Chapter 3

Doubt intuitionistic fuzzy H-ideals in BCK/BCI-algebras*

3.1 Introduction

Several concepts in FS theory are accepted in IFS theory such as IF relations, intuitionistic L-FSs, IF implications, DIFSs etc.

Khalid and Ahmad [50] in 1999 brought the idea of FH-ideals in BCI-algebras. In BCK-algebras, characterization of DFH-ideals and concept of IFH-ideals are investigated in 2003 by Zhan and Tan [92] and in 2010 by Satyanarayan et al. [66, 67].

Jun [47], in 2001 made the DP and T-product of T-fuzzy SAs. Abdulla et al. [5, 6], gave some interesting results on DP of fuzzy ideals in different algebraic structures. Furthermore in BCK-algebras, the notion of DP of IFH-ideals are proposed by Abdullah et al [7] in 2012,.

In this current chapter, we have introduced DIFH-ideals in BCK/BCI-algebras and have made a detailed study of its properties. The outcomes made us conclude that in BCK/BCI-algebras, an IFS is a DIFH-ideal if the complement of this IFS is an IFH-ideal. Besides we have also investigated relations among DIF-ideals and DIFH-ideals.

Another unique inclusion that we have made in this chapter the DP of two DIFSAs and two DIFH-ideals of two BCK/BCI-algebras. We have also studied few important

*Part of the works presented in this chapter are published in

(1) Annals of Fuzzy Mathematics and Informatics, **8**(4) 593-605 (2014), *(2)International Journal of Pure and Applied Researches*, **2**(1) (2016), 11-21.

properties and relationships among them. It is noted that the DP of two IFSs appears as DIFH-ideals and DIFSAs if and only if for any $a, b \in [0,1]$, UC of level a and LC of level b of that IFSs are H-ideals or SAs in BCK/BCI-algebra $U \times V$.

3.2 DIFH-ideals in BCK/BCI-algebras

In the current section, in BCK/BCI-algebras the notion of DIFH-ideals is initiated and different features connected to these are investigated.

Definition 3.2.1. *Let $M = (\alpha_M, \zeta_M)$ be an IFS of a BCK/BCI-algebra U, then M is termed as a* ***DIFH-ideal*** *in U if*

(i) $\alpha_M(0) \leq \alpha_M(q^{'}), \zeta_M(0) \geq \zeta_M(q^{'})$

(ii) $\alpha_M(q^{'} * s^{'}) \leq \alpha_M(q^{'} * (r^{'} * s^{'})) \bigvee \alpha_M(r^{'})$

(iii) $\zeta_M(q^{'} * s^{'}) \geq \zeta_M(q^{'} * (r^{'} * s^{'})) \bigwedge \zeta_M(r^{'})$*, for all* $q^{'}, r^{'}, s^{'} \in U$.

Theorem 3.2.1. *In an associative BCK/BCI-algebra U if the inequility $q^{'} * w \leq x$ meets for a DIFH-ideal, then*

(i) $\alpha_M(q^{'} * w) \leq \alpha_M(x)$

(ii) $\zeta_M(q^{'} * w) \geq \zeta_M(x)$.

Proof. Let $q^{'}, w, x \in U$ with $q^{'} * w \leq x$ then $(q^{'} * w) * x = 0$ and as M is a DIFH-ideal in U, so

$$\begin{aligned}
\alpha_M(q^{'} * w) &\leq max\{\alpha_M(q^{'} * (x * w)), \alpha_M(x)\} \\
&= max\{\alpha_M((q^{'} * x) * w), \alpha_M(x)\} \quad \text{[Since U is associative]} \\
&= max\{\alpha_M((q^{'} * w) * x), \alpha_M(x)\} \\
&= max\{\alpha_M(0), \alpha_M(x)\} \\
&= \alpha_M(x) \quad \text{[because } \alpha_M(0) \leq \alpha_M(x) \text{]}
\end{aligned}$$

Therefore, $\alpha_M(q^{'} * w) \leq \alpha_M(x)$.

Again,

$$\begin{aligned}\zeta_M(q^{'}*w) &\geq min\{\zeta_M(q^{'}*(x*w)),\zeta_M(x)\},\\ &= min\{\zeta_M((q^{'}*x)*w),\zeta_M(x)\} \quad \text{[Since U is associative]}\\ &= min\{\zeta_M((q^{'}*w)*x),\zeta_M(x)\}\\ &= min\{\zeta_M(0),\zeta_M(x)\}\\ &= \zeta_M(x) \text{ [because } \zeta_M(0)\geq\zeta_M(x) \text{]}\end{aligned}$$

Therefore, $\zeta_M(q^{'}*w)\geq\zeta_M(x)$. Thus the theorem proves. □

Proposition 3.2.2. *For a DIFH-ideal $M=(\alpha_M,\zeta_M)$ in a BCK-algebra U. We have $\alpha_M(0*(0*q^{'}))\leq\alpha_M(q^{'})$ and $\zeta_M(0*(0*q^{'}))\geq\zeta_M(q^{'})$, for all $q^{'}\in U$.*

Proof. It can be easily proved. □

Lemma 3.2.1. *If an IFS $M=(\alpha_M,\zeta_M)$ be a DIFH-ideal in a BCK/BCI-algebra U. Then for $q^{'}\leq a$, we have, $\alpha_M(q^{'})\leq\alpha_M(a)$ and $\zeta_M(q^{'})\geq\zeta_M(a)$, for all $q^{'},a\in U$.*

Proof. Let $q^{'},a\in U$ such that $q^{'}\leq a$ then $q^{'}*a=0$. Now, $\alpha_M(q^{'})=\alpha_M(q^{'}*0)\leq max\{\alpha_M(q^{'}*(a*0)),\alpha_M(a)\}=max\{\alpha_M(q^{'}*a),\alpha_M(a)\}=max\{\alpha_M(0),\alpha_M(a)\}=\alpha_M(a)$. Therefore, $\alpha_M(q^{'})\leq\alpha_M(a)$.

Again, $\zeta_M(q^{'})=\zeta_M(q^{'}*0)\geq min\{\zeta_M(q^{'}*(a*0)),\zeta_M(a)\}=min\{\zeta_M(q^{'}*a),\zeta_M(a)\}=min\{\zeta_M(0),\zeta_M(a)\}=\zeta_M(a)$. Therefore, $\zeta_M(q^{'})\geq\zeta_M(a)$. □

EXAMPLE 9. *Let $U=\{0,k,l,m,n\}$ be a BCK-algebra as given in table below:*

$*$	0	k	l	m	n
0	0	0	0	0	0
k	k	0	k	0	0
l	l	l	0	0	0
m	m	m	m	0	0
n	n	m	n	k	0

Let $M=(\alpha_M,\zeta_M)$ be an IFS in U as defined by

U	0	k	l	m	n
α_M	0.1	0.4	0.7	0.8	0.8
ζ_M	0.9	0.6	0.2	0.2	0.2

Then $M=(\alpha_M,\zeta_M)$ is a DIFH-ideal in U.

Theorem 3.2.3. *In a BCK-algebra U, the necesssity operator over a DIFH-ideal M that is, $\bigoplus M = \{\langle q', \alpha_M(q'), \bar{\alpha}_M(q')\rangle / q' \in U\}$ is also a DIFH-ideal.*

Proof. Since $M = (\alpha_M, \zeta_M)$ is a DIFH-ideal in U, then $\alpha_M(0) \leq \alpha_M(q')$ and $\alpha_M(q' * s') \leq \alpha_M(q' * (r' * s')) \bigvee \alpha_M(r')$.

Now, $\alpha_M(0) \leq \alpha_M(q')$, or $1 - \bar{\alpha}_M(0) \leq 1 - \bar{\alpha}_M(q')$, or $\bar{\alpha}_M(0) \geq \bar{\alpha}_M(q')$, for any $q' \in U$. Now for any $q', r', s' \in U$, $\alpha_M(q' * s') \leq max\{\alpha_M(q' * (r' * s')), \alpha_M(r')\}$. This gives, $1 - \bar{\alpha}_M(q' * s') \leq max\{1 - \bar{\alpha}_M(q' * (r' * s')), 1 - \bar{\alpha}_M(r')\}$ or, $\bar{\alpha}_M(q' * s') \geq 1 - max\{1 - \bar{\alpha}_M(q' * (r' * s')), 1 - \bar{\alpha}_M(r')\}$. Finally, $\bar{\alpha}_M(q' * s') \geq min\{\bar{\alpha}_M(q' * (r' * s')), \bar{\alpha}_M(r')\}$. Hence, $\bigoplus M = \{(q', \alpha_M(q'), \bar{\alpha}_M(q'))/q' \in U\}$ is a DIFH-ideal in U. □

Theorem 3.2.4. *For a DIFH-ideal $M = (\alpha_M, \zeta_M)$ in U. The possibility operator over M, which is $\bigotimes M = \{\langle q', \bar{\zeta}_M(q'), \zeta_M(q')\rangle / q' \in U\}$ is also a DIFH-ideal in U.*

Proof. Since $M = (\alpha_M, \zeta_M)$ is a DIFH-ideal in U, then $\zeta_M(0) \geq \zeta_M(q')$.
Also, $\zeta_M(q' * z) \geq \zeta_M(q' * (r' * z) \bigwedge \zeta_M(r')$.

