Chapter 7

Fuzzy translation of a doubt interval-valued fuzzy ideals in BCK-algebras *

7.1 Introduction

In [89], Zadeh made an extension of the concept of a FS by an IVFS (i.e., a FS with an IV MSF). An IVFS is a FS whose MSF is many-valued and forms an interval with respect to the membership scale. IVFSs have many applications in several areas. In various algebraic structures, see [**?**, 16, 69, 70], the concept of IVFSs have been studied.

Biswas [16, 17], defined IVFSG of Rosenfeld's nature and anti fuzzy subgroup and in 2000, Jun [46] introduced the notion of IVFSAs/IVF-ideals in BCK-algebras.

Intertwining Zadeh's notion of IVFS and Jun's(1994) idea of DFSAs/DF-ideals in BCK/BCI–algebras, in this chapter we have introduced the concept of a DIVFSA /DIVF-ideal of BCK algebra and worked on its properties. Further, we have tried to develop into the relations among DIVFSAs and DIVF-ideals. We have also introduced F-translation, F-multiplication of a DIVFSA/DIVF-ideal in a BCK-algebra and have discussed the product of DIVF-ideals in BCK algebras.

---

*A part of the work presented in this chapter is accepted for publication as a special issue in *International Journal of Fuzzy System Applications*, to appear in **8**(4) (2019).

7.2 DIVFSA in BCK-algebras

This section contains the idea of DIVFSA in BCK-algebras with its relevant properties.

Definition 7.2.1. *An IVFS M in V is called a DIVFSA of BCK-algeba V if $\bar{\zeta}_M(v_1 * v_2) \leq rmax\{\bar{\zeta}_M(v_1), \bar{\zeta}_M(v_2)\}$, for all $v_1, v_2 \in V$.*

EXAMPLE 31. *Let consider $V = \{0, s, t, u, v\}$ as a BCK-algebra given in Example 5, Chapter 2 as follows:*

$*$	0	s	t	u	v
0	0	0	0	0	0
s	s	0	s	0	s
t	t	t	0	t	0
u	u	s	u	0	u
v	v	v	t	v	0

Let $M = \{(v_1, \bar{\zeta}_M(v_1)) : v_1 \in V\}$ be an IVFS defined as

V	0	s	t	u	v
$\bar{\zeta}_M$	$[0, 0.5]$	$[0.1, 0.5]$	$[0.1, 0.5]$	$[0.3, 0.6]$	$[0.1, 0.5]$

Then M is a DIVFSA in V.

Theorem 7.2.1. *If R and S be two DIVFSAs of a BCK-algebra V. Then the union of R and S is also a DIVFSA in V.*

Rroof: Let R and S be two DIVFSAs in V. Again let $v_1, v_2 \in R \cup S$. Then $v_1, v_2 \in R$ and $v_1, v_2 \in S$. Now

$$\begin{aligned}
\bar{\zeta}_{R\cup S}(v_1 * v_2) &= [\zeta^L_{R\cup S}(v_1 * v_2), \zeta^U_{R\cup S}(v_1 * v_2)] \\
&= [max(\zeta^L_R(v_1 * v_2), \zeta^L_S(v_1 * v_2)), max(\zeta^U_R(v_1 * v_2), \zeta^U_S(v_1 * v_2))] \\
&\leq [max(\zeta^L_{R\cup S}(v_1), \zeta^L_{R\cup S}(v_2)), max(\zeta^U_{R\cup S}(v_1), \zeta^U_{R\cup S}(v_2))] \\
&= rmax\{\bar{\zeta}_{R\cup S}(v_1), \bar{\zeta}_{R\cup S}(v_2)\}
\end{aligned}$$

Thus the proof ends.

Also, the theorem is interpreted with the help of the example given below.

EXAMPLE 32. *Consider the BCK-algebra V that was taken in Example 6 as follows:*

$*$	0	d	e	f
0	0	0	0	0
d	d	0	d	d
e	e	d	0	0
f	f	d	f	0

Let $M = (\alpha_M, \zeta_M)$ be an IFS of V defined by

V	0	d	e	f
$\bar{\zeta}_M$	$[0.1, 0.2]$	$[0.3, 0.7]$	$[0.3, 0.7]$	$[0.5, 0.8]$

Then M is a DIVFSA in V.

Again, let $N = \{(v_1, \bar{\zeta}_N(v_1)) : v_1 \in V\}$ be an IVFS defined as

V	0	d	e	f
$\bar{\zeta}_N$	$[0.2, 0.4]$	$[0.4, 0.6]$	$[0.5, 0.7]$	$[0.6, 0.8]$

Then N is a DIVFSA in V.

