PG/IIIS/MITM-301/19

M.Sc. 3rd Semester Examination, 2019

MATHEMATICS

(*Partial Differential Equations and Generalized Functions*)

PAPER—MITM-301

Full Marks : 50

Time : 2 hours

Answer **all** questions

The figures in the right-hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

Notations and Symbols have their usual meanings

1. Answer any *four* questions from the following : 2×4

(*a*) State the basic existence theorem for Cauchy problem.

(*Turn Over*)

(*b*) Find the solution of $p^2 + q^2 = x + y$.

(*c*) Show by using weak maximum and weak minimum principle that the Dirichlet problem for the Poisson's equation has unique solution.

(*d*) Define characteristic curve and base characteristics of a first order quasi linear PDE.

(*e*) Give an example of a harmonic function in a domain D which has neither a maximum value nor a minimum value in D.

(*f*) Find the adjoint of the differential operator of $L(u) = 5u_{xx} + 8u_{tt} - 2u_x$.

(*g*) Define the domain of dependence of the Cauchy problem for the wave equation.

(*h*) Solve the following :

$$\left(D^2 + 6DD' + D'^2\right)z = 0$$

where $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$.

(*i*) Show that $\delta(-t) = \delta(t)$ where δ is the Dirac delta function.

2. Answer any *four* questions from the following : 4×4

(*a*) Let $u(x, y)$ be an integral surface of the equation $a(x, y)\, u_x + b(x, y)\, u_y + u = 0$, where $a(x, y)$ and $b(x, y)$ are positive differentiable functions in the entire plane. Let the domain D is defined' as $D = \{(x, y) : |x| < 1, |y| < 1\}$.

(*i*) Show that if u be positive on the boundary of D, then it is positive at every point in D.

(*ii*) Suppose that u attains a local minimum (maximum) at a point $(x_0, y_0) \in D$. Find $u(x_0, y_0)$. $2 + 2$

(*b*) Show that the type of a linear second order partial differential equation is invariant under a change of coordinates. 4

(*c*) Let $u \in C^2(D)$ be a function satisfying the mean value property in D. Show that u is harmonic in D. 4

(*d*) Solve the problem d'Alembert's formula : 4

$$u_{tt} - u_{xx} = xt, -\infty < x < \infty, t \geq 0$$
$$u(x, 0) = 0, -\infty < x < \infty$$
$$u_t(x, 0) = e^x, -\infty < x < \infty.$$

(*e*) Find a formal solution of the problem : 4

$$u_{tt} = u_{xx}, 0 < x < \pi, t \geq 0,$$
$$u(0, t) = u(\pi, t) = 0, t \geq 0,$$
$$u(x, 0) = \sin^3 x, 0 \leq x \leq \pi,$$
$$u_t(x, 0) = \sin^2 x, 0 \leq x \leq \pi.$$

(*f*) Prove that every non-negative harmonic function in the disk of radius a satisfies the following : 4

$$\frac{a-r}{a+r} u(0,0) \leq u(r,\theta) \leq \frac{a+r}{a-r} u(0,0).$$

(*g*) Solve : 4

$$\left(x^2D^2 - 2xyDD' + y^2D'^2 - xD + 3yD'\right)u = 8\frac{y}{x}.$$

(*h*) Solve the equation $\Delta u = 0$ in the disc $D = \{(x, y) : x^2 + y^2 < a^2\}$ with the boundary condition $u = 1 + 3\sin\theta$ on the circle $r = a$. 4

3. Answer any *two* questions from the following : 8 × 2

(*a*) (*i*) Find the derivative of the Heaviside unit step function. 2

(*ii*) Establish the d'Alembert's formula of the Cauchy problem for the non-homogeneous wave equation. 6

(*b*) (*i*) State and prove the strong maximum principle for elliptic pole. 5

(*ii*) Transform the Laplace equation in polar coordinates from cartesian coordinates. 3

(*c*) (*i*) Prove that $\delta(at) = \frac{1}{a}\delta(t)$. 3

(*ii*) Solve the following problem : 5

$$u_t = u_{xx} - u,\ 0 < x < 1,\ t \geq 0$$
$$u(0, t) = u_x(1, t) = 0,\ t \geq 0$$
$$u(x, 0) = x(2 - x),\ 0 \leq x \leq 1.$$

(*d*) Consider the equation

$$u_{xx} - 2u_{xy} + 4e^{y} = 0,\ y > 0.$$

(*i*) Find the canonical form of the above equation. 4

(*ii*) Find the general solution $u(x, y)$ of the equation. 2

(*iii*) Find the solution $u(x, y)$ which satisfies $u(0, y) = f(y)$ and $u_x(0, y) = g(y)$. 2

[*Internal Assessment* : 10 Marks]