Chapter 5

Doubt intuitionistic fuzzy translation of doubt intuitionistic fuzzy Sub-implicative ideals in BCI-algebras*

5.1 Introduction

Intuitionistic theory is comperatively a new notion in the field of the FS theory.. The elements of such IFSs are sets whose elements have DMS and DNMS which is an extension of Zadeh's notion of FS. Atanassov [3] carried out rigorous research based on the theory and applications of IFSs.

In $Chapter 4$ authors have studied DIF SI-ideals in BCI-algebras. Lee et al. [52] and Jun [45] investigated the notion of F-translations, F-extension and F-multiplications of FSAs and F-ideals in BCK/BCI-algebras.

Also in 2013, Senapati et al. [[68], [71]] have investigated IF-translation, IF-extension and IF-multiplication of IFSA, IF-ideals and IFH-ideals in BCK/BCI-algebras.

Being inspired by our earlier works [[11], [12], [13]] and taking clue from the works of Jun [[48], [45]] and Senapati et al.[[68], [71]] in this chapter we have defined

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Communicated

DIF- translation, DIF- extension, DIF- multiplication and DIF- magnified translation of DIFSA and DIF SI-ideals in BCI-algebras and have established some interesting relations among them.

5.2 DIF-translations of DIF SI-ideals in BCI-algebras

Throughout this paper, We take $\chi := 1 - sup\{\alpha_A(v_1)/v_1 \in V\}$ for any IFS $A = (\alpha_A, \zeta_A)$ in V.

In this section, DIF-translations of DIF SI-ideals in BCI-algebras is defined and proved some essential theorems of it.

Definition 5.2.1. *Suppose $M = (\alpha_M, \zeta_M)$ be an IFS in a BCI-algebra V, and $\gamma \in [0, \chi]$. An object of the form $M_\gamma^T = ((\alpha_M)_\gamma^T, (\zeta_M)_\gamma^T)$ is said to be a DIF-γ translation of M if $(\alpha_M)_\gamma^T(v_1) = \alpha_M(v_1) - \gamma$ and $(\zeta_M)_\gamma^T(v_1) = \zeta_M(v_1) + \gamma$, for all $v_1 \in V$.*

Theorem 5.2.1. *An IFS $M = (\alpha_M, \zeta_M)$ in V is a DIF SI-ideal of a BCI-algebra V iff $M_\gamma^T = ((\alpha_M)_\gamma^T, (\zeta_M)_\gamma^T)$ is a DIF SI-ideal in V.*

Proof. Suppose that M is a DIF SI-ideal in V and $v_1, v_2, v_3 \in V$. Then,

(1) $\alpha_M(0) \leq \alpha_M(v_1), \zeta_M(0) \geq \zeta_M(v_1)$,

(2) $\alpha_M(v_2 * (v_2 * v_1)) \leq mav_1\{\alpha_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), \alpha_M(v_3)\}$,

(3) $\zeta_M(v_2 * (v_2 * v_1)) \geq min\{\zeta_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), \zeta_M(v_3)\}$.

Now, we have $(\alpha_M)_\gamma^T(0) = \alpha_M(0) - \gamma \leq \alpha_M(v_1) - \gamma = (\alpha_M)_\gamma^T(v_1)$, and $(\zeta_M)_\gamma^T(0) = \zeta_M(0) + \gamma \geq \zeta_M(v_1) + \gamma = (\zeta_M)_\gamma^T(v_1)$, for all $v_1 \in V$.

Again, we have

$$\begin{aligned}
(\alpha_M)_\gamma^T(v_2 * (v_2 * v_1)) &= \alpha_M(v_2 * (v_2 * v_1)) - \gamma \\
&\leq max\{\alpha_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), \alpha_M(v_3)\} - \gamma \\
&= max\{\alpha_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3) - \gamma, \alpha_M(v_3) - \gamma\} \\
&= max\{(\alpha_M)_\gamma^T(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), (\alpha_M)_\gamma^T(v_3)\}
\end{aligned}$$

and

$$\begin{aligned}(\zeta_M)^T_\gamma(v_2*(v_2*v_1)) &= \zeta_M(v_2*(v_2*v_1))+\gamma\\ &\geq min\{\zeta_M(((v_1*(v_1*v_2))*(v_2*v_1))*v_3),\zeta_M(v_3)\}+\gamma\\ &= min\{\zeta_M(((v_1*(v_1*v_2))*(v_2*v_1))*v_3)+\gamma,\zeta_M(v_3)+\gamma\}\\ &= min\{(\zeta_M)^T_\gamma(((v_1*(v_1*v_2))*(v_2*v_1))*v_3),(\zeta_M)^T_\gamma(v_3)\}\end{aligned}$$

for all $v_1, v_2, v_3 \in V$.

Hence, M^T_γ is DIF SI-ideal in V.

Now, $\alpha_M(0)-\gamma=(\alpha_M)^T_\gamma(0)\leq(\alpha_M)^T_\gamma(v_1)=\alpha_M(v_1)-\gamma$, and $\zeta_M(0)+\gamma=(\zeta_M)^T_\gamma(0)\geq(\zeta_M)^T_\gamma(v_1)=\zeta_M(v_1)+\gamma$. That implies, $\alpha_M(0)\leq\alpha_M(v_1)$ and $\zeta_M(0)\geq\zeta(v_1)$.

