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On H-hyper Connected Spaces

P. Vimala Devi

The Standard Fireworks Rajaratnam College for Women
Sivakasi, Tamilnadu, India
E-mail: vimalkathir15@gmail.com

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ABSTRACT

In this paper, we define H-hyper connected spaces in generalized topological spaces with hereditary class H and we prove the characterizations of H-hyper connected spaces using $\mu-$open, $\mu-$rare and $\sigma-$open sets. Also, we give the relation between $\mu-$hyper connected and H-hyper connected spaces.

Keywords: $\mu-$hyper connected spaces, H-resolvable spaces, H-hyper connected spaces, $\mu-$rare and $\mu-$dense set.

Mathematical Subject Classification (2010): 54D05

1. Introduction

In 2007, Csaszar [1] introduced a class of subsets of a nonempty set called hereditary class and studied modification of generalized topology via hereditary classes. A subfamily μ of $\mathrm{P}(X)$ is called a generalized topology (GT) [2] if $\phi \in \mu$ and μ is closed under arbitrary union. The pair (X,μ) is called a generalized topological space (GTS). The elements of μ are called as $\mu-$ open sets. The compliments of $\mu-$ open sets are called as $\mu-$ closed sets. The largest $\mu-$open set contained in a subset A of X is denoted by $i_\mu(A)$ [1] and is called $\mu-$interior of A. The smallest $\mu-$ closed set containing A is called the $\mu-$closure of A and is denoted by $c_\mu(A)$. If (X,μ) is a generalized topological space, then M_μ denotes the union of all elements of μ [3]. A GTS (X,μ) is said to be strong if $X \in \mu$. A GT μ is said to be a quasi topology on X if $M, N \in \mu$ implies $M \cap N \in \mu$ [4]. A subset A of X is said to be $\mu-$rare or $\mu-$nowhere dense if $i_\mu c_\mu(A)=\phi$. The family of all $\mu-$rare sets is denoted by $\mathrm{H}_r(\mu)$. For a subset A of X, $A\in \sigma$ (resp. $A\in \pi$) if $A \subset c_\mu i_\mu(A)$ (resp. $A \subset i_\mu c_\mu(A)$. A hereditary class H of X is a nonempty collection of subset of X such that $A \subseteq B, B \in H$ implies $A \in H$ [1]. A hereditary class H of X is an ideal [5,6] if $A \cup B \in \mathrm{H}$ whenever $A \in \mathrm{H}$ and $B \in \mathrm{H}$. With respect to the generalized topology μ and a hereditary class H, for each subset A of X, a subset $A^*(\mathrm{H})$ or simply A^* of X is defined by

$A^* = \{x \in X / M \cap A \notin \mathrm{H}\}$ for every $M \in \mu$ containing x [1]. Also, for every $A \subseteq X, c_\mu{}^*(A)$ is defined as $c_\mu{}^*(A) = A \cup A^*$ which induces a GTS μ^* finer that μ and is defined as $\mu^* = \{A \subseteq X / c_\mu{}^*(X - A) = X - A\}$[1]. The elements of μ^* are known as $\mu^* -$open sets and a subset A of (X, μ^*) is called $\mu^* -$closed if X – A is $\mu^* -$open. A subset A of X is said to be $\sigma - \mathrm{H} -$open (resp. $\pi - \mathrm{H} - open$) if $A \subset c_\mu{}^* i_\mu(A)$ (resp. $A \subset i_\mu c_\mu{}^*(A)$)[1]. A hereditary class H is said to be $\mu -$codense [8] if $\pi(\mu) \cap \mathrm{H} = \{\phi\}$. A subset A of a GTS (X, μ) with a hereditary class H is called $\mu -$dense (resp. $\mu^* -$dense) if $c_\mu(A) = X$ (resp. $c_\mu{}^*(A) = X$). A GTS (X, μ) is called $\mu -$hyper connected [7] if every nonempty $\mu -$open set G of X is $\mu -$dense. The following Lemmas will be useful in the sequel and we use some of the results without mentioning it, when the context is clear.

Lemma 1.1. [4, Theorem 2.2] If (X, μ) is a quasi topological space, then $i_\mu(A \cap B) = i_\mu(A) \cap i_\mu(B)$ for $A, B \subset X$.

Lemma 1.2. [8, Theorem 2.1] Let (X, μ) be a GTS with a hereditary class H. Then the following are equivalent.

(a) H is $\mu -$condense.

(b) $X = X^*$

Lemma 1.3. [3] Let (X, μ) be a GTS with a hereditary class H. If $A \in \mathrm{H}$, then $A^* = X - \mathrm{M}_\mu$.