We also assume that $G = M \cup N$ so G is given by:

V	0	d	e	f
$\bar{\zeta}_G$	$[0.2, 0.4]$	$[0.4, 0.7]$	$[0.5, 0.7]$	$[0.6, 0.8]$

Then G is a DIVFSA in V.

Theorem 7.2.2. *An IVFS $M = (\zeta_M^L, \zeta_M^U)$ in V is a DIVFSA of a BCK-algebra V if and only if the FSs ζ_M^L and ζ_M^U are DFSAs in V.*

Proof: Let ζ_M^L and ζ_M^U are DFSAs in V and $v_1, v_2 \in V$, then

$$\begin{aligned}
\bar{\zeta}_M(v_1 * v_2) &= [\zeta_M^L(v_1 * v_2), \zeta_M^U(v_1 * v_2)] \\
&\leq [max(\zeta_M^L(v_1), \zeta_M^L(v_2)), max(\zeta_M^U(v_1), \zeta_M^U(v_2))] \\
&= rmax\{\bar{\zeta}_M(v_1), \bar{\zeta}_M(v_2)\}.
\end{aligned}$$

Hence, $M = (\zeta_M^L, \zeta_M^U)$ in V is an DIVFSA of a BCK-algebra V.

Conversely, let $M = (\zeta_M^L, \zeta_M^U)$ in V be a DIVFSA of a BCK-algebra V. Then for any $v_1, v_2 \in V$, we have

$$\begin{aligned} [\zeta_M^L(v_1 * v_2), \zeta_M^U(v_1 * v_2)] &= \bar{\zeta}_M(v_1 * v_2) \\ &\leq rmax\{\bar{\zeta}_M(v_1), \bar{\zeta}_M(v_2)\} \\ &= [max(\zeta_M^L(v_1), \zeta_M^L(v_2)), max(\zeta_M^U(v_1), \zeta_M^U(v_2))]. \end{aligned}$$

Therefore, $\zeta_M^L(v_1 * v_2) \leq max(\zeta_M^L(v_1), \zeta_M^L(v_2))$ and $\zeta_M^U(v_1 * v_2) \leq max(\zeta_M^U(v_1), \zeta_M^U(v_2))$, hence ζ_M^L and ζ_M^U are DFSAs in V. Thus the proof ends.

Lemma 7.2.1. *If $M = \{(v_1, \bar{\zeta}_M(v_1)) : v_1 \in V\}$ is a DIVFSA in V, then for all $v_1 \in V$, $\bar{\zeta}_M(0) \leq \bar{\zeta}_M(v_1)$.*

Proof:

$$\begin{aligned} \bar{\zeta}_M(0) &= \bar{\zeta}_M(v_1 * v_1) \\ &\leq rmax\{\bar{\zeta}_M(v_1), \bar{\zeta}_M(v_1)\} \\ &= \bar{\zeta}_M(v_1). \end{aligned}$$

Corollary 7.2.1. *Let $M = \{(v_1, \bar{\zeta}_M(v_1)) : v_1 \in V\}$ be a DIVFSA in V. Then the set, $D_{\bar{\zeta}_M} = \{v_1 \in V / \bar{\zeta}_M(v_1) = \bar{\zeta}_M(0)\}$, is a BCK-SA in V.*

Proof: Here $M = \{(v_1, \bar{\zeta}_M(v_1)) : v_1 \in V\}$ is a DIVFSA in V. Obviously, $0 \in D_{\bar{\zeta}_M}$. Now, let $v_1, v_2 \in D_{\bar{\zeta}_M}$, then $\bar{\zeta}_M(v_1) = \bar{\zeta}_M(0) = \bar{\zeta}_M(v_2)$. Also, $\bar{\zeta}_M(v_1 * v_2) \leq rmax\{\bar{\zeta}_M(v_1), \bar{\zeta}_M(v_2)\} = rmax\{\bar{\zeta}_M(0), \bar{\zeta}_M(0)\} = \bar{\zeta}_M(0)$.

Again, by Lemma 7.2.1, $\bar{\zeta}_M(0) \leq \bar{\zeta}_M(v_1 * v_2)$. So, $\bar{\zeta}_M(v_1 * v_2) = \bar{\zeta}_M(0)$. This means that $v_1 * v_2 \in D_{\bar{\zeta}_M}$. Hence, $D_{\bar{\zeta}_M}$ is a BCK-SA in V.

7.3 DIVF-ideal in BCK-algebras

In this section, we define DIVF-ideal in BCK-algebras and investigated its properties.