Again, we have, $\zeta_M(0) \geq \zeta_M(q')$, or $1 - \bar{\zeta}_M(0) \geq 1 - \bar{\zeta}_M(q')$, or $\bar{\zeta}_M(0) \leq \bar{\zeta}_M(q')$, for any $q' \in U$. Also for any $q', r', z \in U$, $\zeta_M(q' * z) \geq min\{\zeta_M(q' * (r' * z), \zeta_M(r')\}$ This implies, $1 - \bar{\zeta}_M(q' * z) \geq min\{1 - \bar{\zeta}_M(q' * (r' * z), 1 - \bar{\zeta}_M(r')\}$. That is, $\bar{\zeta}_M(q' * z) \leq 1 - min\{1 - \bar{\zeta}_M(q' * (r' * z), 1 - \bar{\zeta}_M(r')\}$ or, $\bar{\zeta}_M(q' * z) \leq max\{\bar{\zeta}_M(q' * (r' * z), \bar{\zeta}_M(r')\}$. Hence, $\bigotimes M = \{\langle q', \bar{\zeta}_M(q'), \zeta_M(q')\rangle / q' \in U\}$ is a DIFH-ideal in U. □

Theorem 3.2.5. *Let $M = (\alpha_M, \zeta_M)$ be an IFS in U. Then $M = (\alpha_M, \zeta_M)$ is a DIFH-ideal in U iff $\bigoplus M$, and $\bigotimes M$ are DIFH-ideals in U.*

Proof. The proof is same as Theorem 3.2.3 and Theorem 3.2.4. □

Let us illustrate the Theorem 3.2.3, Theorem 3.2.4 and Theorem 3.2.5 by the help of the example as defined by.

Example 10. *Let $U = \{0, t, u, v, w\}$ be a BCK-algebra as follows:*

$*$	0	t	u	v	w
0	0	0	0	0	0
t	t	0	t	t	t
u	u	u	0	u	u
v	v	v	v	0	v
w	w	w	w	w	0

Let $M = (\alpha_M, \zeta_M)$ be a DIFH-ideal in U as defined by

U	0	t	u	v	w
α_M	0.2	0.4	0.5	0.5	0.6
ζ_M	0.8	0.6	0.4	0.5	0.4

Then $\bigoplus M$, is defined as follows:

U	0	t	u	v	w
α_M	0.2	0.4	0.5	0.5	0.6
$\bar{\alpha}_M$	0.8	0.6	0.5	0.5	0.4

Also $\bigotimes M$ is defined by

U	0	t	u	v	w
$\bar{\zeta}_M$	0.2	0.4	0.6	0.5	0.6
ζ_M	0.8	0.6	0.4	0.5	0.4

So, it can be verified that $\bigoplus M = \{\langle q', \alpha_M(q'), \bar{\alpha}_M(q')\rangle / q' \in U\}$ and $\bigotimes M = \{\langle q', \bar{\zeta}_M(q'), \zeta_M(q')\rangle / q' \in U\}$ are DIFH-ideals in U.

Theorem 3.2.6. *An IFS $M = (\alpha_M, \zeta_M)$ is a DIFH-ideal in U iff the FSs α_M and $\bar{\zeta}_M$ are DFH-ideals in U.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIFH-ideal in U. Then it is obvious that α_M is a DFH-ideal in U, and from Theorem 3.8, we conclude that $\bar{\zeta}_M$ is a DFH-ideal in U.

Conversely, let α_M be a DFH-ideal in U. Therefore $\alpha_M(0) \leq \alpha_M(q')$, $\alpha_M(q') \leq max\{\alpha_M(q' * (r' * s')), \alpha_M(r')\}, \forall q', r', s' \in U$. Again, since $\bar{\zeta}_M$ is a DFH-ideal in U, so, $\bar{\zeta}_M(0) \leq \bar{\zeta}_M(q')$, gives $1 - \zeta_M(0) \leq 1 - \zeta_M(q')$, implies $\zeta_M(0) \geq \zeta_M(q')$.

Also, $\bar{\zeta}_M(q' * s') \leq max\{\bar{\zeta}_M(q' * (r' * s')), \bar{\zeta}_M(r')\}$ or, $1 - \zeta_M(q' * s') \leq max\{1 - \zeta_M(q' * (r' * s')), 1 - \zeta_M(r')\}$ or, $\zeta_M(q' * s') \geq 1 - max\{1 - \zeta_M(q' * (r' * s')), 1 - \zeta_M(r')\}$. Finally, $\zeta_M(q' * s') \geq min\{\zeta_M(q' * (r' * s')), \zeta_M(r')\}$, for all $q', r' \in U$. Hence, $M = (\alpha_M, \zeta_M)$ is a DIFH-ideal in U. □

Corollary 3.2.1. *Let $M = (\alpha_M, \zeta_M)$ be a DIFH-ideal in U. Then the sets, $D_{\alpha_M} = \{q' \in U / \alpha_M(q') = \alpha_M(0)\}$, and $D_{\zeta_M} = \{q' \in U / \zeta_M(q') = \zeta_M(0)\}$ are H-ideals in U.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIFH-ideal in U. Obviously, $0 \in D_{\alpha_M}$ and D_{ζ_M}. Now, let $q^{'}, r^{'}, s^{'} \in U$, such that $q^{'} * (r^{'} * s^{'})$, $r^{'} \in D_{\alpha_M}$. Then $\alpha_M(q^{'} * (r^{'} * s^{'})) = \alpha_M(0) = \alpha_M(r^{'})$. Now, $\alpha_M(q^{'} * s^{'}) \leq max\{\alpha_M(q^{'} * (r^{'} * s^{'})), \alpha_M(r^{'})\} = \alpha_M(0)$.

Again, since α_M is a DFH-ideal in U, $\alpha_M(0) \leq \alpha_M(q^{'} * s^{'})$. Therefore, $\alpha_M(0) = \alpha_M(q^{'} * s^{'})$. It follows that, $q^{'} * s^{'} \in D_{\alpha_M}$, for all $q^{'}, r^{'}, s^{'} \in U$. Therefore, D_{α_M} is a H-ideal in U. In the similar fashion we can conclude that D_{ζ_M} is also a H-ideal in U. □

Definition 3.2.2. *Let $M = (\alpha_M, \zeta_M)$ be an IFS in U, and $c, d \in [0, 1]$, then UC of level c and LC of level d of M, is as follows:*

$$\alpha^{\leq}_{M,c} = \{q^{'} \in U / \alpha_M(q^{'}) \leq c\}$$
$$and \ \ \zeta^{\geq}_{M,d} = \{q^{'} \in U / \zeta_M(q^{'}) \geq d\}.$$

Theorem 3.2.7. *If $M = (\alpha_M, \zeta_M)$ be a DIFH-ideal in U, then $\alpha^{\leq}_{M,c}$ and $\zeta^{\geq}_{M,d}$ are H-ideals in U for any $c, d \in [0, 1]$.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIFH-ideal in U, and let $c \in [0, 1]$ with $\alpha_M(0) \leq c$. Then we have, $\alpha_M(0) \leq \alpha_M(q^{'})$, for all $q^{'} \in U$, but $\alpha_M(q^{'}) \leq c$, for all $q^{'} \in \alpha^{\leq}_{M,c}$. So, $0 \in \alpha^{\leq}_{M,c}$. Let $q^{'}, r^{'}, s^{'} \in U$ with $q^{'} * (r^{'} * s^{'}) \in \alpha^{\leq}_{M,c}$ and $r^{'} \in \alpha^{\leq}_{M,c}$, then, $\alpha_M(q^{'} * (r^{'} * s^{'})) \in \alpha^{\leq}_{M,c}$ and $\alpha_M(r^{'}) \in \alpha^{\leq}_{M,c}$. Therefore, $\alpha_M(q^{'} * (r^{'} * s^{'})) \leq c$ and $\alpha_M(r^{'}) \leq c$. Since α_M is a DFH-ideal in U, it follows that, $\alpha_M(q^{'} * s^{'}) \leq \alpha_M((q^{'} * (r^{'} * s^{'})) \bigvee \alpha_M(r^{'}) \leq c$ and hence $q^{'} * s^{'} \in \alpha^{\leq}_{M,c}$, for all $q^{'}, r^{'}, s^{'} \in U$. Therefore, $\alpha^{\leq}_{M,c}$ is a H-ideal in U for $c \in [0, 1]$. In the similar fashion it can also be proved that $\zeta^{\geq}_{M,d}$ is a H-ideal in U for $d \in [0, 1]$. □