Lemma 1.4. [9, Lemma 1.3] If (X, μ) is a quasi topological space and $A \subset X$, then $G \cap c_\mu(A) \subset c_\mu(G \cap A)$ for every $G \in \mu$.

Lemma 1.5. [3, Theorem 3.8] The set $M - H(M \in \mu, H \in \mathrm{H})$ constitute a base B for μ^*.

Lemma 1.6. [8, Theorem 2.9] Let (X, μ) be a GTS with hereditary classes H and S on X. If $\mathrm{H} \subset S$, then $A^*(S) \subset A^*(\mathrm{H})$ for every subset A of X.

Lemma 1.7. [11, Theorem 2.14] Let (X, μ) be a quasi topological space and H be an ideal. Then $A^* - B^* = (A - B)^* - B^*$.

2. $\mathrm{H} -$Hyper connected Spaces

A subset A of a GTS (X, μ) is said to be $\mathrm{H} -$dense if $A^* = X$.

Theorem 2.1. Every $\mathrm{H} -$dense set is $\mu^* -$dense.

Proof: Let A be a $\mathrm{H}-$ dense subset of X. Then $A^* = X$ and so $c_\mu{}^*(A) = A \cup A^* = X$. Therefore, A is μ^*-dense.

The following example shows that the converse of theorem 2.1 need not be true in general.

Example 2.2. Let $X = \{a,b,c,d\}, \mu = \{\{\phi\},\{a\},\{a,b\},\{b,c\},\{a,b,c\}\}$ and $\mathrm{H} = \{\{\phi\},\{a\},\{b\}\}$.If $A = \{a,b,c\}$, then A is μ^*-dense and $A^* = \{b,c,d\}$.

Definition 2.3. A space (X,μ) with a hereditary class H is called $\mathrm{H}-$hyper connected if every nonempty $\mu-$open set is $\mathrm{H}-$dense.

Clearly, every $\mathrm{H}-$hyper connected space is $\mu-$ hyper connected. Example 2.4 shows that the converse need not be true in general and Theorem 2.5 shows that if X is $\mu-$ hyper connected space with a $\mu-$condense hereditary class H, then X is $\mathrm{H}-$hyper connected.

Example 2.4. Let $X = \{a,b,c,d\}, \mu = \{\{\phi\},\{a\},\{a,b\},\{a,b,c\}\}$ and $\mathrm{H} = \{\{\phi\},\{a\},\{b\}\}$. Clearly, X is $\mu-$ hyper connected space. If $A = \{a\}$, then A is a nonempty $\mu-$open set and $c_\mu{}^*(A) = \{a,d\}$. Hence X is not a $\mathrm{H}-$hyper connected space.

Theorem 2.5. Let (X,μ) be a quasi topological space with a hereditary class H. Then the following are equivalent.

(a) X is $\mathrm{H}-$hyper connected.

(b) X is $\mu-$ hyper connected and H is $\mu-$codense.

Proof: $(a) \Rightarrow (b)$ Suppose X is $\mathrm{H}-$hyper connected. Clearly, X is $\mu-$ hyper connected. Let U be a nonempty set such that $U \in \mu \cap \mathrm{H}$. Since $U \in \mu$, by hypothesis, $U^* = X$. Also, by Lemma 1.3, $U \in \mathrm{H}$ implies that $U^* = X - \mathrm{M}_\mu$. Therefore, $X = X - \mathrm{M}_\mu$ and so $X \cap \mathrm{M}_\mu = \phi$ which implies that $\mathrm{M}_\mu = \phi$, a contradiction to $\phi \neq U \in \mu$ and so $U \notin \mu \cap \mathrm{H}$. Hence H is a $\mu-$codense hereditary class.

$(b) \Rightarrow (a)$ Let $\phi \neq U \in \mu$ and $x \in X$. Suppose $x \notin U^*$. Then there exists a $\mu-$ open set G containing x such that $U \cap G \in \mathrm{H}$. Also, $U \cap G \in \mu$ and so $U \cap G = \phi$. Hence $x \notin c_\mu(U)$, a contradiction to X is $\mu-$ hyper connected. Therefore, $U^* = X$ and hence X is $\mathrm{H}-$hyper connected.