Definition 7.3.1. *An IVFS M in V is called a **DIVF-ideal** of BCK-algebra V if*
(i) $\bar{\zeta}_M(0) \leq \bar{\zeta}_M(v_1)$
(ii) $\bar{\zeta}_M(v_1) \leq rmax\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\}$ for all $v_1, v_2 \in V$.

EXAMPLE 33. *Consider V as a BCK-algebra, that was given in Example 3 Chapter 2, as in the table below:*

$*$	0	d	e	f
0	0	0	0	0
d	d	0	0	d
e	e	d	0	e
f	f	f	f	0

Let $M = (\alpha_M, \zeta_M)$ is an IFS of V defined by

V	0	d	e	f
$\bar{\zeta}_M$	$[0.1, 0.5]$	$[0.2, 0.7]$	$[0.2, 0.7]$	$[0.5, 0.8]$

Then M is a DIVF-ideal in V.

Proposition 7.3.1. *For a DIVF-ideal in a BCK-algebra V, we have the followings:*
(a) If $v_1 \leq v_2$ then $\bar{\zeta}_M(v_1) \leq \bar{\zeta}_M(v_2)$.
(b) If $v_1 * v_2 \leq v_3$ then $\bar{\zeta}_M(v_1) \leq rmax\{\bar{\zeta}_M(v_2), \bar{\zeta}_M(v_3)\}$, for all $v_1, v_2, v_3 \in V$.

Proof: (a) Let $v_1 \leq v_2$, then $v_1 * v_2 = 0$. Now

$$\begin{aligned} \bar{\zeta}_M(v_1) &\leq rmax\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\} \quad [\text{Since } \bar{\zeta}_M(v_1) \text{ is a DIVF-ideal}] \\ &= rmax\{\bar{\zeta}_M(0), \bar{\zeta}_M(v_2)\} \\ &= \bar{\zeta}_M(v_2) \quad [\text{Since } \bar{\zeta}_M(0) \leq \bar{\zeta}_M(v_2) \text{ for DIVF-ideal}]. \end{aligned}$$

So, $\bar{\zeta}_M(v_1) \leq \bar{\zeta}_M(v_2)$.

(b) Here $v_1 * v_2 \leq v_3$, therefore $(v_1 * v_2) * v_3 = 0$. Now,

$$\begin{aligned} \bar{\zeta}_M(v_1 * v_2) &\leq rmax\{\bar{\zeta}_M((v_1 * v_2) * v_3), \bar{\zeta}_M(v_3)\} \quad [\text{Since } \bar{\zeta}_M(v_1) \text{ is a DIVF-ideal}] \\ &= rmax\{\bar{\zeta}_M(0), \bar{\zeta}_M(v_3)\} \\ &= \bar{\zeta}_M(v_3) \quad [\text{Since } \bar{\zeta}_M(0) \leq \bar{\zeta}_M(v_3) \text{ for DIVF-ideal}]. \end{aligned}$$

Therefore, $\bar{\zeta}_M(v_1 * v_2) \leq \bar{\zeta}_M(v_3)$.

Again, as $\bar{\zeta}_M(v_1)$ is a DIVF-ideal, so

$$\begin{aligned} \bar{\zeta}_M(v_1) &\leq rmax\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\} \\ &\leq rmax\{\bar{\zeta}_M(v_3), \bar{\zeta}_M(v_2)\} \quad [\text{Since } \bar{\zeta}_M(v_1 * v_2) \leq \bar{\zeta}_M(v_3)]. \end{aligned}$$

Hence, $\bar{\zeta}_M(v_1) \leq rmax\{\bar{\zeta}_M(v_2), \bar{\zeta}_M(v_3)\}$, for all $v_1, v_2, v_3 \in V$.

Theorem 7.3.2. *For a DIVF-ideal $M = \{(v_1, \bar{\zeta}_M(v_1)) : v_1 \in V\}$ in a BCK-algebra V, the set, $D_{\bar{\zeta}_M} = \{v_1 \in V / \bar{\zeta}_M(v_1) = \bar{\zeta}_M(0)\}$, becomes an ideal in V.*

Proof: $M = \{(v_1, \bar{\zeta}_M(v_1)) : v_1 \in V\}$ is an DIVF-ideal in V. Obviously, $0 \in D_{\bar{\zeta}_M}$. Now, let $v_1 * v_2, v_2 \in D_{\bar{\zeta}_M}$, then $\bar{\zeta}_M(v_1 * v_2) = \bar{\zeta}_M(0) = \bar{\zeta}_M(v_2)$.