Again, we have

$$\begin{aligned}\alpha_M(v_2*(v_2*v_1))-\gamma &= (\alpha_M)^T_\gamma(v_2*(v_2*v_1))\\ &\leq max\{(\alpha_M)^T_\gamma(((v_1*(v_1*v_2))*(v_2*v_1))*v_3),(\alpha_M)^T_\gamma(v_3)\}\\ &= max\{\alpha_M(((v_1*(v_1*v_2))*(v_2*v_1))*v_3)-\gamma,\alpha_M(v_3)-\gamma\}\\ &= max\{\alpha_M(((v_1*(v_1*v_2))*(v_2*v_1))*v_3),\alpha_M(v_3)\}-\gamma\end{aligned}$$

and

$$\begin{aligned}\zeta_M(v_2*(v_2*v_1))+\gamma &= (\zeta_M)^T_\gamma(v_2*(v_2*v_1))\\ &\geq min\{(\zeta_M)^T_\gamma(((v_1*(v_1*v_2))*(v_2*v_1))*v_3),(\zeta_M)^T_\gamma(v_3)\}\\ &= min\{\zeta_M(((v_1*(v_1*v_2))*(v_2*v_1))*v_3)+\gamma,\zeta_M(v_3)+\gamma\}\\ &= min\{\zeta_M(((v_1*(v_1*v_2))*(v_2*v_1))*v_3),\zeta_M(v_3)\}+\gamma\end{aligned}$$

The result follows. □

Theorem 5.2.2. *Let $M=(\alpha_M,\zeta_M)$ be a DIF SI-ideal of a BCI-algebra V and let $\gamma\in[0,\chi]$. Then, the DIF- γ-translation M^T_γ is a DIFSA in V.*

Proof. Let $M=(\alpha_M,\zeta_M)$ be a DIF SI-ideal in V, then (1) $\alpha_M(v_2*(v_2*v_1))\leq\alpha_M(((v_1*(v_1*v_2))*(v_2*v_1))*v_3)\bigvee\alpha_M(v_3)$, and (2) $\zeta_M(v_2*(v_2*v_1))\geq\zeta_M(((v_1*(v_1*v_2))*(v_2*v_1))*v_3)\bigwedge\zeta_M(v_3)$, for all $v_1,v_2,v_3\in V$.

Again, we obtain $((v_1*v_3)*v_3)\leq(v_1*v_3)*(v_3*v_3)=v_1*v_3\leq v_1$ [by using (P6), (M3), (P5), (P2)].

Now, we have $(\alpha_M)^T_\gamma(v_2*(v_2*v_1)) = \alpha_M(v_2*(v_2*v_1))-\gamma \leq max\{\alpha_M(((v_1*(v_1*v_2))*(v_2*v_1))*v_3), \alpha_M(v_3)\}-\gamma$. If $v_2 = v_1$, then $(\alpha_M)^T_\gamma(v_1) = max\{\alpha_M(v_1*v_3), \alpha_M(v_3)\}-\gamma$. Thus it implies that, $(\alpha_M)^T_\gamma(v_1 * v_3) \leq max\{\alpha_M((v_1 * v_3) * v_3), \alpha_M(v_3)\} - \gamma$.That is, $(\alpha_M)^T_\gamma(v_1 * v_3) \leq max\{\alpha_M(v_1), \alpha_M(v_3)\} - \gamma$. Finally, it implies that, $(\alpha_M)^T_\gamma(v_1 * v_3) \leq max\{(\alpha_M)^T_\gamma(v_1), (\alpha_M)^T_\gamma(v_3)\}$. Similarly, it can be proved that

$$(\zeta_M)^T_\gamma(v_1 * v_3) \geq min\{(\zeta_M)^T_\gamma(v_1), (\zeta_M)^T_\gamma(v_3)\}.$$

Hence, M^T_γ is a DIFSA in V. □

Let us illustrate the above theorem using the example given below.

Example 23. *Here we consider the BCI-algebra V, given in Example 3, Chapter 2 as follows:*

$*$	0	d	e	f
0	0	0	0	0
d	d	0	0	d
e	e	d	0	e
f	f	f	f	0

Let $M = (\alpha_M, \zeta_M)$ is an IFS of V defined by

V	0	d	e	f
α_M	0.39	0.44	0.52	0.61
ζ_M	0.61	0.56	0.48	0.39

which is a DIF SI-ideal in V and $\chi = 0.39$.

Let $\gamma = 0.21$, then the IF-γ-translation M^T_γ is given by

V	0	d	e	f
$(\alpha_M)^T_\gamma$	0.18	0.23	0.31	0.40
$(\zeta_M)^T_\gamma$	0.82	0.77	0.69	0.60

Hence, M^T_γ is a DIFSA in V.

Proposition 5.2.3. *Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in a BCI-algebra V. Then, $(\alpha_M)^T_\gamma(0 * (0 * v_1)) \leq (\alpha_M)^T_\gamma(v_1)$ and $(\zeta_M)^T_\gamma(0 * (0 * v_1)) \geq (\zeta_M)^T_\gamma(v_1)$, for all $v_1 \in V$.*

Proof. Since $M = (\alpha_M, \zeta_M)$ is a DIF SI-ideal in V, it follows that $M_\gamma^T = ((\alpha_M)_\gamma^T, (\zeta_M)_\gamma^T)$ is also a DIF SI-ideal in V by Theorem 5.2.1. Now, $(\alpha_M)_\gamma^T(0*(0*v_1)) \leq (\alpha_M)_\gamma^T(((v_1 * (v_1*0))*(0*v_1))*v_3) \bigvee (\alpha_M)_\gamma^T(v_3) = (\alpha_M)_\gamma^T(((v_1*v_1)*(0*v_1))*v_3) \bigvee (\alpha_M)_\gamma^T(v_3) = (\alpha_M)_\gamma^T((0*(0*v_1))*v_3) \bigvee (\alpha_M)_\gamma^T(v_3)$.

