [1,1] & [0,0] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \\ [1,1] & [0,0] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix}$$

Since the number of columns of the $IVFS$-matrix $(\bar{a}_{ij})$ is equal to the number of rows of the choice matrix $(\xi_{ij})_A$, they are conformable for the product. Therefore

$$U \begin{pmatrix} & e_A & & \\ [0.8,1] & [0.2,0.3] & [0.7,0.9] & [0.3,0.5] \\ [0.3,0.4] & [0.7,0.9] & [0.4,0.6] & [0.8,1] \\ [0.7,1] & [0.4,0.5] & [0.5,0.8] & [0.6,0.7] \\ [0.5,0.7] & [0.1,0.3] & [0.9,1] & [0.2,0.5] \end{pmatrix} \otimes e_A \begin{pmatrix} & e_A & & \\ [1,1] & [0,0] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \\ [1,1] & [0,0] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix}$$

$$= U \begin{pmatrix} & E & & \\ [0.8,1] & [0,0] & [0.8,1] & [0,0] \\ [0.4,0.6] & [0,0] & [0.4,0.6] & [0,0] \\ [0.7,1] & [0,0] & [0.7,1] & [0,0] \\ [0.9,1] & [0,0] & [0.9,1] & [0,0] \end{pmatrix}.$$

4 Group Decisions

Group Decision is the academic and professional field that aims to improve collective decision process by taking into account the needs and opinions of every group members. Group decision includes the development and study of methods for assisting groups or individuals within groups, as they interact and collaborate to reach a collective decision. But till now there does not exist any method to solve interval-valued fuzzy soft set(IVFSs) based group decision making problem. In the following subsection at first a general IVFSs based decision making problem is defined, then a new approach is developed to solve such types of problems.

A Generalized Interval-Valued Fuzzy Soft Set Based Group Decision Making Problem: Let N number of decision makers want to select an object jointly from the m number of objects which have n number of features ie., parameters(E). Suppose that each decision maker has freedom to take his decision of inclusion and evaluation of parameters associated with the selected object, ie., each decision maker has his own choice parameters belonging to the parameter set E and has his own view of evaluation. Here it is assumed that parameter evaluation of the objects must be interval-valued fuzzy. Now the problem is to find out the object out of these m objects which satisfies all the choice parameters of all decision makers jointly as much as possible.

A New Approach to Solve $IVFS$-Matrix Based Group Decision Making Problems: This new approach is specially based on choice matrices and its operations. The stepwise procedure of this new approach is presented in the following $IVFSM$-algorithm.

$IVFSM$-**Algorithm**:

Step-I: First construct the combined choice matrices with respect to the choice parameters of the decision makers.

Step-II: Compute the product $IVFS$-matrices by multiplying each given $IVFS$-matrix with the combined choice matrix as per the rule of multiplication of $IVFS$-matrices.

Step-III: Compute the sum of these product $IVFS$-matrices to have the resultant $IVFS$-matrix($\bar{R}_{IVFS}$).

Step-IV: Then compute the weight of each object(O_i) by adding the membership values of the entries of its concerned row(i-th row) of $\bar{R}_{IVFS}$ and denote it as $W(O_i)$.

Step-V: $\forall O_i \in U$, compute the score r_i of O_i such that,

$$r_i = \sum_{O_j \in U} ((\mu_i^- - \mu_j^-) + (\mu_i^+ - \mu_j^+)$$

Step-VI: The object having the highest score becomes the jointly selected object according to all decision makers. If more than one object have the highest score then any one of these highest scorers may be chosen as the jointly selected object.

To illustrate the basic idea of the $IVFSM$-algorithm, now we apply it to the following $IVFS$-matrix based decision making problem.

Example 4.1. Let the set of universe U consist of four cities C_1, C_2, C_3, C_4 and the set of parameters E = { industrialization, greeneries, modern facilities, government pollution control policy } = $\{e_1, e_2, e_3, e_4\}$. Now two Mr.X and Mrs.X wants to live in a city among these four. Mr.X likes an industrialized city but having greeneries and Government pollution control policy. Whereas Mrs.X likes greeneries but a city with modern facilities. So the sets of choice parameters of Mr.X and Mrs.X are respectively, $A = \{e_1, e_2, e_4\} \subset E$ and $B = \{e_2, e_3\} \subset E$. Now let according to the choice parameters of Mr.X and Mrs.X, we have the interval-valued fuzzy soft sets $(\bar{F}_A, E)$ and $(\bar{G}_B, E)$, both describing "the attractiveness of the cities "according to Mr.X and Mrs.X respectively. Let the $IVFS$-matrices of the interval-valued fuzzy soft sets $(\bar{F}_A, E)$ and $(\bar{G}_B, E)$ are respectively,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0.8,1] & [0.1,0.3] & [0,0] & [0.3,0.5] \\ [0,0.2] & [0.7,0.9] & [0,0] & [0.6,0.7] \\ [0.6,0.9] & [0.3,0.4] & [0,0] & [0.4,0.6] \\ [0.7,0.8] & [0.1,0.2] & [0,0] & [0.2,0.3] \end{pmatrix}, (\bar{b}_{ik}) = \begin{pmatrix} [0,0] & [0.2,0.3] & [0.5,0.7] & [0,0] \\ [0,0] & [0.7,0.9] & [0.3,0.6] & [0,0] \\ [0,0] & [0.3,0.5] & [0.5,0.6] & [0,0] \\ [0,0] & [0.2,0.3] & [0.6,0.8] & [0,0] \end{pmatrix}$$

Now the problem is to select the city among the four cities which satisfies the choice parameters of Mr.X and Mrs.X as much as possible.

1) The combined choice matrix of Mr.X and Mrs.X is,

$$e_B \begin{pmatrix} & & e_A & \\ [0,0] & [0,0] & [0,0] & [0,0] \\ [1,1] & [1,1] & [0,0] & [1,1] \\ [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix}$$

or it may be presented in its transposed form as,

$$e_A \begin{pmatrix} & & e_B & \\ [0,0] & [1,1] & [1,1] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] \end{pmatrix}$$

2) Corresponding product $IVFS$-matrices are,

$$U \begin{pmatrix} & & e_A & \\ [0.8,1] & [0.1,0.3] & [0,0] & [0.3,0.5] \\ [0,0.2] & [0.7,0.9] & [0,0] & [0.6,0.7] \\ [0.6,0.9] & [0.3,0.4] & [0,0] & [0.4,0.6] \\ [0.7,0.8] & [0.1,0.2] & [0,0] & [0.2,0.3] \end{pmatrix} \otimes e_A \begin{pmatrix} & & e_B & \\ [0,0] & [1,1] & [1,1] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] \end{pmatrix}$$

$$= U \begin{pmatrix} & & E & \\ [0,0] & [0.8,1] & [0.8,1] & [0,0] \\ [0,0] & [0.7,0.9] & [0.7,0.9] & [0,0] \\ [0,0] & [0.6,0.9] & [0.6,0.9] & [0,0] \\ [0,0] & [0.7,0.8] & [0.7,0.8] & [0,0] \end{pmatrix}$$

$$U \begin{pmatrix} & & e_B & \\ [0,0] & [0.2,0.3] & [0.5,0.7] & [0,0] \\ [0,0] & [0.7,0.9] & [0.3,0.6] & [0,0] \\ [0,0] & [0.3,0.5] & [0.5,0.6] & [0,0] \\ [0,0] & [0.2,0.3] & [0.6,0.8] & [0,0] \end{pmatrix} \otimes e_B \begin{pmatrix} & & e_A & \\ [0,0] & [0,0] & [0,0] & [0,0] \\ [1,1] & [1,1] & [0,0] & [1,1] \\ [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix}$$

$$= U \begin{pmatrix} & & E & \\ [0,0] & [0.4,0.7] & [0.4,0.7] & [0,0] \\ [0,0] & [0.7,0.9] & [0.7,0.9] & [0,0] \\ [0,0] & [0.5,0.6] & [0.5,0.6] & [0,0] \\ [0,0] & [0.6,0.8] & [0.6,0.8] & [0,0] \end{pmatrix}$$

3) The sum of these product $IVFS$-matrices is,

$$\begin{pmatrix} [0,0] & [0.8,1] & [0.8,1] & [0,0] \\ [0,0] & [0.7,0.9] & [0.7,0.9] & [0,0] \\ [0,0] & [0.6,0.9] & [0.6,0.9] & [0,0] \\ [0,0] & [0.7,0.8] & [0.7,0.8] & [0,0] \end{pmatrix} \oplus \begin{pmatrix} [0,0] & [0.4,0.7] & [0.4,0.7] & [0,0] \\ [0,0] & [0.7,0.9] & [0.7,0.9] & [0,0] \\ [0,0] & [0.5,0.6] & [0.5,0.6] & [0,0] \\ [0,0] & [0.6,0.8] & [0.6,0.8] & [0,0] \end{pmatrix}$$

$$= \begin{pmatrix} [0,0] & [0.8,1] & [0.8,1] & [0,0] \\ [0,0] & [0.7,0.9] & [0.7,0.9] & [0,0] \\ [0,0] & [0.6,0.9] & [0.6,0.9] & [0,0] \\ [0,0] & [0.7,0.8] & [0.7,0.8] & [0,0] \end{pmatrix} = R_{IVFS}$$

4) Now the weights of the cities are,

(i) $W(C_1) = [(0 + 0.8 + 0.8 + 0), (0 + 1 + 1 + 0)] = [1.6, 2]$
(ii) $W(C_2) = [(0 + 0.7 + 0.7 + 0), (0 + 0.9 + 0.9 + 0)] = [1.4, 1.8]$
(iii) $W(C_3) = [(0 + 0.6 + 0.6 + 0), (0 + 0.9 + 0.9 + 0)] = [1.2, 1.8]$
(iv) $W(C_4) = [(0 + 0.7 + 0.7 + 0), (0 + 0.8 + 0.8 + 0)] = [1.4, 1.6]$

5) Now the scores for the cities are,

(i) $r_1 = (0.2 + 0.4 + 0.2) + (0.2 + 0.2 + 0.4) = 1.6$
(ii) $r_2 = (-0.2 + 0.2 + 0) + (-0.2 + 0 + 0.2) = 0$
(iii) $r_3 = (-0.4 - 0.2 - 0.2) + (-0.2 + 0 + 0.2) = -0.8$
(iv) $r_4 = (-0.2 + 0 + 0.2) + (-0.4 - 0.2 - 0.2) = -0.8$

6) Since the score r_1 is maximum (1.6), the highest scorer city C_1 will be their selected city which fulfill the choice parameters of Mr.X and Mrs.X as much as possible.

5 Purpose of introducing $IVFSM$-Algorithm

Till now researchers [17, 19] have worked on finding solution of the interval-valued fuzzy soft set (IVFSs) based decision making problems involving only one decision maker. There does not exist any algorithm for solving an IVFSs-based group decision making problem. $IVFSM$-algorithm can solve the IVFSs-based decision making problems involving any number of decision maker.

Moreover the existing methods for solving IVFSs-based decision making problems have some drawbacks. Feng et al.[19] have pointed out the drawback of Yang's method[17] and shown that their proposed method is more efficient than it. Now by the following example we will find out the drawbacks of Feng's method and show that our proposed $IVFSM$-algorithm is more deterministic than Feng's method.

Example 5.1. Let the set of universe U consist of three dresses d_1, d_2, d_3 and the set of parameters $E = \{$ cheap, comfortable, beautiful, costly $\} = \{e_1, e_2, e_3, e_4\}$. Now two Mr.X wants to buy a dress among these three and his set of choice parameters is, $A = \{e_1, e_2, e_3\}$. Suppose that, according to the choice parameters of Mr.X, the interval-valued fuzzy soft set $(\bar{F}_A, E)$ describing "the attractiveness of the dresses "according to Mr.X is given by,

$$\begin{aligned}(\bar{F}_A, E) \quad = \quad & \{ \text{ cheap dresses} = \{d_1/[.7, .9], d_2/[.5, .7], d_3/[.6, .8]\}, \\ & \text{comfortable dresses} = \{d_1/[.6, .8], d_2/[.8, 1], d_3/[.5, .7]\}, \\ & \text{beautiful dresses} = \{d_1/[.1, .3], d_2/[.5, .7], d_3/[.7, .9]\}\}\end{aligned}$$

Now at first we solve this problem by Feng's method and then by $IVFSM$-Algorithm.