Now,

$$\begin{aligned} \bar{\zeta}_M(v_1) &\leq rmax\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\} \\ &= rmax\{\bar{\zeta}_M(0), \bar{\zeta}_M(0)\} \\ &= \bar{\zeta}_M(0). \end{aligned}$$

Therefore, $\bar{\zeta}_M(v_1) \leq \bar{\zeta}_M(0)$. Also, $\bar{\zeta}_M(0) \leq \bar{\zeta}_M(v_1)$ [Since M is a DIVF-ideal]. Hence, $\bar{\zeta}_M(v_1) = \bar{\zeta}_M(0)$. So, $v_1 \in D_{\bar{\zeta}_M}$. That is, $v_1 * v_2, v_2 \in D_{\bar{\zeta}_M}$, implies that $v_1 \in D_{\bar{\zeta}_M}$. So, $D_{\bar{\zeta}_M}$ is an ideal. Thus the proof ends.

Theorem 7.3.3. *Any DIVF-ideal in V is an DIVFSA in V.*

Proof: Let M be a DIVF-ideal in V. Since $v_1 * v_2 \leq v_1$ for all $v_1, v_2 \in V$. Then from Proposition 7.3.1, $\bar{\zeta}_M(v_1 * v_2) \leq \bar{\zeta}_M(v_1)$. Hence $\bar{\zeta}_M(v_1 * v_2) \leq \bar{\zeta}_M(v_1) \leq rmax\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\} \leq rmax\{\bar{\zeta}_M(v_1), \bar{\zeta}_M(v_2)\}$, for all $v_1, v_2 \in V$. Therefore, M is an DIVFSA in V.

The Theorem 7.3.3 may not hold in reverse direction, to justify it let us consider the Example 31. As $\bar{\zeta}_M(u) \leq rmax\{\bar{\zeta}_M(u * s), \bar{\zeta}_M(s)\} \leq rmax\{\bar{\zeta}_M(s), \bar{\zeta}_M(s)\} \leq \bar{\zeta}_M(s)$. But, $\bar{\zeta}_M(u) \geq \bar{\zeta}_M(s)$. Therefore, M is not an DIVF-ideal in V.

Theorem 7.3.4. *Let $M = \{(v_1, \bar{\zeta}_M(v_1)) : v_1 \in V\}$ be an DIVFSA in V, if $\bar{\zeta}_M(v_1) \leq rmax\{\bar{\zeta}_M(v_2), \bar{\zeta}_M(v_3)\}$, with $v_1 * v_2 \leq v_3$, then M is a DIVF-ideal in V.*

Proof: Here $M = \{(v_1, \bar{\zeta}_M(v_1)) : v_1 \in V\}$ is a DIVFSA in V. We have, $\bar{\zeta}_M(0) \leq \bar{\zeta}_M(v_1)$. As, $v_1 * (v_1 * v_2) \leq v_2$ and by the hypothesis, $\bar{\zeta}_M(v_1) \leq rmax\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\}$. So, M is an DIVF-ideal in V.

Theorem 7.3.5. *Let $M = \{(v_1, \bar{\zeta}_M(v_1)) : v_1 \in V\}$ be an IVFS of a BCK-algebra V, then M is a DIVF-ideal in V iff its complement is IVF-ideal in V.*

Proof: Let M is an IVF-ideal in V, to prove M^c (complement of M) is DIVF-ideal, let $v_1, v_2, v_3 \in V$. Now,

$$\begin{aligned}\bar{\zeta}^c_M(0) &= 1 - \bar{\zeta}_M(0) \\ &\leq 1 - \bar{\zeta}_M(v_1) \quad [\text{As } \bar{\zeta}_M(0) \geq \bar{\zeta}_M(v_1)] \\ &= \bar{\zeta}^c_M(v_1).\end{aligned}$$

So, $\bar{\zeta}^c_M(0) \leq \bar{\zeta}^c_M(v_1)$, for all $v_1 \in V$.

Again,

$$\begin{aligned}\bar{\zeta}^c_M(v_1) &= 1 - \bar{\zeta}_M(v_1) \\ &\leq 1 - rmin\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\} \quad [\text{As } \bar{\zeta}_M(v_1) \geq rmin\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)] \\ &= 1 - rmin\{1 - \bar{\zeta}^c_M(v_1 * v_2), 1 - \bar{\zeta}^c_M(v_2)\} \\ &= rmax\{\bar{\zeta}^c_M(v_1 * v_2), \bar{\zeta}^c_M(v_2)\}.\end{aligned}$$

This implies that, M^c is a DIVF-ideal.

Conversely, let M^c is DIVF-ideal in V. Then, $\bar{\zeta}^c_M(0) \leq \bar{\zeta}^c_M(v_1)$ and $\bar{\zeta}^c_M(v_1) \leq rmax\{\bar{\zeta}^c_M(v_1 * v_2), \bar{\zeta}^c_M(v_2)\}$.