Theorem 3.2.8. *If $\alpha^{\leq}_{M,c}$ and $\zeta^{\geq}_{M,d}$ are either empty or H-ideals in U for $c, d \in [0, 1]$, then $M = [\alpha_M, \zeta_M]$ is a DIFH-ideal in U.*

Proof. Let $\alpha^{\leq}_{M,c}$ and $\zeta^{\geq}_{M,d}$ be either empty or H-ideals in U for $c, d \in [0, 1]$. For any $q^{'} \in U$, let $\alpha_M(q^{'}) = c$ and $\zeta_M(q^{'}) = d$. Then $q^{'} \in \alpha^{\leq}_{M,c} \bigwedge \zeta^{\geq}_{M,d}$, so $\alpha^{\leq}_{M,c} \neq \phi \neq \zeta^{\geq}_{M,d}$. Since $\alpha^{\leq}_{M,c}$ and $\zeta^{\geq}_{M,d}$ are H-ideals of U, therefore $0 \in \alpha^{\leq}_{M,c} \bigwedge \zeta^{\geq}_{M,d}$. Hence, $\alpha_M(0) \leq c = \alpha_M(q^{'})$ and $\zeta_M(0) \geq d = \zeta_M(q^{'})$, where $q^{'} \in U$. If there exist $a^{'}, b^{'}, c^{'} \in U$ to the extent that $\alpha_M(a^{'} * c^{'}) > max\{\alpha_M(a^{'} * (b^{'} * c^{'})), \alpha_M(b^{'})\}$, now considering, $c_0 = \frac{1}{2}(\alpha_M(a^{'} * c^{'}) + max\{\alpha_M(a^{'} * (b^{'} * c^{'})), \alpha_M(b^{'})\})$, We have, $\alpha_M(a^{'} * c^{'}) > c_0 > max\{\alpha_M(a^{'} * (b^{'} * c^{'})), \alpha_M(b^{'})\}$. Hence, $a^{'} * c^{'} \notin \alpha^{\leq}_{M,c_0}, (a^{'} * (b^{'} * c^{'})) \in \alpha^{\leq}_{M,c_0}$ and

$b' \in \alpha^{\leq}_{M,c_0}$, that is $\alpha^{\leq}_{M,c_0}$ is not an H-ideal in U, here a contradiction arises. Therefore, $\alpha_M(q' * s') \leq \alpha_M((q' * (r' * s')) \bigvee \alpha_M(r')$, for any $q', r', s' \in U$.

Eventually, suppose that there exist $e', f', g' \in U$ s.t $\zeta_M(e' * g') < min\{\zeta_M(e' * (f' * g')), \zeta_M(f')\}$. Taking $d_0 = \frac{1}{2}(\zeta_M(e' * g') + min\{\zeta_M(e' * (f' * g')), \zeta_M(f')\})$, then $min\{\zeta_M(e' * (f' * g')), \zeta_M(f')\} > d_0 > \zeta_M(e' * g')$. Therefore, $e' * (f' * g') \in \zeta^{\geq}_{M,d}$ and $f' \in \zeta^{\geq}_{M,d}$ but $e' * g' \notin \zeta^{\geq}_{M,d}$. Again a contradiction springs. Thus the proof ends. □

But, if an IFS $M = (\alpha_M, \zeta_M)$, is not a DIFH-ideal in U, then $\alpha^{\leq}_{M,c}$ and $\zeta^{\geq}_{M,d}$ are not H-ideals in U for $c, d \in [0, 1]$, which is described through the example as given below.

Example 11. *Let consider a BCK-algebra V that was given in Example 6 in the below tabulated form:*

$*$	0	d_1	e_1	f_1
0	0	0	0	0
d_1	d_1	0	d_1	d_1
e_1	e_1	d_1	0	0
f_1	f_1	d_1	f_1	0

Let $M = (\alpha_M, \zeta_M)$ be an IFS in V defined by

V	0	d_1	e_1	f_1
α_M	0.1	0.5	0.7	0.6
ζ_M	0.8	0.4	0.2	0.4

which is not a DIFH-ideal in U .

For $c = 0.671$ and $d = 0.252$, we get $\alpha^{\leq}_{M,c} = \zeta^{\geq}_{M,d} = \{0, d_1, f_1\}$, which are not H-ideals in U, as $e_1 * (d_1 * 0) = e_1 * d_1 = d_1 \in \{0, d_1, f_1\}$, and $d_1 \in \{0, d_1, f_1\}$, but $e_1 * 0 \notin \{0, d_1, f_1\}$.

Theorem 3.2.9. *Union of any two DIFH-ideals in U, is also a DIFH-ideal in U.*

Proof. Let $M = (\alpha_M, \zeta_M)$ and $N = (\alpha_N, \zeta_N)$ be two DIFH-ideals in U. Again let, $C = M \cup N = (\alpha_C, \zeta_C)$, where $\alpha_C = \alpha_M \vee \alpha_N$ and $\zeta_C = \zeta_M \wedge \zeta_N$. Let $q' \in U$, then, $\alpha_C(0) = (\alpha_M \vee \alpha_N)(0) = max\{\alpha_M(0), \alpha_N(0)\} \leq max\{\alpha_M(q'), \alpha_N(q')\} = (\alpha_M \vee \alpha_N)(q') = \alpha_C(q')$ and $\zeta_C(0) = (\zeta_M \wedge \zeta_N)(0) = min\{\zeta_M(0), \zeta_N(0)\} \geq min\{\zeta_M(q'), \zeta_N(q')\} =$

$(\zeta_M \wedge \zeta_N)(q^{'}) = \zeta_C(q^{'})$ Also,

$$\begin{aligned}\alpha_C(q^{'} * s^{'}) &= maq^{'}\{\alpha_M(q^{'} * s^{'}), \alpha_N(q^{'} * s^{'})\} \\ &\leq max\{max[\alpha_M(q^{'} * (r^{'} * s^{'}), \alpha_M(r^{'})], max[\alpha_N(q^{'} * (r^{'} * s^{'}), \alpha_N(r^{'})]\} \\ &= max\{max[\alpha_M(q^{'} * (r^{'} * s^{'}), \alpha_N(q^{'} * (r^{'} * s^{'})], max[\alpha_M(r^{'}), \alpha_N(r^{'})]\} \\ &= max[\alpha_C(q^{'} * (r^{'} * s^{'}), \alpha_C(r^{'})].\end{aligned}$$

Similarly, we can prove that, $\zeta_C(q^{'} * s^{'}) \geq min[\zeta_C(q^{'} * (r^{'} * s^{'}), \zeta_C(r^{'})]$.
Thus the proof ends. □

Theorem 3.2.10. *Let consider two IFSs M and N in U, such that one is contained in another. Also M and N are two DIFH-ideals in U. Then intersection of M and N is also DIFH-ideal in U .*

Proof. Consider two DIFH-ideals $M = (\alpha_M, \zeta_M)$ and $N = (\alpha_N, \zeta_N)$ in U. Again let, $D = M \cap N = (\alpha_D, \zeta_D)$, where $\alpha_D = \alpha_M \wedge \alpha_N$ and $\zeta_D = \zeta_M \vee \zeta_N$. Let $q^{'}, r^{'}, s^{'} \in U$, then $\alpha_D(0) = \alpha_M(0) \wedge \alpha_N(0) \leq \alpha_M(q^{'}) \wedge \alpha_N(q^{'}) = \alpha_D(q^{'})$ and $\zeta_D(0) = \zeta_M(0) \vee \zeta_N(0) \geq \zeta_M(q^{'}) \vee \zeta_N(q^{'}) = \zeta_D(q^{'})$. Also,

$$\begin{aligned}\alpha_D(q^{'} * s^{'}) &= \alpha_M(q^{'} * s^{'}) \wedge \alpha_N(q^{'} * s^{'}) \\ &\leq max[\alpha_M(q^{'} * (r^{'} * s^{'})), \alpha_M(r^{'})] \wedge max[\alpha_N(q^{'} * (r^{'} * s^{'})), \alpha_N(r^{'})] \\ &= max\{[\alpha_M(q^{'} * (r^{'} * s^{'})) \wedge \alpha_N(q^{'} * (r^{'} * s^{'}))], [\alpha_M(r^{'}) \wedge \alpha_N(r^{'})]\}, \\ &\quad \text{[because one is contained in another]} \\ &= max[\alpha_D(q^{'} * (r^{'} * s^{'})), \alpha_D(r^{'})].\end{aligned}$$

In similar manner we can proof that, $\zeta_D(q^{'} * s^{'}) \geq min[\zeta_D(q^{'} * (r^{'} * s^{'})), \zeta_D(r^{'})]$.
Thus the proof ends. □

The example given below supports the Theorem 3.2.9 and Theorem 3.2.10.