By Feng's Method:

For a neutral estimation of uncertain membership values we have the Neutral Reduct Fuzzy Soft set(NRFSs) of $(\bar{F}_A, E)$ in the tabular form as,

Table-4(a) : Tabular representation of the NRFSs of $(\bar{F}_A, E)$

	e_1	e_2	e_3
d_1	0.8	0.7	0.2
d_2	0.6	0.9	0.6
d_3	0.7	0.6	0.8

Using Top Level Soft Set:

Table-4(b)

	e_1	e_2	e_3	choice value
d_1	1	0	0	1
d_2	0	1	0	1
d_3	0	0	1	1

As the choice values of all the dresses are same, according to Feng's method, Mr.X may select any one of the three dresses d_1, d_2, d_3. Using Mid Level Soft Set:

Table-4(c)

	e_1	e_2	e_3	choice value
d_1	1	1	0	2
d_2	0	1	1	2
d_3	1	0	1	2

Here also the choice values of all the dresses be the same. Hence **according to Feng's method Mr.X may select either d_1 or, d_2 or, d_3.**

By $IVFSM$-Algorithm:

The interval-valued fuzzy soft matrix associated with $(\bar{F}_A, E)$ is,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0.7,0.9] & [0.6,0.8] & [0.1,0.3] & [0,\ 0] \\ [0.5,0.7] & [0.8,1] & [0.5,0.7] & [0,\ 0] \\ [0.6,0.8] & [0.5,0.7] & [0.7,0.9] & [0,\ 0] \end{pmatrix}$$

The choice matrix of Mr.X is,

$$e_A \begin{pmatrix} & & e_A & \\ [1,1] & [1,1] & [1,1] & [0,0] \\ [1,1] & [1,1] & [1,1] & [0,0] \\ [1,1] & [1,1] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix}$$

The product $IVFS$-matrix is,

$$U \begin{pmatrix} & & e_A & \\ [0.7,0.9] & [0.6,0.8] & [0.1,0.3] & [0,\ 0] \\ [0.5,0.7] & [0.8,1] & [0.5,0.7] & [0,\ 0] \\ [0.6,0.8] & [0.5,0.7] & [0.7,0.9] & [0,\ 0] \end{pmatrix} \otimes e_A \begin{pmatrix} & & e_A & \\ [1,1] & [1,1] & [1,1] & [0,0] \\ [1,1] & [1,1] & [1,1] & [0,0] \\ [1,1] & [1,1] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix}$$

$$= U \begin{pmatrix} & & & E \\ [0.7,0.9] & [0.7,0.9] & [0.7,0.9] & [0,\ 0] \\ [0.8,1] & [0.8,1] & [0.8,1] & [0,\ 0] \\ [0.7,0.9] & [0.7,0.9] & [0.7,0.9] & [0,\ 0] \end{pmatrix}$$

Now the weights of the dresses are,

(i) $W(d_1) = [(0.7 + 0.7 + 0.7 + 0), (0.9 + 0.9 + 0.9 + 0)] = [2.1, 2.7]$
(ii) $W(d_2) = [(0.8 + 0.8 + 0.8 + 0), (1 + 1 + 1 + 0)] = [2.4, 3]$
(iii) $W(d_3) = [(0.7 + 0.7 + 0.7 + 0), (0.9 + 0.9 + 0.9 + 0)] = [2.1, 2.7]$

Now the scores for the dresses are,
(i) $r_1 = (-0.3 + 0) + (-0.3 + 0) = -0.6$
(ii) $r_2 = (0.3 + 0.3) + (0.3 + 0.3) = 1.2$
(iii)$r_3 = (-0.3 + 0) + (-0.3 + 0) = -0.6$

Since the score r_2 is maximum (1.2), **Mr.X select the dress** d_2 which fulfill his choice parameters as much as possible. From this example we can see that $IVFSM$**-Algorithm is more deterministic than Feng's method** as it can properly solve the problem by giving an unique result. Feng's method also may give this unique result but for obtaining this we have to find out the proper level to form the level soft set by try and error method which is laborious.

6 Conclusion

In this paper first we have proposed the concept of $IVFS$-matrix and defined different types of matrices in interval-valued fuzzy soft set theory. Then we have introduced some new operations on these matrices and discussed all these definitions and operations by appropriate examples. Moreover we have proven some theorems along with few properties on these matrices. At last the new efficient $IVFSM$-

algorithm has been developed to solve interval-valued fuzzy soft set based real life group decision making problems.

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Neutrosophic Soft Matrix And It's Application in Solving Group Decision Making Problems from Medical Science

Tanushree Mitra Basu [1]
Department of Applied Mathematics with Oceanology and Computer Programming,
Vidyasagar University, Midnapore -721 102,W.B., India,
E-mail: tanushreemitra13@gmail.com

Shyamal Kumar Mondal
Department of Applied Mathematics with Oceanology and Computer Programming,
Vidyasagar University, Midnapore -721 102,W.B., India,
E-mail: shyamal_260180@yahoo.com

Abstract: The main purpose of this paper is to introduce the concept of Neutrosophic Soft Matrix(NS-Matrix). We have proposed different types of NS-Matrix along with various operations on them. A new methodology, named as NSM -Algorithm based on some of these new matrix operations, has been developed to solve neutrosophic soft set based real life group decision making problems efficiently. Finally NSM-Algorithm has been applied to solve the problems of diagnosis of a disease from the myriad of symptoms as well as to evaluate the effectiveness of different habits of human being responsible for a disease from medical science.

Keywords: Choice Matrix, Group Decision, Neutrosophic Soft Matrix(NS-Matrix), NSM -Algorithm

1 Introduction

The concept of fuzzy sets was introduced by Zadeh(1965). Since then the fuzzy sets and fuzzy logic have been applied in many real life problems in uncertain, ambiguous environment. The traditional fuzzy sets is characterized by the membership value or the grade of membership value. Some times it may be very difficult to assign the membership value for a fuzzy sets. Consequently the concept of interval valued fuzzy sets was proposed (Turksen 1986) to capture the uncertainty of grade of membership value. In some real life problems in expert system, belief system, information fusion and so on , we must consider the truth-membership as well as the falsity-membership for proper description of an object in uncertain, ambiguous environment. Neither the fuzzy sets nor the interval valued fuzzy sets is appropriate for such a situation. Intuitionistic fuzzy sets(IFS) introduced by Atanassov (1986) is appropriate for such a situation.

But the intuitionistic fuzzy sets can only handle the incomplete information considering both the truth-membership (or simply membership) and falsity-membership (or non-membership) values. It does not handle the indeterminate and inconsistent information which exists in belief system. Smarandache (2005) introduced the concept of neutrosophic set(NS) which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data.

[1] To whom all correspondence should be addressed: E-mail : tanushreemitra13@gmail.com, shyamal_260180@yahoo.com

In our regular everyday life we face situations which require procedures allowing certain flexibility in information processing capacity. Soft set theory (Molodtsov 1999; Maji et al. 2002, 2003) addressed those problems successfully. In their early work soft set was described purely as a mathematical method to model uncertainties. The researchers can pick any kind of parameters of any nature they wish in order to facilitate the decision making procedure as there is a varied way of picturing the objects. Maji et al.(2002, 2003) have done further research on soft set theory. Presence of vagueness demanded Fuzzy Soft Set (FSS) (Maji et al. 2001; Basu et al. 2012) to come into picture. But satisfactory evaluation of membership values is not always possible because of the insufficiency in the available information (besides the presence of vagueness) situation. Evaluation of non-membership values is also not always possible for the same reason and as a result there exists an indeterministic part upon which hesitation survives. Certainly fuzzy soft set theory is not suitable to solve such problems. In those situations Intuitionistic Fuzzy Soft Set theory (IFSS)(Maji et al. 2001) may be more applicable. Now in the perlance of soft set theory there is hardly any limitation to select the nature of the criteria and as most of the parameters or criteria(which are words or sentences) are neutrosophic in nature, Maji (2012, 2013) has been motivated to combine the concept of soft set and neutrosophic set to make the new mathematical model neutrosophic soft set and has given an algorithm to solve a decision making problem. But till now there does not exist any method for solving neutrosophic soft set based group decision making problem.

Group Decision is the academic and professional field that aims to improve collective decision process by taking into account the needs and opinions of every group assisting groups or individuals within groups, as they interact and collaborate to reach a collective decision.

Cagman and Enginoglu(2010) introduced a new soft set based decision making method which selects a set of optimum elements from the alternatives. In the same year, the same authors have proposed the definition of soft matrices which are representations of soft sets. This representation has several advantages. It is easy to store and manipulate matrices and hence the soft sets represented by them in a computer. Then Basu et al.(2012) have done further work on soft matrices.

Since there does not exist any method for solving neutrosophic soft set based group decision making problem and soft matrices have been shown to be very efficient to solve group decision making problems we are motivated to introduce the concept of neutrosophic soft matrix and their various operations to solve neutrosophic soft set based group decision making problems.

In this paper we have proposed the concept of neutrosophic soft matrix. Then we have defined different types of neutrosophic soft matrices with giving proper examples. Here we have also proposed the concept of choice matrix associated with a neutrosophic soft set. Moreover we have introduced some operations on neutrosophic soft matrices and choice matrices. Then based on some of these new matrix operations a new efficient solution procedure, named as NSM-Algorithm, has been developed to solve neutrosophic soft set based real life decision making problems which may contain more than one decision maker. The speciality of this new approach is that it may solve any neutrosophic soft set based decision making problem involving a large number of decision makers very easily and the computational procedure is also very simple. At last we have applied the NSM-Algorithm to solve some interesting problems of medical science.

2 Preliminaries

Definition: (Molodtsov 1999)

Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U. Let $A \subseteq E$. A pair (F_A, E) is called a **soft set** over U, where F_A is a mapping given by, $F_A : E \rightarrow P(U)$ such that $F_A(e) = f$ if $e \notin A$.

Here F_A is called approximate function of the soft set (F_A, E). The set $F_A(e)$ is called e-approximate value set which consists of related objects of the parameter $e \in E$. In other words, a soft set over U is a parameterized family of subsets of the universe U.

Definition (Smarandache 2005)

A **neutrosophic set** A on the universe of discourse X is defined as $A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\},\ where\ T, I, F : X \rightarrow \left]^{-}0,1^{+}\right[$ and $^{-}0 \le T_A(x) + I_A(x) + F_A(x) \le 3^{+}$; T, I, F are called neutrosophic components.

"Neutrosophic" etymologically comes from "neutro-sophy" (French *neutre* < Latin *neuter*, neutral and Greek *sophia*, skill/wisdom) which means knowledge of neutral thought.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $\left]^{-}0,1^{+}\right[$. The non-standard finite numbers $1^{+} = 1 + d$, where 1 is the standard part and d is the non-standard part and $^{-}0 = 0 - d$, where 0 is its standard part and d is non-standard part. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $\left]^{-}0,1^{+}\right[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0,1]$.

Any element neutrosophically belongs to any set, due to the percentages of truth/indeterminacy/falsity involved, which varies between 0 and 1 or even less than 0 or greater than 1.

Thus $x(0.5,0.2,0.3)$ belongs to A (which means, with a probability of 50 percent x is in A, with a probability of 30 percent x is not in A and the rest is undecidable); or $y(0,0,1)$ belongs to A (which normally means y is not for sure in A); or $z(0,1,0)$ belongs to A (which means one does know absolutely nothing about z's affiliation with A); here $0.5+0.2+0.3=1$; thus A **is a NS and an IFS too**.

The subsets representing the appurtenance, indeterminacy and falsity may overlap, say the element

$z(0.30,0.51,0.28)$ and in this case $0.30+0.51+0.28>1$; then **B is a NS but is not an IFS; we can call it paraconsistent set** (from paraconsistent logic, which deals with paraconsistent information).

Or, another example, say the element $z(0.1,0.3,0.4)$ belongs to the set C, and here $0.1+0.3+0.4<1$; then B **is a NS but is not an IFS; we can call it intuitionistic set** (from intuitionistic logic, which deals with incomplete information).

Remarkably, in a NS one can have elements which have paraconsistent information (sum of components >1), or, incomplete information (sum of components <1), or, consistent information (in the case when the sum of components $=1$).

Definition (Maji 2012, 2013)

Let U be an initial universe set and E be a set of parameters. Consider $A \subseteq E$. Let N^U denotes the set of all neutrosophic sets of U. The collection $(\breve{F}, A)$ is termed to be the **neutrosophic soft set** over U, where $\breve{F}$ is a mapping given by $\breve{F} : A \to N^U$.

For illustration we consider an example.

Example 2.1

Let U be the set of houses under consideration and E is the set of parameters. Each parameter is a neutrosophic word or sentence involving neutrosophic words.