Now, $\bar{\zeta}^c_M(0) \leq \bar{\zeta}^c_M(v_1)$, which implies that, $1 - \bar{\zeta}_M(0) \leq 1 - \bar{\zeta}_M(v_1)$, that is, $\bar{\zeta}_M(0) \geq \bar{\zeta}_M(v_1)$.

Again, $\bar{\zeta}^c_M(v_1) \leq rmax\{\bar{\zeta}^c_M(v_1 * v_2), \bar{\zeta}^c_M(v_2)\}$, then $1 - \bar{\zeta}_M(v_1) \leq rmax\{1 - \bar{\zeta}_M(v_1 * v_2), 1 - \bar{\zeta}_M(v_2)\}$, next $\bar{\zeta}_M(v_1) \geq rmin\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\}$, which implies that, M is an IVF-ideal in V.

Theorem 7.3.6. *If the FSs ζ^L_M and ζ^U_M are DF-ideals in V then the IVFS $M = (\zeta^L_M, \zeta^U_M)$ in V is a DIVF-ideal and vice-versa.*

Proof: Let ζ^L_M and ζ^U_M are DF-ideals in V and $v_1, v_2 \in V$, then $\zeta^L_M(0) \leq \zeta^L_M(v_1)$ and $\zeta^U_M(0) \leq \zeta^U_M(v_1)$.

Also, $\zeta^L_M(v_1) \leq max[\zeta^L_M(v_1 * v_2), \zeta^L_M(v_2)]$ and $\zeta^U_M(v_1) \leq max[\zeta^U_M(v_1 * v_2), \zeta^U_M(v_2)]$. Now,

$$\begin{aligned}\bar{\zeta}_M(v_1) &= [\zeta^L_M(v_1), \zeta^U_M(v_1)] \\ &\leq [max(\zeta^L_M(v_1 * v_2), \zeta^L_M(v_2)), max(\zeta^U_M(v_1 * v_2), \zeta^U_M(v_2))] \\ &= rmax\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\}.\end{aligned}$$

Therefore, $\bar{\zeta}_M(v_1) \leq rmax\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\}$. Hence, $M = (\zeta^L_M, \zeta^U_M)$ in V is a DIVF-ideal of a BCK-algebra V.

Conversely, let $M = (\zeta_M^L, \zeta_M^U)$ in V be a DIVF-ideal of a BCK-algebra V. Then for any $v_1, v_2 \in V$, we have

$$\begin{aligned}
[\zeta_M^L(v_1), \zeta_M^U(v_1)] &= \bar{\zeta}_M(v_1) \\
&\leq rmax\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\} \\
&= [max(\zeta_M^L(v_1 * v_2), \zeta_M^L(v_2)), max(\zeta_M^U(v_1 * v_2), \zeta_M^U(v_2))].
\end{aligned}$$

Therefore, $\zeta_M^L(v_1) \leq max[\zeta_M^L(v_1*v_2), \zeta_M^L(v_2)]$ and $\zeta_M^U(v_1) \leq max[\zeta_M^U(v_1*v_2), \zeta_M^U(v_2)]$, hence we get ζ_M^L and ζ_M^U are DF-ideal in V. Thus the proof ends.

Theorem 7.3.7. *Union of any two DIVF-ideals in a BCK-algebra V, is also a DIVF-ideal in V.*

Proof: Let M and N be two doubt DIVF-ideals in V. Now let $v_1, v_2 \in M \cup N$. Then $v_1, v_2 \in M$ and $v_1, v_2 \in N$. Again let, $C = M \cup N = (\bar{\zeta}_C)$, where $\bar{\zeta}_C = \bar{\zeta}_M \vee \bar{\zeta}_N$. Let $v_1, v_2 \in V$, then,

$$\begin{aligned}
\bar{\zeta}_C(0) &= \bar{\zeta}_M(0) \vee \bar{\zeta}_N(0) \\
&\leq \bar{\zeta}_M(v_1) \vee \bar{\zeta}_N(v_1) \\
&= \bar{\zeta}_C(v_1).
\end{aligned}$$

Also,

$$\begin{aligned}
\bar{\zeta}_{M\cup N}(v_1) &= [\zeta_{M\cup N}^L(v_1), \zeta_{M\cup N}^U(v_1)] \\
&= [max(\zeta_M^L(v_1), \zeta_N^L(v_1)), max(\zeta_M^U(v_1), \zeta_N^U(v_1))] \\
&\leq [max(\zeta_{M\cup N}^L(v_1 * v_2), \zeta_{M\cup N}^L(v_2)), max(\zeta_{M\cup N}^U(v_1 * v_2), \zeta_{M\cup N}^U(v_2))] \\
&= rmax\{\bar{\zeta}_{M\cup N}(v_1 * v_2), \bar{\zeta}_{M\cup N}(v_2)\}.
\end{aligned}$$

Thus the proof ends.