When $v_3 = v_1$ we get, $(\alpha_M)_\gamma^T(0*(0*v_1)) \leq (\alpha_M)_\gamma^T((0*(0*v_1))*v_1) \bigvee (\alpha_M)_\gamma^T(v_1)$ or, $(\alpha_M)_\gamma^T(0*(0*v_1)) \leq (\alpha_M)_\gamma^T(0) \bigvee (\alpha_M)_\gamma^T(v_1)$ [by using M2]. Therefore, $(\alpha_M)_\gamma^T(0*(0*v_1)) \leq (\alpha_M)_\gamma^T(v_1)$, for all $v_1 \in V$.

Again, $(\zeta_M)_\gamma^T(0*(0*v_1)) \geq (\zeta_M)_\gamma^T(((v_1*(v_1*0))*(0*v_1))*v_3) \bigwedge (\zeta_M)_\gamma^T(v_3) = (\zeta_M)_\gamma^T(((v_1*v_1)*(0*v_1))*v_3) \bigwedge (\zeta_M)_\gamma^T(v_3) = (\zeta_M)_\gamma^T((0*(0*v_1))*v_3) \bigwedge (\zeta_M)_\gamma^T(v_3)$.
When $v_3 = v_1$ we get, $(\zeta_M)_\gamma^T(0*(0*v_1)) \leq (\zeta_M)_\gamma^T((0*(0*v_1))*v_1) \bigwedge (\zeta_M)_\gamma^T(v_1)$ or, $(\zeta_M)_\gamma^T(0*(0*v_1)) \leq (\zeta_M)_\gamma^T(0) \bigwedge (\zeta_M)_\gamma^T(v_1)$ [by using M2]. Therefore, $(\zeta_M)_\gamma^T(0*(0*v_1)) \geq (\zeta_M)_\gamma^T(v_1)$, for all $v_1 \in V$. □

Theorem 5.2.4. *DIF-translation of every DIF SI-ideal in V is a DIF-ideal in V.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V, then γ-translation of M is also DIF SI-ideal in V. Hence,

(1) $(\alpha_M)_\gamma^T(0) \leq (\alpha_M)_\gamma^T(v_1); (\zeta_M)_\gamma^T(0) \geq (\zeta_M)_\gamma^T(v_1)$,
(2) $(\alpha_M)_\gamma^T(v_2*(v_2*v_1)) \leq (\alpha_M)_\gamma^T(((v_1*(v_1*v_2))*(v_2*v_1))*v_3) \bigvee (\alpha_M)_\gamma^T(v_3)$,
and (3) $(\zeta_M)_\gamma^T(v_2*(v_2*v_1)) \geq (\zeta_M)_\gamma^T(((v_1*(v_1*v_2))*(v_2*v_1))*v_3) \bigwedge (\zeta_M)_\gamma^T(v_3)$, for all $v_1, v_2, v_3 \in V$.
If $v_2 = v_1$, then from (2) and (3), $(\alpha_M)_\gamma^T(v_1) \leq (\alpha_M)_\gamma^T(v_1*v_3) \bigvee (\alpha_M)_\gamma^T(v_3)$ and $(\zeta_M)_\gamma^T(v_1) \geq (\zeta_M)_\gamma^T(v_1*v_3) \bigwedge (\zeta_M)_\gamma^T(v_3)$, for all $v_1, v_2, v_3 \in V$.

Hence, M_γ^T is a DIF-ideal in V. □

But the converse may not be true. That is DIF-translation of every DIF-ideal in V is not a DIF SI-ideal in V. which is described through the example below:

EXAMPLE 24. *Consider the BCI-algebra V that was taken in Example 23 as follows.*

$*$	0	d	e	f
0	0	0	0	0
d	d	0	0	d
e	e	d	0	e
f	f	f	f	0

Let $M = (\alpha_M, \zeta_M)$ is an IFS of V defined by

V	0	d	e	f
α_M	0.35	0.52	0.52	0.65
ζ_M	0.65	0.48	0.48	0.35

which is a DIF-ideal in V. And $\chi = 0.35$. Letting $\gamma = 0.15$, then the IF γ-translation M_γ^T of M is given as follows:

V	0	d	e	f
$(\alpha_M)_\gamma^T$	0.2	0.37	0.37	0.50
$(\zeta_M)_\gamma^T$	0.8	0.63	0.63	0.50

Hence, M_γ^T is a DIF-ideal of V.

But here, $(\alpha_M)_\gamma^T(e*(e*d)) \leq max\{(\alpha_M)_\gamma^T(((d*(d*e))*(e*d))*0), (\alpha_M)_\gamma^T(0)\}$. It implies that, $(\alpha_M)_\gamma^T(d) \leq (\alpha_M)_\gamma^T(0)$, thus contradiction arises. So, M_γ^T is not a DIF SI-ideal in V.

Now we formulate a condition for the DIF-translation of a DIF-ideal in V to be a DIF SI-ideal in V.