Consider $E = \{$ beautiful, wooden, costly, very costly, moderate, green surroundings, in good repair, in bad repair, cheap, expensive $\}$. In this case, to define a neutrosophic soft set means to point out beautiful houses, wooden houses, houses in the green surroundings and so on. Suppose that, there are five houses in the universe U given by, $U = \{h_1, h_2, h_3, h_4, h_5\}$ and the set of parameters $A = \{e_1, e_2, e_3, e_4\}$, where e_1 stands for the parameter beautiful, e_2 stands for the parameter wooden, e_3 stands for the parameter costly and the parameter e_4 stands for moderate. Suppose that,

$$F(beautiful) = \{<h_1,0.5,0.6,0.3>,<h_2,0.4,0.7,0.6>,<h_3,0.6,0.2,0.3>, <h_4,0.7,0.3,0.2>,<h_5,0.8,0.2,0.3>\},$$
$$F(wooden) = \{<h_1,0.6,0.3,0.5>,<h_2,0.7,0.4,0.3>,<h_3,0.8,0.1,0.2>, <h_4,0.7,0.1,0.3>,<h_5,0.8,0.3,0.6>\},$$
$$F(costly) = \{<h_1,0.7,0.4,0.3>,<h_2,0.6,0.7,0.2>,<h_3,0.7,0.2,0.5>, <h_4,0.5,0.2,0.6>,<h_5,0.7,0.3,0.4>\},$$
$$F(moderate) = \{<h_1,0.8,0.6,0.4>,<h_2,0.7,0.9,0.6>,<h_3,0.7,0.6,0.4>, <h_4,0.7,0.8,0.6>,<h_5,0.9,0.5,0.7>\}.$$

The neutrosophic soft set (NSS) $(\breve{F}, E)$ is a parameterized family $\{\breve{F}(e_i); i = 1,..,10\}$ of all neutrosophic sets of U and describes a collection of approximation of an object. The mapping $\breve{F}$ here is houses(.), where dot(.) is to be filled up by a parameter $e \in E$.

Therefore, $\breve{F}(e_1)$ means houses(beautiful) whose functional-value is the neutrosophic set

$$\{< h_1,0.5,0.6,0.3 >,< h_2,0.4,0.7,0.6 >,< h_3,0.6,0.2,0.3 >,< h_4,0.7,0.3,0.2 >, < h_5,0.8,0.2,0.3 >\}.$$

Thus we can view the neutrosophic soft set (NSS) (F, A) as a collection of approximation as below:

$$(\breve{F}, A) = \{beautiful\ houses = \{< h_1,0.5,0.6,0.3 >,< h_2,0.4,0.7,0.6 >,< h_3,0.6,0.2,0.3 >,$$
$$< h_4,0.7,0.3,0.2 >,< h_5,0.8,0.2,0.3 >\}, wooden\ houses = \{< h_1,0.6,0.3,0.5 >,$$
$$< h_2,0.7,0.4,0.3 >,< h_3,0.8,0.1,0.2 >,< h_4,0.7,0.1,0.3 >,< h_5,0.8,0.3,0.6 >\},$$
$$costly\ houses = \{< h_1,0.7,0.4,0.3 >,< h_2,0.6,0.7,0.2 >,< h_3,0.7,0.2,0.5 >,$$

$< h_4,0.5,0.2,0.6 >, < h_5,0.7,0.3,0.4 >\}, \textit{moderate houses} = \{< h_1,0.8,0.6,0.4 >,$
$< h_2,0.7,0.9,0.6 >, < h_3,0.7,0.6,0.4 >, < h_4,0.7,0.8,0.6 >, < h_5,0.9,0.5,0.7 >\}\}$

where each approximation has two parts: (i) a predicate p, and (ii) an approximate value-set v (or simply to be called value-set v).

For example, for the approximation

$\textit{beautiful houses} = \{< h_1,0.5,0.6,0.3 >, < h_2,0.4,0.7,0.6 >, < h_3,0.6,0.2,0.3 >, < h_4,0.7,0.3,0.2 >,$
$< h_5,0.8,0.2,0.3 >\}$

we have (i) the predicate part *beautiful houses* and (ii) the approximate value-set is $\{< h_1,0.5,0.6,0.3 >, < h_2,0.4,0.7,0.6 >, < h_3,0.6,0.2,0.3 >, < h_4,0.7,0.3,0.2 >, < h_5,0.8,0.2,0.3 >\}$.

Thus, a neutrosophic soft set $(\tilde{F}, E)$ can be viewed as a collection of approximations like $(\tilde{F}, E) = \{p_1 = v_1, p_2 = v_2, \ldots, p_{10} = v_{10}\}$. For the purpose of storing a neutrosophic soft set in a computer, we could represent it in the form of a table as shown below (corresponding to the neutrosophic soft set in the above example). In this table, the entries are c_{ij} corresponding to the house h_i and the parameter e_j , where c_{ij} **= (true-membership value of** h_i**, indeterminacy-membership value of** h_i**, falsity-membership value of** h_i **) in** $\tilde{F}(e_j)$. The tabular representation of the neutrosophic soft set $(\tilde{F}, A)$ is as follow:

Table-I: Tabular representation of the NSS $(\tilde{F}, A)$

	beautiful	*wooden*	*costly*	*moderate*
h_1	(0.5, 0.6, 0.3)	(0.6, 0.3, 0.5)	(0.7, 0.4, 0.3)	(0.8, 0.6, 0.4)
h_2	(0.4, 0.7, 0.6)	(0.7, 0.4, 0.3)	(0.6, 0.7, 0.2)	(0.7, 0.9, 0.6)
h_3	(0.6, 0.2, 0.3)	(0.8, 0.1, 0.2)	(0.7, 0.2, 0.5)	(0.7, 0.6, 0.4)
h_4	(0.7, 0.3, 0.2)	(0.7, 0.1, 0.3)	(0.5, 0.2, 0.6)	(0.7, 0.8, 0.6)
h_5	(0.8, 0.2, 0.3)	(0.8, 0.3, 0.6)	(0.7, 0.3, 0.4)	(0.9, 0.5, 0.7)

Definition: (Cagman and Enginoglu 2010)

Let (F_A, E) be a soft set over U. Then a subset of $U \times E$ is uniquely defined by

$$R_A = \{(u,e) : e \in A, u \in F_A(e)\}$$

which is called a relation form of (F_A, E). Now the **characteristic function** of R_A is written by,

$$c_{R_A} : U \times E \to \{0,1\},\ c_{R_A} = \begin{cases} 1, (u,e) \in R_A \\ 0, (ij,e) \notin R_A \end{cases}$$

Let $U = \{u_1, u_2, ..., u_m\}$, $E = \{e_1, e_2, ..., e_n\}$, then R_A can be presented by a table as in the following form

Table-II: Tabular representation of R_A of the soft set (F_A, E)

	e_1	e_2	....	e_n
u_1	$c_{R_A}(u_1, e_1)$	$c_{R_A}(u_1, e_2)$	....	$c_{R_A}(u_1, e_n)$
u_2	$c_{R_A}(u_2, e_1)$	$c_{R_A}(u_2, e_2)$	....	$c_{R_A}(u_2, e_n)$
....	....	....	....	....
u_m	$c_{R_A}(u_m, e_1)$	$c_{R_A}(u_m, e_2)$	....	$c_{R_A}(u_m, e_n)$

If $a_{ij} = c_{R_A}(u_i, e_j)$, we can define a matrix

$$[a_{ij}]_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{m1} & a_{m2} & .... & a_{mn} \end{pmatrix}$$

which is called a **soft matrix** of order $m \times n$ corresponding to the soft set (F_A, E) over U. A soft set (F_A, E) is uniquely characterized by the matrix $[a_{ij}]_{m \times n}$. Therefore we shall identify any soft set with its soft matrix and use these two concepts as interchangeable.

3 Some New Concepts of Matrices in Neutrosophic Soft Set Theory

3.1 Definitions

Neutrosophic Soft Matrix:

Let $(\tilde{F}_A, E)$ be a neutrosophic soft set over U, where $\tilde{F}_A$ is a mapping given by, $\tilde{F}_A : E \to N^U$ such that $\tilde{F}_A(e) = f$ if $e \notin A$ where N^U is the set of all neutrosophic sets over U and f is a null neutrosophic set. Then a subset of $U \times E$ is uniquely defined by

$$\tilde{R}_A = \{(u,e) : e \in A, u \in \tilde{F}_A(e)\}$$

which is called a relation form of $(\breve{F}_A, E)$. Now the relation $\breve{R}_A$ is characterized by the truth-membership function $T_A : U \times E \to [0,1]$, indeterminacy-membership function $I_A : U \times E \to [0,1]$ and the falsity-membership function $F_A : U \times E \to [0,1]$, where $T_A(u,e)$ is the true-membership value, $I_A(u,e)$ is the indeterminacy-membership value and $F_A(u,e)$ is the falsity-membership value of the object u associated with the parameter e.

Now if the set of universe $U = \{u_1, u_2, ..., u_m\}$ and the set of parameters $E = \{e_1, e_2, ..., e_n\}$, then $\breve{R}_A$ can be presented by a table as in the following form

Table-III: Tabular representation of $\breve{R}_A$ of the NSS $(\breve{F}_A, E)$

	e_1	e_2	....	e_n
u_1	$(T_{A11}, I_{A11}, F_{A11})$	$(T_{A12}, I_{A12}, F_{A12})$	....	$(T_{A1n}, I_{A1n}, F_{A1n})$
u_2	$(T_{A21}, I_{A21}, F_{A21})$	$(T_{A22}, I_{A22}, F_{A22})$	....	$(T_{A2n}, I_{A2n}, F_{A2n})$
....	....	....	....	....
u_m	$(T_{Am1}, I_{Am1}, F_{Am1})$	$(T_{Am2}, I_{Am2}, F_{Am2})$	....	$(T_{Amn}, I_{Amn}, F_{Amn})$

where $(T_{Amn}, I_{Amn}, F_{Amn}) = (T_A(u_m, e_n), I_A(u_m, e_n), F_A(u_m, e_n))$

If $a_{ij} = (T_A(u_i, e_j), I_A(u_i, e_j), F_A(u_i, e_j))$, we can define a matrix

$$(\breve{a}_{ij})_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{m1} & a_{m2} & .... & a_{mn} \end{pmatrix}$$

which is called an **neutrosophic soft matrix** of order $m \times n$ corresponding to the neutrosophic soft set $(\breve{F}_A, E)$ over U. A neutrosophic soft set $(\breve{F}_A, E)$ is uniquely characterized by the matrix $(\breve{a}_{ij})_{m \times n}$. Therefore we shall identify any neutrosophic soft set with its neutrosophic soft matrix and use these two concepts as interchangeable.

Example 3.1

Let U be the set of five towns, given by, $U = \{t_1, t_2, t_3, t_4, t_5\}$.

Let E be the set of parameters (each parameter is a neutrosophic word), given by,

$E = \{ \textit{highly}, \textit{immensely}, \textit{moderately}, \textit{average}, \textit{less} \} = \{e_1, e_2, e_3, e_4, e_5\}$ (say)

Let $A \subset E$, given by,

$A = \{e_1, e_2, e_3, e_5\}$ (say)

Now suppose that, $\breve{F}_A$ is a mapping, defined as populated towns(.) and given by,

$$\breve{F}_A(e_1) = \{t_1/(.2,.3,.7), t_2/(.8,.2,.1), t_3/(.4,.5,.2), t_4/(.6,.2,.3), t_5/(.7,.3,.2)\},$$
$$\breve{F}_A(e_2) = \{t_1/(0,0,1), t_2/(.9,.4,.1), t_3/(.3,.2,.6), t_4/(.4,.3,.6), t_5/(.6,.4,.3)\},$$
$$\breve{F}_A(e_3) = \{t_1/(.3,.3,.5), t_2/(.4,.2,.6), t_3/(.8,.5,.2), t_4/(.1,.4,.8), t_5/(.3,.5,.7)\},$$
$$\breve{F}_A(e_5) = \{t_1/(.9,.5,.2), t_2/(.1,.4,.8), t_3/(.5,.2,.4), t_4/(.3,.3,.5), t_5/(.2,.4,.8)\}$$

Then the Neutrosophic Soft Set

$$(\breve{F}_A, E) = \{\textit{highly populated town} = \{t_1/(.2,.3,.7), t_2/(.8,.2,.1), t_3/(.4,.5,.2), t_4/(.6,.2,.3), t_5/(.7,.3,.2)\},$$
$$\textit{immensely populated town} = \{t_1/(0,0,1), t_2/(.9,.4,.1), t_3/(.3,.2,.6), t_4/(.4,.3,.6), t_5/(.6,.4,.3)\},$$
$$\textit{moderately populated town} = \{t_1/(.3,.3,.5), t_2/(.4,.2,.6), t_3/(.8,.5,.2), t_4/(.1,.4,.8), t_5/(.3,.5,.7)\},$$
$$\textit{less populated town} = \{t_1/(.9,.5,.2), t_2/(.1,.4,.8), t_3/(.5,.2,.4), t_4/(.3,.3,.5), t_5/(.2,.4,.8)\}\}$$