EXAMPLE 34. *Let V be a BCK-algebra as given in Example 3 in Chapter 2 with the table below:*

$*$	0	d	e	f
0	0	0	0	0
d	d	0	0	d
e	e	d	0	e
f	f	f	f	0

Let $M = (\alpha_M, \zeta_M)$ *is an IFS of* V *defined by*

V	0	d	e	f
$\bar{\zeta}_M$	$[0.2, 0.6]$	$[0.3, 0.8]$	$[0.3, 0.8]$	$[0.4, 0.8]$

and

V	0	d	e	f
$\bar{\zeta}_N$	$[0.2, 0.2]$	$[0.3, 0.4]$	$[0.3, 0.4]$	$[0.5, 0.8]$

We also assume that $C = M \cup N$ *and* C *is defined as:*

V	0	d	e	f
$\bar{\zeta}_C$	$[0.2, 0.6]$	$[0.3, 0.8]$	$[0.3, 0.8]$	$[0.5, 0.8]$

Then clearly, C *is a DIVF-ideal in* V*.*

This theorem can also be generalised as follows:

Corollary 7.3.1. *Let* $\{M_\iota | \iota = 1, 2, 3, ...\}$ *be a family of DIVF-ideal in* BCK*-algebra* V*, then* $\bigcup_{\iota=1}^{m} M_\iota$ *is also a DIVF-ideal of* BCK*-algebra* V*, where* $\bigcup(M_\iota) = max\{M_\iota(v_1) : \iota = 1, 2, 3, ...\}$.

Proposition 7.3.8. *Let* $M = \{(v_1, \bar{\zeta}_M(v_1)) : v_1 \in V\}$ *be a DIVF-ideal of a* BCK*-algebra* V*. Then* $\bar{\zeta}_M(0 * (0 * v_1)) \leq \bar{\zeta}_M(v_1)$.

Proof:

$$\begin{aligned}
\bar{\zeta}_M(0 * (0 * v_1)) &\leq rmax\{\bar{\zeta}_M\{(0 * (0 * v_1)) * v_1\}, \bar{\zeta}_M(v_1)\} \\
&= rmax\{\bar{\zeta}_M(0), \bar{\zeta}_M(v_1)\} \\
&= \bar{\zeta}_M(v_1), \text{ for all } v_1 \in V.
\end{aligned}$$

Therefore, $\bar{\zeta}_M(0 * (0 * v_1)) \leq \bar{\zeta}_M(v_1)$, for all $v_1 \in V$.

7.4 F-translation and F-multiplication of DIVFSA/DIVF-ideal in BCK-algebras

This section deals with the notion of F-translation and F-multiplication of a DIVFSA/ DIVF-ideal in BCK-algebras. For any IVFS $M = \{(v_1, \bar{\zeta}_M(v_1)) : v_1 \in V\}$, let $\bar{\delta} = [\delta^L, \delta^U]$ where $\delta^U = 1 - sup\{\bar{\zeta}_M^U(v_1) : v_1 \in V\}$ and $\delta^L \leq \delta^U$.

Definition 7.4.1. *Let $M = (\zeta_M^L, \zeta_M^U)$ be an IVFS in V and $0 \leq \beta^U \leq \delta^U$, where $\bar{\beta} = [\beta^L, \beta^U]$. An object of the form $M_{\bar{\beta}}^T = [\zeta_{M\bar{\beta}}^{LT}, \zeta_{M\bar{\beta}}^{UT}]$ is identified as fuzzy β-translation of M if $M_{\bar{\beta}}^T(v_1) = \bar{\zeta}_M(v_1) + \bar{\beta}$ for all $v_1 \in V$.*

Definition 7.4.2. *Let $M = (\zeta_M^L, \zeta_M^U)$ be an IVFS in V and $\bar{\rho} \in [0, 1]$. An object having the form $M_{\bar{\rho}}^m = [\zeta_{M\bar{\rho}}^{Lm}, \zeta_{M\bar{\rho}}^{Um}]$ is called fuzzy ρ-multiplication of M if $M_{\bar{\rho}}^m(v_1) = \bar{\zeta}_M(v_1).\bar{\rho}$ for all $v_1 \in V$.*