Theorem 5.2.5. *The DIF-translation of a DIF-ideal in V is a DIF SI-ideal in V if V is an implicative BCI-algebra.*

Proof. Let $M_\gamma^T = ((\alpha_M)_\gamma^T, (\zeta_M)_\gamma^T)$ be a DIF- translation of a DIF-ideal M satisfying the inequalities, $(\alpha_M)_\gamma^T(v_2*(v_2*v_1)) \leq (\alpha_M)_\gamma^T((v_1*(v_1*v_2))*(v_2*v_1))$, and $(\zeta_M)_\gamma^T(v_2*(v_2*v_1)) \geq (\zeta_M)_\gamma^T((v_1*(v_1*v_2))*(v_2*v_1))$. Now, $(\alpha_M)_\gamma^T(v_2*(v_2*v_1)) \leq (\alpha_M)_\gamma^T((v_1*(v_1*v_2))*(v_2*v_1)) \leq (\alpha_M)_\gamma^T(((v_1*(v_1*v_2))*(v_2*v_1))*v_3) \bigvee (\alpha_M)_\gamma^T(v_3)$, and $(\zeta_M)_\gamma^T(v_2*(v_2*v_1)) \geq (\zeta_M)_\gamma^T((v_1*(v_1*v_2))*(v_2*v_1)) \geq (\zeta_M)_\gamma^T(((v_1*(v_1*v_2))*(v_2*v_1))*v_3) \bigwedge (\zeta_M)_\gamma^T(v_3)$, for all $v_1, v_2, v_3 \in V$, [because M_γ^T is a DIF-ideal]. Hence, M_γ^T is a DIF SI-ideal in V.

Second proof. Let in an implicative BCI-algebra V, $M_\gamma^T = ((\alpha_M)_\gamma^T, (\zeta_M)_\gamma^T)$ be a DIF-ideal. Then

$(\alpha_M)_\gamma^T(v_1) \leq max(\alpha_M)_\gamma^T(v_1*v_3), (\alpha_M)_\gamma^T(v_3)\}$, for all $v_1, v_3 \in V$. So, $(\alpha_M)_\gamma^T(v_2*(v_2*v_1)) \leq max\{(\alpha_M)_\gamma^T(v_2*((v_2*v_1))*v_3), (\alpha_M)_\gamma^T(v_3)\}$, but V is implicative BCI-algebra, then $((v_1*(v_1*v_2))*(v_2*v_1)) = (v_2*(v_2*v_1))$. Hence, $(\alpha_M)_\gamma^T(v_2*(v_2*v_1)) \leq max\{(\alpha_M)_\gamma^T(((v_1*(v_1*v_2))*(v_2*v_1))*v_3), (\alpha_M)_\gamma^T(v_3)\}$. Similarly, $(\zeta_M)_\gamma^T(v_2*(v_2*v_1)) \geq$

$min\{(\zeta_M)^T_\gamma(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), (\zeta_M)^T_\gamma(v_3)\}$. Thus the proof is completed . □

Now the Theorem 5.2.5 is illustrated using the example as follows.

EXAMPLE 25. *Let us consider an implicative BCI-algebra U, given in Example 13, Chapter 3 as follows:*

$*$	0	q	r
0	0	r	q
q	q	0	r
r	r	q	0

Let $M = (\alpha_M, \zeta_M)$ be an IFS in U as defined by

U	0	q	r
α_M	0.71	0.79	0.79
ζ_M	0.21	0.16	0.16

which is a DIF-ideal in U. And $\chi = 0.21$. Fitting $\gamma = 0.2$, we have the DIF-γ-translation M^T_γ of M as given below.

U	0	q	r
$(\alpha_M)^T_\gamma$	0.51	0.59	0.59
$(\zeta_M)^T_\gamma$	0.41	0.36	0.36

Hence, M^T_γ is a DIF-ideal as well as DIF SI-ideal in U.

Theorem 5.2.6. *Let M^T_γ be the DIF-translation of a DIF SI-ideal M in V. Then, for any $v_1 \in V$, we have*

(1)$(\alpha_M)^T_\gamma(v_1^n * v_1) \leq (\alpha_M)^T_\gamma(v_1)$, and $(\zeta_M)^T_\gamma(v_1^n * v_1) \leq (\zeta_M)^T_\gamma(v_1)$, if n is odd

(2)$(\alpha_M)^T_\gamma(v_1^n * v_1) = (\alpha_M)^T_\gamma(v_1)$, and $(\zeta_M)^T_\gamma(v_1^n * v_1) = (\zeta_M)^T_\gamma(v_1)$, if n is even

Proof. Let $v_1 \in V$, then $(\alpha_M)^T_\gamma(v_1 * v_1) = (\alpha_M)^T_\gamma(0) \leq (\alpha_M)^T_\gamma(v_1)$. For odd n, let $n = 2q - 1$ for $q \in Z^+$.

Now assume that $(\alpha_M)^T_\gamma(v_1^{2q-1} * v_1) \leq (\alpha_M)^T_\gamma(v_1)$ for some positive integer q.