Therefore the relation form of $(\breve{F}_A, E)$ is written by,

$$\breve{R}_A = \{(\{t_1/(.2,.3,.7), t_2/(.8,.2,.1), t_3/(.4,.5,.2), t_4/(.6,.2,.3), t_5/(.7,.3,.2)\}, e_1),$$
$$(\{t_1/(0,0,1), t_2/(.9,.4,.1), t_3/(.3,.2,.6), t_4/(.4,.3,.6), t_5/(.6,.4,.3)\}, e_2),$$
$$(\{t_1/(.3,.3,.5), t_2/(.4,.2,.6), t_3/(.8,.5,.2), t_4/(.1,.4,.8), t_5/(.3,.5,.7)\}, e_3),$$
$$(\{t_1/(.9,.5,.2), t_2/(.1,.4,.8), t_3/(.5,.2,.4), t_4/(.3,.3,.5), t_5/(.2,.4,.8)\}, e_5)\}$$

Hence the neutrosophic soft matrix $(\breve{a}_{ij})$ is written by,

$$(\breve{a}_{ij}) = \begin{pmatrix} (.2,.3,.7) & (0,0,1) & (.3,.3,.5) & (0,0,1) & (.9,.5,.2) \\ (.8,.2,.1) & (.9,.4,.1) & (.4,.2,.6) & (0,0,1) & (.1,.4,.8) \\ (.4,.5,.2) & (.3,.2,.6) & (.8,.5,.2) & (0,0,1) & (.5,.2,.4) \\ (.6,.2,.3) & (.4,.3,.6) & (.1,.4,.8) & (0,0,1) & (.3,.3,.5) \\ (.7,.3,.2) & (.6,.4,.3) & (.3,.5,.7) & (0,0,1) & (.2,.4,.8) \end{pmatrix}$$

Row-Neutrosophic Soft Matrix:

A neutrosophic soft matrix of order $1 \times n$ i.e., with a single row is called a row-neutrosophic soft matrix. Physically, a row-neutrosophic soft matrix formally corresponds to a neutrosophic soft set whose universal set contains only one object.

Column-Neutrosophic Soft Matrix:

A neutrosophic soft matrix of order $m \times 1$ i.e., with a single column is called a **column-neutrosophic soft matrix**. Physically, a column-neutrosophic soft matrix formally corresponds to a neutrosophic soft set whose parameter set contains only one parameter.

Square Neutrosophic Soft Matrix:

A neutrosophic soft matrix of order $m \times n$ is said to be a **square neutrosophic soft matrix** if $m = n$ ie., the number of rows and the number of columns are equal. That means a square-neutrosophic soft matrix is formally equal to a neutrosophic soft set having the same number of objects and parameters.

Example 3.2

Consider the example 3.1

Here since the neutrosophic soft matrix $(\tilde{a}_{ij})$ contains five rows and five columns, so it is a square-neutrosophic soft matrix.

Complement of a neutrosophic soft matrix:

Let $(\tilde{a}_{ij})$ be an $m \times n$ neutrosophic soft matrix, where $\tilde{a}_{ij} = (T_{ij}, I_{ij}, F_{ij}) \forall i, j$. Then the **complement** of $(\tilde{a}_{ij})$ is denoted by $(\tilde{a}_{ij})^o$ and is defined by,

$(\tilde{a}_{ij})^o = (\tilde{c}_{ij})$, where $(\tilde{c}_{ij})$ is also a neutrosophic soft matrix of order $m \times n$ and $\tilde{c}_{ij} = (F_{ij}, I_{ij}, T_{ij}) \forall i, j$.

Example 3.3

Consider the example 3.1

Then the complement of $(\tilde{a}_{ij})$ is,

$$(\tilde{a}_{ij})^o = \begin{pmatrix} (.7,.3,.2) & (1,0,0) & (.5,.3,.3) & (1,0,0) & (.2,.5,.9) \\ (.1,.2,.8) & (.1,.4,.9) & (.6,.2,.4) & (1,0,0) & (.8,.4,.1) \\ (.2,.5,.4) & (.6,.2,.3) & (.2,.5,.8) & (1,0,0) & (.5,.3,.3) \\ (.3,.2,.6) & (.6,.3,.4) & (.8,.4,.1) & (1,0,0) & (.5,.3,.3) \\ (.2,.3,.7) & (.3,.4,.6) & (.7,.5,.3) & (1,0,0) & (.8,.4,.2) \end{pmatrix}$$

Null Neutrosophic Soft Matrix:

A neutrosophic soft matrix of order $m \times n$ is said to be a **null neutrosophic soft matrix or zero neutrosophic soft matrix** if all of its elements are $(0,0,1)$. A null neutrosophic soft matrix is denoted by, $\breve{\Phi}$. Now the neutrosophic soft set associated with a null neutrosophic soft matrix must be a null neutrosophic soft set.

Complete or Absolute Neutrosophic Soft Matrix:

A neutrosophic soft matrix of order $m \times n$ is said to be a **complete or absolute neutrosophic soft matrix** if all of its elements are $(1,0,0)$. A complete or absolute neutrosophic soft matrix is denoted by, C_A. Now the neutrosophic soft set associated with an absolute neutrosophic soft matrix must be an absolute neutrosophic soft set.

Diagonal Neutrosophic Soft Matrix:

A square neutrosophic soft matrix of order $m \times n$ is said to be a **diagonal-neutrosophic soft matrix** if all of its non-diagonal elements are $(0,0,1)$.

Choice Matrix:

It is a square matrix whose rows and columns both indicate parameters(which are neutrosophic words or sentences involving neutrosophic words). If $\breve{x}$ is a choice matrix, then its element $\breve{x}_{ij}$ is defined as follows:

$$\breve{x}_{ij} = \begin{cases} (1,0,0) \text{ when } i^{th} \text{ and } j^{th} \text{ parameters are both choice parameters of the decision makers} \\ (0,0,1) \text{ otherwise, i.e. when atleast one of the } i^{th} \text{ or } j^{th} \text{ parameters be not under choice} \end{cases}$$

There are different types of choice matrices according to the number of decision makers. We may realize this by the following example.

Example 3.4

Suppose that U be a set of four shopping malls, say, $U = \{s_1, s_2, s_3, s_4\}$

Let E be a set of parameters, given by,

$E = \{\textit{costly}, \textit{excellent work culture}, \textit{assured sale}, \textit{good location}, \textit{cheap}\} = \{e_1, e_2, e_3, e_4, e_5\}$ (say)

Now let the neutrosophic soft set $(\breve{F}, A)$ describing the quality of the shopping malls, is given by,

$$\begin{aligned}
&(\breve{F}, E) \\
&= \{\textit{costly shopping malls} = \{s_1/(0.9,0.1,0.3), s_2/(0.2,0.3,0.7), s_3/(0.4,0.3,0.5), s_4/(0.8,0.4,0.1)\}, \\
&\quad \textit{shop. wt. exclnt. wrkculture.} = \{s_1/(0.8,0.2,0.1), s_2/(0.3,0.4,0.5), s_3/(0.5,0.3,0.4), s_4/(0.4,0.2,0.5)\}, \\
&\quad \textit{shop. wt. asrd. sale} = \{s_1/(0.9,0.4,0.1), s_2/(0.2,0.2,0.7), s_3/(0.4,0.3,0.5), s_4/(0.8,0.5,0.1)\},
\end{aligned}$$

$shop.\,wt.\,good\ location = \{s_1/(0.7,0.3,0.3), s_2/(0.9,0.6,0.1), s_3/(0.4,0.3,0.5), s_4/(0.8,0.1,0.2)\},$
$cheap\ shop. = \{s_1/(0.1,0.4,0.8), s_2/(0.7,0.3,0.1), s_3/(0.5,0.2,0.3), s_4/(0.2,0.4,0.7)\}\}$

Suppose Mr.Ram wants to buy a shopping mall on the basis of his choice parameters excellent work culture, assured sale and cheap which form a subset A of the parameter set E.

Therefore $A = \{e_2, e_3, e_5\}$

Now **the choice matrix of Mr.Ram** is,

$$(\breve{x}_{ij})_A = e_A \begin{pmatrix} & & e_A & & \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,0,0) & (1,0,0) & (0,0,1) & (1,0,0) \\ (0,0,1) & (1,0,0) & (1,0,0) & (0,0,1) & (1,0,0) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,0,0) & (1,0,0) & (0,0,1) & (1,0,0) \end{pmatrix}$$

Now suppose Mr.Ram and Mr.Shyam together wants to buy a shopping mall according to their choice parameters. Let the choice parameter set of Mr.Shyam be, $B = \{e_1, e_2, e_3, e_4\}$ Then **the combined choice matrix of Mr.Ram and Mr.Shyam** is

$$(\breve{x}_{ij})_{(A,B)} = e_A \begin{pmatrix} & & e_B & & \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (1,0,0) & (1,0,0) & (1,0,0) & (1,0,0) & (0,0,1) \\ (1,0,0) & (1,0,0) & (1,0,0) & (1,0,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (1,0,0) & (1,0,0) & (1,0,0) & (1,0,0) & (0,0,1) \end{pmatrix}$$

[Here the entries $\breve{e}_{ij} = (1,0,0)$ indicates that e_i is a choice parameter of Mr.Ram and e_j is a choice parameter of Mr.Shyam. Now $\breve{e}_{ij} = (0,0,1)$ indicates either e_i fails to be a choice parameter of Mr.Ram or e_j fails to be a choice parameter of Mr.Shyam.]

Again the above combined choice matrix of Mr.Ram and Mr.Shyam may be also presented in its transpose form as,

$$(\breve{x}_{ij})_{(B,A)} = e_B \begin{pmatrix} & & e_A & & \\ (0,0,1) & (1,0,0) & (1,0,0) & (0,0,1) & (1,0,0) \\ (0,0,1) & (1,0,0) & (1,0,0) & (0,0,1) & (1,0,0) \\ (0,0,1) & (1,0,0) & (1,0,0) & (0,0,1) & (1,0,0) \\ (0,0,1) & (1,0,0) & (1,0,0) & (0,0,1) & (1,0,0) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \end{pmatrix}$$

Now let us see the form of the combined choice matrix associated with three decision makers. Suppose that Mr.Kartik is willing to buy a shopping mall together with Mr.Ram and Mr.Shyam on the basis of his choice parameters excellent work culture, assured sale and good location which form a subset C of the parameter set E.

Therefore $C = \{e_2, e_3, e_4\}$

Then **the combined choice matrix of Mr.Ram, Mr.Shyam and Mr.Kartik** will be of three different types which are as follows,

(i) $$(\breve{x}_{ij})_{(C,A\wedge B)} = e_C \begin{pmatrix} & & e_{(A\wedge B)} & & \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,0,0) & (1,0,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,0,0) & (1,0,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,0,0) & (1,0,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \end{pmatrix}$$

[Since the set of common choice parameters of Mr.Ram and Mr.Shyam is, $A \wedge B = \{e_2, e_3\}$. Here the entries $\breve{e}_{ij} = (1,0,0)$ indicates that e_i is a choice parameter of Mr.Kartik and e_j is a common choice parameter of Mr.Ram and Mr.Shyam. Now $\breve{e}_{ij} = (0,0,1)$ indicates either e_i fails to be a choice parameter of Mr.Kartik or e_j fails to be a common choice parameter of Mr.Ram and Mr.Shyam.]

$$(\breve{x}_{ij})_{(A,B\wedge C)} = e_A \begin{pmatrix} & & e_{(B\wedge C)} & & \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,0,0) & (1,0,0) & (1,0,0) & (0,0,1) \\ (0,0,1) & (1,0,0) & (1,0,0) & (1,0,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,0,0) & (1,0,0) & (1,0,0) & (0,0,1) \end{pmatrix} [\, Since\ B\wedge C = \{e_2, e_3, e_4\}]$$

(ii)

$$(\breve{x}_{ij})_{(B,C\wedge A)} = e_B \begin{pmatrix} & & e_{(C\wedge A)} & & \\ (0,0,1) & (1,0,0) & (1,0,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,0,0) & (1,0,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,0,0) & (1,0,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,0,0) & (1,0,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \end{pmatrix} [\, Since\ C\wedge A = \{e_2, e_3\}]$$

(iii)

Symmetric Neutrosophic Soft Matrix:

A square neutrosophic soft matrix $\breve{A}$ of order $n\times n$ is said to be a **symmetric neutrosophic soft matrix**, if its transpose be equal to it, i.e., if $\breve{A}^T = \breve{A}$. Hence the neutrosophic soft matrix $(\breve{a}_{ij})$ is symmetric, if $\breve{a}_{ij} = \breve{a}_{ji}, \forall i, j$.

