Chapter 8

Doubt interval-valued fuzzy ideals in BF-algebras*

8.1 Introduction

Andrzej Walendziak introduced the BF-algebra [83] in 2007.

" [83] An algebra $(V; *, 0)$ is said to be a BF-algebra if it fulfills the below stated postulates:

(I) $u_1 * u_1 = 0$

(II) $u_1 * 0 = u_1$

(III) $0 * (u_1 * u_2) = (u_2 * u_1)$, for all $u_1, u_2 \in U$."

Let $M = [0; \infty)$. Define the binary operation $*$ on M as follows: $u_1 * u_2 = |u_1 - u_2|, \forall u_1, u_2 \in M$. Then $(M; *, 0)$ is a BF-algebra.

In [89], Zadeh made an extension of a FS by an IVFS as stated previous chapter.

Saeid and Rezvani [64] extends the concepts of BF-algebra to fuzzy set. Several researchers investigated properties of FSAs and ideals in BCK/BCI-algebras and other algebraic structers see [[11], [12], [41], [43], [49], [53], [56], [58], [68], [69], [70], [74]].

Extending the idea of fuzzy BF-SAs, Zarandi and Saeid [90] defined IVF BF-SAs

[90] (IVF BF-SA) An IVFS M in U is called an IVF BF-SA in U if $\bar{\mu}_M(u_1 * u_2) \geq rmin\{\bar{\mu}_M(u_1), \bar{\mu}_M(u_2)\}$, for all $u_1, u_2 \in U$.

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Communicated

In this chapter we are going to investigate the concept of DIVFSA and DIVF-ideals in BF-algebras. After a detailed study of its properties, we come to this conclusion that in BF-algebras, an IVFS is a DIVF-ideal in BF-algebras if and only if the complement of this IVFS is an IVF-ideal. Finally, results based on DIVFSAs of BF-algebras and DIVF-ideals of BF-algebras are established. At the same time the product of DIVF-ideals in BF-algebras has been introduced.

8.2 DIVFSA in BF-algebras

In this section, we define DIVFSA in BF-algebras and investigated its properties.

Definition 8.2.1. *An IVFS M in U is named as a DIVFSA in U if $\bar{\Upsilon}_M(u_1 * u_2) \leq rmax\{\bar{\Upsilon}_M(u_1), \bar{\Upsilon}_M(u_2)\}$ for all $u_1, u_2 \in U$.*

Example 35. *Let $U = \{0, k, l, m, n\}$ be a BF-algebra as follows:*

$*$	0	k	l	m	n
0	0	n	m	l	k
k	k	0	n	m	l
l	l	k	0	n	m
m	m	l	k	0	n
n	n	m	l	k	0

Let $M = \{(u_1, \bar{\Upsilon}_M(u_1)) : u_1 \in U\}$ be an IVFS defined as $\bar{\Upsilon}_M(0) = [0, 0.2]$, $\bar{\Upsilon}_M(k) = [0.2, 0.4]$, $\bar{\Upsilon}_M(l) = [0.3, 0.4]$, $\bar{\Upsilon}_M(m) = [0.5, 0.6]$, $\bar{\Upsilon}_M(n) = [0.3, 0.4]$. Then M is a DIVF BF-SA in U.

Theorem 8.2.1. *An IVFS $M = (\Upsilon_M^L, \Upsilon_M^U)$ in U is a DIVFSA of a BF-algebra U if and only if the FSs Υ_M^L and Υ_M^U are DFSAs in U.*

Proof: Let Υ_M^L and Υ_M^U are DFSAs in U and $u_1, u_2 \in U$, then

$$\begin{aligned}
\bar{\Upsilon}_M(u_1 * u_2) &= [\Upsilon_M^L(u_1 * u_2), \Upsilon_M^U(u_1 * u_2)] \\
&\leq [max(\Upsilon_M^L(u_1), \Upsilon_M^L(u_2)), max(\Upsilon_M^U(u_1), \Upsilon_M^U(u_2))] \\
&= rmax([\Upsilon_M^L(u_1), \Upsilon_M^U(u_1)], [\Upsilon_M^L(u_2), \Upsilon_M^U(u_2)]) \\
&= rmax\{\bar{\Upsilon}_M(u_1), \bar{\Upsilon}_M(u_2)\}.
\end{aligned}$$

Hence, $M = (\Upsilon_M^L, \Upsilon_M^U)$ in U is a DIVFSA in a BF-algebra U.

