PG/IS/MTM-103/19

M.Sc. 1st Semester Examination, 2019

MATHEMATICS

(*ODE and Special Functions*)

PAPER –MTM-103

Full Marks : 50

Time : 2 hours

The figures in the right-hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

1. Answer any *four* questions : 2 × 4

(*a*) Examine that whether infinity is a regular singular point for Bessel's differential equation or not.

(*b*) Define the utility of Wronskian in connection with ODE.

(*c*) Find all the singularities of the following differential equation and then classify them :

$$(z-1)\frac{d^2w}{dz^2}+(\cot \pi z)\frac{dw}{dz}+(\operatorname{cosec}^2 \pi z)w=0$$

(*d*) Define ordinary point and regular singular point in connection with a second order differential equation.

(*e*) Define eigenvalues and eigenfunctions associated with Sturm-Liouville problem.

(*f*) When a boundary problem is a Sturm-Liouville problem.

(*g*) Write down the hypergeometric series represented by $F(a, b, c; z)$. Prove that

$$F(1,b,b;z)=\frac{1}{1-z}.$$

(*h*) Show that

$$\int_{-1}^{1} P_n(z)dz=\begin{cases}0, & n\neq 0\\ 2, & n=0\end{cases}$$

where the symbol has the usual meaning.

2. Answer any *four* questions : 4 × 4

(*a*) Show that $J_0(kz)$ where k is a constant, satisfies the differential equation

$$xy''(x)+y'(x)+k^2xy=0.$$

(*b*) Construct Green's function for the differential equation $xy''(x)+y'(x)=0$, with the following conditions : $y(x)$ is bounded as $x\to 0$, $y(1)=ay'(1)$, $a\neq 0$.

(*c*) Prove that

$$\int_{-1}^{1} P_m(z)P_n(z)dz=\frac{2}{2n+1}\delta_{mn},$$

where δ_{mn} and $P_n(z)$ are the Kroneker delta and Legendre's polynomial respectively.

(*d*) Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of $x^2\ddot{y}-2x\dot{y}-4y=0$, for all x in $[0, 10]$. Consider the Wronskian $W(x)=y_1(x)y_2'(x)-y_1'(x)\,y_2(x)$, If $W(1)=1$ *then find the value of* $W(3)-W(2)$.

(*e*) Show that

$$J_{\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\sin(z) \text{ and}$$

$$J_{-\frac{3}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left[-\frac{\cos(z)}{z}-\sin(z)\right].$$

(*f*) Let $P_n(z)$ and $Q_n(z)$ be the two solutions of the Legendre's differential equation. Prove that

$$(1-z^2)\left[P_n(z)\frac{d}{dz}\{Q_n(z)\}-Q_n(z)\frac{d}{dz}\{P_n(z)\}\right]$$

is a constant.

(*g*) If the vector functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ defined as follows :

$$\varphi_1=\begin{bmatrix}\varphi_{11}\\ \varphi_{21}\\ \cdot\\ \varphi_{n1}\end{bmatrix}, \varphi_2=\begin{bmatrix}\varphi_{12}\\ \varphi_{22}\\ \cdot\\ \varphi_{n2}\end{bmatrix}, \ldots \varphi_n=\begin{bmatrix}\varphi_{1n}\\ \varphi_{2n}\\ \cdot\\ \varphi_{nn}\end{bmatrix}$$

be n solutions of the homogeneous linear differential equation $\frac{dx}{dt} = A(t)x(t)$ in the interval $a \leq t \leq b$, then these n solutions are linearly dependent in $a \leq t \leq b$ iff Wronskian $W[\varphi_1, \varphi_2, \ldots, \varphi_n] = 0 \forall t$, on $a \leq t \leq b$.

(*h*) Find the characteristics values and characteristic functions of the Sturm-Liouville problem

$$(x^3 y')' + \lambda xy = 0; \; y(1) = 0, y(e) = 0.$$

3. Answer any *two* questions : 8 × 2

(*a*) (*i*) Deduce the integral formula for confluent hypergeometric function. 3

(*ii*) Find the general solution of the non-homogeneous system

$$\frac{dX}{dt} = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} X + \begin{pmatrix} e^{5t} \\ 4 \end{pmatrix}$$

where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. 5

(*b*) (*i*) Let the Legendre's differential equation is of the form $(1 - z^2)w''(z) - 2zw'(z) + n(n+1)w(z) = 0$. The nth degree polynomial solution $w_n(z)$ such that $w_n(1) = 3$. If

$$\int_{-1}^{1}\left[w_n^2(z) + w_{n-1}^2(z)\right]dz = \frac{144}{15},$$

then find the value of n. 2

(*ii*) If α and β are the roots of the equation $J_n(z) = 0$ then show that

$$\int_0^1 J_n(\alpha z)J_n(\beta z)dz = \begin{cases} 0, & \text{if } \alpha \neq \beta \\ \frac{1}{2}[J_n'(z)]^2, & \text{if } \alpha = \beta \end{cases}$$ 6

(*c*) (*i*) Obtain the first five terms in the expansion of the following function f in terms of Legendre's polynomial

$$f(x) = \begin{cases} 0, & \text{if } -1 < x < 0 \\ x, & \text{if } 0 < x < 1 \end{cases}$$ 4

(*ii*) Prove that

$$\frac{d}{dz}\left[z^{-n}J_n(z)\right] = -z^{-n}J_{n+1}(z),$$

where $J_n(z)$ is the Bessel's function. 4

(*d*) (*i*) Find the general solution of the equation

$$2z(1-z)w''(z) + w'(z) + 4w(z) = 0$$

by the method of solution in series about $z = 0$, and show that the equation has a solution which is polynomial in z. 5

(*ii*) Show that

$$nP_n(z) = zP_n'(z) - P_{n-1}'(z),$$

where $P_n(z)$ denotes the Legendre Polynomial of degree n. 3

[*Internal Assessment* : 10 Marks]
