Chapter 6

Doubt intuitionistic fuzzy hyper filters in hyper BE-algebras*

6.1 Introduction

Naturally, system identification problems involve non probabilistic characteristics and as a solution Zadeh [86] first introduced theory of FS as an innovative alternative to probability theory. Since then the theory of FSs gaining massive attention among the researchers and the scope of its applications has already been expanded in several fields like coding theory, engineering, graph theory medical science, social science etc.

The theory of algebric hyperstructure, in recent times has become a well-recognised branch in algebric theory due to its wide applications. Generalizing the concept of algebric structures, Marty [55] first developed the theory of algebraic hyperstructures. The uniqueness of this algebraic hyperstructures is that a set is generated due to the composition of two elements, having meaningful applications in several fields like automata, probability, lattices, geometry, binary relations, codes, graphs, hypergraphs, cryptography etc.

After the introduction of BE-algebra by Kim and Kim as a generalization of a dual BCK-algebra [51], Radfar et al. [61] applied the hyperstructures theory to BE-algebra and define the notion of a Hyp BE-algebra. After that in 2015, Rezaei et al. [62] introduced commutative Hyp BE-algebra, also see [8, 9]. At the same time they proved that every commutative(row diagonal, column row, very thin) Hyp BE-

*A part of the work presented in this chapter is accepted in
International Journal of fuzzy Systems

algebra is a BE-algebra.

Hundred of papers were written to established the relationship between the FSs and algebraic hyperstructures as fuzzy hyperstructure is an interesting topic of research. In 2015 Tang et al. [82], introduced the concept of Hyp-filterand fuzzy Hyp-filterof an ordered semihypergroup.

In 2009, Shabir and Khan [78] introduced the notion of IF-filters of ordered semigroups. They established a relation between IF-filters and IF-prime ideals of ordered semigroups. Next in 2012, Palaniappen et al. [60] applied the notion of IF-ideals in Hyp BCI-algebras. And in recent past Cheng and Xin [20], focused on investigating implicative Hyp-filters and PI Hyp-filters on Hyp BE-algebras.

In 2016, Hamidi et al. [25] introduced the notion of fuzzy homomorphisms in fuzzy Hyp BE-SAs and thus made a connection between fuzzy BE-algebras and Hyp BE-algebras. They also defined the concept of normal fuzzy Hyp BE-SA and investigated some of its properties. Moreover they introduced fuzzy Hyp-filters on Hyp BE-algebras.

The present study considers the DIF Hyp-filters of Hyp BE-algebras and its related properties. Applying some important conditions on IFSs in Hyp BE-filters, we have formed DIF Hyp-filters in Hyp BE-algebras and have given the characterizations of DIF Hyp-filters in Hyp BE-algebras. Besides the authors also give characterizations of DIF Hyp-filters in commutative Hyp BE-algebras. At the same time we deal with the doubt intuitionistic fuzzification of the notion of implicative Hyp-filters in Hyp BE-algebras. We show that every DIFI Hyp-filters in Row-Hyp BE-algebras are DIF Hyp-filters in Hyp BE-algebras and also give the condition such thmt a DIF Hyp-filters in Hyp BE-algebras to be a DIFI Hyp-filters in Hyp BE-algebras. Thus the aim of this study is to extend the IF results to hyperstructures.

6.2 DIF Hyp-filters in hyper BE-algebras

The DIF Hyp-filters in Hyp BE-algebras can be defined in the following two ways;

Definition 6.2.1. *An IFS $M = (\alpha_M, \zeta_M)$ of a Hyp BE-algebra V is called a **DIF Hyp-filter** in Hyp BE-algebra V if*

(1) $\alpha_M(1) \leq \alpha_M(v_2)$, $\zeta_M(1) \geq \zeta_M(v_2)$;

(2) $\alpha_M(v_1) \leq \max\{\sup_{w\in v_1 \circ v_2} \alpha_M(w), \alpha_M(v_2)\}$;

(3) $\zeta_M(v_1) \geq \min\{\inf_{w\in v_1 \circ v_2} \zeta_M(w), \zeta_M(v_2)\}$, *for all* $v_1, v_2 \in V$.

Definition 6.2.2. *Let* $M = (\alpha_M, \zeta_M)$ *be an IFS of a Hyp* BE*-algebra* V*, then* M *is called a* ***DIF Hyp-filter*** *in Hyp* BE*-algebra* V *if*

(1) $\alpha_M(1) \leq \alpha_M(v_1)$, $\zeta_M(1) \geq \zeta_M(v_1)$;

(2) $\alpha_M(v_2) \leq \max\{\sup_{w\in v_1 \circ v_2} \alpha_M(w), \alpha_M(v_1)\}$;

(3) $\zeta_M(v_2) \geq \min\{\inf_{w\in v_1 \circ v_2} \zeta_M(w), \zeta_M(v_1)\}$, *for all* $v_1, v_2 \in V$.

For a DIF Hyp-filter in V, if $v_1 << v_2$ then $\alpha_M(v_2) \leq \alpha_M(v_1)$, $\zeta_M(v_2) \geq \zeta_M(v_1)$.

An IFS of a Hyp BE-algebra V is said to satisfy the inf-sup property if for any SS S of V there exist $u_\circ, v_\circ \in S$ such that $\alpha_M(u_\circ) = inf_{u\in S}\alpha_M(u)$, and $\zeta_M(v_\circ) = sup_{v\in S}\zeta_M(v)$.

Example 29. *Let* $V = \{1, r', s'\}$. *Then (V; o, 1) is a Hyp* BE*-algebra with the table below:*

o	1	r'	s'
1	$\{1\}$	$\{r', s'\}$	$\{s'\}$
r'	$\{1\}$	$\{1, r'\}$	$\{1, s'\}$
s'	$\{1\}$	$\{1, r', s'\}$	$\{1\}$

Let $M = (\alpha_M, \zeta_M)$ *be an IFS of* V *as defined by*

V	1	r'	s'
α_M	0.4	0.5	0.7
ζ_M	0.6	0.5	0.3

Then $M = (\alpha_M, \zeta_M)$ *is a DIF Hyp-filter in Hyp* BE*-algebra* V.