Therefore if $(\breve{a}_{ij})$ be a symmetric neutrosophic soft matrix then the neutrosophic soft sets associated with $(\breve{a}_{ij})$ and $(\breve{a}_{ij})^T$ both be the same.

3.2 Operations

Transpose of a Square Neutrosophic Soft Matrix:

The transpose of a square neutrosophic soft matrix $(\breve{a}_{ij})$ of order $m\times n$ is another square neutrosophic soft matrix of order $n\times m$ obtained from $(\breve{a}_{ij})$ by interchanging its rows and columns. It is denoted by $(\breve{a}_{ij})^T$. Therefore the neutrosophic soft set associated with $(\breve{a}_{ij})^T$ becomes a new neutrosophic soft set over the same universe and over the same set of parameters.

Example 3.5

Consider the example 3.1. Here $(\breve{F}_A, E)$ be a neutrosophic soft set over the universe U and over the set of parameters E, given by,

$$
\begin{aligned}
(\tilde{F}_A, E) = \{ & \textit{highly populated town} = \{t_1/(.2,.3,.7), t_2/(.8,.2,.1), t_3/(.4,.5,.2), t_4/(.6,.2,.3), t_5/(.7,.3,.2)\}, \\
& \textit{immensely populated town} = \{t_1/(0,0,1), t_2/(.9,.4,.1), t_3/(.3,.2,.6), t_4/(.4,.3,.6), t_5/(.6,.4,.3)\}, \\
& \textit{moderately populated town} = \{t_1/(.3,.3,.5), t_2/(.4,.2,.6), t_3/(.8,.5,.2), t_4/(.1,.4,.8), t_5/(.3,.5,.7)\}, \\
& \textit{less populated town} = \{t_1/(.9,.5,.2), t_2/(.1,.4,.8), t_3/(.5,.2,.4), t_4/(.3,.3,.5), t_5/(.2,.4,.8)\}\}
\end{aligned}
$$

whose associated neutrosophic soft matrix is,

$$
(\tilde{a}_{ij}) = \begin{pmatrix}
(.2,.3,.7) & (0,0,1) & (.3,.3,.5) & (0,0,1) & (.9,.5,.2) \\
(.8,.2,.1) & (.9,.4,.1) & (.4,.2,.6) & (0,0,1) & (.1,.4,.8) \\
(.4,.5,.2) & (.3,.2,.6) & (.8,.5,.2) & (0,0,1) & (.5,.2,.4) \\
(.6,.2,.3) & (.4,.3,.6) & (.1,.4,.8) & (0,0,1) & (.3,.3,.5) \\
(.7,.3,.2) & (.6,.4,.3) & (.3,.5,.7) & (0,0,1) & (.2,.4,.8)
\end{pmatrix}
$$

Now its transpose neutrosophic soft matrix is,

$$
(\tilde{a}_{ij})^T = \begin{pmatrix}
(.2,.3,.7) & (.8,.2,.1) & (.4,.5,.2) & (.6,.2,.3) & (.7,.3,.2) \\
(0,0,1) & (.9,.4,.1) & (.3,.2,.6) & (.4,.3,.6) & (.6,.4,.3) \\
(.3,.3,.5) & (.4,.2,.6) & (.8,.5,.2) & (.1,.4,.8) & (.3,.5,.7) \\
(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\
(.9,.5,.2) & (.1,.4,.8) & (.5,.2,.4) & (.3,.3,.5) & (.2,.4,.8)
\end{pmatrix}
$$

Therefore the neutrosophic soft set associated with $(\tilde{a}_{ij})^T$ is,

$$
\begin{aligned}
(\tilde{G}_B, E) = \{ & \textit{highly populated town} = \{t_1/(.2,.3,.7), t_2/(0,0,1), t_3/(.3,.3,.5), t_4/(0,0,1), t_5/(.9,.5,.2)\}, \\
& \textit{immensely populated town} = \{t_1/(.8,.2,.1), t_2/(.9,.4,.1), t_3/(.4,.2,.6), t_4/(0,0,1), t_5/(.1,.4,.8)\}, \\
& \textit{moderately populated town} = \{t_1/(.4,.5,.2), t_2/(.3,.2,.6), t_3/(.8,.5,.2), t_4/(0,0,1), t_5/(.5,.2,.4)\}, \\
& \textit{average populated town} = \{t_1/(.6,.2,.3), t_2/(.4,.3,.6), t_3/(.1,.4,.8), t_4/(0,0,1), t_5/(.3,.3,.5)\}, \\
& \textit{less populated town} = \{t_1/(.7,.3,.2), t_2/(.6,.4,.3), t_3/(.3,.5,.7), t_4/(0,0,1), t_5/(.2,.4,.8)\}\}
\end{aligned}
$$

where $B = \{$ highly, immensely, moderately, average, less $\} \subseteq E$ and G_B is a mapping from B to N^U.

Addition of Neutrosophic Soft Matrices:

Two neutrosophic soft matrices $\tilde{A}$ and $\tilde{B}$ are said to be conformable for addition, if they be of the same order. The addition of two neutrosophic soft matrices $(\tilde{a}_{ij})$ and $(\tilde{b}_{ij})$ of order $m \times n$ is defined by,

$(\breve{a}_{ij}) \oplus (\breve{b}_{ij}) = (\breve{c}_{ij})$, where $(\breve{c}_{ij})$ is also an $m \times n$ neutrosophic soft matrix and

$$\breve{c}_{ij} = (max\{T_{\breve{a}_{ij}}, T_{\breve{b}_{ij}}\}, \frac{I_{\breve{a}_{ij}} + I_{\breve{b}_{ij}}}{2}, min\{F_{\breve{a}_{ij}}, F_{\breve{b}_{ij}}\}) \forall i, j.$$

Example 3.6

Consider the neutrosophic soft matrix of example 3.1,

$$(\breve{a}_{ij}) = \begin{pmatrix} (.2,.3,.7) & (0,0,1) & (.3,.3,.5) & (0,0,1) & (.9,.5,.2) \\ (.8,.2,.1) & (.9,.4,.1) & (.4,.2,.6) & (0,0,1) & (.1,.4,.8) \\ (.4,.5,.2) & (.3,.2,.6) & (.8,.5,.2) & (0,0,1) & (.5,.2,.4) \\ (.6,.2,.3) & (.4,.3,.6) & (.1,.4,.8) & (0,0,1) & (.3,.3,.5) \\ (.7,.3,.2) & (.6,.4,.3) & (.3,.5,.7) & (0,0,1) & (.2,.4,.8) \end{pmatrix}$$

Now consider another neutrosophic soft matrix $(\breve{b}_{ij})$ associated with the neutrosophic soft set $(\breve{G}_B, E)$ (also describing the pollution of the cities) over the same universe U.

Let $B = \{e_1, e_4, e_5\} \subset E$ and

$$\begin{aligned}(\breve{G}, B) = \{ & \textit{highly populated town} = \{t_1/(.3,.5,.7), t_2/(.9,.3,.1), t_3/(.4,.4,.5), t_4/(.7,.3,.2), t_5/(.6,.4,.2)\}, \\ & \textit{average populated town} = \{t_1/(.2,.4,.7), t_2/(.3,.5,.7), t_3/(.7,.2,.1), t_4/(.2,.6,.8), t_5/(.3,.4,.6)\}, \\ & \textit{less populated town} = \{t_1/(.8,.3,.1), t_2/(.2,.4,.7), t_3/(.6,.5,.4), t_4/(.3,.2,.5), t_5/(.2,.4,.6)\}\}\end{aligned}$$

and then the relation form of $(\breve{G}_B, E)$ is written by,

$$\begin{aligned}\breve{R}_B = \{ & (\{t_1/(.3,.5,.7), t_2/(.9,.3,.1), t_3/(.4,.4,.5), t_4/(.7,.3,.2), t_5/(.6,.4,.2)\}, e_1), \\ & (\{t_1/(.2,.4,.7), t_2/(.3,.5,.7), t_3/(.7,.2,.1), t_4/(.2,.6,.8), t_5/(.3,.4,.6)\}, e_2), \\ & (\{t_1/(.8,.3,.1), t_2/(.2,.4,.7), t_3/(.6,.5,.4), t_4/(.3,.2,.5), t_5/(.2,.4,.6)\}, e_5)\}\end{aligned}$$

Hence the neutrosophic soft matrix $(\breve{b}_{ij})$ is written by,

$$(\breve{b}_{ij}) = \begin{pmatrix} (.3,.5,.7) & (0,0,1) & (0,0,1) & (.2,.4,.7) & (.8,.3,.1) \\ (.9,.3,.1) & (0,0,1) & (0,0,1) & (.3,.5,.7) & (.2,.4,.7) \\ (.4,.4,.5) & (0,0,1) & (0,0,1) & (.7,.2,.1) & (.6,.5,.4) \\ (.7,.3,.2) & (0,0,1) & (0,0,1) & (.2,.6,.8) & (.3,.2,.5) \\ (.6,.4,.2) & (0,0,1) & (0,0,1) & (.3,.4,.6) & (.2,.4,.6) \end{pmatrix}$$

Therefore the sum of the neutrosophic soft matrices $(\tilde{a}_{ij})$ and $(\tilde{b}_{ij})$ is,

$$(\tilde{a}_{ij}) \oplus (\tilde{b}_{ij}) = \begin{pmatrix} (0.3,0.4,0.7) & (0,0,1) & (0.3,0.15,0.5) & (0.2,0.2,0.7) & (0.9,0.4,0.1) \\ (0.9,0.25,0.1) & (0.9,0.2,0.1) & (0.4,0.1,0.6) & (0.3,.25,0.7) & (0.2,0.4,0.7) \\ (0.4,0.45,0.2) & (0.3,0.1,0.6) & (0.8,0.25,0.1) & (0.7,0.1,0.1) & (0.6,0.35,0.4) \\ (0.7,0.25,0.2) & (0.4,0.15,0.6) & (0.1,0.2,0.8) & (0.2,0.3,0.8) & (0.3,0.25,0.5) \\ (0.7,0.35,0.2) & (0.6,0.2,0.3) & (0.3,.25,0.7) & (0.3,0.2,0.6) & (0.2,0.4,0.6) \end{pmatrix}$$

Subtraction of Neutrosophic Soft Matrices:

Two neutrosophic soft matrices $\tilde{A}$ and $\tilde{B}$ are said to be conformable for subtraction, if they be of the same order. For any two neutrosophic soft matrices $(\tilde{a}_{ij})$ and $(\tilde{b}_{ij})$ of order $m \times n$, the subtraction of $(\tilde{b}_{ij})$ from $(\tilde{a}_{ij})$ is defined as,

$(\tilde{a}_{ij})!(\tilde{b}_{ij}) = (\tilde{c}_{ij})$, where $(\tilde{c}_{ij})$ is also an $m \times n$ neutrosophic soft matrix and

$$\tilde{c}_{ij} = (min\{T_{\tilde{a}_{ij}}, T_{\tilde{b}^o_{ij}}\}, \frac{I_{\tilde{a}_{ij}} + I_{\tilde{b}_{ij}}}{2}, max\{F_{\tilde{a}_{ij}}, F_{\tilde{b}^o_{ij}}\}) \forall i, j$$ where $(\tilde{b}^o_{ij})$ is the complement of $(\tilde{b}_{ij})$

Example 3.7

Consider the neutrosophic soft matrices $(\tilde{a}_{ij})$ and $(\tilde{b}_{ij})$ of example 3.6. Now,

$$(\tilde{a}_{ij}) = \begin{pmatrix} (.2,.3,.7) & (0,0,1) & (.3,.3,.5) & (0,0,1) & (.9,.5,.2) \\ (.8,.2,.1) & (.9,.4,.1) & (.4,.2,.6) & (0,0,1) & (.1,.4,.8) \\ (.4,.5,.2) & (.3,.2,.6) & (.8,.5,.2) & (0,0,1) & (.5,.2,.4) \\ (.6,.2,.3) & (.4,.3,.6) & (.1,.4,.8) & (0,0,1) & (.3,.3,.5) \\ (.7,.3,.2) & (.6,.4,.3) & (.3,.5,.7) & (0,0,1) & (.2,.4,.8) \end{pmatrix}$$ and

$$(\tilde{b}_{ij})^o = \begin{pmatrix} (.7,.5,.3) & (1,0,0) & (1,0,0) & (.7,.4,.2) & (.1,.3,.8) \\ (.1,.3,.9) & (1,0,0) & (1,0,0) & (.7,.5,.3) & (.7,.4,.2) \\ (.5,.4,.4) & (1,0,0) & (1,0,0) & (.1,.2,.7) & (.4,.5,.6) \\ (.2,.3,.7) & (1,0,0) & (1,0,0) & (.8,.6,.2) & (.5,.2,.3) \\ (.2,.4,.6) & (1,0,0) & (1,0,0) & (.6,.4,.3) & (.6,.4,.2) \end{pmatrix}$$

