Chapter 1

Introduction

1.1 General introduction

The 21st century world is ultra modern and is too much dependent on technology. Now-a-days a sound growth in science and technology determines the fortune of an individual as well as of a whole nation. This unprecedented advancements in technological fields have made our modern society very complex. It has changed our mode of thinking as well as of living, and having a deep impact on our decision making faculties making it more vague and hard to analyze. In present time, computers have overtaken and replaced human actions and thoughts in multitude of fields still it fails to acquire the charms and uniqueness of human brain in respect of decision making based on impreciseness and qualitative data which is still under a an exclusive hegemony of a human brain. Undoubtedly, even today, human intelligence excels computer intelligence in many complex cognitive domains, though the flourishment of fuzzy concepts have made some significant contributions towards the development of tools that are capable of handling humanistic type of problems. Mathematical models are being developed to defend several kinds of systems involving uncertainty. These models mostly are the latest extensions of the ordinary set theory, namely fuzzy sets(FSs).

Fuzzy logic, was a unique invention in respect of mechanization of exclusive human potentialities of reasoning and decision making in a situation of aproximate infomation and uncertainty as well. Fuzzy logic based control system could replace almost any control system simplifying the design of many complicated cases. Still it had some limitations. It could not answer all types of uncertainty rather it had limited acceptability in cases having a better control system.

To represent uncetainty mathematically, Zadeh [86] made some path-braking contributions that provide standardized instruments to deal with imprecisison inherent in many problems like AI, knowledge discovery, information processing, system modelling, control system, multi-agent systems, decision sciences, economics, engineering and medicine.

Extending his notion of FS, Zadeh (1975) [89], in order to treat vagueness and uncertainty with more precision and intuition introduced the concept of IVFS which is efficient enough in combining FS theory with interval mathematics. Following him many other authors (e.g., [[24], [36]]) worked on this notion in 1970s.

FS theory holds single membership value of the element to a FS in [0, 1] that represents the DMS of the element to the FS. But reality may not be always true as there may be some degree of hesitation. This led to a generalization of FSs as intuitionistic fuzzy sets(IFSs) proposed by Atanassov [2]. Intuitionistic theory is comperatively a new notion in the field of the FS theory. the FS theory. The elements of IFSs are sets whose elements have DMS and DNMS which is an extension of Zadeh's notion of FS. Studies show that Atanassov [3] and Bustince et al. [19] made some rigorous research based on the theory and applications of IFSs. As a generalization of the FSs, IFSs has received more and more attention and has extensively being applied to decision making problems such as graph theory, coding theory, data analysis, automata theory, signal processing, formal language theory etc.

Fuzzy algebraic structures were first studied by Rosenfeld in 1971 that unlatched a unique and innovative way of thinking to others. Since then a lot of vital discrete structures have been fuzzified by subsequent authors in a number of ways. Though all of these fuzzifications have not been placed in applications but it is certain that several among them going to be nice tools with wide application areas in coming time.

Imai and Iseki, two great Japanese mathematicians made a momentous contribution to fuzzy algebraic structures in 1966 [[31], [32]] when they raised the notion of two B-algebras BCK-algebras and BCI-algebras. They were derived from two different sources- one from set theory and the other from propositional calculi.

Three most rudiment and elementary operations in set theory are set- difference, union and intersection. From generalization of these three basic operations, we have the concept of Boolean algebras. But if we consider the union and intersection we have the notion of distributive lattices. And again if we consider the union or the intersection only then we get the concept of upper semilattices or lower semilattices. It was Iseki who first highlighted set difference systematically.

Some systems carry within them implicational function among logical functions, as instance we have system of positive implicational calculus, BCK-system and BCI-system, system of weak positive implicational calculus. Undoubtedly these systems bear common properties among them.

In between the notions of set difference in set theory and the implication function in logical systems, there exists a close relationship. For example, we have this following simple inclusion relations in set theory:

$$\begin{aligned}(\Lambda - \psi) &- (\Lambda - \Omega) \subseteq \Omega - \psi, \\ \Lambda &- (\Lambda - \psi) \subseteq \psi.\end{aligned}$$

These are as same as the propositional formulas in propositional calculi:

$$\begin{aligned}(\iota \to \kappa) &\to ((\kappa \to \eta) \to (\iota \to \eta)), \\ \iota &\to ((\iota \to \kappa) \to \kappa).\end{aligned}$$

Iseki constructed notion of two B-algebras in which BCI-algebras are a wider class than BCK-algebras by attending all the questions raised from these relationships. From BCK and BCI-systems in combinatory logic their names evolves. From the point of view of ordering, the essential difference between BCK-algebras and BCI-algebras is element 0 is the least element in BCK-algebras, while in BCI-algebras, it is a minimal element.

Andrzej Walendziak [83] defined the BF-algebra in 2007 that is related to classes of BCK/BCI-algebras. To develop the theory of BCK/BCI-algebras, the role of ideal theory is very crucial and significant. Meng, Liu and several other authors made remarkable efforts in investigating the properties of FSAs and F-ideals in BCK/BCI-algebras.

Notion of FSG was pioneered by Rosenfled [63]. Motivated by these concepts, various kinds of advanced works were done in the field of abstract algebra by many mathematicians in the context of FS.

Xi [84] in 1991 introduced the notion of FS to BCK-algebras. Another idea of FS to BCI-algebras was given by Huang [27] in 1992. Following him Jun [43] in 1994 established the definition of DFSA and DF-ideal of BCK/BCI-algebras to avoid the perplexity created in Huang's [27] definition of fuzzy BCI-algebras. Thus a new type of FSA and F-ideal are defined. Zhan and Tan [92] in 2003 discussed characterization of DFH-ideals in BCK-algebras. Jun [48] in 2002 introduced F SI-ideals in BCI-algebras. Satyanarayan et al. [66] in 2010 introduced IFH-ideals in BCK-algebras. Palaniappan et al. [60] in 2012 introduced the concepts of IF SI-ideals and IF SC -ideals of BCI-algebras. Zarandi and Saeid [90] defined IVF BF-subalgebra.