Definition 6.2.3. *For an IFS* M *in* V *and* $c, d \in [0, 1]$*, the set*

$$U(\alpha_M; c) := \{v_1 \in V \mid \alpha_M(v_1) \leq c\},$$
$$L(\zeta_M; d) := \{v_1 \in V \mid \zeta_M(v_1) \geq d\}.$$

are the UC of level c *and LC of level* d *of the set* M.

Or

$$M_{<c,d>} = \{v_1 \in X/\alpha_M(v_1) \leq c, \zeta_M(v_1) \geq d\}.$$

Theorem 6.2.1. *Let $M = (\alpha_M, \zeta_M)$ be a DIF Hyp-filter in Hyp BE-algebra V. Then, for $v_1 \leq v_2$*

(1) $\alpha_M(v_2) \leq \alpha_M(v_1)$;

(2) $\zeta_M(v_2) \geq \zeta_M(v_1)$, for all $v_1, v_2 \in V$.

Proof. Since $v_1 \leq v_2$, it follows that $1 \in v_1 \circ v_2$ for all $v_1, v_2 \in V$. Then, we have, $\sup_{a\in v_1 \circ v_2} \alpha_M(a) \leq \alpha_M(1)$ and $\inf_{b\in v_1\circ v_2} \zeta_M(a) \geq \zeta_M(1)$. Also from hypothesis, $\alpha_M(1) \leq \alpha_M(v_1)$, $\zeta_M(1) \geq \zeta_M(v_1)$. Hence

$$\begin{aligned} \alpha_M(v_2) &\leq \max\{\sup_{a\in v_1\circ v_2} \alpha_M(a), \alpha_M(v_1)\} \\ &\leq \max\{\alpha_M(1), \alpha_M(v_1)\} \\ &= \alpha_M(v_1). \end{aligned}$$

Therefore, $\alpha_M(v_2) \leq \alpha_M(v_1)$. Similarly, we obtain

$$\begin{aligned} \zeta_M(v_2) &\geq \max\{\inf_{a\in v_1\circ v_2} \zeta_M(a), \zeta_M(v_1)\} \\ &\geq \max\{\zeta_M(1), \zeta_M(v_1)\} \\ &= \zeta_M(v_1). \end{aligned}$$

So, $\zeta_M(v_2) \geq \zeta_M(v_1)$. Thus the proof ends. □

Proposition 6.2.2. *If $M = (\alpha_M, \zeta_M)$ is a DIF Hyp-filter in Hyp BE-algebra V with the condition $v_1 \circ v_2 < v_3$, for all $v_1, v_2, v_3 \in V$. Then, $\alpha_M(v_1) \leq \max\{\alpha_M(v_3), \alpha_M(v_2)\}$ and $\zeta_M(v_1) \geq \min\{\zeta_M(v_3), \zeta_M(v_2)\}$.*

Proof. Let M be a DIF Hyp BE-filter.

Then, $\alpha_M(v_1) \leq \max\{\sup_{v_3\in v_1\circ v_2} \alpha_M(v_3), \alpha_M(v_2)\} = \max\{\alpha_M(v_3), \alpha_M(v_2)\}$ [as $v_1 \circ v_2 < v_3$ implies that $\alpha_M(v_1 \circ v_2) \leq \alpha_M(v_3)$]. Similarly, we have, $\zeta_M(v_1) \geq \min\{\inf_{v_3\in v_1\circ v_2} \zeta_M(v_3), \zeta_M(v_2)\}$ $= \min\{\zeta_M(v_3), \zeta_M(v_2)\}$ [as $v_1 \circ v_2 < v_3$ implies that $\zeta_M(v_1 \circ v_2) \geq \zeta_M(v_3)$]. □

The converse of the above proposition also true if $v_1 \circ v_2 < v_3$.

Proposition 6.2.3. *If $\alpha_M(v_1) \leq \max\{\alpha_M(v_3), \alpha_M(v_2)\}$ and $\zeta_M(v_1) \geq \min\{\zeta_M(v_3), \zeta_M(v_2)\}$, with the condition $v_1 \circ v_2 < v_3$, for all $v_1, v_2, v_3 \in V$. Then $M = (\alpha_M, \zeta_M)$ is a DIF Hyp-filter in Hyp BE-algebra V.*

Proof. Let $\alpha_M(v_1) \leq \max\{\alpha_M(v_3), \alpha_M(v_2)\} = \max\{\sup_{v_3\in v_1\circ v_2} \alpha_M(v_3), \alpha_M(v_2)\}$ [as $v_1 \circ v_2 < v_3$ implies that $\alpha_M(v_1 \circ v_2) \leq \alpha_M(v_3)$]. Similarly, we have, $\zeta_M(v_1) \geq$

$\min\{\zeta_M(v_3), \zeta_M(v_2)\} = \min\{\inf_{v_3 \in v_1 \circ v_2} \zeta_M(v_3), \zeta_M(v_2)\}$ [as $v_1 \circ v_2 < v_3$ implies that $\zeta_M(v_1 \circ v_2) \geq \zeta_M(v_3)$].

Hence M is a DIF Hyp-filter in Hyp BE-algebra V.

