2018

MATHEMATICS

[Honours]

PAPER – V

Full Marks : 90

Time : 4 hours

The figures in the right hand margin indicate marks

Notations have their usual meanings.

[NEW SYLLABUS]

GROUP – A

(*Real Analysis*-II)

1. Answer any *two* questions : 15 × 2

(*a*) (*i*) Show that the necessary and sufficient conditions for the Riemann integrability of a bounded function $f(x)$ is

(*Turn Over*)

$$\lim_{\|P\| \to 0} \{U(P,f) - L(P,f)\} = 0.$$

where $\| P \|$ is the norm of the partition P on $[a, b]$.

(*ii*) Let a function $f:[a, b] \to R$ be integrable on $[a, b]$. Then show that

$$\int_a^b |f(x)| dx \geq | \int_a^b f(x) | dx$$

Give an example to show that the converse of the above result may not be true.

(*iii*) Let f be defined on $[-2, 2]$ by

$$f(x) = 3x^2 \cos\frac{\pi}{x^2} + 2\pi \sin\frac{\pi}{x^2}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

Show that $f(x)$ is integrable on $[-2, 2]$. Evaluate

$$\int_{-2}^{2} f(x) dx.$$ 5 + (3 + 2) + (3 + 2)

(*b*) (*i*) A function $f(x)$ is defined on [0, 1] by

$$f(x) = \frac{1}{2^n}, \quad \frac{1}{2^{n+1}} \le x \le \frac{1}{2^n}, n = 0, 1, \ldots\ldots\ldots$$
$$= 0, \qquad x = 0$$

prove that $f(x)$ is Riemann integrable on [0, 1] and evaluate $\int_0^1 f(x)dx$.

(*ii*) Let a function $f : [a, b] \to R$ be bounded on $[a, b]$ and P be a partition of $[a, b]$. If Q be a refinement of P then show that

$$U(P, f) > U(Q, f) \text{ and } L(P, f) < L(Q, f).$$

(*iii*) State First Mean Value Theorem of Integral calculus. Use this theorem to show that

$$\frac{\pi}{6} \le \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6\sqrt{1-\frac{k^2}{4}}} < 1,$$

where $k^2 < 1$.

(3 + 2) + 5 + (3 + 2)

(c) (i) Let a be the only point of infinite discontinuity of a function f in $[a, b]$ and f be integrable for every ε satisfying $0 < \varepsilon < b - a$. If

$\int_a^b f(x)dx$ is absolutely convergent

and a function ϕ be bounded and integrable on $[a+\varepsilon, b]$ for every ε satisfying $0 < \varepsilon < b - a$ then prove that the integral

$$\int_a^b f(x)\phi(x)dx$$

is absolutely convergent.

(ii) Show that

$$\left| \int_p^q \frac{\sin x}{x} dx \right| \leq \frac{2}{p}, \text{ if } q > p > 0.$$

(iii) Give an example of a function which is Riemann integrable although it has an infinite numbers of points of discontinuity. 5 + 5 + 5

2. Answer any *two* questions : 8 × 2

(*a*) (*i*) If a function $f:[a, b] \to R$ be integrable on $[a, b]$. Then prove that the function F defined by

$$F(x) = \int_a^x f(t)dt, \quad x \in [a, b]$$

is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all x in $[a, b]$.

(*ii*) Prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx^2)}{1+n^3}, \; x \in R$$

is continuously differentiable on R. 4 + 4

(*b*) (*i*) Evaluate $\iint_R f(x,y)\,dx\,dy$ over the rectangle

$R = [0, 1; 0, 1]$ where

$$f(x,y) = \begin{cases} x+y & \text{if } \; x^2 < y < 2x^2 \\ 0 \;, & \text{elsewhere} \end{cases}$$