EXAMPLE 12. *Let a BCI-algebra $U = \{0, i, j, k\}$ be considered as given in below tabulated form:*

$*$	0	i	j	k
0	0	i	j	k
i	i	0	k	j
j	j	k	0	i
k	k	j	i	0

Let $M = (\alpha_M, \zeta_M)$ be an IFS in U as follows:

U	0	i	j	k
α_M	0	0.3	0.2	0.3
ζ_M	1	0.7	0.8	0.7

Then $M = (\alpha_M, \zeta_M)$ is a DIFH-ideal in U.

Again, let $N = (\alpha_N, \zeta_N)$ be an IFS in U as defined by

U	0	i	j	k
α_N	0.2	0.4	0.5	0.5
ζ_N	0.8	0.6	0.5	0.5

Then $N = (\alpha_N, \zeta_N)$ is a DIFH-ideal in U.

We also assume that $P = M \cup N = (\alpha_P, \zeta_P)$ where $\alpha_P = \alpha_M \vee \alpha_N$ and $\zeta_P = \zeta_M \wedge \zeta_N$ and P is defined as:

U	0	i	j	k
α_P	0.2	0.4	0.5	0.5
ζ_P	0.8	0.6	0.5	0.5

Then $P = (\alpha_P, \zeta_P)$ is a DIFH-ideal in U.

Now let, $Q = M \cap N = (\alpha_Q, \zeta_Q)$ where $\alpha_Q = \alpha_M \wedge \alpha_N$ and $\zeta_Q = \zeta_M \vee \zeta_N$.

Then Q is an IFS in U which can be defined as:

U	0	i	j	k
α_Q	0	0.3	0.2	0.3
ζ_Q	1	0.7	0.8	0.7

So, $Q = (\alpha_Q, \zeta_Q)$ is a DIFH-ideal in U.

Theorem 3.2.11. *Every DIFH-ideal in U is a DIF-ideal in U.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIFH-ideal in U, then (i) $\alpha_M(0) \leq \alpha_M(q'); \zeta_M(0) \geq \zeta_M(q')$, (ii) $\alpha_M(q' * s') \leq \alpha_M(q' * (r' * s')) \bigvee \alpha_M(r')$, and (iii) $\zeta_M(q' * s') \geq \zeta_M(q' * (r' * s')) \bigwedge \zeta_M(r'), \forall q', r', s' \in U$. If we put $s' = 0$, then from (ii) and (iii), we get $\alpha_M(q') \leq \alpha_M(q' * r') \bigvee \alpha_M(r')$ and $\zeta_M(q') \geq \zeta_M(q' * r') \bigwedge \zeta_M(r')$, for all $q', r', s' \in U$, since $q' * 0 = q'$, for all $q' \in U$.

Hence, M is a DIF-ideal in U. □

The theorem may not hold in reverse direction. That is every DIF-ideal in U is not a DIFH-ideal in U and this can be substantiated by example below:

EXAMPLE 13. *Let $U = \{0, q, r\}$ be a BCI-algebra given by the table below:*

$*$	0	q	r
0	0	r	q
q	q	0	r
r	r	q	0

Let $M = (\alpha_M, \zeta_M)$ be an IFS in U as defined by

U	0	q	r
α_M	0	0.8	0.8
ζ_M	1	0.2	0.2

Then $M = (\alpha_M, \zeta_M)$ is a DIF-ideal in U.

Where M is not a DIFH-ideal in U, since $\alpha_M(q*r) \nleq max\{\alpha_M(q*(0*r)), \alpha_M(0)\}$. Because, $\alpha_M(q*r) = 0.8$ and $max\{\alpha_M(q*(0*r)), \alpha_M(0)\} = \alpha_M(0) = 0$.

For the IFS $M = (\alpha_M, \zeta_M)$, which is a DIF-ideal in U to be a DIFH-ideal in U, now a condition is created.

Theorem 3.2.12. *In an associative BCK/BCI-algebra U, every DIF-ideal becomes a DIFH-ideal in U.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIFI in U. Then, $\alpha_M(0) \leq \alpha_M(q^{'}); \zeta_M(0) \geq \zeta_M(q^{'})$. Now, since U is associative, then for $q^{'}, r^{'}, s^{'} \in U$, $q^{'} * (r^{'} * s^{'}) = (q^{'} * r^{'}) * s^{'}$. Now,

$$\begin{aligned} \alpha_M(q^{'} * (r^{'} * s^{'})) \bigvee \alpha_M(r^{'}) &= \alpha_M((q^{'} * r^{'}) * s^{'}) \bigvee \alpha_M(r^{'}) \\ &= \alpha_M((q^{'} * s^{'}) * r^{'}) \bigvee \alpha_M(r^{'}) \\ &\geq \alpha_M(q^{'} * s^{'}) \end{aligned}$$

[because M is a DIF-ideal.]

Therefore, $\alpha_M(q^{'} * s^{'}) \leq \alpha_M(q^{'} * (r^{'} * s^{'})) \bigvee \alpha_M(r^{'})$.

Similarly we can prove that, $\zeta_M(q^{'} * s^{'}) \geq \zeta_M(q^{'} * (r^{'} * s^{'})) \bigwedge \zeta_M(r^{'}), \forall q^{'}, r^{'}, s^{'} \in U$. Hence, M is a DIFH-ideal in U. Thus the proof ends. □

Let illustrate the Theorem 3.2.12 with the help of the example below.

EXAMPLE 14. *Let consider a BCK-algebra U that was given in Example 10, with the table below:*

$*$	0	t	u	v	w
0	0	0	0	0	0
t	t	0	t	t	t
u	u	u	0	u	u
v	v	v	v	0	v
w	w	w	w	w	0

Here U is an associative BCK-algebra. Let $M = (\alpha_M, \zeta_M)$ be an IFS in U as represented by

U	0	t	u	v	w
$\bar{\zeta}_M$	0	0.6	0.4	0.8	0.9
ζ_M	1	0.4	0.6	0.2	0.1

Hence, M is a DIF-ideal as well as DIFH-ideal in U.

3.3 DP of DIFH-ideals in BCK/BCI-algebras

In BCK/BCI-algebras the DP of DIFSAs and DIFH-ideals are initiated in this current section, and to study this we first define the product of DIFSs in $U \times V$. Some properties connected to these are investigated .

Definition 3.3.1. *Let U, V be two BCK/BCI-algebras. Again let $K = (\alpha_K, \zeta_K)$ and $M = (\alpha_M, \zeta_M)$ be two DIF sets in U and V respectively. Then the DP of DIFSAs K and M is symbolized by $K \times M = (\alpha_{K\times M}, \zeta_{K\times M})$, here $\alpha_{K\times M} : U \times V \to [0, 1]$ with $\alpha_{K\times M}(k, m) = max\{\alpha_K(u), \alpha_M(v)\}$ and $\zeta_{K\times M} : U \times V \to [0, 1]$ with $\zeta_{K\times M}(k, m) = min\{\zeta_K(u), \zeta_M(v)\} \forall (k, m) \in U \times V$.*

Definition 3.3.2. *An IFS $K \times M = (\alpha_{K\times M}, \zeta_{K\times M})$ of BCK/BCI-algebra $U \times V$ is named as a DIFSA in $U \times V$ if*

$(K_1)\alpha_{K\times M}((k_1, m_1) * (k_2, m_2)) \leq max\{\alpha_{K\times M}(k_1, m_1), \alpha_{K\times M}(k_2, m_2)\}$

$(K_2)\zeta_{K\times M}((k_1, m_1)*(k_2, m_2)) \geq min\{\zeta_{K\times M}(k_1, m_1), \zeta_{K\times M}(k_2, m_2)\}$, for all $(k_1, m_1), (k_2, m_2) \in U \times V$.