Biswas [16] in 1994 introduced the concept of IVFSG. After that in various algebraic structures, see [[46], Senapati et. al.([[69], [70]])], the concept of IVFSs have been studied.

Generalizing the concept of algebric structures, Marty [55] first developed the theory of algebraic hyperstructures. The uniqueness of this algebraic hyperstructures is that a set is generated due to the composition of two elements, having meaningful applications in several fields like automata, probability, lattices, geometry, binary relations, codes, graphs, hypergraphs, cryptography etc.

BE-algebra, introduced by Kim and Kim [51] is a generalization of a dual BCK-algebra. Radfar et al. [61] applied the hyperstructures theory to BE-algebra and define the notion of a hyper BE-algebra. In 2015, Rezaei et al. [62] introduced commutative hyper BE-algebra and studied to the effect that every commutative hyper BE-algebra is a BE-algebra.

In all the above algebras, researchers have dealt with subalgebras, ideals, subimplicative ideals, P-ideals, H-ideals, implicative ideals, hyper structure etc. and their fuzzification. Taking queues from the above fuzzifications we have also intruduced in this thesis seversal concepts and theorems of algebraic structures in the larger framework of doubt fuzzy settings.

1.2 Basic notations and definitions

1.2.1 Fuzzy Sets

Zadeh, an Iranian-American Mathematician and Professor of computer science, introduced FS theory in 1965 as a generalization of Cantor's set theory. Literally the word 'fuzzy' stands for vague, formless or unclear.

Definition 1.2.1. *A classical set is a collection of well defined objects with a crisp boundary. Every classical set, A, is associated with a function called characteristic function and it is denoted by $\chi_A(p)$, where p is an element of A, then*

$$\chi_A(p) = \begin{cases} 1, & \text{if } p \in A \\ 0, & \text{otherwise.} \end{cases}$$

The idea of membership function of a FS is coming from the characteristic function of crisp set.

Definition 1.2.2. *"Let V be a collection of objects, then a FS [86] M in V is defined as $M = \{< v, \alpha_M(v) >: v \in V\}$ where $\alpha_M(v)$ is the measure of membership of v in M and $0 \leq \alpha_M(v) \leq 1$. The complement of M is symbolized by M^c and is provided by $M^c = \{< v, \alpha_M^c(v) >: v \in V\}$ where $\alpha_M^c(v) = 1 - \alpha_M(v)$. "*

The DMS is the degree of belongingness of elements to a FS. In the theory of FS the DMSs of the elements lie in the interval $[0, 1]$ where the DMS 1 ensures complete belongingness of the element in its corresponding FS and the DMS 0 denotes the element does not belong in the FS and the DMS lies in $(0, 1)$ indicates partial belongingness in the FS.

1.2.2 Operations on Fuzzy Sets

The FS theory can be expanded with supporting definitions to set theoretic operations. It was Zadeh who proposed basic operations. In subsequent time several other authors have demonstrated additional and alternative operations as in [22, 23, 85, 94]. The definitions below suggest an overview of a selection of basic operations on FS and its characteristics that provide a general comprehension of FS theory along with various types of set operations that comprise FSs.

Definition 1.2.3. *" [86] Let $M = \{< v, \alpha_M(v) >: v \in V\}$ and $N = \{< v, \alpha_N(v) >: v \in V\}$ be two FSs in V, then following operations are defined as:*

(i) $M \subseteq N \Rightarrow \alpha_M(v) \leq \alpha_N(v)$

(ii) $M = N \Rightarrow \alpha_M(v) = \alpha_N(v)$

(iii) $M \cap N = \min(\alpha_M(v), \alpha_N(v))$

(iv) $M \cup N = \max(\alpha_M(v), \alpha_N(v))$*, for all $v \in V$. "*

Definition 1.2.4. *“* [22] For $\varsigma \in [0, 1]$, the set $U(\alpha_M : \varsigma) = \{v \in V : \alpha_M \geq \varsigma\}$ is called upper ς-level of M. *”*

Let us denote a mapping from U into V by h. Let N be a FS in V, then the *inverse image* [63] of N, symbolized by $h^{-1}(N)$ in U and is provided by $h^{-1}(\alpha_N)(u) = \alpha_N(h(u))$.

Conversely, let M be a FS in U having MSF α_M. Then the *inverse image* [63] of M, symbolized by $h(M)$ in V and is provided by

$$\alpha_{h(M)}(v) = \begin{cases} \sup\limits_{u \in h^{-1}(v)} \alpha_M(u), & \text{if } h^{-1}(v) \neq \phi \\ 1, & \text{otherwise.} \end{cases}$$

Definition 1.2.5. *“* [94] Let $M = \{< v, \alpha_M(v) >: v \in V\}$ and $N = \{< v, \alpha_N(v) >: v \in V\}$ be two FSs of V. The *Cartesian product(briefly, CP)* $M \times N = \{< (v_1, v_2), \alpha_M \times \alpha_N(v_1, v_2) >: v_1, v_2 \in V\}$ is defined by $(\alpha_M \times \alpha_N)(v_1, v_2) = \min\{\alpha_M(v_1), \alpha_N(v_2)\}$, where $\alpha_M \times \alpha_N : V \times V \to [0, 1]$ for all $v_1, v_2 \in V$. *”*

1.2.3 Interval-Valued Fuzzy Sets

Again Zadeh [89] himself introduced the concept of IVFSs in 1975 as an extention of FSs. Initiation of IVFSs is the most simple method to capture the slightest tinge of lack of accuracy of the DMS for a FS because interval is a continuous system that can catch each and every point accurately due to dense property of real number system.. The MSV of an element in IVFSs is represented by an interval instead of by a single number. This MSV always belongs to $P([0, 1])$ whenever $P([0, 1])$ is supposed to be the power set consisting of all subintervals drawn from $[0, 1]$. It is necessary to mention that MSV is always a closed interval whereas the collection $P([0, 1])$ contains all types of subintervals of $[0, 1]$. So, it can be stated that collection of all MSVs is a proper subset of $P([0, 1])$. The formal definition of IVFSs is defined below.