Conversely, let $M = (\Upsilon_M^L, \Upsilon_M^U)$ be a DIVFSA in a BF-algebra U. Then for any $u_1, u_2 \in U$, we have

$$\begin{aligned}
[\Upsilon_M^L(u_1 * u_2), \Upsilon_M^U(u_1 * u_2)] &= \bar{\Upsilon}_M(u_1 * u_2)\\
&\leq rmax\{\bar{\Upsilon}_M(u_1), \bar{\Upsilon}_M(u_2)\}\\
&= rmax([\Upsilon_M^L(u_1), \Upsilon_M^U(u_1)], [\Upsilon_M^L(u_2), \Upsilon_M^U(u_2)])\\
&= [max(\Upsilon_M^L(u_1), \Upsilon_M^L(u_2)), max(\Upsilon_M^U(u_1), \Upsilon_M^U(u_2))].
\end{aligned}$$

Therefore, $\Upsilon_M^L(u_1 * u_2) \leq max(\Upsilon_M^L(u_1), \Upsilon_M^L(u_2))$ and $\Upsilon_M^U(u_1 * u_2) \leq max(\Upsilon_M^U(u_1), \Upsilon_M^U(u_2))$, hence Υ_M^L and Υ_M^U are DFSAs in U. Thus the proof ends.

Lemma 8.2.1. *If $M = \{(u_1, \bar{\Upsilon}_M(u_1)) : u_1 \in U\}$ is a DIVFSA in U, then for all $u_1 \in U$, $\bar{\Upsilon}_M(0) \leq \bar{\Upsilon}_M(u_1)$.*

Proof:

$$\begin{aligned}
\bar{\Upsilon}_M(0) &= \bar{\Upsilon}_M(u_1 * u_1)\\
&\leq rmax\{\bar{\Upsilon}_M(u_1), \bar{\Upsilon}_M(u_1)\}\\
&= rmax([\Upsilon_M^L(u_1), \Upsilon_M^U(u_1)], [\Upsilon_M^L(u_1), \Upsilon_M^u(u_1)])\\
&= \bar{\Upsilon}_M(u_1).
\end{aligned}$$

This completes the proof.

Corollary 8.2.1. *Let $M = \{(u_1, \bar{\Upsilon}_M(u_1)) : u_1 \in U\}$ be a DIVFSA in U. Then the set, $D_{\bar{\Upsilon}_M} = \{u_1 \in U/\bar{\Upsilon}_M(u_1) = \bar{\Upsilon}_M(0)\}$, is a BF-SA in U.*

Proof: Here $M = \{(u_1, \bar{\Upsilon}_M(u_1)) : u_1 \in U\}$ is a DIVFSA in U. Obviously, $0 \in D_{\bar{\Upsilon}_M}$. Now, let $u_1, u_2 \in D_{\bar{\Upsilon}_M}$, then $\bar{\Upsilon}_M(u_1) = \bar{\Upsilon}_M(0) = \bar{\Upsilon}_M(u_2)$. Also, $\bar{\Upsilon}_M(u_1 * u_2) \leq rmax\{\bar{\Upsilon}_M(u_1), \bar{\Upsilon}_M(u_2)\} = rmax\{\bar{\Upsilon}_M(0), \bar{\Upsilon}_M(0)\} = \bar{\Upsilon}_M(0)$.

Again, by Lemma 3.4, $\bar{\Upsilon}_M(0) \leq \bar{\Upsilon}_M(u_1 * u_2)$. So, $\bar{\Upsilon}_M(u_1 * u_2) = \bar{\Upsilon}_M(0)$. This means that $u_1 * u_2 \in D_{\bar{\Upsilon}_M}$. Hence, $D_{\bar{\Upsilon}_M}$ is a BF-SA in U.

Definition 8.2.2. *Let M and N be two IVDF BF-SAs in U. Then union of M and N is defined as $M \cup N = \bar{\Upsilon}_{M\cup N}(u_1) = [\Upsilon_{M\cup N}^L(u_1), \Upsilon_{M\cup N}^U(u_1)] = [max\{\Upsilon_M^L(u_1), \Upsilon_N^L(u_1)\}, max\{\Upsilon_M^U(u_1),$ for all $u_1 \in U$.*

Theorem 8.2.2. *Union of any two DIVFSAs in a BF-algebra U, is also a DIVFSA in U.*

