M.Sc. 1st Semester Examination, 2019

MATHEMATICS

(*Real Analysis*)

PAPER —MTM-101

Full Marks : 50

Time : 2 hours

The figures in the right-hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

1. Answer any *four* questions from the following : 2 × 4

   (*a*) When a metric space (X, d) is said to be connected.

   (*b*) Check whether the hyperbola

   $$\left\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\right\}$$

   is path connected or not.

(Turn Over)

(*c*) Show that the product metric space of two complete metric spaces is also complete.

(*d*) For every $\in > 0$ and $f \in L^1(\mu)$, show that

$$\mu\{x \in X : |f(x)| \geq \in\} \leq \frac{1}{\in} \int f \, fd\mu.$$

(*e*) Show that addition of two complex measurable functions on a measurable set X is a measurable function.

(*f*) If $f(x) = 5$ for all $x \in E$, prove that

$$L \int_E f(x) d\mu = 5\mu(E)$$

(*g*) Give an open cover of the set $(0, 1) \bigcup [2, 5]$.

(*h*) If α is continuous and β is of bounded variation on $[a, b]$, show that

$$\int_a^b \alpha d\beta \quad \text{exists.}$$

2. Answer any *four* questions from the following : 4 × 4

(*a*) Show by an example that composition of two functions of bounded variation may not be a function of bounded variation.

(*b*) Let (X, d) be a discrete metric space such that X has at least two elements. Show that (X, d) is not connected.

(*c*) Show that

$$f(x) = 2x \sin \frac{3\pi}{x}, \quad 0 < x \le 1$$
$$= 0 \qquad\qquad , \quad x = 0$$

is not of bounded variation.

(*d*) Show that l^p is separable metric space for $1 \le p < \infty$.

(*e*) Show that every finite sum of real numbers can be expressed as the R-S integral over some interval.

(*f*) Let $f_n : X \to [0, \infty]$ be measurable for $n = 1, 2, 3, \ldots$, $f_1 \ge f_2 \ge f_3 \ge \ldots \ge 0$, $f_n(x) \to f(x)$ as

$n \to \infty$ for every $x \in X$, and $f_1 \in L^1(\mu)$. Show that

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

(*g*) Let μ be a measure on a σ-algebra $\mathscr{M}$. Then show that

$$\mu(A_n) \to \mu(A) \quad \text{as} \quad n \to \infty$$

if $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathscr{M}$ and $A_1 \subset A_2 \subset A_3 \subset \cdots$.

(*h*) If $f : [a, b] \to R$ is continuous and $g : [a, b] \to R$ is monotonically increasing then prove that there is a point c between a and b such that

$$\int_a^b f dg = f(c)[g(b) - g(a)].$$

3. Answer any *two* questions from the following : 8 × 2

(*a*) (*i*) Show that every bounded Riemann integrable function is Lebesgue integrable and the two integrals are equal in this case. 4

(*ii*) Let μ be a measure on a σ-algebra of subsets of X. Show that the outer measure μ^* induced by μ is countably subadditive. 4

(*b*) (*i*) Define Lebesgue integrable of an unbounded function. Evaluate the Lebesgue integral $\int_0^1 f(x)dx$ where

$$f(x) = \frac{35}{14\sqrt[3]{x}}, \quad 0 < x \le 1$$
$$= 0 \quad , \quad x = 0$$ 5

(*ii*) Evaluate the following :

$$\int_{-1}^{3} 8\sin^3 x d(9x^3 + 6[x])$$ 3

(*c*) (*i*) If $\{G_n\}$ is a sequence of connected sets in a metric space with $G_n \cap G_{n+1} \neq \phi$ for all n, then show that

$\bigcup_{n=1}^{\infty} G_n$ is connected. 4

(*ii*) Prove that continuous image of a compact metric space is compact. 4

(*d*) (*i*) Let s_1 and s_2 be non-negative simple measurable functions on a measurable space X. For every $E \in \mathscr{M}$, let $\psi(E) = \int_E s_1 d\mu$. Show that ψ is a measure on $\mathscr{M}$ and

$$\int_X (s_1 + s_2) d\mu = \int_X s_1 d\mu + \int_X s_2 d\mu$$ 5

(*ii*) Let $f : X \to [0, \infty]$ be measurable, $E \in \mathscr{M}$ and $\int_E f d\mu = 0$. Show that $f = 0$ a.e. on E. 3

[*Internal Assessment* : 10 Marks]