Definition 1.2.6. *“* [89]*(Interval-Valued Fuzzy Sets (briefly, IVFSs)) An IVFS* M *defined on* V *is given by* $M = \{(v, [\nu_M^L(v), \nu_M^U(v)]) : v \in V\}$. *Briefly, denoted by* $M = [\nu_M^L, \nu_M^U]$ *where* ν_M^L *and* ν_M^U *are any two FSs in* V *such that* $\nu_M^L(v) \leq \nu_M^U(v)$*, for all* $v \in V$ *. ”*

Let $\bar{\nu}_M(v) = [\nu_M^L(v), \nu_M^U(v)]$, for all $v \in V$ and let $P([0, 1])$ is supposed to be the power set consisting of all subintervals drawn from $[0, 1]$. It is clear that if $\nu_M^L(v) =$

$\nu_M^U(v) = p$, where $0 \leq p \leq 1$ then $\bar{\nu}_M(v) = [p, p]$ is in $P([0,1])$. Thus $\bar{\nu}_M(v) \in P([0,1])$, for all $v \in V$. Therefore, the IVFS M is given by $M = \{(v, \bar{\nu}_M(v)) : v \in V\}$ where $\bar{\nu}_M : V \to P([0,1])$.

For our convenience, the symbol form $M = (\bar{\nu}_M(v))$ or $(\bar{\nu}_M)$ for the IVFS $M = [\nu_M^L, \nu_M^U]$ are used.

It is very simple to determine the maximum and minimum between two real numbers though not so simple for two intervals. Biswas [16] described the method of finding max/sup and min/inf between two intervals or a set of intervals.

Definition 1.2.7. [16] *"Consider two elements $I_1, I_2 \in P([0,1])$. If $I_1 = [c_1, d_1]$ and $I_2 = [c_2, d_2]$, then rmax$(I_1, I_2) = [\max(c_1, c_2), \max(d_1, d_2)]$ which is denoted by $I_1 \vee^r I_2$ and rmin$(I_1, I_2) = [\min(c_1, c_2), \min(d_1, d_2)]$ which is denoted by $I_1 \wedge^r I_2$. Thus, if $I_k = [c_k, d_k] \in P([0,1])$ for k= 1, 2, 3, 4,..., then we define rsup$_k(I_k) = [\sup_k(c_k), \sup_k(d_k)]$, i.e, $\vee_k^r I_k = [\vee_k c_k, \vee_k d_k]$. Similarly, we define rinf$_k(I_k) = [\inf_k(c_k), \inf_k(d_k)]$ i.e, $\wedge_k^r I_k = [\wedge_k c_k, \wedge_k d_k]$. Now we call $I_1 \geq I_2$ iff $c_1 \geq c_2$ and $d_1 \geq d_2$. Similarly, the relations $I_1 \leq I_2$ and $I_1 = I_2$ are defined ."*

In this thesis, we assumed that any two intervals of $P([0,1])$ are comparable.

1.2.4 Intuitionistic Fuzzy Sets

The notion of IFS is a unique invention by Atanassov [2] that generalizes FSs by adding few new components (which determines the DNMS) in the definition of FS. The main difference between FS and IFS is that the FS measures DMS of an element to a given set but IFS measures both the DMS and DNMS to a given set whereas sum of DMS and DNMS does not exceed 1. So IFS can be thought as the generalization of FS due to adding an extra component in the msv of an element to a given set. IFSs is now being used widely in several secientific fields like medical science, mathematics, computer science, engineering, chemistry, economics, astrology etc. as a important instrument in the modelling of some uncertain phenomena.

Definition 1.2.8. [2](IFS)

"An IFS N over a universe U is of the form $N = \{\langle u, \alpha_N(u), \zeta_N(u)\rangle : u \in U\}$, where $\alpha_N(u) : U \to [0,1]$ and $\zeta_N(u) : U \to [0,1]$, with the condition $0 \leq \alpha_N(u) + \zeta_N(u) \leq 1$

for all $u \in U$. The numbers $\alpha_N(u)$ and $\zeta_N(u)$ denote, respectively, the DMS and the DNMS of the element u in the set N. Obviously, when $\zeta_N(u) = 0$ for every $u \in U$,then $\alpha_N(u) \leq 1$. So IFS reduces to FS due to removing DNMS completely.

The value $s_N(u) = 1 - \alpha_N(u) - \zeta_N(u)$ is the measure of suspicion of the elements $u \in U$ to the IFSs N."

For our convenience, the symbol $N = (\alpha_N, \zeta_N)$ for IFS $N = \{\langle u, \alpha_N(u), \zeta_N(u)\rangle : u \in U\}$ is used.

1.2.5 Operations on IFS

In this section, we introduce operations and CP of IFSs.

Definition 1.2.9. " [2] Let $M = (\alpha_M, \zeta_M)$ and $N = (\alpha_N, \zeta_N)$ be two IFSs in V, then following operations are defined

(i) $M \cap N =< \min(\alpha_M, \alpha_N), \max(\zeta_M, \zeta_N) >$

(ii) $M \cup N =< \max(\alpha_M, \alpha_N), \min(\zeta_M, \zeta_N) >$

(iii) $\overline{M} = (\zeta_M, \alpha_M)$

(iv) $\bigoplus M = (\alpha_M, \overline{\alpha}_M)$

(v) $\bigotimes M = (\overline{\zeta}_M, \zeta_M)$."