Then,

$$\begin{aligned}(\alpha_M)^T_\gamma(v_1^{2(q+1)-1} * v_1) &= (\alpha_M)^T_\gamma(v_1^{2q+1} * v_1)\\ &= (\alpha_M)^T_\gamma(v_1^{2q-1} * (v_1 * (v_1 * v_1)))\\ &= (\alpha_M)^T_\gamma(v_1^{2q-1} * (v_1 * 0))\\ &= (\alpha_M)^T_\gamma(v_1^{2q-1} * v_1)\\ &\leq (\alpha_M)^T_\gamma(v_1)\end{aligned}$$

Hence, $(\alpha_M)^T_\gamma(v_1^n * v_1) \leq (\alpha_M)^T_\gamma(v_1)$, if n is odd.

(2) Again, let n be even, and $n = 2q$.

Now for $q = 1$, $(\alpha_M)^T_\gamma(v_1^2 * v_1) = (\alpha_M)^T_\gamma(v_1 * (v_1 * v_1)) = (\alpha_M)^T_\gamma(v_1 * 0) = (\alpha_M)^T_\gamma(v_1)$.

Also assume that, $(\alpha_M)^T_\gamma(v_1^{2q} * v_1) = (\alpha_M)^T_\gamma(v_1)$ for some positive integer q.

Then,

$$\begin{aligned}(\alpha_M)^T_\gamma(v_1^{2(q+1)} * v_1) &= (\alpha_M)^T_\gamma(v_1^{2q} * (v_1 * (v_1 * v_1))),\\ &= (\alpha_M)^T_\gamma(v_1^{2q} * v_1)\\ &= (\alpha_M)^T_\gamma(v_1).\end{aligned}$$

Hence, $(\alpha_M)^T_\gamma(v_1^n * v_1) = (\alpha_M)^T_\gamma(v_1)$, if n is even.

This proves the first part. In the similar manner we can proof the second part. □

For an IFS $M = (\alpha_M, \zeta_M)$ in V, $\gamma \in [0, \chi]$ and $s_1, s_2 \in [0, 1]$ with $s_2 \geq \gamma$, let

$$U_\gamma(\alpha_M; s_1) = \{v_1 \in V | \alpha_M(v_1) \leq s_1 + \gamma\}$$
$$L_\gamma(\zeta_M; s_2) = \{v_1 \in V | \zeta_M(v_1) \geq s_2 - \gamma\}$$

are the UC of level s_1 and LC of level s_2 of M^T_γ.

Theorem 5.2.7. *If $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V, then $U_\gamma(\alpha_M; s_1)$ and $L_\gamma(\zeta_M; s_2)$ are SI-ideals in V for any $s_1, s_2 \in [0, 1]$.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V, and let $s_1 \in [0, 1]$ with $\alpha_M(0) \leq s_1+\gamma$. Also we have, $\alpha_M(0) \leq \alpha_M(v_1)$, for all $v_1 \in V$, but $\alpha_M(v_1) \leq s_1+\gamma$, for all $v_1 \in U_\gamma(\alpha_M; s_1)$. So, $0 \in U_\gamma(\alpha_M; s_1)$. Let $v_1, v_2, v_3 \in V$ with $(((v_1*(v_1*v_2))*(v_2*v_1))*v_3) \in U_\gamma(\alpha_M; s_1)$ and $v_3 \in U_\gamma(\alpha_M; s_1)$, then, $\alpha_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3) \in U_\gamma(\alpha_M; s_1)$ and $\alpha_M(v_3) \in U_\gamma(\alpha_M; s_1)$. Therefore, $\alpha_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3) \leq s_1 + \gamma$ and

$\alpha_M(v_3) \leq s_1 + \gamma$. Since α_M is a DF SI-ideal in V, it follows that, $\alpha_M(v_2 * (v_2 * v_1)) \leq \alpha_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3) \bigvee \alpha_M(v_3) \leq s_1 + \gamma$ and hence $(v_2 * (v_2 * v_1)) \in U_\gamma(\alpha_M; s_1)$, for all $v_1, v_2, v_3 \in V$. Therefore, $U_\gamma(\alpha_M; s_1)$ is a SI-ideal in V for $s_1 \in [0, 1]$. In the similar manner we can proof that $L_\gamma(\zeta_M; s_2)$ is a SI-ideal in V for $s_2 \in [0, 1]$. □

But, if M is not a DIF SI-ideal then it can not be happened. It can be supported by the example below.

EXAMPLE 26. *Let us consider an implicative BCI-algebra V, given in Example 20, Chapter 4 as follows::*

$*$	0	r	s	t
0	0	0	t	s
r	r	0	t	s
s	s	s	0	t
t	t	t	s	0

Now let consider a DIFS $M = (\alpha_M, \zeta_M)$ in V as follows:

V	0	r	s	t
α_M	0	0.5	0.4	0.6
ζ_M	1	0.5	0.6	0.4

Then, $M = (\alpha_M, \zeta_M)$ is not a DIF SI-ideal in V, since $\alpha_M(r * (r * t)) \nleq \alpha_M(((t * (t * r)) * (r * t)) * s) \cup \alpha_M(s)$.

Now, let $\gamma = 0.2$, then for $s_1 = 0.3$, $U_\gamma(\alpha_M; s_1) = \{0, r, s\}$, which is not a SI-ideal in V as $0 * s = t \notin U_\gamma(\alpha_M; s_1)$.

Also, for $\gamma = 0.2$ and $s_2 = 0.7$, $L_\gamma(\zeta_M; s_2) = \{0, r, s\}$. This is not a SI-ideal in V as $0 * s = t \notin L_\gamma(\zeta_M; s_2)$.