Proposition 2.6. Let (X,μ) be a generalized topological space with a hereditary class H. Then A is $\sigma-\mathrm{H}-$open if and only if there exists $\mu-$open set G such that

$G \subset A \subset c_\mu{}^*(G).$

Proof: Suppose A is $\sigma - \mathrm{H} -$ open. Then $A \subset c_\mu{}^* i_\mu(A).$ Let $G = i_\mu(A).$ Then $i_\mu(A) \subset A \subset c_\mu{}^* i_\mu(A)$ implies that $G \subset A \subset c_\mu{}^*(G).$ Conversely, suppose that there exists $\mu -$ open set G such that $G \subset A \subset c_\mu{}^*(G).$ Then $G \subset A \subset c_\mu{}^* i_\mu(G) \subset c_\mu{}^* i_\mu(A)$ implies that A is a $\sigma - \mathrm{H} -$ open set.

The following Theorem 2.7, Theorem 2.8 and Theorem 2.9 give the properties of $\mathrm{H} -$ hyper connected spaces.

Theorem 2.7. Let (X, μ) be $\mathrm{H} -$ hyper connected. If A contains a nonempty $\mu -$ open set, then A is $\sigma - \mathrm{H} -$ open.

Proof: Let A be a nonempty set such that A contains a nonempty $\mu -$ open set G. Now X is $\mathrm{H} -$ hyper connected implies that $G^* = X$. Hence $c_\mu{}^*(G) = X$. Therefore, $G \subset A \subset c_\mu{}^*(G)$ and so A is $\sigma - \mathrm{H} -$ open, by Proposition 2.6.

Theorem 2.8. If (X, μ, H) is $\mathrm{H} -$ hyper connected and $\mu_1 \subset \mu$ is a GT on X, then (X, μ_1, H) is $\mathrm{H} -$ hyper connected.

Proof: Suppose that (X, μ, H) is $\mathrm{H} -$ hyper connected. Therefore, every nonempty $\mu -$ open set is $\mathrm{H} -$ dense. Let $\mu_1 \subset \mu$. Let G be a $\mu_1 -$ open set and so it is a $\mu -$ open set. Therefore, $G^*(\mu, \mathrm{H}) = X$. Since $\mu_1 \subset \mu$, by Lemma 1.6, $G^*(\mu, \mathrm{H}) \subset G^*(\mu_1, \mathrm{H})$. Therefore, $G^*(\mu_1, \mathrm{H}) = X$. Hence (X, μ_1, H) is $\mathrm{H} -$ hyper connected.

Theorem 2.9. Let (X, μ) be a quasi topological space with an ideal H. Then (X, μ, H) is $\mathrm{H} -$ hyper connected if and only if (X, μ^*, H) is $\mathrm{H} -$ hyper connected.

Proof: Suppose that (X, μ^*, H) is $\mathrm{H} -$ hyper connected . Since $\mu \subset \mu^*,$ by Theorem 2.8, (X, μ, H) is $\mathrm{H} -$ hyper connected. On the other hand , let G be a nonempty $\mu^* -$ open set. Since $\beta = \{U - I : U \in \mu, I \in \mathrm{H}\}$ is a base for $\mu^*, U - I \subset G$ for some $U \in \mu$ and $I \in \mathrm{H}$. Now (X, μ, H) is $\mathrm{H} -$ hyper connected implies that $\mu^* = X$. Therefore, by Lemma 1.7,

$X - I^* = U^* - I^* = (U - I)^* - I^* = (U - I)^* - (X - \mathrm{M}_\mu)$

$= (U - I)^* \cap \mathrm{M}_\mu \subset (U - I)^* \subset G^*.$ Therefore, $(X - (X - \mathrm{M}_\mu)) \subset G^*$ and so $\mathrm{M}_\mu \subset G^*.$ Also, $X - \mathrm{M}_\mu \subset G^*.$ Hence $\mathrm{M}_\mu \cup X - \mathrm{M}_\mu \subset G^*$ and so $X \subset G^*$ which implies that $X = G^*$ where G is nonempty $\mu^* -$ open set. Hence (X, μ^*, H) is $\mathrm{H} -$ hyper connected.

The following Theorem 2.10 gives the characterization of $\mathrm{H}-$hyper connected spaces using $\mu-$open sets

Theorem 2.10. Let (X,μ) be a GTS with a hereditary class H. Then X is $\mathrm{H}-$hyper connected if and only if for each nonempty $\mu-$open sets U and V of X, $U\cap V\notin \mathrm{H}$.
Proof: Let U and V be any nonempty $\mu-$open subset of X. Now X is $\mathrm{H}-$hyper connected implies that $U^{*}=X$. Hence $U\cap V\notin \mathrm{H}$. Conversely, suppose that X is not $\mathrm{H}-$hyper connected. Therefore, there exists a nonempty $\mu-$open set U such that $U^{*}\neq X$. Hence there exists $x\in X$ such that $x\notin U^{*}$ and so there exists a $\mu-$open set V containing x such that $U\cap V\in \mathrm{H}$, a contradiction. Therefore, U is $\mathrm{H}-$dense. Hence X is $\mathrm{H}-$hyper connected.