Proof: Let P and Q be two DIVF BF-SAs in U. Again let $u_1, u_2 \in P \cup Q$. Then $u_1, u_2 \in P$ and $u_1, u_2 \in Q$. Now

$$\begin{aligned}
\bar{\Upsilon}_{P\cup Q}(u_1 * u_2) &= [\Upsilon^L_{P\cup Q}(u_1 * u_2), \Upsilon^U_{P\cup Q}(u_1 * u_2)] \\
&= [max(\Upsilon^L_P(u_1 * u_2), \Upsilon^L_Q(u_1 * u_2)), max(\Upsilon^U_P(u_1 * u_2), \Upsilon^U_Q(u_1 * u_2))] \\
&\leq [max(\Upsilon^L_{P\cup Q}(u_1), \Upsilon^L_{P\cup Q}(u_2)), max(\Upsilon^U_{P\cup Q}(u_1), \Upsilon^U_{P\cup Q}(u_2))] \\
&= rmax\{\bar{\Upsilon}_{P\cup Q}(u_1), \bar{\Upsilon}_{P\cup Q}(u_2)\}
\end{aligned}$$

Thus the proof ends.

Also, the above theorem is interpreted with the help of the example given below.

Example 36. *Consider a BF-algebra $U = \{0, i, j, k\}$ with the table as follows:*

$*$	0	i	j	k
0	0	k	0	i
i	i	0	i	k
j	j	k	0	i
k	k	i	k	0

Let $M = \{(u_1, \bar{\Upsilon}_M(u_1)) : u_1 \in U\}$ be an IVFS defined as

U	0	i	j	k
$\bar{\Upsilon}_M$	$[0.1, 0.2]$	$[0.2, 0.3]$	$[0.3, 0.5]$	$[0.2, 0.3]$

Then M is a DIVFSA in U.

Again, let $N = \{(u_1, \bar{\Upsilon}_N(u_1)) : u_1 \in U\}$ be an IVFS defined as

U	0	i	j	k
$\bar{\Upsilon}_N$	$[0, 0.1]$	$[0.1, 0.2]$	$[0.4, 0.5]$	$[0.1, 0.2]$

Then N is a DIVFSA in U.

We also assume that $R = M \cup N$ and R is defined as:

U	0	i	j	k
$\bar{\Upsilon}_P$	$[0.1, 0.2]$	$[0.2, 0.3]$	$[0.4, 0.5]$	$[0.2, 0.3]$

Then R is a DIVFSA in U.

Theorem 8.2.3. *Let $M = \{(u_1, \bar{\Upsilon}_M(u_1)) : u_1 \in U\}$ be a DIVFSA in U. Then for any $u_1 \in U$, we have*
(i) $\bar{\Upsilon}_M(u_1^m * u_1) \leq \bar{\Upsilon}_M(u_1)$, if m is odd
(ii) $\bar{\Upsilon}_M(u_1^m * u_1) = \bar{\Upsilon}_M(u_1)$, if m is even.

Proof: (i) Let $u_1 \in U$, then $\bar{\Upsilon}_M(u_1 * u_1) = \bar{\Upsilon}_M(0) \leq \bar{\Upsilon}_M(u_1)$ [by Lemma8.2.1].
If m is odd then let $m = 2p - 1$, where p is some positive integer .
Now, assume that $\bar{\Upsilon}_M(u_1^{2p-1} * u_1) \leq \bar{\Upsilon}_M(u_1)$, for some positive integer p.

Then,

$$\begin{aligned}
\bar{\Upsilon}_M(u_1^{2(p+1)-1} * u_1) &= \bar{\Upsilon}_M(u_1^{2p+1} * u_1), \\
&= \bar{\Upsilon}_M(u_1^{2p-1} * (u_1 * (u_1 * u_1))) \\
&= \bar{\Upsilon}_M(u_1^{2p-1} * (u_1 * 0)) \\
&= \bar{\Upsilon}_M(u_1^{2p-1} * u_1) \\
&\leq \bar{\Upsilon}_M(u_1).
\end{aligned}$$

Hence, $\bar{\Upsilon}_M(u_1^m * u_1) \leq \bar{\Upsilon}_M(u_1)$, if m is odd.

(ii) Again, let m be even, and $m = 2q$.

Now for $q = 1$, $\bar{\Upsilon}_M(u_1^2 * u_1) = \bar{\Upsilon}_M(u_1 * (u_1 * u_1)) = \bar{\Upsilon}_M(u_1 * 0) = \bar{\Upsilon}_M(u_1)$.

Also assume that, $\bar{\Upsilon}_M(u_1^{2q} * u_1) = \bar{\Upsilon}_M(u_1)$, for some positive integer q.