Theorem 5.2.8. *If $U_\gamma(\alpha_M; s_1)$ and $L_\gamma(\zeta_M; s_2)$ are SI-ideals in V for $s_1 \in Im(\alpha_M)$ and $s_2 \in Im(\zeta_M)$ with $s_2 \geq \gamma$, then M_γ^T be a DIF SI-ideal in V.*

Proof. Let $U_\gamma(\alpha_M; s_1)$ and $L_\gamma(\zeta_M; s_2)$ be SI-ideals in V for $s_1 \in Im(\alpha_M)$ and $s_2 \in Im(\zeta_M)$ with $s_2 \geq \gamma$. If there exist $a \in V$ such that $(\alpha_M)_\gamma^T(0) > \beta \geq (\alpha_M)_\gamma^T(a)$, then $\alpha_M(a) \leq \beta + \gamma$ but $\alpha_M(0) > \beta + \gamma$. This shows that $a \in U_\gamma(\alpha_M; s_1)$ and $0 \notin U_\gamma(\alpha_M; s_1)$. This is a contradiction, and $(\alpha_M)_\gamma^T(0) \leq (\alpha_M)_\gamma^T(v_1)$, for all $v_1 \in V$. Again let there exist $b \in V$ such that $(\zeta_M)_\gamma^T(0) < \lambda \leq (\alpha_M)_\gamma^T(b)$, then $\zeta_M(b) \geq \lambda - \gamma$, but $\zeta_M(0) < \lambda - \gamma$. This shows that $b \in L_\gamma(\zeta_M; s_2)$ and $0 \notin L_\gamma(\zeta_M; s_2)$. Again a contradiction arises. so, $(\zeta_M)^T_\gamma(0) \leq (\zeta_M)^T_\gamma(v_1)$, for all $v_1 \in V$.

Now let $p, q, r \in V$ with $(\alpha_M)^T_\gamma(q*(q*p)) > \eta \geq max\{(\alpha_M)^T_\gamma(((p*(p*q))*(q*p))*r), (\alpha_M)^T_\gamma(r)\}$, then $\alpha_M(((p*(p*q))*(q*p))*r) \leq \eta+\gamma$ and $\alpha_M(r) \leq \eta+\gamma$. But $\alpha_M(q*(q*p)) > \eta+\gamma$. This is a contradiction. So, $(\alpha_M)^T_\gamma(q*(q*p)) \leq max\{(\alpha_M)^T_\gamma(((p*(p*q))*(q*p))*r), (\alpha_M)^T_\gamma(r)\}$, for any $p, q, r \in V$.

Finally, let there exist $u, v, w \in V$, such that $(\zeta_M)^T_\gamma(v*(v*u)) < \tau \leq min\{(\zeta_M)^T_\gamma(((u*(u*v))*(v*u))*w), (\zeta_M)^T_\gamma(w)\}$, then $\zeta_M(((u*(u*v))*(v*u))*w) \geq \tau-\gamma$ and $\zeta_M(w) \geq \tau-\gamma$. But $\zeta_M(v*(v*u)) < \tau-\gamma$. This is a contradiction. So, $(\zeta_M)^T_\gamma(v*(v*u)) \geq min\{(\zeta_M)^T_\gamma(((u*(u*v))*(v*u))*w), (\zeta_M)^T_\gamma(w)\}$, for any $u, v, w \in V$.

Hence, M^T_γ be a DIF SI-ideal in V. □

5.3 DIF-extensions of DIF SI-ideals in BCI-algebras

Definition 5.3.1. *Let P and Q be two IFSs in V. If $\alpha_P(v_1) \geq \alpha_Q(v_1)$ and $\zeta_P(v_1) \leq \zeta_Q(v_1)$ hold for all $v_1 \in V$, then Q is a DIF-extension of P.*

Definition 5.3.2. *Let P and Q be two IFSs in V. Then, Q is termed as a DIF SI-ideal extension of P if the below stated postulates meet:*

(1) Q is a DIF-extension of P,

(2) P is a DIF SI-ideal in V.

Theorem 5.3.1. *Let M be a DIF SI-ideal of V and $\gamma \in [0, \chi]$. Then, the DIF-γ-translation $M^T_\gamma = ((\alpha_M)^T_\gamma, (\zeta_M)^T_\gamma)$ of M is a DIF SI-ideal extension of M.*

Proof. Straightforward. □

But the above theorem may not hold in reverse direction that can be established through the example below.

Example 27. *Let us consider the BCI-algebra V that was taken in Example 25.*

$*$	0	q	r
0	0	r	q
q	q	0	r
r	r	q	0

Let $M = (\alpha_M, \zeta_M)$ be an IFS in V as defined by

V	0	q	r
α_M	0.51	0.59	0.59
ζ_M	0.41	0.36	0.36

which is a DIF SI-ideal in V. Let $N = (\alpha_N, \zeta_N)$ be an IFS in V as defined by

V	0	q	r
α_N	0.5	0.57	0.57
ζ_N	0.43	0.39	0.39

Hence, N is a DIF SI-ideal extension of M yet it is not the DIF-γ-translation of M, $\forall \gamma \in [0, \chi]$.