Theorem 7.4.1. *Let $M = (\zeta_M^L, \zeta_M^U)$ be a DIVFSA/DIVF-ideal in V. Then the fuzzy β-translation $M_{\bar{\beta}}^T(v_1)$ of M is also a DIVFSA/DIVF-ideal in V for all $\bar{\beta} \in [\bar{0}, \bar{\delta}]$ and vice-versa.*

Proof: Let $M = (\zeta_M^L, \zeta_M^U)$ be a DIVFSA in V and let $\beta \in [0, \bar{\delta}]$. Then $\bar{\zeta}_M(v_1 * v_2) \leq rmax\{\bar{\zeta}_M(v_1), \bar{\zeta}_M(v_2)\}$ for all $v_1, v_2 \in V$. Now,

$$\begin{aligned} M_{\bar{\beta}}^T(v_1 * v_2) &= \bar{\zeta}_M(v_1 * v_2) + \bar{\beta} \\ &\leq rmax\{\bar{\zeta}_M(v_1), \bar{\zeta}_M(v_2)\} + \bar{\beta} \\ &= rmax\{\bar{\zeta}_M(v_1) + \bar{\beta}, \bar{\zeta}_M(v_2) + \bar{\beta}\} \\ &= rmax\{M_{\bar{\beta}}^T(v_1), M_{\bar{\beta}}^T(v_2)\} \end{aligned}$$

Hence, $M_{\bar{\beta}}^T$ is a DIVFSA.

Conversely, let $M_{\bar{\beta}}^T$ is a DIVFSA. Then

$$\begin{aligned} M_{\bar{\beta}}^T(v_1 * v_2) &\leq rmax\{M_{\bar{\beta}}^T(v_1), M_{\bar{\beta}}^T(v_2)\} \\ &= rmax\{\bar{\zeta}_M(v_1) + \bar{\beta}, \bar{\zeta}_M(v_2) + \bar{\beta}\} \\ &= \bar{\zeta}_M(v_1 * v_2) + \bar{\beta} \\ &= rmax\{\bar{\zeta}_M(v_1), \bar{\zeta}_M(v_2)\} + \bar{\beta} \end{aligned}$$

So, $\bar{\zeta}_M(v_1 * v_2) + \bar{\beta} \leq rmax\{\bar{\zeta}_M(v_1), \bar{\zeta}_M(v_2)\} + \bar{\beta}$, implies that $\bar{\zeta}_M(v_1 * v_2) \leq rmax\{\bar{\zeta}_M(v_1), \bar{\zeta}_M(v_2)\}$, for all $v_1, v_2 \in V$. Hence, M is a DIVFSA in V.

Again let M is a DIVF-ideal of a BCK-algebra V and let $\bar{\beta} \in [\bar{0}, \bar{\delta}]$. Then for all $v_1, v_2 \in V$, $M_{\bar{\beta}}^T(0) = \bar{\zeta}_M(0) + \bar{\beta} \leq \bar{\zeta}_M(v_1) + \bar{\beta} = M_{\bar{\beta}}^T(v_1)$. That is $M_{\bar{\beta}}^T(0) \leq M_{\bar{\beta}}^T(v_1)$. And,

$$\begin{aligned}
M_{\bar{\beta}}^{T}(v_1) &= \bar{\zeta}_M(v_1) + \bar{\beta} \\
&\leq rmax\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\} + \bar{\beta} \\
&= rmax\{\bar{\zeta}_M(v_1 * v_2) + \bar{\beta}, \bar{\zeta}_M(v_2) + \bar{\beta}\} \\
&= rmax\{M_{\bar{\beta}}^{T}(v_1 * v_2), M_{\bar{\beta}}^{T}(v_2)\}
\end{aligned}$$

Hence, $M_{\bar{\beta}}^{T}$ is a DIVF-ideal .

Conversely, let $M_{\bar{\beta}}^{T}$ is a DIVF-ideal. Then $M_{\bar{\beta}}^{T}(0) \leq M_{\bar{\beta}}^{T}(v_1)$, implies that $\bar{\zeta}_M(0) \leq \bar{\zeta}_M(v_1)$. Also,

$$\begin{aligned}
M_{\bar{\beta}}^{T}(v_1) &= \bar{\zeta}_M(v_1) + \bar{\beta} \\
&\leq rmax\{M_{\bar{\beta}}^{T}(v_1 * v_2), M_{\bar{\beta}}^{T}(v_2)\} \\
&= rmax\{\bar{\zeta}_M(v_1 * v_2) + \bar{\beta}, \bar{\zeta}_M(v_2) + \bar{\beta}\} \\
&= rmax\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\} + \bar{\beta}
\end{aligned}$$

So, $\bar{\zeta}_M(v_1)+\bar{\beta} \leq rmax\{\bar{\zeta}_M(v_1*v_2), \bar{\zeta}_M(v_2)\}+\bar{\beta}$, implies that $\bar{\zeta}_M(v_1) \leq rmax\{\bar{\zeta}_M(v_1* v_2), \bar{\zeta}_M(v_2)\}$, for all $v_1, v_2 \in V$. Hence, M is a DIVF-ideal in V.

