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Lattice Points on the Cone $z^2 = 15x^2 - 6y^2$

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ABSTRACT

This paper concerns with the problem of obtaining non-zero distinct integer solutions to the ternary quadratic Diophantine equation representing a cone given by $z^2 = 15x^2 - 6y^2$. Different sets of solutions are presented. A few interesting relations between the solutions and special polygonal numbers are obtained. Given a solution, a formula for generating sequence of solutions is illustrated.

Keywords: Ternary quadratic, homogeneous quadratic, cone, integer solutions.

Mathematical Subject Classification (2010): 11D09

Notations used:

- $t_{m,n}$ - Polygonal Number of rank n with sides m
- p_n^m - Pyramidal Number of rank n with sides m
- $\Pr_n$ - Pronic Number of rank n
- J_n - Jacobsthal Number of rank n
- gn_n - Gnomonic Number of rank n

1. Introduction

There is a great interest for mathematicians since ambiguity in homogeneous and non-homogeneous quadratic Diophantine Equations [1-3]. In this context, one may refer [4-13] for varieties of problems on the quadratic Diophantine equations with two or three variables. In this paper, ternary quadratic equation given by $z^2 = 15x^2 - 6y^2$ is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special numbers are exhibited.

2. Method of analysis

The homogeneous Quadratic Diophantine equation with three unknowns to be solved for its non-zero distinct integral solutions is

$$15x^2 - 6y^2 = z^2 \tag{1}$$

Different patterns of solutions of (1) are presented below.

Pattern –I

Introducing the linear transformations

$$x = X + 6t \ , \ y = X + 15t \ , \ z = 3w \tag{2}$$

in (1), it leads to $X^2 = w^2 + 90t^2$ (3)

which is satisfied by

$$\left.\begin{aligned} t &= 2ab \\ w &= 90a^2 - b^2 \\ X &= 90a^2 + b^2 \end{aligned}\right\} \tag{4}$$

From (2), the corresponding non zero distinct integral solutions to (1) are

$$x(a,b) = 90a^2 + b^2 + 12ab$$

$$y(a,b) = 90a^2 + b^2 + 30ab$$

$$z(a,b) = 270a^2 - 3b^2$$

Properties:

1. $y(a,a+1) - x(a,a+1) = 36t_{3,a}$
2. $3\{y(a,a) - x(a,a)\}$ is a nasty number.
3. $3x(a^2,a+1) + z(a^2,a+1) - 72P_a^5$ is a nasty number.
4. $3x(a,b) + z(a,b) \equiv 0 \pmod{6b}$
5. $z(a+1,a) - 3y(a+1,a) + 6\{t_{4,a} + 15\,\mathrm{Pr}_a\} = 0$

Pattern-II:

Write (3) in the form of ratio as

$$\frac{45t}{X - w} = \frac{X + w}{2t} = \frac{p}{q}, q \neq 0 \tag{5}$$

which is equivalent to system of double equations

$$45qt - px + pw = 0$$

$$2pt - qx - qw = 0$$

Applying the method of cross multiplication, we have

$$t = 2pq$$

$$X = 2p^2 + 45q^2$$

$$w = 2p^2 - 45q^2$$

Substituting the values of X ,t and w in (2), the values of x , y and z are

$$\left.\begin{aligned} x(p,q) &= 2p^2 + 45q^2 + 12pq \\ y(p,q) &= 2p^2 + 45q^2 + 30pq \\ z(p,q) &= 6p^2 - 135q^2 \end{aligned}\right\} \quad (6)$$

Thus (6) represents the non-zero distinct integral solutions to (1).

Properties:

1. $y(p,p+1) - x(p,p+1) - 18\,\mathrm{Pr}_p = 0$
2. $12\{y(p^2,p) - x(p^2,p)\}$ is a Cubical integer.
3. $7\{x(p,p) + y(p,p) - z(p,p)\}$ is a Perfect square.
4. $z(1,q) - 3y(1,q) + 270t_{4,q} + q(J_8 + 5J_2) = 0$
5. $x(1,2p+1) - y(1,2p+1) + 18gn_p = 0$

It is worth to mention here that (5) may also be expressed in the form of ratio as follows:

Ratio 1:

$$\frac{90t}{X+w} = \frac{X-w}{t} = \frac{p}{q}, q \neq 0$$

Ratio 2:

$$\frac{30t}{X-w} = \frac{X+w}{3t} = \frac{p}{q}, q \neq 0$$

Ratio 3:

$$\frac{18t}{X-w} = \frac{X+w}{5t} = \frac{p}{q}, q \neq 0$$

Ratio 4:

$$\frac{15t}{X-w} = \frac{X+w}{6t} = \frac{p}{q}, q \neq 0$$

Ratio 5:

$$\frac{10t}{X-w} = \frac{X+w}{9t} = \frac{p}{q}, q \neq 0$$

Following the procedure presented above , the corresponding Gaussian integer solutions of (1) are given below:

Solution for Ratio 1:

$$x(p,q) = -p^2 - 90q^2 - 12pq$$
$$y(p,q) = -p^2 - 90q^2 - 30pq$$
$$z(p,q) = 3p^2 - 270q^2$$