Theorem 5.3.2. *Let M be a DIF SI-ideal in V and $\gamma, \zeta \in [0, \chi]$. Then, the DIF-$\gamma$-translation $M^T_\gamma = ((\alpha_M)^T_\gamma, (\zeta_M)^T_\gamma)$ of M is a DIF SI-ideal extension of the DIF-ζ-translation $M^T_\zeta = ((\alpha_M)^T_\zeta, (\zeta_M)^T_\zeta)$ of M, when $\gamma \leq \zeta$.*

Proof. Straightforward. □

Theorem 5.3.3. *The intersection of DIF SI-ideal extensions of a DIF SI-ideal M is also a DIF SI-ideal extension of M.*

Let explain the above theorem using the example below.

Example 28. *Consider the BCI-algebra V that was taken in Example 19 as follows:*

$*$	0	s	t	u
0	0	0	0	u
s	s	0	0	u
t	t	t	0	u
u	u	u	u	0

Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V defined by

V	0	s	t	u
α_M	0	0.32	0.52	0.62
ζ_M	0.8	0.58	0.48	0.38

This is a DIF SI-ideal in V. Let $B = (\alpha_B, \zeta_B)$ be an IFS in V as represented by

V	0	s	t	u
α_B	0	0.3	0.5	0.6
ζ_B	1	0.7	0.5	0.4

Also, let $C = (\alpha_C, \zeta_C)$ be an IFS in V as defined by

V	0	s	t	u
α_C	0	0.3	0.5	0.6
ζ_C	0.8	0.6	0.5	0.4

Hence, B and C are DIF SI-ideal extensions of M.

We also assume that $P = B \cap C$ then P is defined as:

V	0	s	t	u
α_P	0	0.3	0.5	0.6
ζ_P	1	0.7	0.5	0.4

Obviously, the intersection of B and C is a DIF SI-ideal extension of M.

5.4 DIF-multiplications and DIF-magnified translations of DIF SI-ideals in BCI-algebras

Definition 5.4.1. *Let M be a DIFS in V and $\beta \in [0,1]$. An object having of the form M_β^M is called a DIF-β multiplication of M if $(\alpha_M)_\beta^M(v_1) = \beta.\alpha_M(v_1)$ and $(\zeta_M)_\beta^M(v_1) = \beta.\zeta_M(v_1)$ for all $v_1 \in V$.*

Theorem 5.4.1. *If $M = (\alpha_M, \zeta_M)$ is a DIF SI-ideal in V, then the DIF-β-multiplication of M is a DIF SI-ideal in V for all $\beta \in [0,1]$, and vice-versa.*

Proof. Straightforward. □

Definition 5.4.2. *Let M be a DIFS in V and $\gamma \in [0,\chi]$ and $\beta \in [0,1]$. An object having of the form $M_{\beta\gamma}^{M\chi}$ is called a DIF-magnified translation of M if $(\alpha_M)_{\beta\gamma}^{M\chi}(v_1) = \beta.\alpha_M(v_1) - \gamma$ and $(\zeta_M)_{\beta\gamma}^{M\chi}(v_1) = \beta.\zeta_M(v_1) + \gamma$ for all $v_1 \in V$.*

Theorem 5.4.2. *If $M = (\alpha_M, \zeta_M)$ is a DIFS in V and $\gamma \in [0,\chi]$ and $\beta \in [0,1]$ then M is a DIF SI-ideal in V if and only if $M_{\beta\gamma}^{M\chi}$ is a DIF SI-ideal in V for all $v_1 \in V$.*

Proof. Let M be a DIF SI-ideal in V and $v_1, v_2, v_3 \in V$. Then,

(1) $\alpha_M(0) \leq \alpha_M(v_1), \zeta_M(0) \geq \zeta_M(v_1)$,

(2) $\alpha_M(v_2 * (v_2 * v_1)) \leq max\{\alpha_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), \alpha_M(v_3)\}$,

(3) $\zeta_M(v_2 * (v_2 * v_1)) \geq min\{\zeta_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), \zeta_M(v_3)\}$.

Now, $(\alpha_M)^{M\chi}_{\beta\gamma}(0) = \beta.\alpha_M(0) - \gamma \leq \beta.\alpha_M(v_1) - \gamma = (\alpha_M)^{M\chi}_{\beta\gamma}(v_1)$, and $(\zeta_M)^{M\chi}_{\beta\gamma}(0) = \beta.\zeta_M(0) + \gamma \geq \beta.\zeta_M(v_1) + \gamma = (\zeta_M)^{M\chi}_{\beta\gamma}(v_1)$, for all $v_1 \in V$.

Again, we obtain

$$\begin{aligned}
(\alpha_M)^{M\chi}_{\beta\gamma}(v_2 * (v_2 * v_1)) &= \beta.\alpha_M(v_2 * (v_2 * v_1)) - \gamma \\
&\leq max\{\beta.\alpha_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), \beta.\alpha_M(v_3)\} - \gamma \\
&= max\{\beta.\alpha_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3) - \gamma, \beta.\alpha_M(v_3) - \gamma\} \\
&= max\{(\alpha_M)^{M\chi}_{\beta\gamma}(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), (\alpha_M)^{M\chi}_{\beta\gamma}(v_3)\}
\end{aligned}$$

and

$$\begin{aligned}
(\zeta_M)^{M\chi}_{\beta\gamma}(v_2 * (v_2 * v_1)) &= \beta.\zeta_M(v_2 * (v_2 * v_1)) + \gamma \\
&\geq min\{\beta.\zeta_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), \beta.\zeta_M(v_3)\} + \gamma \\
&= min\{\beta.\zeta_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3) + \gamma, \beta.\zeta_M(v_3) + \gamma\} \\
&= min\{(\zeta_M)^{M\chi}_{\beta\gamma}(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), (\zeta_M)^{M\chi}_{\beta\gamma}(v_3)\}
\end{aligned}$$

for all $v_1, v_2, v_3 \in V$.