□

Theorem 6.2.4. *If $M = (\alpha_M, \zeta_M)$ is a DIF Hyp-filter in Hyp BE-algebra V, then the non empty sets $U(\alpha_M; c)$ and $L(\zeta_M; d)$ are Hyp-filters of V that is $M_{<c,d>}$ is a Hyp-filter on Hyp BE-algebra V for any $c, d \in [0, 1]$ and vice-versa.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF Hyp-filter of V and $U(\alpha_M; c)$ and $L(\zeta_M; d)$ are non empty set for any $c, d \in [0, 1]$. Since $\alpha_M(1) \leq \alpha_M(v_1) \leq c$ and $\zeta_M(1) \geq \zeta_M(v_1) \geq d$, for any $v_1 \in M_{<c,d>}$, it follows that $1 \in M_{<c,d>}$. Now let, $v_1, v_2 \in V$ such that $v_1 \circ v_2 < M_{<c,d>}$ and $v_2 \in M_{<c,d>}$ implies that $\alpha_M(v_2) \leq c$ and $\zeta_M(v_2) \geq d$. For any $w \in v_1 \circ v_2$ there exist $v_3 \in M_{<c,d>}$ such that $v_3 < w$, which implies that $c \geq \alpha_M(w) \geq \alpha_M(v_3)$ and $d \leq \zeta_M(w) \leq \zeta_M(v_3)$ then, $\alpha_M(v_1) \leq \max\{\sup_{w \in v_1 \circ v_2} \alpha_M(w), \alpha_M(v_2)\} \leq \max\{c, c\} = c$. And $\zeta_M(v_1) \geq \min\{\inf_{w \in v_1 \circ v_2} \zeta_M(w), \zeta_M(v_2)\} \geq \min\{d, d\} = d$. This implies that $v_1 \in M_{<c,d>}$. So, $M_{<c,d>}$ is a Hyp-filter on Hyp BE-algebra.

Conversely, let, $M_{<c,d>}$ is a Hyp-filter of V. For any $v_1 \in V$ let $\alpha_M(v_1) = c$ and $\zeta_M(v_1) = d$. Then $v_1 \in M_{<c,d>}$. Since $1 \in M_{<c,d>}$, so $\alpha_M(1) \leq \alpha_M(v_1)$ and $\zeta_M(1) \geq \zeta_M(v_1)$ for all $v_1 \in V$. Again for any $v_1, v_2 \in V$, let $c = max\{sup_{w \in v_1 \circ v_2} \alpha_M(w), \alpha_M(v_2)\}$ and $d = \min\{inf_{w \in v_1 \circ v_2} \zeta_M(w), \zeta_M(v_2)\}$. Then for $v_2 \in M_{<c,d>}$ and $v_3 \in v_1 \circ v_2$, we have

$$\alpha_M(v_3) \leq \{\sup_{w \in v_1 \circ v_2} \alpha_M(w)\} \leq \max\{\sup_{w \in v_1 \circ v_2} \alpha_M(w), \alpha_M(v_2)\} = c;$$
$$\zeta_M(v_3) \geq \{\inf_{w \in v_1 \circ v_2} \zeta_M(w)\} \geq \min\{\inf_{w \in v_1 \circ v_2} \zeta_M(w), \zeta_M(v_2)\} = d.$$

This implies that, $v_3 \in M_{<c,d>}$, so $v_1 \circ v_2 < M_{<c,d>}, v_2 \in M_{<c,d>}$ implies that $v_1 \in M_{<c,d>}$. Therefore, we have

$$\alpha_M(v_1) \leq c = \max\{\sup_{w \in v_1 \circ v_2} \alpha_M(w), \alpha_M(v_2)\};$$
$$\zeta_M(v_1) \geq d = \min\{\inf_{w \in v_1 \circ v_2} \zeta_M(w), \zeta_M(v_2)\}.$$

Hence M is a DIF Hyp-filter in Hyp BE-algebra V. □

Theorem 6.2.5. *Let M be a DIF Hyp-filter of a R-Hyp BE-algebra V. If $v_1 \circ v_2 = v_2 \circ v_1 = \{1\}$, then $\alpha_M(v_1 \circ v_2) \leq \alpha_M(v_2)$ and $\zeta_M(v_1 \circ v_2) \leq \zeta_M(v_2)$.*

Proof. Let M be a DIF Hyp-filter of a R-Hyp BE-algebra V. Then,

$$\begin{aligned}\alpha_M(v_1 \circ v_2) &\leq \max\{\sup_{w\in(v_1\circ v_2)\circ v_2} \alpha_M(w), \alpha_M(v_2)\}\\ &= \max\{\sup_{w\in 1\circ v_2} \alpha_M(w), \alpha_M(v_2)\} \text{ [by the given condition]}\\ &= \max\{\alpha_M(v_2), \alpha_M(v_2)\} \text{ [as } V \text{ is } R\text{-Hyp } BE\text{-algebra]}\\ &= \alpha_M(v_2).\end{aligned}$$

So, $\alpha_M(v_1 \circ v_2) \leq \alpha_M(v_2)$. Also, we have

$$\begin{aligned}\zeta_M(v_1 \circ v_2) &\geq \min\{\inf_{w\in(v_1\circ v_2)\circ v_2} \zeta_M(w), \zeta_M(v_2)\}\\ &= \min\{\inf_{w\in 1\circ v_2} \zeta_M(w), \zeta_M(v_2)\} \text{ [by the given condition]}\\ &= \min\{\zeta_M(v_2), \zeta_M(v_2)\} \text{ [as } V \text{ is } R\text{-Hyp } BE\text{-algebra]}\\ &= \zeta_M(v_2).\end{aligned}$$

Hence, $\zeta_M(v_1 \circ v_2) \leq \zeta_M(v_2)$. □

Theorem 6.2.6. *Assume that M is a DIF Hyp-filter of a R-Hyp BE-algebra V. Then the assertions below are equivalent-*

(1) $\alpha_M(v_1 \circ v_2) \leq \alpha_M(v_1 \circ (v_1 \circ v_2))$, $\zeta_M(v_1 \circ v_2) \geq \zeta_M(v_1 \circ (v_1 \circ v_2))$;

(2) $\alpha_M(v_1 \circ v_2) \leq \max\{\sup_{a\in w\circ(v_1\circ(v_1\circ v_2))} \alpha_M(a), \alpha_M(w)\}$ *and* $\zeta_M(v_1 \circ v_2) \geq \min\{\inf_{a\in w\circ(v_1\circ(v_1\circ v_2))} \zeta_M(a), \zeta_M(w)\}$.