Definition 1.2.10. " [2] Let $M = (\alpha_M, \zeta_M)$ be an IFSS of V. For $s_1, s_2 \in [0, 1]$, the sets $U(\alpha_M : s_1) = \{v \in V : \alpha_M(v) \geq s_1\}$ is named as UC of level s_1 of M and $L(\zeta_M : s_2) = \{v \in V : \zeta_M(v) \leq s_2\}$ is named as LC of level s_2 of M. "

Definition 1.2.11. *An IFS M in V is said to have the max-property and min-property if for any SS $R \subseteq V$ there exist $r_0 \in R$ such that $\alpha_A(r_0) = \sup_{r_0 \in R} \alpha_A(r)$ and $\zeta_A(r_0) = \inf_{r_0 \in R} \zeta_A(r)$ respectively.*

Definition 1.2.12. [4] Let $M = (\alpha_M, \zeta_M)$ and $N = (\alpha_N, \zeta_N)$ be two IFSs of V. The cartesian product(CP) $M \times N = (\alpha_M \times \alpha_N, \zeta_M \times \zeta_N)$ is defined by

$$(\alpha_M \times \alpha_N)(v_1, v_2) = \min\{\alpha_M(v_1), \alpha_N(v_2)\} \text{ and}$$
$$(\zeta_M \times \zeta_N)(v_1, v_2) = \max\{\zeta_M(v_1), \zeta_N(v_2)\},$$

where $\alpha_M \times \alpha_N : V \times V \to [0, 1]$ and $\zeta_M \times \zeta_N : V \times V \to [0, 1]$ for all $v_1, v_2 \in V$.

1.2.6 Basic-algebras

Definition 1.2.13. *"An algebra* $(V, *, 0)$ *of type* $(2, 0)$ *is named as* BCI*-algebra [29] if the following conditions are fulfilled for all* $f, g, h \in V$*:*

$(A1)$ $((f * g) * (f * h)) * (h * g) = 0$

$(A2)$ $(f * (f * g)) * g = 0$

$(A3)$ $f * f = 0$

$(A4)$ $f * g = 0$ *and* $g * f = 0$ *imply* $f = g$*."*

"A BCI-algebra is termed as *associative* [26] if $(f * g) * h = f * (g * h)$ for all $f, g, h \in V$."

Binary operation $*$ on V is recognized as the $*$ *multiplication* on V, and constant 0 of V the *zero element* of V.

Following axioms are contented by any BCI-algebra V for all $f, g, h \in V$:

(P1) $(f * g) * h = (f * h) * g$

(P2) $f * g \leq f$

(P3) $(f * h) * (g * h) \leq (f * g)$

(P4) $f \leq g \Rightarrow f * h \leq g * h, h * g \leq h * f$.

(P5) $f * 0 = f$

(P6) $0 * (f * g) = (0 * f) * (0 * g)$

(P7) $f * (f * (f * g)) = f * g$

A BCI-algebra is named as implicative if it fulfills: $(f * (f * g)) * (g * f) = g * (g * f)$.

Definition 1.2.14. *If a* BCI*-algebra* V *fulfillss* $0 * f = 0$ *for all* $f \in V$*, then we say that* V *is a* BCK*-algebra [32]. Any* BCK*-algebra* V *meets the below stated postulates for all* $f, g, h \in V$*:*

(I) $(f * g) * h = (f * h) * g$

(II) $((f * h) * (g * h)) * (f * g) = 0$

(III) $f * 0 = f$

(IV) $f * g = 0 \Rightarrow (f * h) * (g * h) = 0, (h * g) * (h * f) = 0$.

We recommend the book to the reader [29] and [57] as further reading for information about BCK/BCI-algebras.

Definition 1.2.15. *" [83] If an algebra $(V, *, 0)$ of type $(2, 0)$ fulfills the following assertions for all $f, g \in V$ that is:*

(I) $f * f = 0$

(II) $f * 0 = f$

(III) $0 * (f * g) = (g * f)$*, for all* $f, g \in V$*.*

Then it becomes a BF*-algebra. "*

A partial ordering "$\leq$" on V can be described by $f \leq g$ iff $f * g = 0$.

EXAMPLE 1. *" [83] Let $M = (\mathbb{R}_1; *, 0)$ be an algebra, where $\mathbb{R}_1$ denotes the real numbers set and the binary operation $*$ is defined by*

$$l * m = \begin{cases} l, & if \quad m = 0 \\ m, & if \quad l = 0 \\ 0, & otherwise \end{cases}$$

Then M is a BF-algebra . "

EXAMPLE 2. *Let $M = [0; \infty)$. By defining the binary operation $*$ on M as follows: $f * g = |f - g|$, for all $f, g \in M$. We get a BF-algebra $(M; *, 0)$.*

Definition 1.2.16. *[51] "An algebra $(V; *, 1)$ of type (2,0) is named as BE-algebra if it fulfills the following identities for all $f, g, h \in V$:*

(E1) $f * (g * h) = g * (f * h)$*;*

(E2) $f * f = 1$*;*

(E3) $f * 1 = 1$*;*

(E4) $1 * f = f$*."*

A binary relation "$\leq$" on a BE-algebra U can be defined by $g \leq h$ iff $g * h = 1$. for all $g, h \in U$.

Throughout this thesis, U or V always means a $BCK/BCI/BF$-algebra, a hyper BE-algebra without any specification.

Definition 1.2.17. *"A non-empty SS S_1 in U is identified as a subalgebra(SA) [33, 57, 83] in U if $f * g \in S_1$ for any $f, g \in S_1$."*

From the definition it is noted that, when a SS S_1 in U fulfills only the closer property, then it appears as a SA.