Hence, $M^{M\chi}_{\beta\gamma}$ is DIF SI-ideal in V.

Now, $\beta.\alpha_M(0) - \gamma = (\alpha_M)^{M\chi}_{\beta\gamma}(0) \leq (\alpha_M)^{M\chi}_{\beta\gamma}(v_1) = \beta.\alpha_M(v_1) - \gamma$, and $\beta.\zeta_M(0) + \gamma = (\zeta_M)^{M\chi}_{\beta\gamma}(0) \geq (\zeta_M)^{M\chi}_{\beta\gamma}(v_1) = \beta.\zeta_M(v_1) + \gamma$. That implies, $\alpha_M(0) \leq \alpha_M(v_1)$ and $\zeta_M(0) \geq \zeta(v_1)$.

Again,

$$\begin{aligned}
\beta.\alpha_M(v_2 * (v_2 * v_1)) - \gamma &= (\alpha_M)^{M\chi}_{\beta\gamma}(v_2 * (v_2 * v_1)) \\
&\leq max\{(\alpha_M)^{M\chi}_{\beta\gamma}(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), (\alpha_M)^{M\chi}_{\beta\gamma}(v_3)\} \\
&= max\{\beta.\alpha_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3) - \gamma, \beta.\alpha_M(v_3) - \gamma\} \\
&= \beta.max\{\alpha_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), \alpha_M(v_3)\} - \gamma
\end{aligned}$$

So, $\alpha_M(v_2 * (v_2 * v_1)) \leq max\{\alpha_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), \alpha_M(v_3)\}$.

and,

$$\begin{aligned}
\beta.\zeta_M(v_2 * (v_2 * v_1)) + \gamma &= (\zeta_M)^{M\chi}_{\beta\gamma}(v_2 * (v_2 * v_1)) \\
&\geq min\{(\zeta_M)^{M\chi}_{\beta\gamma}(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), (\zeta_M)^{M\chi}_{\beta\gamma}(v_3)\} \\
&= min\{\beta.\zeta_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3) + \gamma, \beta.\zeta_M(v_3) + \gamma\} \\
&= \beta.min\{\zeta_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), \zeta_M(v_3)\} + \gamma
\end{aligned}$$

So, $\zeta_M(v_2 * (v_2 * v_1)) \geq min\{\zeta_M(((v_1 * (v_1 * v_2)) * (v_2 * v_1)) * v_3), \zeta_M(v_3)\}$, for all $v_1, v_2, v_3 \in V$.

Hence, M is DIF SI-ideal in V.

Thus the proof ends. □

Theorem 5.4.3. *Let M be a DIFS in V and $\gamma \in [0, \chi]$ and $\beta \in [0, 1]$. Then, every DIF-magnified translation $M^{M\chi}_{\beta\gamma}$ of M is a DIF SI-ideal extension of the DIF-β-multiplication M^M_β of M.*

Proof. Let M be an DIFS in V and $\gamma \in [0, \chi]$ and $\beta \in [0, 1]$.

So, $(\alpha_M)^{M\chi}_{\beta\gamma}(v_1) = \beta.\alpha_M(v_1) - \gamma \leq \beta.\alpha_M(v_1) = (\alpha_M)^M_\beta(v_1)$. This implies that, $(\alpha_M)^M_\beta(v_1) \geq (\alpha_M)^{M\chi}_{\beta\gamma}(v_1)$. Again, $(\zeta_M)^{M\chi}_{\beta\gamma}(v_1) = \beta.\zeta_M(v_1) + \gamma \geq \beta.\zeta_M(v_1) = (\zeta_M)^M_\beta(v_1)$. This implies that, $(\zeta_M)^M_\beta(v_1) \leq (\zeta_M)^{M\chi}_{\beta\gamma}(v_1)$. Thus we have, $M^{M\chi}_{\beta\gamma}$ is a DIF-extension of M^M_β.

Now assume that M^M_β is a DIF SI-ideal in V. Then, M is a DIF SI-ideal in V for all $v_1 \in V$. Next, we can prove that $M^{M\chi}_{\beta\gamma}$ of M is a DIF SI-ideal in V, for $\gamma \in [0, \chi]$ and $\beta \in [0, 1]$.

Therefore, every DIF-magnified translation $M^{M\chi}_{\beta\gamma}$ of M is a DIF SI-ideal extension of the DIF β-multiplication M^M_β of M. □

If we put $\beta = 1$ in DIF-magnified translation then it reduces to DIF-translation.

5.5 Summary

In this chapter, we inquired into further properties of DIF SI-ideal in a BCI-algebra. DIF- translation, DIF-extension, DIF-multiplication and DIF-magnified translation of DIFSAs and DIF SI-ideals in BCI-algebras are also introduced here. Relations between DIF- translation of DIF SI-ideal and DIFSAs are presented. Conditions for a DIF-translation of a DIF SI-ideals in a BCI-algebras to be a DIF-translation of

DIF-ideal are provided. Here we show that the DIF-magnified translation of a DIFS is a DIF SI-ideal extension of DIF-multiplication of that DIFS.