Proof. (1⇒2) Let M be a DIF Hyp-filter in Hyp BE-algebra V and (1) holds. Then

$$\alpha_M(v_1 \circ v_2) \leq \alpha_M(v_1 \circ (v_1 \circ v_2)) \text{ and } \zeta_M(v_1 \circ v_2) \geq \zeta_M(v_1 \circ (v_1 \circ v_2)). \tag{6.1}$$

Again since M is a DIF Hyp-filter of V, it follows that

$$\begin{aligned}\alpha_M(v_1 \circ (v_1 \circ v_2)) \leq \max\{\sup_{a\in w\circ(v_1\circ(v_1\circ v_2))} \alpha_M(a), \alpha_M(w)\};\\ \zeta_M(v_1 \circ (v_1 \circ v_2)) \geq \min\{\inf_{a\in w\circ(v_1\circ(v_1\circ v_2))} \zeta_M(a), \zeta_M(w)\}.\end{aligned} \tag{6.2}$$

By using (6.1) and (6.2), we obtain

$$\alpha_M(v_1 \circ v_2) \leq \alpha_M(v_1 \circ (v_1 \circ v_2)) \leq \max\{\sup_{a\in w\circ(v_1\circ(v_1\circ v_2))} \alpha_M(a), \alpha_M(w)\};$$
$$\zeta_M(v_1 \circ v_2) \geq \zeta_M(v_1 \circ (v_1 \circ v_2)) \geq \min\{\inf_{a\in w\circ(v_1\circ(v_1\circ v_2))} \zeta_M(a), \zeta_M(w)\}.$$

Hence the desired result.

(2⇒1) Suppose that (2) holds. Then

$$\alpha_M(v_1 \circ v_2) \leq \max\{\sup_{a\in w\circ(v_1\circ(v_1\circ v_2))} \alpha_M(a), \alpha_M(w)\};$$
$$\zeta_M(v_1 \circ v_2) \geq \min\{\inf_{a\in w\circ(v_1\circ(v_1\circ v_2))} \zeta_M(a), \zeta_M(w)\}.$$

Putting $w = 1$, we get $\alpha_M(v_1 \circ v_2) \leq \max\{\sup_{a \in 1\circ(v_1 \circ(v_1 \circ v_2))} \alpha_M(a), \alpha_M(1)\}$ and $\zeta_M(v_1 \circ v_2) \geq \min\{\inf_{a \in 1\circ(v_1 \circ(v_1 \circ v_2))} \zeta_M(a), \zeta_M(1)\}$. As V is R-Hyp BE-algebra and $\alpha_M(1) \leq \alpha_M(v_1)$, $\zeta_M(1) \geq \zeta_M(v_1)$, so $\alpha_M(v_1 \circ v_2) \leq \alpha_M(v_1 \circ (v_1 \circ v_2))$, $\zeta_M(v_1 \circ v_2) \geq \zeta_M(v_1 \circ (v_1 \circ v_2))$. Thus the proof ends. □

Theorem 6.2.7. *Let M be a DIF Hyp-filter of a RD-Hyp BE-algebra V, with the conditions, $\alpha_M((v_1 \circ v_2) \circ (v_1 \circ w)) \leq \alpha_M(v_1 \circ (v_2 \circ w))$, and $\zeta_M((v_1 \circ v_2) \circ (v_1 \circ w)) \geq \zeta_M(v_1 \circ (v_2 \circ w))$. Then $\alpha_M(v_1 \circ v_2) \leq \max\{\sup_{a \in w\circ(v_1 \circ(x \circ v_2))} \alpha_M(a), \alpha_M(w)\}$, and $\zeta_M(v_1 \circ v_2) \geq \min\{\inf_{a \in w\circ(v_1 \circ(v_1 \circ v_2))} \zeta_M(a), \zeta_M(w)\}$ for all $v_1, v_2, w \in V$.*

Proof. As M is a DIF Hyp-filter of a RD-Hyp BE-algebra V, we have

(1) $\alpha_M(v_2) \leq \max\{\sup_{a \in v_1 \circ v_2} \alpha_M(a), \alpha_M(v_1)\}$;

(2) $\zeta_M(v_2) \geq \min\{\inf_{a \in v_1 \circ v_2} \zeta_M(a), \zeta_M(v_1)\}$, for all $v_1, v_2 \in V$. Also, $\alpha_M(v_1 \circ v_2) \leq \max\{\sup_{a \in w\circ(v_1 \circ v_2)} \alpha_M(a), \alpha_M(w)\}$;

(3) $\zeta_M(v_1 \circ v_2) \geq \min\{\inf_{a \in w\circ(v_1 \circ v_2)} \zeta_M(a), \zeta_M(w)\}$, for all $v_1, v_2, w \in V$.

Since V is a RD-Hyp BE-algebra, it follows that

$$\begin{aligned}
\alpha_M(w \circ (v_1 \circ v_2)) &= \alpha_M((v_1 \circ (w \circ v_2)\\
&= \alpha_M((v_1 \circ v_1) \circ (v_1 \circ (w \circ v_2)))\\
&\leq \alpha_M(v_1 \circ (v_1 \circ (w \circ v_2))) \text{ [by the given condition]}\\
&= \alpha_M(v_1 \circ (w \circ (v_1 \circ v_2)))\\
&= \alpha_M(w \circ (v_1 \circ (v_1 \circ v_2))).
\end{aligned}$$

So, $\alpha_M(v_1 \circ v_2) \leq \max\{\sup_{a \in w\circ(v_1 \circ v_2)} \alpha_M(a), \alpha_M(w)\}$. Similarly, we have

$$\begin{aligned}
\zeta_M(w \circ (v_1 \circ v_2)) &= \zeta_M((v_1 \circ (w \circ v_2)\\
&= \zeta_M((v_1 \circ v_1) \circ (v_1 \circ (w \circ v_2)))\\
&\geq \zeta_M(v_1 \circ (v_1 \circ (w \circ v_2))) \text{ [by the given condition]}\\
&= \zeta_M(v_1 \circ (w \circ (v_1 \circ v_2)))\\
&= \zeta_M(w \circ (v_1 \circ (v_1 \circ v_2))).
\end{aligned}$$