Definition 1.2.18. *"Let I' be non-empty SS in U, then it is named as an ideal [28, 32, 83] in U if it fulfils*

(I_1) $0 \in I'$ and

(I_2) $f * g \in I'$ and $g \in I'$ imply $f \in I'$."

U and $\{0\}$ are ideals of U, called the trivial ideals in U. If an ideal in U is exactly contained in U, we call it a proper ideal of U, thus $\{0\}$ is a proper ideal of U whenever $U \neq \{0\}$.

Definition 1.2.19. *"A non-empty subset H' in U is named as a H-ideal [50, 91] of BCK/BCI-algebra U if it satisfies (I_1) and*

(I_3) $f * (g * h) \in H'$ and $g \in H'$ imply $f * h \in H'$ for all $f, g, h \in U$."

Definition 1.2.20. *"Let I' be a non-empty SS in U, then it is named as a sub-implicative ideal [54] in U if*

(i) $0 \in I'$

(ii) $((f*(f*g))*(g*f))*h \in I'$ and $h \in I'$ imply $g*(g*f) \in I'$ for all $f, g, h \in U$."

Definition 1.2.21. *"Let I' be a non-empty SS in U, then it is named as a P-ideal [56] in U if*

(i) $0 \in I'$

(ii) $(f * h) * (g * h) \in I'$ and $g \in I'$ then $f \in I'$, for all $f, g, h \in U$."

1.2.7 Fuzzy set concept on algebras

Definition 1.2.22. *"A FS $M = \{< f, \alpha_M(f) >: f \in U\}$ in U is named as a FSA [27, 48, 64, 84] of U if it meets the inequality $\alpha_M(f * g) \geq \min\{\alpha_M(f), \alpha_M(g)\}$ for all $f, g \in U$."*

Definition 1.2.23. *"An IVFS M in U is named as an IVFSA [46, 90] of U if $\bar{\nu}_M(f * g) \geq rmin\{\bar{\nu}_M(f), \bar{\nu}_M(g)\}$, for all $f, g \in U$."*

TO make it simple, we use $f \bigvee g$ for $\max(f, g)$, and $f \bigwedge g$ for $\min(f, g)$.

Definition 1.2.24. *"A FS $M = \{< f, \alpha_M(f) >: f \in V\}$ in U is named as a fuzzy ideal [1, 10, 84] of V if it fulfils*

(F_1) $\alpha_M(0) \geq \alpha_M(f)$ and

(F_2) $\alpha_M(f) \geq min\{\alpha_M(f * g), \alpha_M(g)\}$ for all $f, g \in V$."

Definition 1.2.25. *"A FS $M = \{< f, \alpha_M(f) >: f \in U\}$ in U is called a fuzzy H-ideal [50, 91] of BCK/BCI-algebra U if it fulfils (F_1) and*

(F_3) $\alpha(f * h) \geq \min\{\alpha(f * (g * h)), \alpha(g)\}$ for all $f, g, h \in U$."

Definition 1.2.26. *"An IVFS M in U is called an IVF-ideal [46, 90] of U if (i) $\bar{\nu}_M(0) \geq \bar{\nu}_M(f)$ and (ii)$\bar{\nu}_M(f) \geq rmin\{\bar{\nu}_M(f * g), \bar{\nu}_M(g)\}$, for all $f, g \in U$. "*

Definition 1.2.27. *" Let $M = \{\langle f, \alpha_M(f)\rangle : f \in U\}$ be a FS of U and let $\xi \in [0, T]$. A mapping $\alpha_\xi^T \to [0, 1]$ is called a fuzzy ξ-translation of α if it satisfies: $(\alpha_M)_\xi^T(f) = \alpha_M(f) + \xi$, for all $f \in U$." Where $T = 1 - sup\{\alpha_M(f)|f \in U\}$*

Definition 1.2.28. *" [65] (DFSA) A FS $M = \{\langle f, \alpha_M(f)\rangle : f \in U\}$ in U is named as DFSA of U if*

$\alpha_M(f * g) \leq \alpha_M(f) \bigvee \alpha_M(g)$, for all $f, g \in U$."

Definition 1.2.29. *" [10] (DF-ideal) A FS $M = \{\langle f, \alpha_M(f)\rangle : f \in V\}$ in V is named as DF-ideal of V if*

(i) $\alpha_M(0) \leq \alpha_M(f)$

(ii) $\alpha_M(f) \leq \alpha_M(f * g) \bigvee \alpha_M(g)$, for all $f, g \in V$. "

Definition 1.2.30. *"A FS $M = \{\langle f, \alpha_M(f)\rangle : f \in V\}$ in U is named as DF SI-ideal [59] of V if*

(i) $\alpha_M(0) \leq \alpha_M(f)$

(ii) $\alpha_M(g * (g * f)) \leq \alpha_M(((f * (f * g)) * (g * f)) * h) \bigvee \alpha_M(h)$, for all $f, g, h \in V$. "

Definition 1.2.31. *"A FS $M = \{\langle f, \alpha_M(f)\rangle : f \in U\}$ in U is called a DF P-ideal [40] of U if*

(i) $\alpha_M(0) \leq \alpha_M(f)$

(ii) $\alpha_M(f) \leq \alpha_M((f * h) * (g * h)) \bigvee \alpha_M(g)$, for all $f, g, h \in U$."

1.2.8 Intuitionistic fuzzy set concept on algebras

IFSA and IF-ideal are the extension of FSA and fuzzy ideal which are defined below-

Definition 1.2.32. *"An IFS $M = (\alpha_M, \zeta_M)$ in U, is called an IFSA [42] of U if it fulfils the assertions given below.*
(i) $\alpha_M(f * g) \geq \alpha_M(f) \bigwedge \alpha_M(g)$
(ii) $\zeta_M(f * g) \leq \zeta_M(f) \bigvee \zeta_M(g)$, *for all* $f, g \in U$."