Then,

$$\begin{aligned}
\bar{\Upsilon}_M(u_1^{2(q+1)} * u_1) &= \bar{\Upsilon}_M(u_1^{2q} * (u_1 * (u_1 * u_1))), \\
&= \bar{\Upsilon}_M(u_1^{2q} * u_1) \\
&= \bar{\Upsilon}_M(u_1).
\end{aligned}$$

Hence, $\bar{\Upsilon}_M(u_1^m * u_1) = \bar{\Upsilon}_M(u_1)$, if m is even.

8.3 DIVF-ideal in BF-algebras

In this section, we define DIVF-ideal in BF-algebras and investigated its properties.

Definition 8.3.1. *An IVFS M in U is named as a* ***DIVF-ideal*** *of BF-algebra U if*
(i) $\bar{\Upsilon}_M(0) \leq \bar{\Upsilon}_M(u_1)$
(ii) $\bar{\Upsilon}_M(u_1) \leq rmax\{\bar{\Upsilon}_M(u_1 * u_2), \bar{\Upsilon}_M(u_2)\}$ for all $u_1, u_2 \in U$.

EXAMPLE 37. *Let $U = \{0, q, r\}$ be a BF-algebra with the table given below:*

$*$	0	q	r
0	0	q	r
q	q	0	q
r	r	r	0

Let M be an IVFS in U as defined by

U	0	q	r
$\bar{\Upsilon}_M$	$[0.2, 0.3]$	$[0.4, 0.6]$	$[0.6, 0.8]$

Then M is a DIVF-ideal in U.

Proposition 8.3.1. *Let $\bar{\Upsilon}_M(u_1)$ be a DIVF-ideal in U. Then*
(a) If $u_1 \leq u_2$ then $\bar{\Upsilon}_M(u_1) \leq \bar{\Upsilon}_M(u_2)$.
(b) If $u_1 * u_2 \leq u_3$ then $\bar{\Upsilon}_M(u_1) \leq rmax\{\bar{\Upsilon}_M(u_2), \bar{\Upsilon}_M(u_3)\}$, for all $u_1, u_2, u_3 \in U$.

Proof: (a) Let $u_1 \leq u_2$, then $u_1 * u_2 = 0$. Now

$$\begin{aligned} \bar{\Upsilon}_M(u_1) &\leq rmax\{\bar{\Upsilon}_M(u_1 * u_2), \bar{\Upsilon}_M(u_2)\} \quad [\text{Since } \bar{\Upsilon}_M(u_1) \text{ is a DIVF-ideal}] \\ &= rmax\{\bar{\Upsilon}_M(0), \bar{\Upsilon}_M(u_2)\} \\ &= \bar{\Upsilon}_M(u_2) \quad [\text{Since } \bar{\Upsilon}_M(0) \leq \bar{\Upsilon}_M(u_2) \text{ for DIVF-ideal}]. \end{aligned}$$

So, $\bar{\Upsilon}_M(u_1) \leq \bar{\Upsilon}_M(u_2)$.

(b) Here $u_1 * u_2 \leq u_3$, therefore $(u_1 * u_2) * u_3 = 0$. Now,

$$\begin{aligned} \bar{\Upsilon}_M(u_1 * u_2) &\leq rmax\{\bar{\Upsilon}_M((u_1 * u_2) * u_3), \bar{\Upsilon}_M(u_3)\} \quad [\text{Since } \bar{\Upsilon}_M(u_1) \text{ is a DIVF-ideal}] \\ &= rmax\{\bar{\Upsilon}_M(0), \bar{\Upsilon}_M(u_3)\} \\ &= \bar{\Upsilon}_M(u_3) \quad [\text{Since } \bar{\Upsilon}_M(0) \leq \bar{\Upsilon}_M(u_3) \text{ for DIVF-ideal}]. \end{aligned}$$

Therefore, $\bar{\Upsilon}_M(u_1 * u_2) \leq \bar{\Upsilon}_M(u_3)$.

Again, as $\bar{\Upsilon}_M(u_1)$ is a DIVF-ideal, so

$$\begin{aligned} \bar{\Upsilon}_M(u_1) &\leq rmax\{\bar{\Upsilon}_M(u_1 * u_2), \bar{\Upsilon}_M(u_2)\} \\ &\leq rmax\{\bar{\Upsilon}_M(u_3), \bar{\Upsilon}_M(u_2)\} \quad [\text{Since } \bar{\Upsilon}_M(u_1 * u_2) \leq \bar{\Upsilon}_M(u_3)]. \end{aligned}$$

Hence, $\bar{\Upsilon}_M(u_1) \leq rmax\{\bar{\Upsilon}_M(u_2), \bar{\Upsilon}_M(u_3)\}$, for all $u_1, u_2, u_3 \in U$.