Thus, $\zeta_M(v_1 \circ v_2) \geq \min\{\inf_{a \in w\circ(v_1 \circ v_2)} \zeta_M(a), \zeta_M(w)\}$ for all $v_1, v_2, w \in V$. Hence the desired result.. □

Theorem 6.2.8. *An IFS $M = (\alpha_M, \zeta_M)$ is a DIF Hyp-filter of a Hyp BE-algebra if and only if $\bar{\alpha}_M$ and ζ_M are fuzzy Hyp-filters of V.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF Hyp-filter of a Hyp BE-algebra V. Then by hypothesis, $\alpha_M(1) \leq \alpha_M(v_1)$ which implies that $1 - \alpha_M(1) \geq 1 - \alpha_M(v_1)$. Thus,

$$\bar{\alpha}_M(1) \geq \bar{\alpha}_M(v_1). \tag{6.3}$$

Again,

$\alpha_M(v_1) \leq \max\{\sup_{w\in v_1\circ v_2} \alpha_M(w), \alpha_M(v_2)\}$;

or, $1 - \alpha_M(v_1) \geq 1 - \max\{\sup_{w\in v_1\circ v_2} \alpha_M(w), \alpha_M(v_2)\}$;

or, $\bar{\alpha}_M(v_1) \geq \min\{1 - \sup_{w\in v_1\circ v_2} \alpha_M(w), 1 - \alpha_M(v_2)\}$;

or, $\bar{\alpha}_M(v_1) \geq \min\{\inf_{w\in v_1\circ v_2}\{1 - \alpha_M(w)\}, \bar{\alpha}_M(v_2)\}$;

that is,

$$\bar{\alpha}_M(v_1) \geq \min\{\inf_{w\in v_1\circ v_2}\{\bar{\alpha}_M(w)\}, \bar{\alpha}_M(v_2)\}. \tag{6.4}$$

Then clearly from (6.3) and (6.4) we see that $\bar{\alpha}_M$ is a fuzzy Hyp-filter of V, for all $v_1, v_2 \in V$.

Now, it is obvious that ζ_M is a fuzzy Hyp-filter of V as, $\zeta_M(1) \geq \zeta_M(v_1)$ and $\zeta_M(v_1) \geq \min\{\inf_{w\in v_1\circ v_2} \zeta_M(w), \zeta_M(v_2)\}$, for all $v_1, v_2 \in V$.

Conversely let, $\bar{\alpha}_M$ and ζ_M are fuzzy Hyp-filters of V. Then $\bar{\alpha}_M(1) \geq \bar{\alpha}_M(v_1)$ and $\bar{\alpha}_M(v_1) \geq \min\{\inf_{w\in v_1\circ v_2}\{\bar{\alpha}_M(w)\}, \bar{\alpha}_M(v_2)\}$, for all $v_1, v_2 \in V$. From which we can conclude, $\alpha_M(1) \leq \alpha_M(v_1)$ and $\alpha_M(v_1) \leq \max\{\sup_{w\in v_1\circ v_2} \alpha_M(w), \alpha_M(v_2)\}$, for all $v_1, v_2 \in V$. Also, $\zeta_M(1) \geq \zeta_M(v_1)$ and $\zeta_M(v_1) \geq \min\{\inf_{w\in v_1\circ v_2} \zeta_M(w), \zeta_M(v_2)\}$, for all $v_1, v_2 \in V$. Thus, M is a DIF Hyp-filter of V. □

Theorem 6.2.9. *Let V be a Hyp BE-algebra and $\{M_i | i \in I\}$ is a family of DIF Hyp-filters of V. Then $\bigcup_{i\in I} M_i$ is also DIF Hyp-filter of V.*

Proof. Let $M = (\alpha_M, \zeta_M)$ and $N = (\alpha_N, \zeta_N)$ be two DIF Hyp-filters of V. Again let, $C = M \cup N = (\alpha_C, \zeta_C)$, where $\alpha_C = \alpha_M \vee \alpha_N$ and $\zeta_C = \zeta_M \wedge \zeta_N$. Let $v_2 \in V$, then, $\alpha_C(1) = (\alpha_M \vee \alpha_N)(1) = max\{\alpha_M(1), \alpha_N(1)\} \leq max\{\alpha_M(v_2), \alpha_N(v_2)\} = (\alpha_M \vee \alpha_N)(v_2) = \alpha_C(v_2)$ and $\zeta_C(1) = (\zeta_M \wedge \zeta_N)(1) = min\{\zeta_M(1), \zeta_N(1)\} \geq min\{\zeta_M(v_2), \zeta_N(v_2)\} =$

$(\zeta_M \wedge \zeta_N)(v_2) = \zeta_C(v_2)$ Also,

$$\begin{aligned}
\alpha_C(v_1) &= max\{\alpha_M(v_1), \alpha_N(v_1)\} \\
&\leq max\{max\{\sup_{w \in v_1 \circ v_2} \alpha_M(w), \alpha_M(v_2)\}, max\{\sup_{w \in v_1 \circ v_2} \alpha_N(w), \alpha_N(v_2)\}\} \\
&= max\{max\{\sup_{w \in v_1 \circ v_2} \alpha_M(w), \sup_{w \in v_1 \circ v_2} \alpha_N(w)\}\}, max\{\alpha_M(v_2), \alpha_N(v_2)\}\} \\
&= max\{\sup_{w \in v_1 \circ v_2} \alpha_C(w), \alpha_C(v_2)\}.
\end{aligned}$$

Alike, it can be proved that, $\zeta_C(v_1) \geq \min\{\inf_{w \in v_1 \circ v_2} \zeta_C(w), \zeta_C(v_2)\}$.