Theorem 3.3.1. *Let $K = (\alpha_K, \zeta_K)$ and $M = (\alpha_M, \zeta_M)$ be two DIFSAs in U and V respectively. Then $K \times M = (\alpha_{K\times M}, \zeta_{K\times M})$ is also a DIFSA in $U \times V$.*

Proof. For any $(k_1, m_1), (k_2, m_2) \in U \times V$. Then

$$\begin{aligned}
\alpha_{K\times M}((k_1, m_1) * (k_2, m_2)) &= \alpha_{K\times M}(k_1 * k_2, m_1 * m_2) \\
&= max\{\alpha_K((k_1 * k_2), \alpha_M((m_1 * m_2)\} \\
&\leq max\{max\{\alpha_K(k_1), \alpha_K(k_2)\}, max\{\alpha_M(m_1), \alpha_M(m_2)\}\} \\
&= max\{max\{\alpha_K(k_1), \alpha_M(m_1)\}, max\{\alpha_K(k_2), \alpha_M(m_2)\}\} \\
&= max\{\alpha_{K\times M}(k_1, m_1), \alpha_{K\times M}(k_2, m_2)
\end{aligned}$$

Again,

$$\begin{aligned}
\zeta_{K\times M}((k_1, m_1) * (k_2, m_2)) &= \zeta_{K\times M}(k_1 * k_2, m_1 * m_2) \\
&= min\{\zeta_K((k_1 * k_2), \zeta_M((m_1 * m_2)\} \\
&\geq min\{min\{\zeta_K(k_1), \zeta_K(k_2)\}, max\{\zeta_M(m_1), \zeta_M(m_2)\}\} \\
&= min\{min\{\zeta_K(k_1), \zeta_M(m_1)\}, max\{\zeta_K(k_2), \zeta_M(m_2)\}\} \\
&= min\{\zeta_{K\times M}(k_1, m_1), \zeta_{K\times M}(k_2, m_2)
\end{aligned}$$

Therefore, for all $(k_1, m_1), (k_2, m_2) \in U \times V$, $K \times M$ is a DIFSA in BCK/BCI-algebra $U \times V$. Thus the proof ends. □

Theorem 3.3.2. *Let $K = (\alpha_K, \zeta_K)$ and $M = (\alpha_M, \zeta_M)$ be two DIFSA in BCK/BCI-algebras U and V. Then*

(i) $\alpha_{K\times M}(0, 0) \leq \alpha_{K\times M}(k, m)$

(ii) $\zeta_{K\times M}(0, 0) \geq \zeta_{K\times M}(k, m), \forall (k, m) \in U \times V$.

Proof. By definition, $\alpha_{K\times M}(0, 0) = \alpha_{K\times M}\{(k, m)*(k, m)\} \leq \alpha_{K\times M}(k, m) \bigvee \alpha_{K\times M}(k, m) \leq \alpha_{K\times M}(k, m)$.

$\therefore \alpha_{K\times M}(0, 0) \leq \alpha_{K\times M}(k, m), \forall (k, m) \in U \times V$.

Again, $\zeta_{K\times M}(0, 0) = \zeta_{K\times M}\{(k, m)*(k, m)\} \geq \zeta_{K\times M}(k, m) \bigwedge \zeta_{K\times M}(k, m) \geq \zeta_{K\times M}(k, m)$.

$\therefore \zeta_{K\times M}(0, 0) \geq \zeta_{K\times M}(k, m), \forall (k, m) \in U \times V$. □

Lemma 3.3.1. *Let $K = (\alpha_K, \zeta_K)$ and $M = (\alpha_M, \zeta_M)$ be two DIFSAs in U and V. Then the assertions below are fulfilled.*

(i) $\alpha_K(0) \leq \alpha_M(v)$ *and* $\alpha_M(0) \leq \alpha_K(u), \forall u \in U, v \in V$.

(ii) $\zeta_K(0) \geq \zeta_M(v)$ *and* $\zeta_M(0) \geq \zeta_K(u), \forall u \in U, v \in V$.

Proof. Let $\alpha_M(v) < \alpha_K(0)$ and $\alpha_K(u) < \alpha_M(0)$, for several $u \in U$ and $v \in V$.

Then, $\alpha_{K\times M}(k, m) = max[\alpha_K(u), \alpha_M(v)] \leq max[\alpha_M(0), \alpha_K(0)] = \alpha_{K\times M}(0, 0)$.

Which is a contradiction.

Similarly, let $\zeta_K(u) > \zeta_M(0)$ and $\zeta_M(v) > \zeta_K(0)$, for some $u \in U$ and $v \in V$. Now, $\zeta_{K\times M}(k, m) = min[\zeta_K(u), \zeta_M(v)] \geq min[\zeta_M(0), \zeta_K(0)] = \zeta_{K\times M}(0, 0)$.

Hence a contradiction arises. Thus the result is proved. □

Theorem 3.3.3. *If $K \times M$ is a DIFSA in $U \times V$, then either K or M is a DIFSA in $U \times V$.*

Proof. Since $K \times M$ is a DIF-SA in $U \times V$, then for all $(k_1, m_1), (k_2, m_2) \in U \times V$, we have, $\alpha_{K\times M}((k_1, m_1) * (k_2, m_2)) \leq max\{\alpha_{K\times M}(k_1, m_1), \alpha_{K\times M}(k_2, m_2)\}$.

By putting $k_1 = k_2 = 0$, we get,

$\alpha_{K\times M}((0, m_1) * (0, m_2)) \leq max\{\alpha_{K\times M}(0, m_1), \alpha_{K\times M}(0, m_2)\} \cdots (i)$.

Also we have, $\alpha_{K\times M}((0, m_1) * (0, m_2)) = \alpha_{K\times M}((0 * 0), (m_1 * m_2)) = max\{\alpha_K(0 * 0), \alpha_M(m_1 * m_2)\} = \alpha_M(m_1 * m_2) \cdots (ii)$.

Again by using Lemma3.3.1 we have, $max\{\alpha_{K\times M}(0, m_1), \alpha_{K\times M}(0, m_2)\}$

$= max\{\alpha_M(m_1), \alpha_M(m_2)\} \cdots (iii)$.

So from $(i), (ii)$and(iii) we get, $\alpha_M(m_1 * m_2) \leq max[\alpha_M(m_1), \alpha_M(m_2)]$.

Similar way we can prove, $\zeta_M(m_1 * m_2) \geq min[\zeta_M(m_1), \zeta_M(m_2)]$. Hence M is a DIFSA in $U \times V$. □

Definition 3.3.3. *An IFS $K\times M = (\alpha_{K\times M}, \zeta_{K\times M})$ in $U\times V$ is named as a DIFH-ideal in $U \times V$ if*

$(K_3)\alpha_{K\times M}(0, 0) \leq \alpha_{K\times M}(k, m)$ *and* $\zeta_{K\times M}(0, 0) \geq \zeta_{K\times M}(k, m)$

$(K_4)\alpha_{K\times M}((k_1, m_1)*(k_3, m_3)) \leq max\{\alpha_{K\times M}((k_1, m_1)*((k_2, m_2)*(k_3, m_3))), \alpha_{K\times M}(k_2, m_2)\}$

$(K_5)\zeta_{K\times M}((k_1, m_1)*(k_3, m_3)) \geq min\{\zeta_{K\times M}((k_1, m_1)*((k_2, m_2)*(k_3, m_3))), \zeta_{K\times M}(k_2, m_2)\}$,

for all $(k_1, m_1), (k_2, m_2), (k_3, m_3) \in U \times V$.

Now, let us study and investigate different marked properties of DPs unreached so far.

Theorem 3.3.4. *Let $K = (\alpha_K, \zeta_K)$ and $M = (\alpha_M, \zeta_M)$ be two DIFH-ideal in BCK/BCI-algebras U and V. Then $K \times M = (\alpha_{K\times M}, \zeta_{K\times M})$ is a DIFH-ideal in $U \times V$.*

Proof. Let $(k, m) \in U \times V$.