(*ii*) Suppose that $f(x)$ is continuous on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$ and that

$$\int_a^b f(x)dx = 0$$

Prove that $f(x) = 0$ for all $x \in [a, b]$. 4 + 4

(*c*) (*i*) Evaluate

$$\iiint_V \frac{dx\, dy\, dz}{(x+y+z+1)^3},$$

where V is the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

(*ii*) If $f(x)$ is continuous in $[0, 1]$, then show that

$$\lim_{n\to\infty} \int_0^1 \frac{n\, f(x)}{1+n^2x^2}\, dx = \frac{\pi}{2} f(0).$$ 4 + 4

3. Answer any *one* questions : 4 × 1

(*a*) Examine the following equations for the existence of unique implicit function near the points indicated 4

$$f(x, y) = y^2 - yx^2 - 2x^5, (1, -1).$$

(*b*) Find the maximum and minimum value of the function

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20.$$ 4

Also find the saddle point of the function.

GROUP –B

(*Metric space*)

4. Answer any *one* question : 8 × 1

(*a*) (*i*) Show that every open sphere is an open set but not conversely.

(*ii*) Let (X, d) be any metric space, show that the function d_1 defined by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \forall x, y \in X$$

is a metric on X. 4 + 4

(*b*) (*i*) In a metric space (X, d) if the sequence $\{x_n\}$ and $\{y_n\}$ converges respectively to x and y, then prove that the sequence $\{d(x_n, y_n)\}$ converges to $d(x, y)$.

(*ii*) Prove that any closed subset of a complete metric space is complete. 4 + 4

5. Answer any *one* question : 4 × 1

(*a*) If A and B are any two subsets of a metric space (X, d) then show that

$$\text{Int}(A \cap B) = \text{Int}(A) \cap \text{Int}(B).$$ 4

(*b*) Prove that limit of a sequence in a metric space, if exists, is unique. 4

6. Answer any *one* question : 3 × 1

(*a*) Show that union of arbitrary number of closed set is a closed set. 3

(*b*) When is a metric said to be complete ? Give an example of an incomplete metric space. 1 + 2

GROUP – C

(*Complex Analysis*)

7. Answer any *one* question : 8 × 1

(*a*) (*i*) Prove Cauchy-Riemann equation in polar form.

(*ii*) Show that the real and imaginary parts of an analytic function satisfy Laplaces equation. 4 + 4

(*b*) (*i*) Prove that the function

$$u = \frac{1}{2}\log(x^2 + y^2)$$

is harmonic. Find the corresponding analytic function in terms of z.

(*ii*) Show that the function $e^x(\cos y + i\sin y)$ is holomorphic and find its derivative. 4 + 4

8. Answer any *one* question : 2 × 1

(*a*) Give an example of a function $f(z)$ which is continuous everywhere. Justify. 2

(*b*) Examine whether $f(z) = |z|^2$ is differentiable at the origin or not. 2

GROUP – D

(*Tensor Calculus*)

9. Answer any *one* question : 8 × 1

(*a*) (*i*) Define Kronecker delta and outer multiplication of tensors.

(*ii*) Prove that the partial derivative of a covariant vector B_k is not a tensor.

(2 + 2) + 4

(*b*) (*i*) A covariant tensor of rank one has components xy, $2y = z^2$, xz in rectangular co-ordinates. Find its covariant components in spherical coordinates.

(*ii*) Prove that the contraction of a tensor of type (1, 1) is a scalar. 4 + 4

10. Answer any *one* question : 4 × 1

(*a*) If $(ds)^2 = g_{jk}\, dx^j\, dx^k$, show that g_{jk} is a covariant tensor of rank two. 4

(*b*) Define Christoffel symbol of 1st kind and hence find the value of $[ij, k] + [kj, i]$. 4

11. Answer any *one* question : 3 × 1

(*a*) Show that Ricci tensor is a symmetric tensor. 3

(*b*) Define Riemann Curvature tensor R^h_{ijk} and hence find the value of R^h_{hjk}. 1 + 2