Thus the union of any two DIF Hyp-filters of V is also DIF Hyp-filters of V.

Following the same root, it can be proved that $\bigcup_{i \in I} M_i$ is a DIF Hyp-filter of V.

□

Theorem 6.2.10. *If M be a DIF Hyp-filter in a commutative Hyp BE-algebra V. Then, $\alpha_M(v_2 \circ (v_1 \circ v_1)) \leq \alpha_M(v_1)$ and $\zeta_M(v_2 \circ (v_1 \circ v_1)) \geq \zeta_M(v_1)$.*

Proof. Suppose M be a DIF Hyp-filter in a commutative Hyp BE-algebra V, then by the hypothesis (P6), we have, $v_1 < (v_1 \circ v_2) \circ v_2$. Also, as V is a commutative Hyp BE-algebra so, $(v_1 \circ v_2) \circ v_2 = (v_2 \circ v_1) \circ v_1$, Which implies that $v_1 < (v_2 \circ v_1) \circ v_1$. Thus, $\alpha_M((v_2 \circ v_1) \circ v_1) \leq \alpha_M(v_1)$. And $\zeta_M((v_2 \circ v_1) \circ v_1) \geq \zeta_M(v_1)$. Hence the required results. □

Theorem 6.2.11. *If M be a DIF Hyp-filter in a commutative Hyp BE-algebra V. Then,*

$$\alpha_M((v_2 \circ w) \circ (v_1 \circ w)) \leq \max\{\sup_{b \in (w \circ v_2) \circ ((v_1 \circ v_2) \circ (v_1 \circ v_2))} \alpha_M(b), \alpha_M(v_1 \circ v_2)\};$$
$$\zeta_M((v_2 \circ w) \circ (v_1 \circ w)) \geq \min\{\inf_{b \in (w \circ v_2) \circ ((v_1 \circ v_2) \circ (v_1 \circ v_2))} \zeta_M(b), \zeta_M(v_1 \circ v_2)\}.$$

Proof. Assume that M be a DIF Hyp-filter in a commutative Hyp BE-algebra V. So, we have

$$\alpha_M((v_2 \circ w) \circ (v_1 \circ w)) \leq \max\{\sup_{b \in (v_1 \circ v_2) \circ ((v_2 \circ w) \circ (v_1 \circ w))} \alpha_M(b), \alpha_M(v_1 \circ v_2)\};$$
$$\zeta_M((v_2 \circ w) \circ (v_1 \circ w)) \geq \min\{\inf_{b \in (v_1 \circ v_2) \circ ((v_2 \circ w) \circ (v_1 \circ w))} \alpha_M(b), \alpha_M(v_1 \circ v_2)\}.$$

Since V is a commutative Hyp BE-algebra, it follows that

$$\begin{aligned}
\alpha_M((v_1 \circ v_2) \circ ((v_2 \circ w) \circ (v_1 \circ w))) &= \alpha_M((v_1 \circ v_2) \circ (v_1 \circ ((v_2 \circ w) \circ w))) \\
&= \alpha_M((v_1 \circ v_2) \circ (v_1 \circ ((w \circ v_2) \circ v_2))) \\
&= \alpha_M((v_1 \circ v_2) \circ ((w \circ v_2) \circ (v_1 \circ v_2))) \\
&= \alpha_M((w \circ v_2) \circ ((v_1 \circ v_2) \circ (v_1 \circ v_2))).
\end{aligned}$$

Hence, $\alpha_M((v_2 \circ w) \circ (v_1 \circ w)) \leq \max\{\sup_{b\in(w\circ v_2)\circ((v_1\circ v_2)\circ(v_1\circ v_2))} \alpha_M(b), \alpha_M(v_1 \circ v_2)\}$. Also,

$$\begin{aligned}
\zeta_M((v_1 \circ v_2) \circ ((v_2 \circ w) \circ (v_1 \circ w))) &= \zeta_M((v_1 \circ v_2) \circ (v_1 \circ ((v_2 \circ w) \circ w))) \\
&= \zeta_M((v_1 \circ v_2) \circ (v_1 \circ ((w \circ v_2) \circ v_2))) \\
&= \zeta_M((v_1 \circ v_2) \circ ((w \circ v_2) \circ (v_1 \circ v_2))) \\
&= \zeta_M((w \circ v_2) \circ ((v_1 \circ v_2) \circ (v_1 \circ v_2))).
\end{aligned}$$

Hence, $\zeta_M((v_2 \circ w) \circ (v_1 \circ w)) \geq \min\{\inf_{b\in(w\circ v_2)\circ((v_1\circ v_2)\circ(v_1\circ v_2))} \zeta_M(b), \zeta_M(v_1 \circ v_2)\}$. Thus the proof ends. □

Theorem 6.2.12. *If M be a DIF Hyp-filter in a commutative R-Hyp BE-algebra V, and $v_2 \circ v_1 = 1$. Then,*

$$\alpha_M(v_1 \circ v_2) \leq \alpha_M(v_2);$$
$$\zeta_M(v_1 \circ v_2) \geq \zeta_M(v_2).$$

Proof. Let us consider a DIF Hyp-filter M in a commutative R-Hyp BE-algebra V. Then, we obtain

$$\alpha_M(v_1 \circ v_2) \leq \max\{\sup_{w\in(v_1\circ v_2)\circ v_2} \alpha_M(w), \alpha_M(v_2)\};$$
$$\zeta_M(v_1 \circ v_2) \geq \min\{\inf_{w\in(v_1\circ v_2)\circ v_2} \zeta_M(w), \zeta_M(v_2)\}.$$

Since V is a commutative R-Hyp BE-algebra, it follows that $\{v_2\} = 1 \circ v_2 = (v_2 \circ v_1) \circ v_1 = (v_1 \circ v_2) \circ v_2$. That implies, $\alpha_M(v_1 \circ v_2) \leq \alpha_M(v_2)$. In a similar way, it can be proved that $\zeta_M(v_1 \circ v_2) \geq \zeta_M(v_2)$. □

6.3 Product of DIF Hyp-filters in Hyp BE-algebras

This section deals with the product of DIF Hyp-filters in Hyp BE-algebras.