Therefore the subtraction of the neutrosophic soft matrix $(\breve{b}_{ij})$ from the neutrosophic soft matrix $(\breve{a}_{ij})$ is,

$$(\breve{a}_{ij})!(\breve{b}_{ij}) = \begin{pmatrix} (.2,.4,.7) & (0,0,1) & (0.3,.15,.5) & (0,0.2,1) & (.1,.4,.8) \\ (.1,.25,.9) & (0.9,.2,.1) & (0.4,.1,.6) & (0,.25,1) & (.1,.4,.8) \\ (.4,.45,.4) & (0.3,.1,.6) & (0.8,.25,.2) & (0,.1,1) & (.4,.35,.6) \\ (.2,.25,.7) & (0.4,.15,.6) & (0.1,.2,.8) & (0,0.3,1) & (.3,.25,.5) \\ (.2,.35,.6) & (0.6,.2,.3) & (0.3,.25,.7) & (0,0.2,1) & (.2,.4,.8) \end{pmatrix}$$

Product of a Neutrosophic Soft Matrix with a Choice Matrix:

Let U be the set of universe and E be the set of parameters. Suppose that $\breve{A}$ be any neutrosophic soft matrix and $\breve{b}$ be any choice matrix of a decision maker concerned with the same universe U and E. Now if the number of columns of the neutrosophic soft matrix $\breve{A}$ be equal to the number of rows of the choice matrix $\breve{b}$, then $\breve{A}$ and $\breve{b}$ are said to be conformable for the product ($\breve{A}\otimes\breve{b}$) and the product ($\breve{A}\otimes\breve{b}$) becomes a neutrosophic soft matrix. We may denote the product by $\breve{A}\otimes\breve{b}$ or simply by $\breve{A}\breve{b}$.

If $\breve{A} = (\breve{a}_{ij})_{m\times n}$ and $\breve{b} = (\breve{b}_{jk})_{n\times p}$, then $\breve{A}\breve{b} = (\breve{c}_{ik})$

where $\breve{c}_{ik} = (max^{n}_{j=1} min\{T_{\breve{a}_{ij}}, T_{\breve{b}_{jk}}\}, average^{n}_{j=1} average\{I_{\breve{a}_{ij}}, I_{\breve{b}_{jk}}\}, min^{n}_{j=1} max\{F_{\breve{a}_{ij}}, F_{\breve{b}_{jk}}\})$

It is to be noted that, $\breve{b}\breve{A}$ cannot be defined here.

Example 3.8

Let U be the set of four dresses, given by, $U = \{d_1, d_2, d_3, d_4\}$. Let E be the set of parameters, given by, $E = \{$ *cheap*, *beautiful*, *comfortable*, *gorgeous* $\} = \{e_1, e_2, e_3, e_4\}$ (say). Suppose that the set of choice parameters of Mr.X be, $A = \{e_1, e_3\}$. Now let according to the choice parameters of Mr.X, we have the neutrosophic soft set $(\breve{F}, A)$ which describes the attractiveness of the dresses and the neutrosophic soft matrix of the neutrosophic soft set $(\breve{F}, A)$ be,

$$(\breve{a}_{ij}) = \begin{pmatrix} (0.8,0.4,0.1) & (0.2,0.3,0.7) & (0.7,0.2,0.2) & (0.3,0.1,0.5) \\ (0.3,0.4,0.6) & (0.7,0.3,0.1) & (0.4,0.2,0.6) & (0.8,0.5,0.1) \\ (0.7,0.3,0.2) & (0.4,0.2,0.5) & (0.5,0.1,0.3) & (0.6,0.4,0.2) \\ (0.5,0.2,0.4) & (0.1,0.5,0.8) & (0.9,0.6,0.1) & (0.2,0.3,0.7) \end{pmatrix}$$

Again the choice matrix of Mr.X is,

$$(\breve{x}_{ij})_A = e_A\begin{pmatrix} & e_A & & \\ (1,0,0) & (0,0,1) & (1,0,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (1,0,0) & (0,0,1) & (1,0,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \end{pmatrix}$$

Since the number of columns of the neutrosophic soft matrix $(\breve{a}_{ij})$ is equal to the number of rows of the choice matrix $(\breve{x}_{ij})_A$, they are conformable for the product. Therefore

$$\begin{pmatrix} (0.8,0.4,0.1) & (0.2,0.3,0.7) & (0.7,0.2,0.2) & (0.3,0.1,0.5) \\ (0.3,0.4,0.6) & (0.7,0.3,0.1) & (0.4,0.2,0.6) & (0.8,0.5,0.1) \\ (0.7,0.3,0.2) & (0.4,0.2,0.5) & (0.5,0.1,0.3) & (0.6,0.4,0.2) \\ (0.5,0.2,0.4) & (0.1,0.5,0.8) & (0.9,0.6,0.1) & (0.2,0.3,0.7) \end{pmatrix} \otimes \begin{pmatrix} (1,0,0) & (0,0,1) & (1,0,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (1,0,0) & (0,0,1) & (1,0,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \end{pmatrix}$$

$$= \begin{pmatrix} (0.8,0.2375,0.1) & (0.0,0.2375,1.0) & (0.8,0.2375,0.1) & (0.0,0.2375,1.0) \\ (0.4,0.175,0.6) & (0.0,0.175,1.0) & (0.4,0.175,0.6) & (0.0,0.175,1.0) \\ (0.7,0.2375,0.2) & (0.0,0.2375,1.0) & (0.7,0.2375,0.2) & (0.0,0.2375,1.0) \\ (0.9,0.2,0.1) & (0.0,0.2,1.0) & (0.9,0.2,0.1) & (0.0,0.2,1.0) \end{pmatrix}$$

3.3 Properties

Let $\breve{A}$ and $\breve{B}$ be two neutrosophic soft matrices of order $m \times n$. Then

(i) $\breve{A} \oplus \breve{B} = \breve{B} \oplus \breve{A}$

(ii) $(\breve{A} \oplus \breve{B}) \oplus \breve{C} = \breve{A} \oplus (\breve{B} \oplus \breve{C})$

(iii) $\breve{A} ! \breve{B} \neq \breve{B} ! \breve{A}$

(iv) $(\breve{A} ! \breve{B}) ! \breve{C} \neq \breve{A} ! (\breve{B} ! \breve{C})$

(v) $\breve{A} \oplus \breve{A}^o \neq C_A$

(vi) $\breve{A} ! \breve{A} \neq \breve{\Phi}$

Proof: The proofs of (i)-(vi) are directly obtained from the definitions of addition, subtraction and complement.

3.4 Theorems

Theorem 1: If A be a square neutrosophic soft matrix of order $n\times n$, then $(A^T)^T = A$

Proof: Let $A=(\breve{a}_{ij})_{n\times n}$.

Then by definition, $A^T = (\breve{b}_{ij})_{n\times n}$ *where* $b_{ij} = a_{ji} \forall i, j.$ *ie.,* $A^T = (\breve{a}_{ji})_{n\times n}$.

Therefore $(A^T)^T = (\breve{c}_{ij})_{n\times n}$ *where* $c_{ij} = a_{ij}$ *ie.,* $(A^T)^T = (\breve{a}_{ij})_{n\times n} = A$ (Proved)

Theorem 2: If A and B be two square neutrosophic soft matrices of order $n\times n$, then $(A\oplus B)^T = A^T \oplus B^T$

Proof: Let $A=(\breve{a}_{ij})_{n\times n}$ and $B=(\breve{b}_{ij})_{n\times n}$. Then $L.H.S = (A\oplus B)^T = C^T$ *where* $C=(\breve{c}_{ij})_{n\times n}$

$$= (\breve{c}_{ji})_{n\times n} \text{ where } c_{ji} = (max\{T_{a_{ji}}, T_{b_{ji}}\}, \frac{I_{a_{ji}} + I_{b_{ji}}}{2}, min\{F_{a_{ji}}, F_{b_{ji}}\}) \forall i, j$$

and

$$R.H.S = A^T \oplus B^T = (\breve{a}_{ji})_{n\times n} \oplus (\breve{b}_{ji})_{n\times n}$$

$$= (\breve{d}_{ji})_{n\times n} \text{ where } d_{ji} = (max\{T_{a_{ji}}, T_{b_{ji}}\}, \frac{I_{a_{ji}} + I_{b_{ji}}}{2}, min\{F_{a_{ji}}, F_{b_{ji}}\}) \forall i, j = C^T = L.H.S$$

Hence $(A\oplus B)^T = A^T \oplus B^T$ (Proved)

Theorem 3: If A be a square neutrosophic soft matrix of order $n\times n$, then $(A\oplus A^T)$ is symmetric.

Proof: Let $A=(\breve{a}_{ij})_{n\times n}$.

Then by definition, $A^T = (\breve{a}_{ji})_{n\times n}$. Now

$$A\oplus A^T = (\breve{a}_{ij})_{n\times n} \oplus (\breve{a}_{ji})_{n\times n}$$

$$= (\breve{c}_{ij})_{n\times n} \text{ where } c_{ij} = (max\{T_{a_{ij}}, T_{a_{ji}}\}, \frac{I_{a_{ij}} + I_{a_{ji}}}{2}, min\{F_{a_{ij}}, F_{a_{ji}}\}) \forall i, j.$$

Now $c_{ji} = (max\{T_{a_{ji}}, T_{a_{ij}}\}, \frac{I_{a_{ji}} + I_{a_{ij}}}{2}, min\{F_{a_{ji}}, F_{a_{ij}}\}) = c_{ij} \forall i, j$

Therefore $(c_{ij})_{n\times n}$ ie., $(A\oplus A^T)$ is symmetric. (Proved)

Theorem 4:

If A and B be two square neutrosophic soft matrices of order $n\times n$ and if A and B be symmetric, then $A\oplus B$ is symmetric.

Proof: Since A and B be symmetric,

$$A^T = A \text{ and } B^T = B$$

Therefore $A^T \oplus B^T = A\oplus B$

Thus from Theorem 2 we have,

$$(A\oplus B)^T = A^T \oplus B^T = A\oplus B$$

Hence $A\oplus B$ is symmetric. (Proved)

4 A Generalized Neutrosophic Soft Set Based Group Decision Making Problem

Let N number of decision makers want to select an object jointly from the m number of objects which have n number of features i.e., parameters(E). Suppose that each decision maker has freedom to take his decision of inclusion and evaluation of parameters associated with the selected object, i.e., each decision maker has his own choice parameters belonging to the parameter set E and has his own view of evaluation. Here it is assumed that the parameter evaluation of the objects by the decision makers must be neutrosophic and may be presented in linguistic form or neutrosophic soft set format, alternatively, in the form of neutrosophic soft matrix. Now the problem is to find out the object out of these m objects which satisfies all the choice parameters of all decision makers jointly as much as possible.

5 A New Approach to Solve Neutrosophic Soft Set Based Group Decision Making Problems

This new approach is specially based on choice matrices. These choice matrices represent the choice parameters of the decision makers and also help us to solve the neutrosophic soft set (or neutrosophic soft matrix) based decision making problems with least computational complexity.

The Stepwise Solving Procedure: To solve such type of neutrosophic soft set (or neutrosophic soft matrix) based decision making problems, we are presenting the following stepwise procedure which comprises of the newly proposed choice matrices, neutrosophic soft matrices and the operations on them.

NSM **-Algorithm**

Step 1: If the parameter evaluation of the objects by the decision makers are not given in neutrosophic soft matrix form, then first construct the neutrosophic soft matrices according to the given evaluations.

Step 2: Construct the combined choice matrix with respect to the choice parameters of the decision makers.

Step 3: Compute the product neutrosophic soft matrices by multiplying each given neutrosophic soft matrix with the combined choice matrix as per the rule of multiplication of neutrosophic soft matrices.

Step 4: Compute the sum of these product neutrosophic soft matrices to have the resultant neutrosophic soft matrix(R_{NS}).

Step 5: Then compute the weight of each object(O_i) by adding the true-membership values of the entries of its concerned row(i-th row) of R_{NS} and denote it as $W(O_i)$.

Step 6: The object having the highest weight becomes the optimal choice object. If more than one object have the highest weight then go to the next step.