7.5 Product of DIVF-ideal in BCK-algebras

Definition 7.5.1. *Let M and N be two DIVF-ideals of a BCK-algebra V. Then the CP $M \times N$ is $(M \times N) : V \times V \to [0, 1]$, and is defined by*
$(M \times N)(u, v) = rmax\{\bar{\zeta}_M(u), \bar{\zeta}_N(v)\}$, for all $u, v \in V$.

Theorem 7.5.1. *Let M and N be two DIVF-ideal of a BCK-algebra V. Then the CP of M and N is also a DIVF-ideal of $V \times V$.*

Proof: Let $(u, v) \in V \times V$, then by definition,

$$\begin{aligned}
(\bar{\zeta}_M \times \bar{\zeta}_N)(0,0) &= rmax\{\bar{\zeta}_M(0), \bar{\zeta}_N(0)\} \\
&= rmax\{(\zeta_M^L(0), \zeta_M^U(0)), (\zeta_N^L(0), \zeta_N^U(0))\} \\
&= max[(\zeta_M^L(0), \zeta_N^L(0)), (\zeta_M^U(0), \zeta_N^U(0))] \\
&\leq max[(\zeta_M^L(u), \zeta_N^L(v)), (\zeta_M^U(u), \zeta_N^U(v))] \\
&= rmax\{(\zeta_M^L(u), \zeta_M^U(u)), (\zeta_N^L(v), \zeta_N^U(v))\} \\
&= rmax\{\bar{\zeta}_M(u), \bar{\zeta}_N(v)\} \\
&= (\bar{\zeta}_M \times \bar{\zeta}_N)(u,v).
\end{aligned}$$

Therefore, $(\bar{\zeta}_M \times \bar{\zeta}_N)(0,0) \leq (\bar{\zeta}_M \times \bar{\zeta}_N)(u,v)$. (i)

Again let, $(u_1,u_2),(v_1,v_2) \in V \times V$. Then,

$$\begin{aligned}
(\bar{\zeta}_M \times \bar{\zeta}_N)(u_1,u_2) &= rmax\{\bar{\zeta}_M(u_1), \bar{\zeta}_N(u_2)\} \\
&\leq rmax\{rmax\{\bar{\zeta}_M(u_1 * v_1), \bar{\zeta}_M(v_1)\}, rmax\{\bar{\zeta}_N(u_2 * v_2), \bar{\zeta}_N(v_2)\}\} \\
&= rmax\{[max[(\zeta_M^L(u_1 * v_1), \zeta_M^L(v_1))], max[(\zeta_M^U(u_1 * v_1), \zeta_M^U(v_1))]], \\
&\quad [max[(\zeta_N^L(u_2 * v_2), \zeta_N^L(v_2))], max[(\zeta_N^U(u_2 * v_2), \zeta_N^U(v_2))]]\} \\
&= [max\{max[(\zeta_M^L(u_1 * v_1), (\zeta_N^L(u_2 * v_2)], max[\zeta_M^L(v_1), \zeta_N^L(v_2)]\}, \\
&\quad max\{max[(\zeta_M^U(u_1 * v_1), (\zeta_N^U(u_2 * v_2)], max[\zeta_M^U(v_1), \zeta_N^U(v_2)]\} \\
&= rmax\{(\bar{\zeta}_M \times \bar{\zeta}_N)[(u_1 * v_1, u_2 * v_2), (\bar{\zeta}_M \times \bar{\zeta}_N)(v_1, v_2)\}.
\end{aligned}$$

Hence, $(\bar{\zeta}_M \times \bar{\zeta}_N)(u_1,u_2) \leq rmax\{(\bar{\zeta}_M \times \bar{\zeta}_N)[(u_1 * v_1, u_2 * v_2), (\bar{\zeta}_M \times \bar{\zeta}_N)(v_1, v_2)\}$. (ii)

From (i) and (ii) see that $(M \times N)$ is also a DIVF-ideal of $V \times V$.

7.6 Summary

In this chapter, we have defined DIVFSAs and DIVF-ideals in BCK-algebras and have developed various characterizations of DIVFSAs and DIVF-ideals in BCK-algebras. Besides, we have introduced F-translation and F-multiplication of a DIVFSA/DIVF-ideal in BCK-algebras and have defined the product of DIVF-ideal in BCK-algebras.