Definition 6.3.1. *Let $C = (\alpha_C, \zeta_C)$ and $D = (\alpha_D, \zeta_D)$be two DIF Hyp-filters in Hyp BE-algebras U and V, respectively. Then the CP $C \times D = (U \times V, \alpha_C \times \alpha_D, \zeta_C \times \zeta_D)$ is defined by*

$(\alpha_C \times \alpha_D)(a', b') = \max\{\alpha_C(a'), \alpha_D(b')\}$ *and*

$(\zeta_C \times \zeta_D)(a', b') = \min\{\zeta_C(a'), \zeta_D(b')\}$,

where $\alpha_C \times \alpha_D : U \times V \to [0, 1]$ and $\zeta_C \times \zeta_D : U \times V \to [0, 1]$, for all $(a', b') \in U \times V$.

Based on Hyp BE-algebras Cheng et al. [20] defined the product of Hyp BE-algebras as followes:

Let $(U; \circ_1; 1_1)$ and $(V; \circ_2; 1_2)$ be two Hyp BE-algebras and $Y = U \times V$. Hyperoperation $\circ$ on Y is defined by $(a', b') \circ (c', d') = (a' \circ c', b' \circ d')$ for every (a', b') *and* $(c', d') \in U \times V$. Then the product of Hyp BE-algebra Y is also a Hyp BE-algebra.

Theorem 6.3.1. *Let $(U \times V; \circ, (1_1, 1_2))$ be the product of Hyp BE-algebras $(U; \circ_1, 1_1)$ and $(V; \circ_2, 1_2)$. If M and N are DIF Hyp-filters in Hyp BE-algebras U and V respectively, then $M \times N$ is a DIF Hyp-filter of $U \times V$.*

Proof. Suppose that M and N are DIF Hyp-filters of U and V respectively. Now for any $(u, v) \in U \times V$, we have

$$\begin{aligned} (\alpha_M \times \alpha_N)(1_1, 1_2) &= \max\{\alpha_M(1_1), \alpha_N(1_2)\} \\ &\leq \max\{\alpha_M(u), \alpha_N(v)\} \\ &= (\alpha_M \times \alpha_N)(u, v) \end{aligned}$$

and

$$\begin{aligned} (\zeta_M \times \zeta_N)(1_1, 1_2) &= \min\{\zeta_M(1_1), \zeta_N(1_2)\} \\ &\geq \min\{\zeta_M(u), \zeta_N(v)\} \\ &= (\zeta_M \times \zeta_N)(u, v) \end{aligned}$$

Let (u_1, v_1) and $(u_2, v_2) \in U \times V$. Then

$$\begin{aligned} &(\alpha_M \times \alpha_N)(u_1, v_1) \\ &= \max\{\alpha_M(u_1), \alpha_N(v_1)\} \\ &\leq \max\{\max\{\sup_{m \in u_1 \circ u_2} \alpha_M(m), \alpha_M(u_2)\}, \max\{\sup_{n \in v_1 \circ v_2} \alpha_N(n), \alpha_N(v_2)\}\} \\ &= \max\{\max\{\sup_{m \in u_1 \circ u_2} \alpha_M(m), \sup_{n \in v_1 \circ v_2} \alpha_N(n)\}, \max\{\alpha_M(u_2), \alpha_N(v_2)\}\} \\ &= \max\{\sup_{w \in (u_1 \circ u_2, v_1 \circ v_2)} (\alpha_M \times \alpha_N)(w), (\alpha_M \times \alpha_N)(u_2, v_2)\} \\ &= \max\{\sup_{w \in (u_1, v_1) \circ (u_2, v_2)} (\alpha_M \times \alpha_N)(w), (\alpha_M \times \alpha_N)(u_2, v_2)\} \end{aligned}$$

Again, we have

$$\begin{aligned}
&(\zeta_M \times \zeta_N)(u_1, v_1) = \min\{\zeta_M(u_1), \zeta_N(v_1)\} \\
&\geq \min\{\min\{\inf_{m\in u_1\circ u_2} \zeta_M(m), \zeta_M(u_2)\}, \min\{\inf_{n\in v_1\circ v_2} \zeta_N(n), \zeta_N(v_2)\}\} \\
&= \min\{\min\{\inf_{m\in u_1\circ u_2} \zeta_M(m), \inf_{n\in v_1\circ v_2} \zeta_N(n)\}, \min\{\zeta_M(u_2), \zeta_N(v_2)\}\} \\
&= \min\{\inf_{w\in(u_1\circ u_2, v_1\circ v_2)}(\zeta_M \times \zeta_N)(w), (\zeta_M \times \zeta_N)(u_2, v_2)\} \\
&= \min\{\inf_{w\in(u_1,v_1)\circ(u_2,v_2)}(\zeta_M \times \zeta_N)(w), (\zeta_M \times \zeta_N)(u_2, v_2)\}.
\end{aligned}$$

$\therefore M \times N$ is a DIF Hyp-filter of $U \times V$. □

Theorem 6.3.2. *Let M and N are DIF Hyp-filters in Hyp BE-algebras $U \times V$. Then*

$$\bigotimes(M \times N) = (\alpha_M \times \alpha_N, \bar{\alpha}_M \times \bar{\alpha}_N).$$

is a DIF Hyp-filter of $U \times V$.

Proof. The proof is straightforward. □

6.4 DIFI Hyp-filters in hyper BE-algebras

In this section, DIFI Hyp-filter in Hyp BE-algebras is defined and proved some essential theorems of it.