$\alpha_{K\times M}(0,0) = max\{\alpha_K(0), \alpha_M(0)\} \leq max\{\alpha_K(u), \alpha_M(v)\} = \alpha_{K\times M}(k,m)$.

And, $\zeta_{K\times M}(0,0) = min\{\zeta_K(0), \zeta_M(0)\} \geq min\{\zeta_K(u), \zeta_M(v)\} = \zeta_{K\times M}(k,m)$.

Now for any $(k_1,m_1),(k_2,m_2),(k_3,m_3) \in U \times V$,

$$\begin{aligned}
&\alpha_{K\times M}((k_1,m_1)*(k_3,m_3))\\
=&\alpha_{K\times M}(k_1*k_3, m_1*m_3)\\
=&max\{\alpha_K(k_1*k_3), \alpha_M(m_1*m_3)\}\\
\leq&max\{max\{\alpha_K(k_1*(k_2*k_3)), \alpha_K(k_2)\}, max\{\alpha_M(m_1*(m_2*m_3)), \alpha_M(m_2)\}\}
\end{aligned}$$

$$\begin{aligned}
=&max\{max\{\alpha_K(k_1*(k_2*k_3)), \alpha_M(m_1*(m_2*m_3))\}, max\{\alpha_K(k_2), \alpha_M(m_2)\}\}\\
=&max\{\alpha_{K\times M}\{(k_1*(k_2*k_3)),(m_1*(m_2*m_3))\}, \alpha_{K\times M}(k_2,m_2)\}\\
\leq&max\{\alpha_{K\times M}((k_1,m_1)*((k_2,m_2)*(k_3,m_3))), \alpha_{K\times M}(k_2,m_2)\}.
\end{aligned}$$

And

$$\begin{aligned}
&\zeta_{K\times M}((k_1,m_1)*(k_3,m_3))\\
=&\zeta_{K\times M}(k_1*k_3, m_1*m_3)\\
=&min\{\zeta_K(k_1*k_3), \zeta_M(m_1*m_3)\}\\
\geq&min\{min\{\zeta_K(k_1*(k_2*k_3)), \zeta_K(k_2)\}, min\{\zeta_M(m_1*(m_2*m_3)), \zeta_M(m_2)\}\}\\
=&min\{min\{\zeta_K(k_1*(k_2*k_3)), \zeta_M(m_1*(m_2*m_3))\}, min\{\zeta_K(k_2), \zeta_M(m_2)\}\}\\
=&min\{\zeta_{K\times M}\{(k_1*(k_2*k_3)),(m_1*(m_2*m_3))\}, \zeta_{K\times M}(k_2,m_2)\}\\
\geq&min\{\zeta_{K\times M}((k_1,m_1)*((k_2,m_2)*(k_3,m_3))), \zeta_{K\times M}(k_2,m_2)\}.
\end{aligned}$$

Hence for all $(k_1,m_1),(k_2,m_2),(k_3,m_3) \in U \times V$, $K \times M = (\alpha_{K\times M}, \zeta_{K\times M})$ is a DIFH-ideal in $U \times V$. □

The above Theorem is interpreted by the help of the example given below.

EXAMPLE 15. *Let a BCI-algebra $U = \{0, i, j, k\}$ be considered in below tabulated form:*

$*$	0	i	j	k
0	0	i	j	k
i	i	0	k	j
j	j	k	0	i
k	k	j	i	0

Let $K = (\alpha_K, \zeta_K)$ be a DIFH-ideal in U as defined by

U	0	i	j	k
α_K	0	0.3	0.2	0.3
ζ_K	1	0.7	0.8	0.7

Again, let $M = (\alpha_M, \zeta_M)$ be a DIFH-ideal in U as defined by

U	0	i	j	k
α_M	0.2	0.4	0.5	0.5
ζ_M	0.8	0.6	0.5	0.5

Obviously, $U \times U$ is also a BCI-algebra.

Here we get, $\alpha_{K\times M}(0,0) = \alpha_{K\times M}(j,0) = 0.2$, also, $\alpha_{K\times M}(0,i) = \alpha_{K\times M}(j,i) = \alpha_{K\times M}(i,i) = \alpha_{K\times M}(k,i) = 0.4$, again, $\alpha_{K\times M}(0,j) = \alpha_{K\times M}(0,k) = \alpha_{K\times M}(j,j) = \alpha_{K\times M}(j,k) = \alpha_{K\times M}(i,j) = \alpha_{K\times M}(i,k) = \alpha_{K\times M}(k,k) = \alpha_{K\times M}(k,j) = 0.5$.

Again, $\alpha_{K\times M}(i,0) = \alpha_{K\times M}(k,0) = 0.3$.

Also, $\zeta_{K\times M}(0,0) = \zeta_{K\times M}(j,0) = 0.8$, also, $\zeta_{K\times M}(0,i) = \zeta_{K\times M}(j,i) = \zeta_{K\times M}(i,i) = \zeta_{K\times M}(k,i) = 0.6$, again, $\zeta_{K\times M}(0,j) = \zeta_{K\times M}(0,k) = \zeta_{K\times M}(j,j) = \zeta_{K\times M}(j,k) = \zeta_{K\times M}(i,j) = \zeta_{K\times M}(i,k) = \zeta_{K\times M}(k,k) = \zeta_{K\times M}(k,j) = 0.5$.

Again, $\zeta_{K\times M}(i,0) = \zeta_{K\times M}(k,0) = 0.7$.

Then clearly $K \times M$ is a DIFH-ideal in $U \times U$.

Theorem 3.3.5. *Let $K = (\alpha_K, \zeta_K)$ and $M = (\alpha_M, \zeta_M)$ be two DIFH-ideal in U and V respectively. If $K \times M$ is a DIFH-ideal in $U \times V$, then $K \times M$ must be a DIFSA in $U \times V$.*

Proof. Since $K \times M$ is a DIFH-ideal in $U \times V$, then for all $(k_1, m_1), (k_2, m_2), (k_3, m_3) \in U \times V$, we have,

$\alpha_{K\times M}((k_1, m_1)*(k_3, m_3)) \leq max\{\alpha_{K\times M}((k_1, m_1)*((k_2, m_2)*(k_3, m_3))), \alpha_{K\times M}(k_2, m_2)\}$.

Bv putting $k_3 = m_3 = 0$, we get,

$\alpha_{K\times M}(k_1, m_1) \leq max\{\alpha_{K\times M}((k_1, m_1) * (k_2, m_2)), \alpha_{K\times M}(k_2, m_2)\} \cdots (i)$

Again since, $((k_1, m_1) * (k_2, m_2)) \leq (k_1, m_1)$, for all $(k_1, m_1), (k_2, m_2) \in U \times V$.

Then, $\alpha_{K\times M}((k_1, m_1) * (k_2, m_2)) \leq \alpha_{K\times M}(k_1, m_1) \cdots (ii)$.

Hence from (i) and (ii) we get, $\alpha_{K\times M}((k_1, m_1) * (k_2, m_2)) \leq \alpha_{K\times M}(k_1, m_1) \leq max\{\alpha_{K\times M}((k_1, m_1)*(k_2, m_2)), \alpha_{K\times M}(k_2, m_2)\} \leq max\{\alpha_{K\times M}(k_1, m_1), \alpha_{K\times M}(k_2, m_2)\}$,

for all $(k_1, m_1), (k_2, m_2) \in U \times V$.
In the similar manner we can prove that, $\zeta_{K\times M}((k_1, m_1)*(k_2, m_2)) \geq min\{\zeta_{K\times M}(k_1, m_1), \zeta_{K\times M}(k_2, m_2)\}$, for all $(k_1, m_1), (k_2, m_2) \in U \times V$. Thus $K \times M$ is a DIFSA in $U \times V$. □

But the reverse of Theorem 3.3.5 may not be hold in general.

Lemma 3.3.2. *Let $K = (\alpha_K, \zeta_K)$ and $M = (\alpha_M, \zeta_M)$ be two DIFH-ideal in U and V respectively. If $K \times M$ is a DIFH-ideal in $U \times V$, then the followings are true.*
(i) $\alpha_K(0) \leq \alpha_M(v)$ and $\alpha_M(0) \leq \alpha_K(u)$, for all $u \in U$, $v \in V$.
(ii) $\zeta_K(0) \geq \zeta_M(v)$ and $\zeta_M(0) \geq \zeta_K(u)$, for all $u \in U$, $v \in V$.