Theorem 8.3.2. *Let $M = \{(u_1, \bar{\Upsilon}_M(u_1)) : u_1 \in U\}$ be a DIVF-ideal of a BF-algebra U. Then the set, $D_{\bar{\Upsilon}_M} = \{u_1 \in U / \bar{\Upsilon}_M(u_1) = \bar{\Upsilon}_M(0)\}$, is an ideal in U.*

Proof: $M = \{(u_1, \bar{\Upsilon}_M(u_1)) : u_1 \in U\}$ is a DIVF-ideal in U. Obviously, $0 \in D_{\bar{\Upsilon}_M}$. Now, let $u_1 * u_2, u_2 \in D_{\bar{\Upsilon}_M}$, then $\bar{\Upsilon}_M(u_1 * u_2) = \bar{\Upsilon}_M(0) = \bar{\Upsilon}_M(u_2)$.

Now,

$$\begin{aligned} \bar{\Upsilon}_M(u_1) &\leq rmax\{\bar{\Upsilon}_M(u_1 * u_2), \bar{\Upsilon}_M(u_2)\} \\ &= rmax\{\bar{\Upsilon}_M(0), \bar{\Upsilon}_M(0)\} \\ &= \bar{\Upsilon}_M(0). \end{aligned}$$

Therefore, $\bar{\Upsilon}_M(u_1) \leq \bar{\Upsilon}_M(0)$. Also, $\bar{\Upsilon}_M(0) \leq \bar{\Upsilon}_M(u_1)$ [Since M is a DIVF-ideal]. Hence, $\bar{\Upsilon}_M(u_1) = \bar{\Upsilon}_M(0)$. So, $u_1 \in D_{\bar{\Upsilon}_M}$. That is, $u_1 * u_2, u_2 \in D_{\bar{\Upsilon}_M}$, implies that $u_1 \in D_{\bar{\Upsilon}_M}$. So, $D_{\bar{\Upsilon}_M}$ is an ideal. Hence the proof ends.

Theorem 8.3.3. *Let $M = \{(u_1, \bar{\Upsilon}_M(u_1)) : u_1 \in U\}$ be a DIVFS Of a BF-algebra U, then M is a DIVF-ideal in U iff its complement is DIVF-ideal in U.*

Proof: Let M is an IVF-ideal in U, to prove M^c (complement of M) is DIVF-ideal, let $u_1, u_2, u_3 \in U$. Now,

$$\begin{aligned} \bar{\Upsilon}^c_M(0) &= 1 - \bar{\Upsilon}_M(0) \\ &\leq 1 - \bar{\Upsilon}_M(u_1) \quad [\text{As } \bar{\Upsilon}_M(0) \geq \bar{\Upsilon}_M(u_1)] \\ &= \bar{\Upsilon}^c_M(u_1). \end{aligned}$$

So, $\bar{\Upsilon}^c_M(0) \leq \bar{\Upsilon}^c_M(u_1)$, for all $u_1 \in U$.

Again,

$$\begin{aligned} \bar{\Upsilon}^c_M(u_1) &= 1 - \bar{\Upsilon}_M(u_1) \\ &\leq 1 - rmin\{\bar{\Upsilon}_M(u_1 * u_2), \bar{\Upsilon}_M(u_2)\} \quad [\text{As } \bar{\Upsilon}_M(u_1) \geq rmin\{\bar{\Upsilon}_M(u_1 * u_2), \bar{\Upsilon}_M(u_2)] \\ &= 1 - rmin\{1 - \bar{\Upsilon}^c_M(u_1 * u_2), 1 - \bar{\Upsilon}^c_M(u_2)\} \\ &= rmax\{\bar{\Upsilon}^c_M(u_1 * u_2), \bar{\Upsilon}^c_M(u_2)\}. \end{aligned}$$

This implies that, M^c is a DIVF-ideal.

Conversely, let M^c is DIVF-ideal in U. Then, $\bar{\Upsilon}^c_M(0) \leq \bar{\Upsilon}^c_M(u_1)$ and $\bar{\Upsilon}^c_M(u_1) \leq rmax\{\bar{\Upsilon}^c_M(u_1 * u_2), \bar{\Upsilon}^c_M(u_2)\}$.