Definition 6.4.1. *Let $M = (\alpha_M, \zeta_M)$ be an IFS in a Hyp BE-algebra V, then M is called a DIFI Hyp-filter in Hyp BE-algebra V if*

(1) $\alpha_M(1) \leq \alpha_M(u)$, $\zeta_M(1) \geq \zeta_M(u)$;

(2) $\alpha_M(u \circ w) \leq \max\{\sup_{a\in u\circ(v\circ w)} \alpha_M(a), \sup_{b\in u\circ v} \alpha_M(b)\}$;

(3) $\zeta_M(u \circ w) \geq \min\{\inf_{a\in u\circ(v\circ w)} \zeta_M(a), \inf_{b\in u\circ v} \zeta_M(b)\}$, *for all* $u, v, w \in V$.

Definition 6.4.2. *An IFS M of a Hyp BE-algebra V is called a DIF PI Hyp-filter in Hyp BE-algebra V if*

(1) $\alpha_M(1) \leq \alpha_M(u)$, $\zeta_M(1) \geq \zeta_M(u)$;

(2) $\alpha_M(v) \leq \max\{\sup_{v_3\in u\circ((v\circ w)\circ v)} \alpha_M(v_3), \alpha_M(u)\}$;

(3) $\zeta_M(v) \geq \min\{\inf_{v_3\in u\circ((v\circ w)\circ v)} \zeta_M(v_3), \zeta_M(u)\}$, *for all* $u, v, w \in V$.

EXAMPLE 30. *Consider a Hyp BE-algebra $(V; o, 1)$ that was given in Example 29 with the table below:*

o	1	r'	s'
1	$\{1\}$	$\{r', s'\}$	$\{s'\}$
r'	$\{1\}$	$\{1, r'\}$	$\{1, s'\}$
s'	$\{1\}$	$\{1, r', s'\}$	$\{1\}$

Let $M = (\alpha_M, \zeta_M)$ be an IFS of V as defined by

V	1	r'	s'
α_M	0.3	0.3	0.4
ζ_M	0.7	0.6	0.6

Then $M = (\alpha_M, \zeta_M)$ is DIFI Hyp-filter in Hyp BE-algebra V.

Theorem 6.4.1. *Every DIFI Hyp-filter in R-Hyp BE-algebra V is a DIF Hyp-filter in V.*

Proof. In Hyp BE-algebra V let M be a DIFI Hyp-filter. So,

(1) $\alpha_M(1) \leq \alpha_M(u)$, $\zeta_M(1) \geq \zeta_M(u)$;

(2) $\alpha_M(u \circ w) \leq \max\{\sup_{a \in u \circ (v \circ w)} \alpha_M(a), \sup_{b \in u \circ v} \alpha_M(b)\}$;

(3) $\zeta_M(u \circ w) \geq \min\{\inf_{a \in u \circ (v \circ w)} \zeta_M(a), \inf_{b \in u \circ v} \zeta_M(b)\}$,

for all $u, v, w \in V$. Putting $u = 1$ in (2) we obtain that $\alpha_M(1 \circ w) \leq \max\{\sup_{a \in 1 \circ (v \circ w)} \alpha_M(a), \sup_{b \in 1 \circ v} \alpha$ which implies that

$$\alpha_M(w) \leq \max\{\sup_{a \in (v \circ w)} \alpha_M(a), \alpha_M(v)\}.$$

Similarly, we get

$$\zeta_M(w) \leq \min\{\inf_{a \in (v \circ w)} \zeta_M(a), \zeta_M(v)\}.$$

Hence the proof of the theorem ends. □

Now a required condition is given such that a DIF Hyp-filter in V becomes a DIFI Hyp-filter in V.

Theorem 6.4.2. *Let $(V; \circ, 1)$ be a transitive Hyp BE-algebra and M be a DIF Hyp-filter of V such that*

$$\alpha_M(u \circ v) \leq \max\{\sup_{a \in w \circ (u \circ (u \circ v))}, \alpha_M(w)\};$$
$$\zeta_M(u \circ v) \geq \min\{\inf_{a \in w \circ (u \circ (u \circ v))}, \zeta_M(w)\}.$$

Then, M is a DIFI Hyp-filter of V.

Proof. Let M is a DIF Hyp-filter of a transitive Hyp BE-algebra V satisfying the given condition. So, $u \circ (v \circ w) = v \circ (u \circ w) << ((u \circ v) \circ (u \circ (u \circ w)))$. Thus, $\alpha_M((u \circ v) \circ (u \circ (u \circ w))) \leq \alpha_M(u \circ (v \circ w))$. Hence by the given condition we get

$$\begin{aligned} \alpha_M(u \circ w) &\leq \max\{\sup_{a \in ((u \circ v) \circ (u \circ (u \circ w)))} \alpha_M(a), \sup_{b \in u \circ v} \alpha_M(b)\} \\ &= \max\{\sup_{a \in u \circ (v \circ w)} \alpha_M(a), \sup_{b \in u \circ v} \alpha_M(b)\}. \end{aligned}$$

Therefore, M is a DIFI Hyp-filter of V. □

6.5 Summary

The notion of intuitionistic fuzzification of Hyp-filters and implicative Hyp-filters of Hyp BE-algebras are introduced in this chapter as a further study of IF-hyperstructures. To develope the theory of algebraic hyperstructures IF Hyp-filters play a pivotal role. Applying some important conditions on Hyp BE-filters, here, we have formed DIF Hyp-filters in Hyp BE-algebras and have presented characterizations of DIF Hyp-filters in Hyp BE-algebras. We give characteristics of DIF Hyp-filters in commutative Hyp BE-algebras. At the same time we deal with the doubt intuitionistic fuzzification of the notion of implicative Hyp-filters in Hyp BE-algebras. We show that every DIFI Hyp-filters in Row-Hyp BE-algebras are DIF Hyp-filters in Hyp BE-algebras and give the condition such that a DIF Hyp-filters in Hyp BE-algebras to be a DIFI Hyp-filters in Hyp BE-algebras.