M.Sc. 1st Semester Examination, 2019

MATHEMATICS

(*Complex Analysis*)

PAPER – MTM-102

Full Marks : 50

Time : 2 hours

Answer **all** questions

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

1. Answer any *four* questions out of *eight* questions : 2×4

(*a*) Show that $f(z) = \ln z$ has a branch point at $z = 0$.

(*b*) Define zero of an analytic function.

(*c*) Define Jordan arc with an example which is not a Jordan arc.

(*d*) $f(z)$ is defined by means of the equation

$$f(z) = \begin{cases} 1 & \text{when } y < 0, \\ 4y & \text{when } y > 0, \end{cases}$$

and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^5$. Find the value of $\int_C f(z)dz$.

(*e*) Write the Taylor's and Laurent's series representation of a function $f(z)$ by stating necessary condition/s for each of the series.

(*f*) Find the Mobious transformation that maps $0, 1, \infty$ to the respective points $0, i, \infty$.

(g) State the Rouches theorem.

(*h*) Find the points at which $w = \sin(z)$ is not conformal.

2. Answer any *four* questions out of *eight* questions : 4 × 4

(*a*) When α is a fixed real number. Show that the function

$$f(z) = r^{\frac{1}{3}} e^{i\theta/3} (r > 0, \alpha < \theta < \alpha + 2\pi)$$

has derivative everywhere in its domain of definition.

(*b*) Prove that $f(z)$ has a pole of order m at $z = 20$ if and only if $\frac{1}{f(z)}$ has a zero of order m at $z = 20$.

(*c*) Show that

$$\log(i^2) \neq 2\log(i) \quad \text{when } \frac{3\pi}{4} < \theta < \frac{11\pi}{4}$$

and that

$$\log(i^2) = 2\log(i) \quad \text{when } \frac{\pi}{4} < \theta < \frac{9\pi}{4}.$$

(*d*) Without evaluating, find an upper bound for the absolute value of the integral $\int_C e^{\bar{z}^2} dz$ where $C : |z| = 1$, traversed in anti clockwise direction.

(*e*) Find the condition(s) where the transformation $w = \dfrac{az+b}{cz+d} (ad - bc \neq 0)$ transforms the unit circle with center at the origin in the w-plane into a straight line in z-plane.

(*f*) State and prove Cauchy's inequality. Use this inequality to prove Liouville's theorem.

(*g*) Using Rouche's Theorem, find the number of zeroes of $z^{10} + a_1 z^4 + a_2 z^3 + a_3 z^2 + a_4 z + a_5 = 0$ in $|z| = 1$ if $|a_1| > |a_2| + |a_3| + |a_4| + |a_5| + 1$.

(*h*) Find the Taylor or Laurent series expansion of the function $f(z) = \dfrac{3}{z(z-i)}$ with center at $z = -I$ and region of convergence : $1 < |z+i| < 2$.

3. Answer any *two* questions out of *four* questions : 8 × 2

(*a*) (*i*) State and prove the Argument Principle.

(*ii*) If z_0 is pole of order $m > 1$ of $f(z)$, then derive the following formula for Residue of $f(z)$ at $z = z_0$

$$\frac{1}{(m-1)!}\lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)\right].$$ 6 + 2

(*b*) (*i*) Using the method of residues, evaluate

$$\int_0^\infty \frac{x^2 dx}{(x^2+1)(x^2+4)}.$$

(*ii*) Let $f(z)$ be continuous in a simply connected open region D and for any closed curve $C \subseteq D$, $\int_C f(s)ds = 0$. Prove that the integral $\int_{z_0}^{z} f(s)ds$ is independent of the path joining z_0 and z. 6 + 2 (*c*) Use the Schwarz-Chritoffel transformation to arrive at the transformation $w = z^m$ $(0 < m < 1)$, which maps the half plane $y \geq 0$ onto the wedge $|w| \geq 0, 0 \leq \arg w \leq m\pi$ and transforms the point $z = 1$ onto the point $w = 1$. 8

(*d*) (*i*) Show that the transformation

$$w = \frac{a}{2}\left(z + \frac{1}{z}\right), \quad a > 0$$

transforms the region $|z| > 1$, $0 < \text{Arg}\{z\} < \pi$ on to the half-plane $Im\{w\} > 0$.

(*ii*) Prove that if $f(z)$ is analytic at z_0, it must be continuous at z_0. 5 + 3

[*Internal Assessment*—10 Marks]