Now, $\bar{\Upsilon}^c_M(0) \leq \bar{\Upsilon}^c_M(u_1)$, which implies that, $1-\bar{\Upsilon}_M(0) \leq 1-\bar{\Upsilon}_M(u_1)$, that is, $\bar{\Upsilon}_M(0) \geq \bar{\Upsilon}_M(u_1)$.

Again, $\bar{\Upsilon}^c_M(u_1) \leq rmax\{\bar{\Upsilon}^c_M(u_1 * u_2), \bar{\Upsilon}^c_M(u_2)\}$, then $1-\bar{\Upsilon}_M(u_1) \leq rmax\{1-\bar{\Upsilon}_M(u_1 * u_2), 1-\bar{\Upsilon}_M(u_2)\}$, next $\bar{\Upsilon}_M(u_1) \geq rmin\{\bar{\Upsilon}_M(u_1 * u_2), \bar{\Upsilon}_M(u_2)\}$, which implies that, M is an IVF-ideal in U.

Theorem 8.3.4. *Let $M = \{(u_1, \bar{\Upsilon}_M(u_1)) : u_1 \in U\}$ be a DIVFSA in U, if $\bar{\Upsilon}_M(u_1) \leq rmax\{\bar{\Upsilon}_M(u_2), \bar{\Upsilon}_M(u_3)\}$, with $u_1 * u_2 \leq u_3$, then M is a DIVF-ideal in U.*

Proof: Here $M = \{(u_1, \bar{\Upsilon}_M(u_1)) : u_1 \in U\}$ is a DIVFSA in U. We have, $\bar{\Upsilon}_M(0) \leq \bar{\Upsilon}_M(u_1)$. Also since, $u_1 * (u_1 * u_2) \leq u_2$ and by the hypothesis, $\bar{\Upsilon}_M(u_1) \leq rmax\{\bar{\Upsilon}_M(u_1 * u_2), \bar{\Upsilon}_M(u_2)\}$. So, M is a DIVF-ideal in U.

Theorem 8.3.5. *An IVFS $M = (\Upsilon^L_M, \Upsilon^U_M)$ in U is a DIVF-ideal of a BF-algebra U iff the FSs Υ^L_M and Υ^U_M are DF-ideals in U.*

Proof: Let Υ^L_M and Υ^U_M are DF-ideals in U and $u_1, u_2 \in U$, then $\Upsilon^L_M(0) \leq \Upsilon^L_M(u_1)$ and $\Upsilon^U_M(0) \leq \Upsilon^U_M(u_1)$.

Also, $\Upsilon^L_M(u_1) \leq max[\Upsilon^L_M(u_1 * u_2), \Upsilon^L_M(u_2)]$ and $\Upsilon^U_M(u_1) \leq max[\Upsilon^U_M(u_1 * u_2), \Upsilon^U_M(u_2)]$.

Now,

$$\begin{aligned}\bar{\Upsilon}_M(u_1) &= [\Upsilon^L_M(u_1), \Upsilon^U_M(u_1)] \\ &\leq [max(\Upsilon^L_M(u_1 * u_2), \Upsilon^L_M(u_2)), max(\Upsilon^U_M(u_1 * u_2), \Upsilon^U_M(u_2))] \\ &= rmax\{\bar{\Upsilon}_M(u_1 * u_2), \bar{\Upsilon}_M(u_2)\}.\end{aligned}$$

Therefore, $\bar{\Upsilon}_M(u_1) \leq rmax\{\bar{\Upsilon}_M(u_1 * u_2), \bar{\Upsilon}_M(u_2)\}$. Hence, $M = (\Upsilon^L_M, \Upsilon^U_M)$ in U is a DIVF-ideal of a BF-algebra U.

Conversely, let $M = (\Upsilon^L_M, \Upsilon^U_M)$ in U be a DIVF-ideal of a BF-algebra U. Then for any $u_1, u_2 \in U$, we have

$$\begin{aligned}[\Upsilon^L_M(u_1), \Upsilon^U_M(u_1)] &= \bar{\Upsilon}_M(u_1) \\ &\leq rmax\{\bar{\Upsilon}_M(u_1 * u_2), \bar{\Upsilon}_M(u_2)\} \\ &= [max(\Upsilon^L_M(u_1 * u_2), \Upsilon^L_M(u_2)), max(\Upsilon^U_M(u_1 * u_2), \Upsilon^U_M(u_2))].\end{aligned}$$

Therefore, $\Upsilon^L_M(u_1) \leq max[\Upsilon^L_M(u_1 * u_2), \Upsilon^L_M(u_2)]$ and $\Upsilon^U_M(u_1) \leq max[\Upsilon^U_M(u_1 * u_2), \Upsilon^U_M(u_2)]$, hence we get Υ^L_M and Υ^U_M are DF-ideal in U. Thus the proof is completed.