Solution for Ratio 2:

$$x(p,q) = 3p^2 + 30q^2 + 12pq$$
$$y(p,q) = 3p^2 + 30q^2 + 30pq$$
$$z(p,q) = 9p^2 - 90q^2$$

Solution for Ratio 3:

$$x(p,q) = 5p^2 + 18q^2 + 12pq$$
$$y(p,q) = 5p^2 + 18q^2 + 30pq$$
$$z(p,q) = 15p^2 - 54q^2$$

Solution for Ratio 4:

$$x(p,q) = 6p^2 + 15q^2 + 12pq$$
$$y(p,q) = 6p^2 + 15q^2 + 30pq$$
$$z(p,q) = 18p^2 - 45q^2$$

Solution for Ratio 5:

$$x(p,q) = 9p^2 + 10q^2 + 12pq$$
$$y(p,q) = 9p^2 + 10q^2 + 30pq$$
$$z(p,q) = 27p^2 - 30q^2$$

Remarkable observations

If the non-zero integer triplet (x_0, y_0, z_0) is any solution of (1) then each of the following three triplets of integer based on x_0, y_0 and z_0 also satisfies (1).

Triplet 1: (x_n, y_n, z_0)

Let $x_1 = h - 9x_0, y_1 = 9y_0 + h, z_1 = 9z_0, h \neq 0$ (7)

be the solutions of (1).

Substitute (7) in (1) and performing a few calculations, we have

$$h = 30x_0 + 12y_0$$

and then $x_1 = 21x_0 + 12y_0, y_1 = 30x_0 + 21y_0$ which is written in the form of matrix as

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

where $M = \begin{pmatrix} 21 & 12 \\ 30 & 21 \end{pmatrix}$

Replacing the above process, the general solution (x_n, y_n) to (1) is given by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

The Eigen values of M are $\alpha = 21 + 6\sqrt{10}, \beta = 21 - 6\sqrt{10}$, it is well known that

$$M^n = \frac{\alpha^n}{\alpha - \beta}(M - \beta I) + \frac{\beta^n}{\beta - \alpha}(M - \alpha I)$$

Using the above formula, we have

$$M^n = \frac{1}{12\sqrt{10}} \begin{bmatrix} 6\sqrt{10}[(21+6\sqrt{10})^n + (21-6\sqrt{10})^n] & 12[(21+6\sqrt{10})^n - (21-6\sqrt{10})^n] \\ 30[(21+6\sqrt{10})^n - (21-6\sqrt{10})^n] & 6\sqrt{10}[(21+6\sqrt{10})^n + (21-6\sqrt{10})^n] \end{bmatrix}$$

Thus, the general solution (x_n, y_n, z_n) to (9.10) is given by

$$x_n = \frac{1}{12\sqrt{10}}\left[6\sqrt{10}[(21+6\sqrt{10})^n + (21-6\sqrt{10})^n]x_0 + 12[(21+6\sqrt{10})^n - (21-6\sqrt{10})^n]y_0\right]$$

$$y_n = \frac{1}{12\sqrt{10}}\left[30[(21+6\sqrt{10})^n - (21-6\sqrt{10})^n]x_0 + 6\sqrt{10}[(21+6\sqrt{10})^n + (21-6\sqrt{10})^n]y_0\right]$$

$$z_n = 9^n z_0$$

Triplet 2: (x_n, y_0, z_n)

Let $x_1 = h - x_0, y_1 = y_0, z_1 = z_0 + 4h, h \neq 0$

Following the procedure presented in triplet: 1, the corresponding general solution to (1) is given by

$$x_n = \frac{1}{16\sqrt{15}}\left[8\sqrt{15}[(-31+8\sqrt{15})^n + (-31-8\sqrt{15})^n]x_0 - 8[(-31+8\sqrt{15})^n - (-31-8\sqrt{15})^n]z_0\right]$$

$$y_n = y_0$$

$$z_n = \frac{1}{16\sqrt{15}}\left[-120[(-31+8\sqrt{15})^n - (-31-8\sqrt{15})^n]x_0 + 8\sqrt{15}[(-31+8\sqrt{15})^n + (-31-8\sqrt{15})^n]z_0\right]$$

Triplet 3: (x_0, y_n, z_n)

Let $x_1 = 5x_0, y_1 = 5y_0 + h, z_1 = 3h - 5z_0, h \neq 0$

Repeating the above process, the corresponding general solution to (1) is given by

$$x_n = 5^n x_0$$

$$y_n = \frac{1}{i4\sqrt{6}}\left[-i2\sqrt{6}[(1+i2\sqrt{6})^n + (1-i2\sqrt{6})^n]y_0 + 2[(1+i2\sqrt{6})^n - (1-i2\sqrt{6})^n]z_0\right],$$

$$z_n = \frac{1}{i4\sqrt{6}}\left[-12[(1+i2\sqrt{6})^n - (1-i2\sqrt{6})^n]y_0 - i2\sqrt{6}[(1+i2\sqrt{6})^n + (1-i2\sqrt{6})^n]z_0\right]$$

3. Conclusion

In this paper, we have made an attempt to find different patterns of non-zero distinct integer solutions to the ternary cubic equation, given by $z^2 = 15x^2 - 6y^2$. As ternary quadratic equations are rich in variety, one may search for integer solutions to other choices of quadratic equations with multivariates along with suitable properties.

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