Definition 1.2.33. *"An IFS $M = (\alpha_M, \zeta_M)$ in U is called an IF-ideal [42] of U, if the following axioms are fulfilled:*
(i) $\alpha_M(0) \geq \alpha_M(f), \zeta_M(0) \leq \zeta_M(f)$
(ii) $\alpha_M(f) \geq \alpha_M(f * g) \bigwedge \alpha_M(g)$
(iii) $\zeta_M(f) \leq \zeta_M(f * g) \bigvee \zeta_M(g)$, *for all* $f, g \in U$."

Definition 1.2.34. *"An IFS $M = (\mu_M, \lambda_M)$ in U is called an IF H-ideal [67] of U, if the following axioms are fulfilled:*
(i) $\mu_M(0) \geq \mu_M(f), \lambda_M(0) \leq \lambda_M(f)$,
(ii) $\mu_M(f * h) \geq \mu_M(f * (g * h)) \bigwedge \mu_M(g)$,
(iii) $\lambda_M(f * h) \leq \lambda_M(f * (g * h)) \bigvee \lambda_M(g)$, *for all* $f, g, h \in U$."

Definition 1.2.35. *"An IFS $M = (\mu_M, \lambda_M)$ in U is named as an IF SI- ideal [60] of U if the following axioms are fulfilled:*
(i) $\alpha_M(0) \geq \alpha_M(f), \zeta_M(0) \leq \zeta_M(f)$
(ii) $\alpha_M(g * (g * f)) \geq \alpha_M(((f * (f * g)) * (g * f)) * h) \bigwedge \alpha_M(h)$
(iii) $\zeta_M(g * (g * f)) \leq \zeta_M(((f * (f * g)) * (g * f)) * h) \bigvee \zeta_M(h)$, *for all* $f, g, h \in U$."

Definition 1.2.36. *An IFS $M = (\mu_M, \lambda_M)$ in U is called an IF SC-ideal [60] of U if it satisfies:*
(i) $\alpha_M(0) \geq \alpha_M(f), \zeta_M(0) \leq \zeta_M(f)$
(ii) $\alpha_M(f * (f * g)) \geq \alpha_M((g * (g * (f * (f * g)))) * h) \bigwedge \alpha_M(h)$
(iii) $\zeta_M(f * (f * g)) \leq \zeta_M((g * (g * (f * (f * g)))) * h) \bigvee \zeta_M(h)$, *for all* $f, g, h \in U$."

1.2.9 Hyper structures on algebras

1.2.10 Hyper BE-algebras

Definition 1.2.37. *[61] "Consider a nonempty set $\mathbb{P}$ and $\circ : \mathbb{P} \times \mathbb{P} \to \mathbb{P}^*(\mathbb{P})$ be a hyperoperation. Then $(\mathbb{P}; \circ, 1)$ is called a hyper BE-algebra provided it fulfils the* *following postulatess: for all $f, g, h \in \mathbb{P}$,*

(AA1) $f < 1$ *and* $f < f$;

(AA2) $f \circ (g \circ h) = g \circ (f \circ h)$;

(AA3) $f \in 1 \circ f$;

(AA4) $1 < f$ *implies* $f = 1$,

where the relation "<" is defined by $f < g \Leftrightarrow 1 \in f \circ g$. *For any two nonempty SSs* U *and* V *of* $\mathbb{P}$, $U < V$ *means that* $f \in U$ *and* $g \in V$ *s.t* $f < g$ *and* $U \circ V = \cup_{f \in U, g \in V} f \circ g$. *Also* $U << V$ *if for arbitrary choice* $f \in U$, *there exists* $g \in V$ *s.t* $f < g$."

Proposition 1.2.1. *[61] "Let* $(\mathbb{P}; \circ, 1)$ *be a hyper BE-algebra. Then*

(H1) $M \circ (N \circ R) = N \circ (M \circ R)$;

(H2) $M < M$;

(H3) $1 < M$ *implies* $1 \in M$;

(H4) $f < g \circ f$;

(H5) $f < g \circ h$ *implies* $g < f \circ h$;

(H6) $f < (f \circ g) \circ g$;

(H7) $h \in f \circ g$ *implies* $f < h \circ g$;

(H8) $g \in 1 \circ f$ *implies* $g < f$, *for all* $f, g, h \in \mathbb{P}$ *and* $M, N, R \subseteq \mathbb{P}$."

Proposition 1.2.2. *[20] "Let* $(\mathbb{P}; \circ, 1)$ *be a hyper BE-algebra. Then*

(C1) $G << R \circ G$;

(C2) $G < R$ *if and only if* $1 \in G \circ R$;

(C3) $G \subseteq 1 \circ G$;

(C4) $G \subseteq R$ *implies* $G < R$ *and* $G << R$;

(C5) $G << R$ *and* $1 \in G$ *imply* $1 \in R$, *for all* $G, R \subseteq \mathbb{P}$."

Definition 1.2.38. *[61] "A hyper BE-algebra* $(\mathbb{P}; \circ, 1)$ *is called*

(I) row hyper BE-algebra (briefy, R-hyper BE-algebra), if $1 \circ f = \{f\}$, *for all* $f \in \mathbb{P}$;

(II) diagonal hyper BE-algebra (briefly, D-hyper BE-algebra), if $f \circ f = 1$, *for all* $f \in \mathbb{P}$;

(III) RD-hyper BE-algebra, if $\mathbb{P}$ *is both R-hyper BE-algebra and D-hyper BE-algebra."*

Definition 1.2.39. *" [20] A hyper BE-algebra* $(\mathbb{P}; \circ, 1)$ *is named as transitive if the following holds: (T1)* $g \circ h << (f \circ g) \circ (f \circ h)$ *and (T2)* $f \circ g << (g \circ h) \circ (f \circ h)$, *for all* $f, g, h \in \mathbb{P}$."