Step 7: Now we have to consider the sum of the falsity-membership values (Θ) of the entries of the rows associated with those equal weighted objects. The object with the minimum Θ-value will be the optimal choice object. Now if the Θ-values of those objects also be the same, then go to the next step.

Step 8: Now consider the sum of the indeterminacy values (Ψ) of the entries of the rows associated with those equal Θ-valued objects. Now if the Ψ-values of those objects also be the same, any one of them may be chosen as the optimal choice object.

To illustrate the basic idea of the NSM-algorithm, now we apply it to the following neutrosophic soft set (or, neutrosophic soft matrix) based decision making problems.

Example 5.1: Let U be the set of four story books, given by, $U = \{b_1, b_2, b_3, b_4\}$. Let E be the set of parameters, given by, $E = \{\, romantic, thriller, comic, horror \,\} = \{e_1, e_2, e_3, e_4\}$ (say). Suppose that, three sisters Shayana, Dayana and Nayna together want to buy a story book among these four books for their youngest brother Ohm according to their choice parameters, $P = \{e_1, e_3\}, Q = \{e_2, e_3\}, R = \{e_1, e_4\}$ respectively. Now let according to the choice parameter evaluation of the books by Shayana, Dayana and Nayna, we have the neutrosophic soft sets $(\tilde{F}_P, E)$, $(\tilde{G}_Q, E)$, $(\tilde{H}_R, E)$ which describe the nature of the books according to Shayana, Dayana and Nayna respectively and given by,

$$(\tilde{F}_A, E) = \{\, romantic\ books = \{b_1/(0.9,0.6,0.1), b_2/(0.3,0.2,0.5), b_3/(0.7,0.4,0.1), b_4/(0.2,03,0.7)\},$$
$$comic\ books = \{b_1/(1,0,0), b_2/(0.6,0.4,0.3), b_3/(0.3,0.2,0.5), b_4/(0.2,0.3,0.7)\}\}$$

$$(\tilde{G}_B, E) = \{thriller\ books = \{b_1/(0.4,0.3,0.6), b_2/(0.8,0.6,0.1), b_3/(0.5,0.1,0.2), b_4/(0.3,0.2,0.5)\},$$
$$comic\ books = \{b_1/(0.8,0.5,0.1), b_2/(0.6,0.3,0.2), b_3/(0.4,0.2,0.5), b_4/(0.2,0.3,0.8)\}\}$$

$$(\tilde{H}_C, E) = \{\, romantic\ books = \{b_1/(0.9,0.6,0.1), b_2/(0.4,0.2,0.5), b_3/(0.6,0.1,0.3), b_4/(0.3,0.4,0.5)\},$$
$$horror\ books = \{b_1/(0.2,0.3,0.7), b_2/(0.3,0.1,0.5), b_3/(0.6,0.4,0.2), b_4/(0.9,0.2,0)\}\}$$

The problem is to select the story book among the four books which satisfies the choice parameters of Shayana, Dayana and Nayna as much as possible.

Now let us apply our newly proposed NSM -algorithm to solve this problem.

(1) The neutrosophic soft matrices of the neutrosophic soft sets $(\tilde{F}_P, E), (\tilde{G}_Q, E)$ and $(\tilde{H}_R, E)$ are respectively,

$$(\tilde{p}_{ij}) = \begin{pmatrix} (0.9,0.6,0.1) & (0,0,1) & (1,0,0) & (0,0,1) \\ (0.3,0.2,0.5) & (0,0,1) & (0.6,0.4,0.3) & (0,0,1) \\ (0.7,0.4,0.1) & (0,0,1) & (0.3,0.2,0.5) & (0,0,1) \\ (0.2,0.3,0.7) & (0,0,1) & (0.2,0.3,0.7) & (0,0,1) \end{pmatrix},$$

$$(\tilde{q}_{ik}) = \begin{pmatrix} (0,0,1) & (0.4,0.3,0.6) & (0.8,0.5,0.1) & (0,0,1) \\ (0,0,1) & (0.8,0.6,0.1) & (0.6,0.3,0.2) & (0,0,1) \\ (0,0,1) & (0.5,0.1,0.2) & (0.4,0.2,0.5) & (0,0,1) \\ (0,0,1) & (0.3,0.2,0.5) & (0.2,0.3,0.8) & (0,0,1) \end{pmatrix},$$

$$(\tilde{r}_{il}) = \begin{pmatrix} (0.9,0.6,0.1) & (0,0,1) & (0,0,1) & (0.2,0.3,0.7) \\ (0.4,0.2,0.5) & (0,0,1) & (0,0,1) & (0.3,0.1,0.5) \\ (0.6,0.1,0.3) & (0,0,1) & (0,0,1) & (0.6,0.4,0.2) \\ (0.3,0.4,0.5) & (0,0,1) & (0,0,1) & (0.9,0.2,0) \end{pmatrix}$$

(2) The combined choice matrices of Shayana, Dayana and Nayna in different forms are,

$$e_P \begin{pmatrix} & e_{Q\wedge R} & & \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \end{pmatrix} \text{[Since } Q \wedge R = f, P = \{e_1, e_3\}]$$

$$e_Q \begin{pmatrix} & e_{R\wedge P} & & \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (1,0,0) & (0,0,1) & (0,0,1) & (0,0,01) \\ (1,0,0) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \end{pmatrix} \text{[Since } R\wedge P=\{e_1\}, Q=\{e_2,e_3\}]$$

$$e_R \begin{pmatrix} & e_{P\wedge Q} & & \\ (0,0,1) & (0,0,1) & (1,0,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,0,0) & (0,0,1) \end{pmatrix} \text{[Since } P\wedge Q=\{e_3\}, R=\{e_1,e_4\}]$$

(3) Corresponding product neutrosophic soft matrices are,

$$U_P \begin{pmatrix} & e_P & & \\ (0.9,0.6,0.1) & (0,0,1) & (1,0,0) & (0,0,1) \\ (0.3,0.2,0.5) & (0,0,1) & (0.6,0.4,0.3) & (0,0,1) \\ (0.7,0.4,0.1) & (0,0,1) & (0.3,0.2,0.5) & (0,0,1) \\ (0.2,0.3,0.7) & (0,0,1) & (0.2,0.3,0.7) & (0,0,1) \end{pmatrix} \otimes$$

$$e_P \begin{pmatrix} & e_{Q\wedge R} & & \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \end{pmatrix} = \begin{pmatrix} (0,0.75,1) & (0,0.75,1) & (0,0.75,1) & (0,0.75,1) \\ (0,0.75,1) & (0,0.75,1) & (0,0.75,1) & (0,0.75,1) \\ (0,0.75,1) & (0,0.75,1) & (0,0.75,1) & (0,0.75,1) \\ (0,0.75,1) & (0,0.75,1) & (0,0.75,1) & (0,0.75,1) \end{pmatrix}$$

$$U_Q \begin{pmatrix} & e_Q & & \\ (0,0,1) & (0.4,0.3,0.6) & (0.8,0.5,0.1) & (0,0,1) \\ (0,0,1) & (0.8,0.6,0.1) & (0.6,0.3,0.2) & (0,0,1) \\ (0,0,1) & (0.5,0.1,0.2) & (0.4,0.2,0.5) & (0,0,1) \\ (0,0,1) & (0.3,0.2,0.5) & (0.2,0.3,0.8) & (0,0,1) \end{pmatrix} \otimes$$

$$e_Q \begin{pmatrix} & e_{R\wedge P} & & \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (1,0,0) & (0,0,1) & (0,0,1) & (0,0,01) \\ (1,0,0) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \end{pmatrix}$$

$$= \begin{pmatrix} (0.8,0.1,0.1) & (0,0.1,1) & (0,0.1,1) & (0,0.1,1) \\ (0.8,0.1125,0.1) & (0,0.1125,1) & (0,0.1125,1) & (0,0.1125,1) \\ (0.5,0.375,0.2) & (0,0.375,1) & (0,0.375,1) & (0,0.375,1) \\ (0.3,0.625,0.5) & (0,0.625,1) & (0,0.625,1) & (0,0.625,1) \end{pmatrix}$$

$$U_R \begin{pmatrix} & e_R & & \\ (0.9,0.6,0.1) & (0,0,1) & (0,0,1) & (0.2,0.3,0.7) \\ (0.4,0.2,0.5) & (0,0,1) & (0,0,1) & (0.3,0.1,0.5) \\ (0.6,0.1,0.3) & (0,0,1) & (0,0,1) & (0.6,0.4,0.2) \\ (0.3,0.4,0.5) & (0,0,1) & (0,0,1) & (0.9,0.2,0) \end{pmatrix} \otimes$$

$$e_R \begin{pmatrix} & e_{P\wedge Q} & & \\ (0,0,1) & (0,0,1) & (1,0,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,0,0) & (0,0,1) \end{pmatrix} = \begin{pmatrix} (0,0.1125,1) & (0,0.1125,1) & (0.9,0.1125,0.1) & (0,0.1125,1) \\ (0,0.375,1) & (0,0.375,1) & (0.4,0.375,0.5) & (0,0.375,1) \\ (0,0.625,1) & (0,0.625,1) & (0.6,0.625,0.2) & (0,0.625,1) \\ (0,0.15,1) & (0,0.15,1) & (0.9,0.15,0) & (0,0.15,1) \end{pmatrix}$$

[As per the rule of multiplication of neutrosophic soft matrices.]

(4) The sum of these product neutrosophic soft matrices is,

$$\begin{pmatrix} (0,0.75,1) & (0,0.75,1) & (0,0.75,1) & (0,0.75,1) \\ (0,0.75,1) & (0,0.75,1) & (0,0.75,1) & (0,0.75,1) \\ (0,0.75,1) & (0,0.75,1) & (0,0.75,1) & (0,0.75,1) \\ (0,0.75,1) & (0,0.75,1) & (0,0.75,1) & (0,0.75,1) \end{pmatrix}$$

$$\oplus\begin{pmatrix} (0.8,0.1,0.1) & (0,0.1,1) & (0,0.1,1) & (0,0.1,1) \\ (0.8,0.1125,0.1) & (0,0.1125,1) & (0,0.1125,1) & (0,0.1125,1) \\ (0.5,0.375,0.2) & (0,0.375,1) & (0,0.375,1) & (0,0.375,1) \\ (0.3,0.625,0.5) & (0,0.625,1) & (0,0.625,1) & (0,0.625,1) \end{pmatrix}$$

$$\oplus\begin{pmatrix} (0,0.1125,1) & (0,0.1125,1) & (0.9,0.1125,0.1) & (0,0.1125,1) \\ (0,0.375,1) & (0,0.375,1) & (0.4,0.375,0.5) & (0,0.375,1) \\ (0,0.625,1) & (0,0.625,1) & (0.6,0.625,0.2) & (0,0.625,1) \\ (0,0.15,1) & (0,0.15,1) & (0.9,0.15,0) & (0,0.15,1) \end{pmatrix}$$

$$=\begin{pmatrix} (0.8,0.32083,0.1) & (0,0.32083,1) & (0.9,0.32083,0.1) & (0,0.32083,1) \\ (0.8,0.4125,0.1) & (0,0.4125,1) & (0.4,0.4125,0.5) & (0,0.4125,1) \\ (0.5,0.5833,0.2) & (0,0.5833,1) & (0.6,0.5833,0.2) & (0,0.5833,1) \\ (0.3,0.5083,0.5) & (0,0.5083,1) & (0.9,0.5083,0) & (0,0.5083,1) \end{pmatrix} = R_{NS}$$

(5) Now the weights of the story books are,

- $W(b_1) = 0.8+0+0.9+0 = 1.7$
- $W(b_2) = 0.8+0+0.4+0 = 1.2$
- $W(b_3) = 0.5+0+0.6+0 = 1.1$
- $W(b_4) = 0.3+0+0.9+0 = 1.2$

(6) The story book associated with the first row of the resultant neutrosophic soft matrix(R_{NS}) has the highest weight $(W(d_1) = 1.7)$, therefore b_1 be the optimal choice book. Hence Shayana, Dayana and Nayna will buy the story book b_1 for their youngest brother according to their choice parameters.

6 Applications in Medical Science

In medical science there also exist various types of neutrosophic soft set based decision making problems and we may apply the NSM -Algorithm for solving those problems. Now we will discuss two different problems which appear very common in medical science with their solutions.

Problem 6.1 Generally in medical science a patient suffering from a disease may have multiple symptoms. Again it is also observed that there are certain symptoms which may be common to more than one diseases leading to diagnostic dilemma.