Proof: Proof is same as Lemma3.3.1.

Lemma 3.3.3. *In a BCK/BCI-algebra $U \times V$, let $K \times M = (\alpha_{K\times M}, \zeta_{K\times M})$ be a DIFH-ideal. If $(s,t) \leq (k,m)$, then $\alpha_{K\times M}(k,m) \leq \alpha_{K\times M}(s,t)$ and $\zeta_{K\times M}(k,m) \geq \zeta_{K\times M}(s,t)$, for all $(s,t),(k,m) \in U \times V$.*

Proof: Let $(s,t),(k,m) \in U \times V$, such that $(s,t) \leq (k,m)$ implies $(s,t) * (k,m) = (0,0)$. Now,

$$\begin{aligned}
\alpha_{K\times M}(k,m) &= \alpha_{K\times M}((k,m) * (0,0)) \\
&\leq max\{\alpha_{K\times M}((k,m) * ((s,t) * (0,0))), \alpha_{K\times M}(s,t)\} \\
&= max\{\alpha_{K\times M}((k,m) * (s,t)), \alpha_{K\times M}(s,t)\} \\
&= \alpha_{K\times M}(s,t).
\end{aligned}$$

And

$$\begin{aligned}
\zeta_{K\times M}(k,m) &= \zeta_{K\times M}((k,m) * (0,0)) \\
&\geq min\{\zeta_{K\times M}((k,m) * ((s,t) * (0,0))), \zeta_{K\times M}(s,t)\} \\
&= min\{\zeta_{K\times M}((k,m) * (s,t)), \zeta_{K\times M}(s,t)\} \\
&= \zeta_{K\times M}(s,t).
\end{aligned}$$

Thus the proof ends.

Theorem 3.3.6. *Let $K = (\alpha_K, \zeta_K)$ and $M = (\alpha_M, \zeta_M)$ be two DIFH-ideals in U and V. Then $\bigoplus(K \times M) = (\alpha_{K\times M}, \bar{\alpha}_{K\times M})$ is a DIFH-ideal of $U \times V$, where, $\bar{\alpha}_{K\times M} = 1 - \alpha_{K\times M}$.*

Proof. Since by Theorem 3.3.4, $K \times M$ is a DIFH-ideal in $U \times V$. Hence for $(k,m) \in U \times V$.

$\alpha_{K\times M}(0,0) \leq \alpha_{K\times M}(k,m)$. Or, $1-\alpha_{K\times M}(0,0) \geq 1-\alpha_{K\times M}(k,m)$. That is $\bar{\alpha}_{K\times M}(0,0) \geq \bar{\alpha}_{K\times M}(k,m)$

Now for any $(k_1,m_1),(k_2,m_2),(k_3,m_3) \in U \times V$, we have

$\alpha_{K\times M}((k_1,m_1)*(k_3,m_3)) \leq max\{\alpha_{K\times M}((k_1,m_1)*((k_2,m_2)*(k_3,m_3))), \alpha_{K\times M}(k_2,m_2)\}$.

Mext $1-\alpha_{K\times M}((k_1,m_1)*(k_3,m_3)) \geq 1-max\{\alpha_{K\times M}((k_1,m_1)*((k_2,m_2)*(k_3,m_3))), \alpha_{K\times M}(k_2,m_2)\}$.

That is, $\bar{\alpha}_{K\times M}((k_1,m_1)*(k_3,m_3)) \geq min\{1-\alpha_{K\times M}((k_1,m_1)*((k_2,m_2)*(k_3,m_3))), 1-\alpha_{K\times M}(k_2,m_2)\}$.

Finally, $\bar{\alpha}_{K\times M}((k_1,m_1)*(k_3,m_3)) \geq min\{\bar{\alpha}_{K\times M}((k_1,m_1)*((k_2,m_2)*(k_3,m_3))), \bar{\alpha}_{K\times M}(k_2,m_2)\}$.

Hence, $\bigoplus(K \times M)$ is a DIFH-ideal in $U \times V$. □

Theorem 3.3.7. *Let $K = (\alpha_K, \zeta_K)$ and $M = (\alpha_M, \zeta_M)$ be two DIFH-ideals in U and V. Then $\bigotimes(K \times M) = (\bar{\zeta}_{K\times M}, \zeta_{K\times M})$ is a DIFH-ideal in $U \times V$, where, $\bar{\zeta}_{K\times M} = 1-\zeta_{K\times M}$.*

Proof. By Theorem 3.3.4, $K \times M$ is a DIFH-ideal in $U \times V$. So for $(k,m) \in U \times V$. $\zeta_{K\times M}(0,0) \geq \zeta_{K\times M}(k,m)$. Hence, $1-\zeta_{K\times M}(0,0) \leq 1-\zeta_{K\times M}(k,m)$. That is $\bar{\zeta}_{K\times M}(0,0) \leq \bar{\zeta}_{K\times M}(k,m)$.

Now for any $(k_1,m_1),(k_2,m_2),(k_3,m_3) \in U \times V$, we have

$\zeta_{K\times M}((k_1,m_1)*(k_3,m_3)) \geq min\{\zeta_{K\times M}((k_1,m_1)*((k_2,m_2)*(k_3,m_3))), \zeta_{K\times M}(k_2,m_2)\}$.

Mext $1-\zeta_{K\times M}((k_1,m_1)*(k_3,m_3))$
$\leq 1-min\{\zeta_{K\times M}((k_1,m_1)*((k_2,m_2)*(k_3,m_3))), \zeta_{K\times M}(k_2,m_2)\}$.

That is, $\bar{\zeta}_{K\times M}((k_1,m_1)*(k_3,m_3)) \leq max\{1-\zeta_{K\times M}((k_1,m_1)*((k_2,m_2)*(k_3,m_3))), 1-\zeta_{K\times M}(k_2,m_2)\}$

. Finally, $\bar{\zeta}_{K\times M}((k_1,m_1)*(k_3,m_3))$
$\leq max\{\bar{\zeta}_{K\times M}((k_1,m_1)*((k_2,m_2)*(k_3,m_3))), \bar{\zeta}_{K\times M}(k_2,m_2)\}$.

So, $\bigotimes(K \times M)$ is a DIFH-ideal in $U \times V$. □

Theorem 3.3.8. *Let $K = (\alpha_K, \zeta_K)$ and $M = (\alpha_M, \zeta_M)$ be two DIFH-ideals in BCK-algebras respectively U and V. Then $K \times M$ is a DIFH-ideals in BCK-algebras $U \times V$ if and only if $\bigoplus(K \times M) = (\alpha_{K\times M}, \bar{\alpha}_{K\times M})$ and $\bigotimes(K \times M) = (\bar{\zeta}_{K\times M}, \zeta_{K\times M})$ are DIFH-ideals in $U \times V$.*

Proof. The proof can be done at ease taking clues from Theorem 3.3.6 and Theorem 3.3.7. □

Proposition 3.3.9. *Let in a BCK-algebra $U \times V$ an IFS $K \times M = (\alpha_{K\times M}, \zeta_{K\times M})$ be a DIFH-ideal. Then $\alpha_{K\times M}((0,0) * ((0,0) * (k,m))) \leq \alpha_{K\times M}(k,m)$ and $\zeta_{K\times M}((0,0) * ((0,0) * (k,m))) \geq \zeta_{K\times M}(k,m), \forall (k,m) \in U \times V$.*

Proof. This can be proved easily. □

Provided that this proposition does not support for all BCI-algebra $U \times V$.

Corollary 3.3.1. *Let $K \times M = (\alpha_{K\times M}, \zeta_{K\times M})$ be a DIFH-ideal in $U \times V$. Then the sets, $D_{\alpha_{K\times M}} = \{(k,m) \in U \times V / \alpha_{K\times M}(k,m) = \alpha_{K\times M}(0,0)\}$,*
and $D_{\zeta_{K\times M}} = \{(k,m) \in U \times V / \zeta_{K\times M}(k,m) = \zeta_{K\times M}(0,0)\}$ are H-ideals in U.