Theorem 2.11. Let (X,μ) be a GTS with a hereditary class H. If X is $\mathrm{H}-$hyper connected, then the following hold.

(a) $i_{\mu}c_{\mu}{}^{*}(A)=\mathrm{M}_{\mu}$ for every nonempty $\pi-\mathrm{H}-$open set A of X.

(b) Every nonempty $\pi-\mathrm{H}-$open set is $\mu-$dense.

Proof: (a) Let A be a nonempty $\pi-\mathrm{H}-$open set. Then $i_{\mu}c_{\mu}{}^{*}(A)$ is a nonempty $\mu-$open set. Therefore, by hypothesis, $(i_{\mu}c_{\mu}{}^{*}(A))^{*}=X$ and so $c_{\mu}{}^{*}i_{\mu}c_{\mu}{}^{*}(A)=X$. Also, A is a $\pi-\mathrm{H}-$open set implies that $c_{\mu}{}^{*}(A)=c_{\mu}{}^{*}i_{\mu}c_{\mu}{}^{*}(A)$ which in turn implies that $c_{\mu}{}^{*}(A)=X$. Hence $i_{\nu}\,\mathrm{c}_{\mu}{}^{*}(A)=i_{\mu}(X)=\mathrm{M}_{\mu}$.
Therefore, $i_{\mu}c_{\mu}{}^{*}(A)=\mathrm{M}_{\mu}$ for every nonempty $\pi-\mathrm{H}-$open set A of X.
(b)Suppose A is a nonempty $\pi-\mathrm{H}-$open set. Then by (a), $i_{\mu}c_{\mu}{}^{*}(A)=\mathrm{M}_{\mu}$. Also, A is a $\pi-\mathrm{H}-$open set implies that $c_{\mu}(A)=c_{\mu}i_{\mu}c_{\mu}{}^{*}(A)$ and so $c_{\mu}(A)=c_{\mu}i_{\mu}c_{\mu}{}^{*}(A)=c_{\mu}(\mathrm{M}_{\mu})=X$. Therefore, A is $\mu-$dense.

Theorem 2.12. Let (X,μ) be a quasi topological space with a hereditary class H and $A\subset X$. Then the following hold.

(a) $i_{\sigma-\mathrm{H}}(A)=A\cap c_{\mu}{}^{*}i_{\mu}(A)$.

(b) $c_{\sigma-\mathrm{H}}(A)=A\cup i_{\mu}{}^{*}c_{\mu}(A)$.

Proof:

(a) $A\cap c_{\mu}{}^{*}i_{\mu}(A)\subset c_{\mu}{}^{*}i_{\mu}(A)=c_{\mu}{}^{*}(i_{\mu}(A))\cap i_{\mu}c_{\mu}{}^{*}i_{\mu}(A)=c_{\mu}{}^{*}i_{\mu}(A\cap c_{\mu}{}^{*}i_{\mu}(A))$, by Lemma 1.1. Hence $A\cap c_{\mu}{}^{*}i_{\mu}(A)$ is a $\sigma-\mathrm{H}-$open set. Hence $i_{\sigma-\mathrm{H}}(A\cap c_{\mu}{}^{*}i_{\mu}(A))=A\cap c_{\mu}{}^{*}i_{\mu}(A)$.

Now $A \cap c_\mu{}^* i_\mu(A) \subset A$ implies that $i_{\sigma-\text{H}}(A \cap c_\mu{}^* i_\mu(A)) \subset i_{\sigma-\text{H}}(A)$ and so $A \cap c_\mu{}^* i_\mu(A) \subset i_{\sigma-\text{H}}(A)$. Also, $i_{\sigma-\text{H}}(A)$ is $\sigma-\text{H}-$ open implies that $i_{\sigma-\text{H}}(A) \subset c_\mu{}^* i_\mu(i_{\sigma-\text{H}}(A)) \subset c_\mu{}^* i_\mu(A)$ and so $i_{\sigma-\text{H}}(A) \subset A \cap c_\mu{}^* i_\mu(A)$. Hence $i_{\sigma-\text{H}}(A) = A \cap c_\mu{}^* i_\mu(A)$.