Definition 1.2.40. *[62] "A hyper BE-algebra* $(\mathbb{P}; \circ, 1)$ *is termed as commutative if the following holds:* $(f \circ g) \circ g = (g \circ f) \circ f$*, for all* $f, g \in \mathbb{P}$*."*

Definition 1.2.41. *" [61] Let* G *be a nonempty SS of a hyper BE-algebra* $\mathbb{P}$ *and* $1 \in G$*. Then* G *is called a hyper filter of* $\mathbb{P}$*, if* $f \circ g \cap G \neq \phi$*, and* $f \in G$ *imply* $g \in G$*, for all* $f, g \in \mathbb{P}$*."*

Definition 1.2.42. *[25] Let* $(\mathbb{P}; \circ, 1)$ *be a hyper BE-algebra. A FS* $\alpha_M : \mathbb{P} \to [0, 1]$ *is called a fuzzy hyper BE-subalgebra on* $\mathbb{P}$*, if for all* $f, g \in \mathbb{P}$*,* $\inf_{t \in f \circ g} \alpha(t) \geq \min\{\alpha(f), \alpha(g)\}$.

Definition 1.2.43. *[25] Let* $(\mathbb{P}; \circ, 1)$ *be a hyper BE-algebra. A FS* $\alpha_M : \mathbb{P} \to [0, 1]$ *is named as fuzzy hyper filter on* $\mathbb{P}$*, if it satisfied the following properties for all* $f, g \in U$ *:*

(1) $\alpha(1) \geq \alpha(f)$;

(2) $\alpha(g) \geq \min\{\alpha(f), \sup_{t \in f \circ g} \alpha(t)\}$.

Definition 1.2.44. *[20] Let* $(\mathbb{P}; \circ, 1)$ *be a hyper BE-algebra and* G *be a nonempty subset of* $\mathbb{P}$ *containing* 1*. Then* G *is said to be:*

(1) an implicative hyper filter of $\mathbb{P}$*, if* $f \circ (g \circ h) \cap G \neq \phi$ *and* $f \circ g \cap G \neq \phi$ *imply* $f \circ h \cap G \neq \phi$*, for all* $f, g, h \in \mathbb{P}$*.*

(2) a positive implicative hyper filter of $\mathbb{P}$*, if* $f \circ ((g \circ h) \circ g) \cap G \neq \phi$ *and* $f \in G$ *imply* $g \in G$*, for all* $f, g, h \in \mathbb{P}$*.*

1.3 Survey of related literatures

We, human beings always experience a large number of attributes in our lives which are not precise. However, the human brain processes such imprecise terms also. Computers and computing tools are often used to make processing faster than human brain. Though computers have neither intuition nor intelligence, these can be made intelligent artificially, in areas such as chess game, medical diagnosis, robot movement, washing cloth etc. For this reason, a mathematical modelling of vague concepts is necessary. One of the most powerful tools probably for such information processing in the cybernetic-age is FS theory and the fuzzy concept was introduced by Zadeh (1965).

Since then the theory of FSs gaining massive attention among the researchers and the scope of its applications has already been expanded in several fields.

Fuzzy algebraic structures were first studied by Rosenfeld in 1971 that opened a new and innovative way of thinking to mathematicians and others. With the run of time a lot of vital discrete structures have been fuzzified by subsequent authors in a number of ways.

The theory of algebric hyperstructure, in recent times has become a well-recognised branch in algebric theory due to its wide applications in several fields of mathematics and applied sciences. This theory of algebric hyperstructure which was first conceptualised by Marty [55] is basically a generalisation of the concept of algebric structures.

After the introduction of BE-algebra by Kim and Kim as a generalization of a dual BCK-algebra [51], Radfar et al. [61] related the hyperstructures theory to BE-algebra and define the notion of a hyper BE-algebra. After that in 2015, Rezaei et al. [62] introduced commutative hyper BE-algebra, also see [8, 9]. At the same time they proved that every commutative Hyp BE-algebra is a BE-algebra.

Hundred of papers were written to established the relationship between the FSs and algebraic hyperstructures as fuzzy hyperstructure is an interesting topic of research. In 2015 Tang et al. [82], introduced the idea of hyperfilters and fuzzy hyperfilters of an ordered semihypergroup.

In FS theory, there is a single membership value of the element to a FS in $[0, 1]$ and it represents the DMS of the element to the FS. However, in reality it may not always be true as there may be some hesitation degree. Therefore a generalization of FSs was proposed by K. T. Atanassov [2] as IFSs. Thus, intuitionistic theory is comperatively a new notion in the field of the FS theory. The elements of such IFSs are sets whose elements have DMS and DNMS which is an extension of Zadeh's notion of FS. Atanassov [3] carried out rigorous research based on the theory and applications of IFSs.

After that several researchers worked on FSAs and IFSAs and IF-ideals in different algebraic structures [37–39, 44].

In 2009, Shabir and Khan [78] established the idea of IF-filters of ordered semigroups. They established a relation between IF-filters and IF-prime ideals of ordered semigroups. Latter in 2012, Palaniappen et al. [60] applied the notion of IF-ideals in

Hyp BCI-algebras. And in recent past Cheng and Xin [20], focused on investigating implicative and PI Hyp- filters on Hyp BE-algebras.

In recent past, Hamidi et al. [25] established the concept of fuzzy homomorphisms in fuzzy Hyp BE-subalgebras and thus made a connection between fuzzy BE-algebras and Hyp BE-algebras. They also explained the concept of normal fuzzy Hyp BE-subalgebra and investigated some of its properties. Moreover they introduced fuzzy Hyp-filters on Hyp BE-algebras.