Proof. Let $K \times M = (\alpha_{K\times M}, \zeta_{K\times M})$ be a DIFH-ideal in $U \times V$. Obviously, $(0,0) \in D_{\alpha_K \times M}$ and $D_{\zeta_K \times M}$. Now, let $(k_1,m_1),(k_2,m_2),(k_3,m_3) \in U \times V$, such that $(k_1,m_1) * ((k_2,m_2) * (k_3,m_3))$, $(k_2,m_2) \in D_{\alpha_K \times M}$. Then $\alpha_{K\times M}((k_1,m_1) * ((k_2,m_2) * (k_3,m_3))) = \alpha_{K\times M}(0,0) = \alpha_{K\times M}(k_2,m_2)$. Now, $\alpha_{K\times M}((k_1,m_1) * (k_3,m_3)) \leq max\{\alpha_{K\times M}(k_1,m_1) * ((k_2,m_2) * (k_3,m_3)), \alpha_{K\times M}(k_2,m_2)\} = \alpha_{K\times M}(0,0)$.

Again, since $\alpha_{K\times M}$ is a DFH-ideal in $U \times V$, $\alpha_{K\times M}(0,0) \leq \alpha_{K\times M}((k_1,m_1)*(k_3,m_3))$. Therefore, $\alpha_{K\times M}(0) = \alpha_{K\times M}((k_1,m_1)*(k_3,m_3))$. It follows that, $((k_1,m_1)*(k_3,m_3)) \in D_{\alpha_{K\times M}}$, for all $(k_1,m_1),(k_2,m_2),(k_3,m_3) \in U \times V$. Therefore, $D_{\alpha_{K\times M}}$ is a H-ideal in $U \times V$. In the same manner we can prove $D_{\zeta_{K\times M}}$ is also an H-ideal in $U \times V$. □

Theorem 3.3.10. *For a DIFH-ideal $K \times M$ in $U \times V$, either K or M is a DIFH-ideal in $U \times V$.*

Proof. Since $K \times M$ is a DIFH-ideal in $U \times V$, then for all $(k_1,m_1),(k_2,m_2),(k_3,m_3) \in U \times V$, we have

$\alpha_{K\times M}((k_1,m_1)*(k_3,m_3)) \leq max\{\alpha_{K\times M}((k_1,m_1)*((k_2,m_2)*(k_3,m_3))), \alpha_{K\times M}(k_2,m_2)\}$.

By putting $m_1 = m_2 = m_3 = 0$, we get,

$\alpha_{K\times M}((k_1,0)*(k_3,0)) \leq max\{\alpha_{K\times M}((k_1,0)*((k_2,0)*(k_3,0))), \alpha_{K\times M}(k_2,0)\} \cdots (i)$.

Also we have, $\alpha_{K\times M}((k_1,0) * (k_3,0)) = \alpha_{K\times M}((k_1 * k_3),(0 * 0)) = max\{\alpha_K(k_1 * k_3), \alpha_M(0 * 0)\} = \alpha_K(k_1 * k_3) \cdots (ii)$.

similarly, $\alpha_{K\times M}((k_1,0) * ((k_2,0) * (k_3,0))) = \alpha_K(k_1 * (k_2 * k_3)) \cdots (iii)$.

Again by using Lemma 3.3.2 we have, $max\{\alpha_{K\times M}(k_1, 0), \alpha_{K\times M}(k_2, 0)\}$
$= max\{\alpha_K(k_1), \alpha_K(k_2)\} \cdots (iv)$. $\alpha_{K\times M}(k_2, 0) = \alpha_K(k_2) \cdots (v)$.

So from $(i), (ii), (iii), (iv) and (v)$ we get, $\alpha_K(k_1 * k_3) \leq max\{\alpha_K(k_1 * (k_2 * k_3)), \alpha_K(k_2)\}$.

In the similar way we can proof, $\zeta_K(k_1 * k_3) \geq min\{\zeta_K(k_1 * (k_2 * k_3)), \zeta_K(k_2)\}$. Hence K is a DIFH-ideal in $U \times V$. □

3.4 Upper and lower level sets

Definition 3.4.1. *Let in a BCK/BCI-algebra $U \times V$, $K \times M = (\alpha_{K\times M}, \zeta_{K\times M})$ be a DIFH-ideal, and $a, b \in [0, 1]$, Then UC of level a and LC of level b of $K \times M$, are as followes:*

$$\alpha^{\leq}_{K\times M,a} = \{(k, m) \in (U \times V)/\alpha_{K\times M}(k, m) \leq a\}$$
$$and \;\; \zeta^{\geq}_{K\times M,b} = \{(k, m) \in (U \times V)/\zeta_{K\times M}(k, m) \geq b\}.$$

Theorem 3.4.1. *Let $K \times M = (\alpha_{K\times M}, \zeta_{K\times M})$ be an IFS in $U \times V$, then $K \times M$ is a DIFSA in $U \times V$ iff for any $a, b \in [0, 1]$, UC of level a and LC of level b of $K \times M$ are SAs in $U \times V$.*

Proof. Assume that $K \times M$ be an IFS in $U \times V$. Now for any $a, b \in [0, 1]$ and $(k_1, m_1), (k_2, m_2) \in \alpha^{\leq}_{K\times M,a}$, we have $\alpha_{K\times M}((k_1, m_1) \leq a$ and also $\alpha_{K\times M}((k_2, m_2) \leq a$. Again let $K \times M$ is a DIFSA in $U \times V$, then

$$\begin{aligned} \alpha_{K\times M}((k_1, m_1) * (k_2, m_2)) &\leq max\{\alpha_{K\times M}(k_1, m_1), \alpha_{K\times M}(k_2, m_2)\} \\ &\leq max(a, a) \\ &= a \end{aligned}$$

Therefore it implies that $(k_1, m_1) * (k_2, m_2) \in \alpha^{\leq}_{K\times M,a}$.

Similarly, for any $(k_1, m_1), (k_2, m_2) \in \zeta^{\geq}_{K\times M,b}$, we have $\zeta_{K\times M}((k_1, m_1) \geq b$ and also $\zeta_{K\times M}((k_2, m_2) \geq b$. then

$$\begin{aligned} \zeta_{K\times M}((k_1, m_1) * (k_2, m_2)) &\geq min\{\zeta_{K\times M}(k_1, m_1), \zeta_{K\times M}(k_2, m_2)\} \\ &\geq min(b, b) \\ &= b \end{aligned}$$

Therefore it implies that $(k_1, m_1) * (k_2, m_2) \in \zeta^{\geq}_{K\times M,b}$.

Hence, UC of level a and LC of level b of $K \times M$, are SAs in $U \times V$.

Conversely, let UC of level a and LC of level b of $K \times M$, are SAs of BCK/BCI-algebra $U\times V$ and if possible also let $K\times M$ is not a DIFSA in $U\times V$. Then there exist $(k_1, m_1), (k_2, m_2) \in U\times V$, such that $\alpha_{K\times M}((k_1, m_1)*(k_2, m_2)) > max\{\alpha_{K\times M}(k_1, m_1), \alpha_{K\times M}(k_2, m_2)\}$. Now let $a_0 = \frac{1}{2}\{\alpha_{K\times M}((k_1, m_1)*(k_2, m_2))+max\{\alpha_{K\times M}(k_1, m_1), \alpha_{K\times M}(k_2, m_2)\}\}$. This implies, $\alpha_{K\times M}((k_1, m_1) * (k_2, m_2)) > a_0 > max\{\alpha_{K\times M}(k_1, m_1), \alpha_{K\times M}(k_2, m_2)\}$. So $(k_1, m_1) * (k_2, m_2) \notin \alpha^{\leq}_{K\times M,a}$. But $(k_1, m_1), (k_2, m_2) \in \alpha^{\leq}_{K\times M,a}$, which is a contradiction. Thus it proves that $K \times M$ is a DIFSA in $U \times V$. Hence the proof ends. □

Theorem 3.4.2. *If for $a, b \in [0, 1]$, $\alpha^{\leq}_{K\times M,a}$ and $\zeta^{\geq}_{K\times M,b}$ be either contain no elements or form an H-ideals in $U \times V$. Then $K \times M$ is a DIFH-ideal in $U \times V$.*

Proof. Staightforward □

3.5 Summary

The chapter includes introduction of the concept of DIFH-ideals in BCK/BCI-algebras and a study on its essential properties. It also exhibits an extension of the notion of the DP of IFSs to the DP of two DIFSAs and two DIFH-ideals of two BCK/BCI-algebras U and V. For any numbers of BCK-algebra same can be made more widespread. It is mathematically settled that for two DIFH-ideals in U and V, the DP of them is also a DIFH-ideal in $U \times V$. But the reverse may not hold. We have also found that the DP of two IFSs becomes DIFH-ideal and DIFSA if for any $a, b \in [0, 1]$ UC of level a and LC of level b of that IFSs are H-ideals or SAs in BCK-algebra $U \times V$.