(b) $c_{\sigma-\text{H}}(A) = X - i_{\sigma-\text{H}}(X - A)$ Hence by (a),

$c_{\sigma-\text{H}}(A) = X - ((X - A) \cap c_\mu{}^* i_\mu(X - A)) =$

$(X - (X - A) \cup (X - c_\mu{}^* i_\mu(X - A))) = A \cup i_\mu{}^* c_\mu(A)$.

Therefore, $c_{\sigma-\text{H}}(A) = A \cup i_\mu{}^* c_\mu(A)$.

The following Theorem 2.13 gives the characterization of $\mu-$ hyper connected spaces using $\mu-$dense, $\mu-$rare and $\sigma-$open set.

Theorem 2.13. Let (X,μ) be a quasi topological space. Then the following are equivalent.

(a) X is $\mu-$ hyper connected.

(b) For every subset A of X, either A is a $\mu-$dense set or a $\mu-$rare set.

(c) $A \cap B \neq \phi$ for every nonempty $\mu-$open sets A and B.

(d) $A \cap B \neq \phi$ for every $\sigma-$o pen sets A and B of X.

Proof: (a) $\Rightarrow (b)$ Suppose that X is $\mu-$ hyper connected space. Let $A \subset X$. Suppose A is not a $\mu-$rare set. Then $i_\mu c_\mu(A) \neq \phi$. Since $i_\mu c_\mu(A)$ is a nonempty $\mu-$open set by hypothesis $c_\mu \ \mathrm{i}_\mu c_\mu(A) = X$. Hence $X = c_\mu \ \mathrm{i}_\mu c_\mu(A) \subset c_\mu(A)$ implies that A is a $\mu-$dense set

$(b) \Rightarrow (c)$ Let A and B be $\mu-$open sets. Suppose $A \cap B = \phi$. Now by Lemma 1.4, $A \cap c_\mu(B) \subset c_\mu(A \cap B) = c_\mu(\phi) = X - \text{M}_\mu$ implies that $A \cap c_\mu(B) \neq A$. Hence B is not a $\mu-$dense set. Also, $\phi \neq B = i_\mu(B) \subset i_\mu c_\mu(B)$ implies that $i_\mu c_\mu(B) \neq \phi$ and so B is not a $\mu-$rare set, a contradiction to our assumption. Therefore, $A \cap B \neq \phi$.

$(c) \Rightarrow (d)$ Suppose that A and B are nonempty $\sigma-$open sets and $A \cap B = \phi$. Then there exists $\mu-$open sets U and V such that $U \subset A \subset c_\mu(U)$ and $V \subset B \subset c_\mu(V)$. Now $U \cap V \subset A \cap B = \phi$ implies that $U \cap V = \phi$ where U and V are $\mu-$open sets, a contradiction to our assumption. Therefore $A \cap B \neq \phi$.

$(d) \Rightarrow (a)$ Suppose that A and B are $\mu-$open sets. Then A and B are $\sigma-$open sets and so by assumption $A \cap B \neq \phi$. Hence $X \subset c_\mu(A)$ and $X \subset c_\mu(B)$. Therefore, A and B are $\mu-$dense sets. Hence X is $\mu-$ hyper connected.

Theorem 2.14. Let (X,μ) be a quasi topological space. Let the family of all non $\mu-$dense sets is denoted by $\mathcal{H}_0(\mu)$. If X is $\mu-$ hyper connected, then $\mathcal{H}_0(\mu). = \mathcal{H}_r(\mu)$.

Proof: Clearly, $\mathcal{H}_r(\mu) \subset \mathcal{H}_0(\mu)$. If possible, let $A \notin \mathcal{H}_r(\mu)$. Then $i_\mu c_\mu(A) \neq \phi$. Hence there exists $\phi \neq B \in \mu$ such that $B \subset c_\mu(A)$. Clearly, $X - B \in \mathcal{H}_0(\mu)$. Now $c_\mu(A \cup (X - B)) = c_\mu(A) \cup c_\mu(X - B) = c_\mu(A) \cup (X - B) \supset B \cup (X - B) = X$ implies that $A \cup (X - B)$ is $\mu-$dense, a contradiction to Theorem 2.17 of [10]. Hence $\mathcal{H}_0(\mu). = \mathcal{H}_r(\mu)$.

3. Conclusion

The main purpose of this paper was to present $\mathcal{H}-$ hyper connected spaces in generalized topological spaces with hereditary class $\mathcal{H}$, highlighting the properties of $\mathcal{H}-$hyper connected spaces. The possible generalization is plan to extend the study of $\mathcal{H}-$hyper connected spaces using various subsets of generalized topological spaces with hereditary class $\mathcal{H}$.

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