1.4 Organisation of chapters

The thesis is organised in nine chapters.

A general introduction and the complete framework of the thesis are presented in Chapter 1 along with the objective of the work.

In Chapter 2, the idea of DIFSA and DIF-ideals in BCK/BCI-algebras are introduced. Findings of this chapter show that an IFS of BCK/BCI-algebra is DIFSA and DIF-ideal if and only if the complement of this IFS is an IFSA and an IF-ideal. And at the same time some common properties related to them are presented here.

Chapter 3 contains the notion of DIFH-ideals in BCK/BCI-algebras and its essential properties are studied. At the same time this chapter also exhibits an extension of the notion of the DP of IFSs to the notion of the generalized DP of two DIFSAs and two DIFH-ideals of two BCK/BCI-algebras U and V. For any numbers of BCK-algebra same can be made more widespread. Here we conclude that if M and N are two DIFH-ideals of U and V then the DP of M and N is also a DIFH-ideal of $U \times V$. But the converse may not hold.

In Chapter 4, the notion of DIF SI-ideals and DIF P-ideals of BCI-algebras are investigated. Here we conclude that any DIFP-ideal is always a DIFSI-ideal. We show that a DIFSI-ideal may not always be DIFP-ideal with supporting examples. Some related properties of DIFSI-ideals and DIFP-ideals of BCI-algebras are also given.

We have defined DIF translation, DIF extension, DIF multiplication and DIF magnified translation of DIFSAs and DIF SI-ideals in BCI-algebras in Chapter 5 and have established some interesting relations among them.

Chapter 6 bears the introduction of the notion of intuitionistic fuzzification of hyper filters and implicative hyper filters of hyper BE-algebras as a extended study of intuitionistic fuzzy hyperstructures. Here we show characterizations of DIF Hyp-filters in hyper BE-algebras and also DIF Hyp-filters in commutative hyper BE-algebras. We made the doubt intuitionistic fuzzification of the notion of implicative hyper filters in hyper BE-algebras which is unique in itself.

We have investigated the notion of an DIVFSA and DIVF-ideal of BCK algebra and worked on its properties in Chapter 7. Here we have delved deep into the relationships between DIVFS and DIVF-ideal. We have also introduced fuzzy translation, fuzzy multiplication of an DIVFSA/DIVF-ideal of a BCK-algebra and have discussed the product of IVDFIs in BCK algebras.

In Chapter 8, we are going to introduce the concept of DIVF-ideals in BF-algebras. After a detailed study of its properties, we come to the conclusion that in BF-algebras, an IVFS is an DIVF-ideal in BF-algebras iff the complement of this IVFS is an IVF-ideal. Finally, results based on DIVFSAs of BF-algebras and DIVF-ideals of BF-algebras are established. At the same time the product of DIVF-ideals in BF-algebras has been introduced.

And Chapter 9 contains concluding remarks and hints of further researches on the problem that has been studied in the thesis.

1.5 Objectives of the thesis

Union and intersection are two basic operation performed on classical sets. Union and intersection on FSs are defined using max and min operation respectively. opposite operation of min is max in fuzzy sense. So union in fuzzy sense can be thought as the anti operation of intersection.

In classical set theory, when an element $x \in P \cup Q$ then either $x \in P$ or $x \in Q$. But one can not correctly predict in which set the element x certainly belongs. So there arrises a matter of doubt or matter of suspicion. Likewise, a matter of suspicion also arrises when max operation is used in fuzzy sense. So, anti sense always creates a doubtful atmosphere.

Xi [84] in 1991 introduced the notion of FS to BCK-algebras. Another idea of FS to BCI-algebras was given by Huang [27] in 1992. Following him Jun [43] in 1994 established the definition of DFSA and DF-ideal of BCK/BCI-algebras to avoid the perplexity created in Huang's [27] definition of fuzzy BCI-algebras. Thus a new type of FSA and F-ideal are defined.

Fuzzy algebraic structures were first studied by Rosenfeld in 1971 that opened a new and innovative way of thinking to mathematicians and others. With the run of time a lot of vital discrete structures have been fuzzified by subsequent authors in a number of ways.

Despite of existence of huge amount of works in DF environment, there are lot of chances to the researchers to explore BE-algebra and BCK/BCI-algebra in IF atmosphere and IVF atmosphere in doubt sense.

In different algebras, researchers have dealt with subalgebras, ideals, subimplicative ideals, P-ideals, H-ideals, implicative ideals etc. and their fuzzification in doubt sense. Taking queues from the above fuzzifications we have also intruduced in this thesis seversal concepts and theorems of algebraic structures in the larger framework of DIF and DIVF settings.

The concepts of DIFSAs/DIF-ideals, DIFH-ideals, DIF SI-ideals, DIF P-ideal-ideals and DIF translations of DIF SI-ideals in BCK/BCI-algebra are initiated and their properties are characterized. Also DIVFSAs/DIVF-ideals in BCK-algebra and BF-algebra are introduced.

Generalizing the concept of algebric structures, Marty [55] first developed the theory of algebraic hyperstructures. The uniqueness of this algebraic hyperstructures is that a set is generated due to the composition of two elements, having meaningful applications in several fields.

The present study highlights the concept of DIF Hyp-filters in Hyp BE-algebras and some of its properties. Merging the concepts of IFSs and Hyp-filters in BE-algebra, DIF Hyp-filter is obtained. Also, classifications of DIF Hyp-filters in Hyp BE-algebra are given.

Therefore main objective of this theoretical work is to develop the theory of algebraic structures and hyperstructures in the broder framewok of DIF and DIVF settings.

1.6 Summary

An extensive literature survey is presented in this first chapter along with basic notions and definitions. A complete framework of the thesis, arrangement of the chapters and objectives of the thesis are also comprising parts of this chapter.