Definition 8.3.2. *Let M and N be two DIVF BF-ideals in U. Then union of M and N is given as*

$M \cup N = \bar{\Upsilon}_{M\cup N}(u_1)$

$= [\Upsilon^L_{M\cup N}(u_1), \Upsilon^U_{M\cup N}(u_1)] = [max\{\Upsilon^L_M(u_1), \Upsilon^L_N(u_1)\}, max\{\Upsilon^U_M(u_1), \Upsilon^U_N(u_1)\}]$, *for all* $u_1 \in U$.

Theorem 8.3.6. *Union of any two DIVF-ideals of a BF-algebra U, is also a DIVF-ideal in U.*

Proof: Let M and N be two DIVF-ideals in U. Now let $u_1, u_2 \in M \cup N$. Then $u_1, u_2 \in M$ and $u_1, u_2 \in N$. Again let, $C = M \cup N = (\bar{\Upsilon}_C)$, where $\bar{\Upsilon}_C = \bar{\Upsilon}_M \vee \bar{\Upsilon}_N$. Let $u_1, u_2 \in U$, then,

$$\begin{aligned} \bar{\Upsilon}_C(0) &= \bar{\Upsilon}_M(0) \vee \bar{\Upsilon}_N(0) \\ &\leq \bar{\Upsilon}_M(u_1) \vee \bar{\Upsilon}_N(u_1) \\ &= \bar{\Upsilon}_C(u_1). \end{aligned}$$

Also,

$$\begin{aligned} \bar{\Upsilon}_{M\cup N}(u_1) &= [\Upsilon^L_{M\cup N}(u_1), \Upsilon^U_{M\cup N}(u_1)] \\ &= [max(\Upsilon^L_M(u_1), \Upsilon^L_N(u_1)), max(\Upsilon^U_M(u_1), \Upsilon^U_N(u_1))] \\ &\leq [max(\Upsilon^L_{M\cup N}(u_1 * u_2), \Upsilon^L_{M\cup N}(u_2)), max(\Upsilon^U_{M\cup N}(u_1 * u_2), \Upsilon^U_{M\cup N}(u_2))] \\ &= rmax\{\bar{\Upsilon}_{M\cup N}(u_1 * u_2), \bar{\Upsilon}_{M\cup N}(u_2)\}. \end{aligned}$$

Thus the proof ends.

Also, the theorem is interpreted with the help of the example given below.

Example 38. *Let $U = \{0, e, f, g\}$ be a BF-algebra with the table as below:*

$*$	0	e	f	g
0	0	e	f	g
e	e	0	g	f
f	f	g	0	e
g	g	f	e	0

Let M and N be two DIVF-ideals in U as defined by

U	0	e	f	g
$\bar{\Upsilon}_M$	[0.1, 0.2]	[0.2, 0.3]	[0.5, 0.6]	[0.5, 0.6]

and

U	0	e	f	g
$\bar{\Upsilon}_N$	[0.2, 0.2]	[0.3, 0.4]	[0.6, 0.7]	[0.6, 0.7]

We also assume that $C = M \cup N$ and C is defined as:

U	0	e	f	g
$\bar{\Upsilon}_C$	[0.2, 0.2]	[0.3, 0.4]	[0.6, 0.7]	[0.6, 0.7]

Then clearly, C is a DIVF-ideal in U.

The following corollary generalizes the Theorem 8.3.6.

Corollary 8.3.1. *Let $\{M_j | j = 1, 2, 3, ...\}$ be a family of DIVF-ideal of BF-algebra U, then $\bigcup_{j=1}^{n} M_j$ is also a DIVF-ideal of BF-algebra U, where $\bigcup(M_j) = max\{M_j(u_1) : j = 1, 2, 3, ...\}$.*

8.4 Product of DIVF-ideal in BF-algebras

Definition 8.4.1. *Let $\bar{\Upsilon}_M$ and $\bar{\Upsilon}_N$ be two DIVF-ideal in a BF-algebra U. Then the CP $\bar{\Upsilon}_M \times \bar{\Upsilon}_N$ is $(\bar{\Upsilon}_M \times \bar{\Upsilon}_N) : U \times U \to [0, 1]$, and is defined by $(\bar{\Upsilon}_M \times \bar{\Upsilon}_N)(u, v) = rmax\{\bar{\Upsilon}_M(u), \bar{\Upsilon}_N(v)\}$, for all $u, v \in U$.*

