2019

MATHEMATICS

[Honours]

PAPER – II

Full Marks : 90

Time : 4 hours

The figures in the right hand margin indicate marks

GROUP – A

(*Real Analysis*)

[*Marks* : 35]

1. Answer any *one* question : 15 × 1

(*a*) (*i*) State and prove Darboux theorem. 5

(*ii*) If a series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges to a

(*Turn Over*)

real number s, then show that the rearranged series

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\cdots\cdots$$

$$+\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}+\cdots\cdots$$

converges to $\frac{s}{2}$. 5

(*iii*) State Bolzano-Weierstrass Theorem. Verify it for the set $S \subset R$ where

$$S=\left(1+\frac{(-1)^n}{n};n\in\mathbb{N}\right).$$ 1 + 4

(*b*) (*i*) For a sequence $\{x_n\}$, if $\lim_{n\to\infty} x_n = l$. prove that

$$\lim_{n\to\infty}\frac{x_1+x_2+\ldots\ldots+x_4}{n}=l.$$

Hence prove that for a sequence

$\{x_n\}$, if $\lim_{n\to\infty} x_n = l$. 5

where $x_n > 0 \quad \forall_n \in \mathbb{N}$, prove that

$$\lim_{n\to\infty} \sqrt[n]{x_1\, x_2 \ldots\, x_n} = l.$$

(*ii*) Prove that if a set A is open set then its complement A^C is closed. 5

(*iii*) State and prove density property in $\mathbb{R}-\{Q\}$. 5

2. Answer any *two* questions : 8 × 2

(*a*) (*i*) Using Taylor's theorem prove that

$$\cos x \geq 1 - \frac{x^2}{2} \text{ for } -\pi < x < \pi. \qquad 5$$

(*ii*) Examine if the set S is closed in $\mathbb{R}$

$$S = \bigcup_{n=1}^{\infty} I_n \text{ , Where}$$

$$I_n = \left\{ x \in R : \left(\frac{1}{3}\right)^n \leq x \leq 1 \right\}. \qquad 3$$

(*b*) State and prove Taylor's theorem when its call *the Cauchy Remainder form.* 6 + 2 (*c*) (*i*) Prove that the function f defined by

$$f(x)=\frac{1}{x^2+1}, x \in \mathbb{R}$$

is uniformly continuous on $\mathbb{R}$ where $\mathbb{R}$ is the set of all real numbers. 4

(*ii*) Let $I=[a, b]$ and a function $f: I \to \mathbb{R}$ be differentiable on I. Let $f'(a) \neq f'(b)$, If k be a real number lying between $f'(a)$ and $f'(b)$ then there exist a point C in (a, b) s.t $f'(c)=k$. 4

3. Answer any *one* question : 4 × 1

(*a*) If $I_n=\int_0^{\pi/2} x^n \sin x \, dx \; (n \in Z^+)$.

Then prove that

$$I_n+(n^2-n)I_{n-2}=n(\pi/2)^{n-1}.$$ 4

(*b*) If $\lim_{x\to 0}\frac{ae^x+be^{-x}+2\sin x}{\sin x+x\cos x}=2$

find the values of a and b. 4

GROUP – B

(*Several Variables and Applications*)

[*Marks* : 20]

4. Answer any *two* questions : 8 × 2

(*a*) (*i*) Show that the function

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

is continuous at (0,0). 4

(*ii*) Prove that the equation of the tangent at the point '*t*' on the curve

$x=a\dfrac{\phi(t)}{f(t)}$ and $y=a\dfrac{\psi(t)}{f(t)}$ may be written as

$$\begin{vmatrix} x & y & a \\ \phi(t) & \psi(t) & f(t) \\ \phi'(t) & \psi'(t) & f'(t) \end{vmatrix}=0.$$ 4

(*b*) (*i*) Let

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

Show that $f(x, y)$ is not differentiable at (0, 0) although $f(x, y)$ is continuous at (0,0) and f_x and f_y both exist at (0,0). 5

(*ii*) Show that the relation $u=x+y-z$, $v=x-y+z$, $w=x^2+y^2+z^2-2yz$ are not independent. Find the relation between them. 3

(*c*) (*i*) The tangents at two points P and Q on the cycloid $x=a(\theta-\sin\theta)$ $y=a(1-\cos\theta)$ are at right angles. Show that if ρ_1 and ρ_2 be the radii of curvature as these points then $\rho_1^2+\rho_2^2=16a^2$. 4

(*ii*) The evolute of the parabola

$\sqrt{x}+\sqrt{y}=\sqrt{a}$ is $27a(x-y)^2$

$=(2x+2y-3a)^3$. 4

5. Answer any *one* question : 4 × 1

(*a*) Find all the asymptotes of

$$x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0.$$ 4

(*b*) If ρ_1 and ρ_2 are the the radii of curvature at two extremities of any chord of the cardiode $r = a(1 + \cos\theta)$ passing through the pole, prove that $\rho_1^2 + \rho_2^2 = \frac{16}{9}a^2$. 4

GROUP – C

(*Analytical Geometry for two Dimensions*)

[*Marks* : 20]

6. Answer any *two* questions : 8 × 2

(*a*) If the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents two parallel straight lines, show that the distance between them is $2\sqrt{\frac{g^2 - ac}{a(a+b)}}$. 8

(*b*) Reduction the equation 8

$$x^2 + 4xy + y^2 - 2x + 2y + 6 = 0$$

to its Canonical form and show that it represents a hyperbola. Find the latus rectum and the equation of the exist of the hyperbola.

(*c*) Show that the auxiliary circle of the conic

$$\frac{l}{r} = 1 - e\cos\theta \text{ is } r^2(e^2 - 1) + 2ler\cos\theta + l^2 = 0.$$ 8

7. Answer any *one* question : 4 × 1

(*a*) Show that the locus of the poles of the normal chords of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$$ 4

(*b*) Find the equation of two conjugate diameters of the hyperbola $4x^2 - 5y^2 = 20$, if one of them passes through the point (1, 8). 4

GROUP – D

(*Differential Equation*-I)

[*Marks* : 15]

8. Answer any *one* question : 15 × 1

(*a*) (*i*) Solve $x^2 y\,dx - (x^3 + y^3)\,dy = 0.$ 5

(*ii*) Reduce the equation $(px^2 + y^2)(px + y) = (p + 1)^2$ to its Clairaut's form by substitution $u = xy$, $v = x + y$ and find its general and singular solutions. 5

(*iii*) Find the eigen values of eigen function of the differential equation

$$\frac{d^2 y}{dx^2} + \lambda y = 0, \ (\lambda \in \mathbb{R})$$

which satisfies the boundary conditions $y(0) = 0$ and $y(\pi) = 0$. 5

(*b*) (*i*) Show that the given equation

$$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{1}{(1-x^2)^2}, 0<x<1$$

is exact and hence solve it. 5

(*ii*) Show that the orthogonal trajectories of

$$\frac{x^2}{a^2}+\frac{y^2}{\lambda+a^2}=1, (\lambda \text{ being arbitrary})$$

is $x^2+y^2+c=2a^2\log x$. 5

(*iii*) Solve by the method of variation of parameters :

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=\frac{e^{-x}}{x^2}.$$ 5