2019

MATHEMATICS

[Honours]

PAPER —I

Full Marks : 90

Time : 4 hours

The figures in the right-hand margin indicate marks

GROUP – A

(*Classical Algebra*)

[*Marks* : 30]

1. Answer any *one* question : 15 × 1

(*a*) (*i*) If $\alpha, \beta, \gamma, \ldots$ be the roots of the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$, then prove that

$$(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2) \ldots. = (1 - p_2 + p_4 - \ldots.)^2 + (p_1 - p_3 + p_5 - \ldots.)^2$$ 5

(*Turn Over*)

(*ii*) If $\alpha = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$ and if r and p be prime to n, then prove that

$$1 + \alpha^p + \alpha^{2p} + \cdots + \alpha^{(n-1)p} = 0.$$ 5

(*iii*) If each of a, b, c, d be greater than 1 then show that

$$8(abcd + 1) > (a+1)(b+1)(c+1)(d+1).$$ 5

(*b*) (*i*) If $(1+x)^n = a_0 + a_1 x + a_2 x^2 + \cdots$, then show that

$$a_0 - a_2 + a_4 - \cdots = 2^{n/2}\cos\frac{n\pi}{4}$$ and

$$a_1 - a_3 + a_5 - a_7 + \cdots = 2^{n/2}\sin\frac{n\pi}{4}$$ 5

(*ii*) If α, β, γ be roots of the equation

$$x^3 - px^2 + qx - r = 0,$$

then form the equation whose roots are

$$\beta\gamma + \frac{1}{\alpha},\ \gamma\alpha + \frac{1}{\beta},\ \alpha\beta + \frac{1}{\gamma}$$ 5

(*iii*) If a, b, c be three positive numbers any two of which are together greater than the third, then show that

$$\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$ 5

2. Answer any *one* question : 8 × 1

(*a*) (*i*) If z be a complex number and $\frac{z+1}{z-i}$ be purely imaginary, then show that z lies on the circle whose centre is at $\frac{1}{2}(-1+i)$ and the radius is $\frac{1}{\sqrt{2}}$. 4

(*ii*) Find the condition that the cubic $x^3-px^2+qx-r=0$ should have its roots in GP. 4

(*b*) (*i*) If one of the roots of the equation

$$x^3+px^2+qx+r=0$$

equals the sum of the other two, then prove that $p^3+8r=4pq$ 4 (*ii*) Find the least value of $(x^{-2} + y^{-2} + z^{-2})$, when $x^2 + y^2 + z^2 = 9$. 4

3. Answer any *one* question : 4 × 1

(*a*) Solve the equation by Cardan's method

$$x^3 + 3x^2 + 6x + 4 = 0.$$ 4

(*b*) Define special roots of an equation. Find the special roots of the equation $x^{12} - 1 = 0$. 4

4. Answer any *one* question : 3 × 1

(*a*) If n be a positive integer, then prove that

$$(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}} \cos\frac{n\pi}{4}.$$ 3

(*b*) If α, β, γ be the roots of the equation $x^3 + x + 1 = 0$, then prove that

$$(\alpha^2 + 1)(\beta^2 + 1)(\gamma^2 + 1) = 1$$ 3

GROUP – B

(*Abstract Algebra*)

[*Marks* : 35]

5. Answer any *three* questions : 8 × 3

(*a*) (*i*) Write the second principle of mathematical induction. Hence prove that the sum of the cubes of 3 consecutive positive integers is divisible by 9. 4

(*ii*) Let $f : A \to B$ and $g : B \to C$ be two bijective functions. Prove that composite function $gof : A \to C$ is a bijective function. 4

(*b*) (*i*) If $a, b, c \in \mathbb{Z}$ such that $a|bc$ and $gcd(a, b) = 1$ then prove that $a|c$. 4

(*ii*) Prove that every group of prime order is cyclic. 4

(c) (i) Define integral domain. Prove that a commutative ring R with unity is an integral domain if and only if cancellation law holds in R. 4

(ii) Prove that intersection of two subgroups is a subgroup of a group. Give an example to show that union of two subgroups may not be a subgroup. 4

(d) (i) Prove that order of each subgroup of a finite group G is a divisor of the order of the Group G. 4

(ii) Define order of an element in a group G. Prove that $0(a) = 0(xax^{-1})$ for any $a, x \in G$. 4

(e) (i) Prove that the number of positive primes is infinite. 4

(ii) Let G be a cyclic group generated by a of order 15. Compute the order of a^3, a^6, a^8 and a^{10}. 4

6. Answer any *two* questions : 4 × 2

(*a*) Define commutative ring. Prove that a ring R is commutative, iff $(a+b)^2 = a^2 + 2ab + b^2$ for every $a, b \in R$. 4

(*b*) Prove that a finite integral domain is a field. 4

(*c*) Prove that any two cycles in S_n are conjugate if they are of the same length. 4

7. Answer any *one* question : 3 × 1

(*a*) A binary relation on R^2 defined by

$$R = \left\{\left((a,b),(c,d)\right) \in R^2 \times R^2 \mid a^2 + b^2 = c^2 + d^2\right\}.$$

Prove that R is an equivalence relation. 3

(*b*) Express the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 2 & 6 & 1 & 3 \end{pmatrix}$$

as a product of transpositions and hence find whether it is odd or even. 3

GROUP – C

(*Linear Algebra*)

[*Marks* : 25]

8. Answer any *one* question : 15 × 1

(*a*) (*i*) Find the basis and dimension of the subspace S of $\mathbb{R}^3$, where

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0, 2x + y + 3z = 0\}$$ 5

(*ii*) If $\Delta (\neq 0)$ be a determinant of order n and Δ' be its adjoint, then $\Delta' = \Delta^{n-1}$. 5

(*iii*) Find third column, so that the matrix

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & * \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & * \\ \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} & * \end{bmatrix}$$

is orthogonal. 5

(*b*) (*i*) Investigate for what values of λ and μ the following equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have (*I*) no solution (*II*) a unique solution and (*III*) an infinite number of solutions. 5

(*ii*) If α and β be vectors in an inner product space, then show that

$$\| \alpha + \beta \|^2 + \| \alpha - \beta \|^2 = 2\| \alpha \|^2 + 2\| \beta \|^2$$ 5

(*iii*) Prove that eigen values of a real symmetric matrix are real. 5

9. Answer any *one* question : 8 × 1

(*a*) (*i*) Find the eigen values and eigen vectors of the matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$ 4

(*ii*) Reduce the following quadratic form to normal form and examine whether the quadratic form is positive definite or not.

$$6x^2 + y^2 + 18z^2 - 4yz - 12zx$$ 4

(*b*) (*i*) Prove that any orthonormal set of vectors in an inner product space is linearly independent. 4

(*ii*) Examine the linear dependence of the set of vectors $\alpha = (1, 2, -3)$, $\beta = (2, -3, 1)$, $\gamma = (-3, 1, 1)$. Hence find the rank of

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 1 & 1 \end{bmatrix}$$ 4

10. Answer any *one* question : 2 × 1

(*a*) State Cayley-Hamilton theorem and verify the theorem for the matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$. 2

(*b*) If A and B are two square matrices of order n, prove that $\text{trace}(AB) = \text{trace}(BA)$. 2