Theorem 8.4.1. *Let $\bar{\Upsilon}_M$ and $\bar{\Upsilon}_N$ be two DIVF-ideal in a BF-algebra U. Then $\bar{\Upsilon}_M \times \bar{\Upsilon}_N$ is also a DIVF-ideal in $U \times U$.*

Proof: Let $(u, v) \in U \times U$, then by definition,

$$\begin{aligned}
(\bar{\Upsilon}_M \times \bar{\Upsilon}_N)(0, 0) &= rmax\{\bar{\Upsilon}_M(0), \bar{\Upsilon}_N(0)\} \\
&= rmax\{(\Upsilon_M^L(0), \Upsilon_M^U(0)), (\Upsilon_N^L(0), \Upsilon_N^U(0))\} \\
&= max[(\Upsilon_M^L(0), \Upsilon_N^L(0)), (\Upsilon_M^U(0), \Upsilon_N^U(0))] \\
&\leq max[(\Upsilon_M^L(u), \Upsilon_N^L(v)), (\Upsilon_M^U(u), \Upsilon_N^U(v))] \\
&= rmax\{(\Upsilon_M^L(u), \Upsilon_M^U(u)), (\Upsilon_N^L(v), \Upsilon_N^U(v))\} \\
&= rmax\{\bar{\Upsilon}_M(u), \bar{\Upsilon}_N(v)\} \\
&= (\bar{\Upsilon}_M \times \bar{\Upsilon}_N)(u, v).
\end{aligned}$$

Therefore, $(\bar{\Upsilon}_M \times \bar{\Upsilon}_N)(0,0) \leq (\bar{\Upsilon}_M \times \bar{\Upsilon}_N)(u,v)$. (i)

Again let, $(u_1,u_2),(v_1,v_2) \in U \times U$. Then,

$$\begin{aligned}
(\bar{\Upsilon}_M \times \bar{\Upsilon}_N)(u_1,u_2) &= rmax\{\bar{\Upsilon}_M(u_1), \bar{\Upsilon}_N(u_2)\} \\
&\leq rmax\{rmax\{\bar{\Upsilon}_M(u_1 * v_1), \bar{\Upsilon}_M(v_1)\}, rmax\{\bar{\Upsilon}_N(u_2 * v_2), \bar{\Upsilon}_N(v_2)\}\} \\
&= rmax\{[max[(\Upsilon^L_M(u_1 * v_1), \Upsilon^L_M(v_1))], max[(\Upsilon^U_M(u_1 * v_1), \Upsilon^U_M(v_1))]], \\
&\quad [max[(\Upsilon^L_N(u_2 * v_2), \Upsilon^L_N(v_2))], max[(\Upsilon^U_N(u_2 * v_2), \Upsilon^U_N(v_2))]]\} \\
&= [max\{max[(\Upsilon^L_M(u_1 * v_1), (\Upsilon^L_N(u_2 * v_2)], max[\Upsilon^L_M(v_1), \Upsilon^L_N(v_2)]\}, \\
&\quad max\{max[(\Upsilon^U_M(u_1 * v_1), (\Upsilon^U_N(u_2 * v_2)], max[\Upsilon^U_M(v_1), \Upsilon^U_N(v_2)]\} \\
&= rmax\{(\bar{\Upsilon}_M \times \bar{\Upsilon}_N)[(u_1 * v_1, u_2 * v_2), (\bar{\Upsilon}_M \times \bar{\Upsilon}_N)(v_1, v_2)\}.
\end{aligned}$$

Hence, $(\bar{\Upsilon}_M \times \bar{\Upsilon}_N)(u_1,u_2) \leq rmax\{(\bar{\Upsilon}_M \times \bar{\Upsilon}_N)[(u_1 * v_1, u_2 * v_2), (\bar{\Upsilon}_M \times \bar{\Upsilon}_N)(v_1, v_2)\}$. (ii)

From (i) and (ii) see that $\bar{\Upsilon}_M \times \bar{\Upsilon}_N$ is also a DIVF-ideal in $U \times U$.

8.5 Summary

This chapter focuses on the basics of a theory for such an IVFS becoming DIVFSA and DIVF-ideal of BF-algebras. We show that an IVFS of BF-algebras is a DIVFSA and a DIVF-ideal if and only if the complement of this IVFS is an IVFSA and an IVF-ideal. Product of DIVF-ideal in BF-algebras is also established and we have established some common properties related to them.