Now we consider from medical science (Hauser, Stephen, Braunwald, Fauci and Kasper 2001a, 2001b; Tierney 2003) four symptoms such as abdominal pain, fever, nausea vomiting, diarrhea which have more or less contribution in four diseases such as typhoid, peptic ulcer, food poisoning, acute viral hepatitis. Now, from medical statistics, the degree of availability of these four symptoms in these four diseases are observed as follows. The degree of belongingness of all the symptoms abdominal pain, fever, nausea vomiting and diarrhea for the diseases typhoid, peptic ulcer, food poisoning and acute viral hepatitis are $\{(0.3,0.2,0.6),(0.8,0.5,0.1),(0.1,0.4,0.7),(0.2,0.3,0.7)\},\{(0.9,0.6,0.1),(0.2,0.3,0.6),(0.1,0.7,0.8),(0.1,0.2,0.7)\},\{(0.6,0.4,0.2),(0.3,0.2,0.6),(0.6,0.4,0.3),(0.7,0.3,0.2)\}$ and $\{(0.2,0.4,0.6),(0.6,0.1,0.2),(0.5,0.3,0.4),(0.1,0.5,0.7)\}$ respectively.

Suppose a patient who is suffering from a disease, have the symptoms P (abdominal pain, fever and diarrhea). Now the problem is how a doctor detects the actual disease among these four diseases for that patient. Now we will solve this problem by applying NSM -Algorithm.

Here $U=\{typhoid, peptic\ ulcer, food\ poisoning, acute\ viral\ hepatitis\}=\{d_1,d_2,d_3,d_4\}$,

$E=\{abdominal\ pain, fever, nausea\ vomiting, diarrhea\}=\{e_1,e_2,e_3,e_4\}$ and $P=\{e_1,e_2,e_4\}\subset E$

(1) The neutrosophic soft matrix obtained from the given data is,

$$(\breve{a}_{ij})=\begin{pmatrix}(0.3,0.2,0.6) & (0.8,0.5,0.1) & (0.1,0.4,0.7) & (0.2,0.3,0.7)\\ (0.9,0.6,0.1) & (0.2,0.3,0.6) & (0.1,0.7,0.8) & (0.1,0.2,0.7)\\ (0.6,0.4,0.2) & (0.3,0.2,0.6) & (0.6,0.4,0.3) & (0.7,0.3,0.2)\\ (0.2,0.4,0.6) & (0.6,0.1,0.2) & (0.5,0.3,0.4) & (0.1,0.5,0.7)\end{pmatrix}$$

(2) The choice matrix of the patient is,

$$e_P\begin{pmatrix} & e_P & & \\ (1,0,0) & (1,0,0) & (0,0,1) & (1,0,0)\\ (1,0,0) & (1,0,0) & (0,0,1) & (1,0,0)\\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\ (1,0,0) & (1,0,0) & (0,0,1) & (1,0,0)\end{pmatrix}\ [\text{ Since } P=\{e_1,e_2,e_4\}]$$

(3) and (4) Corresponding product neutrosophic soft matrix is,

$$U_A\begin{pmatrix} & e_P & & \\ (0.3,0.2,0.6) & (0.8,0.5,0.1) & (0.1,0.4,0.7) & (0.2,0.3,0.7) \\ (0.9,0.6,0.1) & (0.2,0.3,0.6) & (0.1,0.7,0.8) & (0.1,0.2,0.7) \\ (0.6,0.4,0.2) & (0.3,0.2,0.6) & (0.6,0.4,0.3) & (0.7,0.3,0.2) \\ (0.2,0.4,0.6) & (0.6,0.1,0.2) & (0.5,0.3,0.4) & (0.1,0.5,0.7) \end{pmatrix} \otimes$$

$$e_P\begin{pmatrix} & e_P & & \\ (1,0,0) & (1,0,0) & (0,0,1) & (1,0,0) \\ (1,0,0) & (1,0,0) & (0,0,1) & (1,0,0) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (1,0,0) & (1,0,0) & (0,0,1) & (1,0,0) \end{pmatrix}$$

$$=\begin{pmatrix} (0.8,0.175,0.1) & (0.8,0.175,0.1) & (0,0.175,1) & (0.8,0.175,0.1) \\ (0.9,0.225,0.1) & (0.9,0.225,0.1) & (0,0.225,1) & (0.9,0.225,0.1) \\ (0.7,0.1625,0.2) & (0.7,0.1625,0.2) & (0,0.1625,1) & (0.7,0.1625,0.2) \\ (0.6,0.1625,0.2) & (0.6,0.1625,0.2) & (0,0.1625,1) & (0.6,0.1625,0.2) \end{pmatrix} = R_{NS}$$

[As per the rule of multiplication of neutrosophic soft matrices.]

(5) Now the weights of the diseases are,

- $W(d_1) = 0.8+0.8+0+0.8 = 2.4$
- $W(d_2) = 0.9+0.9+0+0.9 = 2.7$
- $W(d_3) = 0.7+0.7+0+0.7 = 2.1$
- $W(d_4) = 0.6+0.6+0+0.6 = 1.8$

(6) The disease associated with the second row of the resultant neutrosophic soft matrix(R_{NS}) has the highest weight $(W(d_2) = 2.7)$, therefore d_2 be the optimal choice disease. **Hence the patient is suffering from the disease peptic ulcer(d_2).**

Problem 6.2: In medical science (Carranza 2006) there are different types of diseases and various types of reasons are responsible for them. Now suppose that according to Dr.X. personal habits are responsible for dental caries $(0.7,0.1,0.2)$, for gum disease $(0.8,0.2,0.1)$, for oral ulcer $(0.8,0.4,0.2)$; food habits are responsible for dental caries $(0.8,0.4,0.1)$, for gum disease $(0.7,0.3,0.3)$, for oral ulcer $(0.4,0.2,0.5)$. Again let according to Dr.Y personal habits are responsible for dental caries $(0.6,0.2,0.3)$, for gum disease $(0.8,0.1,0.2)$, for oral ulcer $(0.9,0.4,0.1)$; food habits are responsible for dental caries $(0.8,0.2,0.1)$, for gum disease $(0.7,0.3,0.2)$, for oral ulcer $(0.5,0.2,0.4)$ and hereditary factor is also responsible for dental caries $(0.2,0.5,0.7)$, for gum disease $(0.4,0.2,0.3)$, for oral ulcer $(0.6,0.1,0.3)$. Now the problem is to find out the disease which is mostly affected by the personal habits, food habits and hereditary factors of a human being according to both Dr.X and Dr.Y simultaneously.

Now we will solve this problem by applying NSM-Algorithm.

Here $U = \{dental\ caries, gum\ disease, oral\ ulcer\} = \{d_1, d_2, d_3\}$,

$E = \{personal\ habits, food\ habits, hereditary\ factors\} = \{e_1, e_2, e_3\}$. The choice parameter set of Dr.X is, $A = \{e_1, e_2\} \subset E$ and the choice parameter set of Dr.Y is, $A = \{e_1, e_2, e_3\} \subseteq E$

(1) The neutrosophic soft matrices according to Dr.X and Dr.Y are respectively,

$$(\breve{a}_{ij}) = \begin{pmatrix} (0.7,0.1,0.2) & (0.8,0.4,0.1) & (0,0,1) \\ (0.8,0.2,0.1) & (0.7,0.3,0.3) & (0,0,1) \\ (0.8,0.4,0.2) & (0.4,0.2,0.5) & (0,0,1) \end{pmatrix}$$

$$(\breve{b}_{ik}) = \begin{pmatrix} (0.6,0.2,0.3) & (0.8,0.2,0.1) & (0.2,0.5,0.7) \\ (0.8,0.1,0.2) & (0.7,0.3,0.2) & (0.4,0.2,0.3) \\ (0.9,0.4,0.1) & (0.5,0.2,0.4) & (0.6,0.1,0.3) \end{pmatrix}$$

(2) The combined choice matrices of Mr.X and Mr.Y in different forms are,

$$e_A \begin{pmatrix} & e_B & \\ (1,0,0) & (1,0,0) & (1,0,0) \\ (1,0,0) & (1,0,0) & (1,0,0) \\ (0,0,1) & (0,0,1) & (0,0,1) \end{pmatrix}$$

$$e_B \begin{pmatrix} & e_A & \\ (1,0,0) & (1,0,0) & (0,0,1) \\ (1,0,0) & (1,0,0) & (0,0,1) \\ (1,0,0) & (1,0,0) & (0,0,1) \end{pmatrix}$$

(3) Corresponding product neutrosophic soft matrices are,

$$U_A\begin{pmatrix} & e_A & \\ (0.7,0.1,0.2) & (0.8,0.4,0.1) & (0,0,1) \\ (0.8,0.2,0.1) & (0.7,0.3,0.3) & (0,0,1) \\ (0.8,0.4,0.2) & (0.4,0.2,0.5) & (0,0,1) \end{pmatrix} \otimes e_A\begin{pmatrix} & e_B & \\ (1,0,0) & (1,0,0) & (1,0,0) \\ (1,0,0) & (1,0,0) & (1,0,0) \\ (0,0,1) & (0,0,1) & (0,0,1) \end{pmatrix}$$

$$=\begin{pmatrix} (0.8,0.0833,0.1) & (0.8,0.0833,0.1) & (0.8,0.0833,0.1) \\ (0.8,0.0833,0.1) & (0.8,0.0833,0.1) & (0.8,0.0833,0.1) \\ (0.8,0.1,0.2) & (0.8,0.1,0.2) & (0.8,0.1,0.2) \end{pmatrix}$$

$$U_B\begin{pmatrix} & e_B & \\ (0.6,0.2,0.3) & (0.8,0.2,0.1) & (0.2,0.5,0.7) \\ (0.8,0.1,0.2) & (0.7,0.3,0.2) & (0.4,0.2,0.3) \\ (0.9,0.4,0.1) & (0.5,0.2,0.4) & (0.6,0.1,0.3) \end{pmatrix} \otimes e_E\begin{pmatrix} & e_A & \\ (1,0,0) & (1,0,0) & (0,0,1) \\ (1,0,0) & (1,0,0) & (0,0,1) \\ (1,0,0) & (1,0,0) & (0,0,1) \end{pmatrix}$$

$$=\begin{pmatrix} (0.8,0.15,0.1) & (0.8,0.15,0.1) & (0,0.15,1) \\ (0.8,0.1,0.2) & (0.8,0.1,0.2) & (0,0.1,1) \\ (0.9,0.1166,0.1) & (0.9,0.1166,0.1) & (0,0.1166,1) \end{pmatrix}$$

(4) The sum of these product neutrosophic soft matrices is,

$$\begin{pmatrix} (0.8,0.0833,0.1) & (0.8,0.0833,0.1) & (0.8,0.0833,0.1) \\ (0.8,0.0833,0.1) & (0.8,0.0833,0.1) & (0.8,0.0833,0.1) \\ (0.8,0.1,0.2) & (0.8,0.1,0.2) & (0.8,0.1,0.2) \end{pmatrix} \oplus$$

$$\begin{pmatrix} (0.8,0.15,0.1) & (0.8,0.15,0.1) & (0,0.15,1) \\ (0.8,0.1,0.2) & (0.8,0.1,0.2) & (0,0.1,1) \\ (0.9,0.1166,0.1) & (0.9,0.1166,0.1) & (0,0.1166,1) \end{pmatrix}$$

$$=\begin{pmatrix} (0.8,0.11665,0.1) & (0.8,0.11665,0.1) & (0.8,0.11665,0.1) \\ (0.8,0.09165,0.1) & (0.8,0.09165,0.1) & (0.8,0.09165,0.1) \\ (0.9,0.1083,0.1) & (0.9,0.1083,0.1) & (0.8,0.1083,0.2) \end{pmatrix} = R_{NS}$$

(5) Now the weights of the diseases are,

- $W(d_1) = 0.8 + 0.8 + 0.8 = 2.4$
- $W(d_2) = 0.8 + 0.8 + 0.8 = 2.4$
- $W(d_3) = 0.9 + 0.9 + 0.8 = 2.6$

(6) The disease associated with the third row of the resultant neutrosophic soft matrix(R_{NS}) has the highest weight $(W(d_3) = 2.6)$, therefore d_3 be the optimal choice disease. **Hence oral ulcer(d_3) is mostly affected by the personal habits, food habits and hereditary factor according to the both doctors.**

7 Conclusion

In this paper we have proposed the concept of neutrosophic soft matrix and after that different types of matrices in neutrosophic soft set theory have been defined. Then we have introduced here some new operations and properties on these matrices. Furthermore an efficient solution procedure named as NSM -Algorithm has been developed to solve neutrosophic soft set(or neutrosophic soft matrix) based group decision making problems and it has been applied in medical science to the problems of diagnosis of a disease from the myriad of symptoms as well as to evaluate the effectiveness of different habits of human being responsible for a disease.

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